

INFORMATION TO USERS

This was produced from a copy of a document sent to us for microfilming. While the most advanced technological means to photograph and reproduce this document have been used, the quality is heavily dependent upon the quality of the material submitted.

The following explanation of techniques is provided to help you understand markings or notations which may appear on this reproduction.

1. The sign or "target" for pages apparently lacking from the document photographed is "Missing Page(s)". If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting through an image and duplicating adjacent pages to assure you of complete continuity.
2. When an image on the film is obliterated with a round black mark it is an indication that the film inspector noticed either blurred copy because of movement during exposure, or duplicate copy. Unless we meant to delete copyrighted materials that should not have been filmed, you will find a good image of the page in the adjacent frame. If copyrighted materials were deleted you will find a target note listing the pages in the adjacent frame.
3. When a map, drawing or chart, etc., is part of the material being photographed the photographer has followed a definite method in "sectioning" the material. It is customary to begin filming at the upper left hand corner of a large sheet and to continue from left to right in equal sections with small overlaps. If necessary, sectioning is continued again—beginning below the first row and continuing on until complete.
4. For any illustrations that cannot be reproduced satisfactorily by xerography, photographic prints can be purchased at additional cost and tipped into your xerographic copy. Requests can be made to our Dissertations Customer Services Department.
5. Some pages in any document may have indistinct print. In all cases we have filmed the best available copy.

University
Microfilms
International

300 N. ZEEB RD., ANN ARBOR, MI 48106

FERIA, ERLAN HECTOR

MATCHED PROCESSORS FOR OPTIMUM CONTROL

City University of New York

PH.D. 1981

University
Microfilms
International

300 N. Zeeb Road, Ann Arbor, MI 48106

Copyright 1981

by

Feria, Erlan Hector

All Rights Reserved

MATCHED PROCESSORS FOR OPTIMUM CONTROL

by

ERLAN HECTOR FERIA

A dissertation submitted to the Graduate
Faculty in Engineering in partial fulfillment of
the requirements for the degree of Doctor of Philosophy
The City University of New York

1981

COPYRIGHT BY
ERLAN HECTOR FERIA
1981

This manuscript has been read and accepted for the Graduate Faculty in Engineering in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

6/10/81
date

FE Than
Chairman of Examining Committee

6/10/81
date

Paul R Kammef
Executive Officer

Professor Vincent DelToro

Professor George Kranc

Professor Frederick Thau
Supervisory Committee

The City University of New York

ABSTRACT

Optimum and suboptimum control laws for linear (on state) deterministic processes with quadratic (on state) performance criteria and quantized control are investigated using a 'Matched Processors (MPs) approach'. The optimum control law at each stage k is found to consist of $M^{m(N-k)}$ processors and a Comparator Decoder where: a) M , N , and m are the number of control quantization levels, process stages, and process controls, respectively; b) each of the $M^{m(N-k)}$ processors is a single output processor which has a quadratic dependence on the current process state, $x(k)$, and in addition has parameters that depend on a unique control sequence from stage k to $N-1$, $U_{k,N-1}$ (this originates the names 'Quadratic Matched Processor (QUMP)' for each of the $M^{m(N-k)}$ processors, 'Matched Processors (MPs) approach' for the design tool, and 'QUMPs control law' for the designed optimum control law); and c) the Comparator Decoder chooses among the $M^{m(N-k)}$ QUMPs the one that yields the lowest output and then selects from the control sequence that was used in the design of the chosen MP the k -th stage control member, $u^*(k)$ (this is the optimum control action that must be applied to the process). In addition, it is found that for a special case of the general control problem considered in this dis-

sertation, the optimum control law is equivalently given by a simpler Linear Matched Processors (LIMPs) control law which consists of $M^{m(N-k)}$ LIMPs and a Comparator Decoder; the LIMPs control law is simpler than the QUMPs control law since its LIMPs are linear, instead of quadratic, functions of the process state $x(k)$.

Although the implementation at each stage k of each of the $M^{m(N-k)}$ QUMPs or LIMPs is relatively easy to achieve in a digital computer, it is found that the implementation of all the $M^{m(N-k)}$ QUMPs or LIMPs is impossible for typical values of M , m , and $N-k$. To deal with this dimensionality problem we then make use of the structure of the QUMPs and LIMPs control laws to define 'Reduced Order Matched Processors (ROMPs) control laws' which are shown to offer the possibility of having practical control laws with outstanding performances: a ROMPs control law is defined as any QUMPs or LIMPs control law which has had its order of MPs at any stage k , $M^{m(N-k)}$, reduced either by the simplification of the original QUMPs or LIMPs control law or by the outright deletion of some of its $M^{m(N-k)}$ MPs. In particular, we define I Stages Reduced Order Quadratic Matched Processors (ROQUMPs) and Reduced Order Linear Matched Processors (ROLIMPs) control laws with $I \in (1, \dots, N)$ as a class of ROMPs control laws that have the following characteristics: a) the order of their MPs at any stage k can not exceed M^{mI} ; b) N Stages ROQUMPs and ROLIMPs control laws are the same as

the QUMPs and LIMPs control laws, respectively; c) the maximum on-line computation time and memory capacity required by the I Stages POQUMPs and ROLIMPs control laws steadily decrease as I decreases from N to 1; and d) the I Stages ROLIMPs control law corresponding to a general stable scalar example is found to be an optimum control law for all possible values of I. The ROMPs control laws are then used to define ROMPs approaches which search for practical ROMPs control laws with desirable performances: a ROMPs approach is defined as any iterative approach that systematically searches for a physically realizable ROMPs control law with satisfactory on-line computation time and performance. In particular, I Stages ROQUMPs and ROLIMPs approaches are defined as iteration techniques that starting with $I = 1$, i.e. starting with a 1 Stage ROQUMPs or ROLIMPs control law, determine whether the considered I Stages ROMPs control law, besides having a practical on-line computation time and memory capacity, yields a performance that is close to the optimum and/or surpasses the performance of other offered suboptimum control laws. It is found that when the proposed I Stages ROQUMPs and ROLIMPs approaches are applied to two practical processes (with small, intermediate, and large values of N) they yield I Stages ROQUMPs and ROLIMPs control laws with excellent on-line computation time, memory capacity, and performance.

To my wife and my parents

ACKNOWLEDGEMENTS

The author wishes to express grateful acknowledgement to Professor Frederic Thau, his mentor, for the confidence, encouragement, and guidance he afforded him during the course of this research.

The author also wishes to acknowledge the valuable criticisms and suggestions that Professors Vincent DelToro and George Kranc made of this dissertation.

In addition, the author would like to thank the remaining members of his examining committee; Dr. Theodore Raphan (Mount Sinai Hospital of New York) for the time and effort he spent on the reading and constructive evaluation of this dissertation, and Professor Norman Scheinberg for his time and unwavering friendship.

Finally, the author wishes to thank his wife, Yvonne, for her constant support and encouragement.

TABLE OF CONTENTS

ABSTRACT	iv

<u>Chapter</u>	<u>page</u>

I. INTRODUCTION	1

STATEMENT OF PROBLEM	2
HISTORICAL BACKGROUND	4
Dynamic Programming	4
Quantized Linear Quadratic (QLQ) Control Law	7
Sano and N-Nagy (SNN) Control Law	10
ORGANIZATION OF DISSERTATION	15

II. OPTIMUM MATCHED PROCESSORS CONTROL LAWS	18

INTRODUCTION	18
DESIGN OF OPTIMUM CONTROL LAW	19
IMPLEMENTATION FEATURES OF OPTIMUM CONTROL LAW	31
ON-LINE COMPUTATION TIME OF OPTIMUM CONTROL LAW	32
SPECIAL CASE	34
Linear Matched Processors (LIMPs) Control Law	35
Implementation Features of LIMPs Control Law	41
On-Line Computation Time of LIMPs Control Law	42
SUMMARY AND CONCLUSIONS	43
Quadratic Matched Processors (QUMPs) Control Law	43
LIMPs Control Law	45
Conclusions	47

III. REDUCED ORDER MATCHED PROCESSORS CONTROL LAWS	49

INTRODUCTION	49
REDUCED ORDER MATCHED PROCESSORS (ROMPS) CONTROL LAWS	50
I STAGES ROMPS CONTROL LAWS	51
General Features	51
I Stages Reduced Order QUMPs (ROQUMPs) Control Law	55
I Stages Reduced Order LIMPs (ROLIMPs) Control Law	61
TWO SPECIAL CASES OF I STAGES ROMPS CONTROL LAWS	67
A General Stable Scalar Example	68
Equivalent Control Laws	77
SUMMARY AND CONCLUSIONS	86
I Stages ROMPs Control Laws	86

Two Special Cases of I Stages ROMPs Control	
Laws	88
Conclusions	89
IV. REDUCED ORDER MATCHED PROCESSORS APPROACHES	91
INTRODUCTION	91
ROMPS APPROACHES	92
I STAGES ROMPS APPROACHES	93
I Stages ROQUMPs Approach	94
I Stages ROLIMPs Approach	97
I STAGES POMPS APPROACHES FEATURES	99
Increase of Control Law Complexity with each	
Iteration	99
Convergence of Approaches	100
PRACTICAL EXAMPLES	100
General Framework	101
An Armature Controlled DC Motor	112
Longitudinal Aircraft Motion Control	132
SUMMARY AND CONCLUSIONS	160
I Stages ROMPs Approaches	160
Practical Examples	160
Conclusions	161
V. SUMMARY AND CONCLUSIONS	163
SUMMARY	163
RECOMMENDATIONS FOR FUTURE RESEARCH	167
 <u>Appendix</u>	 <u>page</u>
A. DERIVATION OF QUADRATIC MATCHED PROCESSOR	169
B. DERIVATION OF LIMPS CONTROL LAW	177
C. PROOF OF GENERAL STABLE SCALAR EXAMPLE	179
D. PROOF OF THREE EQUIVALENT CONTROL LAWS	248
E. COMPUTER PROGRAMS	259
 BIBLIOGRAPHY	 301

GLOSSARY OF SYMBOLS

n	Number of states for the given process.
m	Number of inputs for the given process.
M	Number of quantization levels in the space of each control component.
N	Number of process stages.
I	Number of stages for which the control components are assumed to attain any value in their space.
$x_{k,N}$	State trajectory from stage k to stage $N-1$.
$U_{k,N-1}$	Control trajectory from stage k to stage $N-1$.
$I_{k,N}$	Stage trajectory from stage k to stage N .
$h_N(x_{0,N}, U_{0,N-1}, I_{0,N})$	Performance Criterion.
$f_{N-k}(x(k))$	Optimum performance as measured from stage k .
$h_{N-k}(x(k), U_{k,N-1}, k)$	Quadratic Matched Processor (QUMP) that is a quadratic function of $x(k)$ and is matched to the control sequence $U_{k,N-1}$.
$h'_{N-k}(x(k), U_{k,N-1}, k)$	Linear Matched Processor (LIMP) that is a linear function of $x(k)$ and is matched to the control sequence $U_{k,N-1}$.

$P_1(U_{k,N-1},k)$	Parameter matrix that is matched to the control sequence $U_{k,N-1}$.
$P_2(U_{k,N-1},k)$	Parameter vector that is matched to the control sequence $U_{k,N-1}$.
$p_3(U_{k,N-1},k)$	Parameter scalar that is matched to the control sequence $U_{k,N-1}$.
ROMP	Reduced order matched processor.
ROQUMP	Reduced order quadratic matched processor.
ROLIMP	Reduced order linear matched processor.

Chapter I

INTRODUCTION

A fundamental problem in the regulation of a linear (on state) deterministic process with quadratic (on state) performance criteria is the design of an optimum control law that assumes quantized controls. This class of problems is often the rule rather than the exception due to the ever increasing use of digital computers in the implementation of control laws for process control. Other types of applications where quantized controls are inherent in the control action, are servomechanism problems that use bang-bang control, and the general class of social, economic, communications, and biomedical applications where only a finite number of choices are available.

In the subsequent sections we first provide a precise statement of the control problem, then make a historical survey of the most important approaches that have been used to solve this control problem, and finally give the organization of this dissertation.

1.1 STATEMENT OF PROBLEM

Consider the discrete time process described by

$$x(k+1) = A(u(k), k)x(k) + B(u(k), k) \quad (1.1.1)$$

$$k = 0, \dots, N-1$$

where:

1. $x(k)$ and $u(k)$ are the n dimensional state vector and the m dimensional control vector, respectively. In addition the state and control spaces are given by

$$x_i(k) \in R, \text{ for all } i \text{ and } k \quad (1.1.2)$$

and

$$u_i(k) \in (c_1^i, \dots, c_M^i), \text{ for all } i \text{ and } k \quad (1.1.3)$$

where $x_i(k)$ is the i -th component of the state vector $x(k)$, R is the space of all real numbers, $u_i(k)$ is the i -th component of the control vector $u(k)$, M is the number of control quantization levels, and c_j^i is real and less than c_{j+1}^i for all j ; and

2. $A(u(k), k)$ and $B(u(k), k)$ are matrix functions of appropriate dimensions.

In addition consider the performance criterion

$$\begin{aligned}
 & h_N(x_{0,N}, u_{0,N-1}, I_{0,N}) \\
 &= \sum_{k=0}^N [x^T(k) Q_1(u(k), k) x(k) + q_3(u(k), k)]
 \end{aligned}
 \tag{1.1.4}$$

where:

1. $Q_1(u(k), k)$ is a matrix function of appropriate dimensions and $q_3(u(k), k)$ is a scalar function;
2. $Q_1(u(k), k)$ is symmetric; and
3. $x_{i,j}$, $u_{i,j}$ and $I_{i,j}$ are the state, control, and stage trajectories from stage i to j , i.e.,

$$x_{i,j} = (x(i), \dots, x(j)) \text{ where } i \leq j$$

$$u_{i,j} = (u(i), \dots, u(j)) \text{ where } i \leq j$$

$$I_{i,j} = (i, \dots, j) \text{ where } i \leq j$$

(1.1.5)

The control problem consists of finding the optimum control law, $u^*(x(k))$, for all k that minimizes $h_N(x_{0,N}, u_{0,N-1}, I_{0,N})$ subject to (1.1.1)-(1.1.3) as constraints, i.e., $u^*(x(k))$ for each stage k must be derived from the minimization

$$f_N(x(0)) = \min_{u(0) \dots u(N-1)} h_N(x_{0,N}, u_{0,N-1}, I_{0,N}) \quad (1.1.6)$$

subject to (1.1.1)-(1.1.3) as constraints, where $f_N(x(0))$ denotes the optimum performance.

1.2 HISTORICAL BACKGROUND

This historical survey is divided in three subsections. First in Section 1.2.1 we mention the Dynamic Programming approach and its basic drawbacks. In Sections 1.2.2 and 1.2.3 two suboptimum control laws for special cases of the control problem described in Section 1.1 and that have small computation time and memory capacity are reviewed.

1.2.1 Dynamic Programming

To solve feedback control problems with quantized control, such as the problem of Section 1.1, the approach that generally has been used by investigators is Dynamic Programming (Lewis and Tou[1963:229-233], Havira and Lewis[1972:191-196], Arora and Pierre[1973:85-91], etc.).

The Dynamic Programming approach consists of the following steps:

STEP 1. From the principle of optimality of Bellman[1957:83] it follows that the control law $u^*(x(k))$ at each stage k is found from the minimization

$$\begin{aligned} & f_{N-k}(x(k)) \\ &= \min_{u(k) \dots u(N-1)} h_{N-k}(x_{k,N}, u_{k,N-1}, I_{k,N}) \end{aligned} \quad (1.1.6a)$$

where

$$\begin{aligned} & h_{N-k}(x_{k,N}, u_{k,N-1}, I_{k,N}) \\ &= \sum_{i=k}^N [x^T(i) Q_1(u(i), i) x(i) + q_3(u(i), i)], \end{aligned} \quad (1.1.6b)$$

$f_{N-k}(x(k))$ is the optimum performance as measured from stage k , and the space of $u(k)$ for all k is given by (1.1.3).

STEP 2 Making use of Equations (1.1.6a) and (1.1.6b) the following recursive relation is then obtained

$$\begin{aligned} & f_{N-k}(x(k)) \\ &= \min_{u(k)} [x^T(k) Q_1(u(k), k) x(k) + q_3(u(k), k) + f_{N-(k+1)}(x(k+1))] \end{aligned} \quad (1.1.6c)$$

where $f_0(x(N)) = 0$.

STEP 3 Finally the optimum control law, $u^*(x(k))$, at each stage k is found by solving the recursive equation (1.1.6c) backwards in time starting at stage $N-1$ and then iteratively moving on to stage k .

The schemes that have been derived using the Dynamic Programming approach have been found to have two basic drawbacks (Sano and N-Nagy[1979:97]):

1. They do not yield a general analytical optimum control law. It is noted that even if the analytical optimum control law cannot be implemented physically, analytical knowledge about its structure may be used for the following two purposes:
 - a) To obtain theoretical insights that can sharpen our intuition and understanding of the control problem; and
 - b) To use the analytical structure of the optimum control law for the design of practical control laws with reasonable performance.

In the subsequent chapters it will be shown how a 'Matched Processors approach' yields an analytical optimum control law that is later used to find practical control laws with outstanding performance.

2. They require a large amount of computation time and memory capacity for most practical applications.

With the objective of alleviating the second drawback, suboptimum control laws have been proposed in the literature

that do not require a large amount of computation time and memory capacity¹. Next we will describe two of these control laws.

1.2.2 Quantized Linear Quadratic (QLQ) Control Law

One kind of suboptimum control law is the Quantized Linear Quadratic (QLQ) control law, $u^{QLQ}(x(k))$. Its derivation is based on methods that have been developed for a continuous control linear process with quadratic performance criterion (Athans[1971:529-552]) and quantizing the optimum control law that results.

Consider the process

$$x(k+1) = A(k)x(k) + B(k)u(k) \quad (1.2.2.1)$$

and performance criteria

$$\begin{aligned} & h_N(x_{0,N}, u_{0,N-1}, I_{0,N}) \\ &= \sum_{k=0}^N [x^T(k)Q(k)x(k) + u^T(k)R(k)u(k)] \quad (1.2.2.2) \end{aligned}$$

where: a) $x(k)$ and $u(k)$ are n and m dimensional state and control vectors, respectively; b) $A(k)$, $B(k)$, $Q(k)$, and $R(k)$ are matrices of appropriate dimensions; c) $Q(k)$ is symmetric; and d) the space of $x(k)$ is given by (1.1.2) and that of $u(k)$ by

¹ However, in Section 4.5 it is demonstrated with several physical examples that these approaches often provide bad performance.

$$u_i(k) \in R, \text{ for all } i \text{ and } k. \quad (1.2.2.3)$$

The optimum control law, $u^*(x(k))$, that minimizes Equation (1.2.2.2) subject to constraints (1.2.2.1), (1.2.2.3), and (1.1.2) is given by (Kwakernaak and Sivan[1972:490-494]),

$$u^*(x(k)) = -K(k)x(k) \quad (1.2.2.4)$$

where $K(k)$ is a $m \times n$ gain matrix that is found off-line from

$$K(k) = [B^T(k)P(k+1)B(k) + R(k)]^{-1}B^T(k)P(k+1)A(k) \quad (1.2.2.5)$$

and $P(k+1)$ results from the backward solution of the Ricatti difference equation,

$$\begin{aligned} P(k) = & A^T(k)P(k+1)A(k) + Q(k) \\ & - A^T(k)P(k+1)B(k)[B^T(k)P(k+1)B(k) + R(k)]^{-1} \\ & \cdot B^T(k)P(k+1)A(k) \end{aligned} \quad (1.2.2.6)$$

with terminal condition

$$P(N) = Q(N).$$

Using the quantized control space (1.1.3) the above optimum control law, $u^*(x(k))$, is then quantized yielding the QLQ control law, $u^{QLQ}(x(k))$, where the components of the m

m-dimensional vector $u^{QLQ}(x(k))$ are given by

$$u_i^{QLQ}(x(k)) = Cl_i[-K(k)x(k)] \quad (1.2.2.7)$$

$$i = 1, \dots, m$$

where $Cl_i[-K(k)x(k)]$ denotes the element of the set $(c_1^i, \dots, c_M^i)$ that is closest to the value of the i -th component of the m dimensional vector $-K(k)x(k)$, and $K(k)$ is found off-line from Equations (1.2.2.5) and (1.2.2.6).

The following memory capacity and on-line computation time are found for the QLQ control law:

MEMORY CAPACITY. Since $K(k)$ is a $m \times n$ gain matrix that can be easily evaluated off-line and in addition the set $(c_1^i, \dots, c_M^i)$ for all i must be saved so that it can be used in the evaluation of $Cl_i[.]$, it follows from Equation (1.2.2.7) that the memory capacity over N stages, C_{QLQ} , that is required by the QLQ control law is given by $m(nN + M)$ words², i.e.,

$$C_{QLQ} = m(nN + M) \text{ words.} \quad (1.2.2.8)$$

ON-LINE COMPUTATION TIME. Since the product $-K(k)x(k)$ requires nm multiplications and $(n - 1)m$ additions it follows from Equation (1.2.2.7) that the on-line computation time, T_{QLQ} , required by the QLQ control law at any stage k is given by

² A word is defined as the number of bits that a digital computer uses to represent a number.

$$T_{QLQ} = \{T_{C1} + T_a(n - 1) + T_m n\} m \quad (1.2.2.9)$$

where: a) T_{C1} is the time required by a digital computer to find the element in set $(c_1^i, \dots, c_M^i)$ that is closest to a given real number; and b) T_a and T_m are the times taken by a digital computer to perform an addition and a multiplication, respectively.

It is noted from Equations (1.2.2.8) and (1.2.2.9) that C_{QLQ} and T_{QLQ} are small for typical values of M , N , m , n , T_{C1} , T_a , and T_m . For instance, when $M = 64$, $N = 100$, $m = 2$, $n = 3$, $T_{C1} = 2$ microseconds, $T_a = 200$ nanoseconds, and $T_m = 2$ microseconds we have that C_{QLQ} and T_{QLQ} are given by

$$C_{QLQ} = 664 \text{ words}$$

and

$$T_{QLQ} = 16.8 \text{ microseconds:}$$

these values for C_{QLQ} and T_{QLQ} are small enough for most practical applications.

1.2.3 Sano and N-Nagy (SNN) Control Law

A second kind of suboptimum control law has been derived by Sano and N-Nagy (SNN) and is denoted by $u^{SNN}(x(k))$ (Sano and N-Nagy[1979:97-100]).

Consider the process

$$x(k+1) = Ax(k) + bu(k) \quad (1.2.3.1)$$

and performance criterion

$$\begin{aligned}
 & h_N(x_{0,N}, u_{0,N-1}) \\
 &= \sum_{k=0}^N [x^T(k) Q x(k) + r u^2(k)] \quad (1.2.3.2)
 \end{aligned}$$

where: a) $x(k)$ is an n dimensional state vector and $u(k)$ is a scalar control; b) A , b , and Q are matrices of appropriate dimensions and r is a scalar; c) Q is symmetric; and d) the space of $x(k)$ is given by (1.1.2) and that of $u(k)$ by

$$u(k) \in (-d_S, \dots, -d_1, 0, d_1, \dots, d_S), \text{ for all } k, \quad (1.2.3.3)$$

where the control quantization levels are uniformly distributed between $-d_S$ and d_S and S is a positive integer.

Sano and N-Nagy then used the subsequent approach to find a suboptimum control law that has relatively small computation time and memory capacity while 'hopefully' yielding a performance that is close to that of the unknown optimum control law that minimizes Equation (1.2.3.2) subject to (1.2.3.1), (1.2.3.3), and (1.1.2) as constraints: it is shown in Section 4.5 that the performance that is obtained when using the Sano and N-Nagy control law is in many ways unsatisfactory.

First using the principle of optimality

(Bellman[1957:83]) they obtained the recursive equation

$$f_{N-k}(x(k)) = \min_{u(k)} [x^T(k) Q x(k) + r u^2(k) + f_{N-(k+1)}(x(k+1))] \quad (1.2.3.4)$$

with

$$f_0(x(N)) = 0$$

where $f_{N-k}(x(k))$ denotes the minimum performance that can be attained when the initial condition is $x(k)$.

Then starting with

$$k = N-1$$

and proceeding on to

$$k = 0,$$

they solved Equation (1.2.3.4) at each stage k by making the simplifying assumption

$$f_{N-(k+1)}(x(k+1)) = x^T(k+1) P(k+1) x(k+1) \quad (1.2.3.5)$$

where $P(k+1)$ is derived by minimizing the integral criterion,

$$g = \int_{R^*} [x^T(k) P(k) x(k) - f_{N-k}(x(k))]^2 dx(k) \quad (1.2.3.6)$$

where R^* represents an assumed finite closed region in the space of $x(k)$ which makes the necessary integrations feasible.

Finally using the above approximations the following SNN control law, $u^{SNN}(x(k))$, was derived:

$$u^{SNN}(x(k)) = Cl[-K(k)x(k)] \quad (1.2.3.7)$$

where $Cl[-K(k)x(k)]$ denotes the element in the set (1.2.3.3) that is closest to the value of the scalar $-K(k)x(k)$, and $K(k)$ is a n dimensional gain vector that is evaluated offline from

$$K(k) = b^T P(k+1) A / w(k+1) \quad (1.2.3.8)$$

with

$$w(k+1) = b^T P(k+1) b + r \quad (1.2.3.9)$$

and $P(k+1)$ results from the backward solution of the difference equation, .

$$\begin{aligned} P(k) &= A^T P(k+1) A + Q \\ &+ e A^T P(k+1) b b^T P(k+1) A / w(k+1) \\ &+ f b^T P(k+1) A A^T P(k+1) b I / w(k+1) \end{aligned} \quad (1.2.3.10)$$

with terminal condition

$$P(N) = Q$$

where

$$e = (225n + 180) (k_2 - k_1) / (5n + 4), \quad (1.2.3.11)$$

$$f = 45 (5k_1 - 9k_2) / (5n + 4), \quad (1.2.3.12)$$

$$k_1 = \text{Sigm}/12 (2S + 1)^5 d_1^5 - d_S^2/3 (2S + 1)^2 d_1^2 + d_S/4 (2S + 1) d_1, \quad (1.2.3.13)$$

$$k_2 = \text{Sigz}/6 (2S + 1)^3 d_1^3 - d_S^2/3 (2S + 1)^2 d_1^2 + d_S/6 (2S + 1) d_1, \quad (1.2.3.14)$$

$$\begin{aligned} \text{Sigm} = & -\sum_{i=1}^{S-1} d_i [(d_i + d_{i+1})^3 (d_i - 3d_{i+1}) \\ & + (d_{i-1} + d_i)^3 (3d_{i-1} - d_i)] \\ & - d_S (d_{S-1} + d_S)^3 (3d_{S-1} - d_S), \end{aligned} \quad (1.2.3.15)$$

$$\text{Sigz} = -\sum_{i=1}^{S-1} d_i (d_{i-1} - d_{i+1}) - d_S (d_{S-1}^2 - d_S^2), \quad (1.2.3.16)$$

and I is a unit matrix.

The following memory capacity and on-line computation time are found for the SNN control law, $u^{\text{SNN}}(x(k))$:

MEMORY CAPACITY. A comparison of Equations (1.2.2.7) and (1.2.3.7) reveal that the memory capacity over N stages, C_{SNN} , that is required by the SNN control law is given by Equation (1.2.2.8) where $m = 1$, i.e.,

$$C_{\text{SNN}} = nN + M \text{ words.} \quad (1.2.3.17)$$

ON-LINE COMPUTATION TIME. Again a comparison of Equations (1.2.2.7) and (1.2.3.7) reveal that the on-line computation time, T_{SNN} , that is required by the SNN control law is given by Equation (1.2.2.9) where $m = 1$, i.e.,

$$T_{\text{SNN}} = T_{\text{cl}} + T_a (n - 1) + T_m n \quad (1.2.3.18)$$

where T_{cl} , T_a , and T_m are as defined for Equation (1.2.2.9).

It is noted that the SNN control law also has excellent memory capacity and on-line computation time.

1.3 ORGANIZATION OF DISSERTATION

The remaining four chapters of the dissertation are organized as follows:

In Chapter II, a general analytical optimum control law is found for the control problem of Section 1.1 via a Matched Processors approach; the optimum control is assigned the name Quadratic Matched Processors (QUMPs) control law. For the special case where $A(u(k), k)$ and $Q_1(u(k), k)$ are given by $A(k)$ and $Q_1(k)$, respectively, it is shown that the QUMPs control law can be replaced by a simpler Linear Matched Processors (LIMPs) control law. In addition, the computation time and memory capacity of both the QUMPs and LIMPs control laws are investigated; it is found that the basic drawback of the QUMPs and LIMPs control laws is the need for a large amount of memory capacity and computation time which in most cases makes their physical implementation impossible.

In Chapter III, the structure of the QUMPs and LIMPs control laws of Chapter II are used to define Reduced Order

Matched Processors (ROMPs) control laws; these ROMPs control laws offer the possibility of having practical control laws with desirable performance which may be used in place of the QUMPs and LIMPs control laws. For two classes of ROMPs control laws, denoted as I Stages Reduced Order Quadratic Matched Processors (ROQUMPs) control laws and I Stages Reduced Order Linear Matched Processors (ROLIMPs) control laws, computation time and memory capacity are investigated; it is found that when I is a small positive integer the corresponding I Stages ROQUMPs and ROLIMPs control laws often need relatively small amounts of computation time and memory capacity. This makes their physical implementation possible. It is also shown in this chapter that for a general stable scalar example its corresponding I Stages ROLIMPs control law is optimum for all possible values of I, and that in addition the 1 Stage ROLIMPs control law require a memory capacity and on-line computation time that besides being relatively small are also smaller than those of the remaining I Stages ROLIMPs control laws.

In Chapter IV, the ROMPs control laws of Chapter III are used to define 'ROMPs approaches' which search for practical ROMPs control laws with desirable performance. In particular, I Stages ROQUMPs and ROLIMPs approaches are defined and applied to two physical examples; for each of these physical examples it is shown that there exist at least a ROMPs control law that requires a small amount of memory capacity and

on-line computation time while yielding outstanding performance.

Finally in Chapter V, summary and conclusions for this dissertation are given including problems for future research.

Chapter II

OPTIMUM MATCHED PROCESSORS CONTROL LAWS

2.1 INTRODUCTION

In Chapter I, a statement of the control problem is given along with a historical survey of the approaches that have been used in its solution. In the historical survey a special mention is made of the fact that one of the basic drawbacks of the offered Dynamic Programming techniques is that they fail to yield an analytical optimum control law (the other drawback being the large amount of computation time and memory capacity needed by these schemes). It was noted that even when the analytical optimum control law is not physically realizable it may be used for the following two purposes:

1. To obtain theoretical insights that sharpen our intuition and understanding of the control problem; and
2. To use the analytical structure of the optimum control law for the design of practical control laws with reasonable performance.

It is the objective of this chapter to find analytical optimum control laws for the problem of Section 1.1 via a Matched Processors (MPs) approach. Although, in most cases the optimum control law is found to be physically unrealizable, its structure is used in subsequent chapters for the design of practical control laws with satisfactory performance.

This chapter is composed of five more sections. In Section 2.2 we first design the desired optimum control law, $u^*(x(k))$, of Section 1.1 via a MPs approach; the designed $u^*(x(k))$ is then discussed. In Sections 2.3 and 2.4 the implementation features, memory capacity, and computation time of $u^*(x(k))$ are investigated. Section 2.5 concerns itself with an important special case of the process and performance criterion Equations (1.1.1) and (1.1.4) (Athans[1972:529-552]) which is found to result in a control law that has a simpler structure than that found in Section 2.2. Finally in Section 2.6 summary and conclusions of this chapter are given.

2.2 DESIGN OF OPTIMUM CONTROL LAW

The optimum control law, $u^*(x(k))$, is promptly derived via a MPs approach which consists of the following three steps:

STEP 1. From (1.1.6) and the principle of optimality which says that "an optimal policy has the property that whatever the initial state and initial decision are the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision" (Bellman[1957:83]), it follows that $u^*(x(k))$ at each stage k can be found from the minimization

$$f_{N-k}(x(k)) = \min_{u(k) \dots u(N-1)} h_{N-k}(x_{k,N}, u_{k,N-1}, I_{k,N}) \quad (2.2.1)$$

where

$$h_{N-k}(x_{k,N}, u_{k,N-1}, I_{k,N}) = \sum_{i=k}^N [x^T(i)Q_1(u(i), i)x(i) + q_3(u(i), i)] \quad (2.2.2)$$

and the minimization is done subject to (1.1.1)-(1.1.3) as constraints.

STEP 2. From the process Equation (1.1.1) we note that it is possible to find an expression for $x(i)$, where

$$i = k+1, \dots, N,$$

as a function of the current state $x(k)$, the control sequence $U_{k,i-1}$, and the stage sequence $I_{k,i-1}$, i.e.,

$$x(i) = y(x(k), U_{k,i-1}, I_{k,i-1}) \quad (2.2.2a)$$

for

$$i = k+1, \dots, N$$

where the vector function $y(.,.,.,.)$ is found by solving the process Equation (1.1.1). Consequently, it is possible to eliminate the functional dependence of $h_{N-k}(X_{k,N}, U_{k,N-1}, I_{k,N})$ in the state sequence $X_{k+1,N}$ by using the solution to the process Equation (1.1.1), i.e., the right hand side (RHS) of Equation (2.2.2a), in the RHS of Equation (2.2.2) for each state $x(i)$ in the state sequence $X_{k+1,N}$.

STEP 3. Finally from STEP 1 and STEP 2 it follows that $u^*(x(k))$ at each stage k is given by

$$u^*(x(k)) = u^*(k) \quad (2.2.3)$$

where $u^*(k)$ is the k -th stage member of the optimum control sequence $U_{k,N-1}^*$ that minimizes

$$\begin{aligned} & f_{N-k}(x(k)) \\ &= \min_{U_{k,N-1} \in US(k)} h_{N-k}(x(k), U_{k,N-1}, k) \end{aligned} \quad (2.2.4)$$

where $US(k)$ is the universal set of all possible control sequences from stage k to stage N , i.e.,

$$\begin{aligned} & US(k) \\ &= [U_{k,N-1} : u_i(s) \in (c_1^i, \dots, c_M^i) \text{ for all } i \text{ and } s], \end{aligned} \quad (2.2.5)$$

and $h_{N-k}(x(k), U_{k,N-1}, k)$ is given by Equation (2.2.2) where $x(i)$ for $i = k+1, \dots, N$ is replaced by the RHS of Equation (2.2.2a).

In Appendix A it is shown that when the solution to Equation (1.1.1) is substituted for $x(i)$ in Equation (2.2.2) we have that $h_{N-k}(x(k), U_{k,N-1}, k)$ (or the Quadratic Matched Processor (QUMP) as named later) is readily derived from

$$\begin{aligned} & h_{N-k}(x(k), U_{k,N-1}, k) \\ &= x^T(k) P_1(U_{k,N-1}, k) x(k) + P_2(U_{k,N-1}, k) x(k) + p_3(U_{k,N-1}, k) \end{aligned} \quad (2.2.6)$$

where $P_1(U_{k,N-1}, k)$ and $P_2(U_{k,N-1}, k)$ are $n \times n$ and $1 \times n$ matrix functions, respectively, and $p_3(U_{k,N-1}, k)$ is a scalar function. These functions are either derived from the closed form equations

$$\begin{aligned} & P_1(U_{k,N-1}, k) \\ &= \sum_{i=k}^N F_1^T(U_{k,i-1}, i, k) Q_1(u(i), i) F_1(U_{k,i-1}, i, k), \end{aligned} \quad (2.2.7)$$

$$\begin{aligned} & P_2(U_{k,N-1}, k) \\ &= \sum_{i=k}^N 2F_2^T(U_{k,i-1}, i, k) Q_1(u(i), i) F_1(U_{k,i-1}, i, k) \end{aligned} \quad (2.2.8)$$

and

$$\begin{aligned} & p_3(U_{k,N-1}, k) \\ &= \sum_{i=k}^N \{ F_2^T(U_{k,i-1}, i, k) Q_1(u(i), i) F_2(U_{k,i-1}, i, k) \\ & \quad + q_3(u(i), i) \} \end{aligned} \quad (2.2.9)$$

where

$$F_1(U_{k,i-1}, i, k) = \prod_{t=i-1}^k A(u(t), t) \quad (2.2.10)$$

with

$$k = 0, \dots, i-1,$$

$$F_1(U_{i,i-1}, i, i) = \text{Unit Matrix for all } i, \quad (2.2.11)$$

$$F_2(U_{k,i-1}, i, k) = \sum_{t=k}^{i-1} F_1(U_{t+1,i-1}, i, t+1) B(u(t), t), \quad (2.2.12)$$

and

$$F_2(U_{i,i-1}, i, i) = \underline{0} \text{ for all } i \quad (2.2.13)$$

or equivalently derived from the backward solution to the linear difference equations,

$$\begin{aligned} & P_1(U_{k,N-1}, k) \\ &= A^T(u(k), k) P_1(U_{k+1,N-1}, k+1) A(u(k), k) + Q_1(u(k), k), \end{aligned} \quad (2.2.14)$$

$$\begin{aligned} & P_2(U_{k,N-1}, k) \\ &= [P_2(U_{k+1,N-1}, k+1) + 2B^T(u(k), k) P_1(U_{k+1,N-1}, k+1)] A(u(k), k) \end{aligned} \quad (2.2.15)$$

and

$$\begin{aligned} & p_3(U_{k,N-1}, k) = p_3(U_{k+1,N-1}, k+1) \\ &+ [P_2(U_{k+1,N-1}, k+1) + B^T(u(k), k) P_1(U_{k+1,N-1}, k+1)] B(u(k), k) \\ &+ q_3(u(k), k) \end{aligned} \quad (2.2.16)$$

where the terminal conditions are given by

$$P_1(U_{N,N-1}, N) = Q_1(N),$$

$$P_2(U_{N,N-1}, N) = 0,$$

and

$$P_3(U_{N,N-1}, N) = 0$$

Discussion of Results:

From Equations (2.2.3)-(2.2.16) and Figures 2.1a-b which give a schematic representation of these equations, it is inferred that the optimum control law, $u^*(x(k))$, requires:

1. The selection of the processor with smallest output, $h_{N-k}(x(k), U_{k,N-1}^*, k)$, among $M^{m(N-k)}$ candidates and then from $U_{k,N-1}^*$ the selection of $u^*(k)$; this operation is represented by the Comparator Decoder of Figure 2.1a and is verified from Equations (2.2.3)-(2.2.5) where in particular we note from Equation (2.2.5) that there are $M^{m(N-k)}$ control sequences in the universal set $US(k)$.
2. The processing of the current state $x(k)$ by $M^{m(N-k)}$ processors as seen from Equations (2.2.4)-(2.2.6) and the bank of $M^{m(N-k)}$ processors of Figure 2.1a: each processor is a quadratic function of $x(k)$ and in addition is matched to a unique control sequence $U_{k,N-1}$ via the matrix functions $P_1(U_{k,N-1}, k)$ and

Figure 2.1a Schematic Diagram of QUMPs Control Law^{1A}
Implementation Requirements (Overall Diagram).

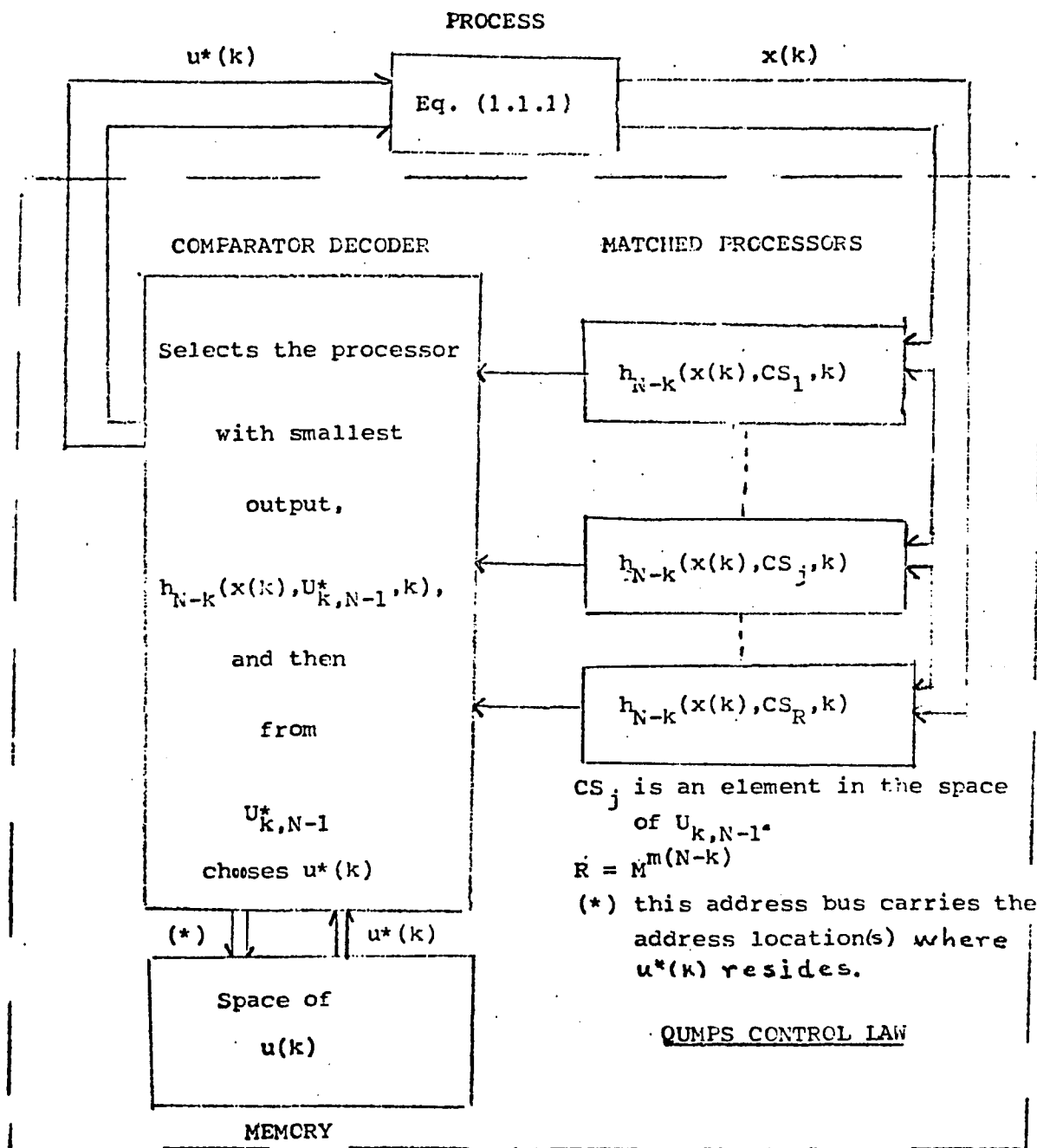
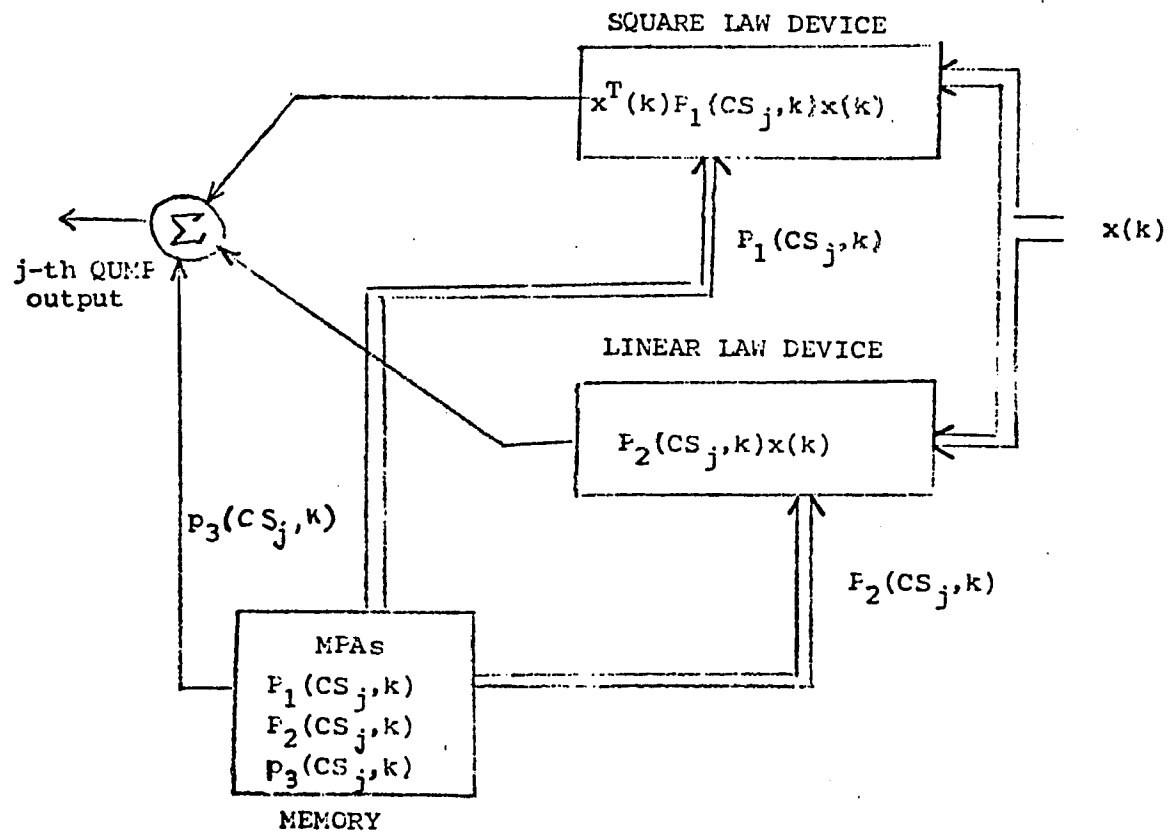


Figure 2.1b Schematic Diagram of QUMF's Implementation Requirements.



$P_2(U_{k,N-1}, k)$ and the scalar function $p_3(U_{k,N-1}, k)$ as seen from Equation (2.2.6) and Figure 2.1b. As a result of these structural features, from now on we call $h_{N-k}(x(k), U_{k,N-1}, k)$, the set

$$\{P_1(U_{k,N-1}, k), P_2(U_{k,N-1}, k), p_3(U_{k,N-1}, k)\},$$

and $u^*(x(k))$ by the names Quadratic Matched Processor (QUMP), Matched Parameters (MPAs), and Quadratic Matched Processors (QUMPs) control law, respectively. It is now clear why the design tool has been named the Matched Processors (MPs) approach.

3. The off-line evaluation and storage of the MPAs from either the closed form Equations (2.2.7)-(2.2.9) or the linear difference Equations (2.2.14)-(2.2.16), for rapid control action.
4. At each stage k the following storage and on-line multiplications, additions, and comparisons are needed as seen from Equations (2.2.3)-(2.2.6) and Figures 2.1a-b:

STORAGE. There are two types of storage requirements, these are: a)

$$(n^2/2 + 3n/2 + 1)M^{m(N-k)} \text{ words}$$

(2.2.17a)

corresponding to the MPAs $P_1(U_{k,N-1}, k)$, $P_2(U_{k,N-1}, k)$, and $P_3(U_{k,N-1}, k)$: note from (2.2.7) that $P_1(U_{k,N-1}, k)$ is symmetric since $Q_1(u(i), i)$ is symmetric for all i ; and b)

$$mM^m \text{ words} \quad (2.2.17b)$$

corresponding to the M^m constant vector values in the space of $u(k)$.

From Expressions (2.2.17a) and (2.2.17b) it then follows that

$$mM^m + (n^2/2 + 3n/2 + 1)M^{m(N-k)} \text{ words} \quad (2.2.18)$$

must be stored by the digital computer that is used to implement $u^*(x(k))$.

MULTIPLICATIONS AND ADDITIONS.

$$n(n + 2)M^{m(N-k)} \text{ multiplications} \quad (2.2.19)$$

and

$$n(n + 1)M^{m(N-k)} \text{ additions} \quad (2.2.20)$$

are carried out by the $M^{m(N-k)}$ QUMPs as can be easily verified from Equation (2.2.6).

COMPARISONS. The minimization (2.2.4) which is attained by performing

$$M^{m(N-k)} \text{ comparisons} \quad (2.2.21)$$

of the $M^{m(N-k)}$ QUMPs outputs.

5. The evaluation of the Optimum Quadratic Matched Processor (OQUMP) at stage $k = 0$, $h_N(x(0), U_{0, N-1}^*, 0)$, to obtain the optimum performance $f_N(x(0))$: This is the case since

$$f_N(x(0)) = h_N(x(0), U_{0, N-1}^*, 0) \quad (2.2.22)$$

as seen from the minimization Equation (2.2.4).

6. Small values for M , m , and $N-k$ so that the required storage and on-line multiplications, additions, and comparisons can be performed by a digital computer; for instance note that the common factor of Expressions (2.2.18)-(2.2.21) given by $M^{m(N-k)}$ is equal to the astronomical value of 16^{20} for a typical set of values of k , m , N , and M such as $k = 0$, $m = 2$, $N = 10$, and $M = 16$. In fact we can say that this is 'the curse of dimensionality' inherited by the MPs approach just as the Dynamic Programming approach of Bellman has its own (Rutherford[1964:33]). However, in the next two chapters it will be shown that it is possible to use the Matched Processors structure of the designed optimum control law to find practical control laws with outstanding performance even for cases where $M^{m(N-k)}$ is astronomically large.

7. The operations performed by a Comparator Decoder, Square Law Device, and Linear Law Device are defined to be analytical operations; as a result, it follows from Figure 2.1 that the optimum QUMPs control law is an analytical control law.

2.3 IMPLEMENTATION FEATURES OF OPTIMUM CONTROL LAW

From Figures 2.1a-b it is noted that the digital computer that is used to implement the QUMPs control law must perform the following tasks:

MEMORY. It must have a memory that accomodates the memory capacity required by the QUMPs control law. From Expression (2.2.18) it is noted that the memory capacity over N stages, C_Q , that is required by the QUMPs control law is given by

$$C_Q = mM^m + (n^2/2 + 3n/2 + 1) \sum_{k=0}^{N-1} M^{m(N-k)} \text{ words}^3$$

or in a more compact form by

$$C_Q = mM^m + (n^2/2 + 3n/2 + 1)M^m(M^{mN} - 1)/(M^m - 1) \text{ words.} \quad (2.3.1)$$

MATCHED PROCESSORS. Using $x(k)$, $P_1(U_{k,N-1},k)$, $P_2(U_{k,N-1},k)$, and $P_3(U_{k,N-1},k)$ as inputs it must find at each stage k the $M^{m(N-k)}$ outputs corresponding to the $M^{m(N-k)}$ QUMPs.

COMPARATOR DECODER. The digital computer must perform the functions of a Comparator Decoder which selects the OQUMP, $h_{N-k}(x(k), U_{k,N-1}^*, k)$, among $M^{m(N-k)}$ QUMPs candidates and then from $U_{k,N-1}^*$ choses $u^*(k)$.

³ Note that the space of $u(k)$ remains the same over the N stages of the process; as a result, it must only be stored once.

2.4 ON-LINE COMPUTATION TIME OF OPTIMUM CONTROL LAW

From Equations (2.2.3)-(2.2.6) and Figures 2.1a-b it follows that the on-line computation time required by the QUMPs control law at each stage k is essentially given by the time that it takes a digital computer to evaluate and compare $M^{m(N-k)}$ MP outputs.

Assuming that it is possible for a digital computer to operate p Matched Processors in parallel, it follows from (2.2.19) and (2.2.20) that the time required by a digital computer to evaluate the $M^{m(N-k)}$ QUMPs outputs at each stage k is given by*

$$(T_a(n+1) + T_m(n+2))n(\text{In}[M^{m(N-k)}/p] + \text{Fr}[M^{m(N-k)}/p]) \quad (2.4.1)$$

where:

1. $\text{In}[X]$ is defined as

$$\text{In}[X] = \text{Integer part of real number } X \quad (2.4.2)$$

and $\text{Fr}[X]$ is given by

$$\text{Fr}[X] = \begin{cases} 0, & \text{when } X \text{ is an integer} \\ 1, & \text{when } X \text{ has a fractional part;} \end{cases} \quad (2.4.3)$$

* It is assumed that the $M^{m(N-k)}$ MP outputs are evaluated in groups of p at a time.

2. T_a is the time taken by the digital computer to execute an addition;
3. T_m is the time used by the digital computer to perform a multiplication; and
4. p is the number of Matched Processors that operate in parallel.

In addition, it follows from Expression (2.2.21) that the time taken by a digital computer to perform $M^{m(N-k)}$ comparisons is given by

$$T_c M^{m(N-k)} \quad (2.4.4)$$

where T_c is the time used by the digital computer to perform a comparison.

When Expressions (2.4.1) and (2.4.4) are added we then have that the on-line computation time at each stage k , $T_Q(k, p)$, that is required in the evaluation of the QUMPs control law is found from

$$\begin{aligned} T_Q(k, p) = & T_c M^{m(N-k)} \\ & + (T_a(n+1) + T_m(n+2))n(\text{In}[M^{m(N-k)}/p] + \text{Fr}[M^{m(N-k)}/p]) \end{aligned} \quad (2.4.5)$$

where $\text{In}[\cdot]$, $\text{Fr}[\cdot]$, T_a , T_m , p , and T_c are as defined for Expressions (2.4.1) and (2.4.4). Note from Expression (2.4.5)

that the maximum on-line computation time with respect to k , $TM_Q(p)$, is given by $T_Q(0,p)$, i.e., it occurs at the first stage of process and $TM_Q(p)$ is given by

$$\begin{aligned}
 & TM_Q(p) \\
 &= T_c M^{mN} + (T_a(n+1) + T_m(n+2))n(\ln[M^{mN}/p] + \text{Fr}[M^{mN}/p])
 \end{aligned}
 \tag{2.4.6}$$

where $\ln[\cdot]$, $\text{Fr}[\cdot]$, T_a , T_m , p , and T_c are as defined for Expressions (2.4.1) and (2.4.4). Also note from Equation (2.4.6) that $TM_Q(p)$ decreases as p increases and its minimum value, TMM_Q , is given by $TM_Q(M^{mN})$ or equivalently by $T_Q(0, M^{mN})$, i.e., TMM_Q is given by

$$\begin{aligned}
 & TMM_Q \\
 &= T_c M^{mN} + (T_a(n+1) + T_m(n+2))n
 \end{aligned}
 \tag{2.4.7}$$

when

$$p = M^{mN}.$$

2.5 SPECIAL CASE

The special case consists of replacing $A(u(k),k)$ and $Q_1(u(k),k)$ by $A(k)$ and $Q_1(k)$ in the process Equation (1.1.1) and performance Equation (1.1.4). In the subsequent subsections simpler results than those found in Sections 2.2, 2.3, and 2.4 are derived for this special case with the aid of Appendix B.

2.5.1 Linear Matched Processors (LIMPs) Control Law

In Appendix B it is demonstrated that when $A(k)$ and $Q_1(k)$ are substituted for $A(u(k),k)$ and $Q_1(U_{k,N-1},k)$, respectively, in the QUMPs control law equations (2.2.3)-(2.2.16) we have that $u^*(x(k))$ is equivalently given by the simpler relations

$$u^*(x(k)) = u^*(k) \quad (2.5.1.1)$$

where $u^*(k)$ is the k -th stage member of the optimum control sequence $U^*_{k,N-1}$ that minimizes

$$\begin{aligned} & f'_{N-k}(x(k)) \\ &= \min_{U_{k,N-1} \in US(k)} h'_{N-k}(x(k), U_{k,N-1}, k) \end{aligned} \quad (2.5.1.2)$$

where $US(k)$ is the universal set of control sequences $U_{k,N-1}$ given by (2.2.5) and $h'_{N-k}(x(k), U_{k,N-1}, k)$ is the Linear Matched Processor (LIMP)

$$\begin{aligned} & h'_{N-k}(x(k), U_{k,N-1}, k) \\ &= P_2(U_{k,N-1}, k)x(k) + p_3(U_{k,N-1}, k) \end{aligned} \quad (2.5.1.3)$$

where the MPAs $P_2(U_{k,N-1}, k)$ and $p_3(U_{k,N-1}, k)$ are derived from either

$$\begin{aligned}
& P_2(U_{k, N-1}, k) \\
&= \sum_{i=k}^N \left\{ 2 \sum_{t=k}^{i-1} B^T(u(t), t) \left[\prod_{j=i-1}^{t+1} A(j) \right]^T Q_1(i) \right\} \\
&\quad \cdot \prod_{t=i-1}^k A(t) \quad (2.5.1.4)
\end{aligned}$$

and

$$\begin{aligned}
& P_3(U_{k, N-1}, k) \\
&= \sum_{i=k}^N \left\{ \left[\sum_{t=k}^{i-1} B^T(u(t), t) \left[\prod_{j=i-1}^{t+1} A(j) \right] Q_1^T(i) \right] \right. \\
&\quad \cdot \sum_{t=k}^{i-1} \prod_{j=i-1}^{t+1} A(j) B(u(t), t) + q_3(u(i), i) \left. \right\} \quad (2.5.1.5)
\end{aligned}$$

or the backward solution to the linear difference equations

$$P_1(k) = A^T(k) P_1(k+1) A(k) + Q_1(k), \quad (2.5.1.6)$$

$$\begin{aligned}
& P_2(U_{k, N-1}, k) \\
&= [P_2(U_{k+1, N-1}, k+1) + 2B^T(u(k), k) P_1(k+1)] A(k) \\
&\quad (2.5.1.7)
\end{aligned}$$

and

$$\begin{aligned}
& p_3(U_{k, N-1}, k) = p_3(U_{k+1, N-1}, k+1) \\
&+ [P_2(U_{k+1, N-1}, k+1) + B^T(u(k), k) P_1(k+1)] B(u(k), k) \\
&\quad + q_3(u(k), k) \quad (2.5.1.8)
\end{aligned}$$

where the terminal conditions are given by

$$P_1(U_{N, N-1}, N) = Q_1(N),$$

$$P_2(U_{N,N-1}, N) = \underline{0},$$

and

$$p_3(U_{N,N-1}, N) = 0$$

Discussion of Results:

A comparison of the QUMPs control law (2.2.3)-(2.2.16) and the control law (2.5.1.1)-(2.5.1.8) or their schematic representations as seen in Figures 2.1a-b and 2.2a-b, yields the following differences:

1. Instead of a QUMP, (2.2.6), we now have a Linear Matched Processor (LIMP), (2.5.1.3). As a result, the control law (2.5.1.1)-(2.5.1.8) is in fact a Linear Matched Processors (LIMPs) control law.
2. Since the LIMP as seen from Equation (2.5.1.3) or Figure 2.2b depends only on the MPAs $P_2(U_{k,N-1}, k)$ and $p_3(U_{k,N-1}, k)$,

$$mM^m + (n + 1)M^{m(N-k)} \text{ words} \quad (2.5.1.9)$$

must only be stored at each stage k for the control law (2.5.1.1)-(2.5.1.8). Therefore, the control law (2.5.1.1)-(2.5.1.8) requires the storage of

$$(n^2/2 + n/2) M^{m(N-k)} \text{ words} \quad (2.5.1.10)$$

less than those required by the QUMPs control law (compare (2.2.18) and (2.5.1.9)).

Figure 2.2a Schematic Diagram of LIMPS Control Law^{1A}
Implementation Requirements (Overall Diagram).

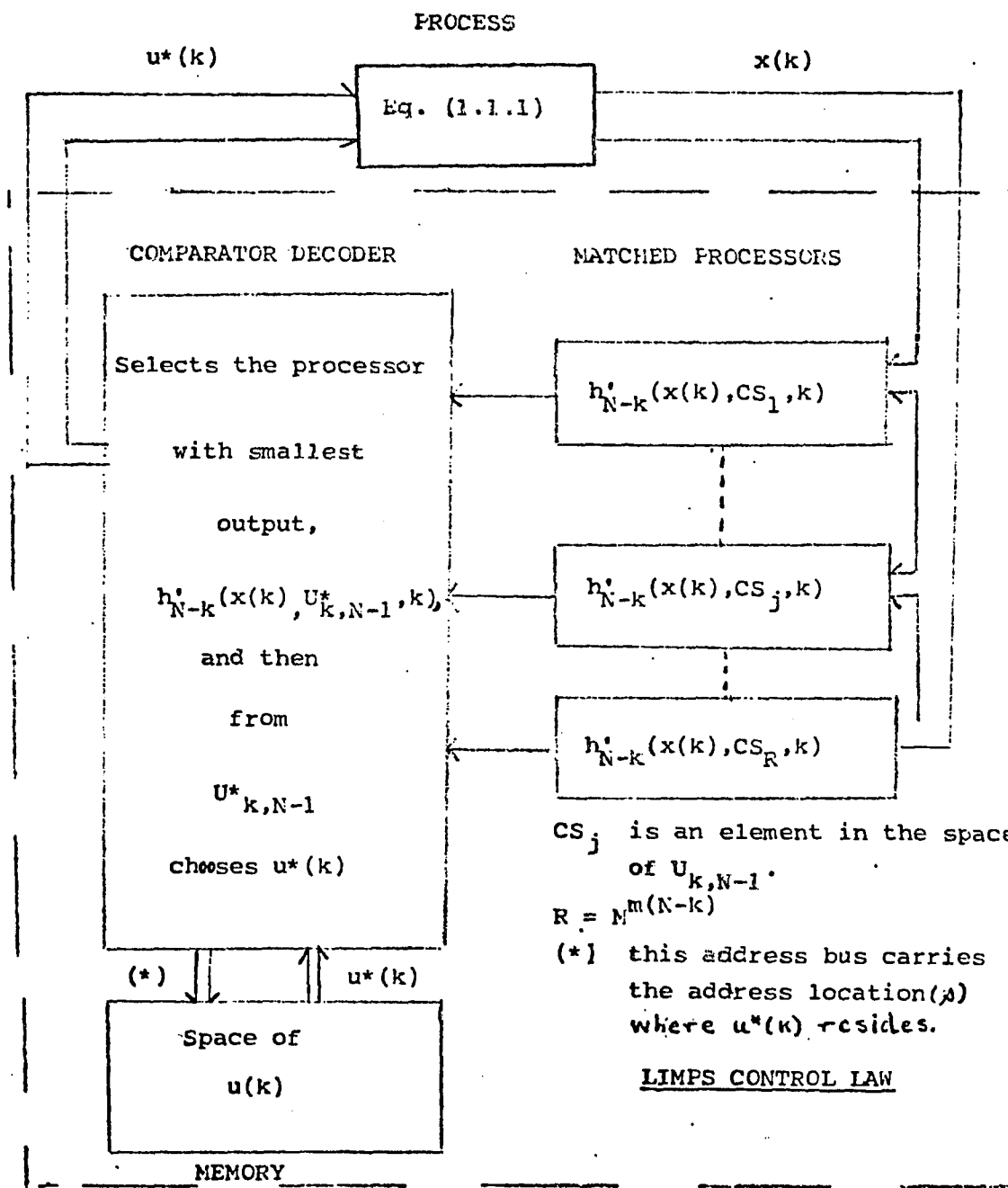
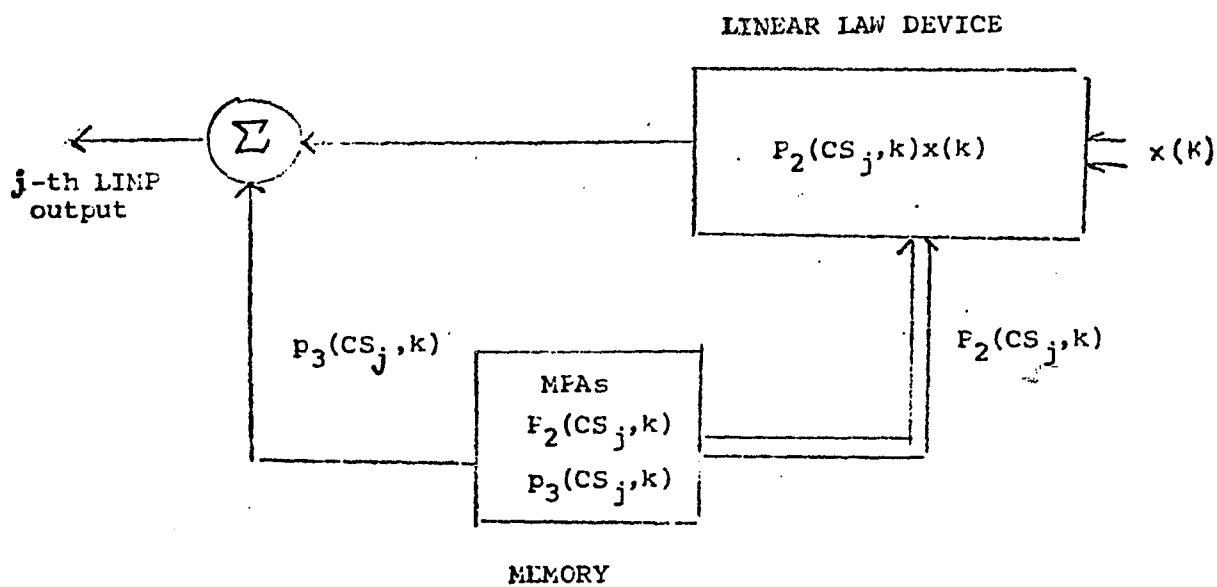


Figure 2.2b Schematic Diagram of LIMP's Implementation Requirements.



