Length spectrum metric and modified length spectrum metric on Teichmüller spaces

by

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A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

2013

This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirements for the degree of Doctor of Philosophy.

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Abstract

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The length spectrum function defines a metric on the reduced Teichmüller space of a Riemann surface which is topologically equivalent, but not metrically equivalent to the Teichmüller metric if the Riemann surface is of finite topological type.

As the first part of this work, in the reduced Teichmüller space of a Riemann surface of finite topological type, we find two points moving towards the boundary of the space along two continuous curves, such that the Teichmüller distance between them approaches infinity while their length spectrum distance approaches zero. Unfortunately, the length spectrum function does not define a metric on the (unreduced) Teichmüller space of a Riemann surface with boundary. In the second part of this work, we introduce a modified length spectrum function that does define a metric on this space. We show that if two points are close with respect to the Teichmüller metric, then they are also close in the modified length spectrum metric. We also show that the converse is not true. Finally, we prove that the (unreduced) Teichmüller space of a Riemann surface of finite topological type with non-empty boundary is not complete under the modified length spectrum metric.

Acknowledgements

I would like to express my sincere gratitude to my advisor Prof. Jun Hu for all the help and guidance that I received from him during the research and writing of this dissertation. I would also like to thank all other members of the dissertation committee, Prof. Keen, Prof. Gardiner and Prof. Chamanara, for their encouragements and enriching comments.

I acknowledge the financial support from The Graduate Center of the City University of New York and Mexico's National Council of Sciences and Technology (CONACyT) ref. 185971.

Last but not least, I would like to dedicate this dissertation to all my family, especially my daughter for being a constant source of motivation; my wife for her love and patience; and my mother for the support she has given me all my life.

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Chapter 1

Introduction

The Teichmüller space parametrizes the space of marked Riemann surfaces of a fixed topological type. In general, two Riemann surfaces of same topological finite type are not conformally equivalent. Nonetheless, there is always an extremal quasiconformal mapping (i.e., a mapping with the smallest deviation from conformal mappings) between them that determines a natural metric in this space. This metric is called the Teichmüller metric and it measures how distant two marked Riemann surfaces are from being conformally equivalent. Several other metrics have been defined on the Teichmüller space, e.g., the Kobayashi metric and the Weil-Peterson metric. In the present work we are interested in the so-called length spectrum metric. This metric measures the ratio of hyperbolic lengths of closed curves and their images to distinguish between points. It was introduced by Sorvali [24] in 1972 and its relation with the Teichmüller metric has been studied extensively ever since. It is known now, due to the work of Liu [18] in 1999, that the Teichmüller metric and the length spectrum metric define the same topology on the reduced Teichmüller space of Riemann surfaces of finite topological type. Nonetheless, In 2003 Li [17] proved that these two metrics are not bilipschitz on Teichmüller spaces of compact Riemann surfaces, a result that was later generalized by Liu, Sun and Wei [19] to surfaces of any topological type.

In chapter 2, we provide preliminary material and introduce Liu's result [18] on the topological equivalence of the length spectrum metric and the Teichmüller metric. Following the main strategy in [18], we organize the proof in a slightly simpler way.

In chapter 3, we give a continuous version of the results by Li [17] and Liu, Sun and Wei [19]. To prove that the Teichmüller metric d_T and the length spectrum metric d_L are not metrically equivalent, Li constructed two sequences $\{\tau_n\}$ and $\{\tau_n^*\}$ of points in the Teichmüller space $T(S_0)$ of a compact Riemann surface S_0, such that $d_L(\tau_n, \tau_n^*) \to 0$ as $n \to \infty$ but $d_T(\tau_n, \tau_n^*) > c$ for a positive constant c and each n. Then Liu, Sun and Wei improved Li's construction to have $d_L(\tau_n, \tau_n^*) \to 0$ but $d_T(\tau_n, \tau_n^*) \to \infty$ as $n \to \infty$ on the reduced Teichmüller space $T^R(S_0)$ of a Riemann surface S_0 of finite topological type. In their proofs, each point τ_n^* is constructed from τ_n by using a big number (depending on n) of Dehn twists along geodesics. We use partial twists to obtain the continuous version of their result. More precisely, we construct two (disjoint) curves $\alpha_1(t)$ and $\alpha_2(t)$ going to the boundary of $T^R(S_0)$, parameterized by a real parameter $t \in [0, 1)$, such that $d_L(\alpha_1(t), \alpha_2(t)) \to 0$ but $d_T(\alpha_1(t), \alpha_2(t)) \to \infty$ as $t \to 1$.

In chapter 4, we first explain why the length spectrum function does not define a metric on the Teichmüller space of Riemann surfaces with boundary. We change the definition of the length spectrum function by including the comparison on the hyperbolic lengths of certain segments of the geodesics connecting any two points on the boundary and show that this modified length spectrum function defines a metric on this Teichmüller space. Then we study the relation of the new metric with the Teichmüller metric. We prove that if two points are close in the Teichmüller metric, then they are also close in the modified length spectrum metric. We also show that the converse is not true and conclude that these metrics do not define the same topology. Finally, we construct a Cauchy sequence with respect to the modified length spectrum that does not converge. This proves the incompleteness of the modified metric.

Unlike finite cases, the length spectrum metric and the Teichmüller metric are in general not topologically equivalent on Teichmüller spaces of Riemann surfaces of infinite type. This was proved by Shiga [23] in 2003. He also proved that the length spectrum metric is not complete in this case. These results led to introducing length spectrum Teichmüller spaces, which have been studied by Alessandrini, Liu, Pappadopoulos, Su, and Zun in [2] and [3] and by Šarić in [21] and [22]. We know the modified length spectrum function defines a metric on the Teichmüller space of this kind of Riemann surfaces. As a continuation of the research presented in this thesis, one can investigate whether or not the results of this new metric proved for Riemann surfaces of finite topological type can be extended to this case. Similarly, one can also explore modified length spectrum Teichmüller spaces of Riemann surfaces of infinite type.

Chapter 2

Topology induced by the length spectrum metric

2.1 Preliminaries

Let S_0 be a Riemann surface. A marked Riemann surface is a pair (S, f), where $f : S_0 \to S$ is a quasiconformal mapping. Two pairs (S_1, f_1) and (S_2, f_2) are equivalent if there exists a conformal mapping $c : S_1 \to S_2$ such that $c \circ f_1$ is homotopic to f_2. The reduced Teichmüller space $T^R(S_0)$ is the set of such equivalence classes $[S, f]$. The homotopy between $c \circ f_1$ and f_2 is said to be a homotopy modulo the boundary if it agrees with $c \circ f_1$ and f_2 on the boundary of S_1. The set of equivalence classes $[S, f]$ under such a homotopy is called the Teichmüller space $T(S_0)$. Clearly, if S_0 has no boundary, then $T^R(S_0) = T(S_0)$.

The Teichmüller metric on $T^R(S_0)$ (resp. $T(S_0)$) is defined by

$$d_T([S_1, f_1], [S_2, f_2]) = \log K(f),$$

where f is the extremal quasiconformal mapping in the homotopy class (resp. the homotopy class modulo the boundary) of $f_2 \circ f_1^{-1}$ and $K(f)$ represents the maximal dilation of f.

Any Riemann surface inherits a metric from its universal covering space. By comparing the lengths of closed curves and their images, another metric, called the length spectrum metric, is defined on reduced Teichmüller spaces. Let Σ_S be the collection of homotopy classes of closed curves in a Riemann surface S that are not homotopic to a puncture or a point. We will employ an abuse of notation by writing γ to denote either a closed curve or a homotopy class. For any $\gamma \in \Sigma_S$, let $l_S(\gamma)$ denote the length of the geodesic in the free homotopy class γ . The length spectrum metric is defined by

$$d_L([S_1, f_1], [S_2, f_2]) = \log \sup_{\gamma \in \Sigma_{S_1}} \left\{ \frac{l_{S_2}(f_2 \circ f_1^{-1}(\gamma))}{l_{S_1}(\gamma)}, \frac{l_{S_1}(\gamma)}{l_{S_2}(f_2 \circ f_1^{-1}(\gamma))} \right\}.$$

This metric was introduced and studied by Tuomas Sorvali [24] in 1972. In 1975, Sorvali [25] proved that the Teichmüller metric d_T and the length spectrum metric d_L are metrically equivalent on the Teichmüller space of a torus and posed the question as to whether or not this is true in general.

Let Σ'_S be the collection of elements of Σ_S corresponding to simple closed curves. Thurston [27] proved that Σ_{S_1} can be replaced by Σ'_{S_1} in the definition of d_L above. In this work, we will use the collection that suits best to our purposes in a particular section.

The following is a well known result by Wolpert (see [1]).

Lemma 1. *Let $f : S_1 \to S_2$ be a quasiconformal mapping between two hyperbolic Riemann surfaces. Then*

$$\frac{l_{S_2}(f(\gamma))}{l_{S_1}(f(\gamma))} \leq K(f) \textit{ for any } \gamma \in \Sigma_{S_1}.$$

As an immediate consequence of this lemma we have:

Lemma 2. *Let $\tau_1, \tau_2 \in T^R(S_0)$, where S_0 is a hyperbolic Riemann surface. Then*

$$d_L(\tau_1, \tau_2) \leq d_T(\tau_1, \tau_2).$$

It follows from this lemma that the mapping

$$Id : (T^R(S_0), d_T) \to (T^R(S_0), d_L)$$

is continuous. In 1986, Li [16] showed that the inverse mapping is continuous as well if S_0 is a compact hyperbolic Riemann surface, proving that the metrics d_L and d_T are topologically equivalent. This result was later generalized by Liu in [18] to the case in which S_0 is a hyperbolic Riemann surface of type (g, m, k), where g, m and k are the number of genus, punctures and ideal boundaries respectively. In the remaining sections of this chapter we give an expository introduction to Liu's result. Using the main ideas presented in [18], we present a slightly simpler proof on the topological equivalence of d_T and d_L on $T^R(S_0)$.

2.2 On Teichmüller spaces of Riemann surfaces of finite conformal type

Throughout this section we will only consider hyperbolic Riemann surfaces of type (g, m).

A foliation on a Riemann surface S is a covering U_i of S together with maps $\phi_i : U_i \to \mathbb{C}$, such that the transition function on overlapping sets U_i and U_j is of the form $\phi_{ij}(x, y) = (\phi_{ij}^1(x, y), \pm y + c)$, for some constant c. The inverse image of horizontal arcs in $\mathbb{C}$ under the ϕ_i are called the leaves of the foliation. The leaves are 1-dimensional except at isolated singularities where p-pronged saddles occur. The value of p is always greater than 2 except when the singularity is a puncture of the surface, a case in which $p = 1$ is also permissible. A measured foliation F on S is a foliation on S, together with a measure on transverse arcs that is invariant in the sense that it is unchanged by homotopies of arcs that keep each endpoint in the same leaf.

For any $\gamma \in \Sigma_S$, define its intersection number with respect to the measured foliation F by

$$i(\gamma, F) = \inf \int_{\gamma'} dF,$$

where $\int_{\gamma'} dF$ represents the integral of γ with respect to the transverse measure and the infimum ranges over all curves γ' in the homotopy class γ. Two measured foliations F_1 and F_2 are called equivalent if $i(\gamma, F_1) = i(\gamma, F_2)$ for any $\gamma \in \Sigma'_S$. We denote the set of equivalence classes of measured foliations by $\mathrm{MF} = \mathrm{MF}(S)$. An equivalence class determined by F induces a map from Σ'_S onto $\mathbb{R}$ by $i(\cdot, F)$, which is then an element of $\mathbb{R}^{\Sigma'_S}$, and the map $\mathrm{MF} \to \mathbb{R}^{\Sigma'_S}$ given by $F \mapsto (\gamma \mapsto i(\gamma, F))$ is injective. It follows that MF inherits a topology if considered as a subspace of $\mathbb{R}^{\Sigma'_S}$ with the product topology. The definition of measured foliation is totally independent of the complex structure of S. Thus it can be defined on the underlaying topological surface. Nonetheless, the complex structure of S can be used to obtain a characterization of measured foliations via quadratic differentials.

Any nonzero holomorphic quadratic differential $\varphi(z)dz^2$ on S with at most single poles at the punctures of S induces a foliation coming from its horizontal trajectories, i.e., curves along which the quadratic differential satisfies $\varphi(z)dz^2 > 0$. The natural parameter

$$\zeta = u + iv = \int \sqrt{\varphi(z)}dz$$

can be constructed on a sufficiently small neighborhood of any point p of S in such a way that ζ is well defined up to a plus or minus sign. The foliation has an invariant measure induced by the Euclidean distance on the parameter ζ.

Let $\mathrm{QD} = \mathrm{QD}(S)$ denote the space of holomorphic quadratic differentials on S with at worst single poles at the punctures of S. The construction used above gives a map $\Theta : \mathrm{QD}(S) \to \mathrm{MF}(S)$ that sends each quadratic differential to the measured foliation defined by its horizontal trajectories. Further, it was proved by Hubbard and Masur [12] that Θ is a homeomorphism.

It is well known that QD has real dimension $6g - 6 + m$. Since the map Θ above is a homeomorphism, it follows that the projectivizations PMF and PQD of MF and QD, respectively, are compact spaces.

Every closed trajectory γ_0 of a quadratic differential φ is embedded in a unique maximal ring domain R_0 swept out by closed trajectories in the same homotopy class. Two ring domains R_0 and R_1 associated with γ_0 and γ_1 are disjoint or identical.

The quadratic differential φ is said to be of *closed trajectories* if its non closed trajectories cover a set of measure zero. The associated ring domains of such a φ divide S into ring domains swept out by closed trajectories. Strebel [26] proved the following theorem.

Theorem 1. *Let $\gamma_1, ..., \gamma_p$ be a collection of disjoint Jordan curves on an arbitrary Riemann surface S which are mutually disjoint and belong to different homotopy classes. Suppose that the ring domains of homotopy type γ_i have bounded moduli for all i. Then, for arbitrary positive numbers b_i, there exists a holomorphic quadratic differential φ on S with closed trajectories such that its ring domains R_i have homotopy type γ_i and heights b_i, $i = 1, ..., p$.*

The quadratic differential φ is uniquely determined and has finite norm $||\varphi|| = \sum b_i^2/M_i$, where M_i is the modulus of R_i.

Let $\gamma \in \Sigma'_S$ and $r > 0$. By Theorem 1, there exists a unique holomorphic quadratic differential $\varphi_{(\gamma,r)}(z)dz^2$, that we will call the simple quadratic differential associated to (γ, r), such that its closed trajectories are of homotopy type γ. They sweep out a cylinder of height r that covers S up to a set of measure zero. Thus, any pair (γ, r) may be considered as the measured foliation associated to $\varphi_{(\gamma,r)}(z)dz^2$. We will denote such a foliation by $r\gamma$. For $r = 1$, we will employ another abuse of notation by writing γ to refer to the measured foliation induced by $(\gamma, 1)$. Notice that for the measured foliation γ, the intersection number $i(\beta, \gamma)$ coincides with the geometric intersection number between β and γ.

