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A STUDY OF A Q-ANALOGUE OF MACROBERT'S E-FUNCTION

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A STUDY OF A q-ANALOGUE OF MACROBERT'S E-FUNCTION

by

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TABLE OF CONTENTS

CHAPTER I

MACROBERT'S E-FUNCTION AND ITS q-EXTENSION (A SURVEY)

1. Introduction

The MacRobert's E-function emerges out of a need to give a meaning to the generalized hypergeometric series ${}_pF_q(\alpha_1,\ldots,\alpha_p;\ \beta_1,\ldots,\beta_q;z)$ when $p \geq q+1$. MacRobert in 1937 [1] defined it as follows:

If $p \geqq q+1$

$$E(p;\alpha_r:q;\rho_s:z) = \Gamma(\alpha_p)\{\prod_{n=1}^{q}(\rho_n-\alpha_n)\}^{-1}$$

$$\times \prod_{n=1}^{q}\int_0^1 \lambda_n^{\alpha_n-1}(1-\lambda_n)^{\rho_n-\alpha_n-1}d\lambda_n \prod_{n=q+1}^{p-2}\int_0^\infty e^{-\lambda_n}\lambda_n^{\alpha_n-1}d\lambda_n$$

$$\times \int_0^\infty e^{-\lambda_{p-1}}\lambda_{p-1}^{\alpha_{p-1}-1}(1+\lambda_1\lambda_2\ldots\lambda_{p-1}/z)^{-\alpha_p}d\lambda_{p-1} \qquad (1)$$

where $|\mathrm{amp}\,z| < \pi$, $R(\alpha_n) > 0$, $n=1,2,\ldots,p-1$ and $R(\rho_n-\alpha_n) > 0$, $n=1,2,\ldots,q$.

If $p \leqq q$, the functions are defined by the equation

$$E(p;\alpha_r:q;\rho_s:z) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2),\ldots,\Gamma(\alpha_p)}{\Gamma(\rho_1)\Gamma(\rho_2),\ldots,\Gamma(\rho_q)}F(p;\alpha_r:q;\rho_s:-\frac{1}{z}), \qquad (2)$$

where $z \neq 0$. 1.1(2) also holds when $p = q+1$, provided that $|z| > 1$. This can be seen by expanding the final binomial expression in 1.1(1) in powers of $1/z$ and integrating term by term. [When $p = q+1$, the infinite integrals do not appear in 1.1(2)].

MacRobert further showed that, if $p \geq q+1$

$$E(p;\alpha_r:q;\rho_s:z) = \sum_{r=1}^{p} P(\alpha_r;p-1,\alpha_s:q,\rho_t:z) \tag{3}$$

where $|\text{amp } z| < \pi$ and

$$P(\alpha_r;p-1;\alpha_s:q;\rho_t:z) = \prod_{s=1}^{p}{}' \Gamma(\alpha_s-\alpha_r)\{\prod_{t=1}^{q}\Gamma(\rho_t-\alpha_r)\}^{-1}$$

$$\Gamma(\alpha_r) \times z^{\alpha_r} F\left\{\begin{matrix} q+1;\alpha_r,\alpha_r-\alpha_p+1,\ldots,\alpha_r-\rho_q+1:(-1)^{p-q}z \\ p-1;\alpha_r-\alpha_1+1,\alpha_r-\alpha_2+1,\ldots,\alpha_r-\alpha_p+1 \end{matrix}\right\}$$

The dash in the product sign signifies that the factor for which s is equal to r is omitted. When $p = q+1$, we must have $|z| < 1$ for convergence in 1.1(3); formula 1.1(3) with form 1.1(2) for the E-function then gives the analytical continuation of the function on the left of 1.1(3) from the region $|z| > 1$ into the region $|z| < 1$.

The particular case $p = 2$, $q = 0$ of formula 1.1(3) is given by formula

$$\sum_{\alpha,\beta} \Gamma(\beta-\alpha)\Gamma(\alpha)z^{\alpha} F(\alpha;\alpha-\beta+1;z) = E(\alpha,\beta::z) \tag{4}$$

where the symbol $\sum_{\alpha,\beta}$ denotes that the expression following is a similar expression with α and β interchanged is to be added.

For $p = 1$, $q = 0$, we have that, if

$$|z|<1,\ |\text{amp } z| < \pi$$

$$z^{\alpha}F(\alpha;-z) = z^{\alpha}(1+z)^{-\alpha} = (1+\frac{1}{z})^{-\alpha},$$

and therefore

$$\Gamma(\alpha)z^{\alpha}F(\alpha;;-z)(=)\Gamma(\alpha)F(\alpha;;-\frac{1}{z}) = E(\alpha::z), \tag{5}$$

where on the right $|z| > 1$.

MacRobert also proved that

$$E(\alpha,\beta::z) = \Gamma(\alpha) \int_0^\infty e^{-\lambda}\lambda^{\beta-1}(1+\frac{\lambda}{z})^{-\alpha}d\lambda \qquad (6)$$

where $R(\beta) > 0$, $|\text{amp } z| < \pi$. Since the function on the left of 1.1(5) is symmetrical in α and β, it follows that

$$E(\alpha,\beta::z) = \Gamma(\beta) \int_0^\infty e^{-\lambda}\lambda^{\alpha-1}(1+\frac{\lambda}{z})^{-\beta}d\lambda,$$

where $R(\alpha) > 0$, $|\text{amp } z| < \pi$.

To derive 1.1(5) consider the integral

$$I = \int_C e^{\xi z}\xi^{\beta-1}(\xi-1)^{-\alpha}d\xi,$$

where z is taken to be real and positive the contour of integration C starts from and ends at $-\infty$ on the real axis, passing in the positive direction round the point $\xi = 1$ and amp ξ, amp $(\xi-1)$ are both equal to $-\pi$, initially. Let the contour be deformed, if necessary, to ensure that $|\xi| > 1$ at all points on it and expand $(\xi-1)^{-\alpha}$ in descending powers of ξ. Then applying the formula

$$\frac{1}{2\pi i} \int_C e^{\xi}\xi^{-z}d\xi = \frac{1}{\Gamma(z)},$$

it is easy to see, that

$$I = \frac{2\pi i}{\Gamma(\alpha-\beta+1)} z^{\alpha-\beta}F(\alpha;\alpha-\beta+1;z). \qquad (7)$$

Again deform the contour into segments of the ξ-axis with small semi-circles at $\xi = 0$ and a small circle at $\xi = 1$. Then if $R(\beta) > 0$, $R(1-\alpha) > 0$ the integral round the circle and the semi-circles tend to zero with their radii and therefore

$$I = 2i \sin \alpha\pi \int_0^1 e^{\xi z}\xi^{\beta-1}(1-\xi)^{-\alpha}d\xi$$

$$+ 2i \sin (\beta-\alpha)\pi \int_0^\infty e^{-\lambda a}\lambda^{\beta-1}(1+\lambda)^{-\alpha}d\lambda$$

$$= 2i \sin \alpha\pi B(\beta,1-\alpha)F(\beta;\beta-\alpha+1;z)$$

$$+ 2i \sin (\beta-\alpha)\pi z^{-\beta}E(\alpha,\beta::z)/\Gamma(\alpha). \qquad (8)$$

From 1.1(7) and 1.1(8) formula 1.1(4) follows, the restriction on z and α being removed by analytic continuation.