3. Again since the LIMP is linear in $x(k)$ it follows from (2.5.1.3) that

$$nM^{m(N-k)} \text{ multiplications} \quad (2.5.1.11)$$

and

$$nM^{m(N-k)} \text{ additions} \quad (2.5.1.12)$$

are executed by the $M^{m(N-k)}$ LIMPs at each stage k . As a result, when (2.2.19) and (2.2.20) are compared with (2.5.1.11) and (2.5.1.12), respectively, we have that the number of multiplications and additions required by the LIMPs control law are smaller than those for the QUMPs control law by

$$n(n+1)M^{m(N-k)} \text{ multiplications} \quad (2.5.1.13)$$

and

$$n^2 M^{m(N-k)} \text{ additions.} \quad (2.5.1.14)$$

4. The optimum performance $f_N(x(0))$ is found from

$$\begin{aligned} f_N(x(0)) &= x^T(0)P_1(0)x(0) + h_N^*(x(0), U_{0,N-1}^*, 0) \\ &\quad (2.5.1.15) \end{aligned}$$

where $h_N^*(x(0), U_{0,N-1}^*, 0)$ is the Optimum Linear Matched Processor (OLIMP) at stage $k = 0$. This result follows from the fact that

$$f_N(x(0)) = h_N(x(0), U_{0,N-1}^*, 0)$$

and Equation (B.4) in Appendix B.

2.5.2 Implementation Features of LIMPs Control Law

A comparison of Figures 2.1a-b and 2.2a-b reveal the following differences and similarities in the practical implementation of the QUMPs and LIMPs control laws by a digital computer:

MEMORY. The memory devices corresponding to the LIMPs control law do not need to store the MPA $P_1(U_{k,N-1}, k)$. Using Expression (2.5.1.9) it is noted that the memory capacity over N stages, C_L , that is required by the LIMPs control law is given by

$$C_L = mM^m + (n + 1) \sum_{k=0}^{N-1} M^{m(N-k)} \text{ words}$$

or in a more compact form by

$$C_L = mM^m + (n + 1)M^m(M^{mN} - 1)/(M^m - 1) \text{ words.} \quad (2.5.2.1)$$

MATCHED PROCESSORS. Each LIMP of the LIMPs control law is a linear function of the current state $x(k)$; therefore, the LIMP does not require the Square Law Device shown in Figure 2.1b.

COMPARATOR DECODER. The Comparator Decoder must select the Optimum Linear Matched Processor (OLIMP), $h'_{N-k}(x(k), U^*_{k,N-1}, k)$, among $M^{m(N-k)}$ LIMPs candidates and then from $U^*_{k,N-1}$ obtain $u^*(k)$; consequently, the Comparator Deco-

ders of the QUMPs and LIMPs control laws perform identical functions.

2.5.3 On-Line Computation Time of LIMPs Control Law

Assuming that it is possible for the digital computer to operate p Matched Processors in parallel, it follows from (2.5.1.11), (2.5.1.12), and the required $M^{m(N-k)}$ comparisons of Equation (2.5.1.2), that the on-line computation time at each stage, $T_L(k,p)$, that is required by the LIMPs control law is given by

$$T_L(k,p) = T_c M^{m(N-k)} + (T_a + T_m)n(\ln[M^{m(N-k)}/p] + \text{Fr}[M^{m(N-k)}/p]) \quad (2.5.3.1)$$

where $\ln[.]$, $\text{Fr}[.]$, T_a , T_m , p , and T_c are as defined for Expressions (2.4.1) and (2.4.4). A comparison of Equations (2.4.5) and (2.5.3.1) reveal that $T_L(k,p)$ is smaller than $T_Q(k,p)$ as expected.

Note from Equation (2.5.3.1) that the maximum on-line computation time with respect to k , $TM_L(p)$, for the LIMPs control law is given by $T_L(0,p)$, i.e., it occurs at the first stage of process and $TM_L(p)$ is given by

$$\begin{aligned}
 & TM_L(p) \\
 &= T_c M^{mN} + (T_a + T_m)n(\ln[M^{mN}/p] + \text{Fr}[M^{mN}/p]) \\
 & \quad (2.5.3.2)
 \end{aligned}$$

where $\ln[\cdot]$, $\text{Fr}[\cdot]$, T_a , T_m , p , and T_c are as defined for Expressions (2.4.1) and (2.4.4). Also note from Equation (2.5.3.2) that $TM_L(p)$ decreases as p increases and its minimum value, TMM_L , is given by $TM_L(M^{mN})$ or equivalently by $T_L(0, M^{mN})$, i.e., TMM_L is given by

$$\begin{aligned}
 & TMM_L \\
 &= T_c M^{mN} + (T_a + T_m)n \\
 & \quad (2.5.3.3)
 \end{aligned}$$

when

$$p = M^{mN}.$$

2.6 SUMMARY AND CONCLUSIONS

2.6.1 Quadratic Matched Processors (QUMPs) Control Law

In this chapter the desired optimum control law $u^*(x(k))$ of Section 1.1 was designed (see (2.2.3)-(2.2.16)) and the following features noted:

1. The design of $u^*(x(k))$ yields a QUMPs control law possessing the MPAs $P_1(U_{k,N-1}, k)$, $P_2(U_{k,N-1}, k)$, and $P_3(U_{k,N-1}, k)$. These MPAs can be evaluated off-line

making use of the closed form Equations (2.2.7)-(2.2.9) or the linear difference Equations (2.2.14)-(2.2.16). Furthermore the memory capacity that is needed by a digital computer to store these MPAs and the M^m constant vector values in the space of $u(k)$ is given by Equation (2.3.1).

2. The QUMPs control law has $M^{m(N-k)}$ QUMPs whose inputs are the current state $x(k)$, and the MPAs $P_1(U_{k,N-1}, k)$, $P_2(U_{k,N-1}, k)$, and $P_3(U_{k,N-1}, k)$. The multiplications and additions that are required by these QUMPs are given by Equations (2.2.19) and (2.2.20), respectively.
3. A Comparator Decoder is present in the QUMPs control law which selects the OQUMP, $h_{N-k}(x(k), U_{k,N-1}^*, k)$, among $M^{m(N-k)}$ QUMPs and then chooses $u^*(k)$ from $U_{k,N-1}^*$. As a result,

$$M^{m(N-k)} \text{ comparisons}$$

are executed by the Comparator Decoder and one simple selection of $u^*(k)$ from $U_{k,N-1}^*$.

4. The on-line computation time at each stage k , $T_Q(k, p)$, and the maximum on-line computation time, $TM_Q(p)$, needed by the QUMPs control law are given by Equations (2.4.5) and (2.4.6), respectively, where p QUMPs are assumed to operate in parallel and $TM_Q(p)$ occurs at the first stage of the process.

5. The minimum value of $TM_Q(p)$, TMM_Q , is given by Equation (2.4.7) and occurs when M^{mN} processors operate in parallel, i.e., $p = M^{mN}$; a comparison of Equations (2.4.6) and (2.4.7) show that $TM_Q(p)$ can decrease substantially as we increase p to its maximum value of M^{mN} .

2.6.2 LIMPs Control Law

When $A(u(k), k)$ and $Q_1(u(k), k)$ are replaced in the process Equation (1.1.1) and performance criterion Equation (1.1.4) by $A(k)$ and $Q_1(k)$, respectively, a LIMPs control law is derived (see (2.5.1.1)-(2.5.1.8)) with the following features:

1. The design of the LIMPs control law requires the evaluation and storage of the MPAs $P_2(U_{k,N-1}, k)$ and $p_3(U_{k,N-1}, k)$. These parameters can be evaluated off-line from the closed form Equations (2.5.1.4) and (2.5.1.5) or the linear difference Equations (2.5.1.6), (2.5.1.7), and (2.5.1.8). The memory capacity that is needed by a digital computer to store these MPAs and the M^m constant vector values in the space of $u(k)$ is given by Equation (2.5.2.1).
2. The LIMPs control law has $M^{m(N-k)}$ LIMPs whose inputs are the current state $x(k)$ and the MPAs $P_2(U_{k,N-1}, k)$ and $p_3(U_{k,N-1}, k)$. The multiplications and additions

that are required by these LIMPs are given by Equations (2.5.1.11) and (2.5.1.12).

3. A Comparator Decoder is present in the LIMPs control law which selects the OLIMP, $h_{N-k}^i(x(k), U_{k, N-1}^*, k)$, among $M^{m(N-k)}$ LIMPs and then chooses $u^*(k)$ from $U_{k, N-1}^*$. As a result,

$$M^{m(N-k)} \text{ comparisons}$$

are executed by the Comparator Decoder and one simple selection of $u^*(k)$ from $U_{k, N-1}^*$.

4. The on-line computation time at each stage k , $T_L(k, p)$, and the maximum on-line computation time, $TM_L(p)$, needed by the LIMPs control law are given by Equations (2.5.3.1) and (2.5.3.2), respectively, where p LIMPs are assumed to operate in parallel and $TM_L(p)$ occurs at the first stage of the process.
5. The minimum value of $TM_L(p)$, TMM_L , is given by Equation (2.5.3.3) and occurs when M^{mN} processors operate in parallel, i.e., $p = M^{mN}$; a comparison of Equations (2.5.3.2) and (2.5.3.3) show that $TM_L(p)$ can decrease substantially as we increase p to its maximum value of M^{mN} .

2.6.3 Conclusions

The following general conclusions can be made concerning the QUMPs and LIMPs control laws derived in this chapter:

1. Off-line calculation of the MPAs $P_1(U_{k,N-1}, k)$ (or $P_1(k)$), $P_2(U_{k,N-1}, k)$, and $p_3(U_{k,N-1}, k)$ for any single realization of $U_{k,N-1}$ can be readily performed from either closed form equations or the backward solution to linear difference equations.
2. Due to the simple structure of the QUMPs and LIMPs, each one of them can be easily implemented in a digital computer.
3. The Comparator Decoder is also easily implemented in a digital computer.
4. The maximum on-line computation time required by the QUMPs and LIMPs control laws can be reduced significantly by increasing the number of QUMPs and LIMPs that operate in parallel, i.e., by increasing p . It is important to note that at the present time there is active research going on in the design of VLSI computing elements whose architecture is identical to the structure of the QUMPs control law shown in Figures 2.1a-b where up to 1000 QUMPs can operate simultaneously (Sugarman[1980:34-35]).

5. The number of QUMPs or LIMPs at each stage k , $M^{m(N-k)}$, limits the practical use of the QUMPs or LIMPs control law due to the fact that even for moderate values for M , m , and $N-k$ the maximum on line computation time, $TM_Q(p)$ or $TM_L(p)$, and memory capacity, C_Q or C_L , are astronomically large. In the next chapter we use the structural knowledge learned in this chapter about the optimum control law to define a class of control laws that are later shown to yield practical control laws with outstanding performance.

Chapter III

REDUCED ORDER MATCHED PROCESSORS CONTROL LAWS

3.1 INTRODUCTION

In the previous chapter it was found that the designed optimum control laws, i.e. the Quadratic Matched Processors (QUMPs) control law for the control problem of Section 1.1 and the equivalent Linear Matched Processors (LIMPs) control law for the special conditions of Section 2.5, had the dimensionality drawback of requiring the evaluation and comparison at each stage k of $M^{m(N-k)}$ Matched Processors (MPs) outputs.

In this chapter we present a class of control laws, based on the QUMPs and LIMPs control laws of Chapter II, which offer the 'possibility' of having control laws that do not have the dimensionality problems of the QUMPs and LIMPs control laws while yielding desirable performance. This is verified with a general stable scalar example in Section 3.4.1 and in the next chapter with a number of practical examples.

The structure of the rest of this chapter is as follows. In Section 3.2 we define a general class of control laws

which we call Reduced Order Matched Processors (ROMPs) control laws that is based on the QUMPs and LIMPs control laws of Chapter II. In Section 3.3 I Stages ROMPs control laws with at most M^{m_1} MPs, where I is a positive integer from 1 to N, are defined and their implementation, memory capacity, and computation time features assessed. In Section 3.4 two special cases of I Stages ROMPs control laws are studied. Finally in Section 3.5 the summary and conclusions of the present chapter are given.

3.2 REDUCED ORDER MATCHED PROCESSORS (ROMPs) CONTROL LAWS

A ROMPs control law is defined as any QUMPs or LIMPs control law which has had its order of MPs at any stage k, $M^{m(N-k)}$, reduced either by the simplification of the original QUMPs or LIMPs control law or by the outright deletion of some of its $M^{m(N-k)}$ MPs. Clearly, when the reduction is attained by 'simplification' the ROMPs control law is also an optimum control law and when the reduction is performed by 'outright deletion' the ROMPs control law can be an optimum or suboptimum control law.

Next a class of ROMPs control laws denoted as I Stages ROMPs control laws is considered.

3.3 I STAGES ROMPS CONTROL LAWS

3.3.1 General Features

As a preamble to the definition of two different types of I Stages ROMPs control laws, we first present two general features of these control laws. These two features are as follows:

1) The order of MPs at each stage k of the I Stages ROMPs control laws is never larger than M^{mI} where I can be any positive integer from 1 to N . This makes the complexity of the I Stages ROMPs control laws directly dependent on the value of I , i.e., the smaller the value of I the simpler the control law and vice versa; and

2) When the order of MPs at a given stage k is given by M^{mI} , each of these MPs is matched to a unique element in the control sequence set of M^{mI} elements given by

$$S(k, I, d)$$

$$= [U_{k, N-1} : U_{k, k-1+I} \in US(k, k-1+I), U_{k+I, N-1} = \{d, \dots, d\}]$$

(3.3.1.1)

with

$$I \in (1, \dots, N)$$

where:

a) $US(k, j)$ is the universal set of control sequences from

stage k to stage j , i.e.,

$$\begin{aligned} & US(k, j) \\ &= [U_{k,j} : u_i(s) \in (c_1^i, \dots, c_n^i) \text{ for all } i \text{ and } s] \end{aligned} \quad (3.3.1.2)$$

where it is noted from (2.2.5) and (3.3.1.2) that

$$US(k, N-1) = US(k); \quad (3.3.1.3)$$

b) d is a constant m dimensional vector with space similar to that of $u(k)$, i.e.,

$$d_i \in (c_1^i, \dots, c_n^i) \text{ for all } i \quad (3.3.1.4)$$

where d_i is the i -th component of the vector d . It is noted from Equations (2.2.5), (3.3.1.1), (3.3.1.2), and (3.3.1.4) that $S(k, I, d)$ is a subset of $US(k)$, i.e.,

$$S(k, I, d) \in US(k); \quad (3.3.1.5)$$

and

c) The value of d is given by any allowable constant value of $u(N-1)$ that either satisfies or comes close to satisfying the desired terminal conditions on the process state $x(N)$. For instance, if the desired terminal condition requires that $x(N)$ be small, i.e., close to a zero vector, it follows from the process Equation (1.1.1) that the constant value of $u(N-1)$ given by

$$u_i(N-1) = 0 \text{ for all } i$$

will satisfy the desired terminal condition provided: i) the process is in the steady state when it reaches stage $N-1$ so that it can be assumed that $x(N-1)$ is also small; ii) $B(u(N-1), N-1)$ is a zero vector when $u(N-1)$ is a zero vector; and iii) there is a zero element in the set $(c_1^i, \dots, c_H^i)$ for all i . It then follows that d is also a zero vector and $S(k, I, \underline{0})$ is given by

$$\begin{aligned} S(k, I, \underline{0}) \\ = [U_{k, N-1} : U_{k, k-1+I} \quad US(k, k-1+I), \\ u(s) = \underline{0} \text{ for } s \in (k+I, \dots, N-1)] \end{aligned}$$

Note that for $I = 1$ we obtain

$$\begin{aligned} S(k, 1, \underline{0}) \\ = [U_{k, N-1} : u_i(k) \in (c_1^i, \dots, c_H^i) \text{ for all } i, \\ u(s) = \underline{0} \text{ for all } s \in (k+1, \dots, N-1)], \end{aligned}$$

for $I = 2$ we obtain

$$\begin{aligned} S(k, 2, \underline{0}) \\ = [U_{k, N-1} : u_i(s) \in (c_1^i, \dots, c_H^i) \text{ for all } i \text{ and } s \in (k, k+1), \\ u(t) = \underline{0} \text{ for all } t \in (k+2, \dots, N-1)], \end{aligned}$$

etc.

Before proceeding on to the definition of the two different I Stages ROMPs control laws we make the following observations concerning the definition of the constant vector 'd' and the name 'I Stages ROMPs control law':

First, in regard to the constant vector d we note that:

i) The present definition of the constant vector d has been selected on the basis of numerous computer simulations that strongly support this definition as a desirable choice; for instance, several practical examples in Section 4.5 and the general stable scalar problem of Section 3.4.1; and

ii) It was noted earlier in an example (see the example given shortly after Equation (3.3.1.5)) that determines d for an assumed terminal condition of the state, $x(N)$, that to find $u(N-1)$ it is necessary that the process be in the steady state when it reaches stage N-1: we can then assume that $x(N-1)$ is the same as $x(N)$ to easily evaluate $u(N-1)$. However, in cases where the steady state condition does not occur at stage N-1, e.g., a two stages process, the steady state condition $x(N-1) = x(N)$ can be assumed nonetheless so that a value for d can be readily available.

Second and last, in regard to the name 'I Stages ROMPs control law' it is noted that the adjective 'I Stages' refers to the fact that at each stage k of the process, the set $S(k, I, d)$ exclusively assumes free values for the current

and future controls over I Stages: this is verified from Equation (3.3.1.1) where we have that $U_{k,k-1+I}$ can be any member of the universal set $US(k,k-1+I)$.

3.3.2 I Stages Reduced Order QUMPs (ROQUMPs) Control Law

The I Stages Reduced Order Quadratic Matched Processors (ROQUMPs) control law, $u^I(x(k))$, is defined in one of two ways depending on how the QUMPs control law's order of QUMPs at each stage k , $M^{m(N-k)}$, compares with a maximum order of M^{mI} . The definition is as follows:

1) When the QUMPs control law's order of QUMPs at stage k , $M^{m(N-k)}$, is less than or equal to M^{mI} where

$$I \in (1, \dots, N)$$

and

$$k \in (0, \dots, N-1),$$

or equivalently when the stage k is such that

$$k \in (N-I, \dots, N-1), \quad (3.3.2.1)$$

the I Stages ROQUMPs control law is the same as the QUMPs control law, i.e.,

$$u^I(x(k)) = u^*(x(k)) \quad (3.3.2.2)$$

where $u^*(x(k))$ is given by Equations (2.2.3)-(2.2.16); and

2) When the QUMPs control law's order of QUMPs at stage k , $M^{m(N-k)}$, is greater than M^{mI} where

$$I \in (1, \dots, N)$$

and

$$k \in (0, \dots, N-1),$$

or equivalently when the stage k is such that

$$k \in (0, \dots, N-I-1), \quad (3.3.2.3)$$

the I Stages ROQUMPs control law is given by

$$u^I(x(k)) = u(I, k) \quad (3.3.2.4)$$

where $u(I, k)$ is the k -th stage element of the control sequence $U_{k, N-1}^I$ that minimizes

$$\begin{aligned} f_{N-k}^I(x(k)) \\ = \min_{U_{k, N-1}} \min_{\in S(k, I, d)} h_{N-k}(x(k), U_{k, N-1}, k) \end{aligned} \quad (3.3.2.5)$$

where $S(k, I, d)$ is given by (3.3.1.1) and $h_{N-k}(x(k), U_{k, N-1}, k)$ is given by Equations (2.2.6)-(2.2.16).

The following characteristics of the I Stages ROQUMPs control laws are immediately noted:

STRUCTURAL CHARACTERISTICS. Two basic structural characteristics of I Stage ROQUMPs control laws are noted:

1) When stage $k \in (N-I, \dots, N-1)$ the I Stages ROQUMPs control law is the same as the QUMPs control law. As a result, the basic structure of the I Stages ROQUMPs control law is made of $M^{m(N-k)}$ QUMPs and a Comparator Decoder as shown in Figure 2.1a-b. Note that Condition (3.3.2.1) includes the last stages of the process.

2) When stage $k \in (0, \dots, N-I-1)$ the I Stages ROQUMPs control law is noted to be composed of M^{mI} QUMPs and a Comparator Decoder where the M^{mI} QUMPs are a subset of the $M^{m(N-k)}$ QUMPs that are present in the QUMPs control law (this follows from (3.3.1.5)) and the Comparator Decoder performs the same functions of the Comparator Decoder that is present in the QUMPs control law. As a result, Figures 2.1a-b also describe the basic components of the I Stages ROQUMPs control law with the important exception that there are only M^{mI} , instead of $M^{m(N-k)}$, QUMPs. Note that Condition (3.3.2.3) includes the stages that are closest to the beginning of process.

MEMORY CAPACITY. Making use of the previously stated structural properties of the I Stages ROQUMPs control law we first have that its memory capacity at each stage k , $CP_{Q,I}(k)$, is given by:

1)

$$CP_{Q,I}(k) = mM^m + (n^2/2 + 3n/2 + 1)M^{m(N-k)} \text{ words} \quad (3.3.2.6)$$

for

$$k \in (N-I, \dots, N-1).$$

This follows from Equation (3.3.2.2) and Expression (2.2.18).

2)

$$CP_{Q,I}(k) = mM^m + (n^2/2 + 3n/2 + 1)M^{mI} \text{ words} \quad (3.3.2.7)$$

for

$$k \in (0, \dots, N-I-1).$$

This follows from the fact that $u^I(x(k))$ and $u^*(x(k))$ are identical except for the total number of QUMPs, i.e., M^{mI} for $u^I(x(k))$ and $M^{m(N-k)}$ for $u^*(x(k))$.

Finally it follows from Equations (3.3.2.6) and (3.3.2.7) and Conditions (3.3.2.1) and (3.3.2.3) that the memory capacity over N stages, $C_{Q,I}$, required by the I Stages ROQUMPs control law is given by

$$C_{Q,I} = mM^m + (n^2/2 + 3n/2 + 1)\{(N-I)M^{mI} + \sum_{k=N-I}^{N-1} M^{m(N-k)}\} \text{ words}$$

or in a more compact form by

$$C_{Q,I} = mM^m + (n^2/2 + 3n/2 + 1)$$

$$\bullet \{(N-I)M^{mI} + M^m(M^{mI} - 1)/(M^m - 1)\} \text{ words.}$$

(3.3.2.8)

ON-LINE COMPUTATION TIME. Again making use of the previously stated structural properties of the I Stages ROQUMPs control law and assuming the M^{mI} outputs are evaluated in groups of p at a time, it follows that the on-line computation time at each stage k , $T_{Q,I}(k,p)$, of the I Stages ROQUMPs control law is given by:

1)

$$T_{Q,I}(k,p) = T_Q(k,p) \quad (3.3.2.9)$$

for

$$k \in (N-I, \dots, N-1)$$

where $T_Q(k,p)$ is given by Equation (2.4.5). This follows from Equation (3.3.2.2).

2) $T_{Q,I}(k,p)$ is given by Equation (2.4.5) for

$$k \in (0, \dots, N-I-1)$$

where $M^{m(N-k)}$ is replaced by M^{mI} . This follows from the fact that $u^I(x(k))$ and $u^*(x(k))$ are identical, except for their number of MPs.

MAXIMUM ON-LINE COMPUTATION TIME. From the previously found on-line computation time at each stage k , $T_{Q,I}(k,p)$,

it follows that the maximum value of $T_{Q,I}(k,p)$ with respect to k , $TM_{Q,I}(p)$, is given by $T_Q(N-I,p)$, i.e., $TM_{Q,I}(p)$ is expressed by

$$\begin{aligned}
 & TM_{Q,I}(p) \\
 &= T_c M^{mI} + (T_\alpha(n+1) + T_m(n+2))n(\ln[M^{mI}/p] + \text{Fr}[M^{mI}/p])
 \end{aligned}
 \tag{3.3.2.10}$$

where $\ln[.]$, $\text{Fr}[.]$, T_α , T_m , p , and T_c are as defined for Expressions (2.4.1) and (2.4.4). Also note from Equation (3.3.2.10) that $TM_{Q,I}(p)$ decreases as p increases and its minimum value, $TMM_{Q,I}$, is given by $TM_{Q,I}(M^{mI})$ or equivalently by $T_{Q,I}(N-I, M^{mI})$, i.e., $TMM_{Q,I}$ is given by

$$\begin{aligned}
 & TMM_{Q,I} \\
 &= T_c M^{mI} + (T_\alpha(n+1) + T_m(n+2))n
 \end{aligned}
 \tag{3.3.2.11}$$

when

$$p = M^{mI}.$$

GENERAL FEATURES. The following general features are observed:

1) The N Stages ROQUMPs control law is the same as the QUMPs control law for all the possible stages. This is verified by noting that when I is equal to N , Condition (3.3.2.1) is satisfied for all possible values of k .

2) An investigation of Equations (3.3.2.10) and (3.3.2.8) reveal that the maximum on-line computation time, $TM_{Q,I}(p)$, and memory capacity, $C_{Q,I}$, decrease steadily as I decreases from N to 1: to show that $C_{Q,I}$ decreases as I decreases from N to 1, first obtain

$$\begin{aligned} C_{Q,I} - C_{Q,I+1} \\ = (n^2/2 + 3n/2 + 1)(N - I)M^{mI}(1 - M^m) \text{ words} \end{aligned}$$

and then note that $C_{Q,I} - C_{Q,I+1}$ is less than or equal to zero for all allowable values of n , M , m , and I . Note that a similar decrease occurs to $TMM_{Q,I}$ as seen from Equation (3.3.2.11).

3.3.3 I Stages Reduced Order LIMPs (ROLIMPs) Control Law

The I Stages Reduced Order Linear Matched Processors (ROLIMPs) control law, $u^I(x(k))$, is defined⁵ in one of two ways depending on how the LIMPs control law's order of LIMPs at each stage k , $M^{m(N-k)}$, compares with a maximum order of M^{mI} . The definition is as follows:

⁵ The I Stages ROLIMPs control law can also be derived from the I Stages ROQUMPs control law using an approach similar to that used in Appendix B to derive the LIMPs control law from the QUMPs control law, i.e., the I Stages ROLIMPs and ROQUMPs control laws are equivalent when the special conditions of Section 2.5 are considered.

1) When the LIMPs control law's order of LIMPs at stage k , $M^{m(N-k)}$, is less than or equal to M^{mI} where

$$I \in (1, \dots, N)$$

and

$$k \in (0, \dots, N-1),$$

or equivalently when the stage k is such that

$$k \in (N-I, \dots, N-1), \quad (3.3.3.1)$$

the I Stages ROLIMPs control law is the same as the LIMPs control law, i.e.,

$$u^I(x(k)) = u^*(x(k)) \quad (3.3.3.2)$$

where $u^*(x(k))$ is given by Equations (2.5.1.1)-(2.5.1.8);
and

2) When the LIMPs control law's order of LIMPs at stage k , $M^{m(N-k)}$, is greater than M^{mI} where

$$I \in (1, \dots, N)$$

and

$$k \in (0, \dots, N-1),$$

or equivalently when the stage k is such that

$$k \in (0, \dots, N-I-1), \quad (3.3.3.3)$$

the I Stages ROLIMPs control law is given by

$$u^I(x(k)) = u(I, k) \quad (3.3.3.4)$$

where $u(I, k)$ is the k -th stage element of the control sequence $U_{k, N-1}^I$ that minimizes

$$f'_{N-k}(x(k)) = \min_{U_{k, N-1} \in S(k, I, d)} h'_{N-k}(x(k), U_{k, N-1}, k) \quad (3.3.3.5)$$

where $S(k, I, d)$ is given by (3.3.1.1) and $h'_{N-k}(x(k), U_{k, N-1}, k)$ is given by Equations (2.5.1.3)-(2.5.1.8).

The following characteristics of the I Stages ROLIMPs control laws are immediately noted:

STRUCTURAL CHARACTERISTICS. Two basic structural characteristics of I Stage ROLIMPs control laws are noted:

1) When stage $k \in (N-I, \dots, N-1)$ the I Stages ROLIMPs control law is the same as the LIMPs control law. As a result, the basic structure of the I Stages ROLIMPs control law is made of $M^{m(N-k)}$ LIMPs and a Comparator Decoder as shown in Figure 2.1a-b. Note that Condition (3.3.3.1) includes the last stages of the process.

2) When stage $k \in (0, \dots, N-I-1)$ the I Stages ROLIMPs control law is noted to be composed of M^{mI} LIMPs and a Comparator Decoder where the M^{mI} LIMPs are a subset of the $M^{m(N-k)}$ LIMPs that are present in the LIMPs control law (this follows from (3.3.1.5)) and the Comparator Decoder performs the

same functions of the Comparator Decoder that is present in the LIMPs control law. As a result, Figures 2.2a-b also describe the basic components of the I Stages ROLIMPs control law with the important exception that there are only M^{mI} , instead of $M^{m(N-k)}$, LIMPs. Note that Condition (3.3.3.3) includes the stages that are closest to the beginning of process.

MEMORY CAPACITY. Making use of the previously stated structural properties of the I Stages ROLIMPs control law we first have that its memory capacity at each stage k , $CP_{L,I}(k)$, is given by:

1)

$$CP_{L,I}(k) = mM^m + (n+1)M^{m(N-k)} \text{ words} \quad (3.3.3.6)$$

for

$$k \in (N-I, \dots, N-1).$$

This follows from Equation (3.3.3.2) and Expression (2.5.1.9).

2)

$$CP_{L,I}(k) = mM^m + (n+1)M^{mI} \text{ words} \quad (3.3.3.7)$$

for

$$k \in (0, \dots, N-I-1).$$

This follows from the fact that $u^I(x(k))$ and $u^*(x(k))$ are identical except for the total number of LIMPs, i.e., M^{mI} for $u^I(x(k))$ and $M^{m(N-k)}$ for $u^*(x(k))$.

It then follows from Equations (3.3.3.6) and (3.3.3.7) and Conditions (3.3.3.1) and (3.3.3.3) that the memory capacity over N stages, $C_{L,I}$, that is required by the I Stages ROLIMPs control law is given by

$$C_{L,I} = mM^m + (n+1)\{(N-I)M^{mI} + \sum_{k=N-I}^{N-1} M^{m(N-k)}\} \text{ words}$$

or in a more compact form by

$$C_{L,I} = mM^m + (n+1)\{(N-I)M^{mI} + M^m(M^{mI} - 1)/(M^m - 1)\} \text{ words.} \quad (3.3.3.8)$$

ON-LINE COMPUTATION TIME. Again making use of the previously stated structural properties of the I Stages ROLIMPs control law and assuming the M^{mI} outputs are evaluated in groups of p at a time, it follows that the on-line computation time at each stage k , $T_{L,I}(k,p)$, of the I Stages ROLIMPs control law is given by:

1)

$$T_{L,I}(k,p) = T_L(k,p) \quad (3.3.3.9)$$

for

$$k \in (N-I, \dots, N-1)$$

where $T_L(k,p)$ is given by Equation (2.5.3.1). This follows from Equation (3.3.3.2).

2) $T_{L,I}(k,p)$ is given by Equations (2.5.3.1) for

$$k \in (0, \dots, N-I-1)$$

where $M^{m(N-k)}$ is replaced by M^{mI} . This follows from the fact that $u^I(x(k))$ and $u^*(x(k))$ are identical, except for their number of MPs.

MAXIMUM ON-LINE COMPUTATION TIME. From the previously found on-line computation time at each stage k , $T_{L,I}(k,p)$, it follows that the maximum value of $T_{L,I}(k,p)$ with respect to k , $TM_{L,I}(p)$, is given by $T_L(N-I,p)$, i.e., $TM_{L,I}(p)$ is given by

$$\begin{aligned} & TM_{L,I}(p) \\ &= T_c M^{mI} + (T_a + T_m)n(\ln[M^{mI}/p] + \text{Fr}[M^{mI}/p]) \end{aligned} \quad (3.3.3.10)$$

where $\ln[.]$, $\text{Fr}[.]$, T_a , T_m , p , and T_c are as defined for Expressions (2.4.1) and (2.4.4). Also note from Equation (3.3.3.10) that $TM_{L,I}(p)$ decreases as p increases and its minimum value, $TMM_{L,I}$, is given by $TM_{L,I}(M^{mI})$ or equivalently by $T_{L,I}(N-I, M^{mI})$, i.e., $TMM_{L,I}$ is expressed by

$$\begin{aligned} & TMM_{L,I} \\ &= T_c M^{mI} + (T_a + T_m)n \end{aligned} \quad (3.3.3.11)$$

when

$$p = M^{mI}.$$

GENERAL FEATURES. The following general features are observed:

1) The N Stages ROLIMPs control law is the same as the LIMPs control law for all the possible stages. This is verified by noting that when I is equal to N, Condition (3.3.3.1) is satisfied for all possible values of k.

2) An investigation of Equations (3.3.3.10) and (3.3.3.8) reveal that the maximum on-line computation time, $TM_{L,I}(p)$, and memory capacity, $C_{L,I}$, decrease steadily as I decreases from N to 1: to show that $C_{L,I}$ decreases as I decreases from N to 1, first obtain

$$\begin{aligned} C_{L,I} - C_{L,I+1} \\ = (n + 1)(N - I)M^{mI}(1 - M^m) \text{ words} \end{aligned}$$

and then note that $C_{L,I} - C_{L,I+1}$ is less than or equal to zero for all allowable values of n, M, m, and I. Note that a similar decrease occurs to $TMM_{L,I}$ as seen from Equation (3.3.3.11).

3.4 TWO SPECIAL CASES OF I STAGES ROMPS CONTROL LAWS

In the following subsections two special cases of I Stages ROMPs control laws are presented. First for a general stable scalar example it is shown that its corresponding I Stages ROLIMPs control law is an optimum control law for all possible values of I i.e., all its I Stages ROLIMPs

control laws starting with the 1 Stage ROLIMPs control law and ending with the N Stages ROLIMPs control law are optimum control laws since they all yield the same control sequence. Then for a single input general example it is demonstrated that its corresponding 1 Stage ROLIMPs control law is equivalently given by two compact control laws with memory capacity and on-line computation time that are generally smaller than those of the 1 Stages ROLIMPs control law.

3.4.1 A General Stable Scalar Example

Consider the general stable scalar process

$$x(k+1) = ax(k) + bu(k) \quad (3.4.1.1)$$

where:

a) $x(k)$ and $u(k)$ are state and control scalars, respectively;

b) the space of $x(k)$ is given by (1.1.2) and that of $u(k)$ by all the integers between the whole number M_1 and $-M_1$, i.e.,

$$u(k) \in (-M_1, \dots, M_1) \quad (3.4.1.2)$$

with

$$M_1 \in (1, 2, \dots);$$

c) the space of the scalar b is given by

$$b \in (y: y > 0); \quad (3.4.1.3)$$

and

d) the space of the scalar a is given by

$$a \in (y: 0 < y < 1). \quad (3.4.1.4)$$

In addition consider the performance criterion

$$\begin{aligned} & h_N(x_{0,N}, u_{0,N-1}) \\ &= \sum_{k=0}^N [q x^2(k) + r |u(k)|] \end{aligned} \quad (3.4.1.5)$$

where q and r are nonnegative scalar weights.

In Appendix C it is shown that the optimum control law, $u^*(x(k))$, that minimizes Equation (3.4.1.5) subject to constraints (1.1.2) and (3.4.1.1)-(3.4.1.4)⁶, besides being a LIMPs control law, is equivalently given by any of the I Stages ROLIMPs control laws which can be defined for this scalar example, i.e.,

$$u^*(x(k)) = u^I(x(k)) \quad (3.4.1.6)$$

for

$$I \in (1, \dots, N)$$

⁶ The proof of Appendix C is actually done for values of a between 0 and 0.98.

where, as found in Appendix C, the I Stages ROLIMPs control law, $u^I(x(k))$, is given by

1)

$$u^I(x(k)) = u^*(k) \quad (3.4.1.7)$$

for

$$k \in (N-I, \dots, N-1)$$

where $u^*(k)$ is the k -th stage member of the optimum control sequence $U_{k,N-1}^*$ that minimizes

$$f_{N-k}'(x(k)) = \min_{U_{k,N-1} \in US(k)} h_{N-k}'(x(k), U_{k,N-1}, k) \quad (3.4.1.8)$$

where

$$US(k) = [U_{k,N-1} : u(s) \in \{-M_1, \dots, M_1\} \text{ for all } s \in (k, \dots, N-1)] \quad (3.4.1.9)$$

and

$$\begin{aligned} & h_{N-k}'(x(k), U_{k,N-1}, k) \\ &= P_2(U_{k,N-1}, k)x(k) + p_3(U_{k,N-1}, k) \end{aligned} \quad (3.4.1.10)$$

where

$$\begin{aligned} & P_2(U_{k,N-1}, k) \\ &= 2qab \sum_{i=k+1}^N a^{2(i-k-1)} \sum_{t=k}^{i-1} a^{k-t} u(t) \end{aligned} \quad (3.4.1.11)$$

and

$$\begin{aligned}
 & p_3(U_{k,N-1}, k) \\
 &= qb^2 \sum_{i=k+1}^N a^{2(i-k-1)} \left[\sum_{t=k}^{i-1} a^{k-t} u(t) \right]^2 + r \sum_{i=k}^{N-1} |u(k)|;
 \end{aligned}
 \tag{3.4.1.12}$$

and

2)

$$u^I(x(k)) = u(I, k) \tag{3.4.1.13}$$

for

$$k \in (0, \dots, N-I-1)$$

where $u(I, k)$ is the k -th stage member of the control sequence $U_{k,N-1}$ that minimizes

$$\begin{aligned}
 f_{N-k}^I(x(k)) = \min_{U_{k,N-1}} h_{N-k}^I(x(k), U_{k,N-1}, k) \\
 \in S(k, I, 0)
 \end{aligned}
 \tag{3.4.1.14}$$

where

$$\begin{aligned}
 & S(k, I, 0) \\
 &= [U_{k,N-1} : u(s) \in \{-M_1, \dots, M_1\} \text{ for all } s \in (k, \dots, k-1+I), \\
 & \quad u(v) = 0 \text{ for all } v \in (k+I, \dots, N-1)]
 \end{aligned}
 \tag{3.4.1.15}$$

(see Table 3.1 for an illustration of the set $S(N-4, 1, 0)$ for $M_1 = 1$) and $h_{N-k}^I(x(k), U_{k,N-1}, k)$ given by Equations (3.4.1.10)-(3.4.1.12).

$$S(N-4, 1, 0) = (CS_1, CS_2, CS_3)$$

where

CS_1 is a unique control sequence of the set $S(N-4, 1, 0)$.

Control Sequence Elements of the Control Sequence $U_{N-4, N-1}$

$U_{N-4, N-1}$ $u(N-4)$ $u(N-3)$ $u(N-2)$ $u(N-1)$

CS_1 -1 0 0 0

CS_2 0 0 0 0

CS_3 1 0 0 0

Table 3.1 The set $S(N-4, 1, 0)$ for $M_1=1$

The following results and observations are noted concerning this scalar example:

ILLUSTRATION OF THE REDUCTION IN CONTROL LAW COMPLEXITY WHEN A LIMPS, 2 STAGE ROLIMPS, AND 1 STAGE ROLIMPS CONTROL LAW ARE COMPARED FOR THE CASE WHERE THE CURRENT STAGE k IS $N-3$ AND THE SPACE OF $u(k)$ IS $\{-1,0,1\}$. Table 3.1a shows:

1. The universal set, $US(N-3)$, of control sequences between stage $N-3$ and $N-1$ where the 27 control sequences in $US(N-3)$ are associated with the 27 matched processors in the LIMPS control law.

2. The 2 Stages set $S(N-3,2,0)$ of control sequences between stage $N-3$ and $N-1$ where the 9 control sequences in $S(N-3,2,0)$ are associated with the 9 matched processors in the 2 Stages ROLIMPS control law.

3. The 1 Stage set $S(N-3,1,0)$ of control sequences between stage $N-3$ and $N-1$ where the 3 control sequences in $S(N-3,1,0)$ are associated with the 3 matched processors in the 1 Stage ROLIMPS control law.

From Table 3.1a it follows that the complexity of the control law increases as I increases from 1 to 3 (note that the 3 Stages ROLIMPS control law is the same as the LIMPS control law), since the only difference between the 1 Stage ROLIMPS, 2 Stages ROLIMPS, and LIMPS control law is the number of MPs that each has.

Table 3.1a Tabulation of control sequences for the sets
 $US(N-3)$, $S(N-3,2,0)$, and $S(N-3,1,0)$ for $M_1=1$.

$US(N-3)$			$S(N-3,2,0)$			$S(N-3,1,0)$		
$u(N-3)u(N-2)u(N-1)$			$u(N-3)u(N-2)u(N-1)$			$u(N-3)u(N-2)u(N-1)$		
-1	-1	-1	-1	-1	0	-1	0	0
-1	-1	0	-1	0	0	0	0	0
-1	-1	1	-1	1	0	1	0	0
-1	0	-1	0	-1	0			
-1	0	0	0	0	0			
-1	0	1	0	1	0			
-1	1	-1	1	-1	0			
-1	1	0	1	0	0			
-1	1	1	1	1	0			
0	-1	-1						
0	-1	0						
0	-1	1						
0	0	-1						
0	0	0						
0	0	1						
0	1	-1						
0	1	0						
0	1	1						
1	-1	-1						
1	-1	0						
1	-1	1						
1	0	-1						
1	0	0						
1	0	1						
1	1	-1						
1	1	0						
1	1	1						

MAXIMUM ON-LINE COMPUTATION TIME AND MEMORY CAPACITY. Substituting $2M_1 + 1$ for M , 1 for n , and 1 for m in Equations (3.3.3.10) and (3.3.3.8), it follows that the maximum on-line computation time, $TM_{L,I}(p)$, and memory capacity, $C_{L,I}$, of the I Stages ROLIMPs control law (3.4.1.7)-(3.4.1.15) are given by

$$TM_{L,I}(p) = T_c(2M_1 + 1)^I + (T_a + T_m)(\ln[(2M_1 + 1)^I/p] + \text{Fr}[(2M_1 + 1)^I/p]) \quad (3.4.1.16)$$

and

$$C_{L,I} = 2M_1 + 1 + 2(N - I)(2M_1 + 1)^I + (2M_1 + 1)[(2M_1 + 1)^I - 1]/M_1 \text{ words} \quad (3.4.1.17)$$

where $\ln[\cdot]$, $\text{Fr}[\cdot]$, T_a , T_m , p , and T_c are as defined for Expressions (2.4.1) and (2.4.4).

1 STAGE ROLIMPS CONTROL LAW. Since $TM_{L,I}(p)$ and $C_{L,I}$ decrease steadily as I decreases from N to 1 it must follow that the optimum 1 Stage ROLIMPs control law is clearly preferred over the other optimum I Stages ROLIMPs control laws. This point is illustrated with Table 3.1b where a tabulation of $TM_{L,I}(p)$ and $C_{L,I}$ for $I = 1, 2, 3$, and 4 is presented for

Table 3.1b Tabulation of $TM_{L,I}(p)$ and $C_{L,I}$ for

$$T_a = 100 \text{ n sec.}$$

$$T_m = 1 \mu\text{sec.}$$

$$T_c = 100 \text{ n sec.}$$

$$p = 1$$

$$M_1 = 64$$

$$N = 100$$

I	$TM_{L,I}(p)$	$C_{L,I}$
1	76.8 μsecs	12,864 words
2	4.915 msec	811,008 words
3	0.3146 sec	51.38×10^6 words
4	20.13 sec	2.26×10^9 words

typical values of T_a , T_m , T_c , p , M_1 , and N ; note how quickly both $TM_{L,I}(p)$ and $C_{L,I}$ increase as I increases from 1 to 4.

A SUGGESTION. This example also suggests that for control problems where m and/or n are greater than one there might also exist I Stages ROMPs control laws with low values of I which yield outstanding performance while having reasonable maximum on-line computation times and memory capacities. Indeed, this is shown to be the case for a number of practical examples in the next chapter.

3.4.2 Equivalent Control Laws

Consider the general linear process

$$x(k+1) = A(k)x(k) + b(k)u(k) \quad (3.4.2.1)$$

where: a) $x(k)$ is a n dimensional vector state and $u(k)$ is a scalar control, respectively; b) the spaces of $x(k)$ and $u(k)$ are given by (1.1.2) and (1.1.3); and c) $A(k)$ and $b(k)$ are matrices of appropriate dimensions.

In addition consider the performance criterion

$$\begin{aligned} & h_N(X_{0,N}, U_{0,N-1}, I_{0,N}) \\ &= \sum_{k=0}^N [x^T(k) Q_1(k) x(k) + r(k) v(u(k))] \end{aligned} \quad (3.4.2.2)$$

where: a) $Q_1(k)$ is a matrix of appropriate dimensions; b) $Q_1(k)$ is symmetric; c) $r(k)$ is a nonnegative scalar; and d) $v(u(k))$ is either $u^2(k)$ or $|u(k)|$.

In the first half of Appendix D it is shown that the 1 Stage ROLIMPs control law, $u^1(x(k))$, for the linear process (3.4.2.1) and performance criterion (3.4.2.2) is given by⁷

$$u^1(x(k)) = u(1,k) \quad (3.4.2.3)$$

where $u(1,k)$ is the value of $u(k)$ that minimizes

$$\begin{aligned} f_{N-k}^1(x(k)) \\ = \min_{u(k)} h_{N-k}^0(x(k), u(k), k) \end{aligned} \quad (3.4.2.4)$$

where the space of $u(k)$ is given by (1.1.3) with $m=1$, and

$$\begin{aligned} h_{N-k}^0(x(k), u(k), k) \\ = P_2(u(k), k)x(k) + p_3(u(k), k) \end{aligned} \quad (3.4.2.5)$$

where

$$\begin{aligned} P_2(u(k), k) \\ = 2b^T(k)P_1(k+1)A(k)u(k), \end{aligned} \quad (3.4.2.6)$$

⁷ When the 1 Stage ROLIMPs control law is derived in Appendix D it is assumed that there is a zero element in the set $(c_1^1, \dots, c_M^1)$.

$$\begin{aligned}
& p_3(u(k), k) \\
& = b^T(k) P_1(k+1) b(k) u^2(k) + r(k) v(u(k)),
\end{aligned}
\tag{3.4.2.7}$$

and $P_1(k+1)$ is found from the backward solution of the linear difference Equation (2.5.1.6).

In the second half of Appendix D it is shown that for each case of $v(u(k))$, i.e., $u^2(k)$ and $|u(k)|$, the 1 Stage ROLIMPs control law, $u^1(x(k))$, can be simplified to yield one of the following compact control laws:

1) When

$$v(u(k)) = u^2(k) \tag{3.4.2.8}$$

we have that $u^1(x(k))$ is equivalently given by the Compact Quadratic (CQ) control law, $u^{cQ}(x(k))$, i.e.,

$$u^1(x(k)) = u^{cQ}(x(k)), \tag{3.4.2.9}$$

where $u^{cQ}(x(k))$ is given by

$$u^1(x(k)) = Cl[-K(k)x(k)] \tag{3.4.2.10}$$

where $Cl[-K(k)x(k)]$ denotes the element in the set $(c_1^1, \dots, c_n^1)$ that is closest to the value of the scalar $-K(k)x(k)$, and $K(k)$ is a n dimensional gain vector that can

be evaluated off-line from

$$K(k) = b^T(k)P_1(k+1)A(k)/[b^T(k)P_1(k+1)b(k) + r(k)] \quad (3.4.2.11)$$

where $P_1(k+1)$ results from the backward solution of the linear difference Equation (2.5.1.6).

2) When

$$v(u(k)) = |u(k)| \quad (3.4.2.12)$$

we have that $u^1(x(k))$ is equivalently given by the Compact Magnitude (CM) control law, $u^{CM}(x(k))$, i.e.,

$$u^1(x(k)) = u^{CM}(x(k)), \quad (3.4.2.13)$$

where $u^{CM}(x(k))$ is given by

$$u^{CM}(x(k)) = \begin{cases} Cl^+ [-K_1(k)x(k) - K_2(k)], & K_1(k)x(k) \leq g(c^-, c^+) \\ Cl^- [-K_1(k)x(k) + K_2(k)], & K_1(k)x(k) > g(c^-, c^+) \end{cases} \quad (3.4.2.14)$$

where:

a) $K_1(k)$ and $K_2(k)$ can be evaluated off-line from

$$K_1(k) = b^T(k)P_1(k+1)A(k)/b^T(k)P_1(k+1)b(k), \quad (3.4.2.15)$$

$$K_2(k) = r(k)/2b^T(k)P_1(k+1)b(k), \quad (3.4.2.16)$$

and $P_1(k+1)$ is found from the backward solution of the linear difference Equation (2.5.1.6);

b) $g(c^-, c^+)$ can also be evaluated off-line from

$$g(c^-, c^+) = [-1 + K_2(k)/(c^- - c^+)](c^- + c^+)/2 \quad (3.4.2.17)$$

where $K_2(k)$ is given by Equation (3.4.2.16), c^+ is the smallest nonnegative element of the set $(c_1^1, \dots, c_M^1)$, and c^- is the smallest, in magnitude, negative element of $(c_1^1, \dots, c_M^1)$ ⁸:

c) $Cl^+[X]$ denotes the nonnegative element in the set $(c_1^1, \dots, c_M^1)$ that is closest to the real element X ; and

d) $Cl^-[X]$ denotes the negative element in the set $(c_1^1, \dots, c_M^1)$ that is closest to the real number X .

⁸ It is assumed that both c^- and c^+ exist, otherwise the approach of Appendix D may be used to show that $u^{CM}(x(k))$ is either given by $Cl^+[-K_1(k)x(k) - K_2(k)]$ or by $Cl^-[-K_1(k)x(k) + K_2(k)]$.

The following results and observations concerning the 1 Stage ROLIMPs, CQ, and CM control laws are noted:

MEMORY STORAGE.

1 Stage ROLIMPs Control Law. Substituting 1 for I and 1 for m in Equation (3.3.3.8) we have that the memory capacity over N stages, $C_{L,I}$, of the 1 Stage ROLIMPs control law is given by

$$C_{L,I} = \{1 + (n + 1)N\}M \text{ words.} \quad (3.4.2.18)$$

CQ Control Law. Since $K(k)$ is a n dimensional gain vector that is evaluated off-line and the set $(c_1^1, \dots, c_M^1)$ must be stored for the purpose of performing the operation $Cl[.]$, it follows from Equation (3.4.2.10) that the memory capacity over N stages, C_{CQ} , that is required by the CQ control law is given by

$$C_{CQ} = nN + M \text{ words.} \quad (3.4.2.19)$$

CM Control Law. Since $K_1(k)$ is a n dimensional gain vector that is evaluated off-line, $K_2(k)$ is a scalar that is evaluated off-line, $g(c^-, c^+)$ is also a scalar that is evaluated off-line, and the set $(c_1^1, \dots, c_M^1)$ must be stored for the purpose of finding $Cl^+[.]$ or $Cl^-[.]$, it follows from Equation (3.4.2.14) that the memory capacity over N stages, C_{CM} , that is required by the CM control law is given by

$$C_{CM} = (n + 2)N + M \text{ words.} \quad (3.4.2.20)$$

MAXIMUM ON-LINE COMPUTATION TIME.

1 Stage ROLIMPs Control Law. Substituting 1 for I and 1 for m in Equation (3.3.3.10) we have that the maximum on-line computation time, $TM_{L,1}(p)$, of the 1 Stage ROLIMPs control law is given by

$$\begin{aligned}
 TM_{L,1}(p) \\
 = T_c M + (T_a + T_m)n(\ln[M/p] + Fr[M/p])
 \end{aligned}
 \tag{3.4.2.21}$$

where $\ln[.]$, $Fr[.]$, T_a , T_m , p , and T_c are as defined for Expressions (2.4.1) and (2.4.4).

CQ Control Law. Since the product $-K(k)x(k)$ require mn multiplications and $m(n-1)$ additions it follows from Equation (3.4.2.10) that the maximum on-line computation time, T_{cQ} , required by the CQ control law is given by

$$\begin{aligned}
 T_{cQ} \\
 = T_{cl} - T_a + (T_a + T_m)n
 \end{aligned}
 \tag{3.4.2.22}$$

where T_a and T_m are as defined for Equation (2.4.1) and T_{cl} is as defined for Equation (1.2.2.9).

CM control law. The following on-line multiplications, additions, and comparisons are noted for the CM control law (3.4.2.14): a) we must perform n multiplications in the

evaluation of $-K_1(k)x(k)$; b) we must perform n additions in the evaluation of $-K_1(k)x(k) + K_2(k)$ or $-K_1(k)x(k) - K_2(k)$; and c) we must perform a comparison of $-K_1(k)x(k)$ with $g(c^-, c^+)$ and a comparison of the elements in the set $(c_1^1, \dots, c_M^1)$ with the value of the scalar $-K_1(k)x(k) + K_2(k)$ or $-K_1(k)x(k) - K_2(k)$. As a result, we have that the maximum on-line computation time, T_{CM} , of the CM control law is given by

$$T_{CM} = T_{ma} + T_c + (T_a + T_m)n \quad (3.4.2.23)$$

where T_a , T_m , and T_c are as defined for Expressions (2.4.1) and (2.4.4) and T_{ma} is either the time taken by a digital computer to find the nonnegative element in the set $(c_1^1, \dots, c_M^1)$ that is closest to the value of the scalar $-K_1(k)x(k) + K_2(k)$ or is the time taken by a digital computer to find the negative element in the set $(c_1^1, \dots, c_M^1)$ that is closest to the value of the scalar $-K_1(k)x(k) + K_2(k)$.

SOME OBSERVATIONS.

1) A comparison of Equations (3.4.2.18), (3.4.2.19), and (3.4.2.20) reveals that for most typical values of n , N , and M the memory capacity of the CQ and CM control laws is less than that of the 1 Stage ROLIMPs control law (see Table 3.2).

Table 3.2 Tabulation of $C_{L,I}$, C_{CQ} , and C_{CM} for different values of n , M , and N .

n	N	M	$C_{L,I}^{(*)}$	$C_{CQ}^{(*)}$	$C_{CM}^{(*)}$
2	10	3	93	23	43
2	10	11	241	31	51
2	50	3	453	103	203
2	50	11	1441	111	211
5	10	3	183	53	73
5	10	11	671	61	81
5	50	3	903	253	353
5	50	11	3311	261	361

(*) Units of words

2) A comparison of Equations (3.4.2.21), (3.4.2.22), and (3.4.2.23) reveal that for most typical values of T_a , T_m , T_c , M , p , n , T_{cl} , and T_{ma} the maximum on-line computation time of the CQ and CM control laws is less than that of the 1 Stage ROLIMPs control law (see Table 3.3).

3) It is noted that the scalar control problem of Section 3.4.1 is a special case of the single input control problem of Section 3.4.2, namely the one corresponding to

$$v(u(k)) = |u(k)|.$$

As a result, the CM control law given by Equation (3.4.2.14) is an optimum control law for the scalar control problem, since it may be used in place of the optimum 1 Stage ROLIMPs control law.

3.5 SUMMARY AND CONCLUSIONS

3.5.1 I Stages ROMPs Control Laws

In this chapter a ROMPs control law was defined as any QUMPs or LIMPs control law which has had its order of MPs at any stage k , $M^{m(N-k)}$, reduced either by the simplification of the original QUMPs or LIMPs control law or by the outright deletion of some of its $M^{m(N-k)}$ MPs. In particular, we defined two classes of ROMPs control laws, the first called an I Stages ROQUMPs control law which is based on the QUMPs control law of Chapter II and the second called an I Stages

Table 3.3 Tabulation of $TM_{L,1}(p)$, T_{CQ} , T_{CM} for various values of M , n and

$$p = 1$$

$$T_a = 100 \text{ nsec}$$

$$T_m = 1 \text{ usec}$$

$$T_c = 100 \text{ nsec}$$

$$T_{cl} = M \cdot T_a$$

$$T_{ma} = M \cdot T_a$$

n	M	$TM_{L,1}(1)^{(*)}$	$T_{CQ}^{(*)}$	$T_{CM}^{(*)}$
2	3	6.9	3.5	3.7
2	11	25.3	4.3	4.5
5	3	16.8	5.7	5.9
5	11	61.6	6.5	6.7

(*) Units of usecs

ROLIMPs control law, which is based on the LIMPs control law of Chapter II. These I Stages ROQUMPs and ROLIMPs control laws were found to have the following features:

1) The structural characteristics of I Stages ROQUMPs and ROLIMPs control laws are the same as those of the ROQUMPs and ROLIMPs control laws with the basic exception that the number of MPs in the I Stages ROQUMPs and ROLIMPs control laws at each stage k are never larger than M^{mI} .

2) The N Stages ROQUMPs and ROLIMPs control laws are the optimum QUMPs and ROLIMPs control laws, respectively.

3) The maximum on-line computation time, and memory capacity of the I Stages ROQUMPs and ROLIMPs control laws steadily decrease as I decreases from N to 1.

3.5.2 Two Special Cases of I Stages ROMPs Control Laws

In Section 3.4.1 and Appendix C it is shown for a general stable scalar example that its corresponding I Stages ROLIMPs control law is optimum for all possible values of I from 1 to N . In addition, since the maximum on-line computation time and memory capacity of the I Stages ROLIMPs control law are steadily increasing functions of I , the 1 Stage ROLIMPs control law is selected over the other I Stages ROLIMPs control laws for implementation.

It was then shown in Section 3.4.2 and Appendix D that for a general single input example its corresponding 1 Stage ROLIMPs control law can be equivalently replaced by one of two compact control laws depending in the type of control penalty function, $v(u(k))$, that we have, i.e., $u^2(k)$ or $|u(k)|$; the control law corresponding to $u^2(k)$ was given the name Compact Quadratic (CQ) control law and that corresponding to $|u(k)|$ the name Compact Magnitude (CM) control law. Both of these compact control laws are shown to yield smaller maximum on-line computation time and memory capacity than those of the 1 Stage ROLIMPs control law.

Finally it is found that the CM control law is an optimum control law for the scalar control problem of Section 3.4.1, since this problem is a special case of the single input control problem of Section 3.4.2: note that the 1 Stage ROLIMPs control law was found to be an optimum control law for the scalar problem.

3.5.3 Conclusions

The following general conclusions result from this chapter:

- 1) The structural characteristics of the I Stages ROQUMPs and ROLIMPs control laws are the same as those of the QUMPs and LIMPs control laws, respectively, varying only in the number of their QUMPs and LIMPs at each stage k .

2) N Stages ROQUMPs and ROLIMPs control laws are the same as the QUMPs and LIMPs control laws, respectively, i.e., they are optimum control laws.

3) The maximum on-line computation time and memory capacity of I Stages ROQUMPs and ROLIMPs control laws steadily increase as a function of I .

4) I Stages ROQUMPs and ROLIMPs control laws have the potential of yielding excellent performance for small values of I while having relatively small maximum on-line computation time and memory capacity; this was verified with the general stable scalar example of Section 3.4.1.

5) In some cases, where the process has a single input, the 1 Stage ROLIMPs control law can be equivalently given by compact control laws with smaller maximum on-line computation time and memory capacity than those of the 1 Stage ROLIMPs control law.

In the next chapter the ROMPs control laws proposed in this chapter are used as the basis of ROMPs approaches that search for practical ROMPs control laws with desirable performance.

Chapter IV

REDUCED ORDER MATCHED PROCESSORS APPROACHES

4.1 INTRODUCTION

In Chapter III general Reduced Order Matched Processors (ROMPs) control laws were defined as well as specific I Stages ROMPs control laws. These control laws were based on the QUMPs and LIMPs control laws of Chapter II and were presented with the objective of offering the possibility of having control laws that do not have the dimensionality problems of the QUMPs and LIMPs control laws while yielding satisfactory performance.