Let ρ be a conformal metric on S, i.e., a metric that is locally of the form $\rho(z)|dz|$, with $\rho \geq 0$. Let $l_\rho(\gamma)$ denote the infimum of the lengths of the simple closed curves of homotopy type γ and set A_ρ to be the area of the S with respect to ρ.

Definition 1. (Analytic) Let $\gamma \in \Sigma'_S$. The extremal length of γ is defined by

$$\mathrm{E}_S(\gamma) = \sup \frac{l_\rho(\gamma)^2}{A_\rho},$$

where the supremum ranges over all conformal metrics ρ in S.

For any element $\gamma \in \Sigma'_S$, let $\mathrm{Mod}(\gamma)$ denote the supremum of the moduli of all annulus in S of homotopy type γ.

Definition 2. (Geometric) Let $\gamma \in \Sigma'_S$. The extremal length of γ is defined by

$$\mathrm{E}_S(\gamma) = \frac{1}{\mathrm{Mod}(\gamma)}$$

Kerckhoff [14] extended the concept of extremal length from Σ'_S to a continuous function $\mathrm{E}_S : \mathrm{MF}(S) \to \mathbb{R}^+$ such that $\mathrm{E}_S(rF) = r^2\mathrm{E}_S(F)$ for any $r > 0$ and any $F \in \mathrm{MF}$. Even more, $\mathrm{E}_S(F) = ||\varphi||$, where φ is the quadratic differential whose measured foliation is equivalent to F. In the same paper, Kerckhoff also proved the following theorem.

Theorem 2. *Let* $[S_1, f_1], [S_2, f_2] \in T(S_0)$. *Then*

$$d_T([S_1, f_1], [S_2, f_2]) = \log \sup_{\gamma \in \Sigma'_{S_1}} \frac{E_{S_2}(f_2 \circ f_1^{-1}(\gamma))}{E_{S_1}(\gamma)}$$

Any measured foliation F induces what is called a geodesic lamination. To construct such an object we straighten all the leaves in F to their corresponding geodesic representatives. Thus all homotopic curves in the foliation collapse to a simple closed geodesic. Each of these simple closed geodesics together with the infinite geodesics coming from non simple closed curves in F form the geodesic lamination. The transverse measure coming from F can be pushed forward to the geodesic lamination to obtain a measured geodesic lamination. The notion of hyperbolic length of a homotopy class $l_S(\gamma)$ can be extended continuously to a function $l_S : \mathrm{MF} \to \mathbb{R}$ such that $l_S(rF) = rl_S(F)$ for any $r > 0$ and $F \in \mathrm{MF}$. This is done by considering the geodesic lamination determined by each $F \in \mathrm{MF}$. In such a geodesic lamination, $l_S(\cdot)$ is locally the product of the transverse measure with the length measure along geodesic leaves.

Lemma 3. *For any* $F \in MF(S)$ *we have*

$$\frac{E_S(F)}{l_S^2(F)} \geq \frac{1}{2\pi|\chi(S)|},$$

where $\chi(S)$ *denotes the Euler characteristic of* S.

Proof. By a well known result by Thurston, the set $\{r\gamma : \gamma \in \Sigma'_S, r > 0\}$ is dense in MF. Thus it is enough to prove that the above inequality holds for any $\gamma \in \Sigma'_S$. Let ρ' denote the hyperbolic metric in S. Then $l_{\rho'}(\gamma) \geq l_S(\gamma)$ and by the Gauss-Bonnet formula $A_{\rho'}(S) = 2\pi|\chi(S)|$. It follows from definition 1 that

$$\mathrm{E}_S(\gamma) \geq \frac{l_S^2(\gamma)}{2\pi|\chi(S)|}$$

for any $\gamma \in \Sigma'_S$. □

Theorem 3. [18] *There exist constants $M_1(S)$ and $M_2(S)$, depending only on S, such that for any $F \in MF(S)$,*

$$M_1(S) \leq \frac{E_S(F)}{l_S^2(F)} \leq M_2(S).$$

Proof. The functions $\mathrm{E}_S(\cdot)$ and $l_S(\cdot)$ are continuous on $\mathrm{MF}(S)$ and take positive values on $\mathrm{MF}(S) \setminus \{0\}$. Moreover, since $\mathrm{E}_S(rF) = r^2\mathrm{E}_S(F)$ and $l_S^2(rF) = r^2 l_S(F)$, the function $E_S(\cdot)/l_S^2(\cdot)$ defines a continuous function on the compact set PMF. Thus it attains a maximum $M_1(S)$ and a minimum $M_2(S)$ which satisfy the statement in the theorem. □

Theorem 4. [18] *Let $[S_1, f], [S_2, f_2] \in T(S_0)$. Then*

$$d_T([S_1, f_1], [S_2, f_2]) \leq C(S_1) + 2d_L([S_1, f], [S_2, f_2]),$$

where $C(S_1)$ is a constant depending only on S_1.

Proof. Let $\gamma \in \Sigma'_S$. By Lemma 3 and Theorem 3, we have

$$\frac{1}{\mathrm{E}_{S_2}(f_2 \circ f_1^{-1}(\gamma))} \leq \frac{2\pi|\chi(S_2)|}{l_{S_2}^2(f_2 \circ f_1^{-1}(\gamma))}, \text{ and}$$

$$\mathrm{E}_{S_1}(\gamma) \leq M_2(S_1) l_{S_1}^2(\gamma).$$

Then by Theorem 2 and the fact that d_T is symmetric we obtain

$$\begin{aligned} d_T([S_1, f_1], [S_2, f_2]) &= \log \sup_{\gamma \in \Sigma'_{S_1}} \frac{\mathrm{E}_{S_1}(\gamma)}{\mathrm{E}_{S_2}(f_2 \circ f_1^{-1}(\gamma))} \\ &\leq \log \sup_{\gamma \in \Sigma'_{S_1}} \frac{2\pi|\chi(S_2)| M_2(S_1) l_{S_1}^2(\gamma)}{l_{S_2}^2(f_2 \circ f_1^{-1}(\gamma))} \\ &= \log 2\pi|\chi(S_2)| M_2(S_1) + 2 \log \sup_{\gamma \in \Sigma'_{S_1}} \frac{l_{S_1}(\gamma)}{l_{S_2}(f_2 \circ f_1^{-1}(\gamma))} \\ &= \log 2\pi|\chi(S_2)| M_2(S_1) + 2 d_L([S_1, f_1], [S_2, f_2]). \end{aligned}$$

Since the Euler characteristic satisfies $\chi(S_2) = \chi(S_1)$, the theorem follows if we let $C(S_1) = \log 2\pi|\chi(S_2)| M_2(S_1)$. □

The following theorem was given by Liu in [18]. He omitted the proof there, but he pointed out that it follows the same argument used by Li in [16] in a similar theorem for compact Riemann surfaces. For the sake of completeness, we include a proof here.

Theorem 5. [18] *Let S_0 be a Riemann surface of finite conformal type. Then the metrics d_T and d_L are topologically equivalent in $T(S_0)$.*

Proof. By Lemma 2, $d_L(\tau_1, \tau_2) \leq d_T(\tau_1, \tau_2)$ for every $\tau_1, \tau_2 \in T(S_0)$. Thus, it is enough to show that any convergent sequence in the metric d_L

converges to the same point in the metric d_T. Let $\{\tau_n\}$ be a sequence in $T(S_0)$ such that $d_L(\tau, \tau_n) \to 0$ as $n \to \infty$. Consider any subsequence $\{\tau_{n_i}\}$ of $\{\tau_n\}$. By Theorem 4 we have

$$d_T(\tau, \tau_{n_i}) \leq C(\tau_1) + 2d_L(\tau_1, \tau_{n_i}).$$

It follows that $\{\tau_{n_i}\}$ is a bounded sequence in the Teichmüller metric. Since $T(S_0)$ is homeomorphic to the unit ball in $\mathbb{R}^{6g-6+2m}$ there exists a subsequence $\{\tau_{n_{i_j}}\}$ of $\{\tau_{n_i}\}$ converging to some point $\tau' \in T(S_0)$. Then by Lemma 2

$$d_L(\tau', \tau_{n_{i_j}}) \leq d_T(\tau', \tau_{n_{i_j}}).$$

Then $d_L(\tau', \tau_{n_{i_j}}) \to 0$ as $j \to \infty$ so we must have $\tau = \tau'$. We have proved that any subsequence of $\{\tau_n\}$ has a subsequence converging to τ in the metric d_T. It follows that $\{\tau_n\}$ converges to τ in the metric d_T. □

2.3 On Teichmüller spaces of Riemann surfaces of finite topological type

All Riemann surfaces considered in this section are of type (g, m, k), with $6g - 6 + m + 3k > 0$ and $k > 0$.

Any Riemann surface S of type (g, m, k), with $k > 0$, can be doubled to obtain a Riemann surface S^d of type $(2g+k-1, 2m, 0)$ in which the boundary curves of S become closed geodesics. The intrinsic metric on S is defined to be the restriction to S of the hyperbolic metric on S^d. The Nielsen kernel $\widetilde{S}$ of S is the Riemann surface of the same type obtained by by removing the k funnels formed by the boundary geodesics and the ideal boundary of S. The surface S is called the Nielsen extension of $\widetilde{S}$ and one of them uniquely determines the other. In fact, S can be recovered from $\widetilde{S}$ as follows: Let $\pi : \mathbb{D} \to \widetilde{S}^d$ be the universal covering map, where $\mathbb{D}$ denotes the unit disk, and let G be the corresponding covering group. If U is a component of $\pi^{-1}(\widetilde{S})$, with stabilizer G' in G, then $S \cong \mathbb{D}/G'$.

Since the collections Σ_S and $\Sigma_{\widetilde{S}}$ are essentially the same, we make no distinction between the corresponding elements in both collections.

Let $\widetilde{f} : \widetilde{S}_1 \to \widetilde{S}_2$ be a quasiconformal mapping. Then $\widetilde{f}$ can be extended to a quasiconformal mapping $f : S_1 \to S_2$ such that $K(f) = K(\widetilde{f})$. The extension is canonical and it can be obtained in the following way: let $\pi_i : \mathbb{D} \to \widetilde{S}_i^d$ be the universal covering map of $\widetilde{S}_i^d$, and let G_i be the corresponding covering group, $i = 1, 2$. Then the double mapping $\widetilde{f}^d : \widetilde{S}_1^d \to \widetilde{S}_2^d$ can be lifted to a mapping $\widetilde{F}^d : \mathbb{D} \to \mathbb{D}$. Let U be a connected component of $\pi_1^{-1}(\widetilde{S}_1)$, and let G_1' be the stabilizer of U in G_1. Then $\widetilde{F}^d(U)$ is a component of $\pi_2^{-1}(\widetilde{S})$ and $G_2' = \widetilde{F}^d G_1' (\widetilde{F}^d)^{-1}$ is its stabilizer in G_2. Hence $\widetilde{F}^d$ induces a mapping $f : \mathbb{D}/G_1' \to \mathbb{D}/G_2'$, but as was mentioned above, $S_i \cong \mathbb{D}/G_i'$.

Bers [4] proved the following result.

Lemma 4. *Let S be a Riemann surface of finite topological type. Then the intrinsic metric on $\widetilde{S}$ and the metric on $\widetilde{S}$ induced by the hyperbolic metric on S are the same.*

For any homotopy class γ in a surface S, let $l^I_S(\gamma)$ denote the length of γ with respect to the intrinsic metric of S. We define a metric on $T^R(S_0)$ similar to d_L using the intrinsic metric:

$$d_{IL}([S_1, f_1], [S_2, f_2]) = \log \sup_{\gamma \in \Sigma_{S_1}} \left\{ \frac{l^I_{S_2}(f_2 \circ f_1^{-1}(\gamma))}{l^I_{S_1}(\gamma)}, \frac{l^I_{S_1}(\gamma)}{l^I_{S_2}(f_2 \circ f_1^{-1}(\gamma))} \right\}.$$

The following theorem is due to Liu [18]. It is a key result that is used to prove the topological equivalence of d_L and d_T. Following the main ideas of the argument given by Liu, we present a slightly different proof.

Theorem 6. [18] *Let $\{[S_n, f_n]\}$ be a sequence in $T^R(S_0)$. Then $\{[S_n, f_n]\}$ converges to $[S, f]$ in the metric d_{IL} if and only if the corresponding sequence $\{[S^d_n, f^d_n]\}$ in $T^R(S^d_0)$ converges to $[S^d, f^d]$ in the metric d_L.*

Proof. Since the intrinsic metric on S_n is the hyperbolic metric induced by the metric on S^d_n, the sufficiency is trivial.

Now we prove the necessity. Let S_0 be decomposed into disjoint pair of pants $\{P_i\}$, $i = 1, 2, ..., k_1$. The pants decomposition gives a collection of decomposition curves $\{L_i\}$, $i = 1, 2, ..., k_2$, a collection of punctures $\{C_j\}$, $j = 1, 2, ..., k_3$, and a collection of ideal boundaries $\{l_k\}$, $k = 1, 2, ..., k_4$. Then we can obtain a pants decomposition for S_0^d given by $\{P_i\} \cup \{\overline{P}_i\}$. The boundary components of this decomposition are given by $\{C_j\} \cup \{\overline{C}_j\} \cup \{l_k\} \cup \{\text{Ł}_i\} \cup \{\overline{L}_i\}$.

The mappings f_n, f_n^d and f induce the corresponding pants decomposition $\{P_i^{(n)}\}$, $\{P_i^{(n)}\} \cup \{\overline{P}_i^{(n)}\}$ and $\{P_i^{(0)}\}$ of S_n, S_n^d and S, respectively.

The intrinsic metric of S_0 induces a metric on any pair of pants P_i and on $\overline{P}_i$, it is induced by the hyperbolic metric of S_0^d. Such a metric is completely determined by the length of its three boundary components. For every Riemann surface, its intrinsic metric is determined by the metric on all the pairs of pants and all the twists about non-punctures and non-ideal boundary components of the pants decomposition. We know, that the twist about a curve is determined by the lengths of some curves that intersect this curve.