MacRobert also derived the asymptotic expansion of $E(\alpha,\beta::z)$ for large values of z by expanding $(1+\frac{\lambda}{z})^{-\alpha}$ in finite series of powers of λ and a remainder. In particular, he obtained the asymptotic expansion in the form $E(\alpha,\beta::z)$ = the first m terms of the series

$$\Gamma(\alpha)\Gamma(\beta)F(\alpha,\beta;-;-\frac{1}{z}) + R_m \qquad (9)$$

where

$$R_m = \frac{(-1)^m\Gamma(\alpha+m)}{m!} z^{-m} \int_0^\infty e^{-\lambda}\lambda^{\beta+m-1}d\lambda$$

$$\times \int_0^1 m(1-t)^{m-1}(1 + \frac{\lambda t}{z})^{-\alpha-m}dt, \qquad (10)$$

by taking m large enough the restriction $R(\beta) > 0$ can be also removed. This expansion holds for $|\text{amp } z| < \frac{3}{2}\pi$.

The $E(\alpha,\beta::z)$ function includes a number of known special functions as special cases. Whittaker functions $W_{k,m}(z)$ and $M_{k,m}$ can be defined by means of the relation

$$E(\tfrac{1}{2}-k+m,\tfrac{1}{2}-k-m::z) = \Gamma(\tfrac{1}{2}-k+m)\Gamma(\tfrac{1}{2}-k-m)z^{-k}e^{\frac{1}{2}z}W_{k,m}(z) \qquad (11)$$

and

$$M_{k,m}(z) = z^{\frac{1}{2}+m} e^{-\frac{1}{2}z} F(\tfrac{1}{2}-k+m;1+2m;z). \tag{12}$$

From 1.1(11) it is obvious that

$$W_{k,-m}(z) = W_{k,m}(z) \text{ and} \tag{13}$$

from 1.1(6) it follows that

$$\Gamma(\tfrac{1}{2}-k+m)W_{k,m}(z) = z^k e^{-\frac{1}{2}z} \int_0^\infty e^{-\lambda}\lambda^{-k+m-\frac{1}{2}}(1+\tfrac{\lambda}{z})^{k+m-\frac{1}{2}} d\lambda, \tag{14}$$

provided that $R(\frac{1}{2}-k+m) > 0$, $|\mathrm{amp}\ z| < \pi$. Also from 1.1(4) it follows that

$$W_{k,m}(z) = \sum_{m,-m} \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-k-m)} M_{k,m}(z), \tag{15}$$

where $2m$ is not an integer.

The MacRobert's generalized E-function has a very elegant Barnes type of contour integral representation, namely

$$E(p;\alpha_r:q;\rho_s:z) = \frac{1}{2\pi i} \int \frac{\Gamma(\xi) \prod_{r=1}^{p} \Gamma(\alpha_r-\xi) z^\xi}{\prod_{t=1}^{q} \Gamma(\rho_t-\xi)} d\xi \ , \tag{16}$$

the integral being taken upward along the η-axis with loops if necessary to ensure that the pole at the origin lie to the left and the poles at $\alpha_1,\alpha_2,\ldots,\alpha_p$ to the right of the contour. Zero and negative integral value of the α's real and ρ's are excluded and the α's must not differ by integral values. (Appendix IV (1)).

Finite and infinite series expansions involving MacRobert's E-function have been studied besides by MacRobert, by F. M. Ragab [(1, 2, 3, 4)] in a series of papers.

2. A q-analogue of $E(\alpha,\beta::z)$

R. P. Agarwal [1] in 1959 defined a basic analogue of MacRobert's E-function and studied some of its fundamental properties. The following notations will be used in what follows.

Let

$$|q| < 1, \log q = -w = -(w_1+iw_2)$$

where w, w_1, w_2 are constants, w_1 and w_2 being real. Also let

$$(q^a)_n \equiv (1-q^a)(1-q^{a+1}) \cdots (1-q^{a+n-1}) \equiv (a)_n,$$

$$(q^a)_o = 1, \ (q^a)_{-n} = (-)^n q^{\frac{n}{2}(n+1)} q^{-na}/(q^{1-a})_n,$$

then the basic generalized hypergeometric function is defined as

$$\begin{aligned} &{}_{r+1}\Phi_r \begin{bmatrix} a_1, a_2, \ldots, a_{r+1}, z \\ b_1, b_2, \ldots, b_r \end{bmatrix} \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n, \ldots, (a_{r+1})_n z^n}{(q)_n (b_1)_n, \ldots, (b_r)_n}, \quad (|z| < 1), \end{aligned} \tag{1}$$

and a confluent hypergeometric function as

$${}_1\Phi_1 \ (a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n (b)_n} z^n q^{\frac{n}{2}(n-1)} .$$

Also

$$E_q(x) = \prod_{n=0}^{\infty} (1-xq^n) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n}{2}(n-1)}}{(q)_n} x^n,$$

$$(x+y)_a \equiv x^{\alpha}(1+y/x)_{\alpha} = \prod_{n=0}^{\infty} \frac{(1+yx^{-1}q^n)}{(1+yx^{-1}q^{a+n})},$$

$$\frac{1}{(1+x)_\alpha} \equiv {}_1\phi_0(\alpha;-x), \text{ for } |x| < 1,$$

$$G(\alpha) = \left\{ \prod_{n=0}^{\infty} (1-q^{\alpha+n}) \right\}^{-1}.$$

Further, following Hahn [1, 2] the q-integral of a function is defined as

$$\left.\begin{aligned} \int_0^x f(y)d(qy) &= x(1-q) \sum_{i=0}^{\infty} q^i f(q^i x) \\ \int_x^\infty f(y)d(qy) &= x(1-q) \sum_{j=1}^{\infty} q^{-j} f(q^{-j} x) \end{aligned}\right\} \tag{2}$$

and thus

$$\int_0^\infty f(y)d(qy) = (1-q) \sum_{j=-\infty}^{\infty} q^j f(q^j).$$

Agarwal [1] defined the basic analogue of E-function by the integral

$$E_q(\alpha,\beta::z) = \frac{G(\alpha)}{(1-q)} \int_0^1 E_q(q\lambda)\lambda^{\beta-1} {}_1\phi_0(\alpha;-\frac{\lambda}{z})d(q\lambda), \tag{3}$$

where $R(\beta) > 0$ and $\arg \lambda = 0$, for simplicity. The integral on the right of 1.2(3) can be evaluated by the help of certain contour integrals used by Watson [1].