In this chapter we consider ROMPs approaches that search for practical ROMPs control laws with satisfactory performance. In particular I Stages ROMPs approaches are presented and illustrated with two practical examples.

The rest of this chapter is organized as follows. In Section 4.2 general ROMPs approaches are defined. In Section 4.3 specific I Stages ROMPs approaches are presented with the aid of a flowchart. In Section 4.4 some features of I Stages ROMPs approaches are discussed. In Section 4.5 the I Stages ROMPs approaches are illustrated with two prac-

tical examples. Finally in Section 4.6 summary and conclusions of the present chapter are given.

4.2 ROMPS APPROACHES

A ROMPs approach is defined as any iterative approach that systematically searches for a ROMPs control law (see Section 3.2) with the following two attributes:

- 1) The ROMPs control law is practical in the sense that it can be implemented in the available digital computer with satisfactory on-line computation time.

- 2) The performance of the ROMPs control law, as measured by (1.1.4), is close to the optimum performance (or is the optimum performance itself as shown for a general scalar example in Section 3.4.1) and/or surpasses the performance derived from other proposed suboptimum control laws.

The following two important comments are made:

- 1) The measure of 'closeness' to the optimum performance is a function of the practical problem at hand, the considered initial conditions of the process, and how close other proposed control laws come to the optimum performance. As a result, it is left to the system designer to use his own measure of 'closeness'. In the practical problem of Section 4.5.2 where the performance of the ROMPs control law is explicitly compared to the optimum performance of the MPs con-

trol law it is found that the performance of the selected ROMPs control law is optimum for the selected set of initial conditions.

2) By 'surpasses the performance derived from other proposed suboptimum control laws' we mean that in the average (with respect to the considered initial conditions) the ROMPs control law yields a performance that is better than that of other proposed suboptimum control laws where the system designer may assign different weights to performances derived from the use of different initial conditions. In the practical examples considered in Sections 4.5.2 and 4.5.3 the ROMPs control laws are found to yield a performance that clearly supasses that of other proposed control laws; in addition the performance of the selected ROMPs control law is found to be relatively close to the optimum performance when compared with how far the performance of other proposed control laws are from the optimum performance.

Next we define a class of ROMPs approaches which are subsequently used in Section 4.5 for the design of practical control laws with outstanding performance.

4.3 I STAGES ROMPS APPROACHES

In the following two subsections we present two I Stages ROMPs approaches based on the I Stages ROQUMPs and ROLIMPs control laws of Section 3.3.2 and 3.3.3. However, it is

noted that before we can use any one of these two I Stages ROMPs approaches it is necessary that we first determine:

1) The number of MPs, p , that can operate in parallel. The value of p depends on the characteristics of the available digital computer (see conclusion 4 in Section 2.6.3);

2) The maximum memory capacity, C_{max} , that the available digital computer can have. This value is often determined from the maximum physical space that can be occupied by the computer's memory and also from financial considerations;

3) The maximum on-line computation time, T_{max} , that a practical control law can have. For instance, T_{max} may be chosen to be 10 percent of the time between two consecutive stages of the process (1.1.1); and

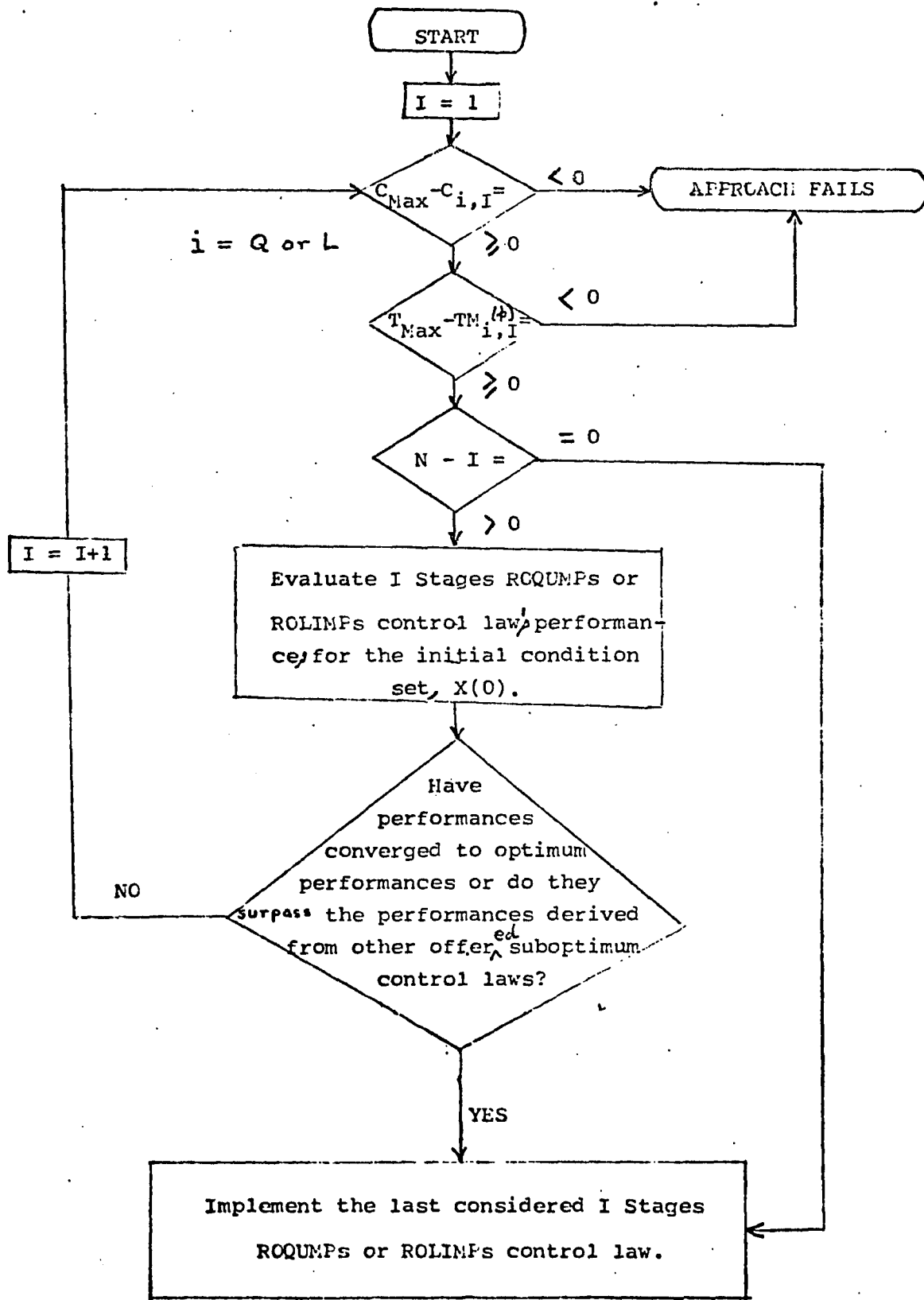
4) A set of initial conditions, $X(0)$, to test the I Stages ROMPs control laws. This set should be selected taking into consideration the physical characteristics of the given process.

4.3.1 I Stages ROQUMPs Approach

The I Stages ROQUMPs approach is an iterative approach which is presented with the aid of the flowchart of Figure 4.3.1.1. From this flowchart it is noted that starting with $I = 1$, each iteration of the I Stages ROQUMPs approach consists of the following five basic steps:

STEP 1. Evaluate

Figure 4.3.1.1 Flowchart of I Stages ROQUMPs and ROLIMFs Approaches.



$$C_{max} - C_{Q,I}$$

where $C_{Q,I}$ is given by Equation (3.3.2.8) and C_{Max} is the maximum memory capacity that the available digital computer can have. If

$$C_{max} - C_{Q,I} \geq 0$$

we proceed to STEP 2, otherwise the I Stages ROQUMPs approach ends without being able to find a practical control law.

STEP 2. Evaluate

$$T_{Max} - TM_{Q,I}(p)$$

where $TM_{Q,I}(p)$ is given by Equation (3.3.2.10) and T_{Max} is the maximum on-line computation time that a practical control law can have. If

$$T_{Max} - TM_{Q,I}(p) \geq 0$$

we proceed to STEP 3, otherwise the I Stages ROQUMPs approach ends without being able to find a practical control law.

STEP 3. Evaluate N-I. If N-I is equal to zero we have that the N Stages ROQUMPs control law is the optimum control law and the design approach ends successfully with the optimum control law implemented in the available digital computer. If N-I is positive we then go to STEP 4.

STEP 4. Find for each element of the selected set of initial conditions, $X(0)$, the performance, as measured by (1.1.4), of the I Stages ROQUMPs control law, then proceed to STEP 5.

STEP 5. Determine if the performances found in STEP 4 converge to the optimum performances and/or surpass the performances derived from other proposed suboptimum control laws. If the answer is NO update I by 1 and go back to STEP 1. If the answer is YES the I Stages ROQUMPs approach ends successfully and the last considered I Stages ROQUMPs control law is implemented in the available digital computer.

4.3.2 I Stages ROLIMPs Approach

The I Stages ROLIMPs approach is an iterative approach which is also presented with the aid of the flowchart of Figure 4.3.1.1. Again from this flowchart it is noted that starting with $I = 1$, each iteration of the I Stages ROLIMPs approach consists of the following five basic steps:

STEP 1. Evaluate

$$C_{Max} - C_{L,I}$$

where $C_{L,I}$ is given by Equation (3.3.3.8) and C_{Max} is the maximum memory capacity that the available digital computer can have. If

$$C_{Max} - C_{L,I} \geq 0$$

we proceed to STEP 2, otherwise the I Stages ROLIMPs approach ends without being able to find a practical control law.

STEP 2. Evaluate

$$T_{Max} - TM_{L,I}(p)$$

where $TM_{L,I}(p)$ is given by Equation (3.3.3.10) and T_{Max} is the maximum on-line computation time that a practical control law can have. If

$$T_{Max} - TM_{L,I}(p) \geq 0$$

we proceed to STEP 3, otherwise the I Stages ROLIMPs approach ends without being able to find a practical control law.

STEP 3. Evaluate N-I. If N-I is equal to zero we have that the N Stages ROLIMPs control law is the optimum control law and the design approach ends successfully with the optimum control law implemented in the available digital computer. If N-I is positive we then go to STEP 4.

STEP 4. Find for each element of the selected set of initial conditions, $X(0)$, the performance, as measured by (1.1.4), of the I Stages ROLIMPs control law, then proceed to STEP 5.

STEP 5. Determine if the performances found in STEP 4 converge to the optimum performances and/or surpass the performances derived from other proposed suboptimum control laws. If the answer is NO update I by 1 and go back to STEP 1. If the answer is YES the I Stages ROLIMPs approach ends successfully and the last considered I Stages ROLIMPs control law is implemented in the available digital computer.

4.4 I STAGES ROMPS APPROACHES FEATURES

The I Stages ROMPs approaches of Section 4.3 are found to have the following two important features.

4.4.1 Increase of Control Law Complexity with each Iteration

Noting from Equations (3.3.2.8), (3.3.2.10), (3.3.3.8), and (3.3.3.10) that the memory capacity and maximum on-line computation time for the I Stages ROQUMPs and ROLIMPs control laws increase as I increases, it follows that the iterative I Stages ROQUMPs and ROLIMPs approaches have the desired property of moving steadily from the fastest and simplest to the slowest and most complex I Stages ROQUMPs and ROLIMPs control laws.

4.4.2 Convergence of Approaches

Since the N Stages ROQUMPs and ROLIMPs control laws are the same as the QUMPs and LIMPs control laws (see GENERAL FEATURES in Sections (3.3.1.1) and (3.3.1.2)), it follows from Figure 4.3.1.1 that the I Stages ROQUMPs and ROLIMPs approaches of Section 4.3.1 and 4.3.2 converge to the optimum QUMPs and LIMPs control laws in N iterations. However, it should be noted that in most cases the N-th iteration is not reached due to at least one of the following events:

1) The I Stages ROQUMPs or ROLIMPs approach ends successfully with a value of I less than N (this is found to be the case for all the practical examples considered in Section 4.5); or

2) The approach ends with a value of I less than N due to a violation by the last considered I Stages ROQUMPs or ROLIMPs control law of either the bound C_{Max} or the bound T_{Max} .

4.5 PRACTICAL EXAMPLES

In this section, we use the I Stages ROMPs approaches of the previous section to search for practical I Stages ROMPs control laws with desirable performance for two physical examples.

The organization of this section is as follows: In Section 4.5.1, we provide the general framework that will be used in the investigation of the practical examples that are

given in subsequent sections. In Section 4.5.2 we consider an armature controlled DC motor example. And finally in Section 4.5.3 we consider a longitudinal aircraft motion control example.

4.5.1 General Framework

We now provide the control system assumptions and mathematical models and techniques that are used in the study of the practical examples of Sections 4.5.2 and 4.5.3.

Block Diagram of Control System

It is assumed that each of the considered physical processes is a continuous-time process commanded by a digital computer as seen in Figure 4.5.1.1 where the following important features are noted:

1) The zero-order hold produces a piecewise constant level for each of the control vector components $u_i(t)$ for $i \in (1, \dots, m)$, that changes value at equidistant instants of time separated by intervals of lengths V , where V is a positive real number;

2) The output of the sampler is constant during an interval of length V ; and

3) When implementing an I Stages ROMPs control law the digital computer obtains the I Stages ROMPs control law's MP

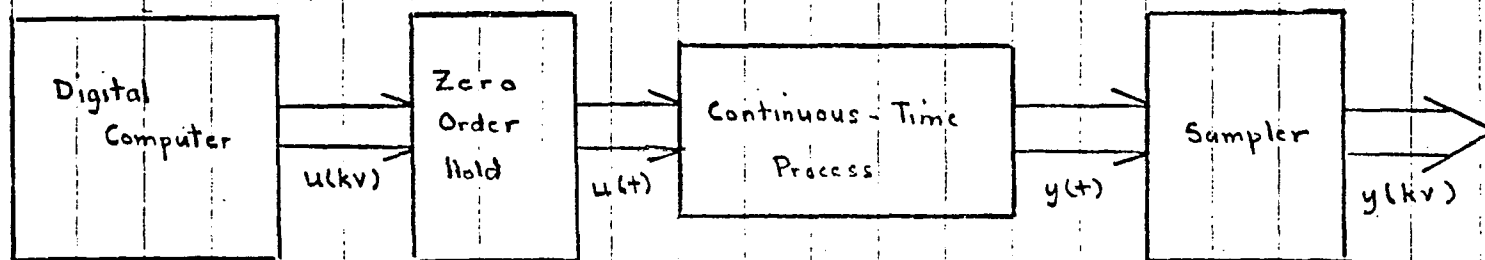


Figure 4.5.1.1 Block Diagram of Control System.

outputs one at a time, i.e., $p = 1$; furthermore, the digital computer has addition, multiplication, and comparison times given by

$$T_{\alpha} = 100 \text{ nsecs}, \quad (4.5.1.1)$$

$$T_m = 1 \text{ usec}, \quad (4.5.1.2)$$

and

$$T_c = 100 \text{ nsecs}, \quad (4.5.1.3)$$

and a maximum memory capacity given by

$$C_{\max} = 100,000 \text{ words} \quad (4.5.1.4)$$

Mathematical Statement of Problem

For each considered practical example we provide process and performance criterion equations that are special cases of the following:

Process

$$\dot{z}(t) = Fz(t) + Gu(t) \quad (4.5.1.5)$$

$$y(t) = H^T z(t) \quad (4.5.1.6)$$

where:

a) $z(t)$ and $y(t)$ are n' dimensional state and m' dimensional output variables, respectively;

b) the space of $z(t)$ and $y(t)$ is given by the space of all real numbers;

c) $u(t)$ is a m dimensional control vector with space given by

$$u_i(t) \in (c_1^i, \dots, c_M^i) \text{ for all } i \quad (4.5.1.8)$$

where the quantization levels of the above control space are uniformly distributed between c_1^i and c_M^i for all i ; and

d) F , G , and H are matrices of appropriate dimensions.

Performance Criterion

PC

$$= \sum_{k=0}^N [(y(k) - tr(k))^T \bar{Q} (y(k) - tr(k)) + u^T(k) R u(k)] \quad (4.5.1.9)$$

where

a) $y(k)$ and $u(k)$ are the sampled output $y(kV)$ and control $u(kV)$ of the process Equations (4.5.1.5) and (4.5.1.6) where V has been dropped from $y(kV)$ and $u(kV)$ to simplify notation;

b) $\bar{Q}$ and R are matrix weights of appropriate dimensions;

c) $\bar{Q}$ is symmetric; and

d) $tr(k)$ is the sampled output $tr(kV)$, with V dropped, of the mathematical model

$$\dot{\hat{z}}(t) = \bar{F} \hat{z}(t) \quad (4.5.1.10)$$

$$\text{tr}(t) = C^T \hat{z}(t) \quad (4.5.1.11)$$

where $\hat{z}(t)$ is a m' dimensional state vector, $\text{tr}(t)$ is a m' dimensional output vector, $\bar{F}$ is a diagonal matrix, C^T is a unit matrix, and $\hat{z}(0)$ is assumed known.

The following important comments are made concerning the mathematical statement of the control problem:

First, the control problem consists in the minimization of PC using as constraints Equations (4.5.1.5), (4.5.1.6), (4.5.1.8), (4.5.1.10), and (4.5.1.11). Note that this control problem is a tracking problem since the performance criterion Equation (4.5.1.9) reflects the desire to have the process sampled output, $y(k)$, follow the signal $\text{tr}(k)$.

Second and last, to be able to design the I Stages ROMPs, QLQ, and SNN control laws of Sections 3.3, 1.2, and 1.3 for the above process and performance criterion it is necessary that the following operations be performed:

1) Reformulation of the given tracking problem as a regulator problem so that the resulting performance criterion becomes a special case of Equations (1.1.4), (1.2.2.2), and (1.2.3.2) when possible (note that the SNN control law is only defined for single input processes); and

2) Time discretization of the reformulated process so that it resembles Equations (1.1.1), (1.2.2.1), and

(1.2.3.1).

Reformulation of Tracking Problem as a Regulator Problem

The given process and performance criterion is reformulated as a regulator problem using the approach of Anderson and Moore[1971;251-253]. When this is done the following continuous time process and performance criterion result:

Process

$$\dot{x}(t) = A'x(t) + B'u(t) \quad (4.5.1.12)$$

where

$$A' = \begin{bmatrix} F & 0 \\ 0 & \bar{F} \end{bmatrix}, \quad (4.5.1.13)$$

$$B' = \begin{bmatrix} G \\ 0 \end{bmatrix}, \quad (4.5.1.14)$$

and $x(t)$ is an n dimensional vector given by

$$x(t) = \begin{bmatrix} z(t) \\ \hat{z}(t) \end{bmatrix} \quad (4.5.1.15)$$

where $u(t)$, F , $\bar{F}$, G , $z(t)$, and $\hat{z}(t)$ are as defined for Equations (4.5.1.5) and (4.5.1.10).

Performance Criterion

$$\begin{aligned} & h_N(X_{0,N}, U_{0,N-1}) \\ &= \sum_{k=0}^N [x^T(k)Qx(k) + u^T(k)Ru(k)] \quad (4.5.1.16) \end{aligned}$$

where

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \quad (4.5.1.17)$$

with

$$Q_{11} = H\bar{Q}H^T, \quad (4.5.1.18)$$

$$Q_{12} = -H\bar{Q}C^T, \quad (4.5.1.19)$$

and

$$Q_{22} = C\bar{Q}C^T, \quad (4.5.1.20)$$

where H , R , $\bar{Q}$, and C are as defined for Equations (4.5.1.6), (4.5.1.9), and (4.5.1.11).

Note from Equation (4.5.1.12) and (4.5.1.16) that the state equation matrix A' and the state weight matrix Q are independent of $u(t)$. As a result, in Sections 4.5.2 and 4.5.3 we search for a satisfactory I Stages ROLIMPs control law for the given physical processes. Furthermore, it will be found in Section 4.5.2 and 4.5.3 that the considered physical examples require $u(t)$ to be small in the steady state. This in turn implies that set $S(k,I,d)$ corresponding to the I Stages ROLIMPs control law is given by

$$S(k,I,d) = S(k,I,0). \quad (4.5.1.21)$$

Process Discretization

A discrete time version of the process Equation (4.5.1.12) is obtained as follows:

First, we note that two intermediate states of the process Equation (4.5.1.12) are given by $x(kV)$ and $x((k+1)V)$ where V is as defined earlier for the zero-order hold of Figure 4.5.1.1, and $k \in (0, \dots, N-1)$.

Second, we note that

$$u(t) = u(kV) \quad (4.5.1.25)$$

for

$$kV < t < (k+1)V$$

as implied by the zero-order hold of Figure 4.5.1.1.

Third and last, we have from Equation (4.5.1.12) and Condition (4.5.1.25) that the process state $x((k+1)V)$ can be found as a function of $x(kV)$ and $u(kV)$ from the relation

$$x(k+1) = Ax(k) + B(u(k)) \quad (4.5.1.26)$$

where

$$A = e^{A'V}, \quad (4.5.1.27)$$

$$B(u(k)) = \int_{kV}^{(k+1)V} e^{A'((k+1)V-\tau)} B'u(k) d\tau, \quad (4.5.1.28)$$

$$e^{A't} = I' + A't + A'^2 t^2 / 2! + \dots, \quad (4.5.1.28a)$$

I' is a unit matrix,

and A' and B' are as defined for Equation (4.5.1.12). Note that in place of $u(kV)$ and $x((k+1)V)$ we have used $u(k)$ and $x(k)$ in Equations (4.5.1.26)-(4.5.1.28).

Control Law Design and Implementation Parameters

Specific values for the following control law design and implementation parameters are given or derived:

1) The process parameters A' and B' . Note that the control law design programs of Appendix E determine A and $B(u(k))$ for all $u(k)$ from A' and B' by making use of Equations (4.5.1.27) and (4.5.1.28);

2) The performance criterion parameters Q and R ;

3) The total number of control stages, N ;

4) The control quantization levels $(c_1^i, \dots, c_M^i)$ for all i ;

5) The sampling time, V ;

6) A set of initial conditions, $X(0)$;

7) The maximum on-line computation time, T_{Max} , and maximum memory capacity, C_{Max} , that are allowed for any control law. Note that C_{Max} was given earlier by Equation (4.5.1.4) and T_{Max} is chosen to be 10 percent of the sampling time V , i.e.,

$$T_{Max} = .1V \quad (4.5.1.29)$$

QLQ, SNN, I Stages ROLIMPs, and Constant I Stages ROLIMPs Control Law Design

After the parameters A' , B' , Q , R , N , $(c_1^i, \dots, c_M^i)$ for all i , and V have been defined we then proceed to use the design programs of Appendix E to derive the corresponding I Stages ROLIMPs, QLQ, and SNN control laws. However, before this is done the following additional feature of the computer programs of Appendix E is discussed.

It was noted in Section 3.3.3 that the I Stages ROLIMPs control law design requires that the Matched Parameters (MPAs) $P_2(U_{k,N-1}, k)$, and $p_3(U_{k,N-1}, k)$ be found and stored for all $k \in (0, \dots, N-1)$: this in turn implies a large memory capacity for large values of N . To alleviate this memory capacity requirement the computer programs of Appendix E allow us to design a Constant I Stages ROLIMPs control law, $u_c^I(x(k))$, that approximates $P_2(U_{k,N-1}, k)$ and $p_3(U_{k,N-1}, k)$ by $P_2(U_{0,N-1}, 0)$ and $p_3(U_{0,N-1}, 0)$, respectively, for all k : this approximation is justified by several computer simulations corresponding to the practical problems of Sections 4.5.2 and 4.5.3 that have shown $P_2(U_{k,N-1}, k)$ and $p_3(U_{k,N-1}, k)$ to remain close to $P_2(U_{0,N-1}, 0)$ and $p_3(U_{0,N-1}, 0)$ for most values of k when N is large. Later in Section 4.5.2 and 4.5.3 these Constant I Stages ROLIMPs control laws are found to yield a performance that is not far from that of the corresponding I Stages ROLIMPs control laws.

The following observations are now made concerning $u_c^I(x(k))$.

From the above definition of $u_c^I(x(k))$ it follows that

$$u_c^I(x(k)) = u^I(x(k)) / \{p_2(U_{k,N-1}, k) = X, p_3(U_{k,N-1}, k) = Y\} \quad (4.5.1.30)$$

where $X = p_2(U_{0,N-1}, 0)$, $Y = p_3(U_{0,N-1}, 0)$, and $u^I(x(k))$ is as defined in Section 3.3.3 for an I Stages ROLIMPs control law. The memory capacity that is associated with this constant control law is easily found from Equation (4.5.1.30) and the definition of $u^I(x(k))$ (see Section 3.3.3) to be given by

$$C_{L,I}^c = mM^m + (n + 1)M^{mI}. \quad (4.5.1.31)$$

Note that the maximum on-line computation time of the Constant I Stages ROLIMPs control law remains the same as that of the I Stages ROLIMPs control law.

I Stages ROMPs Approach

Each of the iterations of the corresponding I Stages ROLIMPs approach of Section 4.4 is then performed by making use of: a) the control law simulation programs of Appendix E; b) the maximum on-line computation time and memory capacity of the corresponding I Stages or Constant I Stages ROLIMPs control laws; c) the maximum on-line computation time allowed for any control law $T_{\max}$; and d) the maximum memory

capacity of the available digital computer $C_{H_{ax}}$.

Summary of Results

The results derived by the use of the I Stages ROLIMPs approach to each of the two practical examples are then summarized via tables that give the performance of the considered control laws, and figures that present a few selected trajectories that result from the use of these control laws. In addition, the maximum on-line computation time and memory capacity of the selected I Stages ROLIMPs control law are given.

4.5.2 An Armature Controlled DC Motor

First, we provide the process and performance criterion equations for an armature controlled DC motor (this example was motivated by Gibson[1963:443]).

Process

$$\dot{z}(t) = Fz(t) + Gu(t) \quad (4.5.2.1)$$

$$y(t) = H^T z(t) \quad (4.5.2.2)$$

with

$$F = \begin{bmatrix} 0 & 1 \\ 0 & -1/T \end{bmatrix}, \quad (4.5.2.3)$$

$$G = \begin{bmatrix} 0 \\ K/T \end{bmatrix}, \quad (4.5.2.4)$$

and

$$H^T = [1 \ 0] \quad (4.5.2.5)$$

where:

a) $z_1(t)$ and $z_2(t)$ are the angular position and angular velocity of a DC motor, respectively;

b) $y(t)$ is the same as $z_1(t)$;

c) $u(t)$ is the electrical voltage applied to a DC motor;

d)

$$T = J/B \quad (4.5.2.6)$$

and

$$K = K'/B \quad (4.5.2.7)$$

where J , B , and K' are the moment of inertia, friction coefficient, and torque coefficient of a DC motor, respectively;

e) T is assumed to be 0.1 second and $K = 200$ rad/sec to yield

$$F = \begin{bmatrix} 0 & 1 \\ 0 & -10 \end{bmatrix} \quad (4.5.2.8)$$

and

$$G = \begin{bmatrix} 0 \\ 2000 \end{bmatrix}; \quad (4.5.2.9)$$

and

f) the space of $u(k)$, in volts, is given by

$$(c_1^1, c_2^1, c_3^1) = (-1/2, 0, 1/2). \quad (4.5.2.9a)$$

Performance Criterion

PC

$$= \sum_{k=0}^N [\bar{Q}(y(k) - tr(k))^2 + Ru^2(k)] \quad (4.5.2.10)$$

where:

a)

$$\bar{Q} = 1; \quad (4.5.2.11)$$

b) R is a nonnegative scalar;

c) tr(k) is the sampled output of the mathematical model

$$\dot{\hat{z}}(t) = \bar{F}\hat{z}(t) \quad (4.5.2.12)$$

$$tr(t) = \bar{C}\hat{z}(t) \quad (4.5.2.13)$$

with

$$\bar{F} = -10, \quad (4.5.2.14)$$

and

$$\bar{C} = 1; \quad (4.5.2.15)$$

and

d) tr(k) for $k \in (0, \dots, N)$ is the trajectory that we desire the sampled angular position, $y(k)$, of the DC motor to follow.

Note from Equations (4.5.2.12)-(4.5.2.15) that tr(t) goes to zero as t goes to ∞ . As a result, it follows from Equations (4.5.2.10), (4.5.2.1)-(4.5.2.2), and

(4.5.2.8)-(4.5.2.9) that both $z(t)$ and $u(t)$ should approach the origin as t goes to ∞ ; this in turn justifies the selection of $S(k, I, 0)$ for the I Stages ROMPs control laws provided 0 is in the space of $u(k)$.

Second, using Equations (4.5.1.12)-(4.5.1.20) the given tracking problem is reformulated as a regulator problem with the following values for A' , B' , and Q :

$$A' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & -10 \end{bmatrix}, \quad (4.5.2.17)$$

$$B' = \begin{bmatrix} 0 \\ 2000 \\ 0 \end{bmatrix}, \quad (4.5.2.18)$$

and

$$Q = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}. \quad (4.5.2.19)$$

Third, the sampling time V is given by

$$V = 0.01 \text{ seconds.} \quad (4.5.2.20)$$

This in turn implies from Equation (4.5.1.29) that

$$T_{\text{Max}} = 0.001 \text{ seconds.} \quad (4.5.2.21)$$

Fourth, an initial condition set $X(0)$ with 32 elements is selected. This set is illustrated by Figures 4.5.2.1a-b where it is noted that $X(0)$ has been partitioned into two mutually exclusive sets $X_1(0)$ and $X_2(0)$ which are respectively illustrated by Figure 4.5.2.1a and Figure 4.5.2.1b. Note that each of the 32 elements in the set $X(0)$, i.e., $x^i(0)$ for $i \in (1, \dots, 32)$, is easily identified by the two coordinates associated with the circled integer i shown on the figure as well as the third coordinate given on the bottom of the figure, e.g., the initial condition $x^9(0)$ is seen from Figure 4.5.2.1a to be given by

$$x^9(0) = [1 \ 0 \ -1.5]^T.$$

Fifth, the following different cases for R and N will be considered for the derivation of a practical I Stages ROLIMPs or Constant I Stages ROLIMPs control law with satisfactory performance.

Case 1

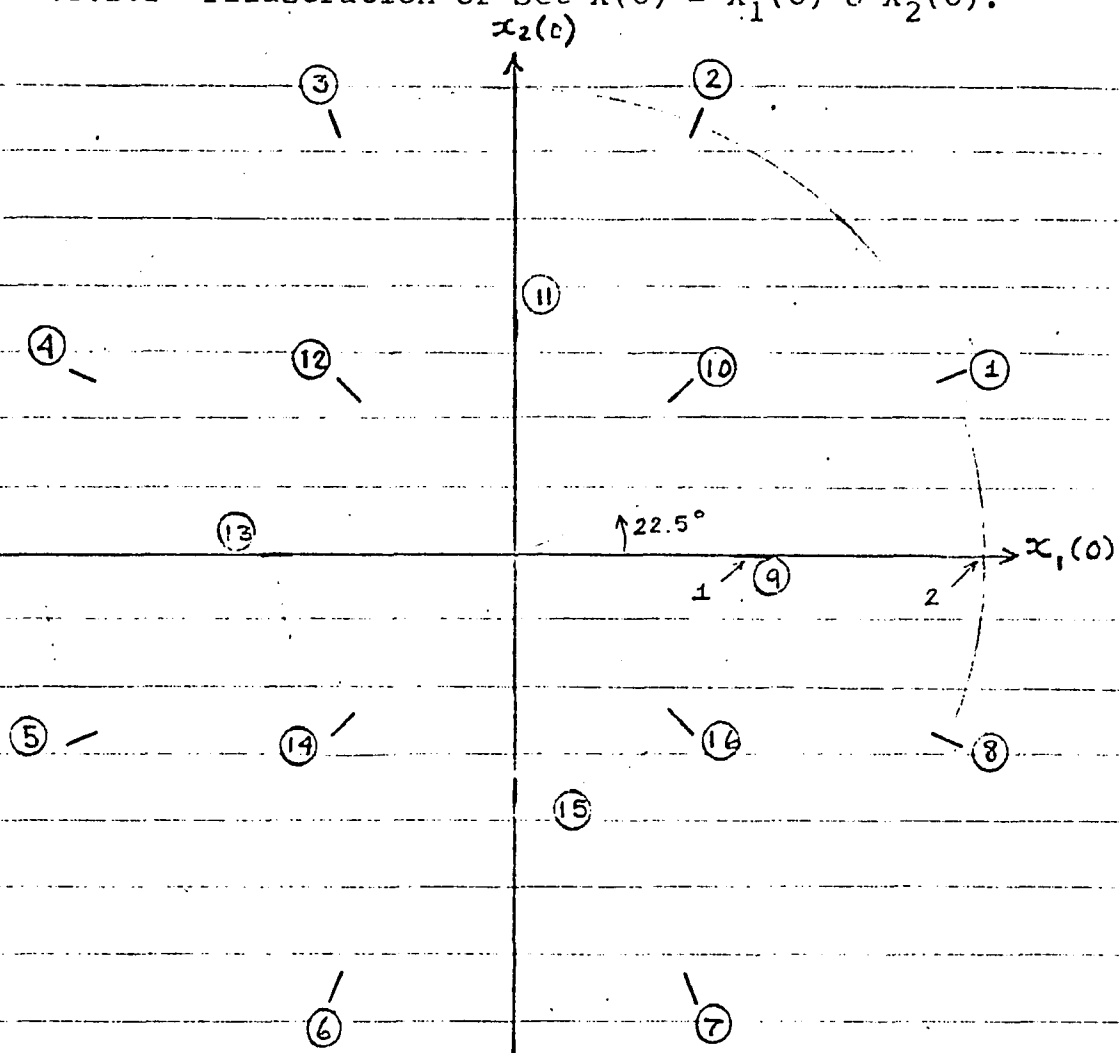
$$R = 0 \quad (4.5.2.22)$$

and

$$N = 7 \quad (4.5.2.23)$$

with $(c_1^1, \dots, c_M^1)$, A^1 , B^1 , Q , V , and $X(0)$ given by Equations (4.5.2.9a), (4.5.2.17)-(4.5.2.20), and Figures 4.5.2.1a-b. Since N is small we are interested in the derivation of a satisfactory I Stages ROLIMPs control law.

Figure 4.5.2.1 Illustration of Set $X(0) = X_1(0) \cup X_2(0)$. 117



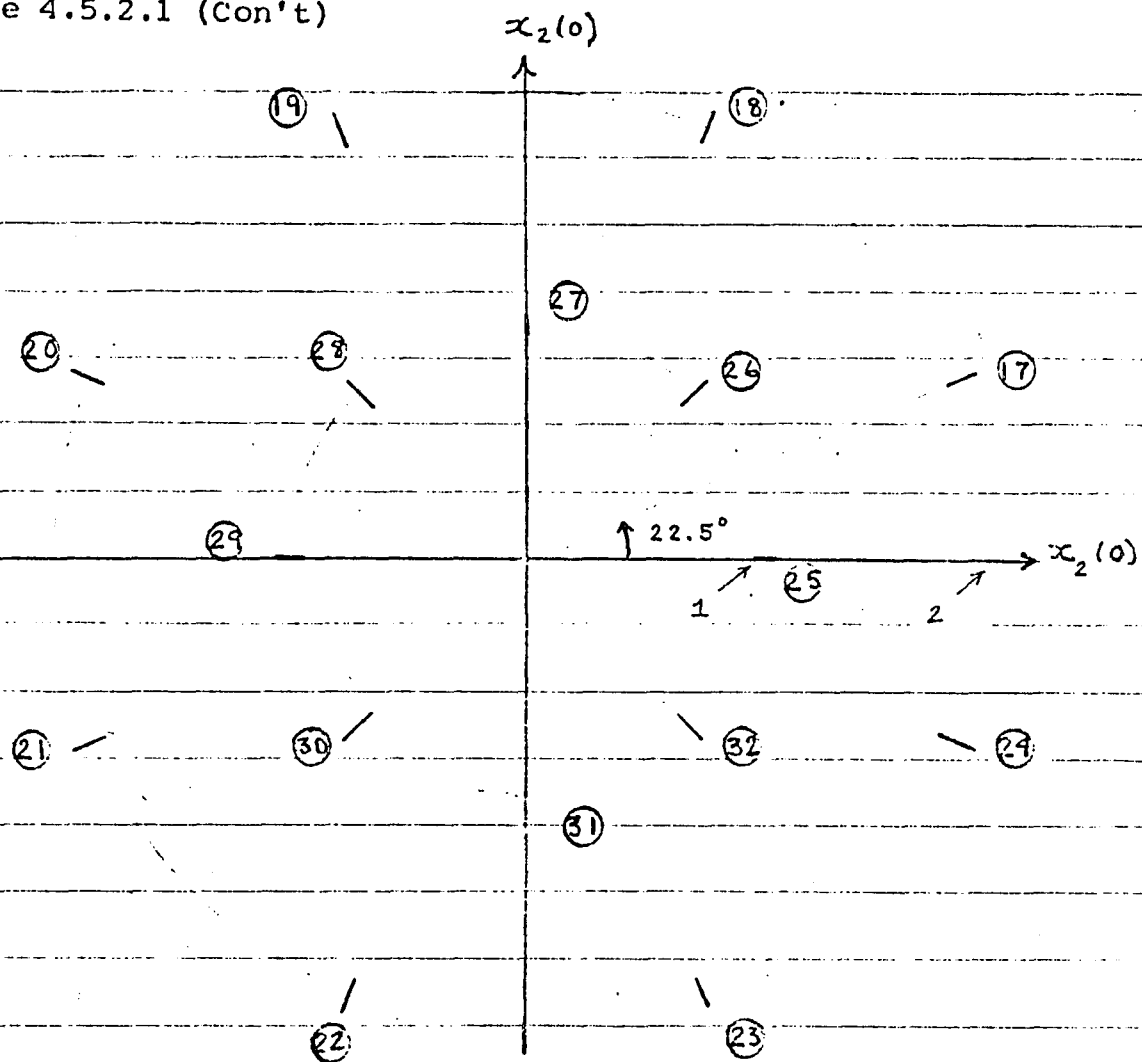
$$X_1(0) = \{x(0), \dot{x}^i(0) \text{ for } i=1, \dots, 16\}$$

where x_i^0 is the initial condition given by the two coordinates shown on the above figure corresponding to the circled integer i , and the third coordinate

$$x_3(0) = -1.5..$$

a) The set $X_1(0)$.

Figure 4.5.2.1 (Con't)



$$X_2(0) = \{x(0) : x^i(0) \text{ for } i = 17, \dots, 32\}$$

where $x^i(0)$ is the initial condition given by the two coordinates shown on the above figure corresponding to the circled integer i , and the third coordinate

$$x_3(0) = -2.5$$

b) The set $X_2(0)$.

Case 2

$$R = 0.1 \quad (4.5.2.24)$$

and

$$N = 7 \quad (4.5.2.25)$$

with $(c_1^1, \dots, c_M^1)$, A^1 , B^1 , Q , V , and $X(0)$ given by Equations (4.5.2.9a), (4.5.2.17)-(4.5.2.20), and Figures 4.5.2.1a-b. Again since N is small we are interested in the derivation of a satisfactory I Stages ROLIMPs control law.

Case 3

$$R = 0 \quad (4.5.2.26)$$

and

$$N = 40 \quad (4.5.2.27)$$

with $(c_1^1, \dots, c_M^1)$, A^1 , B^1 , Q , V , and $X(0)$ given by Equations (4.5.2.9a), (4.5.2.17)-(4.5.2.20), and Figures 4.5.2.1 a-b. In this case N is relatively large, as a result, we are interested in the derivation of a satisfactory Constant I Stages ROLIMPs control law as defined by Equation (4.5.1.30).

Sixth, the I Stages ROLIMPs approach of Section 4.3.2 is then applied to the DC motor yielding the following results

for each different case of R and N.

Case 1

In this first case the corresponding 4 Stages ROLIMPs control law with

$$TM_{L,4}(1) = 275.4 \text{ usecs}$$

and

$$C_{L,4} = 1455 \text{ words}$$

is selected for implementation.

The bases for the selection of the 4 Stages ROLIMPs control law are as follows:

1) The 4 Stages ROLIMPs control law yields the optimum performance for each initial condition in the set $X(0)$. This assertion can be readily verified from Table 4.5.2.1 which presents the performance of 1 thru 7 Stages ROLIMPs control laws for the set of initial conditions $X(0)$ shown in Figures 4.5.2.1a-b. Note that the 7 Stages ROLIMPs control law is the optimum LIMPs control law.

2) The 4 Stages ROLIMPs control law is superior to the 1, 2, and 3 Stages ROLIMPs control law as far as performance is concerned. See Table 4.5.2.1 for a verification of this result.

Table 4.5.2.1 Performance Tabulation for Case 1.

 $R = 0$ $N = 7$

i	P_1	P_2	P_3	P_4	P_5	P_6	P_7
1	34.66559	34.66559	34.66559	34.66559	34.66559	34.66559	34.66559
2	11.87765	11.86880	11.86880	11.86880	11.86880	11.86880	11.86880
3	4.6374150	4.759077	4.189361	4.189361	4.189361	4.189361	4.189361
4	4.9947416	4.9894488	4.9894488	4.9894488	4.9894488	4.9894488	4.9894488
5	5.573542	5.573542	5.523522	5.523522	5.523522	5.523522	5.523522
6	6.411384	6.3778647	6.3778647	6.3778647	6.3778647	6.3778647	6.3778647
7	10.739657	10.73524	10.73524	10.73524	10.73524	10.73524	10.73524
8	33.52995	33.52995	33.52995	33.52995	33.52995	33.52995	33.52995
9	14.93291	14.93291	14.93291	14.93291	14.93291	14.93291	14.93291
10	10.67415	10.67415	10.67415	10.67415	10.67415	10.67415	10.67415
11	3.556401	3.556401	3.556401	3.556401	3.556401	3.556401	3.556401
12	5.778631	5.729109	5.633764	5.633764	5.633764	5.633764	5.633764
13	2.175192	2.1954411	2.1164325	2.1164325	2.1164325	2.1164325	2.1164325
14	4.733516	4.5517370	4.5189303	4.4734317	4.4734317	4.473417	4.473407
15	3.355163	3.352328	3.323944	3.323944	3.323944	3.323944	3.323944
16	10.23217	10.27345	10.27345	10.27345	10.27345	10.27345	10.27345

 i = superscript corresponding to initial condition $x^i(0)$ P_i = Performance of i - Stages ROLHP, Control Law

3) The 4 Stages ROLIMPs control law is superior to the 5 thru 7 Stages ROLIMPs control laws since it yields a smaller maximum on-line computation time and memory capacity. Remember from Section 3.3.3 that $TM_{L,I}(1)$ and $C_{L,I}$ steadily increase as I increases.

4) The maximum on-line computation time and memory capacity of the 4 Stages ROLIMPs control law satisfy the system constraints (4.5.2.21) and (4.5.1.4).

Figures 4.5.2.2a-b present the state and control trajectories that are derived from the use of the 4 Stages ROLIMPs control law for the initial conditions $x^2(0)$ and $x^1(0)$.

Case 2

In this second case the corresponding 3 Stages ROLIMPs control law with

$$TM_{L,3}(1) = 91.8 \text{ usecs}$$

and

$$C_{L,3} = 483 \text{ words}$$

is selected for implementation.

The bases for the selection of the 3 Stages ROLIMPs control law for Case 2 are similar to those used in the selection of the 4 Stages ROLIMPs control law for Case 1. See Table 4.5.2.2 for a verification of this result, and Figures 4.5.2.3a-b for the state and control trajectories that are

Figure 4.5.2.2 State and Control Trajectories Corresponding to the 4 Stages ROLIMPs Control Law.

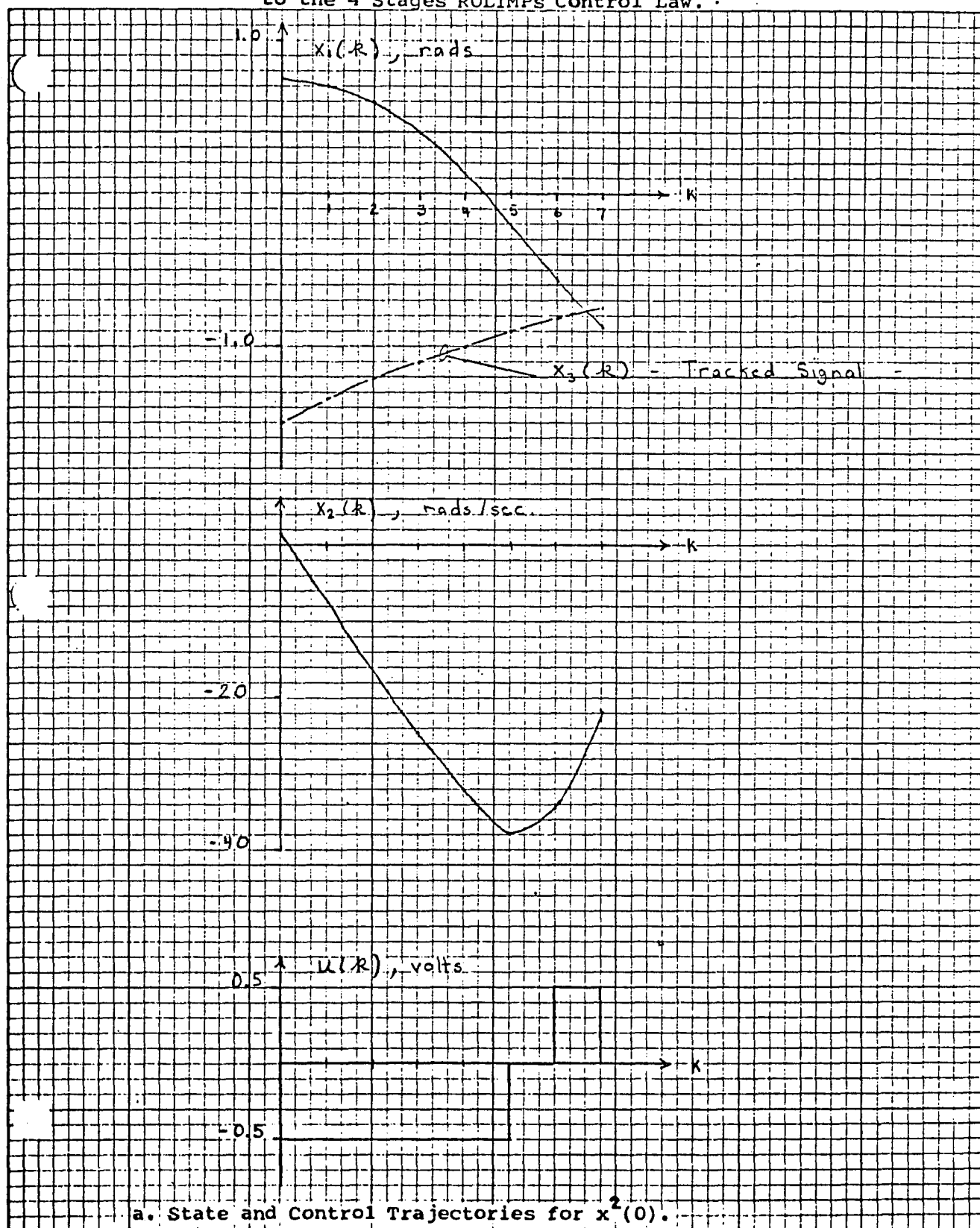


Figure 4.5.2.2 (Con't)

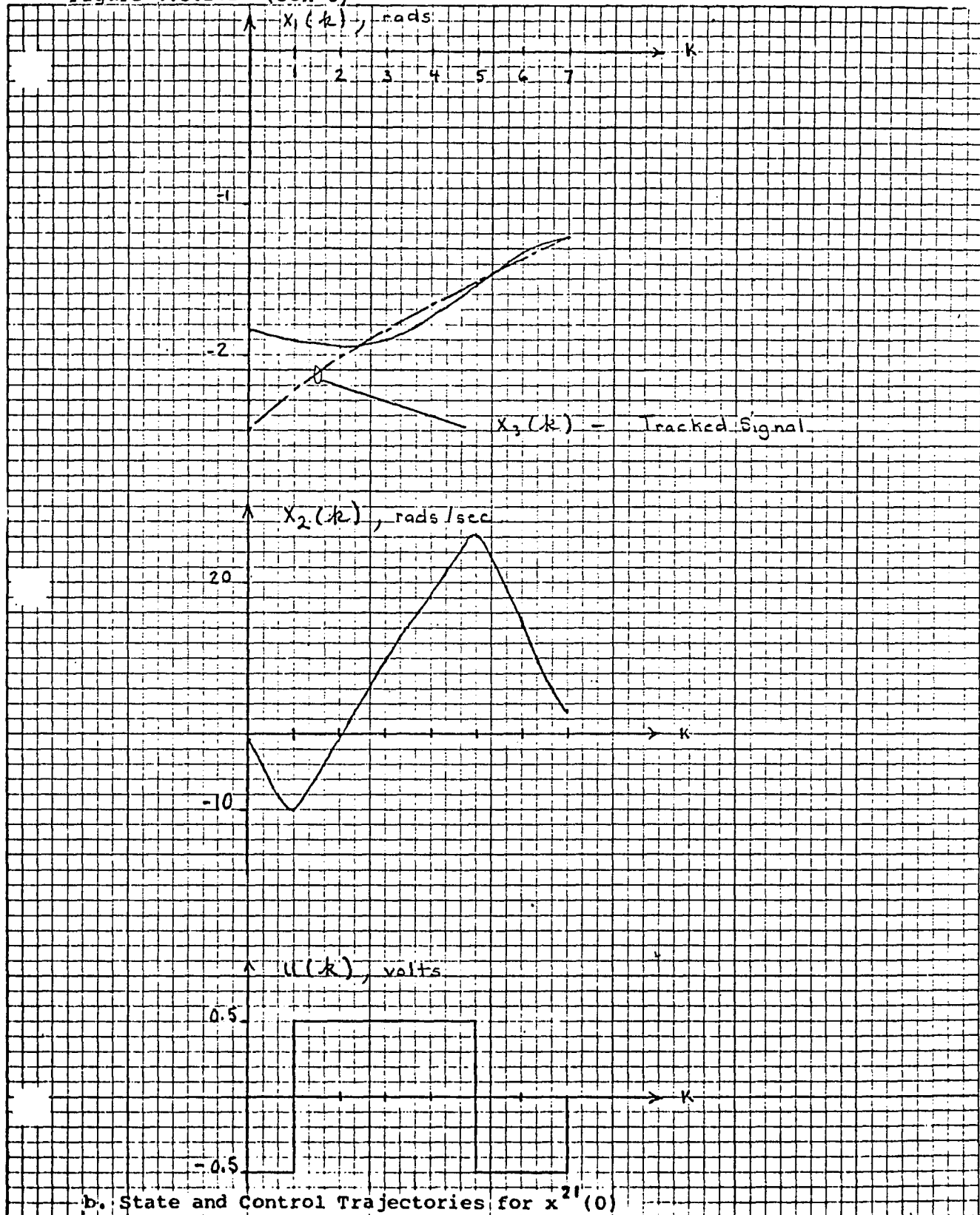


Table 4.5.2.2 Performance Tabulation for Case 2.

$R = 0.1$

$N = 7$

i	P_1	P_2	P_3	P_4	P_5	P_6	P_7
1	35.33223	35.33223	35.33223	35.33223	35.33223	35.33223	35.33223
2	12.38342	12.38342	12.38342	12.38342	12.38342	12.38342	12.38342
3	.6911979	.6911979	.6911979	.6911979	.6911979	.6911979	.6911979
4	.8340096	.8340096	.8340096	.8340096	.8340096	.8340096	.8340096
5	.8908164	.8908164	.8908164	.8908164	.8908164	.8908164	.8908164
6	.6536267	.6536267	.6536267	.6536267	.6536267	.6536267	.6536267
7	11.20781	11.20781	11.20781	11.20781	11.20781	11.20781	11.20781
8	34.18916	34.18916	34.18916	34.18916	34.18916	34.18916	34.18916
9	15.45337	15.45337	15.45337	15.45337	15.45337	15.45337	15.45337
10	11.16586	11.16586	11.16586	11.16586	11.16586	11.16586	11.16586
11	4.018369	4.018369	4.018369	4.018369	4.018369	4.018369	4.018369
12	.7873822	.7873822	.7873822	.7873822	.7873822	.7873822	.7873822
13	.2964147	.2964147	.2964147	.2964147	.2964147	.2964147	.2964147
14	.7659625	.7659625	.7659625	.7659625	.7659625	.7659625	.7659625
15	3.756995	3.756995	3.756995	3.756995	3.756995	3.756995	3.756995
16	10.74377	10.74377	10.74377	10.74377	10.74377	10.74377	10.74377

i - superscript corresponding to the initial conditions $x^i(0)$.

P_i - performance of the i Stages ROLIMP's control law.

Figure 4.5.2.3 State and Control Trajectories Corresponding to the 3 Stages ROLIMPs Control Law.

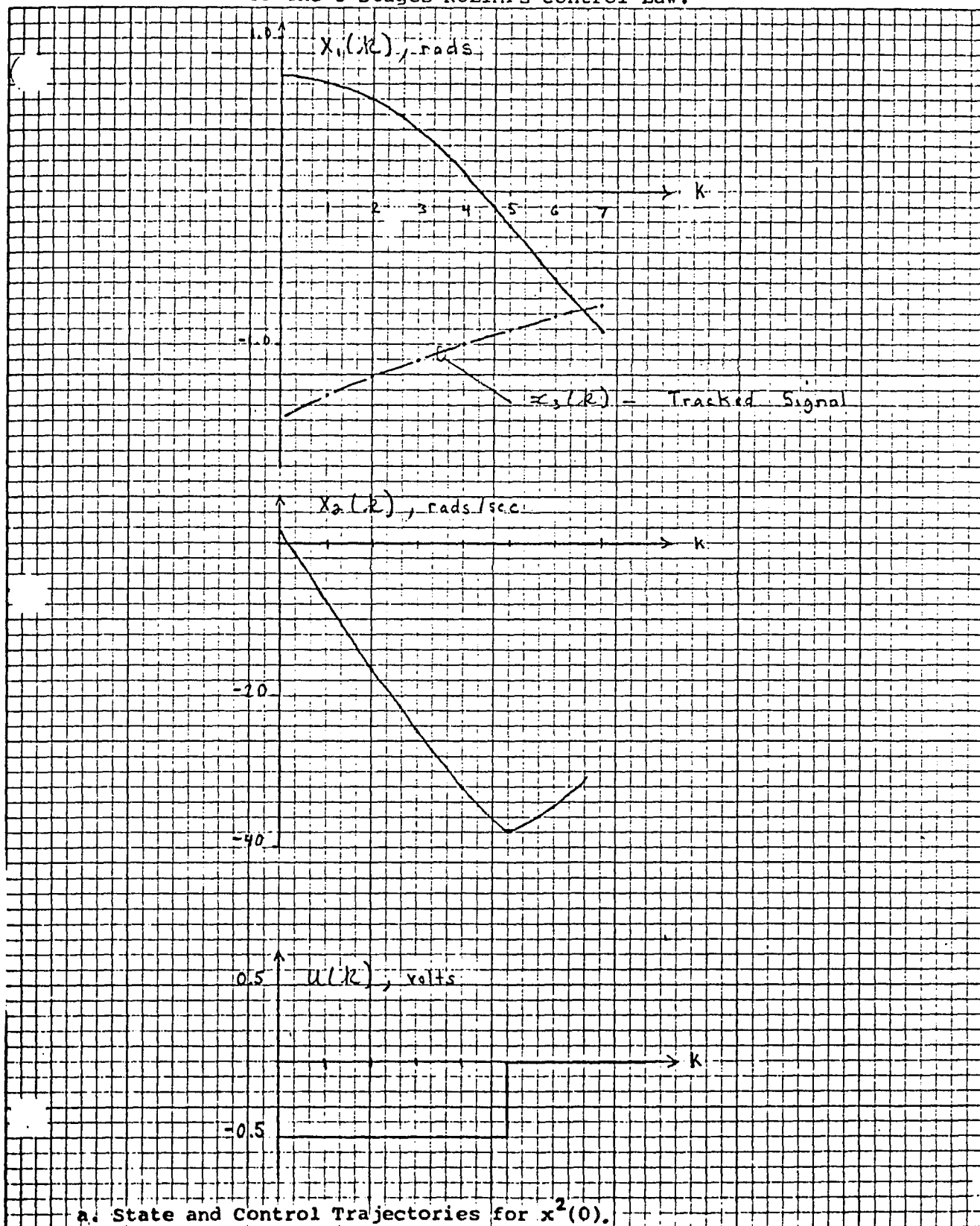
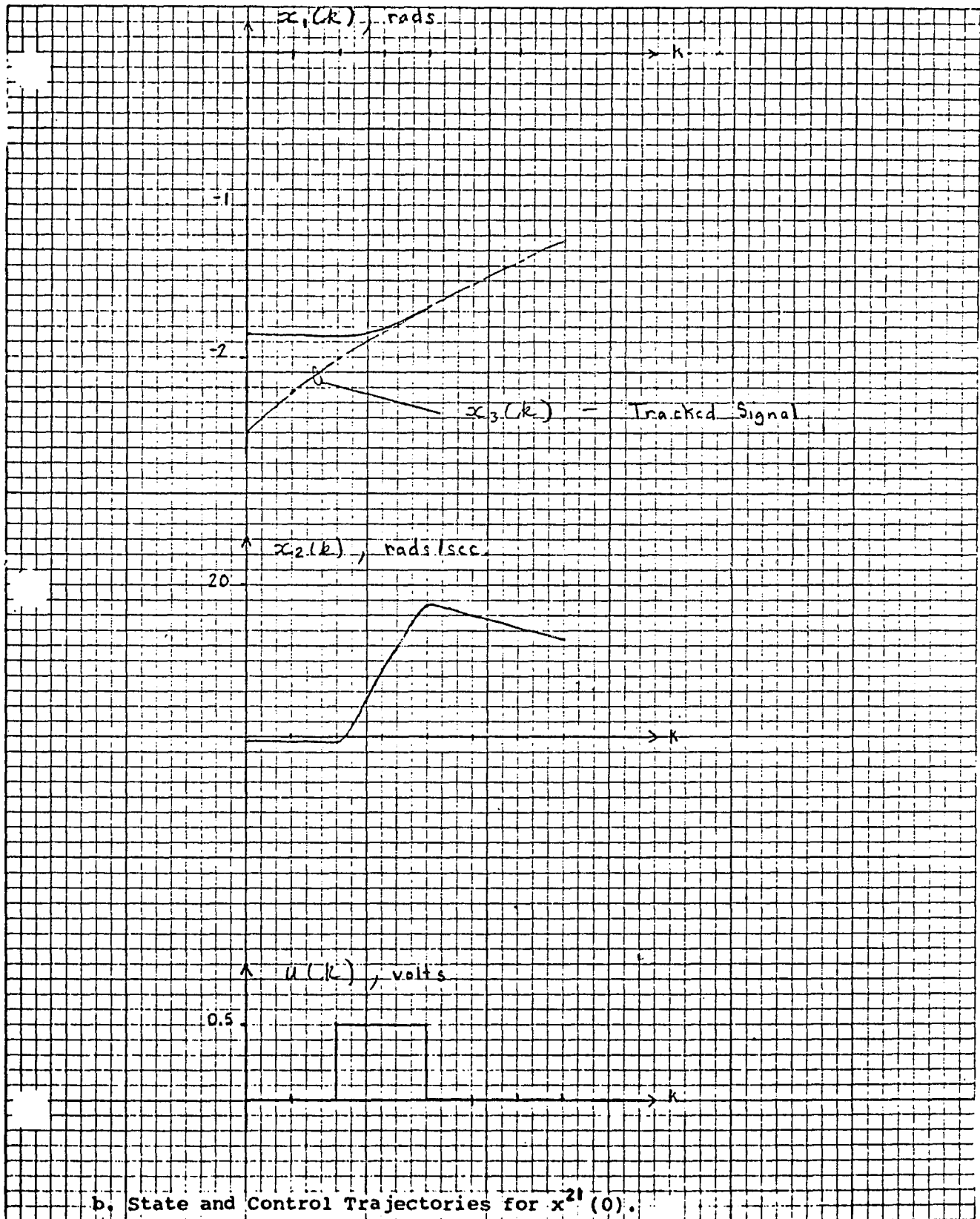


Figure 4.5.2.3 (Con't)



obtained when using the 3 Stages ROLIMPs control law for the initial conditions $x^2(0)$ and $x^{21}(0)$.

Case 3

In this third and last case, the corresponding Constant 5 Stages ROLIMPs control law with

$$TM_{L,5}(1) = 826.2 \text{ usecs}$$

and

$$C_{L,5}^c = 975 \text{ words}$$

is selected for implementation.

The bases for the selection of the Constant 5 Stages ROLIMPs control law are as follows:

1) The performance derived from the use of the Constant 5 Stages ROLIMPs control law for the set $X(0)$ is superior to the performance derived from the use of the QLQ and SNN control laws. This assertion can be readily verified from Table 4.5.2.3 which presents the performance of the Constant 5 Stages ROLIMPs, QLQ, and SNN control laws for the set of initial conditions $X(0)$ shown in Figures 4.5.2.1a-b. A comparison of the optimum performance for Case 1 shown in Table 4.5.2.1 with the performance of the Constant 5 Stages ROLIMPs control law for Case 3 shown in Table 4.5.2.3 reveal that the Constant 5 Stages ROLIMPs control law of Case 3 is very close to being optimum. Note that the optimum perfor-

Table 4.5.2.3 Performance Tabulation for Case 3.

 $R = 0$ $N = 40$

i	P_5^c	P_{QLQ}	P_{SNN}	i	P_5^c	P_{QLQ}	P_{SNN}
1	35.07	44.69	57.09	17	60.30	64.91	90.26
2	12.13	15.04	23.19	18	26.85	35.86	48.15
3	5460	1.463	6.767	19	4.41	5.528	12.33
4	5093	51.52	16.10	20	1721	12.48	10.74
5	5789	5736	16.18	21	1788	2.067	10.66
6	4220	4278	3.02	22	3.72	6.824	10.06
7	11.21	17.10	19.39	23	24.94	31.16	40.79
8	33.83	45.06	80.26	24	58.52	65.22	84.82
9	15.03	17.30	29.40	25	31.66	39.64	56.95
10	10.82	16.42	19.46	26	24.96	35.71	42.23
11	2.58	3.94	11.85	27	12.12	16.62	18.01
12	5119	1512	2.771	28	4.61	5.987	13.69
13	1175	4476	2.978	29	2.50	4.213	7.537
14	4358	1.872	8.923	30	4.21	6.76	14.00
15	3.328	4.071	8.67	31	11.47	13.64	26.06
16	10.57	11.54	18.51	32	24.14	30.31	39.92

 i - Superscript corresponding to the initial condition $x^i(0)$. P_5^c - Performance of the Control 5 Stages Rolimp's control law. P_{QLQ} - Performance of the QLQ control law P_{SNN} - Performance of the SNN control law

mance for Case 1 serves as a lowerbound on the optimum performance for Case 3.