Since the sequence $\{[S_n, f_n]\}$ converges to $[S, f]$ in the metric d_{IL}, we have that $l_{S_n}^I(f_n(\gamma))$ converges to $l_S^I(f(\gamma))$ for every $\gamma \in \Sigma_{S_0}$. By picking all the curves in the pants decomposition and sufficiently many curves intersecting the decomposition curves, we know that for $i = 1, 2, ..., k_1$, the intrinsic metric on $P_i^{(n)}$ converges to the metric on $P_i^{(0)}$ and for $i = 1, 2, ..., k_2$, the twist along the curve $f_n(L_i)$ converges to the twist along $f(L_i)$. On the other hand, the hyperbolic metric on S_0^d is determined by the metrics on $\{P_i\} \cup \{\overline{P}_i\}$ and the twists along $\{l_k\} \cup \{L_i\} \cup \{\overline{L}_i\}$. We know that the metrics on P_i and $\overline{P}_i$ are the same; the twists along L_i and $\overline{L}_i$ are identical, while the twists along the curves l_k are all zero. Thus the Fenchel-Nielsen coordinates of $[S_n^d, f_n^d]$ converge to those of $[S^d, f^d]$. Therefore

$$d_T([S_n^d, f_n^d], [S^d, f^d]) \to 0 \text{ as } n \to \infty.$$

It follows from Lemma 2 that

$$d_L([S_n^d, f_n^d], [S^d, f^d]) \to 0 \text{ as } n \to \infty.$$

□

Let $[S_1, f_1]$ and $[S_2, f_2]$ be two points in $T^R(S_0)$. Consider their corresponding points $[S_1^d, f_1^d]$ and $[S_2^d, f_2^d]$ in $T(S_0^d)$. Then (see [1])

$$d_T([S_1, f_1], [S_2, f_2]) = d_T([S_1^d, f_1^d], [S_2^d, f_2^d]).$$

From this equality, Theorem 5 and Theorem 6 we obtain:

Corollary 1. [18] *The metrics d_{IL} and d_T are topologically equivalent on $T^R(S_0)$.*

Theorem 7. *Let $\widetilde{S}_0$ be the Nielsen kernel of S_0. If the sequence $\{[\widetilde{S}_n, \widetilde{f}_n]\}$ converges to $[\widetilde{S}, \widetilde{f}]$ in the Teichmüller metric of $T^R(\widetilde{S}_0)$, then the corre-*

sponding sequence $\{[S_n, f_n]\}$ *converges to* $[S, f]$ *in the Teichmüller metric of* $T^R(S_0)$.

Proof. As was mentioned above, any mapping $\widetilde{h}_n : \widetilde{S}_n \to \widetilde{S}$ homotopic to $\widetilde{f} \circ \widetilde{f}_n^{-1}$ can be extended to a mapping $h_n : S_n \to S$ homotopic to $f \circ f_n^{-1}$ such that $K(h_n) = K(\widetilde{h}_n)$. It follows that for each $n = 1, 2, ...$ we have that $d_T([S_n, f_n], [S, f]) \leq d_T([\widetilde{S}_n, \widetilde{f}_n], [\widetilde{S}, \widetilde{f}])$ and thus the sequence $\{[S_n, f_n]\}$ must converge to $[S, f]$. □

Theorem 8. [18] *Let* S_0 *be any Riemann surface of finite topological type. Then the metrics* d_L *and* d_T *are topologically equivalent on* $T^R(S_0)$.

Proof. By Lemma 2, it is enough to show that any convergent sequence in the metric d_L converges to the same point in the metric d_T. Let $\{[S_n, f_n]\}$ be a sequence that converges to a point $[S, f]$ in the metric d_L. The sequence $\{[S_n, f_n]\}$ induces a sequence $\{[\widetilde{S}_n, \widetilde{f}_n]\}$ in $T^R(\widetilde{S}_0)$. Using Lemma 4, the metric d_{IL} in $T^R(\widetilde{S}_0)$ can be viewed as induced by the metric d_L in $T^R(S_0)$. Then, by Corollary 1 the sequence $\{[\widetilde{S}_n, \widetilde{f}_n]\}$ converges to the corresponding point $[\widetilde{S}, \widetilde{f}]$ in the Teichmüller metric d_T. Thus by Theorem 7 the sequence $\{[S_n, f_n]\}$ converges to $[S, f]$ in the Teichmüller metric d_T. □

In [18], Liu's does not use Theorem 7 to prove Theorem 8. Here, by using it, we are able to give a simpler argument.

Chapter 3

Comparison between the Teichmüller metric and the length spectrum metric

In 2003 Li [17] proved that the metrics d_L and d_T are not metrically equivalent on the Teichmüller space of a compact Riemann surface of genus $g \geq 2$. In order to prove this, he first constructed a sequence $\{\tau_n = [S_n, f_n]\}$ of points in $T(S_0)$ such that each S_n contains a closed curve β_n with $\lim_{n\to\infty} l_{S_n}(\beta_n) = 0$. Then, by taking $[1/l_{S_n}(\beta_n)]$ number of Dehn twists along each β_n, he obtained another sequence $\{\tau_n^*\}$ of points in $T(S_0)$ such that $d_T(\tau_n, \tau_n^*) \geq d > 0$ for n sufficiently large. On the other hand, since $l_{S_n}(\beta_n) \to 0$, it follows from the Collar Lemma [13] that the effect of that number of Dehn twists on the hyperbolic length of any curve crossing β_n becomes smaller and smaller as n approaches ∞. This means $d_L(\tau_n, \tau_n^*) \to 0$ as $n \to \infty$.

Liu, Sun, and Wei [19], generalized and improved Li's result to the reduced Teichmüller space of any non-elementary Riemann surface by finding two sequences $\{\tau_n\}$ and $\{\tau_n^*\}$ such that $d_T(\tau_n, \tau_n^*) \to \infty$ as $n \to \infty$, while $d_L(\tau_n, \tau_n^*) \to 0$ as $n \to \infty$. Their construction follows Li's idea but uses more Dehn twists to obtain the sequence $\{\tau_n^*\}$. In their constructions, each τ_n^* can be represented by a quasiconformal mapping from S_n onto itself, which comes from cutting S_n along β_n and then gluing the two copies of β_n back after twisting one of these copies a multiple of three hundred and sixty degrees. This feature makes it feasible to estimate the length spectrum metric. This way of finding a representation of a new point in Teichmüller space no longer holds when partial twists are used. On the other hand, the length spectrum metric can also be estimated between τ_n and τ_n^* when partial twists are performed along β_n. This suggests the idea of connecting the points in $\{\tau_n^*\}$ continuously by partial twists. In this chapter, we provide details of such constructions and obtain continuous versions of the results in [17] and [19]. More precisely, we prove the following theorem.

Theorem 9. *Let S_0 be a Riemann surface of type (g, m, k) with $g \geq 2$. Then there exist three curves $\alpha(t)$, $\alpha^*(t)$ and $\hat{\alpha}(t)$, $0 \leq t < 1$, in $T(S_0)$, ($T^R(S_0)$ if $k > 0$) that satisfy:*

1. $\lim_{t\to 1} d_T(\alpha(t), \alpha^*(t)) = \infty$, *while* $\lim_{t\to 1} d_L(\alpha(t), \alpha^*(t)) = 0$, *and*

2. *there exist* $M, m > 0$ *such that* $m \leq d_T(\alpha(t), \hat{\alpha}(t)) \leq M$ *for all* $0 \leq t < 1$, *while* $\lim_{t\to 1} d_L(\alpha(t), \hat{\alpha}(t)) = 0$

In particular, if S_0 *has empty boundary, the curve* α *can be chosen to be a Teichmüller geodesic ray.*

This result is presented in [9].

3.1 Earthquakes and the cross-ratio distortion norm

Earthquakes were introduced by Thurston in [28] to measure the difference between two conformal structures on a surface or two points in the Teichmüller space of a Riemann surface. For background on earthquake maps and relations with quasisymmetric circle homeomorphisms, we refer to [6], [7], [8], or [20].

A geodesic lamination $\mathcal{L}$ on $\mathbb{D}$ is a collection of hyperbolic geodesics that foliates a closed subset of $\mathbb{D}$. The geodesics in $\mathcal{L}$ are called leaves, whereas the connected components of $\mathbb{D} \backslash \mathcal{L}$ are called gaps. The strata of the geodesic lamination $\mathcal{L}$ consists of gaps and leaves. A generalized left earthquake map $\widetilde{E} : \mathbb{D} \to \mathbb{D}$ supported on $\mathcal{L}$ is a possibly discontinuous injective map which is an isometry on each stratum of $\mathcal{L}$. Even more, for any two strata $A \neq B$, the comparison isometry

$$\mathrm{Comp}(A, B) = (\widetilde{E}|_A)^{-1} \circ (\widetilde{E}|_B) \tag{3.1}$$

is a hyperbolic translation whose axis weakly separates A and B and which translates B to the left as viewed from A.

If in addition the map $\widetilde{E}$ is surjective, we call it a left earthquake map.

Thurston [28] showed that each left earthquake map $\widetilde{E} : \mathbb{D} \to \mathbb{D}$ extends to a map defined on $\mathbb{D} \cup \mathbb{S}^1$. The extension is continuous at each point of $\mathbb{S}^1$ and $\widetilde{E}|_{\mathbb{S}^1}$ is a homeomorphism. Conversely, every circle homeomorphism can be realized in this way.

Theorem 10. *Let h be an orientation preserving homeomorphism on the unit circle $\mathbb{S}^1$. Then there exists a geodesic lamination $\mathcal{L}$ and a left earthquake map $\widetilde{E} : \mathbb{D} \to \mathbb{D}$ along the leaves in $\mathcal{L}$ such that $\widetilde{E}|_{\mathbb{S}^1} = h$. The geodesic lamination is uniquely determined by h. Moreover, h determines the isometries on $\widetilde{E}$ on all gaps, and for any leaf L in $\mathcal{L}$, two possibly different isometries on L only differ by a hyperbolic isometry with axis L and translation length between* 0 *and the limit value of the translation lengths of the comparison maps for E on the two sides of L.*

Each generalized earthquake map $\widetilde{E} : \mathbb{D} \to \mathbb{D}$ along the leaves in $\mathcal{L}$ induces a transverse measure σ, called an earthquake measure on $\mathcal{L}$. The measure σ quantifies the amount of relative shearing along the geodesic lamination of the earthquake map.

The following result is due to Thurston [28].

Theorem 11. *Let σ be a transverse measure defined on a geodesic lamination $\mathcal{L}$. Then there exists a generalized earthquake map $\widetilde{E} : \mathbb{D} \to \mathbb{D}$ supported on $\mathcal{L}$ such that σ is the induced earthquake measure by $\widetilde{E}$. Moreover, up to post-composition by a hyperbolic isometry, σ determines the isometries of $\widetilde{E}$ on all gaps, and for any leaf L in $\mathcal{L}$, two possible isometries on L only differ by a hyperbolic isometry with axis L and translation length between 0 and the measure $\sigma(L)$ of L.*

In order to determine whether or not a generalized earthquake map is indeed an earthquake map we need to introduce the concept of the Thurston norm.

Definition 3. Let $\mathcal{L}$ be a geodesic lamination with a transverse measure σ; the Thurston norm of σ is defined to be

$$||\sigma||_{\mathrm{Th}} = \sup \sigma(\alpha),$$

where the supremum is taken over all closed geodesic segments α of length 1 that are transverse to the geodesic lamination.

We say that σ is Thurston bounded if $||\sigma||_{\mathrm{Th}}$ is bounded from above. An earthquake map is called Thurston bounded if the induced earthquake measure is Thurston bounded.

Thurston [28] outlined the proofs of following theorems. Three different complete proofs are given in [6], [7] and [20].

Theorem 12. *Let $\mathcal{L}$ be a geodesic lamination with transverse measure σ. If σ is Thurston bounded, then there exists an earthquake map $\widetilde{E} : \mathbb{D} \to \mathbb{D}$ supported on $\mathcal{L}$ such that σ is the induced earthquake measure by $\widetilde{E}$. Moreover, up to post-composition by a hyperbolic isometry, σ determines the isometries of $\widetilde{E}$ on all gaps, and for any leaf L in $\mathcal{L}$, two possible isometries on L only differ by a hyperbolic isometry with axis L and translation length between* 0 *and the measure $\sigma(L)$ of L.*

Theorem 13. *Let h be an orientation-preserving circle homeomorphism and σ the earthquake measure induced by h. Then σ is Thurston bounded if and only if h is quasisymmetric.*

For situations handled in this work, we are only interested in discrete geodesic laminations, i.e., geodesic laminations consisting of countably many hyperbolic geodesics without any accumulation in $\mathbb{D}$. From now on, we assume that this is the case. We now construct generalized earthquakes for these types of geodesic laminations with transverse measures.

Let $\mathcal{L}$ be discrete geodesic lamination. Two gaps are called neighbors if there is a geodesic in $\mathcal{L}$, called the separating geodesic, that belongs to the boundary of both gaps. Given any two gaps A and B, there exists a unique chain of gaps $A_0 = A$, $A_1, A_2, ..., A_{n+1} = B$ such that A_i and A_{i+1} are neighbors. In this sense, the pattern of neighboring gaps determined by $\mathcal{L}$ is a tree.

Let σ be a measure on a geodesic lamination $\mathcal{L}$, i.e., σ assigns a positive number, called a weight, to each element of $\mathcal{L}$. Fix a gap A of the geodesic lamination $\mathcal{L}$ and define $\widetilde{E}|_A = Id$. Let B be any other gap and let $A_0 = A, A_1, ..., A_{n+1} = B$ be the chain of neighboring gaps between A and B. For each $i = 1, 2, ..., n+1$, let T_i be the hyperbolic translation whose axis is the separating geodesic L_i between A_{i-1} and A_i and that translates A_i to the left a distance λ_i, where λ_i is the weight assigned to L_i. Define $\widetilde{E}|_B = T_1 \circ T_1 \circ ... \circ T_{n+1}$. Since any two gaps are connected by a unique chain of neighboring gaps, this construction gives a map defined on the whole hyperbolic plane. Of course, this construction depends on the gap A, nonetheless, any two maps constructed in this way differ only by pre-composition of a hyperbolic transformation. It is easy to see that this map is injective and that the comparison isometry satisfies condition (3.1), however, $\widetilde{E}$ might not be surjective.

A Thurston bounded earthquake map is a quasi-isometry on $\mathbb{D}$ with respect to the hyperbolic metric. The Thurston norm of the earthquake measure is a quantifier of the quasi-isometry. On the other hand, the cross-ratio distortion norm is a quantifier of the quasisymmetry of the boundary homeomorphism determined by the earthquake map. Now we introduce the definition of the cross-ratio distortion norm and we give its quantitative relationship with the Thurston norm of the measure.