Following, Watson [1], we have

$$\frac{G(\alpha)}{G(1)} {}_1\phi_0(\alpha,-\frac{\lambda}{z}) = \frac{1}{2\pi i} \int_C \frac{G(\alpha-s)\pi \sin(z/\lambda)^s}{G(1-s) \sin \pi s} ds, \tag{4}$$

where the contour C is a line parallel to $R(ws) = 0$ with loops, if necessary, to include the poles of $G(\alpha-s)$; the integral converges if

$$R[s \log(\tfrac{z}{\lambda}) - \log \sin \pi s] < 0,$$

for large values of $|s|$ on the contour, i.e., if

$$|\{\arg z - w_1^{-1}w_2 \log|z|\}| < \pi,$$

since $0 < \lambda < 1$.

Hence 1.2(3), gives

$$E_q(\alpha,\beta::z) = \frac{1}{2\pi i}\frac{G(1)}{(1-q)} \int_0^1 E_q(q\lambda)\lambda^{\beta-1}d(q\lambda) \times \int_C \frac{G(\alpha-s)\pi(\frac{z}{\lambda})^s}{G(1-s)\sin\pi s}\, ds.$$

Changing the order of integration which is justified for $R(\beta-s) > 0$ and the above argument of z, we get

$$E_q(\alpha,\beta::z) = \frac{1}{2\pi i}\frac{G(1)}{(1-q)}\int_C \frac{G(\alpha-s)\pi z^s}{G(1-s)\sin\pi s}\, ds \int_0^1 E_q(q\lambda)\lambda^{\beta-s-1}d(q\lambda)$$

$$= \frac{1}{2\pi i}\int_C \frac{G(\alpha-s)G(\beta-s)\pi z^s}{G(1-s)\sin\pi s}\, ds, \qquad (5)$$

valid by analytic continuation when $R(\beta) > 0$ and

$$|\{\arg z - w_2 w_1^{-1} \log|z|\}| < \pi.$$

The contour integral 1.2(5) gives another integral representation for the E_q-function. Evaluating 1.2(5) by considering the residues at the poles of $G(\alpha-s)$ and $G(\beta-s)$ [see Watson (1)], we get

$$E_q(\alpha,\beta::z) = \sum_{\alpha,\beta}\frac{G(\alpha)G(\beta-\alpha)}{G(1)}\prod_{n=0}^{\infty}\frac{(1+z^{-1}q^{\alpha+n})(1+zq^{1-\alpha+n})}{(1+z^{-1}q^{n})(1+zq^{1+n})}$$

$$\times\ {}_1\Phi_1(\alpha;\alpha-\beta+1:zq^{2-\beta}). \qquad (6)$$

Eq. 1.2(6) gives the series definition for the E_q-function and shows that it is symmetrical in α and β.

Agarwal further derived the recurrence relations

$$(1-q^{\alpha})E_q(\alpha,\beta::z) - E_q(\alpha+1,\beta::z)$$

$$= \frac{q^{\alpha}}{z} E_q(\alpha+1,\beta+1::z), \tag{7}$$

$$(q^{\beta}-q^{\alpha})E_q(\alpha,\beta::z) + q^{\alpha}E_q(\alpha,\beta+1::z)$$

$$= q^{\beta}E_q(\alpha+1,\beta::z), \tag{8}$$

$$(1-q^{\beta})E_q(\alpha,\beta::z) = z^{-1}q^{\beta}(1-q^{\alpha-\beta-1})\ E_q(\alpha,\beta+1::z)$$

$$+ (1-q^{\alpha-1})\ E_q(\alpha-1,\beta+1::z). \tag{9}$$

To prove 1.2(7) he took the left side equal to

$$(1-q^{\alpha})\frac{G(\alpha)}{(1-q)} \int_o^1 E_q(q\lambda)\lambda^{\beta-1}\,{}_1\Phi_0(\alpha;-\frac{\lambda}{z})d(q\lambda)$$

$$- \frac{G(\alpha+1)}{(1-q)} \int_o^1 E_q(q\lambda)\lambda^{\beta-1}\,{}_1\Phi_0(\alpha+1;-\frac{\lambda}{z})d(q\lambda)$$

$$= \frac{G(\alpha+1)}{(1-q)} \int_o^1 E_q(q\lambda)\lambda^{\beta-1}[{}_1\Phi_0(\alpha;-\frac{\lambda}{z})-{}_1\Phi_0(\alpha+1;-\frac{\lambda}{z})]d(q\lambda)$$

$$= \frac{q^{\alpha}G(\alpha+1)}{z(1-q)} \int_o^1 E_q(q\lambda)\lambda^{\beta}\,{}_1\Phi_0(\alpha+1;-\frac{\lambda}{z})d(q\lambda).$$

In order to prove 1.2(8) he took 1.2(7) and a similar relation with α replaced by β. Eliminating $E_q(\alpha+1,\beta+1::z)$ between these two relations, 1.2(8) can be deduced.

To prove 1.2(9) multiply 1.2(7) by $(q^{\beta}-q^{\alpha})$ and 1.2(8) by $(1-q^{\alpha})$ and subtract. Changing α to $\alpha-1$, one gets 1.2(9).

The following finite summation gives a generalization of 1.2(9):

$$\sum_{r=0}^{n} \frac{(q^{-n})_r (q^{\alpha-\beta-n})_r (q^{\alpha-n})_n}{(q^{\alpha-n})_r (q)_r} (-z)^{-r} q^{r(\beta+n)}$$

$$\times E_q(\alpha-n+r, \beta+n::z) = (q^{\beta})_n E_q(\alpha,\beta::z). \tag{10}$$

To prove 1.2(10) consider its left side and use the contour integral 1.2(5) for the E_q-function in it.

This gives

$$\frac{1}{2\pi i} \sum_{r=0}^{n} \frac{(q^{-n})_r (q^{\alpha-\beta-n})_r (q^{\alpha-n})_n}{(q^{\alpha-n})_r (q)_r} (-z)^{-r} q^{r(\beta+n)}$$

$$\times \int_C \frac{G(\alpha-n+r-s)G(\beta+n-s)}{G(1-s)} \frac{\pi z^s}{\sin \pi s} ds. \tag{11}$$

Putting s = t+r and changing the order of integration, one gets on simplification

$$\frac{1}{2\pi i} \int_C \frac{G(\alpha-n-t)G(\beta+n-t)}{G(1-t)} \frac{\pi z^t}{\sin \pi t} (q^{\alpha-n})_n$$

$$\times {}_3\Phi_2 \begin{bmatrix} q^{-n}, q^{\alpha-\beta-n}; q \\ q^{\alpha-n}, q^{1+t-\beta-n} \end{bmatrix} dt.$$

Summing the ${}_3\Phi_2$ by the basic analogue of Saalschutzs theorem, one gets

$$\frac{(q^{\beta})_n}{2\pi i} \int_C \frac{G(\alpha-t)G(\beta-t)}{G(1-t)} \frac{\pi z^t}{\sin \pi t} dt = \frac{(q^{\beta})_n}{2\pi i} \int_C \frac{G(\alpha-t)G(\beta-t)}{G(1-t)} \frac{\pi z^t}{\sin \pi t} dt$$

$$= (q^{\beta})_n E_q(\alpha,\beta::z). \tag{12}$$

This proves 1.2(10). For $n = 1$, 1.2(10) reduces to 1.2(9). Another elegant integral representation for $E_q(\alpha,\beta::z)$ is given by

$$E_q(\alpha,\beta::z) = \frac{1}{(1-q)} \int_0^1 E_q(\alpha,\beta+1::z/[1-q^{\beta}t]_{\beta-1})d(qt), \tag{13}$$

where Re $\alpha > 0$, Re $\beta > 0$ and $|q| < 1$.