2) The maximum on-line computation time and memory capacity of the Constant 5 Stages ROLIMPs control law satisfy the system constraints (4.5.2.21) and (4.5.1.4).

Figures 4.5.2.4.a-d present the state and control trajectories that are derived from the use of the Constant 5 Stages ROLIMPs control law for the initial conditions $x^2(0)$, $x^4(0)$, $x^{14}(0)$, and $x^{21}(0)$.

Seventh and last, the following conclusion are arrived at concerning the results for Cases 1, 2, and 3.

1) Cases 1 and 2 demonstrate for different values of R the fast convergance that occurs to the optimum performance when we steadily move from the 1 Stage to the 7 Stages ROLIMPs control law.

2) Case 3 shows the substantial improvement in performance that is attained with the Constant 5 Stages ROLIMPs control law when compared with the performance of the QLQ and SNN schemes.

4.5.3 Longitudinal Aircraft Motion Control

First, we provide the process and performance criterion equations for the control of longitudinal aircraft motion (this example was selected from Barmish and Schmitendorf[1980:544]):

Figure 4.5.2.4 State and Control Trajectories Corresponding to the Constant 5 Stages, QIQ, and SNN Control Laws.

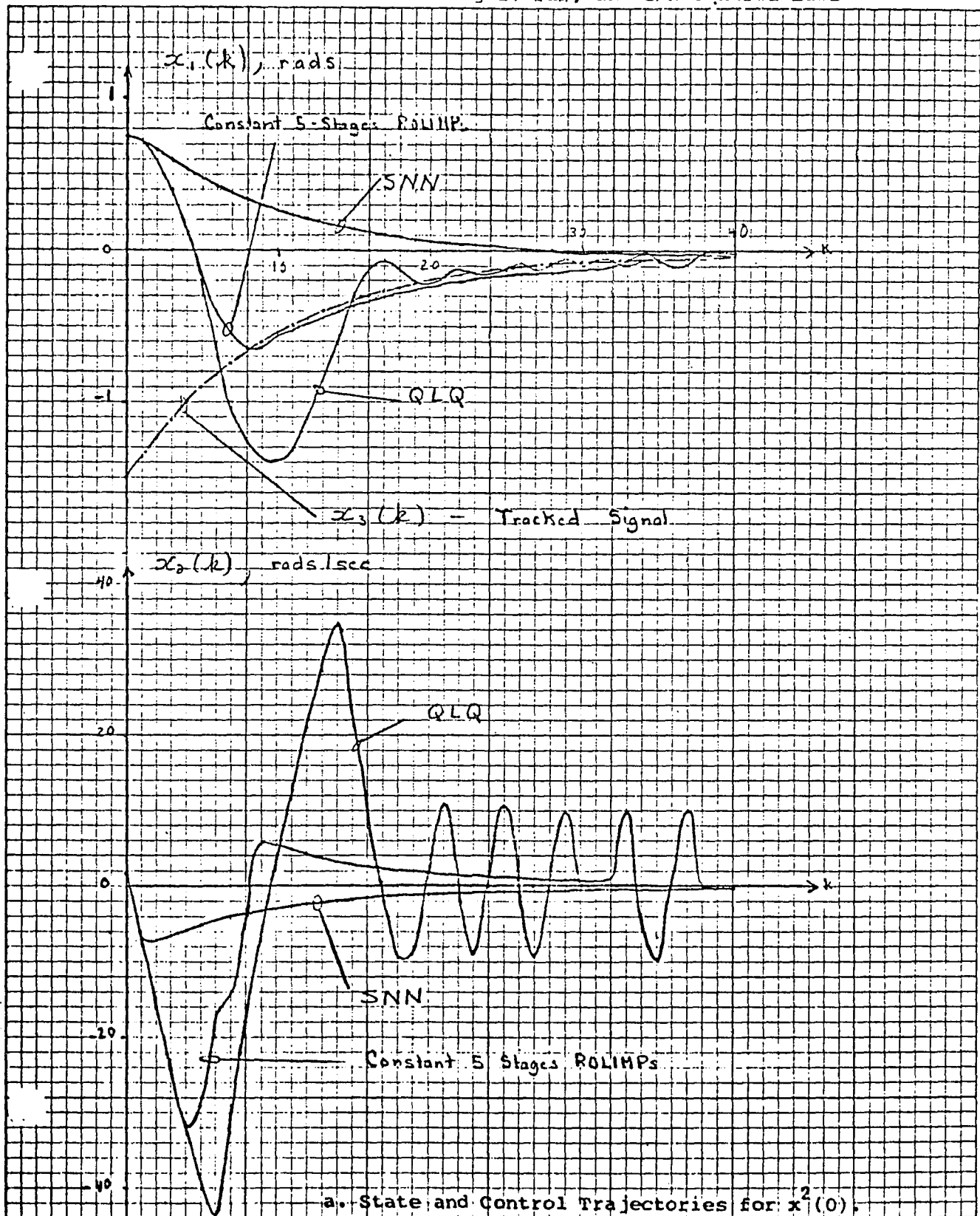
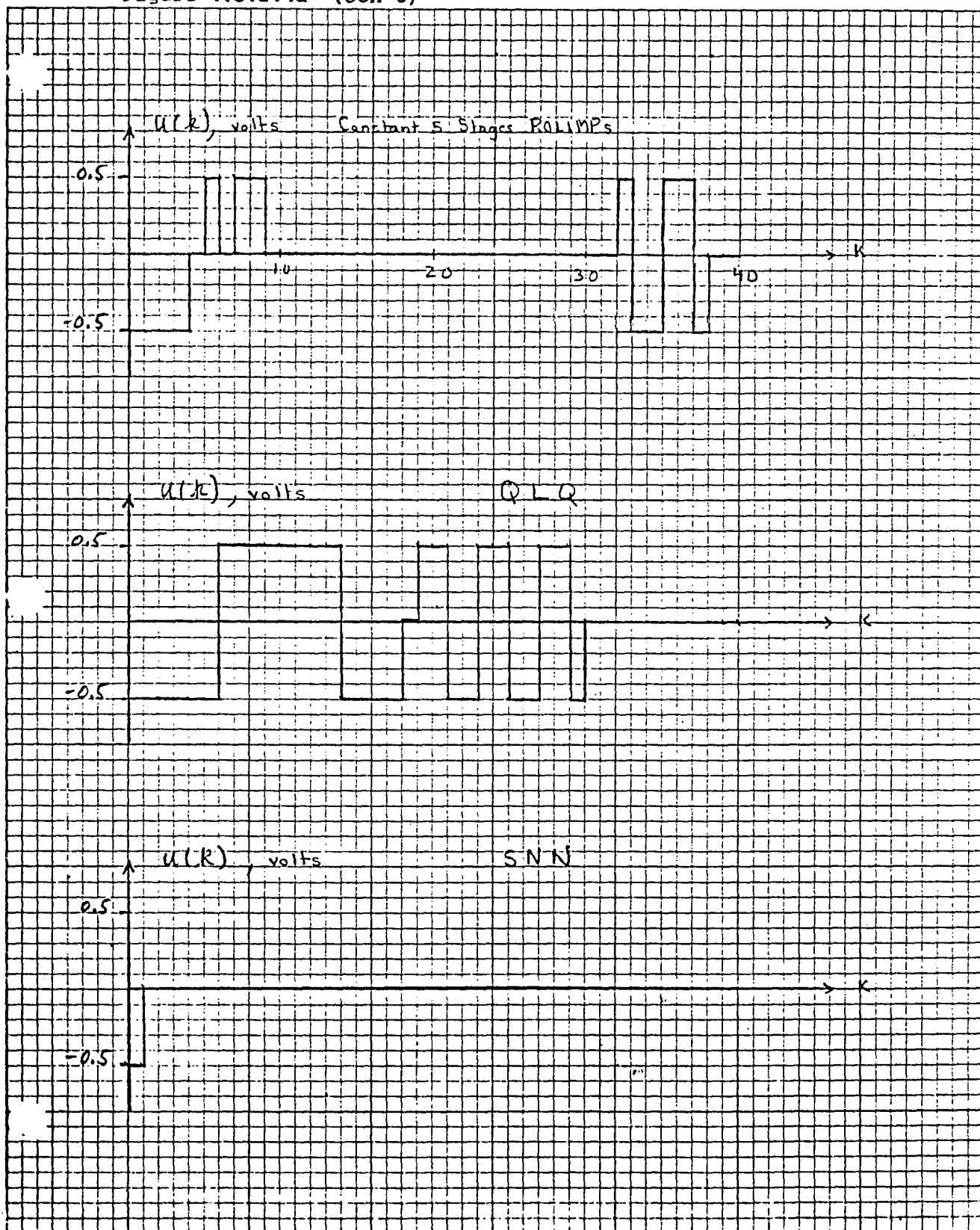
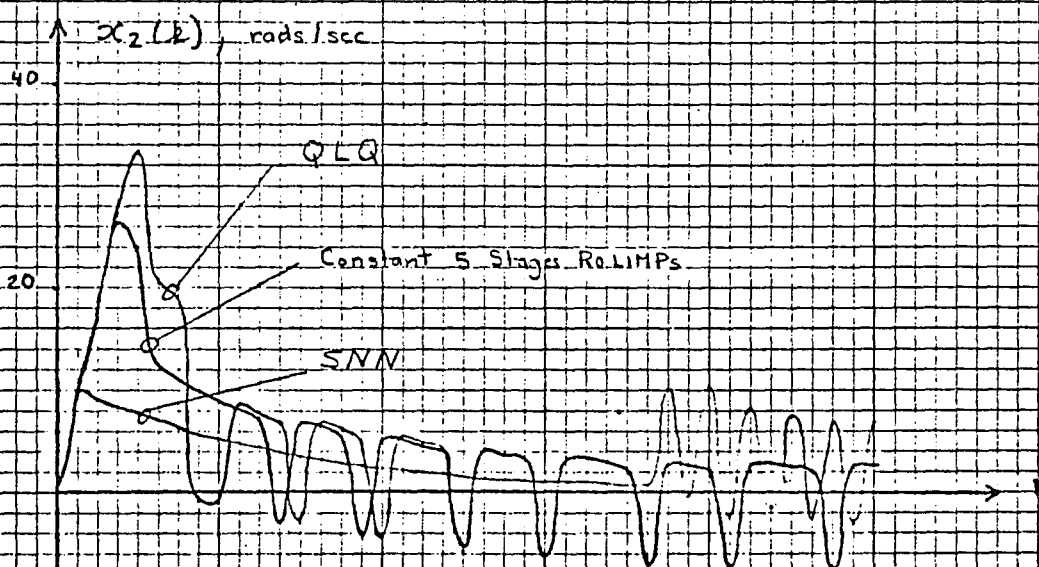
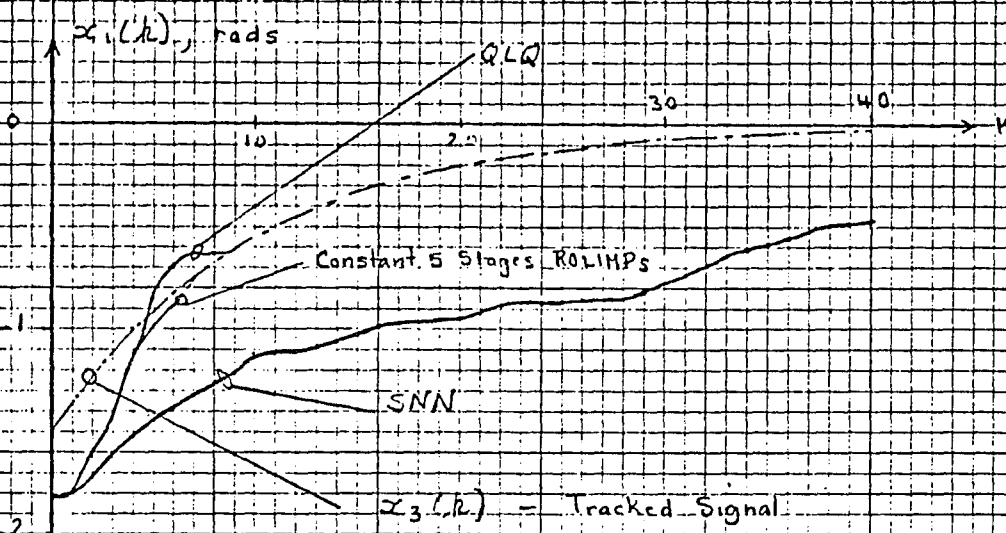


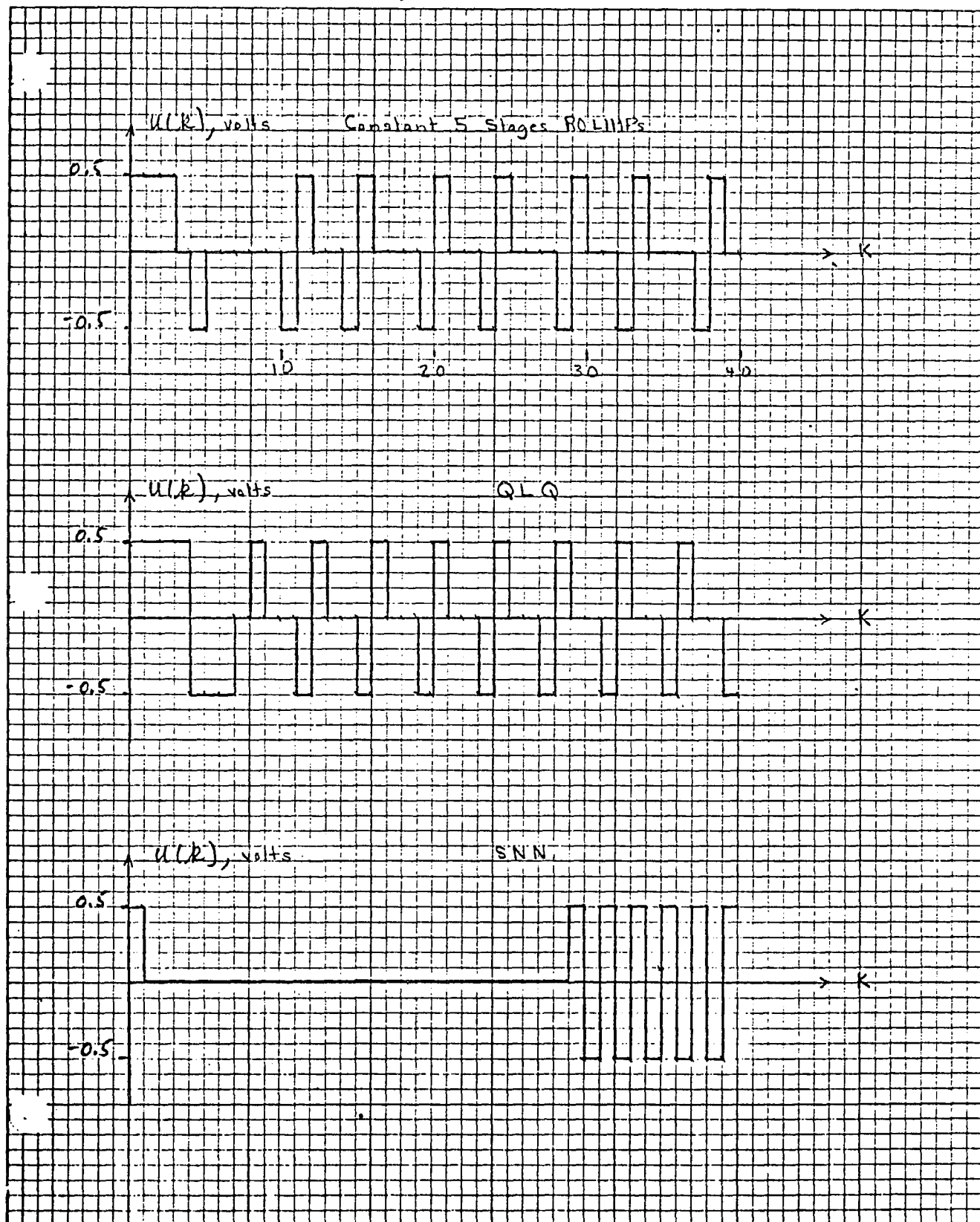
Figure 4.5.2.4a (Con't)





b. State and Control Trajectories for $x^h(0)$.

Figure 4.5.2.4b (Con't)



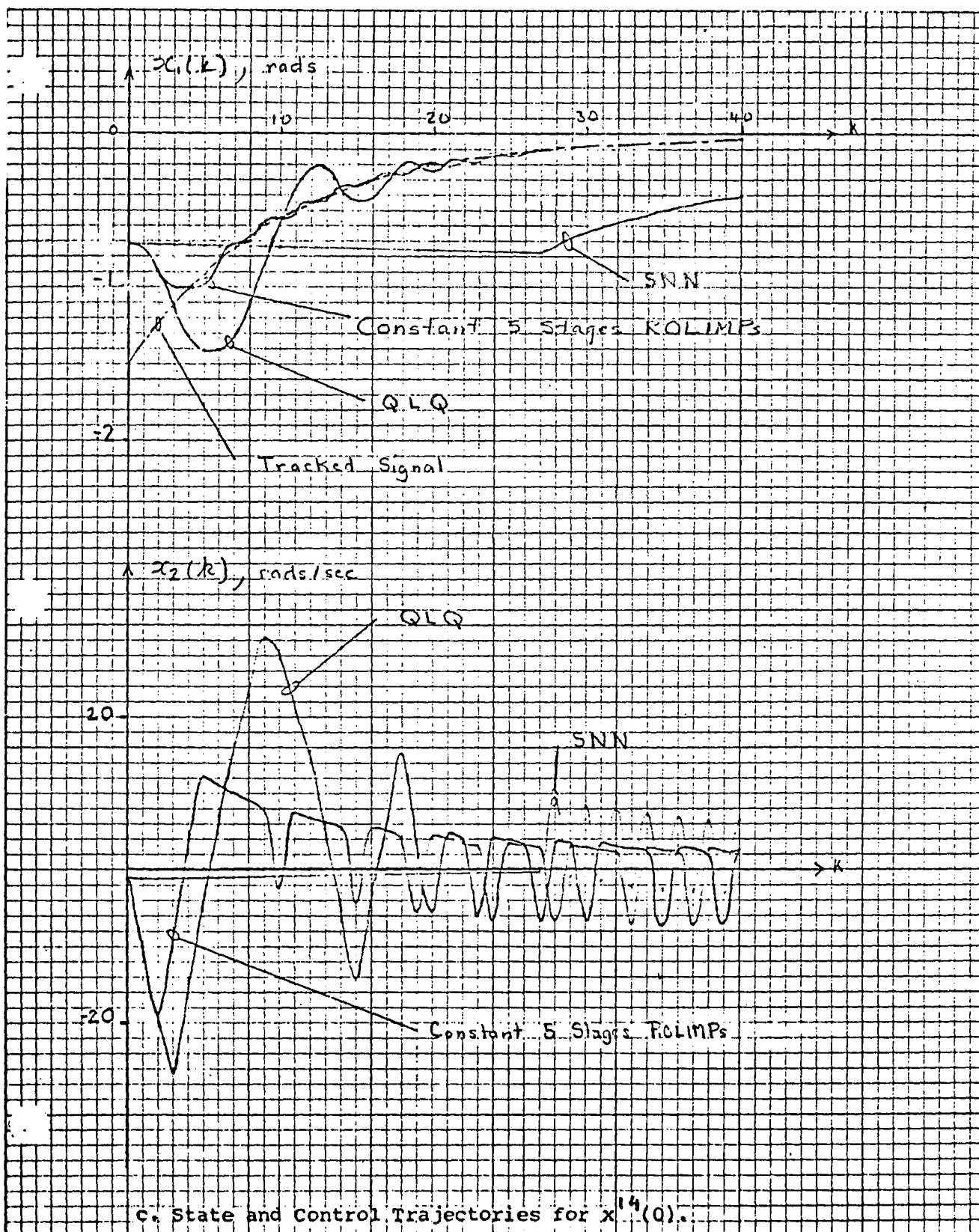
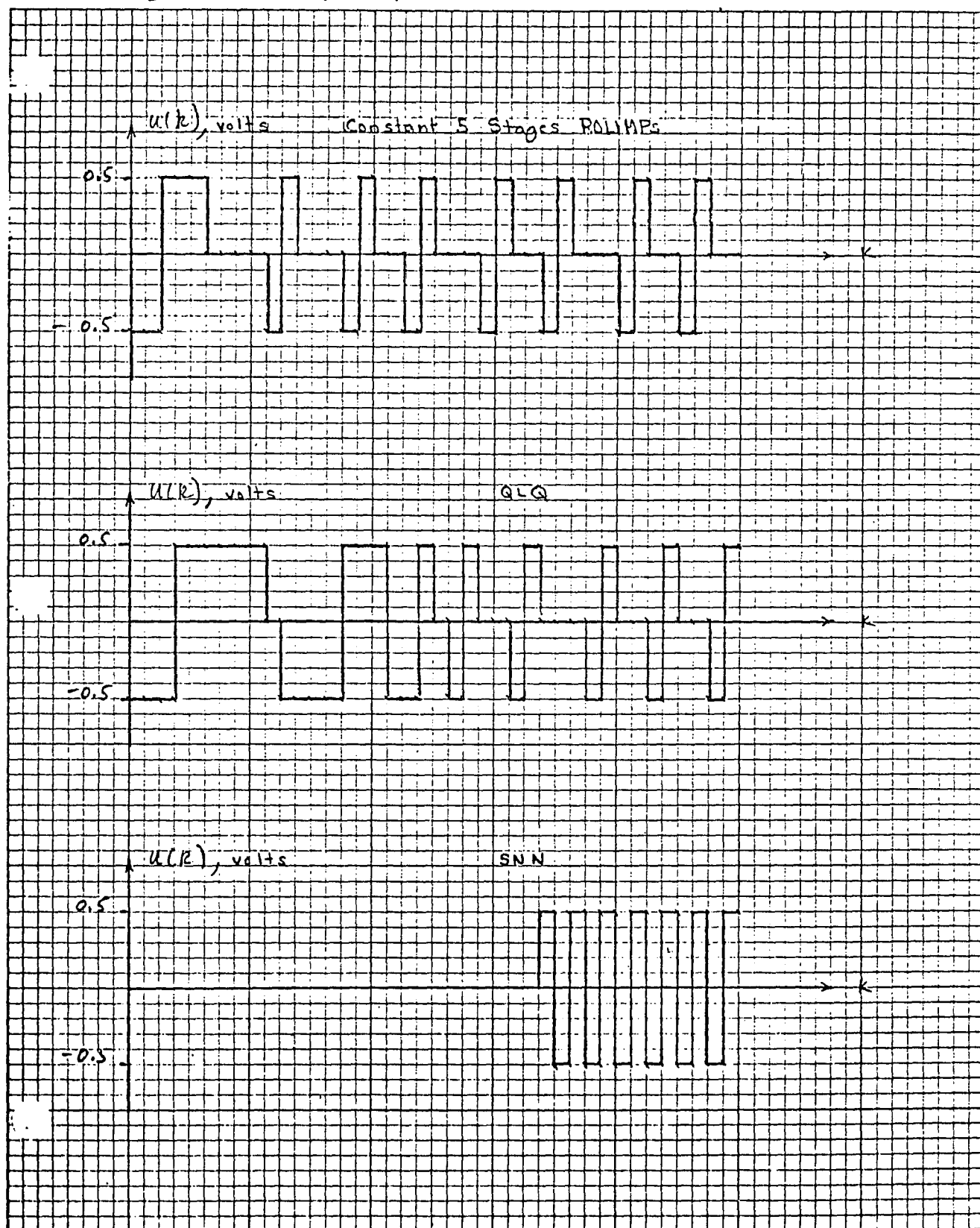


Figure 4.5.2.4c (Con't)



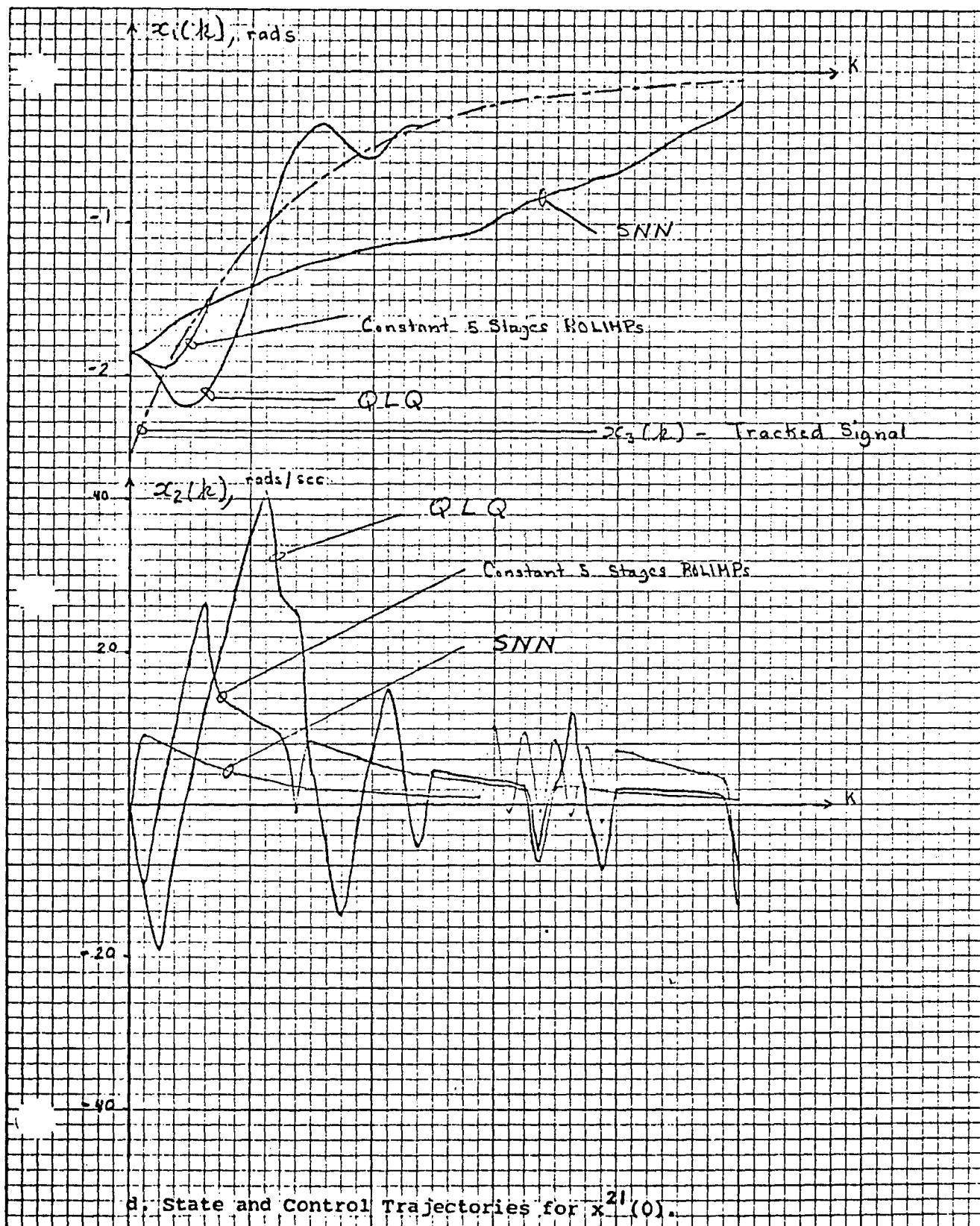
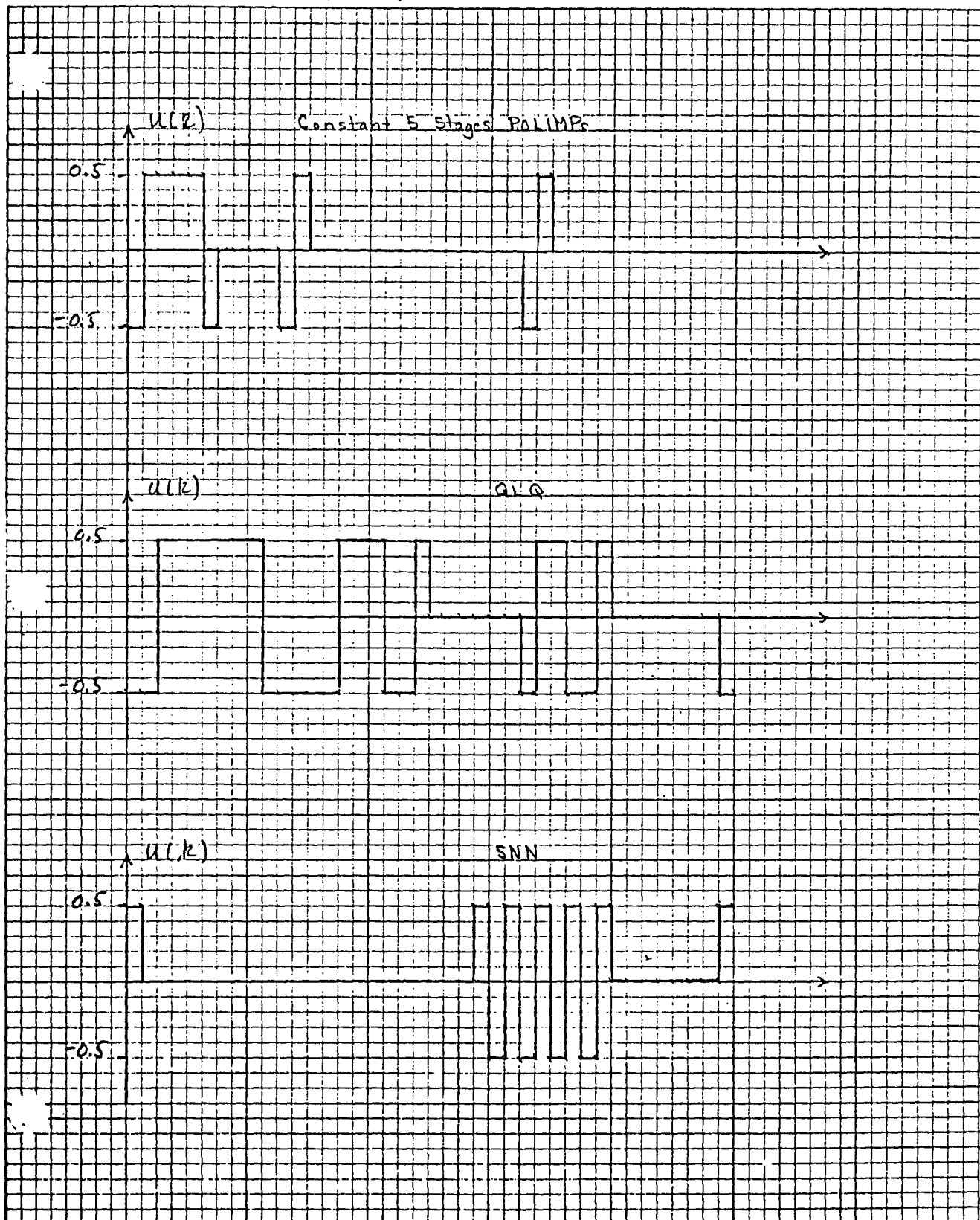


Figure 4.5.2.4d (Con't)



Process

$$\dot{z}(t) = Fz(t) + Gu(t) \quad (4.5.3.1)$$

$$y(t) = H^T z(t) \quad (4.5.3.2)$$

with

$$F = \begin{bmatrix} -2.03 & 1.0 \\ -8.57 & -2.75 \end{bmatrix}, \quad (4.5.3.3)$$

$$G = \begin{bmatrix} -0.021 & -0.156 \\ -1.820 & -0.550 \end{bmatrix}, \quad (4.5.3.4)$$

and

$$H^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (4.5.3.5)$$

where:

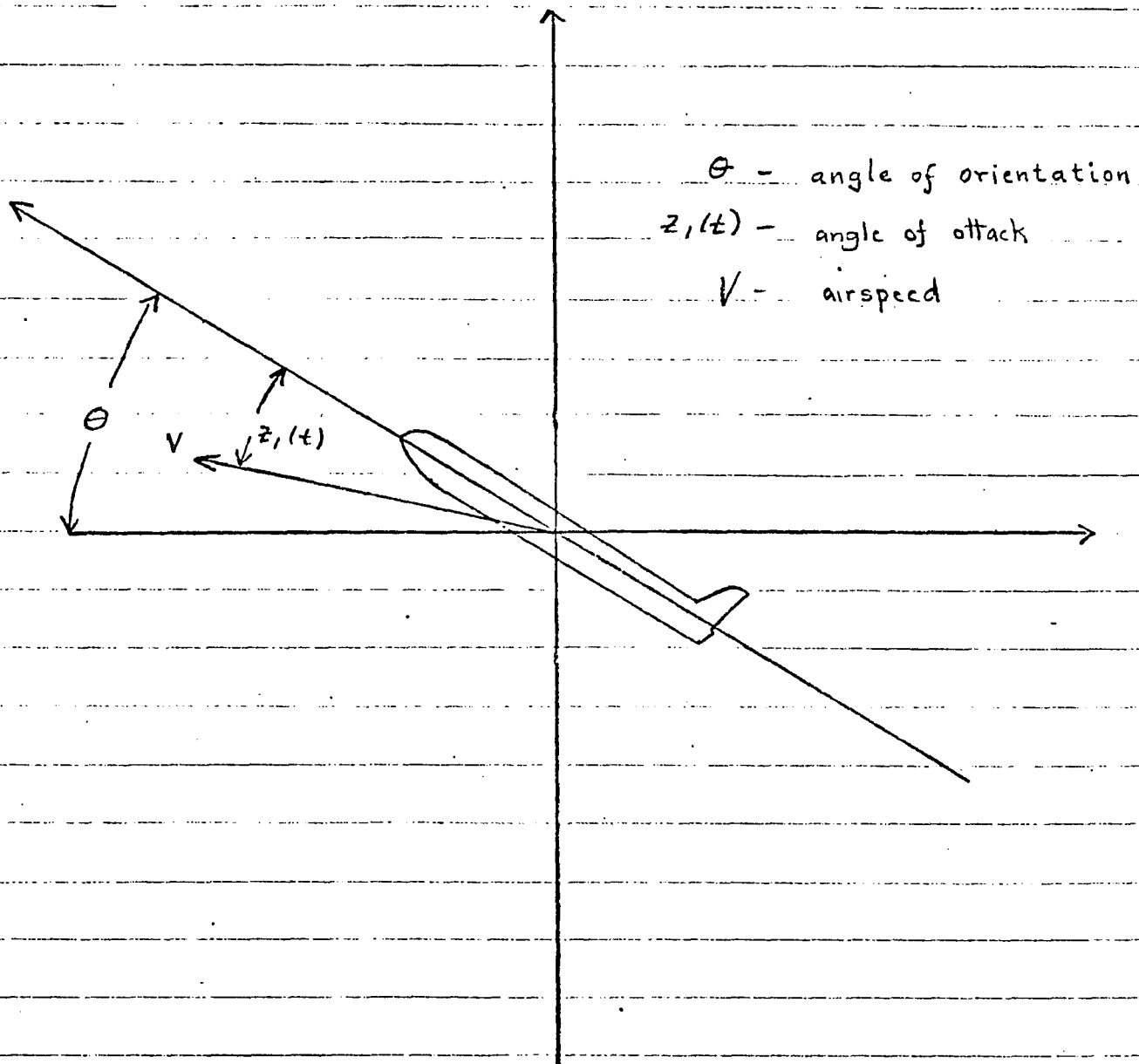
- a) $z_1(t)$ is the angle of attack (see Figure 4.5.3.1);
- b) $z_2(t)$ is the rate of change of the angle of orientation, θ , relative to an inertial line;
- c) $u_1(t)$ is the active elevator control angle; and
- d) $u_2(t)$ is the active aileron control angle.

Performance Criterion

PC

$$= \sum_{k=0}^N [(y(k) - tr(k))^T \tilde{Q} (y(k) - tr(k)) + u^T(t) R u(t)] \quad (4.5.3.6)$$

Figure 4.5.3.1 Airplane Schematic



where:

a)

$$\bar{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad (4.5.3.7)$$

b)

$$R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; \quad (4.5.3.8)$$

c) $tr(k)$ is the sampled output of the mathematical model

$$\dot{\hat{z}}(t) = \bar{F}\hat{z}(t) \quad (4.5.3.9)$$

$$tr(t) = \bar{C}^T \hat{z}(t) \quad (4.5.3.10)$$

with

$$\bar{F} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \quad (4.5.3.11)$$

and

$$\bar{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad (4.5.3.12)$$

and

d) $tr_1(k)$ and $tr_2(k)$ for $k \in (0, \dots, N)$ are the desired trajectories of the angle of attack and the rate of change of the angle of orientation for the airplane, respectively.

Note from Equations (4.5.3.9)-(4.5.3.12) that $tr(t)$ goes to 0 as t goes to ∞ . As a result, it follows from Equations (4.5.3.1)-(4.5.3.8) that both $z(t)$ and $u(t)$ should approach the origin as t goes to ∞ . This in turn justifies the selec-

tion of $S(k, I, Q)$ for the I Stages ROMPs control laws provided 0 is in the space of $u_i(k)$ for $i = 1, 2$.

Second, using Equations (4.5.1.12)-(4.5.1.20) the given tracking problem is reformulated as a regulator problem with the following values for A' , B' , and Q :

$$A' = \begin{bmatrix} -2.03 & 1.0 & 0 & 0 \\ -8.57 & -2.75 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}, \quad (4.5.3.14)$$

$$B' = \begin{bmatrix} -0.021 & -0.156 \\ -1.820 & -0.550 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad (4.5.3.15)$$

and

$$Q = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}. \quad (4.5.3.16)$$

Third, the sampling time V is given by

$$V = 0.07 \text{ secs.} \quad (4.5.3.17)$$

This in turn implies from Equation (4.5.1.29) that

$$T_{\text{Max}} = 0.007 \text{ secs.} \quad (4.5.3.18)$$

Fourth, the number of stages N is given by

$$N = 25.$$

(4.5.3.19)

Fifth, there are 32 elements in the selected set of initial conditions. This set is illustrated by Figures 4.5.3.2a-b where similar comments to those made for Figures 4.5.2.1a-b are also made for Figures 4.5.3.2a-b.

Sixth, the following different cases for $(c_1^i, \dots, c_M^i)$ for $i = 1, 2$ are then considered for the derivation of a practical Constant I Stages ROLIMPs control law with satisfactory performance.

Case 1

$$(c_1^i, c_2^i, c_3^i) = (-10, 0, 10) \text{ for } i=1, 2$$

(4.5.3.20)

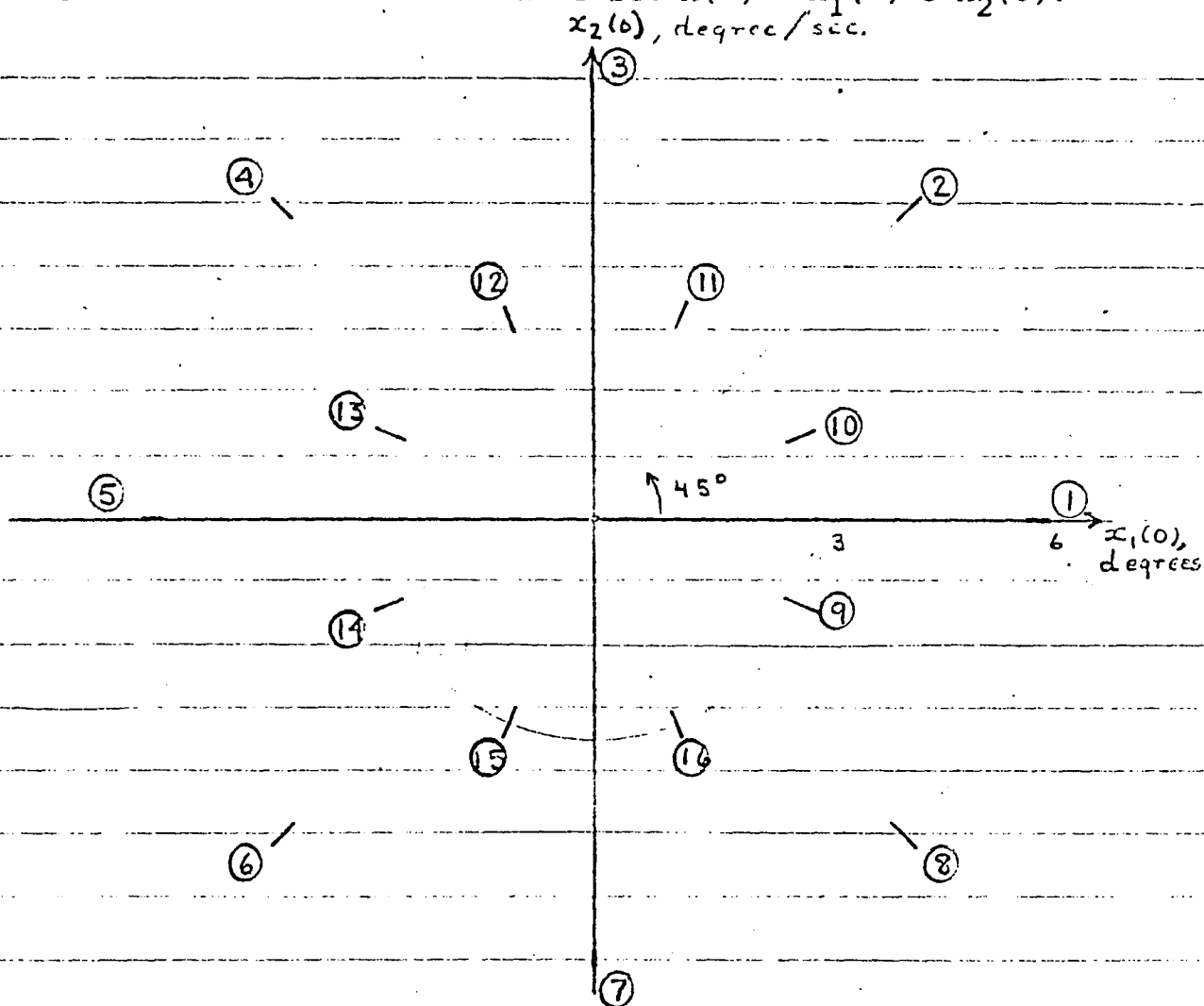
with R , A' , B' , Q , V , N and $X(0)$ given by Equations (4.5.3.8), (4.5.3.14)-(4.5.3.17), (4.5.3.19), and Figures 4.5.3.2a-b. Since N is relatively large we are interested in the derivation of a satisfactory Constant I Stages ROLIMPs control law.

Case 2

$$(c_1^i, \dots, c_{31}^i) = (-150/15, \dots, -10/15, 0, 10/15, \dots, 150/15)$$

for

$$i = 1, 2$$

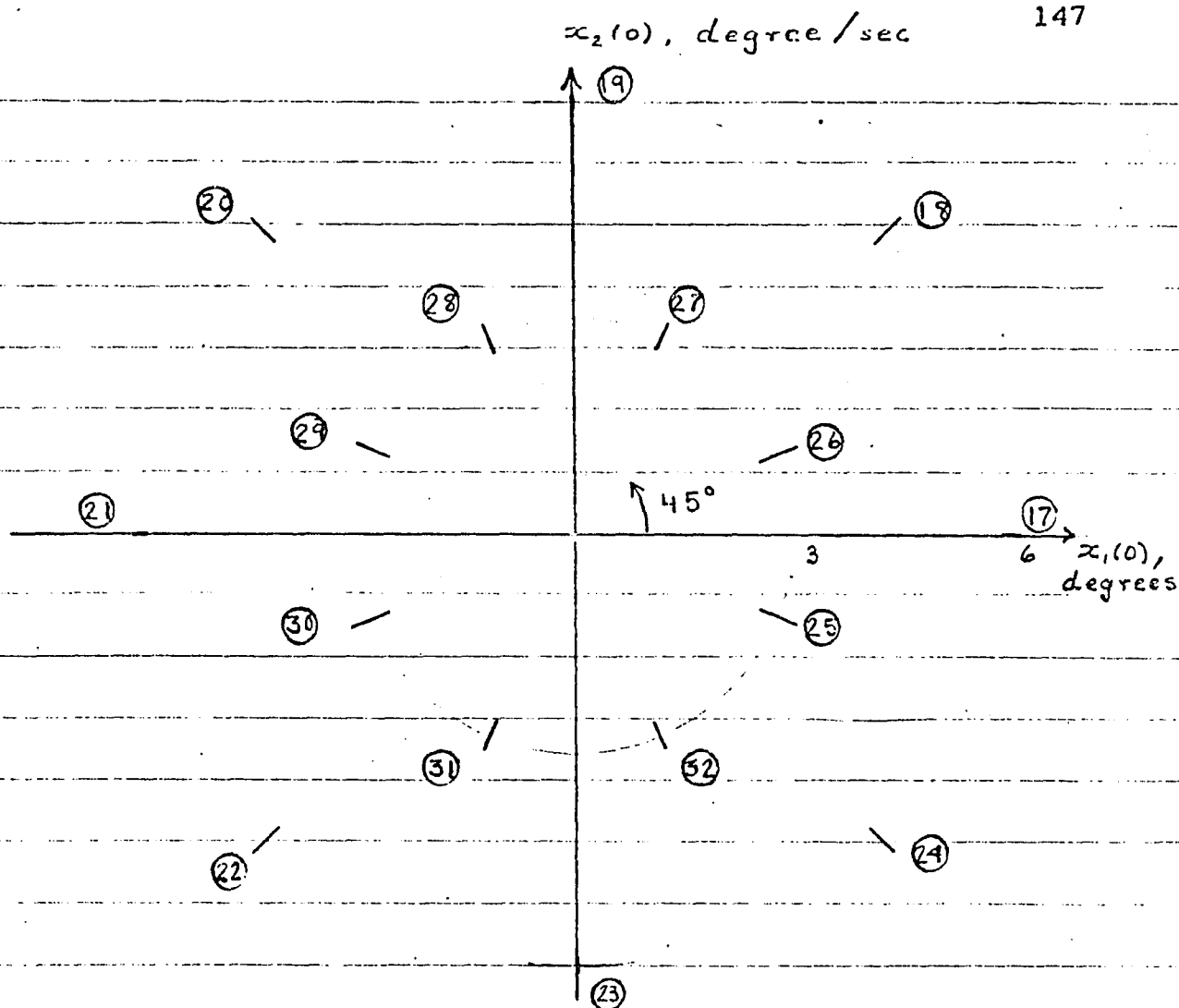
Figure 4.5.3.2 Illustration of Set $X(0) = X_1(0) \cup X_2(0)$.

$$X_1(0) = \{x(0) : x^i(0) \text{ for } i = 1, \dots, 16\}$$

where $x^i(0)$ is the initial condition given by the two coordinates shown on the above figure corresponding to the circled integer i , and the third and fourth coordinates

$$(x_3(0), x_4(0)) = (0, 0).$$

a. The set $X_1(0)$.



$$X_2(0) = \{x(0) : x^i(0) \text{ for } i = 17, \dots, 32\}$$

where $x^i(0)$ is the initial conditions given by the two coordinates shown on the above figure corresponding to the circled integer i , and the third and fourth coordinates

$$(x_3(0), x_4(0)) = (3, 3).$$

b. The set $X_2(0)$.

with R , A' , B' , Q , V , N , and $X(0)$ given by Equations (4.5.3.8), (4.5.3.14)-(4.5.3.17), (4.5.3.19), and Figures 4.5.3.2a-b. Since N is large we are interested in the derivation of a satisfactory Constant I Stages ROLIMPs control law.

Seventh, the I Stages ROLIMPs approach of Section 4.3.2 is then applied to the longitudinal aircraft motion control model yielding the following results for each different case of $(c_1^i, \dots, c_M^i)$ for $i=1,2$.

Case 1

In this first case the corresponding 1-Stage ROLIMPs control law with

$$TM_{t,1}(1) = 30.6 \text{ usecs}$$

and

$$C_{t,1} = 48 \text{ words}$$

is selected for implementation.

The bases for the selection of the Constant 1 Stage ROLIMPs control law are as follows:

1) The Constant 1 Stage ROLIMPs control law yields a performance for each initial condition in the set $X(0)$ that is superior to the performance derived from the use of the QLQ control law. This assertion can be readily verified from Table 4.5.3.1 which presents the performance of the Constant

Table 4.5.3.1 Performance Tabulation for Case 1.

$$(c_1^i, c_2^i, c_3^i) = (-10, 0, 10) \text{ for } i=1, 2$$

i	P_i^c	P_{QLQ}		i	P_i^c	P_{QLQ}
1	131.43	143.77		17	149.86	193.98
2	49.12	57.33		18	28.88	34.35
3	13.15	13.15		19	22.76	35.48
4	86.25	105.02		20	112.93	126.84
5	131.43	143.77		21	190.27	201.75
6	49.12	57.33		22	133.52	214.52
7	13.15	13.15		23	78.71	480.14
8	86.25	105.02		24	135.15	175.29
9	20.05	21.45		25	31.02	33.92
10	17.87	22.56		26	15.40	16.41
11	2.91	51.77		27	11.45	38.44
12	4.98	6.48		28	34.33	115.21
13	20.05	21.45		29	66.37	128.46
14	17.87	22.56		30	71.25	169.59
15	2.91	51.77		31	46.62	295.12
16	4.98	6.48		32	33.60	101.42

i - Superscript corresponding to the initial condition $x^i(0)$

P_i^c - Performance of Constant 1 Stage RLQ control law

P_{QLQ} - Performance of QLQ control law

1 Stage ROLIMPs and QLQ control law for the set of initial conditions $X(0)$ shown in Figures 4.5.3.2.a-b; and

2) The maximum on-line computation time and memory capacity of the Constant 1 Stage ROLIMPs control law satisfy the system constraints (4.5.3.18) and (4.5.1.4).

Figures 4.5.3.3a-b present the state and control trajectories that are derived from the use of the Constant 1 Stage ROLIMPs control law and the QLQ control law for the initial condition $x''(0)$ and $x^{3l}(0)$.

Case 2

In this second case the corresponding Constant 1 Stage ROLIMPs control law with

$$TM_{L,1}(1) = 3267.4 \text{ usecs}$$

and

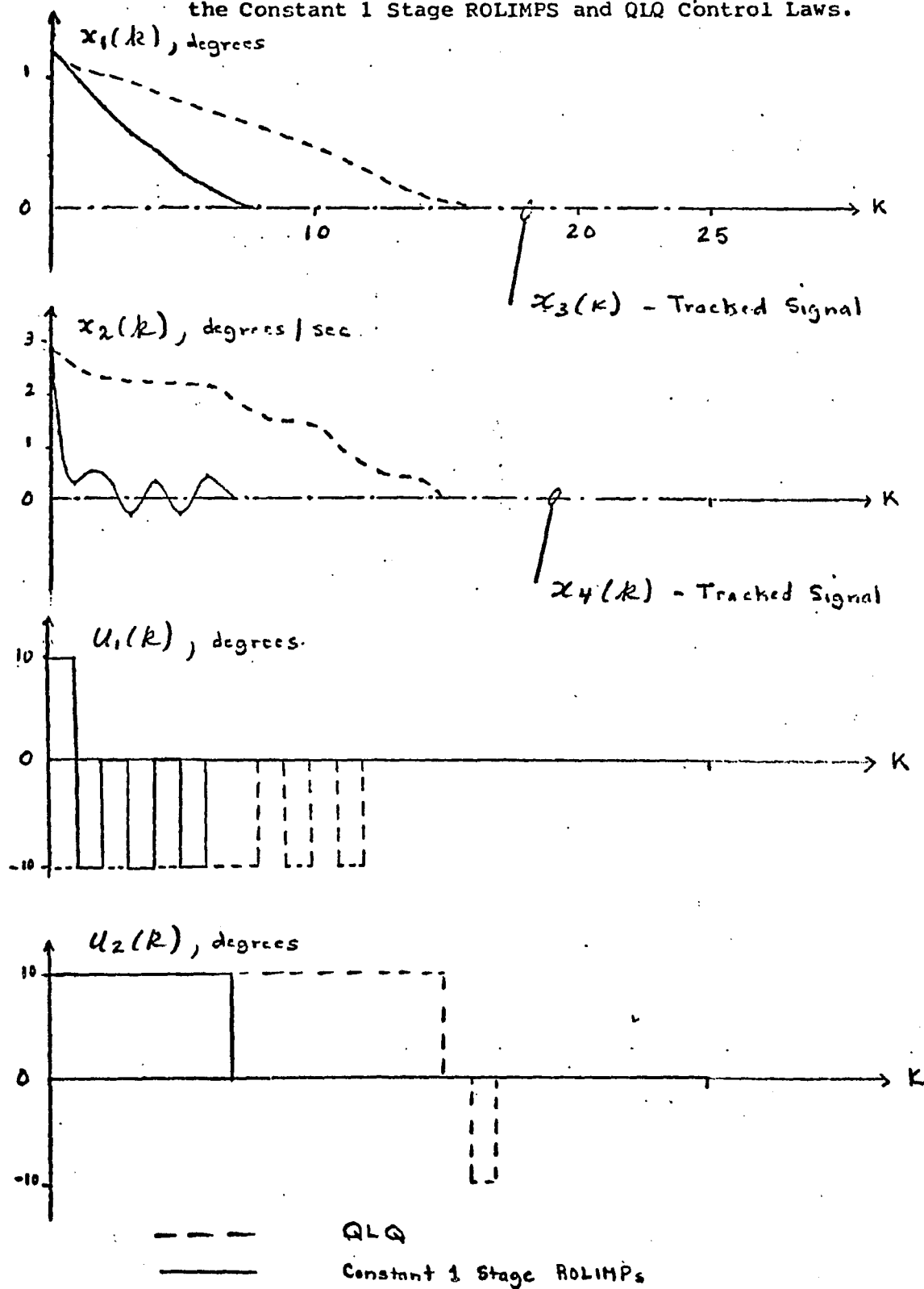
$$C_{L,1}^c = 4808 \text{ words}$$

is selected for implementation.

The bases for the selection of the Constant 1 Stage ROLIMPs control law are as follows:

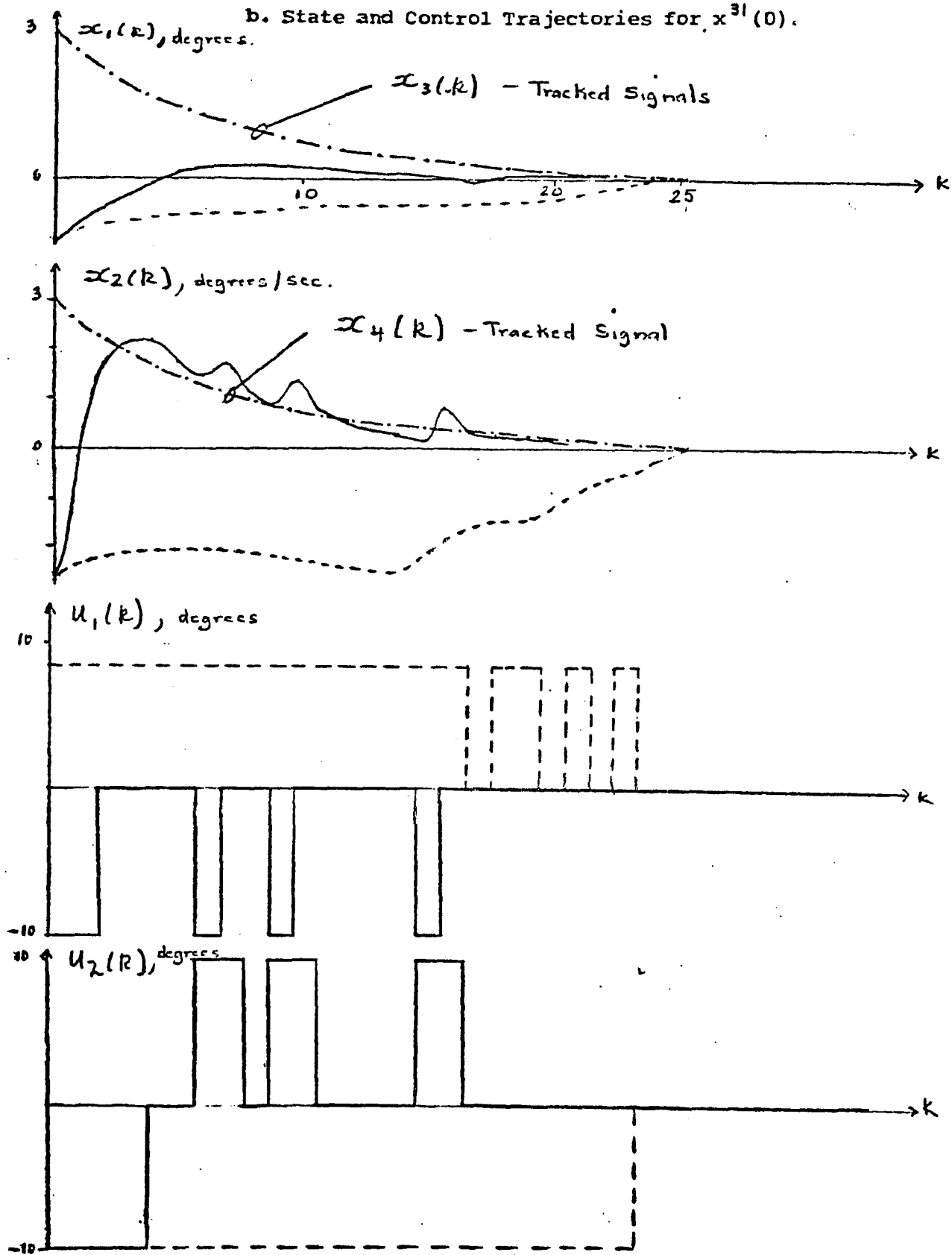
1) The performance derived from the use of the Constant 1 Stage ROLIMPs control law for each initial condition in the set $X(0)$ is superior to the performance derived from the use of the QLQ control law. This assertion can be readily

Figure 4.5.3.3 State and Control Trajectories Corresponding to the Constant 1 Stage ROLIMPS and QLQ Control Laws.



a. State and Control Trajectories for $x''(0)$.

Figure 4.5.3.3 (Con't)

b. State and Control Trajectories for $x^{31}(0)$.

verified from Table 4.5.3.2 which presents the performance of the Constant 1 Stage ROLIMPs and QLQ control laws for the set of initial conditions $X(0)$ shown in Figures 4.5.3.2a-b; and

2) The maximum on-line computation time and memory capacity of the Constant 1 Stage ROLIMPs control law satisfy the system constraints (4.5.1.4) and (4.5.3.18).

Figures 4.5.3.4a-b present the state and control trajectories that are derived from the use of the Constant 1 Stage ROLIMPs control law for the initial conditions $x''(0)$ and $x'''(0)$.

Seventh and last, the following conclusions are arrived at concerning the results of Cases 1 and 2.

1) The performance of the 1 Stage ROLIMPs control law, besides being superior to that of the QLQ control law, is not far from the optimum performance. As supporting evidence we site:

a. Table 4.5.3.4 which shows the performances derived from 1, 2, and 3 Stages ROLIMPs control laws for the case where

$$N = 3$$

and

$$(c_1^i, c_2^i, c_3^i) = (-10, 0, 10) \text{ for } i=1, 2.$$

Table 4.5.3.2 Performance Tabulation for Case 2.

for $(c_1^i, \dots, c_{31}^i) = (-150/15, \dots, -10/15, 0, 10/15, \dots, 150/15)$
 $i = 1, 2$

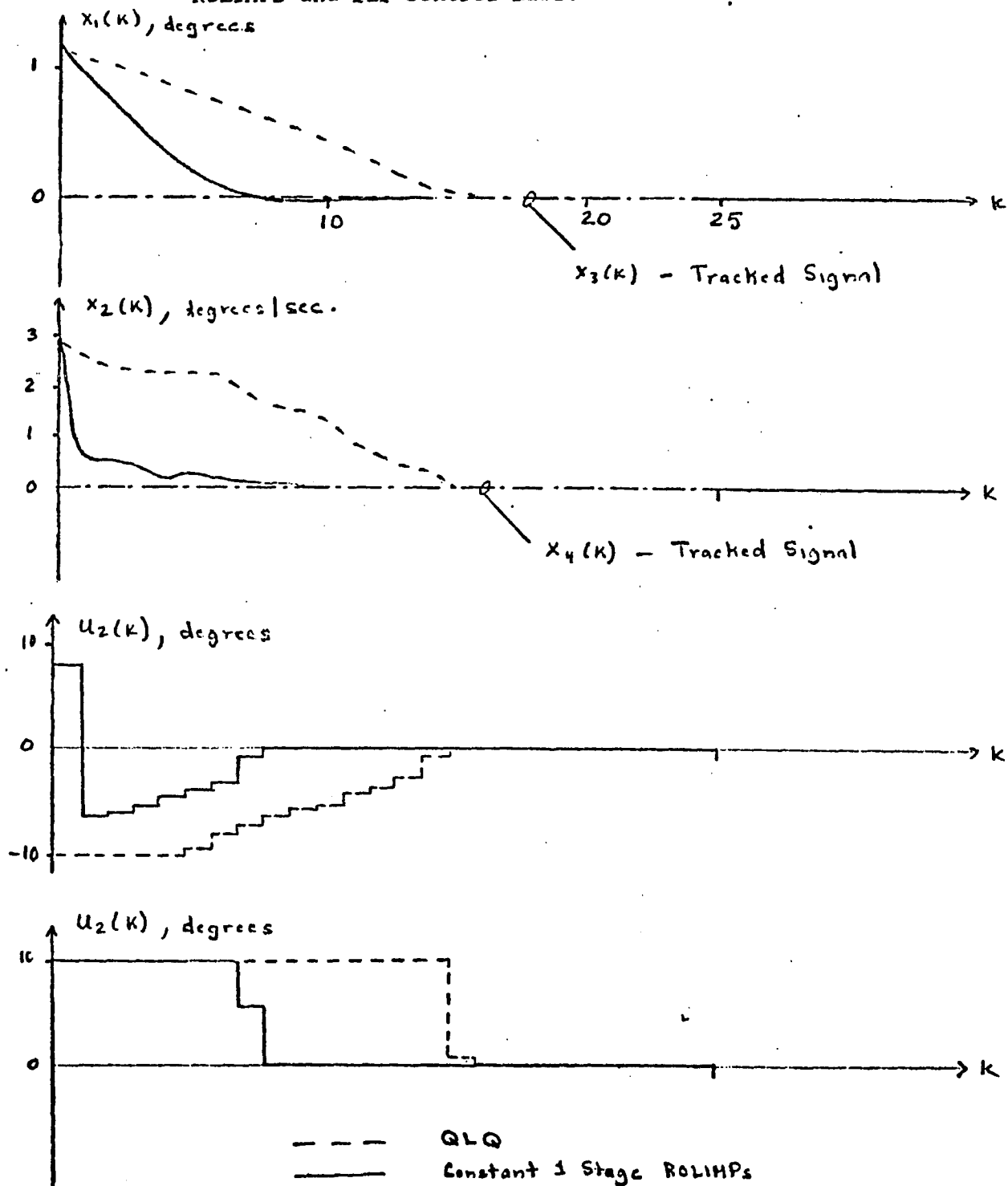
i	P_i^c	P_{QLQ}		i	P_i^c	P_{QLQ}
1	131.22	143.69		17	149.82	192.49
2	50.49	57.49		18	27.31	23.11
3	12.86	12.95		19	22.72	32.04
4	85.92	104.91		20	112.27	127.95
5	131.22	143.69		21	189.95	193.78
6	50.49	57.49		22	130.13	208.78
7	12.86	12.95		23	77.37	462.08
8	85.92	104.91		24	134.36	172.49
9	19.99	20.75		25	31.14	21.18
10	16.65	22.45		26	15.25	13.39
11	2.61	48.87		27	9.97	22.51
12	4.24	6.46		28	32.43	114.00
13	19.99	20.75		29	67.22	128.00
14	16.65	22.45		30	69.69	167.00
15	2.61	48.87		31	46.93	279.66
16	4.24	6.46		32	32.73	92.39

i - Superscript corresponding to the initial condition $x^i(0)$

P_i^c - Performance of Constant 1 Stage ROLIMPs control law

P_{QLQ} - Performance of QLQ control law

Figure 4.5.3.4 State and Control Trajectories for Constant 1 Stage ROLIMPS and QIQ Control Laws.



a. State and Control Trajectories for $x''(0)$.