Definition 4. Let h be an orientation-preserving homeomorphism of the unit circle $\mathbb{S}^1$. The cross-ratio distortion norm $||h||_{\mathrm{cr}}$ of h is defined as

$$||h||_{\mathrm{cr}} = \sup_{Q} \left| \ln \frac{\mathrm{cr}(h(Q))}{\mathrm{cr}(Q)} \right|,$$

where the supremum is taken over all quadruples $Q = \{a, b, c, d\}$ of four points arranged in counter-clockwise order on the unit circle with $\mathrm{cr}(Q) = 1$, and where

$$\mathrm{cr}(Q) = \frac{(b-a)(d-c)}{(c-b)(d-a)}.$$

Theorem 14. *There exists a universal constant $C' > 0$ such that for any measured geodesic lamination $(\mathcal{L}, \sigma)$ we have*

$$\frac{1}{C'} ||\widetilde{E}|_{\mathbb{S}^1}||_{\mathrm{cr}} \leq ||\sigma||_{\mathrm{Th}} \leq C' ||\widetilde{E}|_{\mathbb{S}^1}||_{\mathrm{cr}}.$$

The right inequality in the previous theorem was proved by Gardiner, Hu and Lakic in [6] while the left inequality was proved by Hu in [7].

For any orientation-preserving homeomorphism f of the unit circle $\mathbb{S}^1$ let $\Phi(f)$ and $K_e(f)$ denote its Douady-Earle extension (see [5]) and the infimum of maximal dilatations of quasiconformal extensions of f respectively. Hu and Muzician [11] proved the following two theorems.

Theorem 15. *There exists a universal constant $C'' > 0$ such that for any quasisymmetric circle homeomorphism f*

$$\ln K(\Phi(f)) \leq C''||f||_{\mathrm{cr}}.$$

Theorem 16. *Let f be a quasisymmetric homeomorphism. Then*

$$\lim_{||f||_{\mathrm{cr}}\to\infty} \frac{||f||_{\mathrm{cr}}}{K_e(f)} \leq \pi.$$

Proposition 1. *Let $\{f_n\}$ be a sequence of orientation-preserving homeomorphisms of $\mathbb{S}^1$ such that $K_e(f_n) \to 1$ as $n \to \infty$. Then $||f_n||_{\mathrm{cr}} \to 0$ as $n \to \infty$.*

Proof. Suppose that the proposition is not true. By passing to a subsequence we may assume that there exists $\varepsilon > 0$ such that

$$||f_n||_{\mathrm{cr}} > \varepsilon \text{ for all n.} \tag{3.2}$$

For each n, choose a quadrilateral Q_n such that

$$|\ln \mathrm{cr}(f_n(Q_n))| > \varepsilon.$$

We may assume, by post-composing and pre-composing by a Möbius transformation, that Q_n and $f_n(Q_n)$ are the quadrilaterals in $\mathbb{H}$ given by $\{-1, 0, 1, \infty\}$ and $\{x_n, 0, 1, \infty\}$ respectively. For each n, the conformal modulus $\mathrm{Mod}(Q_n)$ of Q_n is 1 and

$$\frac{\mathrm{Mod}(Q_n)}{K_e(f_n)} \leq \mathrm{Mod}(f_n(Q_n)) \leq K_e(f_n)\mathrm{Mod}(Q_n).$$

Since by assumption $K_e(f_n) \to 1$ as $n \to \infty$. We must have $x_n \to -1$ as $n \to \infty$, which in turn implies that $\mathrm{cr}(f_n(Q_n)) \to 1$, contradicting (3.2). $\square$

3.2 Partial twists on Riemann surfaces

Let β be a simple closed geodesic on a Riemann surface S that is not homotopic to a boundary component of S. As we pointed out at the beginning of this chapter, the point created by a Dehn twist along β is described by a quasiconformal mapping from S onto S, which comes from cutting S along β and then gluing the two copies of β back after twisting one copy of β three hundred and sixty degrees. The first goal of this section is to introduce how partial twists (not equal to multiples of 360^o) along β create new points in the Teichmüller space. This is related to conjugations of the Fuchsian group representing S by earthquake maps corresponding to the partial twists.

Let $\pi : \mathbb{D} \to S$ be the universal covering map. The preimage $\pi^{-1}(\beta)$ consists of a union of non intersecting geodesics. By assigning the weight λ to each one of these geodesics we obtain a discrete measured geodesic lamination $(\mathcal{L}, \sigma_\lambda)$ on $\mathbb{D}$. The Thurston norm $\|\sigma_\lambda\|_{\mathrm{Th}}$ is finite. For if α is any closed geodesic segment of length 1 transversal to the geodesic lamination and $D > 0$ is the length of a collar neighborhood around β, then $\sigma_\lambda(\alpha) = \lambda$ if α intersects only one leaf of the geodesic lamination $\mathcal{L}$. On the other hand, if α intersects $n > 1$ leaves of the geodesic lamination $\mathcal{L}$ we have $1 = l(\alpha) \geq nD/2$. Thus $\sigma_\lambda(\alpha) \leq \frac{\lambda}{D/2}$. It follows that

$$\lambda \leq \|\sigma_\lambda\|_{\mathrm{Th}} \leq \max\left\{\lambda, \frac{\lambda}{D/2}\right\}. \tag{3.3}$$

It follows from Theorems 12 and 13 that the generalized left earthquake map $\widetilde{E} : \mathbb{D} \to \mathbb{D}$ defined by $(\mathcal{L}, \sigma_\lambda)$ is onto and that $\widetilde{E}|_{S^1} : \mathbb{S}^1 \to \mathbb{S}^1$ is a quasisymmetric homeomorphism of the unit circle.

Let G be the Fuchsian group uniformizing S, then for each element $g \in G$, the map $\widetilde{E}g\widetilde{E}^{-1}$ is conformal on every gap of the geodesic lamination $\mathcal{L}$. Moreover, it is continuous on $\mathbb{D}$. In order to prove this, consider a point z in a geodesic L' of the geodesic lamination $E(\mathcal{L})$ and let V be a hyperbolic disk centered at z. since $\widetilde{E}^{-1}$ is a right earthquake, it splits V into two half disks and moves each one to the right with respect to the other. Because g^{-1} is an isometry, it maps $\widetilde{E}^{-1}(V)$ into another two half disks separated by the same distance. These half disks lie on a geodesic of the geodesic lamination $\mathcal{L}$. Finally, since $\widetilde{E}$ is a left earthquake, it maps $(g^{-1} \circ \widetilde{E}^{-1})(V)$ into a connected open set W and the continuity of $\widetilde{E} g \widetilde{E}^{-1}$ follows. We conclude that for every $g \in G$ the mapping $\widetilde{E} g \widetilde{E}^{-1}$ is conformal on $\mathbb{D}$ and that it defines an isomorphism of G onto the Fuchsian group $G' = \widetilde{E} G \widetilde{E}^{-1}$.

The Douady-Earle extension $\Phi(\widetilde{E}|_{\mathbb{S}^1}) : \mathbb{D} \to \mathbb{D}$ of $\widetilde{E}|_{\mathbb{S}^1}$ is a quasiconformal mapping. Moreover, $\Phi(\widetilde{E}|_{\mathbb{S}^1})$ and $\widetilde{E}$ induce the same group isomorphism between G and G'. This is due to the fact that the Douady-Earle extension of a homeomorphism of the unit disk is conformally natural, i.e., $\Phi(T_1 \circ f \circ T_2) = T_1 \circ \Phi(f) \circ T_2$ for any homeomorphism f of the unit circle and any two isometries T_1 and T_2 of $\mathbb{D}$ (for a proof of this result we refer to [5]). Then $\Phi(\widetilde{E}|_{S^1})$ projects to a quasiconformal homeomorphism $E_{\beta,\lambda} : S \to S'$ where $S' = \mathbb{D}/G'$. We call this map *the earthquake map* of length λ along β.

The following results will be useful to determine how the length spectrum changes under earthquake maps on a Riemann surface.

Lemma 5. *Let g_1, g_2 be two hyperbolic transformation of the unit disk $\mathbb{D}$ with axes L_1, L_2 and translation lengths $\tau(g_1), \tau(g_2)$ respectively. If L_1 intersects*

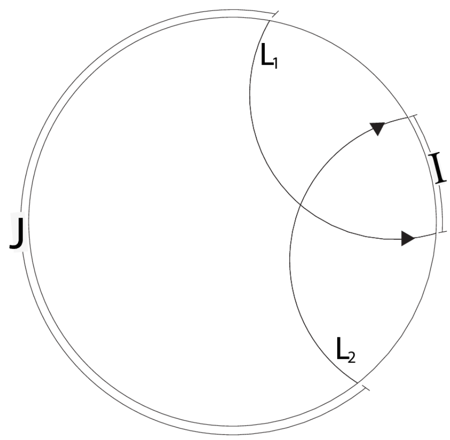

Figure 3.1: A reference figure for the proof of Lemma 5

L_2 at one point, then $g_3 = g_1 \circ g_2$ is hyperbolic. Moreover, the translation length $\tau(g_3)$ of g_3 satisfies

$$\tau(g_2) - \tau(g_1) \leq \tau(g_3) \leq \tau(g_2) + \tau(g_1).$$

Proof. By using the notation given in Figure 3.1, one can easily see that the intervals I and J satisfy $(g_1 \circ g_2)(I) \subset I$ and $J \subset (g_1 \circ g_2)(J)$. Therefore $g_3 = g_1 \circ g_2$ has two fixed points, one in I and another in J. Then g_3 must be a hyperbolic transformation with axis joining the intervals I and J.

Let $p \in L_1 \bigcap L_2$. Then $g_2^{-1}(p) \in L_2$ and

$$\rho(g_2^{-1}(p), g_1(p)) \leq \rho(g_2^{-1}(p), p) + \rho(p, g_1(p)) = \tau(g_2) + \tau(g_1),$$

where $\rho(\cdot,\cdot)$ denotes the hyperbolic metric on $\mathbb{D}$. Since

$$\tau(g_3) = \inf_{z\in\mathbb{D}} \rho(z, g_3(z)) \leq \rho(g_2^{-1}(p), g_1(p))$$

we get

$$\tau(g_3) \leq \tau(g_2) + \tau(g_1).$$

The same argument applied to $g_2 = g_3 \circ g_1^{-1}$ gives the inequality

$$\tau(g_2) \leq \tau(g_3) + \tau(g_1). \quad \square$$

Theorem 17. *Let β be a simple closed geodesic in S and let $E_{\beta,\lambda} : S \to S'$ be the earthquake map of length λ along β. Then*

$$l_S(\gamma) - i(\gamma,\beta)\lambda \leq l_{S'}(E_{\beta,\lambda})(\gamma) \leq l_S(\gamma) + i(\gamma,\beta)\lambda$$

for any $\gamma \in \Sigma'_S$.

Proof. Let $\pi : \mathbb{D} \to S$ be the universal covering map and let G be the Fuchsian group uniformizing S. Then S' is uniformized by $G' = \widetilde{E}G\widetilde{E}^{-1}$ where $\widetilde{E}$ is the earthquake on $\mathbb{D}$ determining $E_{\beta,\lambda}$. Let $g \in G$ be a hyperbolic covering transformation whose axis L projects to a simple closed geodesic γ. Notice that $l_S(\gamma)$ and $l_{S'}(E_{\beta,\lambda}(\gamma))$ equal the translation lengths of g and $g' = \widetilde{E}g\widetilde{E}^{-1}$ respectively. Let's assume that $\gamma \neq \beta$ and $i(\gamma,\beta) = 0$. Then L intersects none of the geodesics in the geodesic lamination $\mathcal{L} = \pi^{-1}(\beta)$.

Without loss of generality we may assume that L is contained in the gap A such that $\widetilde{E}|_A = Id$. Thus, for any point $z \in L$ we have that $g'(z) = \widetilde{E}|_A g \widetilde{E}|_A^{-1}(z) = g(z)$. By the identity principle $g' \equiv g$ and then both elements have the same translation length. It follows that

$$l_S(\gamma) = l_{S'}(E_{\beta,\lambda}(\gamma)) \text{ for } i(\gamma,\beta) = 0. \tag{3.4}$$

If $\gamma = \beta$ a similar argument shows that equation (3.4) also holds.

Assume now that $i(\gamma,\beta) = n > 0$. Let p be a point in $\mathbb{D}$ which projects to a point in the intersection of γ and β. Then the geodesic segment $[p, g(p)]$ crosses exactly $n+1$ lines $L_1, ..., L_{n+1}$ in the geodesic lamination and n gaps $A_1, .., A_n$. Without loss of generality, we may assume $\widetilde{E}|_{A_0} = Id$ where A_0 is the gap which is separated from A_1 by L_1. Then $\widetilde{E}|_{A_n} = T_1 \circ T_2 \circ ... \circ T_n$, where T_k is the hyperbolic transformation whose axis is L_k and has translation length λ. Notice that g maps a little neighborhood V contained in A_0 onto a neighborhood contained in A_n. Thus $g'|_V = (\widetilde{E} g \widetilde{E}^{-1})|_V = (T_1 \circ ...T_{n-1} \circ T_n \circ g)|_V$ and by the identity principle this equality holds everywhere on $\mathbb{D}$. Clearly the axis of g intersects L_n, thus by Lemma 5 we have

$$\tau(g) + \tau(T_n) \leq \tau(T_n \circ g) \leq \tau(g) + \tau(T_n).$$

Denote the attractive fixed points of T_k and g by $e^{i\theta_k}$ and $e^{i\theta_g}$ respectively, and the repulsive ones by $e^{i\omega_k}$ and $e^{i\omega_g}$. Let $I_n = \{e^{i\theta} : \theta_g \leq \theta \leq \theta_n\}$ and

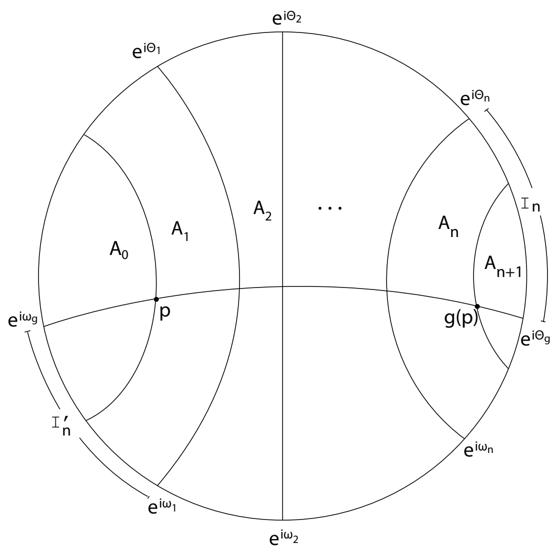

Figure 3.2: Illustration of a step in the proof of Theorem 17

let $I'_n = \{e^{i\theta} : \omega_g \leq \theta \leq \omega_1\}$ (See figure 3.2). Then $(T_n \circ g)(I_n) \subset I_n$ and $I'_n \subset (T_n \circ g)(I'_n)$. This implies that one fixed point of $T_n \circ g$ is contained in the interior of I_n an that the other one is contained in the interior of I'_n. It follows that the axis of $T_n \circ g$ must cross the axis of T_{n-1} and by Lemma 5 we have

$$\tau(T_n \circ g) - \tau(T_{n-1}) \leq \tau(T_{n-1} \circ T_n \circ g) \leq \tau(T_{n-1}) + \tau(T_n \circ g).$$

A similar argument can be applied to T_{n-2} and $T_{n-1} \circ T_n \circ g$. We conclude by induction that

$$\tau(g) - n\lambda = \tau(g) - \sum_{k=1}^{n} \tau(T_k) \leq \tau(g') \leq \tau(g) + \sum_{k=1}^{n} \tau(T_k) = \tau(g) + n\lambda;$$

that is

$$l_S(\gamma) - i(\gamma, \beta)\lambda \leq l_{S'}(E_{\beta,\lambda}(\gamma)) \leq l_S(\gamma) + i(\gamma, \beta)\lambda. \tag{3.5}$$

By combining equations (3.4) and (3.5) the theorem follows. □

3.3 The case when S_0 is conformally finite

In this section all Riemann surfaces are assumed to be of type (g, m).