The $E_q(\alpha,\beta::z)$ function, for $|z| \to \infty$, behaves in a particularly simple manner. Evaluating the integral 1.2(5) and considering the residues at the poles of $\Gamma(s)$ one finds that

$$E_q(\alpha,\beta::z) \sim \frac{G(\alpha)G(\beta)}{G(1)} \, {}_2\Phi_0(\alpha,\beta;-\frac{1}{z}),$$

for $|z| \to \infty$.

R. P. Agarwal [2] further gave a differences equation satisfied by $w = E_q(\alpha,\beta::z)$, namely,

$$zq^{2-\alpha-\beta}(1-q^{\theta})q^{\theta}w = (1-q^{\theta-\alpha})(1-q^{\theta-\beta})w \tag{14}$$

with $\theta \equiv \frac{xd}{dx}$.

Agarwal [2] also gave a new type of a contour integral representation for $E_q(\alpha,\beta::z)$, namely

$$E_q(\alpha,\beta::z) = \frac{1}{2\pi i} \int_{-i\pi/t}^{i\pi/t} \Pi \left[\frac{(1+q^{\alpha-z+s+n})(1+q^{1+z-\alpha-s+n})}{(1-q^{\alpha+s+n})(1-q^{\beta-\alpha-s+n})(1-q^{-s+n})}\right]ds \tag{15}$$

where $q = e^{-t}$, $t > 0$.

Evaluating the integral by the calculus of residues one can find that 1.2(15) after writing z,α,β for q^z, q^{α}, q^{β}, respectively, gives the required result.

3. A Further Generalization of $E_q(\alpha,\beta::z)$

N. Agarwal [1] extended the definition of the basic analogue $E_q(\alpha,\beta::z)$ to generalized q-analogue of MacRobert's E-function and discussed some of its properties. In examining the convergence of the contour integral 1.3(6) the following asymptotic behavior of G(x) due to Littlewood [1] is considered:

(1) When Re(wx) is large and positive, G(x) tends uniformly to unity as $|x|$ tends to infinity.

(2) When Re(wx) is large and negative, and $|x-x_o| > \varepsilon$, where x_o denotes any pole of G(x) and ε is any assigned quantity which is not zero

$$\text{Re of } G(x) = -\frac{1}{2w_1}\{\text{Re}(wx)\}^2 - \frac{1}{2}\text{Re}(wx) - F$$

where $|F|$ does not exceed a finite quantity depending in ε.

She defined the basic analogue of the generalized E-function, $E_q(r,a_p:s;b_t:z)$ as

$$E_q(r,a_p:s;b_t:z) = G(a_r)\{\prod_{n=1}^{s} G(b_n-a_n)\}^{-1}$$

$$\times \prod_{n=1}^{s}\{\frac{G(1)}{(1-q)}\int_0^1 \lambda_n^{a_n-1}(1-q\lambda_n)_{b_n-a_n-1}d(q\lambda_n)\}$$

$$\times \prod_{n=s+1}^{r-2}\{\frac{G(1)}{(1-q)}\int_0^1 E_q(q\lambda_n)\lambda_n^{a_n-1}d(q\lambda_n)\}$$

$$\times \frac{1}{(1-q)}\int_0^1 E_q(q\lambda_{r-1})\lambda_{r-1}^{a_{r-1}-1}\,{}_1\Phi_0(a_r;\frac{-\lambda_1\lambda_2,\ldots,\lambda_{r-1}}{z})d(q\lambda_{r-1}), \quad (1)$$

where

$$\mathrm{Re}(a_n) > 0,\ n=1,2,\ldots,r-1,$$

$$\mathrm{Re}(b_n-a_n) > 0,\ n=1,2,\ldots,s.$$

If $r \leq s$, $z \neq 0$, the functions are defined by

$$E_q(r;a_p:s;b_t:z) = \frac{G(a_1),\ldots,G(a_r)}{G(b_1),\ldots,G(b_s)}\ {}_r\phi_s\left[\begin{matrix}a_1,\ldots,a_r;-\frac{1}{z}\\ b_1,\ldots,b_s\end{matrix}\right]. \tag{2}$$

Then she proceeded to evaluate the integral on the right of 1.3(1) using the result 1.2(5) due to R. P. Agarwal [1], namely

$$E_q(a,b::z) = \frac{G(a)}{(1-q)} \int_0^1 E_q(q\lambda)\lambda^{b-1}\ {}_1\Phi_0(a;-\lambda/z)\,d(q\lambda)$$

$$= \frac{1}{2\pi i}\int_C \frac{G(a-s)G(b-s)\pi z^s}{G(1-s)\ \sin\pi s}\,ds, \tag{3}$$

1(3.1) gives

$$\left\{\prod_{n=1}^{s} G(b_n-a_n)\right\}^{-1} \prod_{n=1}^{s} \frac{G(1)}{(1-q)} \int_0^1 \lambda_n^{a_n-1}(1-q\lambda_n)_{b_n-a_n-1}\,d(q\lambda_n)$$

$$\times \prod_{n=s+1}^{r-2}\left\{\frac{G(1)}{(1-q)}\int_0^1 E_q(q\lambda_n)\lambda_n^{a_n-1}\,d(q\lambda_n)\right.$$

$$\times \frac{1}{2\pi i}\int_C \frac{G(a_r-x)G(a_{r-1}-x)\ (z/\lambda_1\lambda_2,\ldots,\lambda_{r-2})^x}{G(1-x)\ \sin\pi x}\,dx$$

$$= \left\{\prod_{r=1}^{s} G(b_n-a_n)\right\}^{-1} \prod_{n=1}^{s} \frac{G(1)}{(1-q)} \int_0^1 \lambda_n^{a_n-1}(1-q\lambda_n)_{b_n-a_n-1}\,d(q\lambda_n)$$

$$\times \prod_{n=s+1}^{r-2}\left\{\frac{G(1)}{(1-q)}\int_0^1 E_q(q\lambda_n)\lambda_n^{a_n-1}\,d(q\lambda_n)\ \frac{G(1)}{(1-q)}\int_0^1 E_q(q\lambda_{r-2})\right.$$

$$\times \lambda_{r-2}^{a_{r-2}-1}\,d(q\lambda_{r-2}) \times \frac{1}{2\pi i}\int_C \frac{G(a_r-x)G(a_{r-1}-x)\pi(z/\lambda_1\lambda_2,\ldots,\lambda_{r-2})^x}{G(1-x)\ \sin\pi x}\,dx.$$