Figure 4.5.3.4 (Con't)

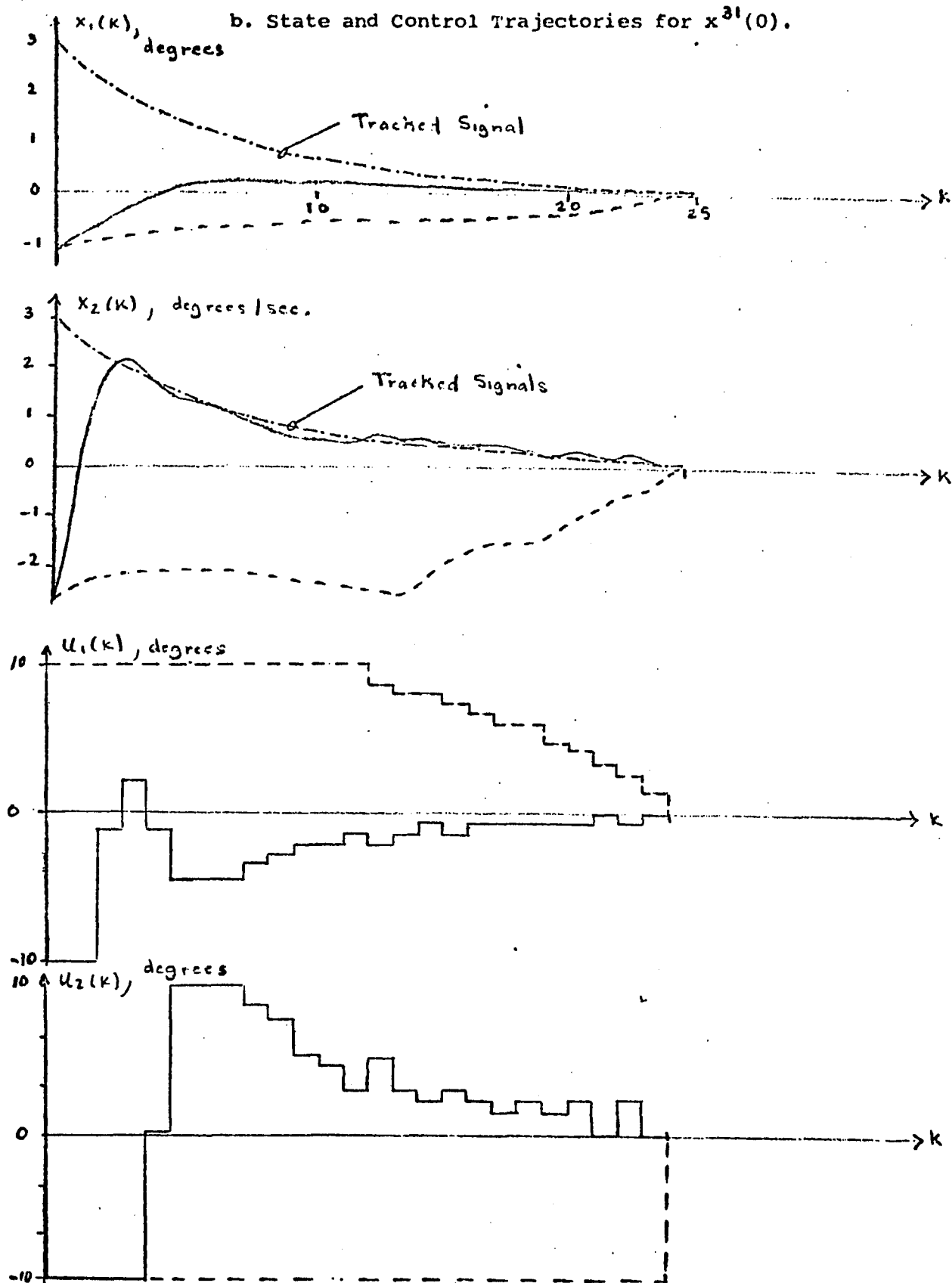
b. State and Control Trajectories for $x^{31}(0)$.

Table 4.5.3.4 Performance Tabulation.

N=3								
$(c_1, c_2, c_3) = (-10, 0, 10)$ for $i = 1, 2$								
i	P ₁	P ₂	P ₃		i	P ₁	P ₂	P ₃
1	79.77	79.42	79.42		17	78.27	78.27	78.27
2	32.97	32.97	32.74		18	7.84	7.43	7.22
3	12.93	12.93	12.93		19	10.96	10.86	10.86
4	65.28	65.28	65.28		20	85.78	85.54	85.54
5	79.77	79.42	79.42		21	131.76	131.76	131.73
6	32.97	32.97	32.74		22	92.51	92.51	92.51
7	12.93	12.93	12.93		23	67.46	67.46	67.46
8	65.28	65.28	65.28		24	105.05	105.05	105.05
9	15.42	14.93	14.93		25	23.46	23.46	23.46
10	12.23	12.23	12.23		26	5.32	5.32	5.32
11	22.64	22.03	22.03		27	3.67	3.12	3.12
12	42.24	39.86	39.86		28	22.29	22.29	22.29
13	15.42	14.93	14.93		29	48.07	48.07	48.07
14	12.23	12.23	12.23		30	51.20	50.62	50.62
15	2.26	2.20	2.20		31	34.93	34.93	34.93
16	4.22	3.99	3.99		32	28.03	28.03	28.03

i = Superscript corresponding to initial condition $x^i(0)$

P₃ = Performance of I Stages ROVMP₃ Control Law

Note that the performances of the 1, 2, and 3 Stages ROLIMPs control laws remain relatively close; and

b. Table 4.5.3.5 which shows the performances derived from 1, 2 and 3 Stages ROLIMPs control laws for the case where

$$N=15$$

and

$$(c_1^i, c_2^i, c_3^i) = (-10, 0, 10) \text{ for } i=1, 2.$$

Note that the performances of the 1, 2, and 3 Stages ROLIMPs control laws remain close to each other and are not far from the performances of the Constant 1 Stage ROLIMPs control law of Tables 4.5.3.1 and 4.5.3.2.

2) The performance of the 1 Stage ROLIMPs control law remains very close as we increase the number of quantization levels from 3 to 31. As a result, it is possible to use a Constant 1 Stage ROLIMPs control law with just 3 quantization levels to approximate the performance of a Constant 1 Stage ROLIMPs control law with a large number of quantization levels. Note that a smooth control action for the 3 level case may be obtained by connecting the output of the sample and hold of Figure 4.5.1.1 to a low pass filter whose output is in turn fed to the controlled process.

Table 4.5.3.5 Performance Tabulation.

N=15

$$(c_1^i, c_2^i, c_3^i) = (-10, 0, 10) \text{ for } i=1, 2$$

i	P_1	P_2	P_3		i	P_1	P_2	P_3
1	131.65	136.22	136.20		17	147.81	148.58	151.45
2	49.04	49.55	49.09		18	24.67	30.30	28.63
3	13.15	13.15	13.15		19	22.57	18.78	15.72
4	86.34	92.30	92.29		20	111.96	110.85	110.85
5	131.65	136.28	136.20		21	189.48	189.42	189.48
6	49.04	49.55	49.04		22	132.82	126.30	126.65
7	13.15	13.15	13.15		23	76.89	72.61	73.44
8	86.34	92.30	92.29		24	133.12	135.16	133.23
9	19.84	19.32	19.32		25	28.06	34.45	31.49
10	17.69	17.49	17.81		26	13.82	22.27	19.08
11	29.09	3.00	3.00		27	11.10	8.407	8.526
12	4.944	4.823	4.823		28	32.72	28.62	28.02
13	19.84	19.32	19.32		29	65.98	64.54	62.47
14	17.69	17.49	17.81		30	70.00	70.00	66.59
15	2.91	3.00	3.00		31	44.96	42.57	42.53
16	4.94	4.82	4.82		32	32.63	31.58	31.95

i - Superscript corresponding to the initial condition. $x^i(0)$

P_i - Performance of i stages RLIMPs control law

4.6 SUMMARY AND CONCLUSIONS

4.6.1 I Stages ROMPs Approaches

In this chapter a ROMPs approach was defined as any iterative approach that systematically searches for a practical ROMPs control law with satisfactory performance. In particular we defined two I Stages ROMPs approaches, the first called an I Stages ROQUMPs approach which is based on the I Stages ROQUMPs control law of Chapter III and the second called an I Stages ROLIMPs approach which is based on the I Stages ROLIMPs control law of Chapter III. These I Stages ROQUMPs and ROLIMPs approaches were found to have the following properties:

1) The iterative I Stages ROQUMPs and ROLIMPs approaches have the desired property of moving steadily from the fastest and simplest to the slowest and most complex I Stages ROQUMPs and ROLIMPs control laws.

2) The I Stages ROQUMPs and ROLIMPs approaches converge to the optimum QUMPs and LIMPs control laws in N iterations.

4.6.2 Practical Examples

An Armature Controlled DC Motor

The I Stages ROLIMPs approach was applied to an armature controlled DC motor and the following general results derived.

1) The iterative I Stages ROLIMPs approach converges quickly to the optimum yielding practical I Stages ROLIMPs control laws with excellent performances.

2) The performances of the QLQ and SNN control laws, were found to be rather poor when compared with the performance of the derived practical Constant I Stages ROMPs control law.

Longitudinal Aircraft Motion Control

The I Stages ROLIMPs approach was applied to an armature controlled DC motor and the following general results derived.

1) A practical 1 Stage ROLIMPs control law is found to yield excellent performance when compared with the performance obtained with the QLQ control law.

2) The performance of the 1 Stage ROLIMPs control law does not seem to vary significantly as we increase the number of quantization levels.

4.6.3 Conclusions

The results of the practical examples of Section 4.5 as well as the scalar example of Section 3.4.1 suggest that the ROMPs approach of the present chapter offers the possibility of finding practical ROMPs control laws that approximate or exactly match the performance of the optimum MPs control laws of Chapter II.

In the next chapter we summarize the most important results of this dissertation and then suggest problems for future research.

Chapter V

SUMMARY AND CONCLUSIONS

5.1 SUMMARY

In this dissertation the design and implementation of optimum and suboptimum control laws for linear (on state) deterministic processes with quadratic (on state) performance criteria and quantized control are investigated.

First, the control law techniques that have been used in the past to solve the considered class of control problems are surveyed. It is found that the proposed techniques belong to one of the following two types:

Type 1

Optimum techniques based on Dynamic Programming that nonetheless have the basic drawback of not yielding explicit analytic control law expressions and in addition require a large amount of computation time and memory capacity.

Type 2

Suboptimum techniques such as the QLQ and SNN control laws that are later found to yield a very poor performance when applied to two practical examples.

Second, using an approach that we designate as the 'Matched Processors (MPs) approach', the optimum control law at each stage k is found. The optimum control law consists of $M^{m(N-k)}$ processors and a Comparator Decoder where: a) M , N , and m are the number of control quantization levels, process stages, and process controls, respectively; b) each of the $M^{m(N-k)}$ processors is a single output processor which has a quadratic dependence in the current state, $x(k)$, and in addition has parameters that depend on a unique control sequence from stage k to $N-1$, $U_{k,N-1}$ (this originates the names 'Quadratic Matched Processor (QUMP)' for each of the $M^{m(N-k)}$ processors, 'Matched Processors (MPs) approach' for the design tool, and 'QUMPs control law' for the designed optimum control law); and c) the Comparator Decoder chooses among the $M^{m(N-k)}$ QUMPs the one that yields the lowest output and then selects from the control sequence that is used in the design of the chosen MP the k -th stage control member, $u^*(k)$ (this is the optimum control action that must be applied to the process). In addition it is found that for a special case of the general control problem considered in this dissertation, the optimum control law is equivalently given by a simpler Linear Matched Processors (LIMPs) control law which consists of $M^{m(N-k)}$ LIMPs and a Comparator Decoder. It is also found that the derived QUMPs control laws have a structure that is identical to an architecture for VLSI computing elements that is being presently implemented at JPL (see reference by Sugarman).

Third, although the implementation at each stage k of each of the $M^{m(N-k)}$ QUMPs or LIMPs is relatively easy to achieve in a digital computer, it is found that the implementation of all the $M^{m(N-k)}$ QUMPs or LIMPs is impossible for typical values of M , m , and $N-k$. To deal with this dimensionality problem we then make use of the basic structure of the QUMPs and LIMPs control laws to define 'Reduced Order Matched Processors (ROMPs) control laws' which are later found to offer the possibility of having practical control laws with outstanding performance: a ROMPs control law is then defined as any QUMPs or LIMPs control law which has its order of MPs at any stage k , $M^{m(N-k)}$, reduced either by the simplification of the original QUMPs or LIMPs control law or by the outright deletion of some of its $M^{m(N-k)}$ MPs. In particular we define I Stages Reduced Order Quadratic Matched Processors (ROQUMPs) and Reduced Order Linear Matched Processors (ROLIMPs) control laws with $I \in (1, \dots, N)$ as a class of ROMPs control laws that have the following characteristics: a) the order of their MPs at any stage k can not exceed M^{mI} ; b) N Stages ROQUMPs and ROLIMPs are the same as the QUMPs and LIMPs control laws, respectively; c) the maximum on-line computation time and memory capacity required by the I Stages ROQUMPs and ROLIMPs steadily decrease as I decreases from N to 1; and d) the I Stages ROLIMPs control law corresponding to a general stable scalar example is found to be an optimum control law for all possible values of I .

Fourth, in order to use the ROMPs control laws in applications we then defined a ROMPs approach: a ROMPs approach is defined as any iterative approach that systematically searches for a practical ROMPs control law with satisfactory performance. In particular we defined two I Stages ROMPs approaches the first called an I Stages ROQUMPs approach which is based on the I Stages ROQUMPs control law and the second called an I Stages ROLIMPs approach which is based on the I Stages ROLIMPs control law. These ROMPs approaches were found to have the following properties:

1) The iterative I Stages ROQUMPs and ROLIMPs approaches have the desired property of moving steadily from the fastest and simplest to the slowest and most complex I Stages ROQUMPs and ROLIMPs control laws.

2) The I Stages ROQUMPs and ROLIMPs approaches converge to the optimum QUMPs or LIMPs control laws in N iterations.

Fifth and last, ROMPs approaches were then applied to two practical examples yielding in both cases practical I Stages ROMPs control laws with excellent performance. In both cases it was also shown that the QLQ and SNN control laws yield very poor performance.

5.2 RECOMMENDATIONS FOR FUTURE RESEARCH

This dissertation deals with a new approach to the solution of optimum control problems which allows, unlike Dynamic Programming, the derivation of analytical optimum control laws for the considered class of nonlinear processes; furthermore, this new approach is found to offer the possibility of obtaining practical control laws with excellent performance. As a result, the further development of the present 'Matched Processors Theory' is recommended by addressing the following problems.

First, the search for a class of ROMPs control laws that approximates the performance of the I Stages ROMPs control laws while requiring fewer number of MPs. Note that a smaller number of MPs implies a smaller maximum on-line computation time and memory capacity for the considered ROMPs control law.

Second, the application of the present theory to engineering, social, economic, communications, and biomedical applications.

Third, the application of the 'Matched Processors approach' and 'ROMPs approach' to other classes of deterministic problems, such as the classes of control problems that deal with: a) processes where the space of the states is bounded and/or quantized; b) processes that are quadratic on the state; and c) performance criteria that places a pe-

nality on the magnitude of the states: the determination of new analytical control laws besides being useful for the design of satisfactory control laws may also be useful in the design of structures for VLSI computing elements.

Fourth, the investigation of the duality that exists between quantized control problems and detection problems. This study will further our theoretical knowledge of information systems as well as suggest the use of new and/or old techniques for the solution of quantized control and/or detection problems.

Fifth, the application of the present theory to stochastic control problems with discrete and/or continuous disturbances.

Sixth, the methodical development of a 'Matched Processors Theory' that will run parallel to that which has been developed for Dynamic Programming.

Appendix A

DERIVATION OF QUADRATIC MATCHED PROCESSOR

Equations (2.2.6)-(2.2.16) which define the Quadratic Matched Processor (QUMP) $h_{N-k}(x(k), U_{k,N-1}, k)$ are derived via the following steps:

STEP 1. It is noted that the solution to the process Equation (1.1.1) is given by

$$x(i) = F_1(U_{k,i-1}, i, k)x(k) + F_2(U_{k,i-1}, i, k) \quad (A.1)$$

$$i = k+1, \dots, N$$

where $F_1(U_{k,i-1}, i, k)$ and $F_2(U_{k,i-1}, i, k)$ are defined by Equations (2.2.10)-(2.2.13).

STEP 2. Substituting Equation (A.1) in the right hand side of Equation (2.2.2) we obtain

$$\begin{aligned} h_{N-k}(x(k), U_{k,N-1}, k) &= x^T(k)P_1(U_{k,N-1}, k)x(k) \\ &+ x^T(k)H_1(U_{k,N-1}, k) + H_2(U_{k,N-1}, k)x(k) + p_3(U_{k,N-1}, k) \end{aligned} \quad (A.2)$$

where $P_1(U_{k,N-1}, k)$, $p_3(U_{k,N-1}, k)$, $H_1(U_{k,N-1}, k)$, and $H_2(U_{k,N-1}, k)$ are, respectively, given by (2.2.7), (2.2.9),

$$\begin{aligned}
& H_1(U_{k,N-1}, k) \\
&= \sum_{i=k}^N F_1^T(U_{k,i-1}, i, k) Q_1(u(i), i) F_2(U_{k,i-1}, i, k), \quad (A.3)
\end{aligned}$$

and

$$\begin{aligned}
& H_2(U_{k,N-1}, k) \\
&= \sum_{i=k}^N F_2^T(U_{k,i-1}, i, k) Q_1(u(i), i) F_1(U_{k,N-1}, i, k) \quad (A.4)
\end{aligned}$$

STEP 3. Since $Q_1(u(i), i)$ is symmetric it follows from Equations (A.3) and (A.4) that

$$\begin{aligned}
& P_2(U_{k,N-1}, k) x(k) \\
&= x^T(k) H_1(U_{k,N-1}, k) + H_2(U_{k,N-1}, k) x(k) \quad (A.5)
\end{aligned}$$

where $P_2(U_{k,N-1}, k)$ is given by Equation (2.2.8). Using the LHS of Equation (A.5) in Equation (A.2) we complete the derivation of Equations (2.2.6)-(2.2.13).

STEP 4. By demonstrating the following five Lemmas we now derive the linear difference Equations (2.2.14)-(2.2.16).

Lemma 1

$$F_1(U_{k,i-1}, i, k) = F_1(U_{k+1,i-1}, i, k+1) A(u(k), k) \quad (A.6)$$

$$k = 0, \dots, i-1$$

with $F_1(U_{i,i-1}, i, i)$ given by Equation (2.2.11). This difference equation is derived as follows:

1. Rewrite Equation (2.2.10) as

$$F_1(U_{k,i-1}, i, k) = \prod_{t=i-1}^{k+1} A(u(t), t) A(u(k), k) \quad (A.7)$$

2. Note from Equation (2.2.10) that

$$F_1(U_{k+1,i-1}, i, k+1) = \prod_{t=i-1}^{k+1} A(u(t), t) \quad (A.8)$$

3. Substitute Equation (A.8) in Equation (A.7) to finally obtain the recursive Equation (A.6).

Lemma 2

$$\begin{aligned} & F_2(U_{k,i-1}, i, k) \\ &= F_2(U_{k+1,i-1}, i, k+1) + F_1(U_{k+1,i-1}, i, k+1) B(u(k), k) \\ & \quad (A.9) \\ & k = 0, \dots, i-1 \end{aligned}$$

where $F_2(U_{i,i-1}, i, i)$ is given by Equation (2.2.13). This difference equation is found as follows:

1. Rewrite Equation (2.2.12) as

$$\begin{aligned} F_2(U_{k,i-1}, i, k) &= \sum_{t=k+1}^{i-1} F_1(U_{t+1,i-1}, i, t+1) B(u(t), t) \\ &+ F_1(U_{k+1,i-1}, i, k+1) B(u(k), k) \quad (A.10) \end{aligned}$$

2. Note from Equation (2.2.12) that

$$F_2(U_{k+1, i-1}, i, k+1) = \sum_{t=k+1}^{i-1} F_1(U_{t+1, i-1}, i, t+1)B(u(t), t) \quad (A.11)$$

3. Substitute Equation (A.11) in Equation (A.10) to obtain the desired result.

Lemma 3. The difference equation for $P_2(U_{k, N-1}, k)$ is given by Equation (2.2.14) and is derived as follows:

1. Rewrite Equation (2.2.7) as

$$\begin{aligned} P_1(U_{k, N-1}, k) &= \sum_{i=k+1}^N F_1^T(U_{k, i-1}, i, k) Q_1(u(i), i) F_1(U_{k, i-1}, i, k) \\ &\quad + F_1^T(U_{k, k-1}, k, k) Q_1(u(k), k) F_1(U_{k, k-1}, k, k) \end{aligned} \quad (A.12)$$

2. Substitute Equations (2.2.11) and (A.6) in Equation (A.12) to obtain

$$\begin{aligned} P_1(U_{k, N-1}, k) &= A^T(u(k), k) \\ &\quad \bullet \sum_{i=k+1}^N F_1^T(U_{k+1, i-1}, i, k+1) Q_1(u(i), i) F_1(U_{k+1, i-1}, i, k+1) \\ &\quad \bullet A(u(k), k) + Q_1(u(k), k) \end{aligned} \quad (A.13)$$

3. Note from Equation (2.2.7) that

$$\begin{aligned}
& P_1(U_{k+1, N-1}, k+1) \\
&= \sum_{i=k+1}^N F_1^T(U_{k+1, i-1}, i, k+1) Q_1(u(i), i) F_1(U_{k+1, i-1}, i, k+1) \\
& \quad (A.14)
\end{aligned}$$

4. Substitute Equation (A.14) in Equation (A.13) to obtain the desired difference equation (2.2.14).

Lemma 4. The difference equation for $P_2(U_{k, N-1}, k)$ is given by Equation (2.2.15) and is derived as follows:

1. Rewrite Equation (2.2.8) as

$$\begin{aligned}
& P_2(U_{k, N-1}, k) \\
&= \sum_{i=k+1}^N 2F_2^T(U_{k+1, i-1}, i, k) Q_1(u(i), i) F_1(U_{k+1, i-1}, i, k) \\
& \quad + 2F_2^T(U_{k, k-1}, k, k) Q_1(u(k), k) F_1(U_{k, k-1}, k, k) \\
& \quad (A.15)
\end{aligned}$$

2. Use Equations (2.2.11), (2.2.13), (A.6), and (A.9) in Equation (A.15) to obtain

$$\begin{aligned}
& P_2(U_{k, N-1}, k) \\
&= \sum_{i=k+1}^N [2F_2^T(U_{k+1, i-1}, i, k+1)Q_1(u(i), i)] \\
&\quad \bullet F_1(U_{k+1, i-1}, i, k+1)A(u(k), k) \\
&\quad + 2B^T(u(k), k) \sum_{i=k+1}^N F_1^T(U_{k+1, i-1}, i, k+1)Q_1(u(i), i) \\
&\quad \bullet F_1(U_{k+1, i-1}, i, k+1)A(u(k), k)
\end{aligned} \tag{A.16}$$

3. From (2.2.7) and (2.2.8) it then follows that Equation (A.16) yields the desired difference Equation (2.2.15) when we substitute $P_1(U_{k+1, N-1}, k+1)$ and $P_2(U_{k+1, N-1}, k+1)$ for their respective defining expressions in Equation (A.16).

Lemma 5. The difference equation for $p_3(U_{k, N-1}, k)$ is given by Equation (2.2.16). This equation is derived as follows;

1. Rewrite Equation (2.2.9) as

$$\begin{aligned}
& p_3(U_{k,N-1}, k) \\
&= \sum_{i=k+1}^N \{ [F_2^T(U_{k,i-1}, i, k) Q_1(u(i), i)] \\
&\quad \bullet F_2(U_{k,i-1}, i, k) + q_3(u(i), i)] \\
&\quad + [F_2^T(U_{k,k-1}, k, k) Q_1(u(k), k)] \\
&\quad \bullet F_2(U_{k,k-1}, k, k) + q_3(u(k), k) \quad (A.17)
\end{aligned}$$

2. Make use of Equations (2.2.13) and (A.9) in Equation (A.17) and also make use of the fact that $Q_1(U_{k,N-1}, k)$ is symmetric to obtain

$$\begin{aligned}
& p_3(U_{k,N-1}, k) \\
&= \sum_{i=k+1}^N \{ [F_2^T(U_{k+1,i-1}, i, k+1) Q_1(u(i), i)] \\
&\quad \bullet F_2(U_{k+1,i-1}, i, k+1) + q_3(u(i), i)] \\
&\quad + \sum_{i=k+1}^N [2F_2^T(U_{k+1,i-1}, i, k+1) Q_1(u(i), i)] \\
&\quad \bullet F_2(U_{k+1,i-1}, i, k+1) B(u(k), k) \\
&\quad + B^T(u(k), k) \sum_{i=k+1}^N F_1^T(U_{k+1,i-1}, i, k+1) Q_1(u(i), i) \\
&\quad \bullet F_1(U_{k+1,i-1}, i, k+1) B(u(k), k) \\
&\quad + q_3(u(k), k) \quad (A.19)
\end{aligned}$$

3. Finally substitute $p_1(U_{k+1, N-1}, k+1)$, $p_2(U_{k+1, N-1}, k+1)$, and $p_3(U_{k+1, N-1}, k+1)$ for their respective expressions in Equation (A.19) to obtain (2.2.16).

Appendix B

DERIVATION OF LIMPS CONTROL LAW

The Linear Matched Processors (LIMPs) control law given by Equations (2.5.1.1)-(2.5.1.8) is derived from the QUMPs control law Equations (2.2.3)-(2.2.16) as follows:

1) Substitute $A(t)$ for $A(u(t), t)$ in Equations (2.2.10) to obtain

$$F_1(U_{k,i-1}, i, k) = \prod_{t=i-1}^k A(t) \quad (B.1)$$

2) Use (B.1) in (2.2.12) to obtain

$$\begin{aligned} F_2(U_{k,i-1}, i, k) \\ = \sum_{t=k}^{i-1} \prod_{j=i-1}^{t+1} A(j) B(u(t), t) \end{aligned} \quad (B.2)$$

3) Substitute $Q_1(t)$ for $Q_1(u(t), t)$ in Equations (2.2.7)-(2.2.9) and in addition use (B.1) and (B.2) in (2.2.7)-(2.2.9) to obtain

$$\begin{aligned} P_1(k) \\ = \sum_{i=k}^N \left[\prod_{t=i-1}^k A(t) \right]^T Q_1(t) \prod_{t=i-1}^k A(t), \end{aligned} \quad (B.3)$$

and the desired Equations (2.5.1.4) and (2.5.1.5). Note that the right hand side of Equation (B.3) is not a function of $U_{k,N-1}$; as a result, $P_1(U_{k,N-1}, k)$ has been replaced by $P_1(k)$.

4) Replace $P_1(U_{k,N-1}, k)$, $A(u(k), k)$, and $Q_1(u(k), k)$ by $P_1(k)$, $A(k)$, and $Q_1(k)$, respectively, in Equations (2.2.14)-(2.2.16) to obtain the linear differential Equations (2.5.1.6)-(2.5.1.8).

5) Substitute $P_1(k)$ for $P_1(U_{k,N-1}, k)$ in (2.2.6) to obtain

$$\begin{aligned} & h_{N-k}(x(k), U_{k,N-1}, k) \\ &= x^T(k) P_1(k) x(k) + h'_{N-k}(x(k), U_{k,N-1}, k) \end{aligned} \quad (B.4)$$

where $h'_{N-k}(x(k), U_{k,N-1}, k)$ is given by Equation (2.5.1.3).

7) Use (B.4) in (2.2.4) to obtain

$$\begin{aligned} & f_{N-k}(x(k)) \\ &= x^T(k) Q_1(k) x(k) + f'_{N-k}(x(k)) \end{aligned} \quad (B.5)$$

where $f'_{N-k}(x(k))$ is given by (2.5.1.2).

8) Note that to find the optimum control sequence $U^*_{k,N-1}$ we only need to consider the minimization (2.5.1.2). This completes the derivation of the LIMPs control law.

Appendix C

PROOF OF GENERAL STABLE SCALAR EXAMPLE

The objective of this appendix is to show that the I Stages ROLIMPs control law for the process Equation (3.4.1.1) and performance criterion Equation (3.4.1.5) is given by Equations (3.4.1.7)-(3.4.1.15) and is in addition an optimum control law for all $I \in (1, \dots, N)$. Since the proof is rather lengthy we first provide, with the aid of a flowchart, a general outline of the proof and then proceed to fill in the necessary steps; however, it should be noted that some of the steps are so lengthy in themselves that we again provide outlines, often with the aid of flowcharts, before giving detailed proofs of some of the steps. A final word of warning, in some cases the details of straight forward algebraic manipulations are not given.

Outline of Proof.

The general outline of the proof is composed of three parts which are given next with the aid of the flowchart of Figure C.1.

Part 1. In this part it is shown that the optimum control law, $u^*(x(k))$, for the process Equation (3.4.1.1) and

Part 1

Show that the optimum control law at any stage k is given by Equations (C.1) - (C.3) and (3.4.1.10) - (3.4.1.12).

Part 2

Use the optimum control law of Part 1 to show that $h'_{N-k}(x(k), u^*_{k,N-1}, k)$ is equivalently given by Equation (C.5).

Part 3

Show that the I Stages RLHMPs control law is given by Equations (3.4.1.7) - (3.4.1.15) and then use the results of Parts 1 and 2 to show that the I Stages RLHMPs control law is an optimum control law for all $I \in (1, \dots, N)$.

Figure C.1 Flow Diagram of Overall Proof.

performance criterion Equation (3.4.1.5) is given by the following LIMPs control law:

$$u^*(x(k)) = u^*(k) \quad (C.1)$$

for all $k \in (0, \dots, N-1)$, where $u^*(k)$ is the k -th stage member of the optimum control sequence, $U_{k,N-1}^*$, which is derived from the minimization

$$h'_{N-k}(x(k), U_{k,N-1}^*, k) = \min_{U_{k,N-1} \in US(k)} h'_{N-k}(x(k), U_{k,N-1}, k) \quad (C.2)$$

where

$$US(k) = \{ U_{k,N-1} : u(s) \in \{-M_1, \dots, M_1\} \text{ for all } s \in (k, \dots, N-1) \}, \quad (C.3)$$

$h'_{N-k}(x(k), U_{k,N-1}, k)$ is given by Equations (3.4.1.10)-(3.4.1.12), and $h'_{N-k}(x(k), U_{k,N-1}^*, k)$ is the Optimum LIMP (OLIMP) that is matched to the optimum control sequence $U_{k,N-1}^*$.

Part 2. In this part it is shown that $h'_{N-k}(x(k), U_{k,N-1}^*, k)$ (see Equation (C.2)) for all $k \in (0, \dots, N-1)$ can be found by just considering the LIMPs

that are associated with the control sequences in the set

$$\begin{aligned}
 S_1(k) &= [U_{k,N-1} : U_{k,S-1} = \{z_{M_1}, \dots, z_{M_1}\}, u(s) = z_j, \\
 U_{S+1, N-1} &= \{0, \dots, 0\} \text{ for all } z \in (-1, +1), j \in (1, \dots, M_1), \\
 s \in (k, \dots, N-1); U_{k, N-1} &= \{0, \dots, 0\}] \dots
 \end{aligned}
 \tag{C.4}$$

(see Table C.1 for an illustration of the set $S_1(N-4)$ for the case where $M_1=1$), i.e., in this part we show that

$$h_{N-k}^*(x(k), U_{k, N-1}^*, k) = \min_{U_{k, N-1} \in S_1(k)} h_{N-k}^*(x(k), U_{k, N-1}, k)
 \tag{C.5}$$

where $S_1(k)$ is given by Equation (C.4) and $h_{N-k}^*(x(k), U_{k, N-1}, k)$ by Equations (3.4.1.10)-(3.4.1.12).

Part 3. In this third and last part of the proof we first show that the I Stages ROLIMPs control law for the process Equation (3.4.1.1) and performance criterion Equation (3.4.1.5) is given by Equations (3.4.1.7)-(3.4.1.15) and then use the results of Parts 1 and 2 to show that the I Stages ROLIMPs control law is an optimum control law for all $I \in (1, \dots, N)$.

Proof

Part 1. Since the process Equation (3.4.1.1) and performance criterion Equation (3.4.1.5) are a special case of Equations (1.1.1) and (1.1.4), respectively, and in addition

Table C.1 The set $S_1(N-4)$ for $M_1 = 1$.

$$S_1(N-4) = \{CS_{1,i} \text{ for } i=1, \dots, 9\}$$

where

$CS_{1,i}$ is a unique control sequence of the set $S_1(N-4)$

Control Sequence

Elements of the Control Sequence $U_{N-4, N-1}$

$U_{N-4, N-1}$	$u(N-4)$	$u(N-3)$	$u(N-2)$	$u(N-1)$
$CS_{1,1}$	-1	-1	-1	-1
$CS_{1,2}$	-1	-1	-1	0
$CS_{1,3}$	-1	-1	0	0
$CS_{1,4}$	-1	0	0	0
$CS_{1,5}$	0	0	0	0
$CS_{1,6}$	1	0	0	0
$CS_{1,7}$	1	1	0	0
$CS_{1,8}$	1	1	1	0
$CS_{1,9}$	1	1	1	1

satisfy the special conditions of Section 2.5, it follows that the optimum control law, $u^*(x(k))$, for this scalar example is found from the LIMPs control law Equations (2.5.1.1)-(2.5.1.8).

Next replacing $B(u(t),t)$, $A(\cdot)$, $Q_1(\cdot)$, $Q_2(u(i),i)$, and $q_3(u(i),i)$ by $bu(t)$, a , q , 0 , and $r|u(i)|$, respectively, in Equations (2.5.1.4) and (2.5.1.5), and after some straight forward algebraic manipulations we obtain Equations (3.4.1.11) and (3.4.1.12): note that Equations (C.1), (C.2), and (3.4.1.10) are the same as Equations (2.5.1.1)-(2.5.1.3), with the exception that $f'_{N-k}(x(k))$ has been replaced by the equivalent expression $h'_{N-k}(x(k), U^*_{k,N-1}, k)$. This ends the demonstration of Part 1.

Part 2. The proof corresponding to Part 2 will be done inductively. First we note from Equations (C.2), (C.3), and (C.4) that $h'_{N-k}(x(k), U^*_{k,N-1}, k)$ for $k = N-1$ is also found from Equation (C.5) since $US(N-1) = S_1(N-1)$. Then assuming $h'_{N-(k+1)}(x(k+1), U^*_{k,N-1}, k+1)$ is given by Equation (C.5) where k is replaced by $k+1$ we show that $h'_{N-k}(x(k), U^*_{k,N-1}, k)$ is also determined from Equation (C.5) to finally complete the induction proof. Since the induction proof is very long we next give an outline of the proof with the aid of the flowchart of Figure C.2, and then fill in the required steps.

Preliminary Results

Note that $h'_{N-k}(x(k), u^*_{k, N-1}, k)$ satisfies Equation (C.5) when $k = N-1$.

Part 2.1

Using the results of Part 1 show that $h'_{N-k}(x(k), u^*_{k, N-1}, k)$ satisfies the recursive Equation (C.8).

Part 2.2

Assume $h'_{N-(k+1)}(x(k+1), u^*_{k+1, N-1}, k+1)$ is given by Equations (C.5) with k replaced by $k+1$ to show, using the recursive Equation (C.8), that $h'_{N-k}(x(k), u^*_{k, N-1}, k)$ must be given by Equation (C.11).

Part 2.3

Using Equation (C.11), show that $h'_{N-k}(x(k), u^*_{k, N-1}, k)$ is also given by Equation (C.5). This completes the induction proof.

Figure C.2 Flow Diagram of Part 2's Induction Proof.

Induction Proof Outline:

Part 2.1. Using the results of Part 1, show that $h'_{N-k}(x(k), U^*_{k, N-1}, k)$ satisfies the recursive relation

$$\begin{aligned} & h'_{N-k}(x(k), U^*_{k, N-1}, k) \\ = & \min_{u(k) \in \{-M_1, \dots, M_1\}} [2abP_1(k+1)u(k)x(k) + r|u(k)| + b^T P_1(k+1)u^2(k) \\ & + h'_{N-(k+1)}(T(x(k), u(k)), U^*_{k+1, N-1}, k+1)] \end{aligned} \quad (C.8)$$

where

$$P_1(k) = q \sum_{i=k}^N a^{2(i-k)} \quad (C.9)$$

and

$$T(x(k), u(k)) = ax(k) + bu(k) \quad (C.10)$$

Part 2.2. Assuming $h'_{N-(k+1)}(x(k+1), U^*_{k+1, N-1}, k+1)$ is given by Equation (C.5) where k is replaced by $k+1$, show that from the recursive Equation (C.8) it must follow that $h'_{N-k}(x(k), U^*_{k, N-1}, k)$ is given by

$$h'_{N-k}(x(k), U^*_{k, N-1}, k) = \min_{U_{k, N-1} \in S_2(k)} h'_{N-k}(x(k), U_{k, N-1}, k) \quad (C.11)$$

where

$$S_2(k) = [U_{k, N-1} : u(k) \in \{-M_1, \dots, M_1\}, U_{k+1, N-1} \in S_1(k+1)] \quad (C.12)$$

with $S_1(k+1)$ given by Equation (C.4) where k is replaced by $k+1$, and $h'_{N-k}(x(k), U_{k, N-1}, k)$ is determined from Equations

(3.4.1.10)-(3.4.1.12) (see Table C.2 for an illustration of $S_2(N-4)$ for $M_1=1$). Note from Equations (C.4) and (C.12) that $S_1(k)$ is a subset of $S_2(k)$, i.e.,

$$S_1(k) \subseteq S_2(k). \quad (C.13)$$

Compare Tables C.1 and C.2 for a verification of (C.13) when $k = N-4$ and $M_1 = 1$.

Part 2.3. Finally demonstrate that $h'_{N-k}(x(k), U^*_{k, N-1}, k)$ is given by Equation (C.5) by making use of the results of Part 2.2.

Induction Proof

Part 2.1. In Part 1 it was noted that the optimum control law for the scalar example is derived from Equations (2.5.1.1)-(2.5.1.8). We now replace the general parameters of Equations (2.5.1.7) and (2.5.1.8) by their corresponding specific parameters in Equations (3.4.1.11) and (3.4.1.12) to obtain the following results:

$$P_2(U_{k, N-1}, k) = aP_2(U_{k+1, N-1}, k+1) + 2abu(k)P_1(k+1) \quad (C.14)$$

and

$$\begin{aligned} P_3(U_{k, N-1}, k) = & P_3(U_{k+1, N-1}, k+1) + bu(k)P_2(U_{k+1, N-1}, k+1) \\ & + b^2u^2(k)P_1(k+1) + r|u(k)| \end{aligned} \quad (C.15)$$

$$S_2(N-4) = (CS_{2,i} \text{ for } i=1, \dots, 21)$$

where

$CS_{2,i}$ is a unique control sequence of the set $S_2(N-2)$.

Control Sequence	Elements of the Control Sequence $U_{N-4, N-1}$			
$U_{N-4, N-1}$	$u(N-4)$	$u(N-3)$	$u(N-2)$	$u(N-1)$
$CS_{2,1}$	-1	-1	-1	-1
$CS_{2,2}$	-1	-1	-1	0
$CS_{2,3}$	-1	-1	0	0
$CS_{2,4}$	-1	0	0	0
$CS_{2,5}$	1	-1	-1	-1
$CS_{2,6}$	1	-1	-1	0
$CS_{2,7}$	1	-1	0	0
$CS_{2,8}$	0	-1	-1	-1
$CS_{2,9}$	0	-1	-1	0
$CS_{2,10}$	0	-1	0	0
$CS_{2,11}$	0	0	0	0
$CS_{2,12}$	0	1	0	0
$CS_{2,13}$	0	1	1	0
$CS_{2,14}$	0	1	1	1
$CS_{2,15}$	-1	1	0	0
$CS_{2,16}$	-1	1	1	0
$CS_{2,17}$	-1	1	1	1
$CS_{2,18}$	1	0	0	0
$CS_{2,19}$	1	1	0	0
$CS_{2,20}$	1	1	1	0
$CS_{2,21}$	1	1	1	1

Table C.2 The set $S_2(N-4)$ when $M_1 = 1$

Next substituting Equations (C.14) and (C.15) in Equation (3.4.1.10) we have after some algebraic manipulations

$$\begin{aligned}
 & h'_{N-k}(x(k), U_{k, N-1}, k) \\
 &= 2abP_i(k+1)u(k)x(k) + r|u(k)| + b^2 u^2(k)P_i(k+1) \\
 &+ h'_{N-(k+1)}(T(x(k), u(k)), U_{k+1, N-1}, k+1) \quad (C.16)
 \end{aligned}$$

where $I(x(k), u(k))$ is given by Equation (C.10) and

$$\begin{aligned}
 & h'_{N-(k+1)}(T(x(k), u(k)), U_{k+1, N-1}, k+1) \\
 &= P_2(U_{k+1, N-1}, k+1)T(x(k), u(k)) + p_3(U_{k+1, N-1}, k+1) \quad (C.17)
 \end{aligned}$$

Finally, noting that

$$\begin{aligned}
 & \min_{U_{k, N-1} \in US(k)} h'_{N-k}(x(k), U_{k, N-1}, k) = \min_{u(k) \in \{-M_1, \dots, M_1\}} \\
 & [\min_{U_{k+1, N-1} \in US(k+1)} h'_{N-k}(x(k), U_{k, N-1}, k) / x(k), u(k), k] \quad (C.18)
 \end{aligned}$$

where $US(k)$ is given by Equation (C.3), and also noting from Equation (C.16) that

$$\begin{aligned}
 & \min_{U_{k+1, N-1} \in US(k+1)} h'_{N-k}(x(k), U_{k, N-1}, k) \\
 &= 2abu(k)P_i(k+1)x(k) + r|u(k)| + b^2 u^2(k)P_i(k+1) \\
 &+ \min_{U_{k+1, N-1} \in US(k+1)} h'_{N-(k+1)}(T(x(k), u(k)), U_{k+1, N-1}, k+1) \quad (C.19)
 \end{aligned}$$

it follows that the desired recursive Equation (C.8) is obtained when the RHS of Equation (C.19) is substituted in Equation (C.18) for the expression inside the brackets. Note that in Equation (C.8) we use $h'_{N-k}(x(k), U^*_{k, N-k}, k)$ in place of

$$\min_{U_{k, N-1} \in US(k)} h'_{N-k}(x(k), U_{k, N-1}, k). \quad (C.20)$$

Part 2.2. Noting that when it is assumed that $h'_{N-(k+1)}(T(x(k), u(k)), U^*_{k+1, N-1}, k+1)$ is given by Equation (C.5) with k replaced by $k+1$, we are also assuming that $U^*_{k+1, N-1} \in S_1(k+1)$, it must follow from the recursive Equation (C.8) that the optimum control sequence $U^*_{k, N-1}$ that is present in $h'_{N-k}(x(k), U^*_{k, N-1}, k)$ must be such that $U^*_{k, N-1} \in S_2(k)$ where $S_2(k)$ is given by (C.12). As a result, we have that Equation (C.11) must follow from the assumed expression for $h'_{N-(k+1)}(T(x(k), u(k)), U^*_{k+1, N-1}, k+1)$.

Part 2.3. In this last part of the induction proof we use the results of Part 2.2 to finally show that $h'_{N-k}(x(k), U^*_{k, N-1}, k)$ is also given by Equation (C.5).

Again the proof of this part will be divided in subparts which are now given with the aid of the flowchart of Figure C.3:

Part 2.3.1

Assume that the plot of $h'_{N-(k+1)}(x(k+1), U^*_{k+1, N-1, k+1})$ vs. $x(k+1)$ has the general characteristics shown on Figure C.4 where k is replaced by $k+1$.

Part 2.3.2

After noting that the proof of both Theorem 1 (see Equation (C.41.)) and Theorem 2 (see Equation (C.26.)) implies the validity of Equation (C.5) as well as its graphical characteristics (see Figure C.4.), perform the required proofs.

Figure C.3 Flow Diagram of the Proof for Part 2.3.

Part 2.3.1. Assume that $h'_{N-(k+1)}(x(k+1), U^*_{k+1, N-1}, k+1)$, besides being given by Equation (C.5) with k replaced by $k+1$, has the graphical characteristics shown on Figure C.4 (see Figure C.4a for the special case corresponding to $a=1/2$, $b=1$, $M_1=1$, $k=N-4$, and $I=3$) with k replaced by $k+1$: when showing that $h'_{N-k}(x(k), U^*_{k, N-1}, k)$ is given by Equation (C.5) it will also be shown implicitly that the general characteristics of Figure C.4, to be discussed next, also hold for $h'_{N-k}(x(k), U^*_{k, N-1}, k)$.

The following important properties of Figure C.4 should be noted:

1) The graph is made of straight line segments, each corresponding to the plotting of $h'_{N-k}(x(k), U^*_{k, N-1}, k)$ vs. $x(k)$ (see Equation (3.4.1.10)) for a specific realization of the control sequence $U^*_{k, N-1}$, namely the optimum control sequence $U^*_{k, N-1}$ associated with each straight line segment on the figure: for instance, for the straight line segment that exists between the values of $x(k)$ given by $x^k_{+1, k, 1}$ and $x^k_{-1, k, 1}$ (these values to be defined later) we have that

$$U^*_{k, N-1} = \{0, \dots, 0\}.$$

Furthermore, it is noted that the k -th stage element of $U^*_{k, N-1}$, $u^*(k)$, is the optimum control that must be applied to the process: for instance, when $x(k)$ lies between $x^k_{+1, k, 1}$ and $x^k_{-1, k, 1}$ we have that $u^*(k) = 0$ must be applied to the process;

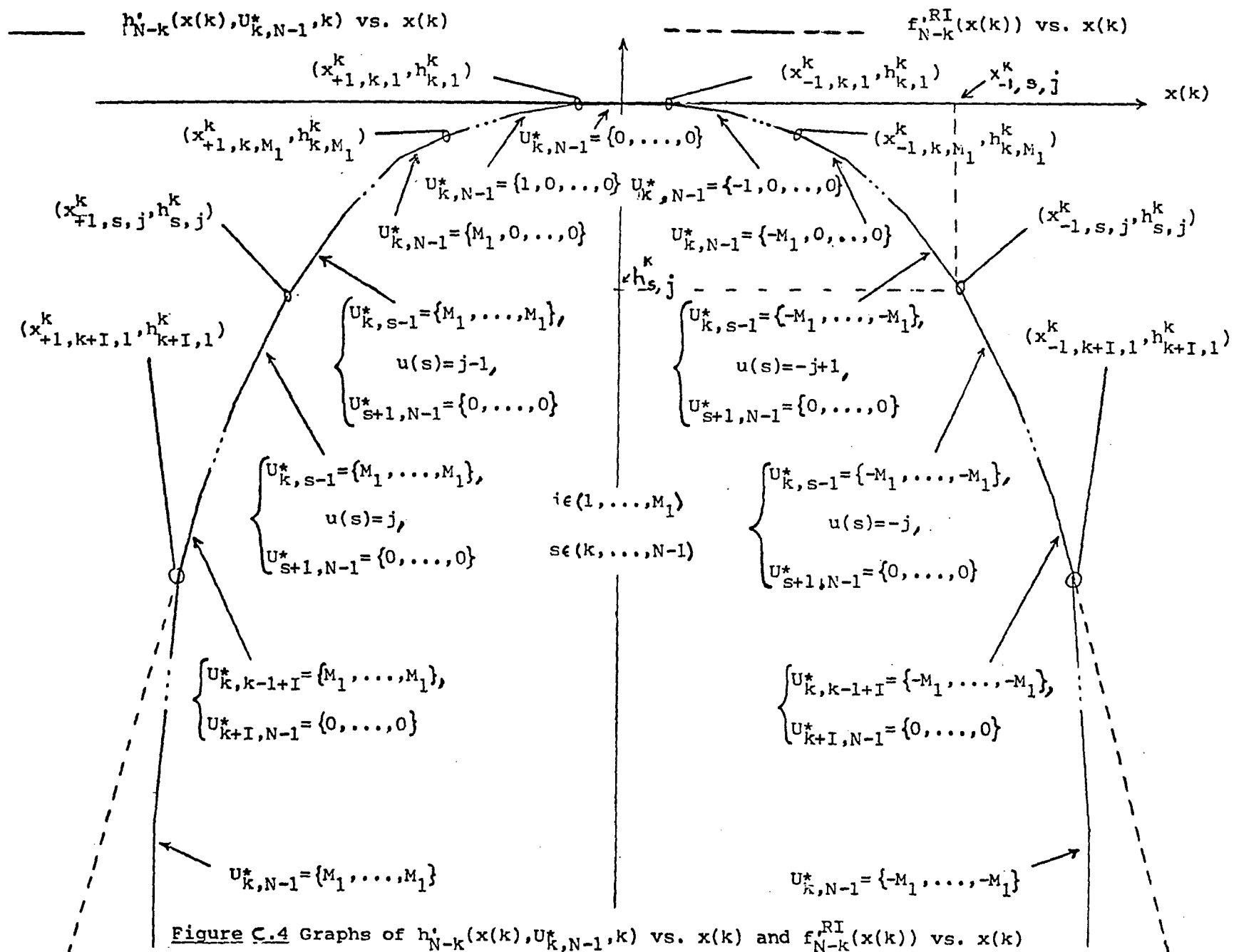


Figure C.4 Graphs of $h_{N-k}^*(x(k), U_{k,N-1}^*, k) \text{ vs. } x(k)$ and $f_{N-k}^{RI}(x(k)) \text{ vs. } x(k)$

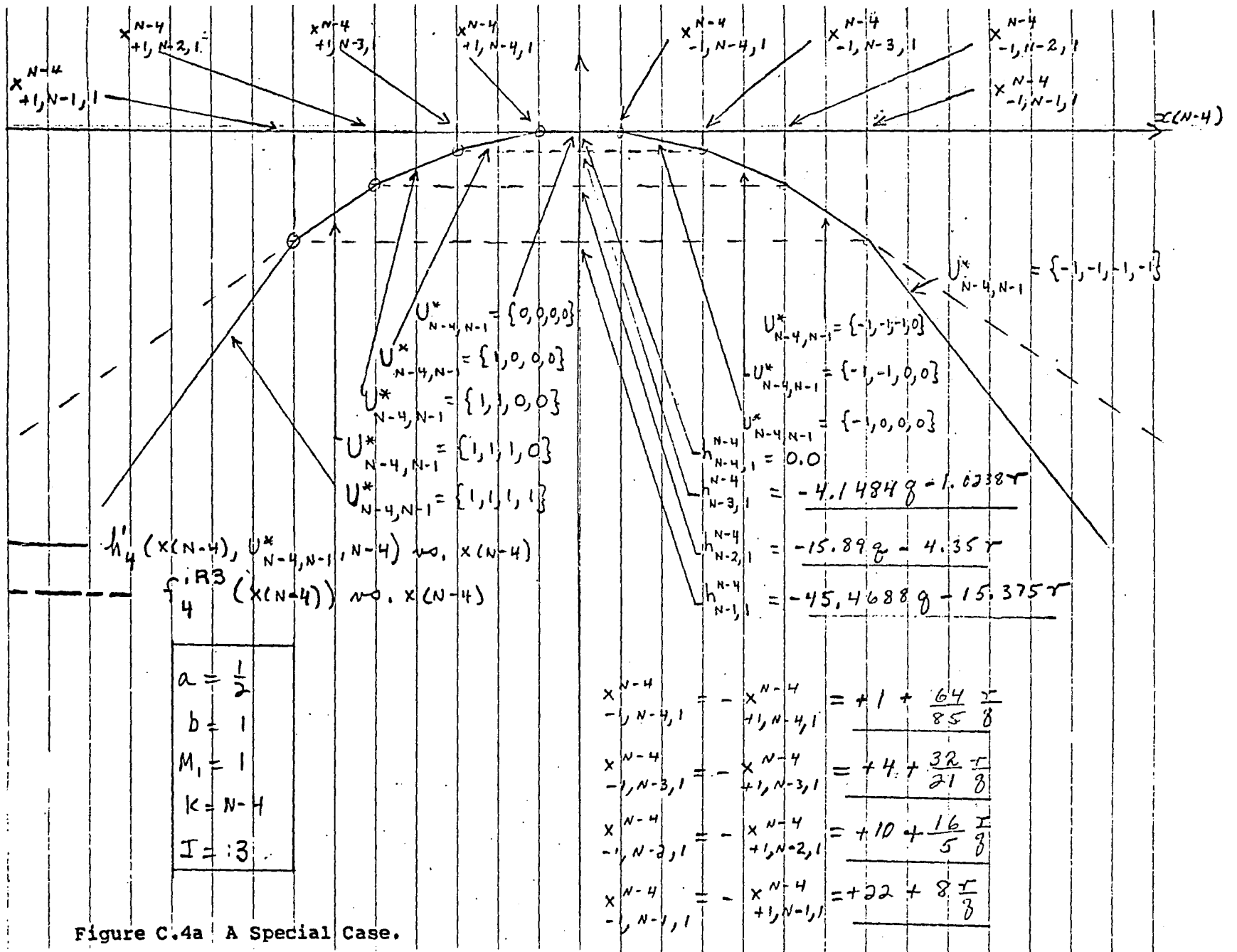


Figure C.4a A Special Case.

2) The graph has a specific distribution of $U_{k,N-1}^*$ vs. $x(k)$. When moving along the graph from left to right starting at $x(k) = 0$ it is noted that the optimum control sequence $U_{k,N-1}^*$ attains nonincreasing negative values for its members and the reverse occurs when moving from right to left starting at $x(k) = 0$: note that two pairs of consecutive straight line segments, on the LHS and RHS of the figure, have been assigned general expressions for $U_{k,N-1}^*$ that reflect this property;

3) It is noted that $U_{k,N-1}^*$ for any straight line segment is always a member of the Set (C.4): this is a necessary property of the plot if we wish to use this plot in the demonstration that $h_{N-k}'(x(k), U_{k,N-1}^*, k)$ is given by Equation (C.5) when $h_{N-(k+1)}'(x(k+1), U_{k+1,N-1}^*, k+1)$ is also given by Equation (C.5) with k replaced by $k+1$; and

4) The two components of the general breakpoint for any two consecutive straight line segments, $(x_{z,s,j}^k, h_{s,j}^k)$, are found by:

i) Solving for the value of $x(k)$, $x_{z,s,j}^k$, that satisfies the equality

$$h_{N-k}'(x(k), U_{k,N-1,1}, k) = h_{N-k}'(x(k), U_{k,N-1,2}, k) \quad (C.21)$$

where: a) $U_{k,N-1,1}$ is the control sequence $U_{k,s-1} = \{z_{M_1}, \dots, z_{M_1}\}$, $u(s) = z(j-1)$, $U_{s+1,N-1} = \{0, \dots, 0\}$;

b) $U_{k,N-1,2}$ is the control sequence $U_{k,s-1} = \{zM_1, \dots, zM_1\}$,
 $u(s) = zj$, $U_{s+1,N-1} = \{0, \dots, 0\}$; c) $z \in \{-1, +1\}$,
 $s \in (k, \dots, N-1)$, $j \in (1, \dots, M_1)$; and d) $h'_{N-k}(x(k), U_{k,N-1}, k)$
 is given by Equations (3.4.1.10)-(3.4.1.12). The equality
 given by Equation (C.21) is found to yield after some
 straight forward algebraic manipulations the expression for
 $x^k_{2,s,j}$ given by

$$x^k_{2,s,j} = -z[bc_1 + r/2qba \sum_{i=s+1}^{N-k} a^{2(i-k-1)}] / a \quad (C.22)$$

where $z \in \{-1, +1\}$, $s \in (k, \dots, N-1)$, $j \in (1, \dots, M_1)$, and

$$c_1 = (j-1/2)a^{k-s} + M_1 \sum_{t=k}^{s-1} a^{k-t}; \quad (C.23)$$

ii) Determining the value of $h^k_{s,j}$ from the substitution
 of Equation (C.22) in the RHS of Equation (C.21) to obtain

$$h^k_{s,j} = qb^2 \left[\sum_{i=k+1}^s a^{2(i-k-1)} (-2c_1 c_2(i) + c_2^2(i)) \right. \\ \left. + \sum_{i=s+1}^{N-k} a^{2(i-k-1)} (-c_1^2 + a^{2(k-s)} / 4) \right] + r(s-k-D_1)M_1 \quad (C.24)$$

for $s \in (k, \dots, N-1)$ and $j \in (1, \dots, M_1)$, where c_1 is given by
 Equation (C.23),

$$c_2(i) = M_1 \sum_{t=k}^{i-1} a^{k-t}, \quad (C.25)$$

and

$$D_1 = \left[\sum_{i=k+1}^S a^{2(i-k-1)} \sum_{t=k}^{i-1} a^{s-t} + \sum_{i=s+1}^N a^{2(i-k-1)} \sum_{t=k}^{s-1} a^{s-t} \right] / \sum_{i=s+1}^N a^{2(i-k-1)} \quad (C.26)$$

Part 2.3.2. From Equations (C.5) and (C.11) (which was established in Part 2.2 when it was assumed that $h'_{N-(k+1)}(T(x(k), u(k)), U^*_{k+1, N-1}, k+1)$ is given by Equation (C.5) with k replaced by $k+1$), it follows that in order to show the equivalence between Equations (C.5) and (C.11) the following condition must be shown to be true: for all $U_{k, N-1}$ such that

$$U_{k, N-1} \in S_2(k) \quad (C.30)$$

we must have that

$$h'_{N-k}(x(k), U_{k, N-1}, k) \geq h'_{N-k}(x(k), U^*_{k, N-1}, k) \quad (C.31)$$

where $U^*_{k, N-1}$ must always be in the set $S_1(k)$, i.e.,

$$U^*_{k, N-1} \in S_1(k). \quad (C.32)$$

As a result, from (C.30)-(C.32), the fact that $h'_{N-k}(x(k), U_{k, N-1}, k)$ is a linear function of $x(k)$, and Figure (C.4), it follows that the following necessary and sufficient conditions must be met when showing that Equation (C.5) as well as its general graphical characteristics of Figure C.4 are true:

1) The value of $h'_{N-k}(x^k_{z,s,j}, U_{k,N-1}, k)$ for any $U_{k,N-1}$ in the set $S_2(k)$ is never lower than $h^k_{s,j}$ for all possible values of z, s , and j , i.e.,

$$h'_{N-k}(x^k_{z,s,j}, U_{k,N-1}, k) > h^k_{s,j} \quad (C.34)$$

for all $U_{k,N-1} \in S_2(k)$ and all $z \in (-1, +1)$, $s \in (k, \dots, N-1)$, and $j \in (1, \dots, M_1)$; and

2) The slope of $h'_{N-k}(x(k), U_{k,N-1}, k)$, i.e., $P_2(U_{k,N-1}, k)$, for any $U_{k,N-1}$ in the set $S_2(k)$ is less than or equal to the slope of the extreme LHS straight line segment of Figure C.4, i.e.,

$$P_2(U_{k,N-1}, k) \leq P_2(U_{k,N-1}, k) / U_{k,N-1} = \{M_1, \dots, M_1\} \quad (C.35)$$

for all $U_{k,N-1} \in S_2(k)$ as easily seen from Figure C.4. Also the slope of $h'_{N-k}(x(k), U_{k,N-1}, k)$ for any control sequence $U_{k,N-1}$ in the set $S_2(k)$ must be greater than or equal to the slope of the extreme RHS straight line segment of Figure C.4, i.e.,

$$P_2(U_{k,N-1}, k) \geq P_2(U_{k,N-1}, k) / U_{k,N-1} = \{-M_1, \dots, -M_1\} \quad (C.38)$$

for all $U_{k,N-1} \in S_1(k)$ as verified from Figure C.4.

Next we establish the validity of (C.34), (C.35), and (C.38) via two theorems. In Theorem C.1 we consider Condi-

tion (C.34) and in Theorem C.2 we consider both Conditions (C.35) and (C.38).