Let β be a simple closed geodesic on S_0 and let φ_β be the unique simple quadratic differential given by Theorem 1, such that its horizontal closed trajectories are freely homotopic to β and

$$||\varphi_\beta|| = \iint_{S_0} |\varphi_\beta| dxdy = 1.$$

We will denote the associated ring domain of φ_β by $R_{S,\beta}$.

For each $t \in (0,1)$, $\mu_t = -t\frac{\overline{\varphi}_\beta}{|\varphi_\beta|}$ defines a Beltrami differential on S_0. We use μ_t to give a new conformal structure on S_0.

For any local parameter $z : U \to V$ on S_0, let $z' : V \to z'(V)$ be a quasiconformal mapping with complex dilation μ_t. Since μ_t is a Beltrami differential the new parameters z' give a new complex structure on S_0. We will denote the new Riemann surface by S_t. The identity mapping $Id_t : S_0 \to S_t$ is a Teichmüller mapping with initial quadratic differential φ_β and some terminal quadratic differential ψ_t. The mapping Id_t can be locally expressed as

$$Id_t : \zeta \mapsto \zeta' = \frac{\zeta - t\overline{\zeta}}{1-t}$$

where ζ and ζ' are natural parameters of φ_β and ψ respectively. It follows that Id_t stretches the vertical trajectories of φ_β by a factor $K_t = \frac{1+t}{1-t}$ and leaves the lengths of the horizontal trajectories invariant. Then the quadratic differential ψ_t is also simple and its associated ring domain $R_{S_t,\beta}$ coincides with $R_{S_0,\beta}$. Nonetheless, the conformal moduli are different; in fact it is easily seen that

$$\mathrm{Mod}(R_{S_t,\beta}) = K_t \mathrm{Mod}(R_{S_0,\beta}).$$

Let β_t be the geodesic on S_t homotopic to β. It follows from Definition 2 and the above equality that

$$E_{S_t}(\beta_t) \leq \frac{1}{K_t \mathrm{Mod}(R_{S_0,\beta})}.$$

Then by Lemma 3 we obtain

$$l_{S_t}(\beta_t) \leq \left(\frac{2\pi|\chi(S_0)|}{K_t \mathrm{Mod}(R_{S_0,\beta})} \right)^{\frac{1}{2}} \leq C\sqrt{1-t}, \tag{3.6}$$

where C is a constant depending only on S_0. Notice that $l_{S_t}(\beta_t) \to 0$ as $t \to 1$.

Using the geodesic ray (in the Teichmüller metric) given by $\alpha(t) = [S_t, Id_t]$ for $t \in (0,1)$, we construct the path $\alpha^*(t) = [S_t', E_{\beta_t,\lambda_t} \circ Id_t]$ where $\lambda_t = \log|\log l_{S_t}(\beta_t)|$.

Theorem 18. *The paths* $\alpha(t)$ *and* $\alpha^*(t)$ *satisfy* $\lim_{t\to 1} d_T(\alpha(t), \alpha^*(t)) = \infty$ *while* $\lim_{t\to 1} d_L(\alpha(t), \alpha^*(t)) = 0$.

Proof. First we estimate the length spectrum metric between $\alpha(t)$ and $\alpha^*(t)$. Suppose γ is a closed geodesic in S_t. Then by Theorem 17 we have

$$l_{S_t'}(E_{\beta_t,\lambda_t}(\gamma)) = l_{S_t}(\gamma) \text{ if } i(\gamma, \beta_t) = 0. \tag{3.7}$$

On the other hand, if $i(\gamma, \beta_t) > 0$, Theorem 17 gives

$$1 - \frac{i(\gamma, \beta_t) \log|\log l_{S_t}(\beta_t)|}{l_{S_t}(\gamma)} \leq \frac{l_{S_t'}(E_{\beta_t,\lambda_t}(\gamma))}{l_{S_t}(\gamma)} \leq 1 + \frac{i(\gamma, \beta_t) \log|\log l_{S_t}(\beta_t)|}{l_{S_t}(\gamma)}.$$

By inequality (3.6), $l_{S_t}(\beta_t)$ is small when t is close to 1. In this case we know by the Collar Lemma that β_t has a collar neighborhood of approximate length $D_t \approx \log(16/l_{S_t}(\beta_t))$. Then $l_{S_t}(\gamma) \geq i(\gamma, \beta) \log(16/l_{S_t}(\beta_t))$ and

$$1 - \frac{\log|\log l_{S_t}(\beta_t)|}{\log \frac{16}{l_{S_t}(\beta_t)}} \leq \frac{l_{S_t'}(E_{\beta_t,\lambda_t}(\gamma))}{l_{S_t}(\gamma)} \leq 1 + \frac{\log|\log l_{S_t}(\beta_t)|}{\log \frac{16}{l_{S_t}(\beta_t)}} \tag{3.8}$$

for t close to 1 and $i(\gamma, \beta) > 0$. By combining (4.2) and (4.5) and letting $t \to 1$ we get

$$\lim_{t\to 1} d_L(\alpha(t), \alpha^*(t)) = 0.$$

Let $\pi_t : \mathbb{D} \to S_t$ be the universal covering map and let G_t be the group uniformizing S_t. Denote by $\widetilde{E}_t : \mathbb{D} \to \mathbb{D}$ the left earthquake corresponding to E_{β_t,λ_t}. By inequality (3.3) and Theorem 14 we have

$$\|\widetilde{E}_t\|_{\text{cr}} \geq \frac{1}{C'} \|\sigma_{\lambda_t}\|_{\text{Th}} \geq \frac{1}{C'} \lambda_t = \frac{1}{C'} \log|\log l_{S_t}(\beta_t)|,$$

and thus the cross-ratio distortion norm is approaching ∞ as $t \to 1$. It follows from Theorem 16 that the the maximal dilatation of any extension of $\widetilde{E}_t$ is approaching ∞ as $t \to 1$. Since any lift F_t of the extremal quasiconformal mapping in the homotopy class of E_{β_t,λ_t} satisfies $F_t|_{\mathbb{R}} = (T \circ \widetilde{E}_t)|_{\mathbb{R}}$ where T is a Möbius transformation, it follows that $K(F_t) \to \infty$ as $t \to 1$. Equivalently

$$\lim_{t\to 1} d_T(\alpha(t), \alpha^*(t)) = \infty. \quad \square$$

Now consider the path $\hat{\alpha}(t) = [S'_t, E_{\beta_t,\hat{\lambda}_t}]$ where $\hat{\lambda}_t = 1$ for all t. We have the following theorem.

Theorem 19. *There exist constants $M, m > 0$ such that*

$$m \le d_T(\alpha(t), \hat{\alpha}(t)) \le M \text{ for all } t, \text{ while } \lim_{t\to 1} d_L(\alpha(t), \hat{\alpha}(t)) = 0.$$

Proof. A similar argument to the one used in Theorem 18 shows that $d_L(\alpha(t), \hat{\alpha}(t)) = 0$. For the sake of simplicity we omit the details. It remains to show the existence of the constants M and m. By inequality (3.3) and Theorem 14 we have

$$C' \max\{1, 2/D_t\} \ge C' \|\sigma_{\hat{\lambda}_t}\|_{\mathrm{Th}} \ge \|\widetilde{E}_t\|_{\mathrm{cr}} \ge \frac{1}{C'} \|\sigma_{\hat{\lambda}_t}\|_{\mathrm{Th}} \ge C' > 0,$$

where as before, $D_t \approx \log(16/l_{S_t}(\beta_t))$ is the length of a collar neighborhood around the geodesic β_t and $\widetilde{E}_t : \mathbb{D} \to \mathbb{D}$ is the earthquake map corresponding to $E_{\beta_t,\hat{\lambda}_t}$. Since the cross-ratio distortion norm of $\widetilde{E}_t$ is bounded from above, it follows from Theorem 15 that there exists a constant M such that $K_e(\widetilde{E}_t) \leq e^M$ for all t. In particular

$$d_T(\alpha(t), \hat{\alpha}(t)) \leq M \text{ for all } t.$$

On the other hand, since the cross-ratio distortion norm of $\widetilde{E}_t$ is bounded from below, we know by Proposition 1, that there exists a constant $m > 0$ such that $e^m \leq K_e(\widetilde{E}_t)$ for all t. Thus

$$m \leq d_T(\alpha(t), \hat{\alpha}(t)) \text{ for all } t. \quad \square$$

3.4 The case when S_0 has boundary

Now we consider the case in which S_0 is of type (g, m, k) with $k > 0$. In order to prove Theorem 9 completely, we just need to construct a curve $\alpha(t) = [S_t, h_t]$, $0 \leq t < 1$, in $T^R(S_0)$ such that there exists a simple closed curve β in S_0 with

$$l_{S_t}(\beta_t) \to 0 \text{ as } t \to 1,$$

where $\beta_t = h_t(\beta)$. Once this is done, we can follow exactly the same construction that was used in the previous section to obtain the curves $\alpha^*(t)$ and $\hat{\alpha}(t)$ with the desired properties. This can be done by using an appropriate quadratic differential in the Nielsen kernel of the base Riemann surface.

Let $\widetilde{S}_0$ be the Nielsen kernel of S_0 and let β be a geodesic in $\widetilde{S}_0$ that is not homotopic to a boundary component. By Theorem 1, we can choose a quadratic differential $\varphi_{\beta,\overline{\beta}}$ in $\widetilde{S}_0^d$ of closed trajectories with exactly two associated cylinders, one of homotopy type β and another of homotopy type $\overline{\beta}$, such that the heights of both cylinders are equal. The quadratic differential is symmetric with respect to the boundary of $\widetilde{S}_0$, along which it is real. This follows from the uniqueness part in Theorem 1, since the quadratic differential $\overline{\varphi_{\beta,\overline{\beta}}(\overline{z})}dz^2$ is also holomorphic and it has the same ring domains as $\varphi_{\beta,\overline{\beta}}(z)dz^2$.

By multiplying $\varphi_{\beta,\overline{\beta}}$ by $1/||\varphi_{\beta,\overline{\beta}}||$ we may assume that it has norm one. Then, as we did in the previous section, we can construct a Teichmüller geodesic ray $t \mapsto \alpha(t)$, $0 \leq t < 1$, on $T(\widetilde{S}_0^d)$. Moreover, since the quadratic differential $\varphi_{\beta,\overline{\beta}}$ is symmetric with respect to the boundary of $\widetilde{S}_0$, for each $0 \leq t < 1$ there exist a Riemann surface $\widetilde{S}_t$ and a quasiconformal mapping $\widetilde{h}_t : \widetilde{S}_0 \to \widetilde{S}_t$ such that $\alpha(t) = [\widetilde{S}_t^d, \widetilde{h}_t^d]$, where the complex dilatation of $\widetilde{h}_t^d$ is given by $\mu_t = -t\overline{\varphi}_{\beta,\overline{\beta}}/|\varphi_{\beta,\overline{\beta}}|$. Just as before, we can apply inequality (3.6) to obtain

$$l_{\widetilde{S}_t^d}(\widetilde{h}_t^d(\beta)) \to 0 \text{ as } t \to 1. \tag{3.9}$$

Recall from section 2.3 that for each t, the mapping $\widetilde{h}_t^d : \widetilde{S}_0 \to \widetilde{S}_t$ induces a mapping $h_t : S_0 \to S_t$, such that $K(h_t) = K(\widetilde{h}_t)$, where S_t denotes the

Nielsen extension of $\widetilde{S}_t$. It follows from Lemma 4 and (3.9) that

$$l_{S_t}(h_t(\beta)) \to 0 \text{ as } t \to 1.$$

This completes the proof of Theorem 9.

Chapter 4

Modified length spectrum metric

When a Riemann surface S_0 has boundary, $T^R(S_0) \neq T(S_0)$. In this case, d_L does not define a metric on $T(S_0)$ since it does not separate points. For if $f : S_0 \to S_0$ is given by a Dehn twist along a boundary geodesic β, i.e., β is a geodesic homotopic to some boundary component of S_0, then $d_T([S_0, f], [S_0, Id]) > 0$. Thus $[S_0, f] \neq [S_0, Id]$ in $T(S_0)$. However, since there is no closed geodesic γ crossing β, the length of every such γ is unchanged under the Dehn twist and so $d_L([S_0, f], [S_0, Id]) = 0$.

In this chapter, we first introduce a modified length spectrum that does define a metric on $T(S_0)$. Then we study properties of this new metric and its relationship with the Teichmüller metric on $T(S_0)$ when S_0 is of finite topological type.

Let S be a Riemann surface with boundary and let Σ''_S be the collection of homotopy classes relative to the endpoints of arcs connecting boundary points of S such that none of them is homotopic to a boundary segment relative to the endpoints. In any class γ of Σ''_S, there exists a unique geodesic arc α with respect to the hyperbolic metric. Let β_1 and β_2 be the two closed geodesics homotopic to the boundary components that contain the endpoints of α (possibly $\beta_1 = \beta_2$), namely, the boundary geodesics of the corresponding boundary components. If $\beta_1 \neq \beta_2$, then α crosses each of them exactly twice, since any geodesic arc entering a funnel cannot escape from it. If $\beta_1 = \beta_2$, then α crosses β_1 exactly twice, probably at the same point. Let $l_S(\gamma)$ denote the length of the geodesic segment of α between β_1 and β_2. We define the modified length spectrum on $T(S_0)$ by

$$d_{ML}([S_1, f_1], [S_2, f_2]) = \log \sup_{\gamma \in \Sigma^*_{S_1}} \left\{ \frac{l_{S_2}(f_2 \circ f_1^{-1}(\gamma))}{l_{S_1}(\gamma)}, \frac{l_{S_1}(\gamma)}{l_{S_2}(f_2 \circ f_1^{-1}(\gamma))} \right\},$$

where $\Sigma^*_{S_1} = \Sigma'_{S_1} \bigcup \Sigma''_{S_1}$.

In this chapter, we first prove the following theorem.