Changing the order of integration, which is valid by absolute convergence of both the integrals, for Re $(a_{r-2}) > 0$ and $|\{\arg z - w_2 w_1^{-1} \log|z|\}| < \pi$ and since $0 < \lambda < 1$, and because of 1.3(3) one gets

$$\left\{\prod_{r=1}^{s} \{G(b_n-a_n)\}\right\}^{-1} \prod_{n=1}^{s} \left\{\frac{G(1)}{(1-q)} \int_0^1 \lambda_n^{a_n-1} (1-q\lambda_n)_{b_n-a_n-1} d(q\lambda_n)\right.$$

$$\times \frac{1}{2i} \int_C \frac{G(a_r-x)G(a_{r-1}-x)(z/\lambda_1\lambda_2,\ldots,\lambda_{r-3})^x}{G(1-x)\sin\pi x} dx$$

$$\times \frac{G(1)}{(1-q)} \int_0^1 E_q(q\lambda_{r-2})\lambda_{r-2}^{a_{r-2}-1-x} d(q\lambda_{r-2}).$$

Evaluating the innermost integral this becomes

$$\left\{\prod_{n=1}^{s} G(b_n-a_n)\right\}^{-1} \prod_{n=1}^{s} \frac{G(1)}{(1-q)} \int_0^1 \lambda_n^{a_n-1} (1-q\lambda_n)_{b_n-a_n-1} d(q\lambda_n)\Big\}$$

$$\times \prod_{n=s+1}^{r-3} \left\{\frac{G(1)}{(1-q)} \int_0^1 E_q(q\lambda_n)\lambda_n^{a_n-1} d(q\lambda_n)\right\}$$

$$\times \frac{1}{2i} \int_C \frac{G(a_r-x)G(a_{r-1}-x)G(a_{r-2}-x)(z/\lambda_1\lambda_2,\ldots,\lambda_{r-2})^x}{G(1-x)\sin\pi x} dx. \qquad (4)$$

Similarly, on using repeatedly the known integral

$$\frac{1}{(1-q)} \int_0^1 x^{\alpha-1}(1-qx)_{\lambda-1} d(qx) = \prod_{n=0}^{\infty} \frac{(1-q^{\alpha+\lambda+n})(1-q^{1+n})}{(1-q^{\alpha+n})(1-q^{\lambda+n})},$$

Re $(\alpha) > 0$, Re $\lambda > 0$ due to Hahn [1], 1.3(4) reduces to

$$\left\{\prod_{n=1}^{s} G(b_n-a_n)\right\}^{-1} \frac{G(1)}{(1-q)} \int_0^1 \lambda_n^{a_n-1} (1-q\lambda_n)_{b_n-a_n-1} d(q\lambda_n)$$

$$\times \frac{G(1)}{(1-q)} \int_0^1 \lambda_s^{a_s-1} (1-q\lambda_s)_{b_s-a_s-1} d(q\lambda_s)$$

$$\times \frac{1}{2i} \int_C \prod_{t=s+1}^{r} \frac{G(a_t-x)(z/\lambda_1\lambda_2,\ldots,\lambda_s)^x}{G(1-x)\ \sin\pi x}\, dx. \tag{5}$$

Next, changing the order of integration which is justified for Re $(a_s) > 0$ and $|\{\arg z - w_2 w_1^{-1} \log|z|\} < \pi$, $0 < \lambda_s < 1$, we get

$$\left\{\prod_{n=1}^{s} G(b_n-a_n)\right\}^{-1} \prod_{n=1}^{s-1} \frac{G(1)}{(1-q)} \lambda_n^{a_n-1} (1-q\lambda_n)_{b_n-a_n-1} d(q\lambda_n)$$

$$\times \frac{1}{2\pi i} \int_C \prod_{t=s+1}^{r} \frac{G(a_t-x)\pi(z/\lambda_1\lambda_2,\ldots,\lambda_{s-1})^x}{G(1-x)\ \sin\pi x}\, dx$$

$$\times \frac{G(1)}{(1-q)} \int_o^1 \lambda_s^{a_s-1-x} (1-q\lambda_s)_{b_s-a_s-1} d(q\lambda_s).$$

Evaluating the inner most integral by means of another known integral due to Hahn (3, 3.16), we get

$$\left\{\prod_{n=1}^{s-1} G(b_n-a_n)\right\}^{-1} \prod_{n=1}^{s-1} \frac{G(1)}{(1-q)} \int_o^1 \lambda_n^{a_n-1} (1-q\lambda_n)_{b_n-a_n-1} d(q\lambda_n)$$

$$\times \frac{1}{2\pi i} \int_C \prod_{t=s}^{r} \frac{G(a_t-x)\pi(z/\lambda_1\lambda_2,\ldots,\lambda_{s-1})^x}{G(b_s-x)\ \sin\pi x}\, dx.$$

Repeating this operation, one gets

$$E_q(r;a_p:s;b_t:z) = \frac{1}{2\pi i} \int_C \prod_{t=1}^{r,s} \frac{G(a_t-x)\pi z^x dx}{G(1-x)G(b_t-x)\ \sin\pi x}, \tag{6}$$

convergent for $|\{\arg z - w_2 w_1^{-1} \log|z|\}| < \pi$.

The contour integral 1.3(6) furnishes a further representation for the E_q-function which is analogous to the Barnes type integral for the E_q-function (MacRobert [2], p. 374) to which it reduces when $q \to 1$. (Appendix IV(2)).

Evaluating the integral 1.3(6) by considering the residues at the poles that lie to the right of the contour, i.e., of $G(a_t-x)$ for

$t=1,2,\ldots,r$, one gets (c.f. Watson [1]), for $r \geqq s+1$,

$$E_q(r;a_p:s;b_t:z) = \frac{G(a_1)G(a_2-a_1),\ldots,G(a_r-a_1)}{G(1)G(b_1-a_1),\ldots,G(b_s-a_1)}$$

$$\times \prod_{n=0}^{\infty} \frac{(1+z^{-1}q^{a_1+n})(1+zq^{1-a_1+n})}{(1+z^{-1}q^n)(1+zq^{1+n})}$$

$$\times {}_{s+1}\phi_{r-1}\left[\begin{matrix} a_1, 1+a_1-b_1,\ldots,1+a_1-b_s; zQ \\ 1+a_1-a_2,\ldots,1+a_1-a_r \end{matrix}\right] + \text{idem}(a_1;a_2,\ldots,a_r),$$

where

$$Q = (-1)^{r-s}\left\{q^{\frac{1}{2}n(n+1)}\right\}^{r-s-1} q^{b_1+\cdots+b_s+1+a_1(r-s-2)-a_2-\cdots-a_r} \tag{7}$$

and $|q| < 1$ and idem $(\alpha;\beta)$ means that the preceeding expression is to be added with α and β interchanged.

Hence, 1.3(7) gives the series definition of $E_q(r;a_p:s;b_t:z)$ which is the basic analogue of a result due to MacRobert (1;(9)).

Also on evaluating the residues at the poles that lie to the left of the given contour (i.e., poles of $\Gamma(x)$), it is found that the integral is equal to

$$\frac{G(a_1),\ldots,G(a_r)}{G(b_1),\ldots,G(b_s)}\ {}_r\phi_s\left[\begin{matrix} a_1,\ldots;a_r;-\frac{1}{z} \\ b_1,\ldots,b_s \end{matrix}\right], \tag{8}$$

which gives the definition of $E_q(r,a_p:s,b_t:z)$ for $r \leq s$, which is the basic analogue of MacRobert's result (1;(20)).