Theorem C.1

$$h_{N-k}^k(x_{z,s,j}^k, U_{k,N-1}, k) > h_{s,j}^k \quad (\text{C.41})$$

for

i) All $U_{k,N-1} \in S_2(k)$ where $S_2(k)$ is given by Equation (C.12);

ii) All breakpoints $(x_{z,s,j}^k, h_{s,j}^k)$ where

$$z \in (-1, +1), \quad (\text{C.42})$$

$$s \in (k, \dots, N-1), \quad (\text{C.43})$$

and

$$j \in (1, \dots, M_1); \quad (\text{C.44})$$

and

iii) All q, r, a, b, M, N , and k such that

$$q \in (y: 0 < y < \infty), \quad (\text{C.45})$$

$$r \in (y: 0 < y < \infty), \quad (\text{C.46})$$

$$a \in (y: 0 < y < .98), \quad (\text{C.47})$$

$$b \in (y: 0 < y < \infty), \quad (\text{C.48})$$

$$M_1 \text{ is any positive integer,} \quad (\text{C.49})$$

$$N \in (k+2, k+3, \dots), \quad (C.50)$$

and

$$k \in (0, \dots, N-1). \quad (C.51)$$

Proof of Theorem C.1

The proof will be performed by first presenting with the aid of the flowchart of Figure C.5 an outline of the proof and then proving each of the required steps.

Step 1. In this step it is shown that the set $S_2(k)$ (see Equations (C.12) and (C.4)) is equal to the union of two sets, namely

$$S_2(k) = S(k) \cup S_2^I(k) \quad (C.52)$$

where

$$\begin{aligned} S(k) = \{ U_{k, N-1} : u(k) = t_k, U_{k+1, N-1} = \{0, \dots, 0\} \\ \text{for all } t_k \in (-M_1, \dots, M_1) \} \end{aligned} \quad (C.53)$$

and

$$\begin{aligned} S_2^I(k) = \{ U_{k, N-1} : u(k) = t_k, U_{k+1, v-1} = \{z^1 M_1, \dots, z^1 M_1\}, u(v) = z^1 p, \\ U_{v+1, N-1} = \{0, \dots, 0\} \text{ for all } t_k \in (-M_1, \dots, M_1), z^1 \in (-1, +1), \\ v \in (k+1, \dots, N-1), p \in (1, \dots, M_1) \} \end{aligned} \quad (C.54)$$

See Tables 3.1, C.2, and C.3 for an illustration of the Sets $S(N-4)$ ⁹, $S_2(N-4)$, and $S_2^I(N-4)$.

⁹ A comparison of Sets (3.4.1.15) and (C.53) reveals that $S(k) = S(k, 1, 0)$ for all $k \in (0, \dots, N-1)$.

Step 1

Show that $S_2(k)$ is given by the union of the subsets $S(k)$ and $S'_2(k)$,
i.e.,

$$S_2(k) = S(k) \cup S'_2(k),$$

where $S(k)$ and $S'_2(k)$ are given by Equations (C.53) and (C.54).

Step 2

Show that the Inequality (C.41) is satisfied for any $U_{k,N-1} \in S(k)$
and all allowable values of $z, s, j, q, r, a, b, M, N$, and k given
by Equations (C.42) - (C.51).

Step 3

Show that the Inequality (C.41) is satisfied for any $U_{k,N-1} \in S'_2(k)$
and all allowable values of $z, s, j, q, r, a, b, M, N$, and k given
by Equations (C.42) - (C.51). This ends the proof of Theorem 1

Figure C.5 Flow Diagram of the Proof for Theorem 1 of
Part 2.3.2.

Table C.3 The set $S_2^1(N-4)$ for $M_1 = 1$.

$$S_2^1(N-4) = \{CS_{2,i}^1 \text{ for } i=1, \dots, 18\}$$

where

 $CS_{2,i}^1$ is a unique control sequence of the set $S_2^1(N-4)$.
Control SequenceElements of the Control Sequence $U_{N-4, N-1}$ $U_{N-4, N-1}$ $U(N-4)$ $U(N-3)$ $U(N-2)$ $U(N-1)$ $CS_{2,1}^1$

-1

-1

-1

-1

 $CS_{2,2}^1$

-1

-1

-1

0

 $CS_{2,3}^1$

-1

-1

0

0

 $CS_{2,4}^1$

1

-1

-1

-1

 $CS_{2,5}^1$

1

-1

-1

0

 $CS_{2,6}^1$

1

-1

0

0

 $CS_{2,7}^1$

0

-1

-1

-1

 $CS_{2,8}^1$

0

-1

-1

0

 $CS_{2,9}^1$

0

-1

0

0

 $CS_{2,10}^1$

0

1

0

0

 $CS_{2,11}^1$

0

1

1

0

 $CS_{2,12}^1$

0

1

1

1

 $CS_{2,13}^1$

-1

1

0

0

 $CS_{2,14}^1$

-1

1

1

0

 $CS_{2,15}^1$

-1

1

1

1

 $CS_{2,16}^1$

1

1

0

0

 $CS_{2,17}^1$

1

1

1

0

 $CS_{2,18}^1$

1

1

1

1

Step 2. Show that Inequality (C.41) is satisfied for all $U_{K, N-1} \in S(k)$ and all allowable values of $z, s, j, q, r, a, b, M_1, N$, and k (see (C.42)-(C.51)).

Step 3. Show that Inequality (C.41) is satisfied for all $U_{K, N-1} \in S_2^1(k)$ and all allowable values of $z, s, j, q, r, a, b, M_1, N$, and k ; as a result completing the proof of Theorem C.1 since $S_2(k)$ is equal to the union of $S(k)$ and $S_2^1(k)$.

Next we proceed to prove Steps 1, 2, and 3.

Step 1 The validity of Equation (C.52) where $S(k)$ and $S_2^1(k)$ are given by the Sets (C.53) and (C.54) can be readily verified from the Sets (C.12) and (C.4):

From (C.12) and (C.4) it follows that $S_2(k)$ can be rewritten as

$$\begin{aligned}
 S_2(k) = & [U_{K, N-1} : u(k) = t_K, U_{K, N-1} = \{z^1 M_1, \dots, z^1 M_1\}, \\
 & u(v) = z^1 p, U_{v+1, N-1} = \{0, \dots, 0\} \text{ for all } t_K \in (-M_1, \dots, M_1), \\
 & z^1 \in (-1, +1), v \in (k+1, \dots, N-1), p \in (1, \dots, M_1); u(k) = t_K, \\
 & U_{K+1, N-1} = \{0, \dots, 0\} \text{ for all } t_K \in (-M_1, \dots, M_1)]
 \end{aligned}
 \tag{C.54a}$$

We then note from Equation (C.54a) that the set of control sequences in the LHS of the semicolon ';' is given by $S_2^1(k)$ and the set of control sequences in the RHS of the semicolon is given by $S(k)$. This concludes the proof of Equation (C.52).

Step 2. Before showing Inequality (C.41) for all $U_{k,N-1} \in S(k)$ we first proceed to find an expression for $h'_{N-k}(x_{a,s,j}^k, U_{k,N-1}, k) - h_{s,j}^k$ as a function of the parameters $z, s, j, t_k, q, r, a, b, M_1, N$, and k . Furthermore, to simplify notation we let the vector p' represent the previously mentioned parameters except for k , i.e.,

$$p' = (z, s, j, t_k, q, r, a, b, M_1, N). \quad (C.55)$$

First, substituting t_k for $u(k)$ and zero for each member of $U_{k+1, N-1}$ in Equations (3.4.1.11)-(3.4.1.12) we obtain (note that the parameters assigned to $U_{k, N-1}$ were derived from the Set (C.53)),

$$P_2(t_k, k) = 2qab t_k \sum_{i=k+1}^N a^{2(i-k-1)} \quad (C.56)$$

and

$$p_3(t_k, k) = qb^2 t_k^2 \sum_{i=k+1}^N a^{2(i-k-1)} + r|t_k| \quad (C.57)$$

Then replacing $F_2(U_{k, N-1}, k)$, $p_3(U_{k, N-1}, k)$, and $x(k)$ by Eqs. (C.56)-(C.57) and (C.22) in Eq. (3.4.1.10) we obtain after some algebraic manipulations

$$h'_{N-k}(p', k) = qb^2 \sum_{i=k+1}^N a^{2(i-k-1)} (t_k^2 - 2zc_1 t_k) + r(|t_k| - zD_2 t_k) \quad (C.58)$$

where c_1 is given by Eq. (C.23), p' is the parameter vector of Eq. (C.55), and

$$D_2 = \frac{\sum_{i=k+1}^N a^{2(i-k-1)}}{\sum_{i=s+1}^N a^{2(i-k-1)}} a^{s-k} \quad (C.59)$$

or equivalently in compact form

$$D_2 = \frac{1 - a^{2(N-k)}}{a^{s-k}(1 - a^{2(N-s)})} \quad (C.60)$$

Next subtracting Eq. (C.24) from Eq. (C.58) we have after some algebraic manipulations

$$\begin{aligned} h'_{N-k}(p', k) - h^k_{s, j} = qb^2 \left\{ \sum_{i=k+1}^s a^{2(i-k-1)} (H_1^2(i) + 2H_2(i)H_1(i)) + \right. \\ \left. + \sum_{i=s+1}^N a^{2(i-k-1)} (H_3^2 - \frac{1}{4}a^{k-s}) \right\} + \\ + r(|t_k| - zD_2 t_k + (D_1 - s + k)M_1) \end{aligned} \quad (C.61)$$

where

$$H_1(i) = M_1 \sum_{t=k}^{i-1} a^{k-t} - zt_k \quad (C.62)$$

$$H_2(i) = (j - \frac{1}{2})a^{k-s} + M_1 \sum_{t=i}^{s-1} a^{k-t} \quad (C.63)$$

$$H_3 = (j - \frac{1}{2}) a^{k-s} + M_1 \sum_{t=k}^{s-1} a^{k-t} - z t_k \quad (C.64)$$

and D_1 and D_2 are as shown on Eqs. (C.26) and (C.60). Note that in the derivation of Eq. (C.61) we have made use of the equalities $C_1 = H_2(i) + C_2(i)$ and $(zz)^2 = 1$ for $z \in (+1, -1)$.

(C.65)

Finally noting that Ineq. (C.41) for $\bigwedge_{k, N-1}^{\text{any}} S(k)$ is demonstrated by satisfying the inequality

$$h'_{N-k}(p', k) - h^k_{s, j} \geq 0 \quad (C.66)$$

for all the parameters present in p' (see Eq. (C.55)) such that

$$q \in (\bar{y}: 0 \leq \bar{y} < \infty), \quad (C.67)$$

$$r \in (\bar{y}: 0 \leq \bar{y} < \infty), \quad (C.68)$$

$$a \in (\bar{y}: 0 < \bar{y} < 4.93), \quad (C.69)$$

$$b \in (\bar{y}: 0 \leq \bar{y} < \infty), \quad (C.70)$$

$$z \in (-1, +1), \quad (C.71)$$

$$j \in (1, \dots, M_1), \quad (C.72)$$

$$t_k \in (-M_1, \dots, 0, \dots, M_1), \quad (C.73)$$

$$s \in (k, \dots, N-1), \quad (C.74)$$

$$N \in (k+2, k+3, \dots), \quad (C.75)$$

$$M_1 \text{ is any positive integer,} \quad (C.76)$$

and

$$k \in (L, \dots, N-2), \quad (C.77)$$

we first proceed to show that Ineq. (C.66) holds for $s=k$ and all other allowable values of the parameters in p' and k (where by 'allowable values' we mean those values of p' and k that belong to the spaces defined in Eqs. (C.67)-(C.77); from now on when using the expression 'allowable values' we will be referring to those values allowed by Eqs. (C.67)-(C.77)) and then for $s > k$ and all other allowable values of p' and k .

Case $s=k$

When $s=k$ we have from Eqs. (C.61)-(C.64), (C.26), and (C.59) that

$$h'_{N-k}(p,k) - h^k_{s,j} = qb^2 \sum_{i=k+1}^N a^{2(i-k-1)} \left(\left(j - \frac{1}{2} - zt_k \right)^2 - \frac{1}{4} \right) + r(|t_k| - zt_k) \quad (C.78)$$

The RHS of

this equation is next shown to be nonnegative for all allowable values of its parameters.

We first note that qb^2 , $a^{2(i-k-1)}$, and r are all nonnegative since q is nonnegative, b is real, a is real, $i-k-1$ is an integer, and r is nonnegative. Therefore it follows that the RHS of Eq. (C.78) will be nonnegative if

$$\left(j - \frac{1}{2} - zt_k \right)^2 - \frac{1}{4} \geq 0 \quad (C.79)$$

and

$$|t_k| - zt_k \geq 0. \quad (C.80)$$

The above Ineqs. (C.79)-(C.80) are easily seen to be satisfied by noting from Eqs. (C.71)-(C.73) that the lowest value that $\left(j - \frac{1}{2} - zt_k \right)^2$ can have is $\frac{1}{4}$, and z is either +1 or -1.

Case $s > k$

For $s > k$ we have that Eqs. (C.61)-(C.64) can be rewritten as follows:

$$\begin{aligned} h'_{N-k}(p,k) - h^k_{s,j} = qb^2 \left\{ \sum_{i=k+1}^s a^{2(i-k-1)} (H_1^2(i) + 2H_2(i)H_1(i)) + \right. \\ \left. + \sum_{i=s+1}^N a^{2(i-k-1)} \left(H_3^2 - \frac{1}{4}a^{2(k-s)} \right) \right\} + \\ + r(|t_k| - t_k + (1 - 2D_2)t_k + (D_2 - 1)M_1 + (D_3 - s + 1 + k)M_1) \quad (C.81) \end{aligned}$$

where D_2 is given by Eq. (C.59) and

$$H_1(i) = (M_1 - zt_k) + M_1 \sum_{t=k+1}^{i-1} a^{k-t} \quad (C.82)$$

$$H_2(i) = \left(j - \frac{1}{2} \right) a^{k-s} + M_1 \sum_{t=i}^{s-1} a^{k-t} \quad (C.83)$$

$$H_3 = \left(j - \frac{1}{2} \right) a^{k-s} + (M_1 - zt_k) + M_1 \sum_{t=k+1}^{s-1} a^{k-t} \quad (C.84)$$

and

$$D_3 = \frac{\sum_{i=k+2}^s a^{2(i-k-1)} \sum_{t=k+1}^{i-1} a^{s-t} + \sum_{i=s+1}^N a^{2(i-k-1)} \sum_{t=k+1}^{s-1} a^{s-t}}{\sum_{i=s+1}^N a^{2(i-k-1)}} \quad (C.85)$$

or equivalently in compact form

$$D_3 = \frac{-a^{2(N-s)+1} + a^{2N-s-k-1} + a^{-s+k+1}}{(1-a)(1-a^{2(N-s)})} \quad (C.86)$$

(note that on the derivation of Eq.(C.81) we made use of the equality

$$D_1 = D_2 + D_3)$$

Again noting that qb^2 , $a^{2(i-k-1)}$, and r are nonnegative and that M_1 is greater than zero (see Cond.(C.76)) we have that Eq. (C.81) will be nonnegative if the following conditions are satisfied:

Condition 1

$$H_1(i) > 0 \text{ for } k+1 \leq i \leq s \quad (C.87)$$

From Eq.(C.82) it is clear that Ineq. (C.87) is true since both

$M_1 - zt_k$ and

$$M_1 \sum_{t=k+1}^{i-1} a^{k-t} \quad (C.88)$$

are nonnegative for all allowable values of M_1 , z , t_k , i , a , and k .

Condition 2

$$H_2(i) > 0 \text{ for } k+1 \leq i \leq s \quad (C.89)$$

From Eq.(C.83) we have that Ineq.(C.89) is true since $j - \frac{1}{2}$, a^{k-s} , and

$$M_1 \sum_{t=i}^{s-1} a^{k-t} \quad (C.90)$$

are nonnegative for all allowable values of j , a , k , s , M_1 and i .

Condition 3

$$H_3 > \frac{1}{2} a^{k-s} \quad (C.91)$$

Noting that $(j - \frac{1}{2})a^{k-s}$ is greater than or equal to $\frac{1}{2}a^{k-s}$,

$M_1 - zt_k$ is nonnegative, and

$$M_1 \sum_{t=k+1}^{s-1} a^{k-t} \quad (C.92)$$

is nonnegative for all allowable values of j , a , k , s , M_1 , z , and t_k it follows from Eq. (C.84) that Ineq. (C.91) is satisfied.

Condition 4

$$|t_k| - t_k + (1 - zD_2)t_k + (D_2 - 1)M_1 \geq 0 \quad (C.93)$$

From Eq. (C.60) we have that D_2 is greater than or equal to one for all allowable values of a , s , k , and N . Now for values of t_k that are nonnegative we have that Ineq. (C.93) becomes

$$(1 + D_2)t_k + (D_2 - 1)M_1 \geq 0 \text{ for } z = -1 \quad (C.94)$$

and

$$(D_2 - 1)(M_1 - t_k) \geq 0 \text{ for } z = +1 \quad (C.95)$$

Ineqs. (C.94)-(C.95) are clearly seen to be satisfied for all allowable values of its parameters D_2 , t_k , and M_1 .

Finally for negative values of t_k we have that Ineq. (C.93) becomes

$$-(1 + D_2)t_k + (D_2 - 1)M_1 \geq 0 \text{ for } z = +1 \quad (C.96)$$

for

$$(D_2 - 1)(M_1 + t_k) \geq 0 \text{ for } z = -1 \quad (C.97)$$

Ineqs. (C.96)-(C.97) are also noticed to be satisfied for all allowable values of its parameters D_2 , t_k , and M_1 . As a result it must follow that Ineq. (C.93) is satisfied for all allowable values of its parameters.

Condition 5

$$D_3(s) - s + k + 1 \geq 0 \text{ for } s \geq k \quad (C.98)$$

where $D_3(s)$ is given by Eq. (C.86).

Using an inductive proof we next show that Ineq. (C.98) must hold. First substituting $k+1$ for s in Eq. (C.86) we have that $D_3(k+1)$ is zero and as a result Ineq. (C.98) is satisfied since $D_3(k+1) - (k+1) + k + 1$ is zero. Next substituting $s+1$ for s in the LHS of Ineq. (C.98) and also making use of Eq. (C.86) we have after some algebraic manipulations

$$D_3(s+1)-(s+1)+k+1 = D_3(s)-s+k+1 + \frac{a^{-s+k-1}}{1-a^{2(N-s)}} + \frac{a^{2(N-s)}}{1-a^{2(N-s)}} ((a^{-2}-1)(s-k-1) + a^{-2}-1-a^{-1}+a^{s-k-1}) \quad (C.99)$$

The induction proof is completed by showing that Eq.(C.99) is nonnegative when $D_3(s)-s+k+1$ is nonnegative.

Since a^{-s+k-1} , $a^{2(N-s)}$, and $1-a^{2(N-s)}$ are all nonnegative for all allowable values of their parameters it follows from Eq. (C.99) that assuming $D_3(s)-s+k+1$ is nonnegative $D_3(s+1)-(s+1)+k+1$ will also be nonnegative if it can be shown that

$$D_3^1(s) \geq 0 \text{ for } s \geq k+1 \quad (C.100)$$

where

$$D_3^1(s) = (a^{-2}-1)(s-k-1) + a^{-2}-1-a^{-1}+a^{s-k-1} \quad (C.101)$$

for all allowable values of a and k and s greater than $k+1$.

Ineq. (C.100) is now readily shown by using an induction approach.

Substituting $k+2$ for s in Eq. (C.101) and after some algebraic manipulations we obtain

$$D_3^1(k+2) = \frac{(2-a)(1-a^2)}{a^2} \quad (C.102)$$

which is nonnegative for all allowable values of a . Then replacing s for $s+1$ in Eq. (C.101) and after some algebraic manipulations we have

$$D_3^1(s+1) = D_3^1(s) + (1-a) \left(\frac{1+a}{a^2} - a^{s-k-1} \right) \quad (C.103)$$

Given $D_3^1(s)$ is nonnegative it follows from Eq. (C.103) that $D_3^1(s+1)$ must be nonnegative since $1-a$ and $(1+a)/a^2 - a^{s-k-1}$ are nonnegative for all allowable values of a , s , and k . This concludes the inductive proof of Ineq. (C.100) and as a result also the inductive proof of Ineq. (C.98).

Step 3 We now proceed to find an expression for

$h_{N-k}^i(x_{z,s,j}^k, U_{k,N-1}, k)$ as a function of the parameters in p' (see Eq. (C.55)), z' , v , p , and k for the case where

$$U_{k,N-1} \in S_2^1(k) \quad (C.104)$$

First substituting t_k for $u(k)$, $z'M_1$ for each of the components of $U_{k+1,v-1}$, $z'p$ for $u(v)$, and 0 for each of the components of $U_{v+1,N-1}$ in Eqs. $\begin{matrix} (3.4.1.11) \\ \lambda \end{matrix} - \begin{matrix} (3.4.1.12) \\ \lambda \end{matrix}$ (note that the parameters assigned to $U_{k,N-1}$ were obtained from the Set (C.54)) we obtain

$$P_2(t_k, z', v, p, k) = 2qab \left\{ \sum_{i=k+1}^v a^{2(i-k-1)} C_3(i) + \sum_{i=v+1}^N a^{2(i-k-1)} C_4 \right\} \quad (C.105)$$

where

$$C_3(i) = t_k + z'M_1 \sum_{t=k+1}^{i-1} a^{k-t} \quad (C.106)$$

and

$$C_4 = t_k + z'(M_1 \sum_{t=k+1}^{v-1} a^{k-t} + pa^{k-v}) \quad (C.107)$$

$$\begin{aligned} \phi_3(t_k, z', v, p, k) = qb^2 \left\{ \sum_{i=k+1}^v a^{2(i-k-1)} C_3^2(i) + \sum_{i=v+1}^N a^{2(i-k-1)} C_4^2 \right\} + \\ + r(|t_k| + (v-k-1)M_1 + p) \end{aligned} \quad (C.108)$$

where $C_3(i)$ and C_4 are given by Eqs. (C.106) - (C.107).

Next replacing $P_2(U_{k,N-1}, k)$, $\phi_3(U_{k,N-1}, k)$, and $x(k)$ by Eqs. (C.105) - (C.108) and $\begin{matrix} (C.102) \\ \lambda \end{matrix}$ in Eq. (3.4.1.10) we obtain after some algebraic manipulations

$$\begin{aligned} h_{N-k}^i(p', z', v, p, k) = qb^2 \left\{ \sum_{i=k+1}^v a^{2(i-k-1)} (C_3^2(i) - 2zC_1C_3(i)) + \sum_{i=v+1}^N a^{2(i-k-1)} (C_4^2 - 2zC_1C_4) \right\} + \\ + r(|t_k| - t_k + (1-zD_2)t_k + (1-zz'D_4)p + (v-k-1-zz'D_5)M_1) \end{aligned} \quad (C.109)$$

where C_1 , $C_3(i)$, C_4 , and D_2 are given by Eqs. (C.23), (C.106) - (C.107), and (C.60) and

$$D_4 = \frac{\sum_{i=v+1}^N a^{2(i-k-1)}}{\sum_{i=s+1}^N a^{2(i-k-1)}} a^{s-v} \quad (C.110)$$

or equivalently in compact form

$$D_4 = \frac{1-a^{2(N-v)}}{a^{s-v}(1-a^{2(N-s)})} \quad (C.111)$$

and

$$D_5 = \frac{\sum_{i=k+2}^v a^{2(i-k-1)} \sum_{t=k+1}^{i-1} a^{s-t} + \sum_{i=v+1}^N a^{2(i-k-1)} \sum_{t=k+1}^{v-1} a^{s-t}}{\sum_{i=s+1}^N a^{2(i-k-1)}} \quad (C.112)$$

Finally noting that Ineq. (C.41) for $\bigcup_{k,N-1}^{any} S_2^1(k)$ is demonstrated by proving the inequality

$$h'_{N-k}(p', z', v, p, k) - h_{s,j}^k \geq 0 \quad (C.113)$$

for all allowable values of the parameters in p' and k (see Eqs. (C.67)-(C.77)) and in addition for

$$z' \in (-1, +1), \quad (C.114)$$

$$v \in (k+1, \dots, N-1), \quad (C.115)$$

and

$$p \in (1, \dots, M_1), \quad (C.116)$$

we next show that Ineq. (C.113) holds for all allowable values of its parameters by considering independently the following cases: i) $s > v$; ii) $s = v$; iii) $s = k < v$; and iv) $k < s < v$. Note that these cases comprise all the possible relations between s and v .

Case i ($s > v$)

Subtracting Eq. (C.24) from Eq. (C.109) where $s > v$ we obtain after some algebraic manipulations

$$\begin{aligned}
& h'_{N-k}(p', z', v, p, k) - h^k_{s,j} \\
& = qb^2 \left\{ \sum_{i=k+1}^v a^{2(i-k-1)} (K_1^2(i) + 2K_2(i)K_1(i)) + \right. \\
& + \sum_{i=v+1}^s a^{2(i-k-1)} (K_3^2(i) + 2K_2(i)K_3(i)) + \sum_{i=s+1}^N a^{2(i-k-1)} (K_4^2 - \frac{1}{4}a^{2(k-s)}) \left. \right\} + \\
& + r \left(\left| t_k \right| - t_k + (1-zD_2)t_k + (D_2-1)M_1 + (1-zz'D_4)p + (D_4-1)M_1 + D_5M_1(1-zz') + \right. \\
& + (v-s+1+R_1)M_1 \left. \right) \tag{C.117}
\end{aligned}$$

where

$$K_1(i) = M_1 \sum_{t=k}^{i-1} a^{k-t-zt_k-zz'M_1} \sum_{t=k+1}^{i-1} a^{k-t} \tag{C.118}$$

$$K_2(i) = (j - \frac{1}{2})a^{k-s+M_1} \sum_{t=i}^{s-1} a^{k-t} \tag{C.119}$$

$$K_3(i) = M_1 \sum_{t=k}^{i-1} a^{k-t-zt_k-zz'} (M_1 \sum_{t=k+1}^{v-1} a^{k-t+pa^{k-v}}) \tag{C.120}$$

$$K_4 = (j - \frac{1}{2})a^{k-s+M_1} \sum_{t=k}^{s-1} a^{k-t-zt_k-zz'} (M_1 \sum_{t=k+1}^{v-1} a^{k-t+pa^{k-v}}) \tag{C.121}$$

$$R_1 = \frac{\sum_{i=v+2}^s a^{2(i-k-1)} \sum_{t=v+1}^{i-1} a^{s-t} + \sum_{i=s+1}^N a^{2(i-k-1)} \sum_{t=v+1}^{s-1} a^{s-t}}{\sum_{i=s+1}^N a^{2(i-k-1)}} \tag{C.122}$$

or equivalently in compact form we have that R_1 is given by

$$R_1 = \frac{-a^{2(N-s)+1} - 1 + a^{2N-s-v} + a^{v-s+1}}{(1-a)(1-a^{2(N-s)})} \tag{C.123}$$

and D_2 , D_4 , and D_5 are given by Eqs. (C.60), (C.111), and (C.112).

Note that in the derivation of Eq. (C.117) we have made use of the following equalities

$$C_1 = K_2(i) + M_1 \sum_{t=k}^{i-1} a^{k-t} \tag{C.124}$$

and

$$D_1 = D_2 + D_4 + D_5 + R_1 \tag{C.125}$$

Noting that qb^2 , $a^{2(i-k-1)}$, and r are nonnegative for all allowable values of their parameters, we have that Eq. (C.117) can

be shown to be nonnegative by demonstrating the following conditions to be true:

Condition 1

$$K_1(i) \geq 0 \text{ for } k+1 \leq i \leq v \quad (C.125)$$

Rewriting Eq. (C.118) as

$$K_1(i) = (M_1 - zt_k) + M_1(1 - zz') \sum_{t=k+1}^{i-1} a^{k-t} \quad (C.126)$$

we have that Ineq. (C.125) is true since $M_1 - zt_k$, M_1 , $1 - zz'$, and

$$\sum_{t=k+1}^{i-1} a^{k-t} \quad (C.127)$$

are all nonnegative for all allowable values of their parameters.

Condition 2

$$K_2(i) \geq 0 \text{ for } k+1 \leq i \leq s \quad (C.128)$$

Noting that $j - \frac{1}{2}$, a^{k-s} , M_1 , and

$$\sum_{t=i}^{s-1} a^{k-t} \quad (C.129)$$

are nonnegative for all allowable values of their parameters we have from Eq. (C.119) that Ineq. (C.128) must be true.

Condition 3

$$K_3(i) \geq 0 \text{ for } v+1 \leq i \leq s \quad (C.130)$$

Rewriting Eq. (C.120) as

$$K_3(i) = M_1 - zt_k + M_1(1 - zz') \sum_{t=k+1}^{v-1} a^{k-t} + (M_1 - zz'p) a^{k-v} + M_1 \sum_{t=v+1}^{i-1} a^{k-t} \quad (C.131)$$

we note that Ineq. (C.130) must hold since $M_1 - zt_k$, M_1 , $1 - zz'$, $M_1 - zz'p$, a^{k-v} ,

$$\sum_{t=k+1}^{v-1} a^{k-t} \text{ and } \sum_{t=v+1}^{i-1} a^{k-t} \quad (C.132)$$

are all nonnegative for all allowable values of their parameters.

Condition 4

$$K_4 \geq \frac{1}{2} a^{k-s} \quad (C.133)$$

Rewriting Eq. (C.121) as

$$K_4 = (j - \frac{1}{2}) a^{k-s} + M_1 - z t_k + M_1 (1 - z z') \sum_{t=k+1}^{v-1} a^{k-t} + (M_1 - z z' p) a^{k-v} + M_1 \sum_{t=v+1}^{s-1} a^{k-t} \quad (C.134)$$

we note that Ineq. (C.133) must be satisfied since $(j - \frac{1}{2}) a^{k-s}$ is greater than or equal to $\frac{1}{2} a^{k-s}$ for all allowable values of j and the rest of the terms on the RHS of Eq. (C.134) are nonnegative for all allowable values of their parameters (including the term $M_1 - z t_k$).

Condition 5

$$|t_k| - t_k + (1 - z D_2) t_k + (D_2 - 1) M_1 \geq 0 \quad (C.93)$$

Ineq. (C.93) was earlier shown to be true for all allowable values of t_k , z , M_1 and D_2 (see Condition 4 on pg. 116).

Condition 6

$$(1 - z z' D_4) p + (D_4 - 1) M_1 \geq 0 \quad (C.136)$$

Since $1/a^{s-v}$ and $(1 - a^{2(N-v)}) / (1 - a^{2(N-s)})$ are both greater than or equal to one for all allowable values of a , N , and $s > v$ it follows from Eq. (C.111) that D_4 is also greater than or equal to one. Next noting that that $z z'$ can only have the values of -1 and $+1$ (see Eqs. (C.71) and (C.114)) it follows that Ineq. (C.136) can be rewritten as

$$(1 + D_4) p + (D_4 - 1) M_1 \geq 0 \text{ for } z z' = -1 \quad (C.137)$$

and

$$(D_4 - 1) (M_1 - p) \geq 0 \text{ for } z z' = +1 \quad (C.138)$$

where Ineqs. (C.137) are noticed to be satisfied for all allowable values of D_4 , p , and M_1 . As a result we have that Ineq. (C.136) is satisfied for all allowable values of its parameters.

Condition 7

$$D_5 M_1 (1-zz') \geq 0 \quad (C.139)$$

From Eq.(C.112) we note that D_5 is nonnegative for all allowable values of its parameters. Furthermore, since M_1 and $1-zz'$ are also nonnegative we have that Ineq.(C.139) must be true.

Condition 8

$$v-s+1+R_1(v,s) \geq 0 \text{ for } s \geq v \quad (C.140)$$

where $R_1(v,s)$ is given by Eq.(C.123).

Ineq.(C.140) will be shown to be true by using an inductive proof. First substituting $s-1$ for v in Eq.(C.123) and also in the LHS of Ineq.(C.140) we obtain

$$(s-1)-s+1+R_1(s-1,s)=0 \quad (C.141)$$

which is seen to satisfy Ineq.(C.140). Next replacing v by $v-1$ in the LHS of Ineq.(C.140) and also using Eq. (C.122) we obtain after some algebraic manipulations

$$(v-1)-s+1+R_1(v-1,s)=v-s+1+R_1(v,s)+ \frac{(a-1)a^{2N-s-v}+(a^{-1}-1)a^{v-s+1}}{(1-a)(1-a^{2(N-s)})}-1 \quad (C.142)$$

Finally assuming $v-s+1+R_1(v,s)$ is nonnegative we have from Eq. (C.142) that $(v-1)-s+1+R_1(v-1,s)$ will be nonnegative if the inequality

$$\frac{(a-1)a^{2N-s-v}+(a^{-1}-1)a^{v-s+1}}{(1-a)(1-a^{2(N-s)})} \geq 1 \quad (C.143)$$

can be shown to hold for all allowable values of its parameters. This is now done to conclude the inductive proof of Ineq. (C.140).

To show Ineq. (C.143) to be true we first perform an algebraic manipulation of Ineq. (C.143) to obtain

$$a^{v-s} \frac{1-a^{2(N-v)}}{1-a^{2(N-s)}} \geq 1 \quad (C.144)$$

Ineq. (C.144) is seen to be true since a^{v-s} and $(1-a^{2(N-v)})/(1-a^{2(N-s)})$ are both greater than or equal to one for all allowable values of a , N , v , and s . This concludes the inductive proof of Ineq. (C.140).

Case ii (s=v)

Subtracting Eq. (C.24) from Eq. (C.109) where $s=v$ we obtain after some algebraic manipulations

$$h_{N-k}^i(p', z', v, p, k) - h_{s,j}^k = qb^2 \bar{K}_1 + r(t_k - t_k + (1-zD_2)t_k + (D_2-1)M_1 + (1-zz')p + R_2M_1(1-zz')) \quad (C.145)$$

where

$$\bar{K}_1 = \sum_{i=k+1}^s a^{2(i-k-1)} (K_1^2(i) + 2K_2(i)K_1(i)) + \sum_{i=s+1}^N a^{2(i-k-1)} (K_5^2 - \frac{1}{4}a^{2(k-s)}) \quad (C.146)$$

with

$$K_5 = (j - \frac{1}{2})a^{k-s} + M_1 \sum_{t=k}^{s-1} a^{k-t} - zt_k - zz'(M_1 \sum_{t=k+1}^{s-1} a^{k-t} + pa^{k-s}) \quad (C.147)$$

and $K_1(i)$ and $K_2(i)$ are given by Eqs. (C.118) - (C.119).

Also

$$R_2 = \frac{\sum_{i=k+2}^s a^{2(i-k-1)} \sum_{t=k+1}^{i-1} a^{s-t} + \sum_{i=s+1}^N a^{2(i-k-1)} \sum_{t=k+1}^{s-1} a^{s-t}}{\sum_{i=s+1}^N a^{2(i-k-1)}} \quad (C.148)$$

and D_2 is given by Eq. (C.60). Note that in the derivation of Eq. (C.145) we have used the equality

$$D_1 = D_2 + R_2 \quad (C.149)$$

Again noting that qb^2 , r , and $a^{2(i-k-1)}$ are nonnegative for all allowable values of their parameters we have that Eq. (C.145) can be shown to be nonnegative by demonstrating the following conditions to be true:

Condition 1

$$R_2 M_1 (1-zz') \geq 0 \quad (C.150)$$

From Eq. (C.148) we note that R_2 is nonnegative for all allowable values of its parameters. Furthermore, since M_1 and $1-zz'$ are also nonnegative for allowable values of M_1 and zz' we have that Ineq. (C.150) must be true.

Condition 2

$$(1-zz')p \geq 0 \quad (C.151)$$

Ineq. (C.151) is easily seen to be true for all allowable values of zz' and p .

Condition 3

$$|t_k| - t_k + (1-zD_2)t_k + (D_2-1)M_1 \geq 0 \quad (C.93)$$

See Condition 4 in pg. 116 for the proof of the above inequality.

Condition 4

$$\bar{K}_1 \geq 0 \quad (C.152)$$

We now proceed to show Ineq. (C.152) to be true by first considering the case where $zz'=-1$ and then the case where $zz'=+1$.

$zz'=-1$: From Eqs. (C.118),^(C.114) and (C.147) we have that when $zz'=-1$ $K_2(i)$ remains the same and $K_1(i)$ and K_5 become

$$K_1(i) = M_1 - zt_k + 2M_1 \sum_{t=k+1}^{i-1} a^{k-t} \quad (C.153)$$

and

$$K_5 = M_1 - zt_k + (j - \frac{1}{2})a^{k-s} + 2M_1 \sum_{t=k+1}^{s-1} a^{k-t} + pa^{k-s} \quad (C.154)$$

It then follows from Eqs. (C.119), (C.153), and (C.154) that

$$K_2(i) \geq 0 \text{ for } k+1 \leq i \leq s, \quad (C.155)$$

$$K_1(i) \geq 0 \text{ for } k+1 \leq i \leq s, \quad (C.156)$$

and

$$K_5 \geq \frac{1}{2}a^{k-s}, \quad (C.157)$$

since $(j - \frac{1}{2})a^{k-s}$ is greater than or equal to $\frac{1}{2}a^{k-s}$, and M_1 , a^{k-t} for $t \in (k+1, \dots, s)$, $M_1 - zt_k$, and p are all nonnegative for all allowable values of their parameters. From Eq. (C.146) and Ineqs. (C.155) - (C.157) it is easily seen that Ineq. (C.152) must be true.

(C.119),
 $zz'=+1$: From Eqs. (C.118), (C.147) we have that when $zz'=+1$
 $K_2(i)$ remains the same and $K_1(i)$ and K_5 become

$$K_1 = M_1 - zt_k \quad (C.158)$$

and

$$K_5 = M_1 - zt_k + (j - \frac{1}{2} - p)a^{k-s} \quad (C.159)$$

Next rewriting Eq. (C.146) as

$$\bar{K}_1 = \sum_{i=k+1}^s a^{2(i-k-1)} 2K_2(i)K_1 + \sum_{i=s+1}^N a^{2(i-k-1)} K_6 \quad (C.160)$$

where

$$K_6 = K_5^2 - \frac{1}{4}a^{2(k-s)} + \frac{1-a^{2(s-k)}}{a^{2(s-k)}(1-a^{2(N-s)})} K_1^2 \quad (C.161)$$

and K_5 and K_1 are defined by Eqs. (C.159) and (C.158) we have
 from Eq. (C.160) that Ineq. (C.152) can be shown to be true by demonstrating
 that $K_2(i)$ for $i \in (k+1, \dots, s)$, K_1 , and K_6 are all nonnegative.

First from Eqs. (C.119) and (C.158) we have that $K_2(i)$ for
 $i \in (k+1, \dots, s)$ and K_1 are nonnegative since $(j - \frac{1}{2})a^{k-s}$, M_1 , a^{k-t}
 for $t \in (i, \dots, s-1)$, and $M_1 - zt_k$ are all nonnegative for all allowable
 values of their parameters. Finally to show that K_6 is also nonnegative
 we do as follows:

Noticing

$$\frac{1-a^{2(s-k)}}{a^{2(s-k)}(1-a^{2(N-s)})} > \frac{1-a^{2(s-k)}}{a^{2(s-k)}} \quad (C.162)$$

for all allowable values of a , s , k , and N we substitute Eqs. (C.158)
 and (C.159) in Eq. (C.161) and also replace

$$\frac{1-a^{2(s-k)}}{a^{2(s-k)}(1-a^{2(N-s)})} \quad (C.163)$$

by $(1-a^{2(s-k)})a^{-2(s-k)}$ in Eq. (C.161) to obtain after some algebraic
 manipulations

$$K_6 > (M_1 - zt_k)^2 a^{2(k-s)} + 2(j - \frac{1}{2} - p)(M_1 - zt_k)a^{k-s} + (j - \frac{1}{2} - p)^2 a^{2(k-s)} - \frac{1}{4}a^{2(k-s)} \quad (C.164)$$

We now proceed to show that K_6 is nonnegative by showing that the RHS of Ineq. (C.164) is nonnegative first for the case where $j - \frac{1}{2} - p$ is greater than zero and then for the case where $j - \frac{1}{2} - p$ is less than zero.

Since $((j - \frac{1}{2} - p)^2 - \frac{1}{4})a^{2(k-s)}$, $2(j - \frac{1}{2} - p)(M_1 - zt_k)a^{k-s}$, and $(M_1 - zt_k)^2 a^{2(k-s)}$ are all nonnegative for all allowable values of their parameters when $j - \frac{1}{2} - p$ is greater than zero, it then follows from Ineq. (C.164) that K_6 is nonnegative when $j - \frac{1}{2} - p$ is greater than zero.

When $j - \frac{1}{2} - p$ is less than zero we have that

$$2(j - \frac{1}{2} - p)(M_1 - zt_k)a^{k-s} > 2(j - \frac{1}{2} - p)(M_1 - zt_k)a^{2(k-s)} \quad (C.165)$$

which then implies from Ineq. (C.164) that

$$K_6 > \{(M_1 - zt_k)^2 + 2(j - \frac{1}{2} - p)(M_1 - zt_k) + (j - \frac{1}{2} - p)^2 - \frac{1}{4}\}a^{2(k-s)} \quad (C.167)$$

Finally rewriting Ineq. (C.167) we have that

$$K_6 > \{(M_1 - zt_k + j - \frac{1}{2} - p)^2 - \frac{1}{4}\}a^{2(k-s)} \quad (C.168)$$

However, since $(M_1 - zt_k + j - \frac{1}{2} - p)^2$ is greater than or equal to $\frac{1}{4}$ and $a^{2(k-s)}$ is nonnegative for all allowable values of their parameters it follows that the RHS of Ineq. (C.168) is nonnegative and as a result K_6 must also be nonnegative. This concludes the proof of Ineq. (C.152).

Case iii ($s=k < v$)

Subtracting Eq. (C.24) from Eq. (C.109) we obtain after some algebraic manipulations

$$\begin{aligned} & h'_{N-k}(p', z', v, p, k) - h^k_{s, j=qb^2 \bar{K}_2} + \\ & + r(|t_k| - zt_k + (1 - zz'R_3)p + (v - k - 1 - zz'R_4)M_1) \end{aligned} \quad (C.169)$$

where

$$\bar{k}_2 = \sum_{i=k+1}^v a^{2(i-k-1)} (K_7^2(i) - \frac{1}{4}) + \sum_{i=v+1}^N a^{2(i-k-1)} (K_8^2 - \frac{1}{4}) \quad (C.170)$$

with

$$K_7(i) = j - \frac{1}{2} - z t_k - z z' M_1 \sum_{t=k+1}^{i-1} a^{k-t} \quad (C.171)$$

$$K_8 = j - \frac{1}{2} - z t_k - z z' (M_1 \sum_{t=k+1}^{v-1} a^{k-t} + p a^{k-v}) \quad (C.172)$$

Also

$$R_3 = \frac{\sum_{i=v+1}^N a^{2(i-k-1)}}{\sum_{i=k+1}^N a^{2(i-k-1)}} a^{k-v} \quad (C.173)$$

or equivalently in compact form

$$R_3 = a^{v-k} \frac{1-a^{2(N-v)}}{1-a^{2(N-k)}} \quad (C.174)$$

and

$$R_4 = \frac{\sum_{i=k+2}^v a^{2(i-k-1)} \sum_{t=k+1}^{i-1} a^{k-t} + \sum_{i=v+1}^N a^{2(i-k-1)} \sum_{i=k+1}^{v-1} a^{k-t}}{\sum_{i=k+1}^N a^{2(i-k-1)}} \quad (C.175)$$

or equivalently in compact form

$$R_4 = \frac{-a^{2N-v-k+1} + a^{2(N-k)} - a^{v-k} + a}{(1-a)(1-a^{2(N-k)})} \quad (C.176)$$

Noting that qb^2 , r , and $a^{2(i-k-1)}$ are nonnegative for all allowable values of their parameters we have that Eq. (C.169) can be shown to be nonnegative by demonstrating the following conditions to be true:

Condition 1

$$|t_k| - z t_k \geq 0 \quad (C.177)$$

The above condition is easily seen to be satisfied for all allowable values of t_k and z .

Finally assuming $v-k-1-R_4(v)$ is nonnegative we have from Eq. (C.182) that $(v+1)-k-1-R_4(v+1)$ will be nonnegative if the inequality

$$\frac{(1-a^{-1})a^{2N-v-k+1} + (1-a)a^{v-k}}{(1-a)(1-a^{2(N-k)})} \leq 1 \quad (C.183)$$

can be shown to hold for all allowable values of its parameters, this will conclude the inductive proof.

To show Ineq. (C.183) to be true we first perform an algebraic manipulation of the inequality to obtain

$$\frac{a^{v-k}(1-a^{2(N-v)})}{1-a^{2(N-k)}} \leq 1 \quad (C.184)$$

The above inequality is easily seen to be true since a^{v-k} and $(1-a^{2(N-v)})/(1-a^{2(N-k)})$ are both less than or equal to one for all allowable values of a , N , and v . This concludes the proof of Ineq. (C.179).

Condition 4

$$\bar{K}_2 \geq 0 \quad (C.185)$$

where $\bar{K}_2$ is given by Eq. (C.170).

The proof of Ineq. (C.185) will be done by first considering the case where $v=k+1$, then $v=k+2$, and finally v greater than $k+2$.

$v=k+1$: Substituting $v=k+1$ in Eqs. (C.171) ^{and (C.172)} _^ we obtain

$$K_7(k+1) = j - \frac{1}{2} - zt_k \quad (C.186)$$

and

$$K_8 = j - \frac{1}{2} - zt_k - zz' \frac{p}{a} \quad (C.187)$$

Furthermore using $v=k+1$ in Eq. (C.170) $\bar{K}_2$ becomes after some straight forward algebraic manipulations

$$\bar{K}_2 = \sum_{i=k+2}^N a^{2(i-k-1)} K_9 \quad (C.188)$$

where

$$K_9 = K_8^2 - \frac{1}{4} + \frac{1-a^2}{a^2(1-a^{2(N-k-1)})} (K_7^2(k+1) - \frac{1}{4}) \quad (C.189)$$

and K_8 and $K_7(k+1)$ are given by Eqs. (C.186) - (C.187).

Since $a^{2(i-k-1)}$ is nonnegative for all allowable values of its parameters it follows from Eq. (C.188) that $\bar{K}_2$ will be nonnegative if we can show that K_9 is nonnegative for all allowable values of its parameters. K_9 is now shown to be nonnegative as follows:

We first note from Eq. (C.186) that the lowest value that $K_7^{2(k+1)}$ can have is $\frac{1}{4}$ therefore it must follow that $K_7^{2(k+1)} - \frac{1}{4}$ is nonnegative. Furthermore, since $1-a^2$ and a^2 are nonnegative and $(1-a^{2(N-k-1)})^{-1}$ is greater than or equal to one for all allowable values of a , N , and k it follows that

$$\frac{1-a^2}{a^2(1-a^{2(N-k-1)})} (K_7^{2(k+1)} - \frac{1}{4}) > \frac{1-a^2}{a^2} (K_7^{2(k+1)} - \frac{1}{4}) \quad (C.190)$$

for all allowable values of its parameters. Next making use of Eq. (C.189) and Ineq. (C.190) we have that

$$K_9 > K_9' \quad (C.191)$$

where

$$K_9' = K_8^2 - \frac{1}{4} + \frac{1-a^2}{a^2} (K_7^{2(k+1)} - \frac{1}{4}) \quad (C.192)$$

and (C.187).

with K_8 and $K_7^{2(k+1)}$ given by Eqs. (C.186). Now we proceed to show K_9' to be nonnegative which in turn implies K_9 to be nonnegative as seen from Ineq. (C.191).

First using Eqs. (C.186) and (C.187) in Eq. (C.192) we obtain

$$K_9' = \frac{(j - \frac{1}{2} - zt_k)^2}{a^2} - 2zz' \frac{P}{a} (j - \frac{1}{2} - zt_k) + \frac{P^2}{a^2} - \frac{1}{4a^2} \quad (C.193)$$

Next we show that K_9' is nonnegative by first considering the case where $-2zz' \frac{P}{a} (j - \frac{1}{2} - zt_k)$ is positive and then the case where $-2zz' \frac{P}{a} (j - \frac{1}{2} - zt_k)$ is nonpositive.

Since

$$\frac{(j - \frac{1}{2} - zt_k)^2}{a^2} - \frac{1}{4a^2} > 0 \quad (C.194)$$

and $\frac{p^2}{a^2}$ is nonnegative for all allowable values of j, z, t_k, a , and p it follows from Eq. (C.193) that K_9' is nonnegative when $-2zz' \frac{p}{a} (j - \frac{1}{2} - zt_k)$ is positive.

Finally since

$$-2zz' \frac{p}{a} (j - \frac{1}{2} - zt_k) \geq -2zz' \frac{p}{a^2} (j - \frac{1}{2} - zt_k) \quad (C.195)$$

when $-2zz' \frac{p}{a} (j - \frac{1}{2} - zt_k)$ is nonpositive we can use Ineq. (C.195) and Eq. (C.193) to derive the inequality

$$K_9' \geq \frac{(j - \frac{1}{2} - zt_k - zz'p)^2 - \frac{1}{4}}{a^2} \quad (C.196)$$

where the RHS of the inequality is easily seen to be nonnegative for all allowable values of j, z, t_k, zz', a , and p . As a result it must follow that K_9' is nonnegative and also K_9 is nonnegative when $-2zz' \frac{p}{a} (j - \frac{1}{2} - zt_k)$ is nonpositive. This concludes the proof of Condition 4 for the case where $v=k+1$.

$v=k+2$: Substituting $v=k+2$ in Eqs. (C.171)-(C.172) we obtain

$$K_7(i) = \begin{cases} j - \frac{1}{2} - zt_k & , i=k+1 \\ j - \frac{1}{2} - zt_k - zz' \frac{M_1}{a} & , i=k+2 \end{cases} \quad (C.197)$$

and

$$K_8 = j - \frac{1}{2} - zt_k - zz' \left(\frac{M_1}{a} + \frac{p}{a^2} \right) \quad (C.198)$$

The above expressions for $K_7(i)$ and K_8 are then used in Eq. (C.170) to yield

$$\bar{K}_2 = K_{10} + \sum_{i=k+3}^N a^{2(i-k-1)} K_{11} \quad (C.199)$$

where

$$K_{10} = K_7^2(k+2) - \frac{1}{4} + \frac{1-a^2}{a^2(1-a^{2(N-k-1)})} (K_7^2(k+1) - \frac{1}{4}) \quad (C.200)$$

and

where

$$-2zz' \left(\frac{M_1}{a} + \frac{p}{a^2} \right) \left(j - \frac{1}{2} - zt_k \right) \geq 0 \quad (C.206)$$

and then for

$$-2zz' \left(\frac{M_1}{a} + \frac{p}{a^2} \right) \left(j - \frac{1}{2} - zt_k \right) < 0 \quad (C.207)$$

When Ineq. (C.206) holds it can be easily seen from Ineq. (C.194) and Eq. (C.205) that K_{11} must be nonnegative for all allowable values of its parameters. This in turn implies from Ineq. (C.204) that K_{11} is nonnegative for all allowable values of its parameters as long as Ineq. (C.206) is satisfied. On the other hand, when Ineq. (C.207) holds we have that

$$-2zz' \left(\frac{M_1}{a} + \frac{p}{a^2} \right) \left(j - \frac{1}{2} - zt_k \right) > -2zz' \frac{M_1+p}{a^2} \left(j - \frac{1}{2} - zt_k \right) \quad (C.208)$$

and

$$\left(\frac{M_1}{a} + \frac{p}{a^2} \right)^2 > \left(\frac{M_1+p}{a^2} \right)^2 \quad (C.209)$$

As a result it must follow from Eq. (C.205) and Ineqs. (C.208)-(C.209) that

$$K_{11} \geq \frac{\left(j - \frac{1}{2} - zt_k - zz' (M_1+p) \right)^2 - \frac{1}{4}}{a^2} \quad (C.210)$$

Since the RHS of Ineq. (C.210) is nonnegative for all allowable values of z, a, j, z, t_k, M_1 , and p it follows that K_{11} is nonnegative and in turn from Eq. (C.204) we have that K_{11} is nonnegative for all allowable values of its parameters. This concludes the proof of Condition 4 for the case where $v=k+2$.

$v \geq k+2$: When $v \geq k+2$ we have from Eqs. (C.171)-(C.172) that

$$K_7(i) = \begin{cases} j - \frac{1}{2} - zt_k & , i=k+1 \\ j - \frac{1}{2} - zt_k - zz' \frac{M_1}{a} & , i=k+2 \\ j - \frac{1}{2} - zt_k - zz' \left(\frac{M_1}{a} + \frac{M_1}{a^2} + M_1 \sum_{t=k+3}^{i-1} a^{k-t} \right) & , k+3 \leq i \leq v \end{cases} \quad (C.211)$$

$$K_{11} = K_8^2 - \frac{1}{4} + \frac{1-a^2}{a^2(1-a^{2(N-k-1)})} (K_7^2(k+1) - \frac{1}{4}) \quad (C.201)$$

with $K_7(k+1)$, $K_7(k+2)$, and K_8 defined by Eqs. (C.197) and (C.198).

Since $a^{2(i-k-1)}$ is nonnegative for all allowable values of its parameters it follows from Eq. (C.199) that $\bar{K}_2$ is nonnegative when K_{10} and K_{11} are both nonnegative. We now show that K_{10} and K_{11} are nonnegative.

Using Ineq. (C.190), Eq. (C.200), and Eq. (C.197) we have that

$$K_{10} \geq K_{10}' \quad (C.202)$$

where

$$K_{10}' = \frac{(j - \frac{1}{2} - zt_k)^2}{a^2} - 2zz' \frac{M_1}{a} (j - \frac{1}{2} - zt_k) + \frac{M_1^2}{a^2} - \frac{1}{4a^2} \quad (C.203)$$

Note that the above result is derived using an approach similar to the one used to derive Eqs. (C.191)-(C.193) for the case where $v=k+1$.

Next making a comparison of Eqs. (C.193) and (C.203) we note that

Eq. (C.193) is similar to Eq. (C.203) for $p=M_1$, therefore, since it was shown earlier for the case where $v=k+1$ that Eq. (C.193) is nonnegative for all allowable values of its parameters including $p=M_1$, it must follow that K_{10}' as given by Eq. (C.203) must be nonnegative. Finally since K_{10}' is nonnegative we have from Ineq. (C.202) that K_{10} is nonnegative for all allowable values of its parameters.

Using similar arguments as those used earlier for K_{10} we have that

$$K_{11} \geq K_{11}' \quad (C.204)$$

where

$$K_{11}' = \frac{(j - \frac{1}{2} - zt_k)^2}{a^2} - 2zz' \left(\frac{M_1}{a} + \frac{p_2}{a^2} \right) (j - \frac{1}{2} - zt_k) + \left(\frac{M_1}{a} + \frac{p_2}{a^2} \right)^2 - \frac{1}{4a^2} \quad (C.205)$$

We next show that K_{11}' is nonnegative by first considering the case

and

$$K_8 = j - \frac{1}{2} - zt_k - zz' \left(\frac{M_1}{a} + \frac{M_1}{a^2} + \sum_{t=k+3}^{v-1} a^{k-t} + pa^{k-v} \right) \quad (C.212)$$

Furthermore, partially expanding Eq. (C.170) we obtain

$$\bar{K}_2 = K_{12} + \sum_{i=k+3}^v a^{2(i-k-1)} (K_7^2(i) - \frac{1}{4}) + \sum_{i=v+1}^N a^{2(i-k-1)} (K_8^2 - \frac{1}{4}) \quad (C.213)$$

where

$$K_{12} = K_7^2(k+1) - \frac{1}{4} + a^2 (K_7^2(k+2) - \frac{1}{4}) \quad (C.214)$$

with $K_7(i)$ and K_8 given by Eqs. (C.211) and (C.212). It then follows from Eq. (C.213) that $\bar{K}_2$ is nonnegative if it can be shown that K_{12} is nonnegative and that both $K_7^2(i)$ for $i \in (k+3, \dots, v)$ and K_8^2 are not less than $\frac{1}{4}$.

Since

$$\left| zz' \left(\frac{M_1}{a} + \frac{M_1}{a^2} + \sum_{t=k+3}^{v-1} a^{k-t} + pa^{k-v} \right) \right| \geq \left| j - \frac{1}{2} - zt_k \right| + \frac{1}{2} \quad (C.215)$$

for all allowable values of zz' , M_1 , a , v , k , p , z , j , and

t_k we have from Eq. (C.212) that

K_8^2 is greater than or equal to $\frac{1}{4}$ for all allowable values of its parameters. On the other hand, we have that

$$\left| j - \frac{1}{2} - zt_k - zz' \left(\frac{M_1}{a} + \frac{M_1}{a^2} + M_1 \sum_{t=k+3}^{i-1} a^{k-t} \right) \right| \geq \frac{1}{2} \quad (C.218)$$

for all allowable values of j , z , t_k , zz' , a , M_1 , i , and

k , as a result it follows

from Eq. (C.211) that $K_7^2(i)$ for $i \in (k+3 \leq i \leq v)$ is greater than or equal to $\frac{1}{4}$ all allowable values of its parameters.

Finally, we show that K_{12} is nonnegative by using

Eq. (C.211) in Eq. (C.214) to obtain

$$K_{12} = (j - \frac{1}{2} - zt_k)^2 - 2zz'aM_1(j - \frac{1}{2} - zt_k) + M_1^2 - \frac{1}{4} + ((j - \frac{1}{2} - zt_k)^2 - \frac{1}{4})a^2 \quad \text{B.220}$$

Eq. (C.220) is first shown to be nonnegative for the case where

$$-2zz'aM_1(j - \frac{1}{2} - zt_k) > 0 \quad (\text{C.221})$$

and then for the case where

$$-2zz'aM_1(j - \frac{1}{2} - zt_k) < 0 \quad (\text{C.222})$$

When Ineq. (C.221) holds we have that since Ineq. (C.79) also holds it follows from Eq. (C.220) that K_{12} is nonnegative for all allowable values of its parameters. On the other hand, when Ineq. (C.222) holds we have that

$$-2zz'aM_1(j - \frac{1}{2} - zt_k) > -2zz'M_1(j - \frac{1}{2} - zt_k) \quad (\text{C.223})$$

which clearly implies from Eq. (C.220) that

$$K_{12} > (j - \frac{1}{2} - zt_k - M_1)^2 - \frac{1}{4} + ((j - \frac{1}{2} - zt_k)^2 - \frac{1}{4})a^2 \quad (\text{C.224})$$

The RHS of Eq. (C.224) is nonnegative for all values of j , z , t_k and M_1 , therefore it follows that K_{12} is nonnegative for all allowable values of its parameters. This concludes the proof of Ineq. (C.185) for the case where $v > k+2$.

Case iv ($k < s < v$)

Subtracting Eq. (C.24) from Eq. (C.109) for the case where $k < s < v$ we have that

$$\begin{aligned} h'_{N-k}(p', z', v, p, k) - h^k_{s,j} &= qb^2 \bar{K}_3 + \\ &+ r(|t_k| - t_k + (1 - zD_2)t_k + (D_2 - 1)M_1 + (1 - zz'D_4)p + R_5M_1) \end{aligned} \quad (\text{C.225})$$

where

$$\begin{aligned} \bar{K}_3 &= \sum_{i=k+1}^s a^{2(i-k-1)} (K_1^2(i) + 2K_2(i)K_1(i)) + \\ &+ \sum_{i=s+1}^v a^{2(i-k-1)} (K_{13}^2(i) - \frac{1}{4}a^{2(k-s)}) + \sum_{i=v+1}^N a^{2(i-k-1)} (K_4^2 - \frac{1}{4}a^{2(k-s)}) \end{aligned} \quad (\text{C.226})$$

with

$$K_{13}(i) = (j - \frac{1}{2})a^{k-s+M_1} \sum_{t=k}^{s-1} a^{k-t} - zt_k - zz'M_1 \sum_{t=k+1}^{i-1} a^{k-t} \quad (C.227)$$

and $K_1(i)$, $K_2(i)$, and K_4 are given by Eqs. (C.115)-(C.119) and (C.121).

Also D_2 and D_4 are given by Eqs. (C.60) and (C.111) and

$$R_5 = v-s + \frac{\sum_{i=k+2}^s a^{2(i-k-1)} \sum_{t=k+1}^{i-1} a^{s-t} + \sum_{i=s+1}^N a^{2(i-k-1)} \sum_{t=k+1}^{s-1} a^{s-t}}{\sum_{i=s+1}^N a^{2(i-k-1)}} - zz' \frac{\sum_{i=k+2}^v a^{2(i-k-1)} \sum_{t=k+1}^{i-1} a^{s-t} + \sum_{i=v+1}^N a^{2(i-k-1)} \sum_{t=k+1}^{v-1} a^{s-t}}{\sum_{i=s+1}^N a^{2(i-k-1)}} \quad (C.228)$$

Noting that qb^2 and r are nonnegative for all allowable values of their parameters, we have that Eq. (C.225) can be shown to be nonnegative by demonstrating the following conditions to be true

Condition 1

$$|t_k| - t_k + (1 - zD_2)t_k + (D_2 - 1)M_1 > 0 \quad (C.229)$$

where D_2 is given by Eq. (C.60).

See Condition 4 in pg. 116 for the proof of the above inequality.

Condition 2

$$(1 - zz'D_4)p > 0 \quad (C.230)$$

Since a^{v-s} and $(1 - a^{2(N-v)})/(1 - a^{2(N-s)})$ are both less than or equal to one for all allowable values of a , N , v , and s (note $s < v$) we have from Eq. (C.111) that D_4 is less than or equal to one. It then follows that Ineq. (C.230) is satisfied for all allowable values of zz' , D_4 , and p since p and $1 - zz'D_4$ are both nonnegative.

Condition 3

$$R_5(v, s) > 0 \text{ for } s < v \quad (C.231)$$

where $R_5(v, s)$ is given by Eq. (C.228).

The proof of Ineq. (C.231) will be done by first considering the case where $zz' = -1$ and then the case where $zz' = +1$.

$zz' = -1$: When $zz' = -1$ we can easily see from Eq. (C.228) that $R_5(v, s)$ is nonnegative for all allowable values of its parameters (note that $v-s$ is positive since $s < v$).

$zz' = +1$: When $zz' = +1$ Eq. (C.228) can be expressed compactly as follows

$$R_5(v, s) = v - s + \frac{-1 - a^{2(N-s)+1} + a^{v-s} + a^{2N-v-s+1}}{(1-a)(1-a^{2(N-s)})} \quad (\text{C.232})$$

We now proceed to show that Eq. (C.232) is nonnegative for all allowable values of its parameters by using an inductive proof.

First introducing $v = s+1$ in Eq. (C.232) we have that $R_5(s+1, s) = 0$, as a result we have that Ineq. (C.231) is satisfied for $v = s+1$.