Theorem 20. *Assume that S_0 is a Riemann surface with boundary. Then the modified length spectrum function d_{ML} defines a metric on $T(S_0)$.*

Then we assume that S_0 is a Riemann surface of type (g, m, k), where g, m and k are the genus, the number of punctures and the number of ideal boundaries, respectively, with $k > 0$ and $6g - 6 + m + 3k > 0$. Under these assumptions, we show the following results.

Theorem 21. *The identity mapping* $Id : (T(S_0), d_T) \to (T(S_0), d_{ML})$ *is continuous, but the inverse mapping is not.*

Corollary 2. *The topologies induced by* d_{ML} *and* d_T *on* $T(S_0)$ *are not equivalent.*

Theorem 22. *The metric space* $(T(S_0), d_{ML})$ *is not complete.*

These results are presented in [10].

4.1 A new metric on the Teichmüller space of Riemann surfaces with boundary

In this section we prove Theorem 20. Notice that from the definition, it is easy to verify that d_{ML} is nonnegative and symmetric, and satisfies the triangle inequality. The main work is to show that it separates points. We first introduce some notation.

1. Let $L_{x,y}$ denote the geodesic in the unit disk $\mathbb{D}$ or the upper half-plane $\mathbb{H}$ with respect to the hyperbolic metric that connects two points x and y on the boundary of $\mathbb{D}$ or $\mathbb{H}$.

2. If a geodesic L intersects two geodesics L_1 and L_2, then $l(L; L_1, L_2)$ denotes the length, in the hyperbolic metric, of the segment of L between L_1 and L_2.

Proposition 2. *Let L_1 and L_2 be two disjoint geodesics in $\mathbb{H}$ without common endpoints and let L_{x_0,y_0} be their common orthogonal. Then*

$$l(L_{x_0,y}; L_1, L_2) > l(L_{x_0,y_0}; L_1, L_2)$$

for any geodesic $L_{x_0,y}$ crossing L_1 and L_2 with $y_0 \neq y$. Moreover, for any given value $l_0 > l(L_{x_0,y_0}; L_1, L_2)$ there exist exactly two geodesics L_{x_0,y_1} and L_{x_0,y_2} crossing L_1 and L_2 such that $l(L_{x_0,y_1}; L_1, L_2) = l(L_{x_0,y_2}; L_1, L_2) = l_0$. These geodesics are contained in different connected components of $\mathbb{H} \backslash L_{x_0,y_0}$.

Without loss of generality, we may assume that $L_{x_0,y_0} = L_{0,\infty}$, $L_1 = L_{-1,1}$, $L_2 = L_{-b,b}$ for some $b > 1$. Let c be a positive number such that $2c > b$. Then the intersection points of $L_{0,2c}$ with L_1 and L_2 are the solutions of the following systems respectively:

$$\begin{cases} x^2 + y^2 &= 1 \\ (x-c)^2 + y^2 &= c^2 \end{cases}$$

and

$$\begin{cases} x^2 + y^2 &= b^2 \\ (x-c)^2 + y^2 &= c^2. \end{cases}$$

Solving these systems we see that the x-coordinates of the intersection points of $L_{0,2c}$ with L_1 and L_2 are given by $1/2c$ and $b^2/2c$ respectively. Parametrize the segment of $L_{0,2c}$ between L_1 and L_2 by the equation

$$\gamma_{0,2c}(t) = t + i\sqrt{c^2 - (t-c)^2} = t + i\sqrt{2ct - t^2},\ t \in [1/2c, b^2/2c].$$

Then

$$\begin{aligned} l(L_{0,2c}; L_1, L_2) &= \int_{\frac{1}{2c}}^{\frac{b^2}{2c}} \frac{|\gamma'_{0,2c}(t)|}{\mathrm{Im}(\gamma_{0,2c}(t))} dt \\ &= \int_{\frac{1}{2c}}^{\frac{b^2}{2c}} \frac{1}{\sqrt{2ct - t^2}} \sqrt{1 + \frac{(c-t)^2}{2ct - t^2}} dt \\ &= \int_{\frac{1}{2c}}^{\frac{b^2}{2c}} \frac{c}{(2c-t)t} dt \\ &= \ln b + \tfrac{1}{2} \ln \tfrac{4c^2 - 1}{4c^2 - b^2}. \end{aligned}$$

The function

$$h(c) = l(L_{0,2c}; L_1, L_2) = \ln b + \frac{1}{2} \ln \frac{4c^2 - 1}{4c^2 - b^2}$$

has derivative given by

$$h'(c) = \frac{4c}{4c^2 - 1} - \frac{4c}{4c^2 - b^2}.$$

Since $b > 1$, then $h'(c) < 0$ for $c > b/2$. It follows that h is a strictly decreasing function for $c > b/2$, which implies that the length $l(L_{0,2c}; L_1, L_2)$ decreases as c goes to ∞. In fact $h(\infty) = \ln b = l(L_{0,\infty}; L_1, L_2)$.

Similarly, if $c < 0$ satisfies $2c < -b$, then the function $g(c) = h(-c)$ gives the length of $\gamma_{0,2c}$. Since $g'(c) = -h'(-c) > 0$ it follows that for $c < -b/2$, g is an strictly increasing function of c with $g(-\infty) = \ln b = l(L; L_{0,\infty}, L_2)$. $\square$

Let S be a a Riemann surface with boundary and let $\pi : \mathbb{D} \to \mathrm{int}(S)$ be the universal covering map. If G is the covering group of $\mathrm{int}(S)$, then the universal covering map extends to a covering map $\pi : \overline{\mathbb{D}} \cup \Lambda(G) \to S$, where $\Lambda(G)$ is the limiting set of G. Every quasiconformal mapping $f : S \to S$ can be lifted to a mapping $F : \overline{\mathbb{D}} \cup \Lambda(G) \to S$ that extends to a homeomorphism of $F : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$. In order to avoid the details of the maps π and F on the boundary of $\mathbb{D}$, from now on we say in brief that $\pi : \mathbb{D} \to S$ is the covering map of S and that $F : \mathbb{D} \to \mathbb{D}$ is a lift of f.

Now we prove that d_{ML} separates points. Given any two points $[S_1, f_1]$ and $[S_2, f_2]$ in $T(S_0)$, we use the same symbols to denote their equivalent classes in $T^R(S_0)$. Then

$$d_L([S_1, f_1], [S_2, f_2]) \leq d_{ML}([S_1, f_1], [S_2, f_2]).$$

Suppose that $d_{ML}([S_1, f_1], [S_2, f_2]) = 0$. Then $d_L([S_1, f_1], [S_2, f_2]) = 0$. Since d_L is a metric in $T^R(S_0)$, it follows that $f_2 \circ f_1^{-1}$ is homotopic to a conformal mapping $c : S_1 \to S_2$. We need to prove that $f = f_2 \circ f_1^{-1}$ is homotopic to c modulo the boundary.

The following is a classical result that can be found in [15].

Theorem 23. *Let S_1 and S_2 be two hyperbolic Riemann surfaces and let $f_i : S_1 \to S_2$, $i = 1, 2$, be two quasiconformal mappings. Assume that the Fuchsian group G_1 representing S_1 is non-elementary. Then*

(1) f_1 is homotopic to f_2 if and only if they can be lifted to mappings of $\mathbb{H}$ which agree on the limit set of G_1; and

(2) f_1 is homotopic to f_2 modulo the boundary if and only if they can be lifted to mappings of $\mathbb{H}$ which agree on $\widehat{\mathbb{R}}$.

Let $F : \mathbb{H} \to \mathbb{H}$ and $C : \mathbb{H} \to \mathbb{H}$ be lifts of f and c respectively that agree on the limit set Λ of the Fuchsian group G_1 uniformizing S_1. Clearly, $F \circ C^{-1}$ maps each connected component of $\mathbb{S}^1 \setminus \Lambda$ onto itself. It remains to show $F \circ C^{-1}$ is the identity on each connected component. Let L be a lift of a boundary geodesic β of S_1. Then one of the arcs bounded by the endpoints of L on $\mathbb{S}^1$ is a connected component of $\mathbb{S}^1 \setminus \Lambda$ and each connected component is bounded by the endpoints of a lift of a boundary geodesic of S_1.

Let L_1 and L_2 be two different lifts of a boundary geodesic β of S_1, and let L_{x_0,y_0} be their common perpendicular. The mapping C, being conformal, maps L_{x_0,y_0} to the common perpendicular between $C(L_1)$ and $C(L_2)$. By assumption, $l_{S_2}(f(\gamma)) = l_{S_1}(\gamma)$ for every $\gamma \in \Sigma''_{S_0}$. On the other hand, since c is an isometry, we must have $l_{S_1}(\gamma) = l_{S_2}(c(\gamma))$ for every $\gamma \in \Sigma''_{S_0}$. It follows that

$$\begin{aligned} l(L_{F(x_0),F(y_0)}; C(L_1), C(L_2)) &= l(L_{x_0,y_0}; L_1, L_2) \\ &= l(L_{C(x_0),C(y_0)}; C(L_1), C(L_2)). \end{aligned}$$

Since the common perpendicular segment between two geodesics is the unique arc of smallest length among all segments connecting them, it follows that $F(x_0) = C(x_0)$ and $F(y_0) = C(y_0)$.

Let I_i be the interval in the real line bounded by the endpoints of L_i. It projects to the component of the ideal boundary of S_0 homotopic to β. Assume $x_0 \in I_1$ and $y_0 \in I_2$. For any point $x \in I_2 \setminus \{y_0\}$, consider the geodesic $L_{x_0,x}$. Then

$$\begin{aligned} l(L_{F(x_0),F(x)}; C(L_1), C(L_2)) &= l(L_{x_0,x}; L_1, L_2) \\ &= l(L_{C(x_0),C(x)}; C(L_1), C(L_2)). \end{aligned}$$

It follows from Proposition 2 that $F(x) = C(x)$. This argument can be applied to any point $x \in \widehat{\mathbb{R}}$ that is not in the limit set of G_1. Thus both maps agree on the whole boundary of $\mathbb{H}$ and by Theorem 23 their projections are homotopic to each other modulo the boundary. Hence d_{ML} separates points and Theorem 20 is now proved.

4.2 Properties of the modified length spectrum metric

From now on we will assume that all Riemann surfaces are of type (g, m, k), with $k > 0$. Recall that as a consequence of Lemma 2 we know that the mapping

$$Id : (T^R(S_0), d_T) \to (T^R(S_0), d_L)$$

is continuous. In this section we will prove a similar result when d_L is replaced by d_{ML}. Before doing so, we introduce some lemmas.

Lemma 6. *Let $L_{a,b}$, $a < b$, be a geodesic in $\mathbb{H}$ and I a closed interval contained in (a, b). Assume $\{L_{a_n,b_n}\}$, $b < a_n < b_n$, is a sequence of geodesics in $\mathbb{H}$ such that the hyperbolic distance between L_{a_n,b_n} and $L_{a,b}$ is bounded below by $\epsilon > 0$ for every n. Then for any sequence $\{x_n\}$ contained in I and every sequence $\{y_n\}$ such that $y_n \in (a_n, b_n)$, $d(L_{x_n,y_n}; L_{a,b}, L_{a_n,b_n}) \to \infty$ if $a_n, b_n \to b$ as $n \to \infty$.*

Proof. For each n, we use the Möbius transformation

$$T_n(z) = \frac{b_n - a}{b_n - b}\frac{z - b}{z - a},$$

to map $L_{a,b}$ and L_{a_n,b_n} to $L_{\infty,0}$ and $L_{T_n(a_n),1}$ respectively. By hypothesis, the distance between $L_{\infty,0}$ and $L_{T_n(a_n),1}$ is bounded below by $\epsilon > 0$. Then, there exists $r > 0$ such that $T_n(a_n) \geq r$ for every n. Then for any $y_n \in (a_n, b_n)$ we have

$$\begin{aligned} l(L_{x_n,y_n}; L_{a,b}, L_{a_n,b_n}) &= l(L_{T_n(x_n),T_n(y_n)}; L_{\infty,0}, L_{T_n(a_n),1}) \\ &\geq l(L_{T_n(x_n),T_n(y_n)}; L_{\infty,0}, L_{r,1}). \end{aligned}$$

Since $T_n(x_n) \to -\infty$ as $n \to \infty$, it follows that

$$l(L_{T_n(x_n),T_n(y_n)}; L_{\infty,0}, L_{r,1}) \to \infty \text{ as } n \to \infty.$$

□

Let x and y be two distinct points of the unit circle $\mathbb{S}^1$. By the interval $\lceil x, y \rceil$ of the unit circle we mean the set of all points $w \in \mathbb{S}^1$ such that w belongs to the circular arc joining x and y described by going around $\mathbb{S}^1$ counterclockwise.

For any $d > 0$, choose $b = b(d) \in \mathbb{S}^1$ such that $0 < \arg(b) < \pi/2$ and the hyperbolic distance between $L_{-i,i}$ and $L_{\bar{b},b}$ is d. We call the geodesic $L_{\bar{b},b}$ is the *d-standard geodesic.*

Let $L_{\bar{b},b}$ be the d-standard geodesic. For any $s > 0$ sufficiently small, let $I_s = \lceil x, y \rceil \subseteq \lceil i, -i \rceil$ and $J_{s,b} = \lceil z, w \rceil \subseteq \lceil \bar{b}, b \rceil$ be the intervals of the unit circle such that the arc length of each of the intervals $\lceil i, x \rceil$, $\lceil y, -i \rceil$, $\lceil \bar{b}, z \rceil$, and $\lceil w, b \rceil$ is s.

Lemma 7. *Assume that* $0 < d_0 < d_1$. *For every* $D > d_1$, *there exists* $s_0 > 0$ *such that for every* $d_0 \leq d \leq d_1$, *if* $l(L_{p,q}; L_{-i,i}, L_{\overline{b(d)},b(d)}) \leq D$, *then* $p \in I_{s_0}$

and $q \in J_{s_0,b(d)}$.

Proof. Let $D > d_1$. For any $d \in [d_0, d_1]$, there exists a maximal $s = s(d)$ such that if $l(L_{p,q}; L_{-i,i}, L_{\overline{b(d)},b(d)}) \le D$ then $p \in I_s$ and $q \in J_{s,b(d)}$. The function $d \mapsto s(d)$ is a continuous function defined on the compact interval $[d_0, d_1]$. Then $s_0 = \min_{d\in[d_0,d_1]} s(d)$ satisfies the conclusion of the lemma. □

Lemma 8. *Assume that* $0 < d_0 < d_1$ *and that* s_0 *is a positive number small enough (only depending on* d_1*). Then for every* $\epsilon > 0$*, there exists* $\delta > 0$ *depending on* d_0*,* d_1*,* s_0 *and* ϵ *such that:*

1. *For every* $d \in [d_0, d_1]$*,*
2. *for every* $x, y, z, w \in \mathbb{S}^1$*,*
3. *for every* $p_1, p_2 \in I_{s_0}$*, and*
4. *for every* $q_1, q_2 \in J_{s_0,b(d)}$*,*

$$\left| \log \frac{l(L_{p_1,q_1}; L_{-i,i}, L_{\overline{b(d)},b(d)})}{l(L_{p_2,q_2}; L_{x,y}, L_{z,w})} \right| < \epsilon$$

provided that each of the numbers $|p_1 - p_2|, |q_1 - q_2|, |x - i|, |y + i|, |z - \overline{b(d)}|$ *and* $|w - b(d)|$ *is less than* δ.