The following recurrence relations are satisfied by $E_q(r;a_p:s;b_t:z)$:

$$(1-q^{a_1})E_q(r;a_p:s;b_t:z) - E_q(a_1+1,a_2,\ldots,a_r:s;b_t:z)$$

$$= q^{a_1}/z\ E_q(r;a_p+1:s;b_t+1;z) \tag{9}$$

$$(q^{a_2}-q^{a_1})\, E_q(r;a_p:s;b_t:z)+q^{a_1}\, E_q(a_1,a_2+1,a_3,\ldots,a_r:s,b_t:z)$$

$$= q^{a_2}\, E_q(a_1+1,a_2,\ldots,a_r:s;b_t:z) \tag{10}$$

$$(1-q^{b_1-1})\, E_q(r;a_p:s;b_t:z) - E_q(r;a_p:b_1-1,b_2;\ldots,b_s:z)$$

$$= (q^{b_1-1})/z\, E_q(r,a_p+1:s;b_t+1:z) \tag{11}$$

$$(q^{b_2}-q^{b_1})\, E_q(r;a_p:s;b_t:z) + q^{b_1}\, E_q(r;a_p:b_1,b_2-1,b_3,\ldots,b_s:z)$$

$$= q^{b_2}\, E_q(r;a_p:b_1-1,b_2,\ldots,b_s:z) \tag{12}$$

and

$$(1-q^{a_2})\, E_q(r;a_p:s,b_t:x) = (1-q^{a_1-1})\, E_q(a_1-1,a_2+1,a_3,\ldots,a_r:s;b_t:z)$$

$$= q^{a_2}/z(1-q^{a_1-a_2-1})\, E_q(a_1,a_2+1,\ldots,a_r+1:s;b_t+1:z). \tag{13}$$

4. Certain Expansions Associated with $E_q(r;a_p:s;b_t:z)$

N. Agarwal [1] has derived the following finite series expansions involving E_q-functions:

$$\sum_{m=0}^{n} \frac{(q^{-n};m)(q^{a_1-a_2-n};m)(q^{a_1-n};n)}{(q;m)(q^{a_1-n};m)}\, z^{-m} q^{m(a_2+n)}$$

$$\times E_q(a_1-n+m,a_2+n,a_3+m,\ldots,a_r+m:s;b_t+m:z)$$

$$= (q^{a_2};n)\, E_q(r;a_p:s;b_t:z), \tag{1}$$

and

$$\sum_{m=0}^{n} \frac{(q^{-n};m)(q^{a};m)(q^{1+\frac{1}{2}a};m)(-q^{1+\frac{1}{2}a};m)(q^{b};m)}{(q;m)(q^{1/2a};m)(-q^{1/2a};m)(q^{1+m-b};m)(q^{1+a+n};m)} (-z)^{-m} q^{1/2m(m-1)}$$

$$\times q^{(a-b+n)m} E_q\begin{bmatrix} a_1+m,\ldots,a_r+m, 1+a-b+m, 1+a+m+n:z \\ b_1+m,\ldots,b_s+m, 1+a-b+n+m, 1+a+2m \end{bmatrix}$$

$$= \frac{(q^{1+a};n)}{(q^{1+a-b};n)} \; E_q(r;a_p:s;b_t:z). \tag{2}$$

In order to prove 1.4(1) she substituted the integral 1.3(6) for the E_q-function on its left hand side and obtained

$$\frac{1}{2\pi i} \sum_{m=0}^{n} \frac{(q^{-n};m)(q^{a_1-n};m)(q^{a_1-a_2-n};m) z^{-m} q^{m(a_2+n)}}{(q;m)(q^{a_1-n};m)}$$

$$\times \int_C \frac{G(a_1-n+m-x)G(a_2+n-x)G(a_2+m-x),\ldots,G(a_r+m-x)}{G(1-x)G(b_1+m-x),\ldots,G(b_s+m-x)\ \sin\pi x} \pi z^x dx.$$

Replacing x by s+m and changing the order of integration and summation, she got

$$\frac{1}{2\pi i} \int_C \frac{G(a_1-n-s)G(a_2+n-s)G(a_s-s),\ldots,G(a_r-s)}{G(1-s)G(b_1-s),\ldots,G(b_s-s)\ \sin\pi s} \pi z^s$$

$$\times (q^{a_1-n};n) \; {}_3\phi_2\begin{bmatrix} q^s, q^{a_1-a_2-n}, q^{-n}; q \\ q^{a_1-n}, q^{1+s-a_2-n} \end{bmatrix} \quad ds.$$

Using the summation theorem for a Saalschutzian ${}_3\Phi_2$ (1.6(4)) one gets the required result. 1.4(1) is a genaralization of 1.3(13) to which it reduces when n=1. Also when r=2 and s=0 it reduces to a result of R. P. Agarwal (1).

Using the summation theorem for a well-poised ${}_6\Phi_5$ [1.6(6)] one can prove 1.4(2).

5. Certain Integrals Involving E_q-functions

We give below certain results that N. Agarwal [1] proved for integrals involving E_q-functions. If Re $(a_{r+1}) > 0$

$$\frac{G(1)}{(1-q)} \int_o^1 \lambda^{a_{r+1}-1} E_q(q\lambda)\, E_q(r;a_p:s;b_t:\tfrac{z}{\lambda})d(q\lambda)$$

$$= E_q(r+1;a_p:s;b_t:z). \tag{1}$$

If Re $(a_{r+1}) > 0$ and Re $(b_{s+1}-a_{r+1}) > 0$, then

$$\frac{G(1)}{(1-q)} \int_o^1 \lambda^{a_{r+1}-1} (1-q\lambda)_{b_{s+1}-a_{r+1}-1}\, E_q(r;a_p:s,b_t:\tfrac{z}{\lambda})d(q\lambda)$$

$$= G(b_{s+1}-a_{r+1})\, E_q(r+1;a_p:s+1;b_t:z) \tag{2}$$

and

$$\int \frac{1}{2\pi i} \int_C e_q(\xi)\xi^{-b_{s+1}}\, E_q(r;a_p:s;b_t:z\xi)d\xi$$

$$= G(1)\, E_q(r;a_p:s+1;b_t:z), \tag{3}$$

where the path of integration C encircles the (zero) null point and also in the usual manner, can be deformed into a loop parallel to the imaginary axis. The main interest of these integrals is that they not only evaluate certain integrals involving E_q-functions but also give certain integral representations of E_q-function.

Since the basic hypergeometric functions have proved to be of increasing importance in the study of combinatory analysis and partition theory, I was inspired to continue the work on E_q-functions, since they are only a finite sum of certain particular ${}_r\phi_s$-functions. The basic hypergeometric functions have been very widely studied during the last three decades or so by W. N. Bailey, L. J. Slater, R. P. Agarwal, G. E. Andrews, and others. A very good account of the applications of these functions has been given recently by G. E. Andrews [1,].