Next substituting $v+1$ for v in Eq. (C.232) we obtain

$$R_5(v+1, s) = R_5(v, s) + 1 - \frac{(a-1)a^{2N-s-v} + (a^{-1}-1)a^{v-s+1}}{(1-a)(1-a^{2(N-s)})} \quad (\text{C.233})$$

It then follows that assuming $R_5(v, s)$ is nonnegative $R_5(v+1, s)$ will also be nonnegative if we can show that inequality

$$\frac{(a-1)a^{2N-s-v} + (a^{-1}-1)a^{v-s+1}}{(1-a)(1-a^{2(N-s)})} < 1 \quad (\text{C.234})$$

is true. To show Ineq. (C.234) to be true we perform an algebraic manipulation of the inequality to obtain

$$a^{v-s} \frac{1-a^{2(N-v)}}{1-a^{2(N-s)}} < 1 \quad (\text{C.235})$$

Ineq. (C.235) is seen to hold since a^{v-s} and $(1-a^{2(N-v)})/(1-a^{2(N-s)})$ are both less than one for all allowable values of a , N , v , and s . This concludes the proof of Ineq. (C.231).

Condition 4

$$\bar{K}_3 \geq 0 \quad (C.236)$$

We now proceed to prove Ineq.(C.236) to be true by first considering the case where $zz'=-1$ and then the case where $zz'=+1$.

$zz'=-1$: Letting $zz'=-1$ in Eqs.(C.118), ^(C.119), (C.121), and (C.227) we have that $K_2(i)$ does not change and

$$K_1(i) = M_1 - zt_k + 2M_1 \sum_{t=k+1}^{i-1} a^{k-t} \quad (C.237)$$

$$K_{13}(i) = M_1 - zt_k + (j - \frac{1}{2})a^{k-s} + 2M_1 \sum_{t=k+1}^{s-1} a^{k-t} + M_1 \sum_{t=s}^{i-1} a^{k-t} \quad (C.238)$$

and

$$K_4 = M_1 - zt_k + (j - \frac{1}{2})a^{k-s} + 2M_1 \sum_{t=k+1}^{s-1} a^{k-t} + M_1 \sum_{t=s}^{v-1} a^{k-t} + pa^{k-v} \quad (C.239)$$

We now show that $\bar{K}_3$ (see Eq.(C.226)) is nonnegative by showing that $K_1(i)$ and $K_2(i)$ are nonnegative for $i \in (k+1, \dots, s)$ and that $K_{13}^2(i)$ for $i \in (s+1, \dots, v)$ and K_4^2 are both greater than or equal to $\frac{1}{4}a^{2(k-s)}$.

First from Eq.(C.119) we have that $K_2(i)$ is nonnegative since $j - \frac{1}{2}$, a^{k-s} , and

$$M_1 \sum_{t=i}^{s-1} a^{k-t} \quad (C.240)$$

are nonnegative for all allowable values of their parameters. On the other hand, we have that $K_1(i)$ as seen from Eq.(C.237) must also be nonnegative since $M_1 - zt_k$, M_1 , and

$$\sum_{t=k+1}^{i-1} a^{k-t} \quad (C.241)$$

are nonnegative for all allowable values of their parameters.

Finally since $(j - \frac{1}{2})a^{k-s}$ is greater than or equal to $\frac{1}{2}a^{k-s}$ and ^{and (C.239)} all the other terms of Eqs.(C.238) _{values} are nonnegative (including the term $M_1 - zt_k$) for all the allowable _{values} of their parameters it follows that both $K_{13}^2(i)$ and K_4^2 are greater than or equal to $\frac{1}{4}a^{2(k-s)}$.

(C.119)

zz'=+1: Letting $zz'=+1$ in Eqs. (C.118), (C.121), and (C.227)

we have that $K_2(i)$ does not change and

$$K_1(i) = M_1 - zt_k \quad (C.242)$$

$$K_{13}(i) = M_1 - zt_k + (j - \frac{1}{2})a^{k-s-M_1} \sum_{t=s}^{i-1} a^{k-t} \quad (C.243)$$

and

$$K_4 = M_1 - zt_k + (j - \frac{1}{2})a^{k-s-M_1} \sum_{t=s}^{v-1} a^{k-t} - pa^{k-v} \quad (C.244)$$

We next show that Ineq. (C.236) holds for $zz'=+1$ by first considering the case where $M_1 - zt_k = 0$, then $M_1 - zt_k = 1$, and finally $M_1 - zt_k \geq 2$.

When $M_1 - zt_k = 0$ we show the desired result by noticing that Eqs. (C.242) - (C.244) become

$$K_1 = 0 \quad (C.245)$$

$$K_{13}(i) = \begin{cases} (j - \frac{1}{2} - M_1)a^{k-s} & , i=s+1 \\ (j - \frac{1}{2} - M_1)a^{k-s-M_1} \sum_{t=s+1}^{i-1} a^{k-t} & , s+1 \leq i \leq v \end{cases} \quad (C.246)$$

and

$$K_4 = (j - \frac{1}{2})a^{k-s-M_1} \sum_{t=s}^{v-1} a^{k-t} - pa^{k-v} \quad (C.247)$$

It then follows from Eqs. (C.245), (C.246), and (C.247) that K_1 is zero, $K_{13}^2(i)$ for $i \in (s+1, \dots, v)$ and K_4^2 for all allowable values of their parameters are greater than or equal to $\frac{1}{4}a^{2(k-s)}$. Also from Eq. (C.119) we have that $K_2(i)$ is nonnegative for all allowable values of its parameters.

All of the previously mentioned properties for K_1 , $K_2(i)$, K_{13}^2 , and K_4^2 clearly show from Eq. (C.226) that $\bar{K}_3$ is nonnegative as desired.

For $M_1 - zt_k = 1$ we show the desired result by noticing that Eqs. (C.242) - (C.244) become

$$K_1=1 \quad (C.248)$$

$$K_{13}(i)=\begin{cases} 1+(j-\frac{1}{2}-M_1)a^{k-s} & , i=s+1 \\ 1+(j-\frac{1}{2}-M_1)a^{k-s}-M_1a^{k-s-1}-M_1\sum_{t=s+2}^{i-1}a^{k-t} & , s+2\leq i\leq v \end{cases} \quad (C.249)$$

$$K_4=1-pa^{k-v}+(j-\frac{1}{2}-M_1)a^{k-s}-M_1\sum_{t=s+1}^{v-1}a^{k-t} \quad (C.250)$$

and (C.249)

Eqs. (C.248) are then used along with Eq. (C.119) in Eq. (C.226) to obtain

$$\begin{aligned} \bar{K}_3 = & 2((j-\frac{1}{2})a^{k-s}+M_1\sum_{t=k+1}^{s-1}a^{k-t})+ \\ & + \sum_{i=k+2}^s a^{2(i-k-1)}(1+2K_2(i))+(1+(j-\frac{1}{2}-M_1)a^{k-s})^2a^{2(s-k)}+\frac{3}{4}+ \\ & + \sum_{i=s+2}^v a^{2(i-k-1)}(K_{13}^2(i)-\frac{1}{4}a^{2(k-s)})+\sum_{i=v+1}^N a^{2(i-k-1)}(K_4^2-\frac{1}{4}a^{2(k-s)}) \end{aligned} \quad (C.251)$$

with $K_{13}(i)$ and K_4 given by Eqs. (C.249). A close investigation of Eq. (C.251) reveals that $\bar{K}_3$ is nonnegative for all allowable values of its parameters if it can be shown that $K_2(i)$ for $i \in (k+2, \dots, s)$ is nonnegative and $K_{13}^2(i)$ for $i \in (s+2, \dots, v)$ and K_4^2 are both greater than or equal to $\frac{1}{4}a^{2(k-s)}$. From Eq. (C.119) we have that $K_2(i)$ for $i \in (k+2, \dots, s)$ is nonnegative for all allowable of its parameters. Finally, since

$$\begin{aligned} \text{the expressions } & (j-\frac{1}{2}-M_1)a^{k-s} + \frac{1}{2}a^{k-s} ; -M_1\sum_{t=s+1}^{v-1}a^{k-t} ; \\ & -M_1\sum_{t=s+2}^{i-1}a^{k-t} \quad \text{for } i=s+2, \dots, v; \end{aligned}$$

$1-M_1a^{k-s-1}$, and $1-pa^{k-v}$ are nonpositive

for all allowable values of their parameters it follows from and (C.250) for $i \in (s+2, \dots, v)$

Eqs. (C.249) that $K_{13}^2(i)$ and K_4^2 are both greater than or equal to $\frac{1}{4}a^{2(k-s)}$. This concludes the proof of Ineq. (C.236) for the case

where $M_1-zt_k=1$.

Finally for the case where $M_1-zt_k \geq 2$ we have that making use of Eqs. (C.119) and (C.242) in Eq. (C.226) we obtain after some algebraic manipulations

$$\begin{aligned} \bar{K}_3 = & \sum_{i=k+2}^8 a^{2(i-k-1)} {}_2M_1 \sum_{t=i}^{s-1} a^{k-t} (M_1 - zt_k) + \sum_{i=s+1}^v a^{2(i-k-1)} K_{13}^2(i) + \\ & + \sum_{i=v+1}^N a^{2(i-k-1)} K_4^2 + \sum_{i=s+1}^N a^{2(i-k-1)} K_{14} \end{aligned} \quad (C.254)$$

where

$$K_{14} = a^{2(k-s)} \left\{ \frac{1-a^{2(s-k)}}{1-a^{2(N-s)}} \left[(M_1 - zt_k)^2 + 2(j - \frac{1}{2}) a^{k-s} (M_1 - zt_k) \right] - \frac{1}{4} \right\} \quad (C.255)$$

from Eq. (C.254)

It is then noted that $\bar{K}_3$ is nonnegative if it can be shown that K_{14} is nonnegative; this is the case since all the other terms of Eq. (C.254) are clearly seen to be nonnegative for all allowable values of their parameters. We now proceed to show that K_{14} is nonnegative for all allowable values of its parameters.

Since $(1-a^{2(N-s)})^{-1}$ is larger than or equal to one it follows from Eq. (C.255) that

$$K_{14} \geq K_{14}' \quad (C.256)$$

where

$$K_{14}' = a^{2(k-s)} \left\{ (1-a^{2(s-k)}) \left[(M_1 - zt_k)^2 + 2(j - \frac{1}{2}) a^{k-s} (M_1 - zt_k) \right] - \frac{1}{4} \right\} \quad (C.257)$$

Furthermore we have that

$$(1-a^{2(s-k)}) \left[(M_1 - zt_k)^2 + 2(j - \frac{1}{2}) a^{k-s} (M_1 - zt_k) \right] \geq (1-a^{2(s-k)}) (4+2a^{k-s}) \quad (C.258)$$

for all allowable values of $M_1 - zt_k$ and j (note that $M_1 - zt_k$ is assumed to be greater than or equal to 2).

From Inequality (C.256) and (C.258) we then have that

$$K_{14} > a^{2(k-s)} [(1-a^{2(s-k)})(4+2a^{k-s}) - 1/4] \quad (C.259)$$

where we now verify that the RHS of Inequality (C.259) is nonnegative for all allowable values of its parameters. First, we substitute z for a^{s-k} in the RHS of (C.259) to obtain after some algebraic manipulations

$$K_{14} > (-4z^3 - z^2 + 15z/4 + 2)/z^3 \quad (C.260)$$

where

$$z = a^{s-k}. \quad (C.260a)$$

It then follows that for

$$0 < z < 0.98 \quad (C.260b)$$

the two factors on the RHS of Inequality (C.260), i.e., $1/z^3$ and $-4z^3 - z^2 + 15z/4 + 2$, are nonnegative: note that the proof for the factor $-4z^3 - z^2 + 15z/4 + 2$ is easily accomplished by plotting $-4z^3 - z^2 + 15z/4 + 2$ vs. z for z between 0 and 0.98.

Making use of Equation (C.260a) it is finally noted that Inequality (C.260b) will be satisfied for $s > k$ and values of a between 0 and 0.98. As a result we have that K_{14} is nonnegative for all possible values of s , N , $M_1 - zt_k$, j and a . This ends the proof of Theorem C.1.

Theorem C.2

$$P_2(U_{K,N-1}, k) / U_{K,N-1} = \{M_1, \dots, M_1\}$$

$$\geq P_2(U_{K,N-1}, k)$$

$$\geq P_2(U_{K,N-1}, k) / U_{K,N-1} = \{-M_1, \dots, -M_1\} \quad (C.261)$$

for all $U_{K,N-1} \in S_2(k)$, where $S_2(k)$ is given by Equation (C.12).

Proof of Theorem C.2

Since $q, a, b, a^{2(i-k-1)}$, and a^{k-t} are all nonnegative for all their allowable values, it readily follows from Equations (3.4.1.11) that $P_2(U_{K,N-1}, k)$ has its smallest value when

$$U_{K,N-1} = (-M_1, \dots, -M_1) \quad (C.262)$$

and its largest value when

$$U_{K,N-1} = (M_1, \dots, M_1). \quad (C.263)$$

As a result, we have that Inequality (C.261) is true for all $U_{K,N-1} \in S_2(k)$. This ends the proof of Theorem C.2 and the proof of Part 2 of the overall proof.

Part 3. Next, with the aid of the flowchart shown on

Figure C.6, we present an outline of the proof for Part 3.

Outline of proof

Part 3.1. In this subpart, we show that the I Stages ROLIMPs control law for the process Equation (3.4.1.1) and performance criterion Equation (3.4.1.5) is given by Equations (3.4.1.7)-(3.4.1.15).

Part 3.2. In this subpart, using the results of Part 1 we note that the I Stages ROLIMPs control law is an optimum control law for all $I \in (1, \dots, N)$ and $k \in (N-I, \dots, N-1)$.

Part 3.3. In this last subpart, using the results of Part 2 we show that the I ROLIMPs control law is an optimum control law for all $I \in (1, \dots, N)$ and $k \in (0, \dots, N-I-1)$.

Proof of Part 3

Part 3.1. The I Stages ROLIMPs control law for the process Equation (3.4.1.1) and performance criterion Equation (3.4.1.5) is readily derived from the results of Part 1 and the general definition of an I Stages ROLIMPs control law given in Section 3.3.3. The derivation is as follows:

First, from Equation (3.3.3.2) for Condition (3.3.3.1), and Equations (C.1)-(C.3) and (3.4.1.10)-(3.4.1.12) it is verified that for all $I \in (1, \dots, N)$ and $k \in (N-I, \dots, N-1)$ the I Stages ROLIMPs control law is given by Equations (3.4.1.7)-(3.4.1.12).

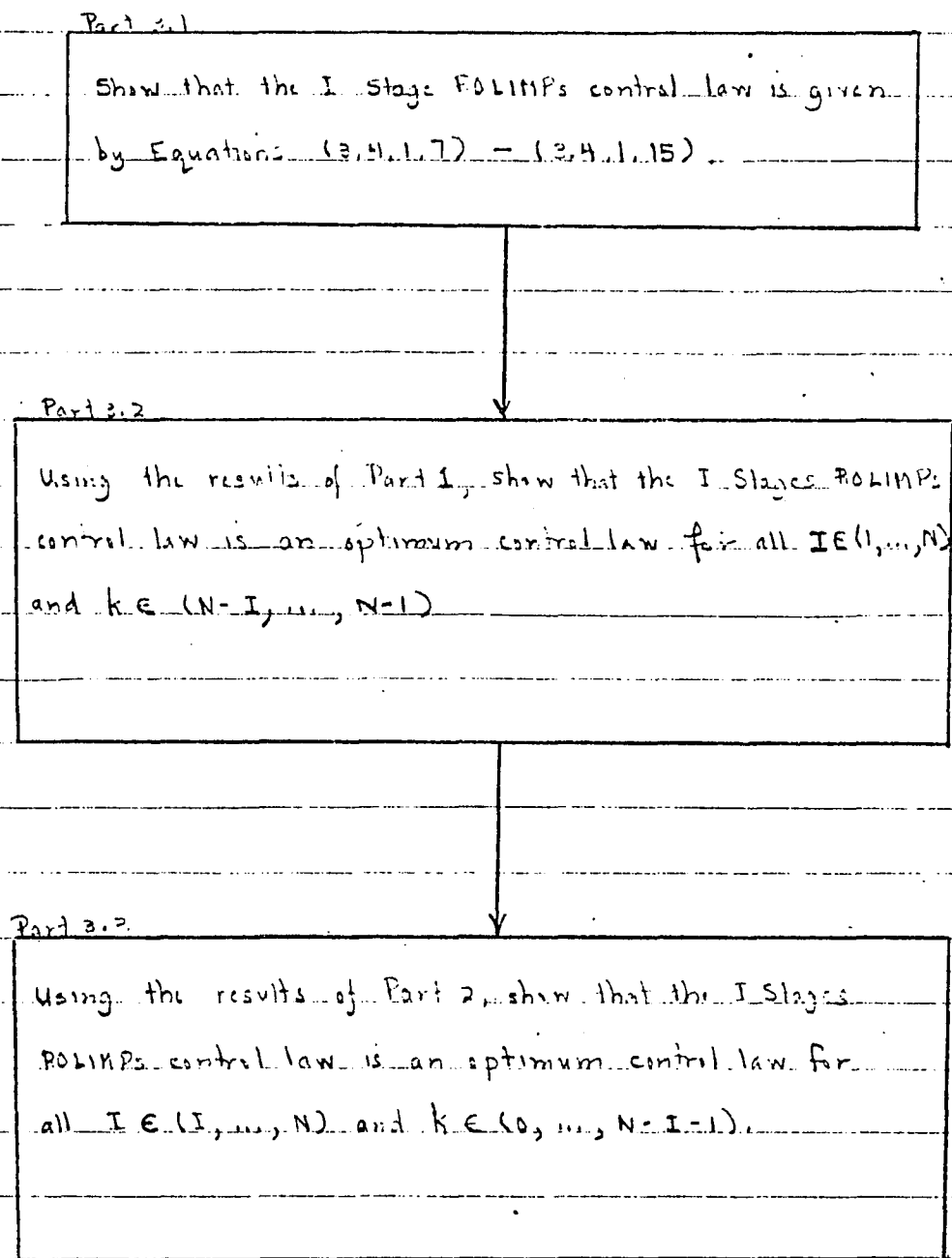


Figure C.6 Flow Diagram of the Proof for Part 3.

Second, from Equations (3.3.3.4) and (3.3.3.5) for Condition (3.3.3.3), it is noted that we need the value of d for the set $S(k, I, d)$ where d is as defined for Equation (3.3.1.1). We start the derivation of d by noting that the performance criterion Equation (3.4.1.5) reflects the desire to have the final state of the process $x(N)$ be close to zero. As a result, we have from the process Equation (3.4.1.1) that the final control $u(N-1)$ that satisfies the desired terminal conditions on $x(N)$, is given by

$$u(N-1)=0,$$

provided the process is assumed to be in the steady state when it reaches stage $N-1$ so that it can be assumed that $x(N-1)$ is also close to zero: note from (3.4.1.2) that the space of $u(N-1)$ contains the value of zero. It then follows from the previous discussion and the definition of d , that

$$d=0 \quad (C.264)$$

provided the process is assumed to be at the steady state when it reaches stage $N-1$.

Third, use Equations (C.264) and (3.3.1.2) and the space of $u(k)$ given by (3.4.1.2), in Equation (3.3.1.1) to obtain Equation (3.4.1.15).

Fourth and last, from Equations (3.3.3.4) and (3.3.3.5) for Condition (3.3.3.3), the previously established Equation

(3.4.1.15), and Equations (3.4.1.10)-(3.4.1.12) which were derived in Part 1, it is verified that for all $I \in (1, \dots, N)$ and $k \in (0, \dots, N-I-1)$ the I Stages ROLIMPs control law is given by Equations (3.4.1.13), (3.4.1.14), (3.4.1.15), and (3.4.1.10)-(3.4.1.12). This ends the proof of Part 3.1.

Part 3.2. A comparison of Equations (C.1)-(C.3) and (3.4.1.7)-(3.4.1.9) reveals that the I Stages ROLIMPs control law for all $I \in (1, \dots, N)$ and $k \in (N-I, \dots, N-1)$ is an optimum control law.

Part 3.3. In Part 2, using an inductive proof, we established that Equations (C.1) and (C.5) are equivalent, and in addition that the plot of $h_{N-k}^T(x(k), U_{k, N-1}^*, k)$ vs. $x(k)$ is as shown on Figure C.4. In this last part of the overall proof we show that when the stage k of the process is such that $k \in (0, \dots, N-I-1)$, the I Stages ROLIMPs control law, $u^T(x(k))$, for all $I \in (1, \dots, N)$ (see Equations (3.4.1.13)-(3.4.1.15) and (3.4.1.10)-(3.4.1.12)) is an optimum control law. Next we present with the aid of the flow-chart shown in Figure C.7 an outline of the proof for Part 3.3.

Outline of Proof

Part 3.3.1. In this part, we show that the plot of $u^*(x(k))$ vs. $x(k)$ that is derived from Figure C.4 is as shown on Figure C.8.

using plot of $h'_{N-k}(x(k), u^*_{k, N-1}, k)$ vs. $x(k)$ shown in Figure C.4 demonstrate that $u^*(x(k))$ vs. $x(k)$ is given by Figure C.8.

Part 3.3.2

Show that the plot of $f'^{RI}_{N-k}(x(k))$ vs. $x(k)$ where $f'^{RI}_{N-k}(x(k))$ is given by Equation (C.265) is as shown on Figure C.4.

Part 3.3.3.

Using plot of $f'^{RI}_{N-k}(x(k))$ vs. $x(k)$ shown on Figure C.4 show that $u^*(x(k))$ vs. $x(k)$ is also given by Figure C.8.

Part 3.3.4

Show that Equations (C.265) and (3.4.1.14) are equivalent for all $I \in (1, \dots, N)$ and $k \in (0, \dots, N-I-1)$ to end the proof of Part 3.3, Part 3, the overall proof, and this appendix.

Figure C.7 Flow Diagram of the Proof for Part 3.3.

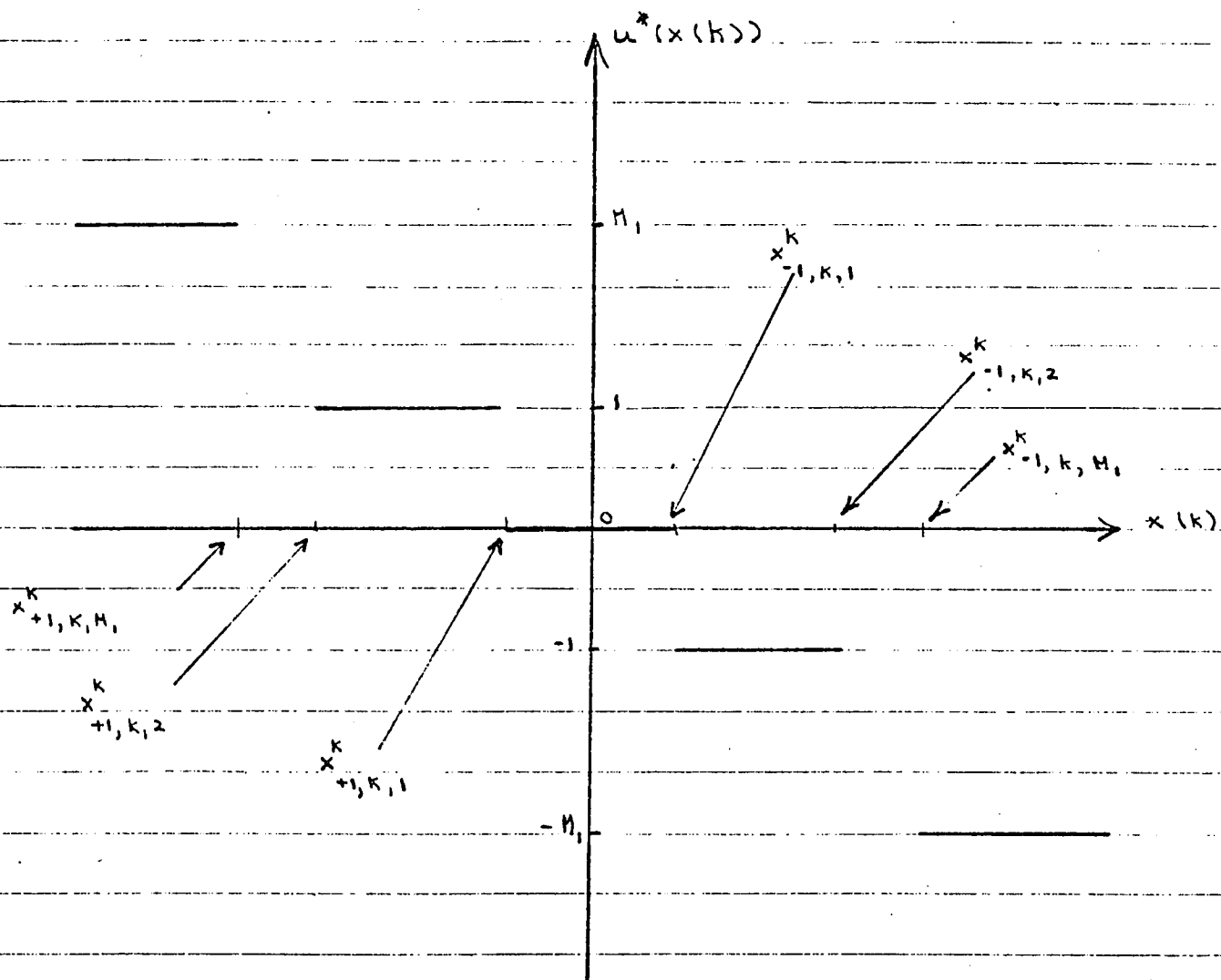


Figure C.8 Graph of $u^*(x(k))$ vs. $x(k)$.

Part 3.3.2. In this part, the expression

$$f_{N-k}^{RI}(x(k)) = \min_{U_{k,N-1} \in S_3(k,I)} h'_{N-k}(x(k), U_{k,N-1}^*, k) \quad (C.265)$$

where

$$S_3(k,I) = \{U_{k,N-1} : U_{k,s-1} = \{z_{M_1}, \dots, z_{M_1}\}, u(s) = z_j, \\ U_{s+1,N-1} = \{0, \dots, 0\} \text{ for all } z \in (-1, +1), j \in (1, \dots, M_1), \\ s \in (k, \dots, k-1+I); U_{k,N-1} = \{0, \dots, 0\}\} \quad (C.266)$$

and $k \in (0, \dots, N-I-1)$, is plotted vs. $x(k)$ and is shown for all $I \in (1, \dots, N)$ to be given by: a) the dashed line segments of Figure C.4; and b) the plot of $h'_{N-k}(x(k), U_{k,N-1}^*, k)$ vs. $x(k)$, also shown in Figure C.4, for

$$x_{+1, k+I, 1}^k < x(k) < x_{-1, k+I, 1}^k \quad (C.267)$$

where $x_{+1, k+I, 1}^k$ and $x_{-1, k+I, 1}^k$ are both derived from Equation (C.22). Note that $S_3(k,I)$ is a subset of both $S_1(k)$ and $S(k,I,0)$ by comparing Equation (C.266), (C.4), and (3.4.1.15), i.e.,

$$S_3(k,I) \subset S_1(k) \quad (C.267a)$$

and

$$S_3(k,I) \subset S(k,I,0). \quad (C.267b)$$

Part 3.3.3. In this part, it is shown that the plot of $u^*(x(k))$ vs. $x(k)$ that results from the plot of $f_{N-k}^{RI}(x(k))$ vs. $x(k)$ is also given by Figure C.8 for all $I \in (1, \dots, N)$.

This implies that Equation (C.265) also yields an optimum control law for all $I \in (1, \dots, N)$.

Part 3.3.4. In this last part, it is demonstrated that Equations (C.265) and (3.4.1.14) are equivalent for all $I \in (1, \dots, N)$: this finally establishes the desired result of Part 3.3 and concludes the overall proof of this appendix.

Proof of Part 3.3

Part 3.3.1. From the plot of $h'_{N-k}(x(k), U^*_{k, N-1}, k)$ vs. $x(k)$ shown in Figure C.4 we note that the optimum control sequence $U^*_{k, N-1}$ is known for each possible value of $x(k)$, therefore since from Equation (C.1) we have that $u^*(x(k))$ is equal to the k -th stage member of the optimum control sequence $U^*_{k, N-1}$ it is possible for us to obtain the plot of $u^*(x(k))$ vs. $x(k)$ from the plot of $h'_{N-k}(x(k), U^*_{k, N-1}, k)$ vs. $x(k)$. For instance, when $x(k)$ lies between $x^{k}_{+1, k, 1}$ and $x^{k}_{-1, k, 1}$ we note from Figure C.4 that $U^*_{k, N-1} = \{0, \dots, 0\}$ which in turn implies

$$u^*(x(k)) = 0 \quad (C.268)$$

for

$$x^{k}_{+1, k, 1} < x(k) < x^{k}_{-1, k, 1} \quad (C.269)$$

Proceeding along the same lines for all the other values of $x(k)$ we finally obtain Figure C.8 where it is observed that $u^*(x(k))$ is equal to $-M_1$ when $x(k)$ is greater than x^{k}_{-1, k, M_1} and equal to M_1 when $x(k)$ is less than x^{k}_{+1, k, M_1} .

Part 3.3.2. From the plot of $h'_{N-k}(x(k), U_{k,N-1}^*, k)$ vs. $x(k)$ shown on Figure C.4 we note that

$$U_{k,N-1}^* \in S_3(k, I) \quad (C.270)$$

for Condition C.267 where $S_3(k, I)$ is given by Equation (C.266). As a result it follows from Equations (C.265) and (C.5) and the above observation that

$$f'_{N-k}^{RI}(x(k)) = h'_{N-k}(x(k), U_{k,N-1}^*, k) \quad (C.271)$$

for Condition (C.267). Furthermore, from Equation (3.4.1.11) it is noted that the inequality

$$\begin{aligned} P_2(U_{k,N-1}, k) / U_{k,k-1+I} &= \{M_1, \dots, M_1\}, U_{k+1,N-1} = \{0, \dots, 0\} \\ &> P_2(U_{k,N-1}, k) \\ &> P_2(U_{k,N-1}, k) / U_{k,k-1+I} &= \{-M_1, \dots, -M_1\}, U_{k+1,N-1} = \{0, \dots, 0\} \end{aligned} \quad (C.272)$$

is satisfied for all $U_{k,N-1}$ such that $U_{k,N-1} \in S_3(k, I)$ and $I \in (1, \dots, N)$.

From Equation (C.265), Inequality (C.272), and the plot of $f'_{N-k}^{RI}(x(k))$ vs. $x(k)$ for Condition (C.267) (see Equation C.271 and Figure C.4) it readily follows that the dashed straight line to the right of $x(k) = x_{-1,k+I}^k$, corresponds to the plot of $h'_{N-k}(x(k), U_{k,N-1}, k)$ vs. $x(k)$ for the control sequence $U_{k,k-1+I} = \{-M_1, \dots, -M_1\}$, $U_{k+1,N-1} = \{0, \dots, 0\}$, and the dashed straight line to the left of $x(k) = x_{+1,k+I}^k$ corres-

ponds to the plot of $h'_{N-k}(x(k), U_{k,N-1}, k)$ vs. $x(k)$ for the control sequence $U_{k,k-1+I} = \{M_1, \dots, M_1\}$, $U_{k+I,N-1} = \{0, \dots, 0\}$. This ends the demonstration of Part 3.3.2.

Part 3.3.3. It is easily seen from the plot of $f'_{N-k}^{RI}(x(k))$ vs. $x(k)$ on Figure C.4 that the plot of $u^*(x(k))$ vs. $x(k)$ for this case is also given by Figure C.8 for all $I \in (1, \dots, N)$.

Part 3.3.4 An investigation of Property (C.267b) and Equations (C.265) and (3.4.1.14) reveals that Equations (C.265) and (3.4.1.14) are equivalent if the following condition can be shown to be true: the condition is that for all $U_{k,N-1}$ such that

$$U_{k,N-1} \in S(k, I, 0) \quad (C.273)$$

it must follow that

$$f'_{N-k}^I(x(k)) \geq f'_{N-k}^{RI}(x(k)) \quad (C.274)$$

where the control sequence, $U_{k,N-1}^I$, that minimizes $f'_{N-k}^I(x(k))$ and $f'_{N-k}^{RI}(x(k))$ is such that

$$U_{k,N-1}^I \in S_3(k, I). \quad (C.275)$$

The above condition can be easily shown to be true as follows:

First, it is noted that in Part 3.2 it was found that Equation (C.271) holds for Condition (C.267). As a result it must follow that the desired Conditions (C.273)-(C.275) must hold for values of $x(k)$ that satisfy Condition C.267.

Second and last, for values of $x(k)$ larger than $x_{-1, k+1, 1}^k$ and smaller than $x_{+1, k+1, 1}^k$ it follows from the plot of $f_{N-k}^{RI}(x(k))$ vs. $x(k)$ in Figure C.4 that Conditions (C.273)-(C.275) hold if and only if

$$\begin{aligned} P_2(U_{k, N-1}, k) / U_{k, k-1+I} &= \{M_1, \dots, M_1\}, U_{k+1} = \{0, \dots, 0\} \\ &\geq P_2(U_{k, N-1}, k) \\ \geq P_2(U_{k, N-1}, k) / U_{k, k-1+I} &= \{-M_1, \dots, -M_1\}, U_{k+1, N-1} = \{0, \dots, 0\} \end{aligned} \quad (C.276)$$

for all $U_{k, N-1}$ such that

$$U_{k, N-1} \in S(k, I, 0).$$

Using Equation (3.4.1.11), the above condition is easily verified. This jointly ends the proof of Part 3.4, Part 3, the overall proof, and this appendix.

Appendix D
PROOF OF THREE EQUIVALENT CONTROL LAWS

The objectives of this appendix are as follows:

1) To show that the 1 Stage ROLIMPs control law for the control problem of Section 3.4.2 is given by Equations (3.4.2.3)-(3.4.2.7) and (2.5.1.6);

2) To demonstrate that when

$$v(u(k))=u^2(k) \quad (D.1)$$

the 1 Stage ROLIMPs control law is equivalent to the Compact Quadratic (CQ) control law given by Equations (3.4.2.10), (3.4.2.11), and (2.5.1.6); and

3) To show that when

$$v(u(k))=|u(k)| \quad (D.2)$$

the 1 Stage ROLIMPs control law is equivalent to the Compact Magnitude (CM) control law given by Equations (3.4.2.14)-(3.4.2.17) and (2.5.1.6).

Next we find the 1 Stage ROLIMPs control law.

1 Stage ROLIMPs Control Law

The 1 Stage ROLIMPs control law is readily derived by making use of the parameters of the process Equation (3.4.2.1) and performance criterion Equation (3.4.2.2) in the I Stages ROLIMPs control law's defining equations of Section 3.3.2:

First, it is noted that Condition (3.3.3.1) is not satisfied by any allowable values of k when $I = 1$, and that on the other hand, Condition (3.3.3.3) is satisfied for all allowable values of k when $I = 1$; as a result, we have that $u^I(x(k))$ is solely given by Equations (3.3.3.4), (3.3.3.5), (3.3.1.1), and (2.5.1.3)-(2.5.1.8).

Second, we find the value of d according to its definition: the definition of d is given when the parameters of Equation (3.3.1.1) are defined. We start by noting that the performance criterion Equation (3.4.2.2) reflects the desire to have the final state of the process, $x(N)$, be close to a zero vector, i.e., $x(N) = 0$; as a result, we have from the process Equation (3.4.2.1) that the final control $u(N-1)$ that satisfies the desired terminal condition on $x(N)$, is given by

$$u(N-1) = 0$$

provided: a) the process is in the steady state when it reaches stage $N-1$ so that it can be assumed that $x(N-1)$ is also close to a zero vector; b) there is a zero element in the set $(c'_1, \dots, c'_M)$. It then follows from the previous discussion and the definition of d , that $d = 0$ provided the about conditions on $x(N-1)$ and the set $(c'_1, \dots, c'_M)$ are valid.

Third, replace d by 0 and I by 1 in Equation (3.3.1.1) to obtain

$$\begin{aligned} & S(k, 1, 0) \\ &= [U_{k, N-1} : u(k) \in (c'_1, \dots, c'_M); u(s) = 0 \text{ for } s \in (k+1, \dots, N)] \end{aligned} \quad (D.3)$$

Fourth, replace I by 1 in Equation (3.3.3.4) to obtain one of the desired equations, Equation (3.4.2.3).

Fifth, replace $S(k, I, d)$ by $S(k, 1, 0)$ and I by 1 in Equation (3.3.3.5) to obtain

$$\begin{aligned} & f^I_{N-k}(x(k)) \\ &= \min_{U_{k, N-1} \in S(k, 1, 0)} h^I_{N-k}(x(k), U_{k, N-1}, k). \end{aligned} \quad (D.4)$$

It then follows from Equation (D.3) that Equation (D.4) is

equivalent to the second desired Equation (3.4.2.4), where

$$h'_{N-k}(x(k), u(k), k) = h'_{N-k}(x(k), U_{k, N-1}, k) \quad (D.5)$$

with

$$u(s) = 0 \text{ for all } s \in (k+1, \dots, N)$$

Sixth and last, let $u(s) = 0$ for $s \in (k+1, \dots, N)$ in Equations (2.5.1.3)-(2.5.1.5), as implied by Equation (D.5); also replace $B(u(t), t)$, $Q_2(u(i), i)$, and $q_3(u(i), i)$ by $b(t)u(t)$, 0 , and $r(i)v(u(i))$, respectively, in Equations (2.5.1.4) and (2.5.1.5): this yields the last desired Equations (3.4.2.5)-(3.4.2.7), when we realize that $P_1(k+1)$ is given by Equation (B.3) in Appendix B where k is replaced by $k+1$.

Next we find the CQ control law corresponding to the case given by Equation (D.1).

Compact Quadratic Control Law

The CQ control law is derived from the 1 Stage ROLIMPs control law as follows:

First, place the RHS of Equation (D.1) in Equation (3.4.2.7) to obtain

$$p_3(u(k), k) = (b^T(k)P_1(k+1)b(k) + r(k))u^2(k) \quad (D.6)$$

Second, replace $P_2(u(k), k)$ and $p_3(u(k), k)$ by the RHS of Equations (3.4.2.6) and (D.6), respectively, in Equation (3.4.2.5) to yield

$$\begin{aligned} h'_{N-k}(x(k), u(k), k) &= 2b^T(k)P_1(k+1)A(k)x(k)u(k) \\ &+ [b^T(k)P_1(k+1)b(k) + r(k)]u^2(k). \end{aligned} \quad (D.7)$$

Third, complete the square of $u(k)$ in (D.7) to obtain

$$\begin{aligned} h'_{N-k}(x(k), u(k), k) &= - [b^T(k)P_1(k+1)A(k)x(k)]^2 / [b^T(k)P_1(k+1)b(k) + r(k)] \\ &+ [\sqrt{b^T(k)P_1(k+1)b(k) + r(k)} u(k) \\ &+ b^T(k)P_1(k+1)A(k)x(k) / \sqrt{b^T(k)P_1(k+1)b(k) + r(k)}]^2 \end{aligned} \quad (D.8)$$

Fourth, use Equation (D.8) in Equation (3.4.2.4) to yield

$$\begin{aligned} f'_{N-k} &= - [b^T(k)P_1(k+1)A(k)x(k)]^2 / [b^T(k)P_1(k+1)b(k) + r(k)] \\ &+ \min_{u(k)} [\sqrt{b^T(k)P_1(k+1)b(k) + r(k)} u(k) \\ &+ b^T(k)P_1(k+1)A(k)x(k) / \sqrt{b^T(k)P_1(k+1)b(k) + r(k)}]^2 \end{aligned} \quad (D.9)$$

Fifth and last, note that when $u(k)$ is given by Equations (3.4.2.10), (3.4.2.11), and (2.5.1.6) the minimization of

Equation (D.9) is accomplished. This concludes the proof of the Compact Quadratic control law.

Next we find the Compact Magnitude control law corresponding to the case given by Equation (D.2).

Compact Magnitude Control Law

The CM control law is derived from the 1 Stage ROLIMPs control law as follows:

First, proceed similarly as for the CQ control law case, except use Equation (D.2) instead of (D.1), to obtain

$$\begin{aligned} h'_{N-k}(x(k), u(k), k) \\ = 2b^T(k)P_1(k+1)A(k)x(k)u(k) + b^T(k)P_1(k+1)b(k)u^2(k) \\ + r(k)|u(k)| \end{aligned} \quad (D.10)$$

Second, show that when $x(k)$ satisfies the inequality

$$K_1(k)x(k) \leq g(c^-, c^+) \quad (D.11)$$

where $K_1(k)$ and $g(c^-, c^+)$ are given by Equations (3.4.2.15) and (3.4.2.17), respectively, it follows that $u^{CM}(x(k))$ is given by

$$u^{CM}(x(k)) = Cl^+[-K_1(k)x(k) - K_2(k)] \quad (D.12)$$

where $Cl^+[\cdot]$, $K_1(k)$, and $K_2(k)$ are as defined for Equation (3.4.2.14). The proof is as follows:

We start the proof by demonstrating that if Inequality (D.11) holds, it then follows that

$$h'_{N-K}(x(k), u(k), k) \geq h'_{N-K}(x(k), c^+, k) \quad (D.13)$$

for all negative values of $u(k)$ in the set $(c_1, \dots, c_M)$. This is demonstrated by first making the following operations in Inequality (D.13):

i) Substitute Equation (D.10), with $|u(k)|$ replaced by $-u(k)$, in the LHS of Inequality (D.13);

ii) Substitute Equation (D.10), with $|u(k)|$ replaced by c^+ , in the RHS of Inequality (D.13); and

iii) Use the result of the previous substitutions to obtain, after some algebraic manipulations, the condition in $x(k)$ given by

$$\begin{aligned} & K_1(k)x(k) \\ & \leq + [-u(k) - c^+ + K_2(k)(u(k) + c^+)/(u(k) - c^+)]/2 \end{aligned} \quad (D.14)$$

where $K_1(k)$ and $K_2(k)$ are given by Equations (3.4.2.15) and (3.4.2.16), and $u(k)$ can only be a negative value of the set $(c_1^-, \dots, c_M^-)$. Then noting that the smallest value that the RHS of Inequality (D.14) can have as a function of $u(k)$ is given by Equation (3.4.2.17) (note that $K_2(k)$ is a nonnegative real number), it must follow that when Inequality (D.11) holds Inequality (D.13) also holds.

2) After showing that Inequality (D.13) follows from Inequality (D.11) we then have from Equations (3.4.2.3) and (3.4.2.4) that $u^1(x(k))$ or equivalently $u^{CM}(x(k))$ is found from the minimization

$$\min_{u(k)} h_{N-k}^{*+}(x(k), u(k), k) \quad (D.15)$$

where $u(k)$ can be any nonnegative element of the set $(c_1^1, \dots, c_M^1)$ and $h_{N-k}^{*+}(x(k), u(k), k)$ is given by the RHS of Equation (D.10) with $\{u(k)\}$ replaced by $+u(k)$, i.e.,

$$\begin{aligned} & h_{N-k}^{*+}(x(k), u(k), k) \\ &= [2b^T(k)P_1(k+1)A(k)x(k) + r(k)]u(k) \\ &+ b^T(k)P_1(k+1)b(k)u^2(k). \end{aligned} \quad (D.16)$$

Note that the minimization of (D.15) does not include as possible values for $u(k)$ the subset of negative values in $(c_1^1, \dots, c_M^1)$: this is the case since negative values of $u(k)$ never yield a unique minimum for $h_{N-k}^{*+}(x(k), u(k), k)$ as implied by Inequality (D.13).

3) Next completing the square for $u(k)$ in (D.16) and then introducing the new expression for (D.16) (with the square

completed) in (D.15), we obtain

$$\begin{aligned}
 & \min_{u(k)} h_{N-k}^+(x(k), u(k), k) \\
 &= - [b^T(k)P_1(k+1)A(k)x(k) + r(k)/2]^2 / b^T(k)P_1(k+1)b(k) \\
 & \quad + \min_{u(k)} [\sqrt{b^T(k)P_1(k+1)b(k)} u(k) \\
 & \quad [b^T(k)P_1(k+1)A(k)x(k) + r(k)/2] / \sqrt{b^T(k)P_1(k+1)b(k)}] \\
 & \hspace{15em} (D.17)
 \end{aligned}$$

Finally, performing the minimization of the RHS of Equation (D.17) we obtain the desired Equation (D.12).

Third and last, show that when $x(k)$ satisfies the inequality

$$K_1(k)x(k) > g(c^-, c^+) \quad (D.18)$$

where $K_1(k)$ and $g(c^-, c^+)$ are given by Equations (3.4.2.15) and (3.4.2.17), it follows that $u^{cm}(x(k))$ is given by

$$u^{cm}(x(k)) = Cl^-[-K_1(k)x(k) + K_2(k)] \quad (D.19)$$

where $Cl^-[\cdot]$, $K_1(k)$, and $K_2(k)$ are as defined for Equation (3.4.2.14). The proof of Equation (D.19) for Inequality (D.18) readily follows by using the same approach used to establish Equation (D.12) for Inequality (D.11). The outline of the proof, without details, is as follows:

1) Demonstrate that when Inequality (D.18) holds, it must follow that

$$h'_{N-k}(x(k), u(k), k) > h'_{N-k}(x(k), c^-, k) \quad (D.20)$$

for all negative values of $u(k)$ in the set $(c_1^-, \dots, c_M^-)$.

2) Show that $u^*(x(k))$ or equivalently $u^{c^*}(x(k))$ is found from the minimization

$$\min_{u(k)} h'^{-}_{N-k}(x(k), u(k), k) \quad (D.21)$$

where

$$\begin{aligned} h'^{-}_{N-k}(x(k), u(k), k) \\ = [2b^T(k)P_1(k+1)A(k)x(k) - r(k)]u(k) \\ + b^T(k)P_1(k+1)b(k)u^2(k) \end{aligned} \quad (D.22)$$

Demonstrate that

$$\begin{aligned} \min_{u(k)} h'^{-}_{N-k}(x(k), u(k), k) \\ = -[b^T(k)P_1(k+1)A(k)x(k) - r(k)/2]^2 / b^T(k)P_1(k+1)b(k) \\ + \min_{u(k)} [\sqrt{b^T(k)P_1(k+1)b(k)} u(k) \\ + [b^T(k)P_1(k+1)A(k)x(k) - r(k)/2] / \sqrt{b^T(k)P_1(k+1)b(k)}]^2 \end{aligned} \quad (D.23)$$

4) Finally note that performing the minimization of the RHS of Equation (D.23), the desired Equation (D.19) is derived to complete the proof of the Compact Magnitude control law and end this appendix.

Appendix E

COMPUTER PROGRAMS

It is the objective of this appendix to show how the computer programs of Sections E.1.3, E.2.4, E.3.3, and E.4.4 may be used to design and implement I Stages ROLIMPs, Constant I Stages ROLIMPs (see Equation (4.5.1.30) for its definition), QLQ, and SNN control laws.

This appendix will be divided in four Sections. In Section E.1, we consider the computer program that designs an I Stages or Constant I Stages ROLIMPs control law. In Section E.2, we present the computer program that simulates the application of an I Stages or Constant I Stages ROLIMPs control law to the process Equation (4.5.1.26) for any given initial condition. In Section E.3, we consider the computer program that designs the QLQ and SNN control laws. And finally in Section E.4, we present the computer program that simulates the application of the QLQ and SNN control laws to the process Equation (4.5.1.26) for any given initial condition.

E.1 COMPUTER PROGRAM THAT DESIGNS AN I STAGES OR CONSTANT I STAGES ROLIMPS CONTROL LAW

E.1.1 Purpose of Program

The general purpose of the computer program of Section E.1.3 is to design an I Stages or Constant I Stages ROLIMPs control law with

$$S(k, I, d) = S(k, I, 0)$$

for the class of process and performance criterion given by Equations (4.5.1.12) and (4.5.1.16) where the process Equation (4.5.1.12) is discretized according to Equation (4.5.1.26). The parameters of the designed I Stages or Constant I Stages ROLIMPs control law, performance criterion Equation (4.5.1.16), and discretized process Equation (4.5.1.26) can then be used by the simulation program of Section E.2.4.

The specific program tasks that accomplish the above general purpose are as follows:

Task 1

To determine A and $B(u(k))$ for all $u(k)$, by evaluating the RHS of Equations (4.5.1.27) and (4.5.1.28) for each member in the space (4.5.1.8). A and $B(u(k))$ for all $u(k)$ are then used for the design of the I Stages or Constant I Stages ROLIMPs control law and are also stored for future use by the simulation program of Section E.2.4;

Task 2

To find $P_2(U_{k,N-1}, k)$ and $p_3(U_{k,N-1}, k)$ for all

$$k \in (0, \dots, N-I-1) \text{ and } U_{k,N-1} \in S(k, I, \underline{0})$$

and also for all

$$k \in (N-I, \dots, N-1) \text{ and } U_{k,N-1} \in US(k)$$

where:

a) $S(k, I, \underline{0})$ and $US(k)$ are given by Equations (3.3.1.1) and (2.2.5); and

b) $P_2(U_{k,N-1}, k)$ and $p_3(U_{k,N-1}, k)$ are derived by using the difference Equations (2.2.14)-(2.2.16) and the process and performance criterion parameters of Equations (4.5.1.26) and (4.5.1.16). $P_2(U_{k,N-1}, k)$ and $p_3(U_{k,N-1}, k)$ are also stored for future use by the simulation program of Section E.2.4. It is noted that when designing a Constant I Stages ROLIMPs control law the MPAs $P_2(U_{k,N-1}, k)$ and $p_3(U_{k,N-1}, k)$ are only found and stored for $k = 0$ and all $U_{0,N-1} \in S(0, I, \underline{0})$;

Task 3

To store, for later use by the simulation program of Section E.2.4, the process and performance criterion parameters given by Q, R, N, V, n, m , and $(c_1^i, \dots, c_n^i)$ for $i = 1, \dots, m$; and

Task 4

To store, for later use by the simulation program of Section E.2.4, the control law parameter that specifies whether we have designed an I Stages or a Constant I Stages ROLIMPs control law.

E.1.2 Program Inputs

The following input arguments must be provided to the design program of Section E.1.3:

N	Number of process stages N.
I	Number of inputs m.
J	Number of states n.
V	Sampling time V in seconds.
C1(i)	Lowest quantization level c_1^i for $i = 1, \dots, m$.
CM(i)	Highest quantization level c_M^i for $i = 1, \dots, m$.
MW	Number of quantization levels M.
IST	Value of I for I Stages or Constant I Stages Control Law.
NR	Control law parameter: 1 Constant I Stages ROLIMPs control law 2 I Stages ROLIMPs control law
B(i,1)	Component of matrix $B'(u(t))$, $B'_i(u(t))$, for all i where $u(t)$ is given by

$(U(1,1), \dots, U(m,1)),$ e.g., if
 $B_2^* (u(t)) = 2u_1(t) + u_2(t)$ we have that
 $B(2,1) = 2*U(1,1) + U(2,1).$

$A(i,j)$ Component of matrix A' , $A'_{i,j}$, for all i and j .

$Q1(i,j)$ Component of matrix Q , $Q_{i,j}$, for all i and j .

$Q3$ $u^T(k)Ru(k)$ where $u(k)$ is given by
 $(U(1,1), \dots, U(m,1)).$

MT Number of possible values for $u(k)$, M^m .

where:

a) N , I , J , V , $C1(.)$, $CM(.)$, MW , IST , and NR are initialized between the comment statement

C BEGINNING OF INITIALIZATION 1

and the comment statement

C END OF INITIALIZATION 1;

b) $A(.,.)$, $B(.,1)$, $Q1(.,.)$, and $Q3$ are initialized between the comment statement

C BEGINNING OF INITIALIZATION 2

and the comment statement

C END OF INITIALIZATION 2;

and

c) The values of N, I, J, and MT are used to dimension the matrices given by the dimension statements at the beginning of the program, according to the comment dimension statements also shown at the beginning of the program.

E.1.3 Design Program

```

C      DIMENSION M(20,I),IV(20,I),MTT(20),MRR(20),SH(I),EX(I)
C      DIMENSION A(J,J),AA1(J,J),AA2(J,J),B(J,1),NA(2),NB(2)
C      A1D(J*J),A2D(J*J),A3D(J*J),DU(2*J*J),ND(2),IVV(20)
C      DIMENSION BT(J,1),BC(20,MT,J),AC(20,1,J,J)
C      DIMENSION Q1(J,J),Q2(1,3),UC(I,1),U(I,1),C1(I),CM(I)
C      DIMENSION QQ1(20,1,J,J),QQ2(20,1,J),QQ3(20,MT)
C      DIMENSION AUC(J,J),BUC(J,1),Q1UC(J,J),Q2UC(1,J)
C      DIMENSION P1(J,J),PP1(N+1,J,J),PP2(N+1,J),PP3(N+1),AA(J,J),
C      AA4(J,J),P2(1,J),BA(1,J),PA(1,J),BA1(1,J),AA3(J,J),BA3(1,J)
C OC
C      DIMENSION AU(I,J,J),BU(I,J),QU1(I,J,J),QU2(I,J),QU3(I)
C OC
      DIMENSION M(1,1),IV(1,1),MTT(1),MRR(1),SH(1),EX(1),IVV(1)
      DIMENSION A(3,3),AA1(3,3),AA2(3,3),B(3,1),NA(2),NB(2),ND(2)
      A1D(9),A2D(9),A3D(9),DU(18)
      DIMENSION BT(3,1),BC(1,3,3),AC(1,1,3,3)
      DIMENSION Q1(3,3),Q2(1,3),UC(1,1),U(1,1),C1(1),CM(1)
      DIMENSION QQ1(1,1,3,3),QQ2(1,1,3),QQ3(1,3)
      DIMENSION AUC(3,3),BUC(3,1),Q1UC(3,3),Q2UC(1,3)
      DIMENSION P1(3,3),PP1(8,3,3),PP2(8,3),PP3(8),AA(3,3),
      AA4(3,3),P2(1,3),BA(1,3),PA(1,3),BA1(1,3),AA3(3,3),BA3(1,3)
C OC
      DIMENSION AU(1,5,5),BU(1,5),QU1(1,5,5),QU2(1,5),QU3(64)
C OC
C OC
      NAU=1
      NBU=1
      NQU1=1
      NQU2=1
      NQU3=2
C OC
C BEGINNING OF INITIALIZATION 1
      N=7
      I=1
      J=3
      V=0.01
      C1(1)=-1.0
      CM(1)=1.0
      MW=3
      IST=1
      NR=2
C END OF INITIALIZATION 1

```

```

      NH=2
      NC=1
      NT=2
      NQ1=1
      NQ2=1
      DO 11 LZ=1,I
      SH(LZ)=(C1(LZ)+CM(LZ))/2.0
      EX(LZ)=(CM(LZ)-C1(LZ))/2.0
11      UC(LZ,1)=0.0
      WRITE(10,102)IST,MW
102     FORMAT(I10,I10)
      WRITE(10,2103)NH,NT,NR,NC,NQ1,NQ2,I,J,N
2103    FORMAT(I2,I2,I2,I2,I2,I2,I2,I2,I4)
      DO 2115 L=1,I
2115    WRITE(10,2125)SH(L)
2125    FORMAT(E20.7)
      DO 2135 L=1,I
2135    WRITE(10,2113)EX(L)
2113    FORMAT(E20.7)
C OC
      IF(NH.NE.1)GO TO 1207
      WRITE(10,1203)NAU,NBU,NQU1,NQU2,NQU3
1203    FORMAT(I4,I4,I4,I4,I4)
1207    CONTINUE
C OC
      MCC=1
C DO LOOPS THAT VARY NSC
      DO 5 NS=IST,1
      DO 5 M1=MW,1
      DO 25 IR=1,NS
      DO 25 IC=1,I
25      M(IR,IC)=M1
C END OF DO LOOPS THAT VARY NSC
C STORING A(U(K)), B(U(K)), Q1(U(K)), Q2(U(K)), AND Q3(U(K))
      MT=1
      DO 35 IR=1,NS
      MR=1
      DO 35 IC=1,I
      MR=MR*M(IR,IC)
      MRR(IR)=MR
      MT=MT*M(IR,IC)
35      MTT(IR)=MT
      DO 55 LS=1,NS
      MT=MRR(LS)
      DO 45 IC=1,I
45      IV(LS,IC)=1
      IV(LS,1)=0
      DO 55 L1=1,MT
      IF(MCC.NE.1)GO TO 1008
      DO 1002 IR=1,I
1002    U(IR,1)=UC(IR,1)
      GO TO 1018
1008    CONTINUE
      IV(LS,1)=IV(LS,1)+1

```

```

DO 65 IC=1,I
IF(IV(LS,IC).EQ.M(LS,IC)+1)IV(LS,IC+1)=IV(LS,IC)+1
65 IF(IV(LS,IC).EQ.M(LS,IC)+1)IV(LS,IC)=1
DO 85 IR=1,I
85 U(IR,1)=SH(IR)+EX(IR)*(1.0*(M(LS,IR)-1-2*(IV(LS,IR)-1)))
/((M(LS,IR)-1)*1.0)
1018 CONTINUE
C BEGINNING OF INITIALIZATION 2
DO 4444 JR=1,J
DO 4444 JC=1,J
4444 A(JR,JC)=0.0
DO 4445 JR=1,J
4445 B(JR,1)=0.0
A(1,2)=1.0
A(2,2)=-10.0
A(3,3)=-10.0
B(2,1)=1000.0*U(1,1)
Q1(1,1)=1.0
Q1(1,3)=-1.0
Q1(3,1)=-1.0
Q1(3,3)=1.0
Q3=0.0
C END OF INITIALIZATION 2
DO 4446 JR=1,J
4446 Q2(1,JR)=0.0
IF(NT.EQ.1)GO TO 67
IF(L1.EQ.1)GO TO 47
IF(NC.EQ.1)GO TO 57
47 CONTINUE
DO 95 JC=1,J
DO 95 JR=1,J
JP=(JC-1)*J+JR
95 A1D(JP)=A(JR,JC)
DO 105 JR=1,2
NA(JR)=J
NB(JR)=J
105 ND(JR)=J
CALL EXPINT(A1D,NA,A2D,NB,A3D,ND,V,0,DU)
DO 115 JC=1,J
DO 115 JR=1,J
JP=(JC-1)*J+JR
AA1(JR,JC)=A2D(JP)
115 AA2(JR,JC)=A3D(JP)
57 CONTINUE
CALL GMPRD(AA2,B,BT,J,J,1)
67 CONTINUE
IF(MCC.NE.1)GO TO 1028
DO 1016 JR=1,J
DO 1016 JC=1,J
IF(NT.EQ.1)GO TO 1038
A(JR,JC)=AA1(JR,JC)
B(JR,1)=BT(JR,1)
1038 AUC(JR,JC)=A(JR,JC)
BUC(JR,1)=B(JR,1)

```

```

1016  Q1UC(JR,JC)=Q1(JR,JC)
      Q2UC(1,JR)=Q2(1,JR)
      Q3UC=Q3
      MCC=2
      GO TO 1008
1028  CONTINUE
      DO 125 JR=1,J
      IF(NT.EQ.1)GO TO 77
      B(JR,1)=BT(JR,1)
77    CONTINUE
125   BC(LS,L1,JR)=B(JR,1)
      IF(L1.EQ.1)GO TO 27
      IF(NC.EQ.1)GO TO 28
27    CONTINUE
      DO 135 JR=1,J
      DO 135 JC=1,J
      IF(NT.EQ.1)GO TO 87
      A(JR,JC)=AA1(JR,JC)
87    CONTINUE
      AC(LS,L1,JR,JC)=A(JR,JC)
135   CONTINUE
28    CONTINUE
      IF(LS.GT.1)GO TO 18
C OC
      IF(NH.NE.1)GO TO 1007
      IF(NC+NT.EQ.4)GO TO 1007
C BEGINNING OF INITIALIZATION 3
      DO 1024 JR=1,J
      DO 1024 JC=1,J
      AU(1,JR,JC)=0.0
      BU(1,JR)=0.0
      QU1(1,JR,JC)=0.0
1024  QU2(1,JR)=0.0
      QU3(1)=2*U(1,1)
      BU(1,5)=30.0
C END OF INITIALIZATION 3
      IF(NT.EQ.1)GO TO 1017
      DO 1005 IR=1,I
      DO 1015 JR=1,J
1015  BT(JR,1)=BU(IR,JR)
      CALL GMPRD(AA2,BT,BB,J,J,1)
      DO 1025 JR=1,J
1025  BU(IR,JR)=BB(JR,1)
1005  CONTINUE
1017  CONTINUE
      IF(L1.GT.1)GO TO 1027
      DO 1035 JR=1,J
      DO 1035 JC=1,J
1035  WRITE(10,1003)Q1UC(JR,JC)
1003  FORMAT(E20.7)
      DO 1045 JC=1,J
1045  WRITE(10,1023)Q2UC(1,JC)
1023  FORMAT(E20.7)
1027  IF(L1.EQ.1)GO TO 1037

```

```

      IF(NAU.EQ.1)GO TO 1047
1037  DO 1055 IR=1,I
      DO 1055 JR=1,J
      DO 1055 JC=1,J
1055  WRITE(10,1013)AU(IR,JR,JC)
1013  FORMAT(E20.7)
1047  IF(L1.EQ.1)GO TO 1057
      IF(NBU.EQ.1)GO TO 1067
1057  DO 1065 IR=1,I
      DO 1065 JR=1,J
1065  WRITE(10,1033)BU(IR,JR)
1033  FORMAT(E20.7)
1067  IF(L1.EQ.1)GO TO 1077
      IF(NQU1.EQ.1)GO TO 1087
1077  DO 1075 IR=1,I
      DO 1075 JR=1,J
      DO 1075 JC=1,J
1075  WRITE(10,1043)QU1(IR,JR,JC)
1043  FORMAT(E20.7)
1087  IF(L1.EQ.1)GO TO 1097
      IF(NQU2.EQ.1)GO TO 1107
1097  DO 1085 IR=1,I
      DO 1085 JC=1,J
1085  WRITE(10,1053)QU2(IR,JC)
1053  FORMAT(E20.7)
1107  IF(L1.EQ.1)GO TO 1117
      IF(NQU3.EQ.1)GO TO 1007
1117  DO 1006 IR=1,I
1006  WRITE(10,1063)QU3(IR)
1063  FORMAT(E20.7)
1007  CONTINUE
C OC
      IF(L1.EQ.1)GO TO 7
      IF(NC.EQ.1)GO TO 17
7      DO 145 JR=1,J
      DO 145 JC=1,J
145  WRITE(10,3)A(JR,JC)
3      FORMAT(E20.7)
17      DO 155 JR=1,J
155  WRITE(10,13)B(JR,1)
13      FORMAT(E20.7)
18      CONTINUE
      IF(L1.EQ.1)GO TO 97
      IF(NQ1.EQ.1)GO TO 98
97      CONTINUE
      DO 165 JR=1,J
      DO 165 JC=1,J
      QQ1(LS,L1,JR,JC)=Q1(JR,JC)
165  CONTINUE
98      CONTINUE
      IF(L1.EQ.1)GO TO 107
      IF(NQ2.EQ.1)GO TO 108
107  CONTINUE
      DO 175 JR=1,J

```

```

      QQ2(LS,L1,JR)=Q2(1,JR)
175  CONTINUE
108  CONTINUE
      QQ3(LS,L1)=Q3
      IF(LS.GT.1)GO TO 55
      IF(L1.EQ.1)GO TO 117
      IF(NQ1.EQ.1)GO TO 127
117  DO 185 JR=1,J
      DO 185 JC=1,J
185  WRITE(10,23)Q1(JR,JC)
23   FORMAT(E20.7)
127  IF(L1.EQ.1)GO TO 137
      IF(NQ2.EQ.1)GO TO 147
137  DO 195 JR=1,J
195  WRITE(10,33)Q2(1,JR)
33   FORMAT(E20.7)
147  WRITE(10,43)Q3
43   FORMAT(E20.7)
55   CONTINUE
C END OF STORING OF A(.), B(.), Q1(.), Q2(.), AND Q3(.)
C IN THIS PART WE DETERMINE P1(K), P2(K), AND P3(K)
      IF(MCC.NE.2)GO TO 308
      DO 305 JR=1,J
      DO 305 JC=1,J
      Q1(JR,JC)=Q1UC(JR,JC)
      Q2(1,JC)=Q2UC(1,JC)
      Q3=Q3UC
      A(JR,JC)=AUC(JR,JC)
      B(JR,1)=BUC(JR,1)
      P1(JR,JC)=Q1(JR,JC)
      PP1(N+1,JR,JC)=Q1(JR,JC)
      PP2(N+1,JC)=Q2(1,JC)
      PP3(N+1)=0.0
305  P2(1,JC)=Q2(1,JC)
      P3=0.0
      DO 315 K=1,N
      CALL GMPRD(P1,A,AA,J,J,J)
      CALL GTPRD(A,AA,AA4,J,J,J)
      CALL GTPRD(B,AA,BA,J,1,J)
      CALL GMPRD(P2,A,PA,1,J,J)
      CALL GMPRD(P1,B,BT,J,J,1)
      CALL GTPRD(B,BT,U1,J,1,1)
      CALL GMPRD(P2,B,U2,1,J,1)
      P3=P3+U2+U1+Q3
      DO 325 JR=1,J
      DO 325 JC=1,J
      P1(JR,JC)=AA4(JR,JC)+Q1(JR,JC)
      P2(1,JC)=PA(1,JC)+2*BA(1,JC)+Q2(1,JC)
      PP1(N+1-K,JR,JC)=P1(JR,JC)
      PP2(N+1-K,JC)=P2(1,JC)
325  PP3(N+1-K)=P3
315  CONTINUE
      MCC=3
308  CONTINUE

```

```

C END OF EVALUATION OF P1(K), P2(K), AND P3(K)
C EVALUATION AND STORAGE OF P1(U(K)), P2(U(K)), AND P3(U(K))
  NN=N
  IF(NR.EQ.1)NN=1
  NSS=NS
  DO 405 K=1,NN
  IF(K.GT.N+1-NS)NSS=NSS-1
  DO 415 IC=1,NSS
415  IVV(IC)=1.0
  IVV(1)=0
  MT=MTT(NSS)
  DO 405 L1=1,MT
  IVV(1)=IVV(1)+1
  DO 425 IR=1,NSS
  IF(IVV(IR).EQ.MRR(IR)+1)IVV(IR+1)=IVV(IR+1)+1
425  IF(IVV(IR).EQ.MRR(IR)+1)IVV(IR)=1
  DO 435 JR=1,J
  DO 435 JC=1,J
  P1(JR,JC)=PP1(K+NSS,JR,JC)
435  P2(1,JC)=PP2(K+NSS,JC)
  P3=PP3(K+NSS)
  DO 475 LS=1,NSS
  LL=IVV(NSS+1-LS)
  DO 455 JR=1,J
  DO 455 JC=1,J
  IF(NC.NE.1)GO TO 407
  A(JR,JC)=AC(NSS+1-LS,1,JR,JC)
  GO TO 417
407  A(JR,JC)=AC(NSS+1-LS,LL,JR,JC)
417  B(JR,1)=BC(NSS+1-LS,LL,JR)
  IF(NQ1.NE.1)GO TO 427
  Q1(JR,JC)=QQ1(NSS+1-LS,1,JR,JC)
  GO TO 437
427  Q1(JR,JC)=QQ1(NSS+1-LS,LL,JR,JC)
437  IF(NQ2.NE.1)GO TO 447
  Q2(1,JR)=QQ2(NSS+1-LS,1,JR)
  GO TO 455
447  Q2(1,JR)=QQ2(NSS+1-LS,LL,JR)
455  CONTINUE
  Q3=QQ3(NSS+1-LS,LL)
  CALL GMPRD(P1,A,AA1,J,J,J)
  CALL GTPRD(A,AA1,AA4,J,J,J)
  CALL GTPRD(B,AA1,BA1,J,1,J)
  CALL GMPRD(P2,A,PA,1,J,J)
  CALL GMPRD(P1,B,BT,J,J,1)
  CALL GTPRD(B,BT,U1,J,1,1)
  CALL GMPRD(P2,B,U2,1,J,1)
  DO 465 JR=1,J
  DO 465 JC=1,J
  P1(JR,JC)=AA4(JR,JC)+Q1(JR,JC)
465  P2(1,JC)=PA(1,JC)+2*BA1(1,JC)+Q2(1,JC)
475  P3=P3+U2+U1+Q3
  IF(NQ1+NC.EQ.2)GO TO 527
  DO 505 JR=1,J

```



```

      DO 505 JC=1,J
505   WRITE(10,503)P1(JR,JC)
503   FORMAT(E20.7)
527   DO 515 JC=1,J
515   WRITE(10,513)P2(1,JC)
513   FORMAT(E20.7)
      WRITE(10,523)P3
523   FORMAT(E20.7)
405   CONTINUE
C END OF EVALUATION AND STORING OF P1(.), P2(.), AND P3(.)
5     CONTINUE
      END

```

E.2 COMPUTER PROGRAM THAT SIMULATES AN I STAGES OR CONSTANT I STAGES ROLIMPS CONTROL LAW

E.2.1 Purpose of Program

The general purpose of the program of Section E.2.4 is to simulate the application of an I Stages or Constant I Stages ROLIMPs control law to the process Equation (4.5.1.26) for any given initial condition. This simulation yields specific state and control trajectories for the process Equation (4.5.1.26) and a numerical value for the RHS of the performance criterion Equation (4.5.1.16).