Proof. Suppose the lemma is false. Then there exists $\epsilon > 0$ such that for every $\delta_n = 1/n$,

1. there exists $d_n \in [d_0, d_1]$,

2. there exist $x_n, y_n, z_n, w_n \in \mathbb{S}^1$ with $|x_n - i|, |y_n + i|, |z_n - \overline{b(d_n)}|, |w - b(d_n)| < 1/n$,
3. there exist $p_1^{(n)}, p_2^{(n)} \in I_{s_0}$ with $|p_1^{(n)} - p_2^{(n)}| < 1/n$, and
4. there exist $q_1^{(n)}, q_2^{(n)} \in J_{s_0, b(d_n)}$ with $|q_1^{(n)} - q_2^{(n)}| < 1/n$

satisfying

$$\left| \log \frac{l(L_{p_1^{(n)}, q_1^{(n)}}; L_{-i,i}, L_{\overline{b(d_n)}, b(d_n)})}{l(L_{p_2^{(n)}, q_2^{(n)}}; L_{x_n, y_n}, L_{z_n, w_n})} \right| \geq \epsilon. \tag{4.1}$$

.

We may assume, by passing to a subsequence, that

1. $d_n \to d^{(0)}$, which implies $b(d_n) \to b(d^{(0)})$,
2. $x_n \to i$, $y_n \to -i$, $z_n \to b(d^{(0)})$, $w_n \to \overline{b(d^{(0)})}$,
3. $p_1^{(n)} \to p_1^{(0)}$, $q_1^{(n)} \to q_1^{(0)}$ and thus $p_2^{(n)} \to p_1^{(0)}$, $q_2^{(n)} \to q_1^{(0)}$.

Since $p_1^{(n)} \in I_{s_0}$ and $q_1^{(n)} \in J_{s_0, b(d_n)}$ it follows that $p_1^{(0)}$ and $q_1^{(0)}$ are contained in the interior of $\lceil i, -i \rceil$ and $\lceil \overline{b(d^{(0)})}, b(d^{(0)}) \rceil$ respectively. Then we can choose n sufficiently large so that

$$\left| \log \frac{l(L_{p_1^{(n)}, q_1^{(n)}}; L_{-i,i}, L_{\overline{b(d_n)}, b(d_n)})}{l(L_{p_1^{(0)}, q_1^{(0)}}; L_{-i,i}, L_{\overline{b(d^{(0)})}, b(d^{(0)})})} \right| < \epsilon/2,$$

and

$$\left| \log \frac{l(L_{p_2^{(n)}, q_2^{(n)}}; L_{x_n, y_n}, L_{z_n, w_n})}{l(L_{p_1^{(0)}, q_1^{(0)}}; L_{-i,i}, L_{\overline{b(d^{(0)})}, b(d^{(0)})})} \right| < \epsilon/2.$$

Combining both inequalities we get

$$\left|\log \frac{l(L_{p_1^{(n)},q_1^{(n)}}; L_{-i,i}, L_{\overline{b(d_n)},b(d_n)})}{l(L_{p_2^{(n)},q_2^{(n)}}; L_{x_n,y_n}, L_{z_n,w_n})}\right| < \epsilon \text{ for } n \text{ sufficiently large.}$$

This contradicts inequality (4.1). □

Lemma 9. *Let S be a Riemann surface and $\widetilde{S}$ its Nielsen kernel. Then for any curve $\gamma \in \Sigma''_S$*

$$\frac{l_{\widetilde{S}^d}(\widetilde{\gamma}^d)}{2} \leq l_S(\gamma) \leq \frac{l_{\widetilde{S}^d}(\widetilde{\gamma}^d)}{2} + 2M,$$

where $\widetilde{\gamma}$ denotes the restriction of the curve γ to $\widetilde{S}$, $\widetilde{\gamma}^d$ is the double of $\widetilde{\gamma}$, and

$$M = \max\{l_S(\beta) : \beta \textit{ is a boundary geodesic in } S\}.$$

Proof. For any curve α on $\widetilde{S}^d$, let $\widetilde{l}_{\widetilde{S}^d}(\alpha)$ denote the length of α in the hyperbolic metric on $\widetilde{S}^d$. Let γ be an arc in Σ''_S; without loss of generality we may assume that it is a geodesic arc. By Lemma 4, $\widetilde{l}_{\widetilde{S}^d}(\widetilde{\gamma}) = l_S(\gamma)$. Since $\widetilde{l}_{\widetilde{S}^d}(\widetilde{\gamma}) = \widetilde{l}_{\widetilde{S}^d}(\widetilde{\gamma}^d)/2 \geq l_{\widetilde{S}^d}(\widetilde{\gamma}^d)/2$, the left-hand side inequality follows.

Recall that γ either crosses two distinct boundary geodesics exactly once or one exactly twice. Let $\beta_1, \beta_2 \in \Sigma'_S$ be the ones crossed by γ at the points p_1 and p_2 respectively. If $\beta_1 = \beta_2$ then p_1 and p_2 belong to the same geodesic boundary an in this case p_1 may be equal to p_2. Let β be the closed geodesic on $\widetilde{S}^d$ in the homotopy class of $\widetilde{\gamma}^d$. Then β crosses β_1 and β_2 in a similar fashion as γ. Denote the intersection points by q_1 and q_2. Let β_i' be one of the two segments of β_i joining p_i to q_i. Then

$$\begin{aligned} l_S(\gamma) = \widetilde{l}_{\widetilde{S}^d}(\widetilde{\gamma}) &\leq \widetilde{l}_{\widetilde{S}^d}(\beta_1') + \widetilde{l}_{\widetilde{S}^d}(\beta \cap \widetilde{S}) + \widetilde{l}_{\widetilde{S}^d}(\beta_2') \\ &= \widetilde{l}_{\widetilde{S}^d}(\beta_1') + \tfrac{1}{2}\widetilde{l}_{\widetilde{S}^d}(\beta) + \widetilde{l}_{\widetilde{S}^d}(\beta_2') \\ &\leq l_{\widetilde{S}^d}(\beta_1) + \tfrac{1}{2}\widetilde{l}_{\widetilde{S}^d}(\beta) + l_{\widetilde{S}^d}(\beta_2) \\ &\leq \tfrac{1}{2}\widetilde{l}_{\widetilde{S}^d}(\beta) + 2M \\ &= \tfrac{1}{2}l_{\widetilde{S}^d}(\widetilde{\gamma}^d) + 2M, \end{aligned}$$

where the second equality follows from the fact that β is homotopic to $\widetilde{\gamma}^d$, which is symmetric with respect to the boundary geodesics of S. The right-hand inequality now follows. □

Lemma 10. *Let $f_n : S_0 \to S_n$, $n = 1, 2, ...$ be a sequence of quasiconformal mappings such that $K(f_n) \to 1$ as $n \to \infty$. Then for each n, there exists a mappings $h_n : S_0 \to S_n$ homotopic to f_n modulo the boundary such that for n sufficiently large, h_n maps Nielsen kernels to Nielsen kernels, i.e., $h_n(\widetilde{S}_0) = \widetilde{S}_n$, and $K(h_n) \to 1$ as $n \to \infty$.*

Proof. For each $n = 0, 1, 2, ..$, let $\pi_n : \mathbb{D} \to S_n$ be the universal covering map, let G_n be the group uniformizing S_n, and let $F_n : \mathbb{D} \to \mathbb{D}$ be a lift of f_n normalized so that it fixes three points. Then $K(F_n) \to 1$ and F_n converges uniformly to the identity map.

Let $D_0 \subseteq \mathbb{D}$ be a Dirichlet fundamental domain for G_0. Then $F_n(D_0)$ is a fundamental domain for G_n. For each edge e of D_0, replace $F_n(e)$ by the geodesic segment connecting the endpoints of $F_n(e)$. For n large, F_n is close to the identity and then the vertices of D_0 are moved very little by F_n. Therefore, if n is sufficiently large, these new edges do not intersect each other except when they have the image of a vertex of D_0 as a common endpoint. Using these new edges we obtain a new fundamental domain D_n for G_n. For each such large n, let $\widetilde{D}_n = \pi_n^{-1}(\widetilde{S}_n) \bigcap D_n$. The region $\widetilde{D}_n$ is a polygon whose vertices are either in $\mathbb{D}$ or in $\partial\mathbb{D}$ such that it projects to the Nielsen kernel $\widetilde{S}_n$ of S_n. Each D_n is the union of hyperbolic triangles with disjoint interior, thus we can construct a piecewise hyperbolic affine map $H_n : \widetilde{D}_0 \to \widetilde{D}_n$ mapping vertices to vertices. In order to extend H_n to D_0, we foliate each connected component of $\overline{D}_0 \setminus \widetilde{D}_0$ by geodesic rays starting at $\partial\widetilde{D}_0$ and ending at $\mathbb{S}^1 \bigcap \overline{D}_0$, where $\overline{D}_0$ denotes the closure of D_0 in the closed unit disk with respect to the Euclidean metric. For each such geodesic ray with endpoints $z \in \partial\widetilde{D}_0$ and $x \in \mathbb{S}^1 \bigcap \overline{D}_0$ we let H_n map it proportionally to the geodesic starting at $H_n(z)$ and ending at $F_n(x)$. Finally we extend H_n to the whole hyperbolic disk by using the actions of G_0 and G_n on $\mathbb{D}$. By theorem 23, H_n projects to a mapping $h_n : S_0 \to S_n$ which is homotopic to f_n modulo the boundary and by construction $h_n(\widetilde{S}_0) = \widetilde{S}_n$. Moreover, since F_n converges uniformly to the identity, the vertices of D_n approach the vertices of D_0 as $n \to \infty$. Thus $H_n \to Id$ and $K(H_n) \to 1$. □

Theorem 24. *The identity function* $Id : (T(S_0), d_T) \to (T(S_0), d_{ML})$ *is continuous.*

Proof. Let $\{\tau_n\}$ be a sequence of points in $T(S_0)$ converging to a point τ in the Teichmüller metric and let $\epsilon > 0$ be given. Without loss of generality, we may assume that $\tau = [S_0, Id]$. Let $\tau_n = [S_n, f_n]$, by Lemma 10, we may assume that $f_n(\widetilde{S}_0) = \widetilde{S}_n$ and $K(f_n) \to 1$ as $n \to \infty$. Consider the points $[S_n, f_n]$ and $[S_0, Id]$ as elements of $T^R(S_0)$. By Theorem 8 we know that d_L and d_T are topologically equivalent in $T^R(S_0)$. Then there exists N_1 such that

$$\left|\log \frac{l_{S_n}(f_n(\gamma))}{l_{S_0}(\gamma)}\right| < \epsilon \text{ for every } n > N_1 \text{ and every } \gamma \in \Sigma'_{S_0}. \tag{4.2}$$

In particular, condition (4.2) implies that the sequence

$$M_n = \max\{l_{S_n}(\beta) : \beta \text{ is a boundary geodesic in } S_n\},\ n = 0, 1, 2, ...$$

converges to M_0 as $n \to \infty$. Thus there exists a constant $M' > 0$ such that $M_n \leq M'$ for every n.

Let $\gamma \in \Sigma''_{S_0}$, since f_n maps $\widetilde{S}_0$ to $\widetilde{S}_n$, we have $\widetilde{f_n(\gamma)} = \widetilde{f}_n(\widetilde{\gamma})$ and $(\widetilde{f}_n(\widetilde{\gamma}))^d = \widetilde{f}_n^d(\widetilde{\gamma}^d)$, where $\widetilde{f}_n = f|_{\widetilde{S}_0}$ and $\widetilde{f}_n^d : \widetilde{S}_0^d \to \widetilde{S}_n^d$ is the double mapping of $\widetilde{f}_n$. Then by Lemma 9 and the fact that $M_n \leq M'$ for any $n = 0, 1, 2, ...$ we conclude that

$$\frac{l_{\widetilde{S}_0^d}(\widetilde{\gamma}^d)}{2} \leq l_{S_0}(\gamma) \leq \frac{l_{\widetilde{S}_0^d}(\widetilde{\gamma}^d)}{2} + 2M', \tag{4.3}$$

and

$$\frac{l_{\widetilde{S}_n^d}(\widetilde{f}_n^d(\widetilde{\gamma}))}{2} \leq l_{S_n}(f_n(\gamma)) \leq \frac{l_{\widetilde{S}_n^d}(\widetilde{f}_n^d(\widetilde{\gamma}^d))}{2} + 2M'. \tag{4.4}$$

By combining inequalities (4.3) and (4.4) we obtain

$$\frac{\frac{1}{2}l_{\widetilde{S}_n^d}(\widetilde{f}_n^d(\widetilde{\gamma}^d))}{\frac{1}{2}l_{\widetilde{S}_0^d}(\widetilde{\gamma}^d) + 2M'} \leq \frac{l_{S_n}(f_n(\gamma))}{l_{S_0}(\gamma)} \leq \frac{\frac{1}{2}l_{\widetilde{S}_n^d}(\widetilde{f}_n^d(\widetilde{\gamma}^d)) + 2M'}{\frac{1}{2}l_{\widetilde{S}_0^d}(\widetilde{\gamma}^d)},$$

or

$$\frac{\frac{l_{\widetilde{S}_n^d}(\widetilde{f}_n^d(\widetilde{\gamma}^d))}{l_{\widetilde{S}_0^d}(\widetilde{\gamma}^d)}}{1 + \frac{4M'}{l_{\widetilde{S}_0^d}(\widetilde{\gamma}^d)}} \leq \frac{l_{S_n}(f_n(\gamma))}{l_{S_0}(\gamma)} \leq \frac{l_{\widetilde{S}_n^d}(\widetilde{f}_n^d(\widetilde{\gamma}^d))}{l_{\widetilde{S}_0^d}(\widetilde{\gamma}^d)} + \frac{4M'}{l_{\widetilde{S}_0^d}(\widetilde{\gamma}^d)}.$$

Lemma 1 and the fact that $K(\widetilde{f}_n^d) = K(\widetilde{f}_n) \leq K(f_n)$ yield

$$\frac{1}{K(f_n)} \leq \frac{l_{\widetilde{S}_n^d}(\widetilde{f}_n^d(\widetilde{\gamma}^d))}{l_{\widetilde{S}_0^d}(\widetilde{\gamma}^d)} \leq K(f_n).$$

Thus we can choose D and N_2 sufficiently large so that for $n > N_2$ and $l_{S_0}(\gamma) > D$, $l_{\widetilde{S}_n^d}(\widetilde{f}_n^d(\widetilde{\gamma}^d))/l_{\widetilde{S}_0^d}(\widetilde{\gamma}^d)$ is close to 1 and $4M'/l_{\widetilde{S}_0^d}(\widetilde{\gamma}^d)$ is sufficiently small. More precisely, we choose D and N_2 such that

$$\left|\log \frac{l_{S_n}(f_n(\gamma))}{l_{S_0}(\gamma)}\right| < \epsilon \text{ for every } n > N_2 \text{ and every } \gamma \in \Sigma''_{S_0}, l_{S_0}(\gamma) > D. \tag{4.5}$$

It remains to consider all arcs $\gamma \in \Sigma''_{S_0}$ such that $l_{S_0}(\gamma) \leq D$.