6. Certain Known Summation Formulae

Basic analogue of binomial expansion.

$$ {}_1\Phi_0[a;z] = \prod_{n=0}^{\infty} \left[\frac{1-azq^n}{1-zq^n}\right]. \tag{1} $$

Basic analogue of Gauss's Theorem.

$$ {}_2\phi_1 \begin{bmatrix} a,b;\frac{c}{ab} \\ c \end{bmatrix} = \prod_{n=0}^{\infty} \left[\frac{(1-cq^n/b)(1-cq^n/a)}{(1-cq^n)(1-cq^n/ab)}\right], \tag{2} $$

provided that Re $(c-a-b) > 0$.

Basic analogue of Dixon's Theorem.

$$ {}_4\Phi_3 \begin{bmatrix} a,-q\sqrt{a},b,c;q\sqrt{a}/bc \\ -\sqrt{a},aq/b,aq/c \end{bmatrix} = \prod_{n=0}^{\infty} \left[\frac{(1-aq^{n+1})(1-\sqrt{a}q^{n+1}/b)(1-\sqrt{a}q^{n+1}/c)(1-aq^{n+1}/bc)}{(1-aq^{n+1}/b)(1-aq^{n+1}/c)(1-\sqrt{a}q^{n+1})(1-\sqrt{a}q^{n+1}/bc)}\right]. \tag{3} $$

Basic analogue of Saalschutz's Theorem.

$$ {}_3\phi_2 \begin{bmatrix} b,c,q^{-N};q \\ d,bcq^{1-N}/d \end{bmatrix} = \frac{(d/b,N)(d/c;N)}{(d,N)(d/bc;N)} , \tag{4} $$

N a non-negative integer.

Basic analogue of Dougall's Theorem.

$$ {}_8\phi_7 \begin{bmatrix} a,q\sqrt{a},-q\sqrt{a},b,c,d,e,q^{-N};q \\ \sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d,aq/e,aq^{n+1} \end{bmatrix} $$

$$ = \frac{(aq;N)(aq/cd;N)(aq/bd;N)(aq/bc,N)}{(aq/b;N)(aq/c;N)(aq/d;N)(aq/bcd;N)} \tag{5} $$

provided that $bcde = a^2q^{N+1}$, and N is a positive integer.

The sum of a well-poised ${}_6\phi_5$.

$$ {}_6\phi_5 \begin{bmatrix} q^a,q^{1+1/2a},-q^{1+1/2a},q^b,q^c,q^d;q^{1+a-b-c-d} \\ q^{1/2a},-q^{1/2a},q^{1+a-b},q^{1+a-c},q^{1+a-d} \end{bmatrix} $$

$$ = \prod_{n=0}^{\infty} \left[\frac{(1-q^{1+a+n})(1-q^{1+a-b-c+n})(1-q^{1+a-b-d+n})(1-q^{1+a-c-d+n})}{(1-q^{1+a-b+n})(1-q^{1+a-c+n})(1-q^{1+a-d+n})(1-q^{1+a-b-c-d+n})} \right] , \tag{6} $$

where $R\,[1+a-b-c-d] > 0$.

The sum of a nearly poised ${}_5\phi_4$.

$$ {}_5\phi_4 \begin{bmatrix} q^a,q^{1+1/2a},-q^{1+1/2a},q^b,q^{-N};q \\ q^{1/2a},-q^{1/2a},q^{1+a-b},q^{2+2b-N} \end{bmatrix} $$

$$ = \frac{[a-2b,N-1][-1-b;N][1-q^{a-2b+2N-1}]}{[1+a-b;N][-1-2b;N]} . \tag{7} $$

CHAPTER II

CERTAIN GENERAL EXPANSION OF BASIC GENERALIZED MACROBERT'S E-FUNCTION

1. Introduction

Earlier Kush [1] gave certain summation formulae involving generalized E_q-functions.

If

$$f(x) = x^{\alpha} \sum_{r=-\infty}^{\infty} a_r x^r,$$

then $f[x+h] = \sum_{r=-\infty}^{\infty} a_r [x+h]_{\alpha+r}$,

where $[x+y]_{\alpha} = x^{\alpha}[1+y/x]_{\alpha} = \prod_{n=0}^{\infty} \frac{(1+y/xq^{n})}{(1+y/xq^{\alpha+n})}$.

2. A Simple Expansion

F. H. Jackson [1] gave a basic analogue of the Taylor's Expansion in the form

$$f[x+h] = \sum_{r=0}^{\infty} \frac{q^{1/2r(r-1)} h^r}{(q)_r} \Delta^r f(x), \tag{1}$$

where $f[x+h]$ is an absolutely convergent series and

$$x^m \Delta^m \equiv [\delta][\delta-1],\ldots,[\delta-m+1],$$

with $\delta \equiv x \frac{\partial}{\partial x}$,

and $[\delta+a] \equiv (1-q^{\delta+a})$.

Taking $h = (\lambda-1)x$, we get,

$$f[x+(\lambda-1)x] = \sum_{r=0}^{\infty} \frac{q^{1/2r(r-1)}(\lambda-1)_r^r x^r}{(q)_r} \Delta^r f(x). \tag{2}$$

Let

$$f(x) = x^{-a_1} E_q(r;a_p:s,b_t:x)$$

and apply the known result of N. Agarwal [1], namely

$$z^m \Delta^m \{z^{-a_1} E_q(r;a_p:s;b_t:z)\}$$
$$= (-)^m q^{-1/2m(m-1)-a_1 m} z^{-a_1} E_q(a_1+m,a_2,\ldots,a_r:s;b_t:zq^m).$$

We obtain

$$[1+(\lambda-1)]_{-a_1} E_q(r;a_p:s;b_t:[x+q^{-1}(\lambda-1)x]$$
$$= \sum_{n=0}^{\infty} \frac{q^{-a_1 n}(1-\lambda)_n^n}{(q)_n} E_q(a_1+n,a_2,\ldots,a_r:s;b_t:xq^n). \tag{3}$$

If we put $(\lambda-1) = y/x$, in 2.2(3), we have the simple expansion

$$[1+y/x]_{-a_1} E_q(r;a_p:s;b_t:[x+q^{-1}y])$$
$$= \sum_{n=0}^{\infty} \frac{q^{-a_1 n}(-y/x)^n}{(q)_n} E_q(a_1+n,a_2,\ldots,a_r:s;b_t:xq^n). \tag{4}$$

Next, taking $y = \lambda x$, we get the simple form

$$\sum_{n=0}^{\infty} \frac{(-\lambda)^n q^{-a_1 n}}{(q)_n} E_q(a_1+n,a_2,\ldots,a_r:s;b_t:xq^n)$$
$$= [1+\lambda]_{-a_1} E_q(r;a_p:s;b_t:x[1+\lambda q^{-1}]). \tag{5}$$

3. A General Expansion

In this section we derive a general expansion associated with $E_q(r;a_p:s;b_t:z)$. In fact, we deduce the following expansion:

$$\sum_{m=0}^{n} \frac{(q^{-n})_m (q^{b_1-a_1})_m (q^{b_1})^{n}_{m} (-z)^{-m} q^{m(a_1+n)}}{(q)_m (q^{b_1})_m}$$

$$\times E_q(a_1+n, a_2+m, \ldots, a_r+m : b_1+n+m, b_2+m, \ldots, b_s+m : z)$$

$$= (q^{a_1})_n \; E_q(r;a_p:s;b_t:z). \tag{1}$$

Consider the left side and substitute there the integral 1.3(6) for the E_q-function.