The specific program tasks that accomplish the above general purpose are as follows:

Task 1

The evaluation of $u^I(x(k))$ or $u_c^I(x(k))$ from their defining equations (see Sections 3.3.3 and 4.5.1) and then $x(k+1)$ from Equation (4.5.1.26) for all possible values of k .

Task 2

The evaluation of the RHS of the performance criterion Equation (4.5.1.16) making use of the derived values of $u^T(x(k))$ or $u_c^T(x(k))$ and $x(k+1)$ for all $k \in (0, \dots, N-1)$.

E.2.2 Input Arguments

The inputs that are required by the simulation program are:

1) The stored outputs of the design program of Section E.1.3;

2) The initial condition, $x(0)$, that we wish to test. The initial condition is placed between the comment statement

C START OF INITIAL CONDITION

and the comment statement

C END OF INITIAL CONDITION

making use of the matrix $X1(j,1)$ for $j = 1, \dots, n$ where

$$X1(j,1) = x_j(0) \text{ for all } j: \text{ and}$$

3) The values of N , I , J , MT , and NT are used to dimension the matrices given by the dimension statements at the beginning of the program, according to the comment dimension statements also shown at the beginning of the program. Note that N , I , J , and MT were defined earlier in Section E.1.2

and NT is equal to the total number of Matched Processors, M^{m1} , that belong to the I Stages or Constant ROLIMPs control law.

E.2.3 Outputs that are Available for Display

Between the comment statement

C START OF DISPLAY

and the comment statement

C END OF DISPLAY

we have the following outputs available for display:

1) The control trajectory $u(k)$ for $k \in (0, \dots, N-1)$ in the form of the matrix

$$U(i,j)$$

for all

$$i \in (1, \dots, N)$$

and

$$j \in (1, \dots, m)$$

where

$$U(i,j) = u_j(i-1) \text{ for all } i \text{ and } j;$$

2) The state trajectory $x(k)$ for $k \in (0, \dots, N)$ in the form of the matrix

$$X0(i,j)$$

for all

$$i \in (1, \dots, N+1)$$

and

$$j \in (1, \dots, n)$$

where

$$XO(i, j) = x_j(i-1) \text{ for all } i \text{ and } j$$

and

3) The value of the RHS of the performance criterion Equation (4.5.1.16), P_I or P_I^c , that results when we apply the considered I Stages or Constant I Stages ROMPs control law to the process Equation (4.5.1.26). This performance is displayed in the form of

$$PER(1)$$

where

$$P_I \text{ or } P_I^c = PER(1)$$

E.2.4 Simulation Program

```

C      DIMENSION X(J,1),PX(J,1),MTT(20),IVV(20),IM(I),IV(1,I+1),
C      M(20,I),MRR(20),PER(20),RPER(20),X1(J,1),X2(J,1)
C      DIMENSION PS1(NT,J,J),PS2(NT,J),PS3(NT),
C      P1(J,J),P2(1,J),BA(J,1),AA(J,J),Q1A(J,J),Q2A(1,J)
C      DIMENSION U(N+1,I),XO(N+1,J),SH(I),EX(I),UC(I)
C      DIMENSION B(MT,J),A(1,J,J),Q1(1,J,J),
C      Q2(1,J),Q3(MT)
C OC
C      DIMENSION Q1UC(J,J),Q2UC(1,J)
C      DIMENSION AU(I,J,J,1),BU(I,J,1),Q1U(I,J,J,1),
C      Q2U(I,J,1),Q3U(I,1)
C      DIMENSION IWU(N,I),IW(N),G(1,J),MY(8)
C      DIMENSION AP(J,J),BP(J,1),PL(1,J),PS(1,J)
C      DIMENSION UA(J,1),P6(J,1),HU(N),MC(N)
C OC
      DIMENSION X(3,1),PX(3,1),MTT(7),IVV(7),IM(1),IV(1,2),
      M(7,1),MRR(7),PER(7),RPER(7),X1(3,1),X2(3,1)

```

```

        DIMENSION PS1(2187,3,3),PS2(2187,3),PS3(2187),
        P1(3,3),P2(1,3),BA(3,1),AA(3,3),Q1A(3,3),Q2A(1,3)
        DIMENSION U(36,1),XO(36,3),SH(1),EX(1),UC(1)
        DIMENSION B(3,3),A(1,3,3),Q1(1,3,3),
        Q2(1,3),Q3(3)
C OC
        DIMENSION Q1UC(5,5),Q2UC(1,5)
        DIMENSION AU(1,5,5,1),BU(1,5,1),Q1U(1,5,5,1),
        Q2U(1,5,1),Q3U(1,64)
        DIMENSION IWU(100,1),IW(100),G(1,5),MY(8)
        DIMENSION AP(5,5),BP(5,1),PL(1,5),P5(1,5)
        DIMENSION UA(5,1),P6(5,1),HU(100),MC(100)
C OC
C OC
        DELTA=0.001
C OC
C START OF INITIAL CONDITIONS
        X1(3,1)=-1.5
        X1(1,1)=0.0
        X1(2,1)=0.0
C END OF INITIAL CONDITIONS
        ITER=0
        REWIND 10
        READ(10,102)IST,MW
102      FORMAT(I10,I10)
        READ(10,103)NH,NT,NR,NC,NQ1,NQ2,I,J,N
103      FORMAT(I2,I2,I2,I2,I2,I2,I2,I2,I4)
        DO 58 IP=1,J
58        XO(1,IP)=X1(IP,1)
        DO 115 L=1,I
115       READ(10,125)SH(L)
125       FORMAT(E20.7)
        DO 135 L=1,I
135       READ(10,113)EX(L)
113       FORMAT(E20.7)
C OC
        IF(NH.NE.1)GO TO 1207
        READ(10,1203)NAU,NBU,NQU1,NQU2,NQU3
1203      FORMAT(I4,I4,I4,I4,I4)
1207      CONTINUE
C OC
C BEGINNING OF DO LOOPS THAT VARY NSC
        DO 5 NS=IST,1
        DO 5 MM=MW,1
        DO 25 IR=1,NS
        DO 25 IC=1,I
25        M(IR,IC)=MM
C END
        ITER=ITER+1
C BEGINNING OF PARAMETERS READING
        MT=1
        DO 136 IR=1,NS
        MR=1
        DO 136 IC=1,I

```

```

MR=MR*M(IR,IC)
MRR(IR)=MR
MT=MT*M(IR,IC)
MTT(IR)=MT
MT=MRR(1)
DO 155 L1=1,MT

C OC
IF(NH.NE.1)GO TO 1007
IF(NC+NT.EQ.4)GO TO 1007
IF(L1.GT.1)GO TO 1027
DO 1035 JR=1,J
DO 1035 JC=1,J
READ(10,1003)Q1UC(JR,JC)
FORMAT(E20.7)
DO 1045 JC=1,J
READ(10,1023)Q2UC(1,JC)
FORMAT(E20.7)
IF(L1.EQ.1)GO TO 1037
IF(NAU.EQ.1)GO TO 1047
DO 1055 IR=1,I
DO 1055 JR=1,J
DO 1055 JC=1,J
READ(10,1013)AU(IR,JR,JC,L1)
FORMAT(E20.7)
IF(L1.EQ.1)GO TO 1057
IF(NBU.EQ.1)GO TO 1067
DO 1065 IR=1,I
DO 1065 JR=1,J
READ(10,1033)BU(IR,JR,L1)
FORMAT(E20.7)
IF(L1.EQ.1)GO TO 1077
IF(NQU1.EQ.1)GO TO 1087
DO 1075 IR=1,I
DO 1075 JR=1,J
DO 1075 JC=1,J
READ(10,1043)Q1U(IP,JR,JC,L1)
FORMAT(E20.7)
IF(L1.EQ.1)GO TO 1097
IF(NQU2.EQ.1)GO TO 1107
DO 1085 IR=1,I
DO 1085 JC=1,J
READ(10,1053)Q2U(IR,JC,L1)
FORMAT(E20.7)
IF(L1.EQ.1)GO TO 1117
IF(NQU3.EQ.1)GO TO 1007
DO 1116 IR=1,I
READ(10,1063)Q3U(IR,L1)
FORMAT(E20.7)
CONTINUE

C OC
IF(L1.EQ.1)GO TO 165
IF(NC.EQ.1)GO TO 175
DO 185 JR=1,J
DO 185 JC=1,J

```

136

C OC

1035
10031045
1023
1027

1037

1055
1013
1047

1057

1065
1033
1067

1077

1075
1043
1087

1097

1085
1053
1107

1117

1116
1063
1007

C OC

165

```

185 READ(10,123)A(L1,JR,JC)
123 FORMAT(E20.7)
175 DO 195 JC=1,J
195 READ(10,133)B(L1,JC)
133 FORMAT(E20.7)
    IF(L1.EQ.1)GO TO 205
    IF(NQ1.EQ.1)GO TO 215
205 DO 225 JR=1,J
    DO 225 JC=1,J
225 READ(10,143)Q1(L1,JR,JC)
143 FORMAT(E20.7)
215 IF(L1.EQ.1)GO TO 235
    IF(NQ2.EQ.1)GO TO 245
235 DO 255 JC=1,J
255 READ(10,265)Q2(L1,JC)
265 FORMAT(E20.7)
245 READ(10,275)Q3(L1)
275 FORMAT(E20.7)
155 CONTINUE
C END
C BEGINNING OF TRAJECTORIES AND PERFORMANCE ACQUISITION
    DO 304 IC=1,J
304 X(IC,1)=X1(IC,1)
    NSS=NS
    DO 305 K=1,N
    IF(NR.EQ.1)GO TO 306
    IF(K.GT.N+1-NS)NSS=NSS-1
306 CONTINUE
    MT=MTT(NSS)
    IVV(1)=0
    DO 315 IC=1,I
315 IV(1,IC)=1
    IV(1,1)=0
    DO 336 L=1,I
336 IM(L)=1
    DO 325 L1=1,MT
    IVV(1)=IVV(1)+1
    IF(IVV(1).EQ.MRR(1)+1)IVV(1)=1
    IV(1,1)=IV(1,1)+1
    DO 335 IC=1,I
    IF(IV(1,IC).EQ.M(1,IC)+1)IV(1,IC+1)=IV(1,IC+1)+1
335 IF(IV(1,IC).EQ.M(1,IC)+1)IV(1,IC)=1
    IF(NR.EQ.2)GO TO 307
    IF(NC+NQ1.EQ.2)GO TO 317
    DO 345 JR=1,J
    DO 345 JC=1,J
    IF(K.GT.1)GO TO 327
    READ(10,303)PS1(L1,JR,JC)
303 FORMAT(E20.7)
327 P1(JR,JC)=PS1(L1,JR,JC)
345 CONTINUE
317 DO 355 JC=1,J
    IF(K.GT.1)GO TO 337
    READ(10,313)PS2(L1,JC)

```

```

313  FORMAT(E20.7)
337  P2(1,JC)=PS2(L1,JC)
355  CONTINUE
      IF(K.GT.1)GO TO 347
      READ(10,323)PS3(L1)
323  FORMAT(E20.7)
347  P3=PS3(L1)
      GO TO 357
307  IF(NC+NQ1.EQ.2)GO TO 367
      DO 365 JR=1,J
      DO 365 JC=1,J
365  READ(10,333)P1(JR,JC)
333  FORMAT(E20.7)
367  DO 375 JC=1,J
375  READ(10,343)P2(1,JC)
343  FORMAT(E20.7)
      READ(10,353)P3
353  FORMAT(E20.7)
357  CONTINUE
      IF(NC+NQ1.EQ.2)U1=0.0
      IF(NC+NQ1.EQ.2)GO TO 377
      CALL GMPRD(P1,X,PX,J,J,1)
      CALL GTPRD(X,PX,U1,J,1,1)
377  CALL GMPRD(P2,X,U2,1,J,1)
      HEQ=U1+U2+P3
      IF(L1.EQ.1)HT=HEQ
      IF(L1.EQ.1)IT1=IVV(1)
      IF(HEQ.GE.HT) GO TO 325
      DO 385 L=1,I
385  IM(L)=IV(1,L)
      HT=HEQ
      IT1=IVV(1)
325  CONTINUE
C OC  IW(K)=IT1
C OC
C OC  DO 395 IR=1,I
C OC
C OC  IWU(K,IR)=IM(IR)
C OC
395  U(K,IR)=(SH(IR)+EX(IR)*(1.0*(M(1,IR)-1-2*(IM(IR)-1)))/((
      -1)*1.0))M(1,IR)
      DO 405 JR=1,J
405  BA(JR,1)=B(IT1,JR)
      Q3A=Q3(IT1)
      M1=IT1
      M2=IT1
      M3=IT1
      IF(NC.EQ.1)M1=1
      IF(NQ1.EQ.1)M2=1
      IF(NQ2.EQ.1)M3=1
      DO 415 JR=1,J
      DO 415 JC=1,J
      AA(JR,JC)=A(M1,JR,JC)

```



```

      Q1A(JR,JC)=Q1(M2,JR,JC)
415    Q2A(1,JC)=Q2(M3,JC)
      CALL GMPRD(AA,X,PX,J,J,1)
      CALL GMADD(PX,BA,X,J,1)
      DO 425 JR=1,J
425    XO(K+1,JR)=(X(JR,1))
      CALL GMPRD(Q1A,X,PX,J,J,1)
      CALL GTPRD(X,PX,U1,J,1,1)
      CALL GMPRD(Q2A,X,U2,1,J,1)
305    PERF=PERF+U1+U2+Q3A
      PER(ITER)=PERF
      PERF=0.0
C END
C START OF DISPLAY
C END OF DISPLAY
C OC
      IF(NH.NE.1)GO TO 5
      IF(NC+NT.EQ.4)GO TO 5
      WRITE(6,2003)
2003    FORMAT(15X,'K',8X,'I',6X,'U',19X,'HU',22X,'MC')
      DO 2005 IR=1,I
      DO 2015 JR=1,J
2015    X(JR,1)=XO(N+1,JR)*2.0
      CALL GTPRD(X,Q1UC,Q2A,J,1,J)
      CALL GMADD(Q2A,Q2UC,G,1,J)
      DO 2035 K=1,N
      K2=N+1-K
      DO 2045 MP=1,8
2045    MY(MP)=IW(N+1-K)
      IF(NC.EQ.1)MY(1)=1
      IF(NQ1.EQ.1)MY(2)=1
      IF(NQ2.EQ.1)MY(3)=1
      IF(NAU.EQ.1)MY(4)=1
      IF(NBU.EQ.1)MY(5)=1
      IF(NQU1.EQ.1)MY(6)=1
      IF(NQU2.EQ.1)MY(7)=1
      IF(NQU3.EQ.1)MY(8)=1
      DO 2055 JR=1,J
      DO 2055 JC=1,J
      AA(JR,JC)=A(MY(1),JR,JC)
      X(JR,1)=XO(K2,JR)
      Q1A(JR,JC)=Q1(MY(2),JR,JC)
      Q2A(1,JC)=Q2(MY(3),JC)
      AP(JR,JC)=AU(IR,JR,JC,MY(4))
      BP(JC,1)=BU(IR,JC,MY(5))
      P1(JR,JC)=Q1U(IR,JR,JC,MY(6))
2055    P2(1,JC)=Q2U(IR,JC,MY(7))
      P3=Q3U(IR,MY(8))
      CALL GMPRD(AP,X,UA,J,J,1)
      CALL GMADD(UA,BP,P6,J,1)
      CALL GMPRD(G,P6,U1,1,J,1)
      CALL GMPRD(P2,X,U2,1,J,1)
      CALL GMPRD(P1,X,UA,J,J,1)
      CALL GTPRD(X,UA,U3,J,1,1)

```

```

      HU(K2)=P3+U1+U2+U3
      CALL GTPRD(X,Q1A,PL,J,1,J)
      CALL GMPRD(G,AA,P5,1,J,J)
      DO 2065 JC=1,J
2065   G(1,JC)=P5(1,JC)+PL(1,JC)*2.0+Q2A(1,JC)
      IF(IWU(K2,IR).EQ.1)GO TO 2037
      IF(IWU(K2,IR).EQ.M(1,IR))GO TO 2047
      HT=SQRT(HU(K2)**2)
      IF(HT-DELTA.GT.0.0)MC(K2)=10
      IF(HT-DELTA.LE.0.0)MC(K2)=0
      GO TO 2036
2037   IF(HU(K2).LE.0.0)MC(K2)=-1
      IF(HU(K2).GT.0.0)MC(K2)=-11
      GO TO 2036
2047   IF(HU(K2).GE.0.0)MC(K2)=1
      IF(HU(K2).LT.0.0)MC(K2)=11
2036   WRITE(6,2013)K2,I,U(K2,IR),HU(K2),MC(K2)
2013   FORMAT(15X,I4,5X,I2,5X,F10.4,10X,E14.7,10X,I3)
2035   CONTINUE
      LJ=0
      DO 2075 K=1,N
      IF(MC(K).GT.5)LJ=LJ+1
2075   IF(MC(K).LT.-5)LJ=LJ+1
      PCV=LJ*100.0/(N*1.0)
      WRITE(6,2023)PCV
2023   FORMAT(30X,'PCV=',F10.4)
2005   CONTINUE
C OC
5     CONTINUE
C BEGINNING OF DISPLAYING OF NORMALIZED PERFORMANCES
      NP=ITER
      DO 525 L=1,NP
      RPER(L)=PER(L)/PER(1)
525   WRITE(6,583)L,PER(L),L,RPER(L)
583   FORMAT(20X,'PER(' ,I2,' )=' ,E14.7,5X,'NPER(' ,I2,' )=' ,E14.7)
C END
      END

```

E.3 COMPUTER PROGRAM THAT DESIGNS QLQ AND SNN CONTROL LAWS

E.3.1 Purpose of Program

The general purpose of the program of Section E.3.3 is to design QLQ and SNN (for single input processes only) control laws for the class of process and performance criterion giv-

en by Equations (4.5.1.12) and (4.5.1.16) where the process Equation (4.5.1.12) is discretized according to Equation (4.5.1.26): the parameters of the designed QLQ control law, SNN control law, performance criterion Equation (4.5.1.16), and discretized process Equation (4.5.1.26) can then be used by the simulation program of Section E.4.4.

The specific program tasks that accomplish the above general purpose are as follows:

Task 1

To determine A and $B(u(k))$ for all $u(k)$, by evaluating the RHS of Equations (4.5.1.27) and (4.5.1.28) for each member in the space given by (4.5.1.8). A and $B(u(k))$ for all $u(k)$ are then used for the design of the QLQ and SNN control laws and are also stored for future use by the simulation program of Section E.4.4;

Task 2

To determine the QLQ control law gain, $K(k)$, for all $k \in \{0, \dots, N-1\}$, where $K(k)$ is derived by using Equation (1.2.2.5), the Ricatti difference Equation (1.2.2.6), the performance criterion parameters of Equation (4.5.1.16), and the parameters of the process Equation (4.5.1.26). $K(k)$ for all k is also stored for future use by the simulation program of Section E.4.4.

Task 3

To determine the SNN control law gain, $K(k)$, for all $k \in (0, \dots, N-1)$, where $K(k)$ is derived by using Equations (1.2.3.8)-(1.2.3.9), the difference Equation (1.2.3.10), the performance criterion parameters of Equation (4.5.1.16), and the parameters of the process Equation (4.5.1.26). $K(k)$ for all k is also stored for future use by the simulation program of Section E.4.4; and

Task 4

To store, for later use by the simulation program of Section E.4.4, the process and performance criterion parameters given by Q , R , N , V , n , m , and $(c_1^i, \dots, c_M^i)$ for $i = 1, \dots, m$;

E.3.2 Program Inputs

The following input arguments must be provided to the design program of Section E.3.3:

N	Number of process stages N .
I	Number of inputs m .
J	Number of states n .
V	Sampling time V in seconds.
$Cl(i)$	Lowest quantization level c_1^i for $i = 1, \dots, m$.
$CM(i)$	Highest quantization level c_M^i for $i = 1, \dots, m$.
NW	Number of quantization levels M .

$A(i,j)$ Component of matrix A' , $A'_{i,j}$, for all i and j .

$B(i,1)$ Component of vector $B'(u(t))$, $B'_L(u(t))$, for $i = 1, \dots, n$ where $B'(u(t)) = B'u(t)$ and $u(t)$ is given by $(U(1,1), \dots, U(m,1))$.

$Q1(i,j)$ Component of matrix Q , $Q_{i,j}$, for all i and j .

$Q3$ $u^T(k)Ru(k)$ where $u(k)$ is given by $(U(1,1), \dots, U(m,1))$.

$AL(i,j)$ Component of matrix A' , $A'_{i,j}$, for all i and j .

$BL(i,j)$ Component of matrix B' , $B'_{i,j}$, for all i and j .

$Q(i,j)$ Component of matrix Q , $Q_{i,j}$, for all i and j .

$R(i,j)$ Component of matrix R , $R_{i,j}$, for all i and j .

where:

a) N , I , J , V , $C1(\cdot)$, $CM(\cdot)$, and NW are initialized between the comment statement

C BEGINNING OF INITIALIZATION 1

and the comment statement

C END OF INITIALIZATION 1;

b) $A(\cdot, \cdot)$, $B(\cdot, 1)$, $Q1(\cdot, \cdot)$, and $Q3$ are initialized between the comment statement

C BEGINNING OF INITIALIZATION 2

and the comment statement

```
C END OF INITIALIZATION 2;
```

c) AL(...), BL(...), Q(...), and R(...) are initialized between the comment statement

```
C BEGINNING OF INITIALIZATION 3
```

and the comment statement

```
C END OF INITIALIZATION 3;
```

and

d) The values of N, I, and J are used to dimension the matrices given by the dimension statements at the beginning of the program, according to the comment dimension statements also shown at the beginning of the program.

E.3.3 Design Program

```
C      DIMENSION IV(1,I),MRR(1),SH(I),EX(I)
C      DIMENSION B(J,1),NA(2),NB(2),ND(2),
C      A1D(J*J),A2D(J*J),A3D(J*J),DU(2*J*J)
C      DIMENSION BT(J,1),C1(I),CM(I)
C      DIMENSION Q1(J,J),Q2(1,J),UC(I,1),U(I,1)
C      DIMENSION AUC(J,J),BUC(J,1),Q1UC(J,J),Q2UC(1,J)
C      DIMENSION FI(J,J),A4D(J*J),AL(J,J),NA1(2),NB1(2),ND1(2)
C      DIMENSION A5D(J*J),A6D(J*J),DU1(2*J*J),BL(J,I),ABL(J,I)
C      DIMENSION Q(J,J),P(J,J),AA(J,J),BAL(I,I),R(I,I),RLAB(I,I)
C      DIMENSION N7(I),N8(I),BLA(I,J),BBA(I,J)
C      DIMENSION M(1,I),A(J,J),AA1(J,J),AA2(J,J),RKLQ(N+1,I,J)
C OC
C      DIMENSION AU(I,J,J),BU(I,J),QU1(I,J,J),QU2(I,J),QU3(I)
C OC
      DIMENSION IV(1,1),MRR(1),SH(1),EX(1)
      DIMENSION B(3,1),NA(2),NB(2),ND(2),
      A1D(9),A2D(9),A3D(9),DU(18)
      DIMENSION BT(3,1),C1(I),CM(I)
      DIMENSION Q1(3,3),Q2(1,3),UC(1,1),U(1,1)
```

```

        DIMENSION AUC(3,3),BUC(3,1),Q1UC(3,3),Q2UC(1,3)
        DIMENSION FI(3,3),A4D(9),AL(3,3),NA1(2),NB1(2),ND1(2)
        DIMENSION A5D(9),A6D(9),DU1(18),BL(3,1),ABL(3,1)
        DIMENSION Q(3,3),P(3,3),AA(3,3),BAL(1,1),R(1,1),PLAB(1,1)
        DIMENSION N7(1),N8(1),BLA(1,3),BBA(1,3)
        DIMENSION M(1,1),A(3,3),AA1(3,3),AA2(3,3),RKLQ(41,1,3)
C OC
        DIMENSION AU(1,5,5),BU(1,5),QU1(1,5,5),QU2(1,5),QU3(64)
C OC
C OC
        NAU=1
        NBU=1
        NQU1=1
        NQU2=1
        NQU3=2
C OC
C BEGINNING OF INITIALIZATION 1
        N=20
        I=1
        J=3
        V=0.01
        C1(1)=-1.0
        CM(1)=1.0
        NW=3
C END OF INITIALIZATION 1
        NH=2
        NR=2
        NC=1
        NT=2
        NQ1=1
        NQ2=1
        DO 11 LZ=1,I
        SH(LZ)=(C1(LZ)+CM(LZ))/2.0
        EX(LZ)=(CM(LZ)-C1(LZ))/2.0
11      UC(LZ,1)=0.0
        WRITE(10,102)MW
102     FORMAT(I20)
        WRITE(10,2103)NH,NT,NR,NC,NQ1,NQ2,I,J,N
2103    FORMAT(I2,I2,I2,I2,I2,I2,I2,I2,I4)
        DO 2115 L=1,I
2115    WRITE(10,2125)SH(L)
2125    FORMAT(E20.7)
        DO 2135 L=1,I
2135    WRITE(10,2113)EX(L)
2113    FORMAT(E20.7)
C OC
        IF(NH.NE.1)GO TO 1207
        WRITE(10,1203)NAU,NBU,NQU1,NQU2,NQU3
1203    FORMAT(I4,I4,I4,I4,I4)
1207    CONTINUE
C OC
        MCC=1
C DO LOOPS THAT VARY NSC
        DO 5 M1=MW,1

```

```

      DO 25 IC=1,I
25      M(1,IC)=M1
C END OF DO LOOPS THAT VARY NSC
C STORING A(U(K)), B(U(K)), Q1(U(K)), Q2(U(K)), AND Q3(U(K))
      MR=1
      DO 35 IC=1,I
      MR=MR*M(1,IC)
35      MRR(1)=MR
      MT=MRR(1)
      DO 45 IC=1,I
45      IV(1,IC)=1
      IV(1,1)=0
      DO 55 L1=1,MT
      IF(MCC.NE.1)GO TO 1008
      DO 1002 IR=1,I
1002     U(IR,1)=UC(IR,1)
      GO TO 1018
1008     CONTINUE
      IV(1,1)=IV(1,1)+1
      DO 65 IC=1,I
      IF(IV(1,IC).EQ.M(1,IC)+1)IV(1,IC+1)=IV(1,IC+1)+1
65      IF(IV(1,IC).EQ.M(1,IC)+1)IV(1,IC)=1
      DO 85 IR=1,I
85      U(IR,1)=SH(IR)+EX(IR)*(1.0*(M(1,IR)-1-2*(IV(1,IR)-1)))
      /((M(1,IR)-1)*1.0)
1018     CONTINUE
C BEGINNING OF INITIALIZATION 2
      DO 4444 JR=1,J
      DO 4444 JC=1,J
4444     A(JR,JC)=0.0
      DO 4445 JR=1,J
4445     B(JR,1)=0.0
      A(1,2)=1.0
      A(2,2)=-10.0
      A(3,3)=-10.0
      B(2,1)=1000.0*U(1,1)
      Q1(1,1)=1.0
      Q1(1,3)=-1.0
      Q1(3,1)=-1.0
      Q1(3,3)=1.0
      Q3=0.0
C END OF INITIALIZATION 2
      DO 4446 JR=1,J
      DO 4446 JC=1,J
4446     Q2(1,JR)=0.0
      IF(NT.EQ.1)GO TO 67
      IF(L1.EQ.1)GO TO 47
      IF(NC.EQ.1)GO TO 57
47      CONTINUE
      DO 95 JC=1,J
      DO 95 JR=1,J
      JP=(JC-1)*J+JR
95      A1D(JP)=A(JR,JC)
      DO 105 JR=1,2

```



```

      NA(JR)=J
      NB(JR)=J
105    ND(JR)=J
      CALL EXPINT(A1D,NA,A2D,NB,A3D,ND,V,0,DU)
      DO 115 JC=1,J
      DO 115 JR=1,J
      JP=(JC-1)*J+JR
      AA1(JR,JC)=A2D(JP)
115    AA2(JR,JC)=A3D(JP)
57     CONTINUE
      CALL GMPRD(AA2,B,BT,J,J,1)
67     CONTINUE
      IF(MCC.NE.1)GO TO 1028
      DO 1016 JR=1,J
      DO 1016 JC=1,J
      IF(NT.EQ.1)GO TO 1038
      A(JR,JC)=AA1(JR,JC)
      B(JR,1)=BT(JR,1)
1038   AUC(JR,JC)=A(JR,JC)
      BUC(JR,1)=B(JR,1)
      Q1UC(JR,JC)=Q1(JR,JC)
1016   Q2UC(1,JR)=Q2(1,JR)
      Q3UC=Q3
      MCC=2
      GO TO 1008
1028   CONTINUE
      DO 125 JR=1,J
      IF(NT.EQ.1)GO TO 77
      B(JR,1)=BT(JR,1)
77     CONTINUE
125    CONTINUE
      IF(L1.EQ.1)GO TO 27
      IF(NC.EQ.1)GO TO 28
27     CONTINUE
      DO 135 JR=1,J
      DO 135 JC=1,J
      IF(NT.EQ.1)GO TO 87
      A(JR,JC)=AA1(JR,JC)
87     CONTINUE
135    CONTINUE
28     CONTINUE
C OC
      IF(NH.NE.1)GO TO 1007
      IF(NC+NT.EQ.4)GO TO 1007
C BEGINNING OF INITIALIZATION
      DO 1024 JR=1,J
      DO 1024 JC=1,J
      AU(1,JR,JC)=0.0
      BU(1,JR)=0.0
      QU1(1,JR,JC)=0.0
1024   QU2(1,JR)=0.0
      QU3(1)=2*U(1,1)
      BU(1,5)=30.0
C END OF INITIALIZATION

```

```

      IF(NT.EQ.1)GO TO 1017
      DO 1005 IR=1,I
      DO 1015 JR=1,J
1015  BT(JR,1)=BU(IR,JR)
      CALL GMPRD(AA2,BT,BB,J,J,1)
      DO 1025 JR=1,J
1025  BU(IR,JR)=BB(JR,1)
1005  CONTINUE
1017  CONTINUE
      IF(L1.GT.1)GO TO 1027
      DO 1035 JR=1,J
      DO 1035 JC=1,J
1035  WRITE(10,1003)Q1UC(JR,JC)
1003  FORMAT(E20.7)
      DO 1045 JC=1,J
1045  WRITE(10,1023)Q2UC(1,JC)
1023  FORMAT(E20.7)
1027  IF(L1.EQ.1)GO TO 1037
      IF(NAU.EQ.1)GO TO 1047
1037  DO 1055 IR=1,I
      DO 1055 JR=1,J
      DO 1055 JC=1,J
1055  WRITE(10,1013)AU(IR,JR,JC)
1013  FORMAT(E20.7)
1047  IF(L1.EQ.1)GO TO 1057
      IF(NBU.EQ.1)GO TO 1067
1057  DO 1065 IR=1,I
      DO 1065 JR=1,J
1065  WRITE(10,1033)BU(IR,JR)
1033  FORMAT(E20.7)
1067  IF(L1.EQ.1)GO TO 1077
      IF(NQU1.EQ.1)GO TO 1087
1077  DO 1075 IR=1,I
      DO 1075 JR=1,J
      DO 1075 JC=1,J
1075  WRITE(10,1043)QU1(IR,JR,JC)
1043  FORMAT(E20.7)
1087  IF(L1.EQ.1)GO TO 1097
      IF(NQU2.EQ.1)GO TO 1107
1097  DO 1085 IR=1,I
      DO 1085 JC=1,J
1085  WRITE(10,1053)QU2(IR,JC)
1053  FORMAT(E20.7)
1107  IF(L1.EQ.1)GO TO 1117
      IF(NQU3.EQ.1)GO TO 1007
1117  DO 1006 IR=1,I
1006  WRITE(10,1063)QU3(IR)
1063  FORMAT(E20.7)
1007  CONTINUE
C OC
      IF(L1.EQ.1)GO TO 7
      IF(NC.EQ.1)GO TO 17
7      DO 145 JR=1,J
      DO 145 JC=1,J

```

```

145  WRITE(10,3)A(JR,JC)
3    FORMAT(E20.7)
17   DO 155 JR=1,J
155  WRITE(10,13)B(JR,1)
13   FORMAT(E20.7)
18   CONTINUE
     IF(L1.EQ.1)GO TO 97
     IF(NQ1.EQ.1)GO TO 98
97   CONTINUE
     DO 165 JR=1,J
     DO 165 JC=1,J
165  CONTINUE
98   CONTINUE
     IF(L1.EQ.1)GO TO 107
     IF(NQ2.EQ.1)GO TO 108
107  CONTINUE
     DO 175 JR=1,J
175  CONTINUE
108  CONTINUE
     IF(L1.EQ.1)GO TO 117
     IF(NQ1.EQ.1)GO TO 127
117  DO 185 JR=1,J
     DO 185 JC=1,J
185  WRITE(10,23)Q1(JR,JC)
23   FORMAT(E20.7)
127  IF(L1.EQ.1)GO TO 137
     IF(NQ2.EQ.1)GO TO 147
137  DO 195 JR=1,J
195  WRITE(10,33)Q2(1,JR)
33   FORMAT(E20.7)
147  WRITE(10,43)Q3
43   FORMAT(E20.7)
55   CONTINUE
C BEGINNING OF INITIALIZATION 3
     DO 4001 JR=1,J
     DO 4001 JC=1,J
     DO 4001 IR=1,I
     DO 4001 IC=1,I
     AL(JR,JC)=0.0
     BL(JR,IR)=0.0
     R(IR,IC)=0.0
4001 Q(JR,JC)=0.0
     AL(1,2)=1.0
     AL(2,2)=-10.0
     AL(3,3)=-10.0
     BL(2,1)=1000.0
     Q(1,1)=1.0
     Q(1,3)=-1.0
     Q(3,1)=-1.0
     Q(3,3)=1.0
C END OF INITIALIZATION 3
     DO 3004 JR=1,J
     DO 3004 JC=1,J
     FI(JR,JC)=0.0

```

```

3004  FI(JR,JR)=1.0
      IF(NT.EQ.1)GO TO 3007
      DO 3005 JC=1,J
      DO 3005 JR=1,J
      JP=(JC-1)*J+JR
3005  A4D(JP)=AL(JR,JC)
      DO 3015 JC=1,2
      NA1(JC)=J
      NB1(JC)=J
3015  ND1(JC)=J
      CALL EXPINT(A4D,NA1,A5D,NB1,A6D,ND1,V,0,DU1)
      DO 3025 JC=1,J
      DO 3025 JR=1,J
      JP=(JC-1)*J+JR
      AL(JR,JC)=A5D(JP)
3025  AA2(JR,JC)=A6D(JP)
      CALL GMPRD(AA2,BL,ABL,J,J,I)
      DO 3035 JR=1,J
      DO 3035 IC=1,I
3035  BL(JR,IC)=ABL(JR,IC)
3007  CONTINUE
      LF=3
      IF(I.GT.1)LF=2
      JD=I-1
      IF(JD.EQ.0)GO TO 3046
      DO 3036 IR=1,JD
3036  IF(M(1,IR).NE.M(1,IR+1))LF=2
3046  JE=1
      IF(MCC.NE.2)JE=LF
      IF(JE**LF.EQ.4)GO TO 3018
      DO 3002 LL=JE,LF
      DO 3045 JR=1,J
      DO 3045 JC=1,J
3045  P(JR,JC)=Q(JR,JC)
      DO 3001 K=1,N
      CALL GMPRD(P,AL,A,J,J,J)
      CALL GTPRD(AL,A,AA,J,J,J)
      CALL GMPRD(P,BL,ABL,J,J,I)
      CALL GTPRD(BL,ABL,BAL,J,I,I)
      CALL GMADD(R,BAL,RLAB,I,I)
      CALL MINV(RLAB,I,D,N7,N8)
      CALL GTPRD(BL,A,BLA,J,I,J)
      IF(LL.GT.1)GO TO 3017
      CALL GMADD(AA,Q,P,J,J)
      GO TO 3027
3017  IF(LL.GT.2)GO TO 3037
      CALL GMADD(AA,Q,AA1,J,J)
      U1=-1.0
      CALL SMPY(RLAB,U1,BAL,I,I,0)
      CALL GTPRD(BLA,BAL,ABL,I,J,I)
      CALL GMPRD(ABL,BLA,AA,J,I,J)
      CALL GMADD(AA1,AA,P,J,J)
      GO TO 3027
3037  IF(K.GT.1)GO TO 3077

```

```

TEST=1
RJ1=((M(1,1)-1)*1.0)/2.0
K1=1
RIS1=2*RJ1**2.0-RJ1
1029 IF(K1.EQ.RJ1)GO TO 1019
RIS1=RIS1+TEST*4*K1**2.0
K1=K1+1
GO TO 1029
1019 K1=1
RIS2=-RJ1*((2*RJ1-1)**3.0)*(2*RJ1-3)
1049 IF(K1.EQ.RJ1)GO TO 1039
RIS2=RIS2+TEST*K1*((2*K1+1)**3.0)*(2*K1+3)+((2*K1-1)**3.0)*
(-2*K1+3)
K1=K1+1
GO TO 1049
1039 RK1=RIS2/(12.0*(2*RJ1+1)**5.0)-(1.0*(RJ1**2.0))/(3.0*(2*
**2.0)+(1.0*RJ1)/(4.0*(2*RJ1+1))
RK2=RIS1/(6.0*(2*RJ1+1)**3.0)-(1.0*(RJ1**2.0))/(3.0*(2*
**2.0)+(1.0*RJ1)/(6.0*(2*RJ1+1))
RK3=(225*J+180.0)*(RK2-RK1)/(1.0*(5*J+4))
RK4=45*(5*RK1-9*RK2)/(1.0*(5*J+4))
3077 CALL GMADD(AA,Q,AA1,J,J)
CALL GTPRD(BLA,BLA,AA2,1,J,J)
CALL MTRA(BLA,ABL,1,J,0)
CALL GMPRD(BLA,ABL,U2,1,J,1)
U3=U2*RK4*RLAB(1,1)
CALL SMPY(FI,U3,A,J,J,0)
U3=RK3*RLAB(1,1)
CALL SMPY(AA2,U3,AA,J,J,0)
CALL GMADD(AA1,AA,AA2,J,J)
CALL GMADD(AA2,A,P,J,J)
3027 CALL GMPRD(RLAB,BLA,BBA,I,I,J)
IF(NR.EQ.2)GO TO 3047
IF(K.NE.N)GO TO 3001
3047 DO 3055 IR=1,I
DO 3055 JC=1,J
3055 RKLQ(N+1-K,IR,JC)=BBA(IR,JC)
3001 CONTINUE
NX=N
IF(NR.EQ.1)NX=1
DO 3056 K=1,NX
DO 3056 IR=1,I
DO 3056 JC=1,J
3056 WRITE(10,3057)RKLQ(K,IR,JC)
3057 FORMAT(E20.7)
3002 CONTINUE
MCC=3
3018 CONTINUE
5 CONTINUE
END

```

E.4 COMPUTER PROGRAM THAT SIMULATES THE QLQ AND SNN CONTROL LAWS.

E.4.1 Purpose of Program

The general purpose of the program of Section E.4.4 is to simulate the application of QLQ and SNN control laws, which have been designed using the design program of Section E.3.3, to the process Equation (4.5.1.26) for some given initial condition. For each of the considered control laws, this simulation yields specific state and control trajectories for the process Equation (4.5.1.26), and a numerical value for the RHS of the performance criterion Equation (4.5.1.16).

The specific program tasks that accomplish the above general purpose are as follows:

Task 1

The evaluation of $u^{QLQ}(x(k))$ for $k \in (0, \dots, N-1)$ from Equation (1.2.2.7). After evaluating $u^{QLQ}(x(k))$ for a specific value of k , $x(k+1)$ is evaluated from Equation (4.5.1.26);

Task 2

The evaluation of $u^{SNN}(x(k))$ for $k \in (0, \dots, N-1)$ from Equation (1.2.3.7). After evaluating $u^{SNN}(x(k))$ for a specific value of k , $x(k+1)$ is evaluated from Equation (4.5.1.26);

Task 3

The evaluation of the RHS of the performance criterion Equation (4.5.1.16) making use of the derived values of $u^{QLQ}(x(k))$ and $x(k+1)$ for all $k \in (0, \dots, N-1)$; and

Task 4

The evaluation of the RHS of the performance criterion Equation (4.5.1.16) making use of the derived values of $u^{SNN}(x(k))$ and $x(k+1)$ for all $k \in (0, \dots, N-1)$.

E.4.2 Input Arguments

The inputs that are required by the simulation program are:

1) The stored parameters of the design program of Section E.3.3;

2) The initial condition, $x(0)$, that we wish to test: the initial condition must be placed between the comment statement

C START OF INITIAL CONDITION

and the comment statement

C END OF INITIAL CONDITION

making use of the matrix $X1(j,1)$ for $j = 1, \dots, n$ where

$X1(j,1) = x(0)$ for all j :

and

3) The values of N , I , and J are used to dimension the matrices given by the dimension statements at the beginning of the program, according to the comment dimension statements also shown at the beginning of the program. Note that N , I , and J were defined earlier in Section E.3.2.

E.4.3 Outputs that are Available for Display

Between the comment statement

C START OF DISPLAY

and the comment statement

C END OF DISPLAY

we have the following outputs available for display:

1) The control trajectory $u(k)$ for $k \in (0, \dots, N-1)$ in the form of the matrix

$$U(i,j)$$

for all

$$i \in (1, \dots, N)$$

and

$$j \in (1, \dots, m)$$

where

$$U(i,j) = u_j(i-1) \text{ for all } i \text{ and } j.$$

It is noted that when the internal program parameter, LL , is given by 2 we have that $U(i,j)$ for all i and j is the control trajectory produced by the QLQ control law and when LL

is given by 3 we have that $U(i,j)$ for all i and j is the control trajectory produced by the SNN control law;

2) The state trajectory $x(k)$ for all $k \in (0, \dots, N)$ in the form of the matrix

$$XO(i,j)$$

for all

$$i \in (1, \dots, N+1)$$

and

$$j \in (1, \dots, n)$$

where

$$XO(i,j) = x_j(i-1) \text{ for all } i \text{ and } j.$$

Again it is noted that when LL is given by 2 we have that $XO(i,j)$ for all i and j is the state trajectory produced by the QLQ control law and when LL is given by 3 we have that $XO(i,j)$ for all i and j is the state trajectory produced by the SNN control law;

3) The value of the RHS of the performance criterion Equation (4.5.1.16), P_{QLQ} , that results when we apply the QLQ control law to the process Equation (4.5.1.26). This performance value is displayed when $LL = 2$ via

$$PER(2)$$

with

$$P_{QLQ} = PER(2)$$

and

4) The value of the RHS of the performance criterion Equation (4.5.1.16), P_{SNN} , that results when we apply the SNN control law to the process Equation (4.5.1.26). This performance value is displayed when LL = 3 via

PER(3)

where

$$P_{SNN} = \text{PER}(3).$$

E.4.4 Simulation Program

```

C      DIMENSION X(J,1),PX(J,1),IM(I),IV(1,I+1),
C      M(20,I),MRR(20),PER(10),X1(J,1),X2(J,1)
C      DIMENSION P1(J,J),P2(1,J),BA(J,1),AA(J,J),Q1A(J,J)
C      DIMENSION U(N+1,I),XO(N+1,J),SH(I),EX(I),UC(I),Q2A(1,J)
C      DIMENSION B(3000,J),A(1,J,J),Q1(1,J,J),
C      Q2(1,J),Q3(3000),RK(3,I,J)
C OC
C      DIMENSION Q1UC(J,J),Q2UC(1,J)
C      DIMENSION AU(I,J,J,1),BU(I,J,1),Q1U(I,J,J,1),
C      Q2U(I,J,1),Q3U(I,1)
C      DIMENSION IWU(N,I),IW(N),G(1,J),MY(8)
C      DIMENSION AP(J,J),BP(J,1),PL(1,J),P5(1,J)
C      DIMENSION UA(J,1),P6(J,1),HU(N),MC(N)
C OC
C      DIMENSION X(3,1),PX(3,1),IM(1),IV(1,2),
C      M(1,1),MRR(1),PER(6),X1(3,1),X2(3,1)
C      DIMENSION P1(3,3),P2(1,3),BA(3,1),AA(3,3),Q1A(3,3)
C      DIMENSION U(21,1),XO(21,3),SH(1),EX(1),UC(1),Q2A(1,3)
C      DIMENSION B(3,3),A(1,3,3),Q1(1,3,3),
C      Q2(1,3),Q3(3),RK(3,1,3)
C OC
C      DIMENSION Q1UC(5,5),Q2UC(1,5)
C      DIMENSION AU(1,5,5,1),BU(1,5,1),Q1U(1,5,5,1),
C      Q2U(1,5,1),Q3U(1,64)
C      DIMENSION IWU(100,1),IW(100),G(1,5),MY(8)
C      DIMENSION AP(5,5),BP(5,1),PL(1,5),P5(1,5)
C      DIMENSION UA(5,1),P6(5,1),HU(100),MC(100)
C OC
C OC
C      DELTA=0.001
C OC
C START OF INITIAL CONDITION
C      X1(3,1)=-1.5
C      X1(1,1)=0.0

```

```

      X1(2,1)=0.0
C END OF INITIAL CONDITION
      MCC=0
      ITER=0
      PERF=0.0
      REWIND 10
      READ(10,102)MW
102   FORMAT(I20)
      READ(10,103)NH,NT,NR,NC,NQ1,NQ2,I,J,N
103   FORMAT(I2,I2,I2,I2,I2,I2,I2,I2,I4)
      DO 56 IY=1,J
56    XO(1,IY)=X1(IY,1)
      DO 115 L=1,I
115   READ(10,125)SH(L)
125   FORMAT(E20.7)
      DO 135 L=1,I
135   READ(10,113)EX(L)
113   FORMAT(E20.7)
C OC
      IF(NH.NE.1)GO TO 1207
      READ(10,1203)NAU,NBU,NQU1,NQU2,NQU3
1203  FORMAT(I4,I4,I4,I4,I4)
1207  CONTINUE
C OC
C BEGINNING OF DO LOOPS THAT VARY NSC
      DO 5 MM=MW,1
      DO 25 IC=1,I
25    M(1,IC)=MM
C END
C BEGINNING OF PARAMETERS READING
      MR=1
      DO 35 IC=1,I
      MR=MR*M(1,IC)
35    MRR(1)=MR
      MT=MRR(1)
      DO 155 L1=1,MT
C OC
      IF(NH.NE.1)GO TO 1007
      IF(NC+NT.EQ.4)GO TO 1007
      IF(L1.GT.1)GO TO 1027
      DO 1035 JR=1,J
      DO 1035 JC=1,J
1035  READ(10,1003)Q1UC(JR,JC)
1003  FORMAT(E20.7)
      DO 1045 JC=1,J
1045  READ(10,1023)Q2UC(1,JC)
1023  FORMAT(E20.7)
1027  IF(L1.EQ.1)GO TO 1037
      IF(NAU.EQ.1)GO TO 1047
1037  DO 1055 IR=1,I
      DO 1055 JR=1,J
      DO 1055 JC=1,J
1055  READ(10,1013)AU(IR,JR,JC,L1)
1013  FORMAT(E20.7)

```

```

1047 IF(L1.EQ.1)GO TO 1057
      IF(NBU.EQ.1)GO TO 1067
1057 DO 1065 IR=1,I
      DO 1065 JR=1,J
1065 READ(10,1033)BU(IR,JR,L1)
1033 FORMAT(E20.7)
1067 IF(L1.EQ.1)GO TO 1077
      IF(NQU1.EQ.1)GO TO 1087
1077 DO 1075 IR=1,I
      DO 1075 JR=1,J
      DO 1075 JC=1,J
1075 READ(10,1043)Q1U(IR,JR,JC,L1)
1043 FORMAT(E20.7)
1087 IF(L1.EQ.1)GO TO 1097
      IF(NQU2.EQ.1)GO TO 1107
1097 DO 1085 IR=1,I
      DO 1085 JC=1,J
1085 READ(10,1053)Q2U(IR,JC,L1)
1053 FORMAT(E20.7)
1107 IF(L1.EQ.1)GO TO 1117
      IF(NQU3.EQ.1)GO TO 1007
1117 DO 1116 IR=1,I
1116 READ(10,1063)Q3U(IR,L1)
1063 FORMAT(E20.7)
1007 CONTINUE
C OC

```

```

      IF(L1.EQ.1)GO TO 165
      IF(NC.EQ.1)GO TO 175
165 DO 185 JR=1,J
      DO 185 JC=1,J
185 READ(10,123)A(L1,JR,JC)
123 FORMAT(E20.7)
175 DO 195 JC=1,J
195 READ(10,133)B(L1,JC)
133 FORMAT(E20.7)
      IF(L1.EQ.1)GO TO 205
      IF(NQ1.EQ.1)GO TO 215
205 DO 225 JR=1,J
      DO 225 JC=1,J
225 READ(10,143)Q1(L1,JR,JC)
143 FORMAT(E20.7)
215 IF(L1.EQ.1)GO TO 235
      IF(NQ2.EQ.1)GO TO 245
235 DO 255 JC=1,J
255 READ(10,265)Q2(L1,JC)
265 FORMAT(E20.7)
245 READ(10,275)Q3(L1)
275 FORMAT(E20.7)
155 CONTINUE
C END

```

```

MCC=MCC+1
LF=3
IF(I.GT.1)LF=2
JD=I-1

```

```

425      XO(K+1,JR)=(X(JR,1))
        CALL GMPRD(Q1A,X,PX,J,J,1)
        CALL GTPRD(X,PX,U1,J,1,1)
        CALL GMPRD(Q2A,X,U2,1,J,1)
305      PERF=PERF+U1+U2+Q3A
        PER(ITER)=PERF
        PERF=0.0
C END
C START OF DISPLAY
C END OF DISPLAY
C OC
        IF(NH.NE.1)GO TO 3075
        IF(NC+NT.EQ.4)GO TO 5
        WRITE(6,2003)
2003      FORMAT(15X,'K',8X,'I',6X,'U',19X,'HU',22X,'MC')
        DO 2005 IR=1,I
        DO 2015 JR=1,J
2015      X(JR,1)=XO(N+1,JR)*2.0
        CALL GTPRD(X,Q1UC,Q2A,J,1,J)
        CALL GMADD(Q2A,Q2UC,G,1,J)
        DO 2035 K=1,N
        K2=N+1-K
        DO 2045 MP=1,8
2045      MY(MP)=IW(N+1-K)
        IF(NC.EQ.1)MY(1)=1
        IF(NQ1.EQ.1)MY(2)=1
        IF(NQ2.EQ.1)MY(3)=1
        IF(NAU.EQ.1)MY(4)=1
        IF(NBU.EQ.1)MY(5)=1
        IF(NQU1.EQ.1)MY(6)=1
        IF(NQU2.EQ.1)MY(7)=1
        IF(NQU3.EQ.1)MY(8)=1
        DO 2055 JR=1,J
        DO 2055 JC=1,J
        AA(JR,JC)=A(MY(1),JR,JC)
        X(JR,1)=XO(K2,JR)
        Q1A(JR,JC)=Q1(MY(2),JR,JC)
        Q2A(1,JC)=Q2(MY(3),JC)
        AP(JR,JC)=AU(IR,JR,JC,MY(4))
        BP(JC,1)=BU(IR,JC,MY(5))
        P1(JR,JC)=Q1U(IR,JR,JC,MY(6))
2055      P2(1,JC)=Q2U(IR,JC,MY(7))
        P3=Q3U(IR,MY(8))
        CALL GMPRD(AP,X,UA,J,J,1)
        CALL GMADD(UA,BP,P6,J,1)
        CALL GMPRD(G,P6,U1,1,J,1)
        CALL GMPRD(P2,X,U2,1,J,1)
        CALL GMPRD(P1,X,UA,J,J,1)
        CALL GTPRD(X,UA,U3,J,1,1)
        HU(K2)=P3+U1+U2+U3
        CALL GTPRD(X,Q1A,PL,J,1,J)
        CALL GMPRD(G,AA,P5,1,J,J)
        DO 2065 JC=1,J
2065      G(1,JC)=P5(1,JC)+PL(1,JC)*2.0+Q2A(1,JC)

```

```

      IF(IWU(K2,IR).EQ.1)GO TO 2037
      IF(IWU(K2,IR).EQ.M(1,IR))GO TO 2047
      HT=SQRT(HU(K2)**2)
      IF(HT-DELTA.GT.0.0)MC(K2)=10
      IF(HT-DELTA.LE.0.0)MC(K2)=0
      GO TO 2036
2037  IF(HU(K2).LE.0.0)MC(K2)=-1
      IF(HU(K2).GT.0.0)MC(K2)=-11
      GO TO 2036
2047  IF(HU(K2).GE.0.0)MC(K2)=1
      IF(HU(K2).LT.0.0)MC(K2)=11
2036  WRITE(6,2013)K2,I,U(K2,IR),HU(K2),MC(K2)
2013  FORMAT(15X,I4,5X,I2,5X,F10.4,10X,E14.7,10X,I3)
2035  CONTINUE
      LJ=0
      DO 2075 K=1,N
      IF(MC(K).GT.5)LJ=LJ+1
2075  IF(MC(K).LT.-5)LJ=LJ+1
      PCV=LJ*100.0/(N*1.0)
      WRITE(6,2023)PCV
2023  FORMAT(30X,'PCV=',F10.4)
2005  CONTINUE
3075  CONTINUE
5     CONTINUE
C BEGINNING OF DISPLAYING OF NORMALIZED PERFORMANCES
      NP=ITER
      DO 525 L=1,NP
525   WRITE(6,583)L,PER(L)
583   FORMAT(20X,'PER('',I2,'')=',E14.7)
C END
      END

```

BIBLIOGRAPHY

- Anderson, B. D. O., and Moore, J. B. Linear Optimum Control. Englewood Cliffs, N. J.: Prentice Hall, 1971.
- Arora, J. K., and Pierre, D. A. "Optimal Trajectories for Multi-dimensional Nonlinear Process by Iterated Dynamic Programming." IEEE Trans. S. M. C. vol. SMC-3, pp. 85-91, Jan. 1973.
- Athans, M. "The Role and Use of Stochastic Linear Quadratic Gaussian Problem in Control System Design." IEEE Trans. A. C., vol. AC-16, pp. 529-552, Dec. 1971.
- Barmish, B. R., and Schmitendorf, W. E. "New Results on Controllability of Systems of the Form $\dot{x}(t) = A(t)x(t) + f(t, u(t))$." IEEE Trans. A. C., vol. AC-25, pp. 540-547, June 1980.
- Bellman, R. Dynamic Programming. New Jersey: Princeton University Press, 1957.
- Gibson, J. E. Nonlinear Automatic Control. New York: McGraw-Hill Book Company, 1963.
- Havira, R. M., and Lewis, J. B. "Computation of Quantized Control using Differential Dynamic Programming." IEEE Trans. A. C., vol. AC-17, pp. 191-196, April 1972.
- Kwakernaak, H., and Sivan, R. Linear Optimal Control Systems. New York: Wiley-Interscience, 1972.
- Lewis, J. B., and Tou, J. T. "Optimum Sampled-Data Systems with Quantized Control Signals." AIEE Trans., vol. 82, pp. 229-233, 1963.
- Rutherford, A. Discrete Dynamic Programming. New York: Blaisdell Publishing Company, 1964.
- Sano, A., N-Nagy, F. L. N. "Suboptimum Regulator Design of Linear Systems with Quantized Controls." IEEE Trans. A. C., vol. AC-24, pp. 97-100, Feb. 1979.
- Sugarman, R. "VLSI Computing: A Tough Nut to Crack." IEEE Spectrum, vol. 17, pp. 34-35, Jan. 1980.