Let G_0 be the Fuchsian group uniformizing S_0 and let $\pi : \mathbb{D} \to S_0$ be the universal covering map. Choose a boundary component B of S_0 and let β the corresponding boundary geodesic. We lift β to a geodesic β^* in $\mathbb{D}$ and choose an interval I covering B once such that it is contained in the connected component of $\overline{\mathbb{D}} \setminus \beta^*$ that projects to B. Assume, without loss of generality, that all elements in Σ''_{S_0} are geodesic arcs. Let Ξ be the collection of those lifts γ^* of the elements $\gamma \in \Sigma''_{S_0}$ with $l(\gamma) \leq D$ such that they have one endpoint on I. Consider the collection of geodesics $\beta_i^* \neq \beta^*$ on $\mathbb{D}$, projecting to a boundary geodesic on S_0 and such that $\beta_j^* \cap \gamma^* \neq \emptyset$ for some $\gamma^* \in \Xi$. We claim that this collection has finitely many elements $\beta_1^*, ..., \beta_r^*$.

Suppose there are infinitely many such β_j^*. For each j, let $\gamma_j^* \in \Xi$ such that $\gamma_j^* \bigcap \beta_j^* \neq \emptyset$. If there is a subsequence $\{\beta_{j_k}^*\}$ that does not accumulate at one of the endpoints of β^*, then it is easily seen that $l(\gamma_{j_k}^*; \beta^*, \beta_{j_k}^*) \to \infty$ as $k \to \infty$, which contradicts the fact that $l(\gamma_{j_k}^*; \beta^*, \beta_{j_k}^*) \leq D$. Suppose now that there is a subsequence $\{\beta_{j_k}^*\}$ accumulating at one of the endpoints of β^*. By the Collar Lemma [13], the distance between β^* and β_j^* is bounded from below by a constant $d_0 > 0$. Then by Lemma 6 we also obtain $l(\gamma_{j_k}^*; \beta^*, \beta_{j_k}^*) \to \infty$ as $k \to \infty$. Again a contradiction to $l(\gamma_{j_k}^*; \beta^*, \beta_{j_k}^*) \leq D$

As we mentioned above, by the Collar Lemma, there exists $d_0 > 0$ such that the hyperbolic distance between β^* and β_j^*, $j = 1, ..., r$, is at least d_0. On the other hand, since $l(\gamma^*; \beta^*, \beta_j^*) \leq D$ for each γ^* in Ξ, this distance is bounded above by some number $d_1 \leq D$. For each $j = 1, ..., r$, we can normalize the group G_0 so that $\beta^* = L_{-i,i}$ and $\beta_j^* = L_{\bar{b},b}$, i.e., β_j^* is the d-standard geodesic for some $d \in [d_0, d_1]$. Let $F_{n,j}$ denote the normalization of the mapping F_n. Let $s_0 > 0$ be the constant that guarantees the conclusion of Lemma 7 and choose $\delta > 0$ so that it makes Lemma 8 true. Recall that F_n converges uniformly to the identity map. Then for each fixed j, the same is true for $F_{n,j}$. Then we can choose $N(j)$ sufficiently large so that for each $n > N(j)$, $|F_{n,j}(x) - x| < \delta$. It follows from Lemma 8 that for every $n > N(j)$ and for every geodesic $\gamma^* = L_{p,q} \in \Xi$ crossing $L_{-i,i}$ and $L_{\bar{b},b}$ we have

$$\left| \log \frac{l(L_{p,q}; L_{-i,i}, L_{\overline{b(d)},b(d)})}{l(L_{F_{n,j}(p),F_{n,j}(q)}; L_{F_{n,j}(-i),F_{n,J}(i)}, L_{F_{n,j}(\bar{b}),F_{n,j}(b)})} \right| < \epsilon. \tag{4.6}$$

We can apply the same argument to any geodesic in the collection $\beta_1^*, ..., \beta_r^*$, and to any boundary component B of S_0, since there are only finitely many of them, we conclude that there exists N_3 such that

$$\left| \log \frac{l_{S_n}(f_n(\gamma))}{l_{S_0}(\gamma)} \right| < \epsilon \text{ for every } n > N_3 \text{ and every } \gamma \in \Sigma''_{S_0}, l_{S_0}(\gamma) \le D. \tag{4.7}$$

By choosing $N = \max\{N_1, N_2, N_3\}$ and combining inequalities (4.2), (4.5), and (4.7) we have

$$\log \sup_{\gamma \in \Sigma_{S_0}} \left\{ \frac{l_{S_n}(f_n(\gamma))}{l_{S_0}(\gamma)}, \frac{l_{S_0}(\gamma)}{l_{S_n}(f_n(\gamma))} \right\} < \epsilon \text{ for every } n > N,$$

that is, $d_{ML}([S_n, fn], [S_0, Id]) \to 0$ as $n \to \infty$ and the continuity of the mapping $Id : (T(S_0), d_T) \to (T(S_0), d_{ML})$ follows. □

As a corollary of the previous proof we have the following result.

Corollary 3. *Let* $\{[S_n, f_n]\}$ *be a sequence in* $T(S_0)$ *such that*

1. *for each* n, $f_n(\widetilde{S}_0) = \widetilde{S}_n$,

2. $K(f_n|_{\widetilde{S}_n}) \to 1$ *as* $n \to \infty$, *and*

3. *for each* n, *there is a lift* $F_n : \mathbb{D} \to \mathbb{D}$ *of* f_n *such that the sequence* $\{F_n\}$ *converges uniformly to the identity on* $\mathbb{S}^1$.

Then $d_{ML}([S_n, f_n], [S_0, Id]) \to 0$ *as* $n \to \infty$.

Unlike the case of $T^R(S_0)$, the metrics d_T and d_{ML} do not define the same topology on $T(S_0)$.

Theorem 25. *There exists a sequence* $\{\tau_n\}$ *in* $T(S_0)$ *such that*

$$d_{ML}(\tau_n, \tau) \to 0 \text{ as } n \to \infty \text{ and}$$

$$d_T(\tau_n, \tau) \to \infty \text{ as } n \to \infty,$$

where $\tau = [S_0, Id]$.

Proof. For each $n = 1, ...,$ we will construct a mapping $F_n : \mathbb{D} \to \mathbb{D}$ that projects to a mapping $f_n : S_0 \to S_0$ such that the sequence $\{[S_0, f_n]\}$ satisfies the hypothesis of Corollary 3.

Let G_0 be the Fuchsian group uniformizing S_0 and $\pi : \mathbb{D} \to S_0$ the covering map. Suppose $D_0 \subseteq \mathbb{D}$ is a Dirichlet fundamental domain for the group G_0 and denote its closure in the unit disk with respect to the Euclidean topology by $\overline{D}$. Let $I \subseteq \overline{D}_0$ be an interval of $\mathbb{S}^1$ that projects to an arc in a boundary component of S_0. Let T be a Möbius transformation from the unit circle on the upper half-plane mapping I to $[0,1]$ and for each n, let $b_n = 1/(2^{n+1}-1), c_n = 1/2^n$. Define $F_n|_I$ to be the mapping $T^{-1} \circ H_n \circ T$, where $H_n : [0,1] \to [0,1]$ is the piecewise linear map that sends $0, b_n, c_n$ and 1 to $0, (2^n-1)/2^{2n}$ and $c_n, 1$ respectively. Note that H_n converges uniformly to the identity. Denote by β the boundary geodesic on S_0 homotopic to the boundary component containing $\pi(I)$ and let A be the connected component of $\overline{D}_0 \setminus \pi^{-1}(\beta)$ containing I. Define $F_n|_{(\overline{D}_0 \cap \mathbb{S}^1 \setminus I) \cup (D_0 \setminus A)}$ to be the identity. Finally, foliate A by geodesic rays that start at $D_0 \cap \pi^{-1}(\beta)$ and end at $\overline{A} \cap \mathbb{S}^1$. For every geodesic ray in the foliation starting at z and ending at $x \in \mathbb{S}^1$, let F_n map this ray onto the one starting at z and ending at at $F_n(x)$ by preserving the hyperbolic distance to z. The mapping $F_n : D_0 \to D_0$ can be extended to the whole hyperbolic plane by pre-composing and post-composing by elements of G_0. Since $F_n|_{\overline{D}_0}$ converges uniformly to the identity, it follows follows from the equicontinuity of the elements in G_0 that $F_n : \mathbb{D} \to \mathbb{D}$ converges uniformly to the identity as well. Moreover, each F_n can be projected to a mapping $f_n : S_0 \to S_0$ such that $f_n|_{\widetilde{S}_0} = Id_{\widetilde{S}_0}$.

Let $\tau_n = [S_0, f_n]$. Then by Corollary 3,

$$d_{ML}(\tau_n, \tau) \to 0 \text{ as } n \to \infty,$$

where $\tau = [S_0, Id]$.

Notice that by construction,

$$\mathrm{cr}(0, b_n, c_n, 1) = 1,$$

while

$$\mathrm{cr}(F_n(0), F_n(b_n), F_n(c_n), F_n(1)) = 2^n - 2 + \frac{1}{2^{n-1}} \to \infty \text{ as } n \to \infty.$$

Theorem 16 then implies that $K_e(f_n) \to \infty$ as $n \to \infty$, i.e.,

$$d_T(\tau, \tau_n) \to \infty \text{ as } n \to \infty.$$

□

Theorems 24 and 25 together imply Theorem 21. Now we prove Theorem 22.

Let G_0, D_0 and $I = [a, b]$ be as in the proof of Theorem 25. Let T be a Möbius transformation mapping I to $[0, 1]$, with $T(a) = 0$ and $T(b) = 1$. For each n, we construct a mapping $F_n : \mathbb{D} \to \mathbb{D}$ as in the proof of Theorem 25, except that the mapping H_n used to define $F_n|_I = T^{-1} \circ H_n \circ T$, is the piecewise linear mapping sending $0, 1/2$ and 1 to $0, 1/2^n$ and 1, respectively. Notice that for $n > m$

$$H_n \circ H_m^{-1}(x) = \begin{cases} 2^{m-n}x & \text{if } 0 \leq x \leq \frac{1}{2^m} \\ \frac{2^m - 2^{m+n}}{2^n - 2^{n+m}}(x-1)+1 & \text{if } \frac{1}{2^m} \leq x \leq 1 \end{cases}$$

Thus

$$\max_{x\in[0,1]} |H_n \circ H_m^{-1}(x) - x| \leq \frac{1}{2^m} - \frac{1}{2^n} \leq \frac{1}{2^m}.$$

It follows that $H_n \circ H_m^{-1}$ is uniformly close to the identity for n and m large, $n > m$. The equicontinuity of G_0 gives that $F_n \circ F_m^{-1}$ is also uniformly close to the identity on $\mathbb{S}^1$ for n and m large. Each F_n projects to a mapping $f_n : S_0 \to S_0$ such that $f_n|_{\widetilde{S}_0} = Id_{\widetilde{S}_0}$. Since $f_n \circ f_m^{-1}$ is the identity on the Nielsen kernel $\widetilde{S}_0$, we can apply Corollary 3 to conclude that

$$d_{ML}([S_0, f_n], [S_0, f_m]) = d_{ML}([S_0, f_n \circ f_m^{-1}], [S_0, Id]) \to 0 \text{ as } n, m \to \infty.$$

Thus $\{[S_0, f_n]\}$ is a Cauchy sequence under d_{ML}. Now we prove this sequence does not converge. Suppose that this sequence satisfies

$$d_{ML}([S_0, f_n], [S, f]) \to 0 \text{ as } n \to \infty \text{ for some } \tau = [S, f] \in T(S_0). \tag{4.8}$$

Then

$$d_L([S_0, f_n], [S, f]) \to 0 \text{ as } n \to \infty \text{ in } T^R(S_0).$$

By construction $[S_0, f_n] = [S_0, Id]$ in $T^R(S_0)$ for each n. It follows that $[S, f]$ and $[S_0, Id]$ determine the same point in the reduced Teichmüller space and hence, S_0 is conformally equivalent to S. By post-composing by an appropriate conformal mapping we may assume that $\tau = [S_0, f]$ and that f_n is homotopic to f for each n.

Let F be a lift of f that agrees with F_n on the limit set of G_0 for all n. We may assume that we chose $I = \lceil a, b \rceil$ in such a way that there exist lifts β_1^* and β_2^* of two (or possible one) boundary geodesics in S_0 such that their common orthogonal is $\gamma_0^* = L_{a,y}$ for some $y \in \mathbb{S}^1$. By construction each F_n fixes a and y. Then $L_{a,y} = L_{F_n(a),F_n(y)}$ for all n. It follows that $l_{S_0}(f_n(\gamma_0)) = l_{S_0}(\gamma_0)$ for all n, where γ_0 is the projection of γ_0^*. Then, condition (4.8) yields $l_{S_0}(\gamma_0) = l_{S_0}(f(\gamma_0))$. Since the common perpendicular segment is the smallest arc joining two geodesics we must have $F(a) = a$ and $F(y) = y$.

Now let $\gamma_1^* = L_{T^{-1}(1/2),x}$. Since by construction, $F_n(T^{-1}(1/2)) \to a$ as $n \to \infty$, we have

$$l(L_{F_n(T^{-1}(1/2)),F_n(y)}; \beta_1^*, \beta_2^*) \to l(L_{a,y}; \beta_1^*, \beta_2^*), \text{ as } n \to \infty,$$

that is

$$l_{S_0}(f_n(\gamma_1)) \to l_{S_0}(\gamma_0) \text{ as } n \to \infty,$$

where γ_1 is the projection of γ_1^*. By condition (4.8) we obtain $l_{S_0}(f(\gamma_1)) = l_{S_0}(\gamma_0)$. Using again the fact that the common orthogonal has smallest length along all curves connecting two geodesics we get $F(T^{-1}(1/2)) = a$. Since

$F : \mathbb{S}^1 \to \mathbb{S}^1$ is one-to-one and $F(a) = a$ we get a contradiction. $\square$

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