This gives

$$\frac{1}{2\pi i} \sum_{m=0}^{n} \frac{(q^{-n})_m (q^{b_1-a_1})_m (q^{b_1})_n (-z)^{-m} q^{m(a_1+n)}}{(q)_m (q^{b_1})_m}$$

$$\times \int_C \frac{G(a_1+n-x)G(a_2+m-x), \ldots, G(a_r+m-x)\pi z^x}{G(1-x)G(b_1+n+m-x)G(b_2+m-x), \ldots, G(b_s+m-x) \; \sin\pi x} \, dx.$$

Replacing x by y+m and then changing the order of integration and summation, we get

$$\frac{1}{2\pi i} \int_C \frac{G(a_1+n-y)G(a_2-y), \ldots, G(a_r-y)\pi z^y}{G(1-y)G(b_1+n-y)G(b_2-y), \ldots, G(b_s-y) \; \sin\pi y}$$

$$\times (q^{b_1})_n \; {}_3\Phi_2 \left[\begin{matrix} q^y, q^{b_1-a_1}, q^{-n}; q \\ q^{b_1}, q^{1+y-a_1-n} \end{matrix} \right] dy.$$

Using the summation theorem for a Saalschustzian ${}_3\Phi_2$, this equals to

$$(q^{a_1})_n \frac{1}{2\pi i} \int_C \frac{G(a_1-y)G(a_2-y),\ldots,G(a_r-y)\pi z^y}{G(1-y)G(b_1-y),\ldots,G(b_s-y)\sin\pi y}\,dy$$

$$= (q^{a_1})_n E_q(r;a_p:s;b_t:z).$$

This proves the result 2.3(1).

4. A more general expansion than 2.3(1) can be given in the following form: for $r \leq s$, Re $(\rho_1-k) > 0$,

$$\sum_{n=0}^{\infty} \frac{(q^{1+1/2\ell})_n(-q^{1+1/2\ell})_n(q^{1-\rho_1+\ell})_n G(\rho_1-k)G(\ell+n)(1-q^{\ell})q^{1/2n(n-1+\rho_1-k)}}{(q^{1/2\ell})_n(-q^{1/2\ell})_n G(\rho_1+n)G(\ell-k+1+n)z^n}$$

$$\times E_q\begin{pmatrix}\ell-k+1+n,\alpha_1+n,\ldots,\alpha_r+n:z\\ \ell+2n+1,\rho_1-k+n,\rho_2+n,\ldots,\rho_s+n\end{pmatrix} = E_q(r;a_p:s;\rho_t:z). \qquad (1)$$

To prove it, substitute the integral 1.3(6) for the E_q-function on the left and replace ξ by $\xi+n$ and change the order of summation and integration, which is satisfied by uniform and absolute convergence of the ${}_6\Phi_5$ for Re $(\rho_1-k) > 0$, $r \leq s$.

The left hand expression then becomes

$$\frac{1}{2\pi i}\int_C \frac{G(\ell-k+1-\xi)G(\ell+1)G(\rho_1-k)\prod_{p=1}^{r}G(\alpha_p-\xi)\pi z^{\xi}}{G(\ell+1-\xi)G(\rho_1-k-\xi)G(1-\xi)G(\rho_1)G(\ell-k+1)\prod_{t=2}^{s}G(\rho_t-\xi)\sin\pi\xi}$$

$$\times\ {}_6\phi_5\begin{pmatrix}q^{\ell},q^{1+1/2\ell},-q^{1+1/2\ell},q^{\ell-\rho_1+1},q^{k},q^{\xi};q^{\rho_1-k-\xi}\\ q^{1/2\ell},-q^{1/2\ell},q^{\rho_1},q^{\ell-k+1},q^{\ell+1-\xi}\end{pmatrix} d\xi.$$

On summing the ${}_6\phi_5$ by a particular case of the basic analogue of the Dougall's theorem 1.6(5) and on simplification, we obtain the result 2.4(1).

5. We deduce in this section another infinite expansion involving the E_q-function, namely

$$\sum_{n=0}^{\infty} \frac{(\rho_1-\sigma)_n q^{n\sigma+1/2n(n-1)}}{(\rho_1)_n (q)_n z^n} E_q\left(\begin{matrix} r;\alpha_p+n:z \\ s;\sigma+n,\rho_2+n,\ldots,\rho_s+n \end{matrix}\right)$$

$$= \frac{G(\rho_1)}{G(\sigma)} E_q(r;\alpha_p:s;\rho_t:z). \qquad (1)$$

As in section 4, we have that left-hand side of 2.5(1) is equal to

$$\sum_{n=0}^{\infty} \frac{(\rho_1-\sigma)_n q^{n\sigma+1/2n(n-1)}}{(\rho_1)_n (q)_n z^n}$$

$$\times \frac{1}{2i}\int_C \frac{\prod_{p=1}^{r} G(\alpha_p-\xi+n) z^{\xi}}{G(1-\xi)G(\sigma-\xi+n) \prod_{t=2}^{s} G(\rho_t-\xi+n) \sin\pi\xi} d\xi$$

$$= \frac{1}{2i}\int_C \frac{\prod_{p=1}^{r} G(\alpha_p-\xi) z^{\xi}}{G(1-\xi)G(\sigma-\xi) \prod_{t=2}^{s} G(\rho_t-\xi) \sin\pi\xi}$$

$$\times {}_2\phi_1\left(\begin{matrix} q^{\rho_1-\sigma}, q^{\xi}; q^{\sigma-\xi} \\ q^{\rho_1} \end{matrix}\right) d\xi .$$

The change in the order of integration and summation is justified as in section 4, for $R\ell(\sigma) > 0$ and $r \le s$.

Summing the ${}_2\phi_1$, by the basic analogue of Gauss's Theorem 1.6(2), we get

$$\frac{1}{2\pi i}\int_C \frac{\prod_{p=1}^{r} G(\alpha_p-\xi)G(\rho_1)G(\sigma-\xi)\pi z^{\xi}}{G(1-\xi)G(\sigma)\prod_{t=1}^{s} G(\rho_t-\xi)\sin\pi\xi}\,d\xi$$

$$= \frac{G(\rho_1)}{G(\sigma)}\; E_q(r;\alpha_p:s;\rho_t:z).$$

This proves the result of 2.5(1).

6. In this section I deduce still another expansion for an E_q-function whose argument contains a basic binomial factor. We shall prove that

$$\sum_{n=0}^{\infty} \frac{(1-\lambda)_n q^{1/2n(n-1)}}{(q)_n(\lambda z)^n}\; E_q\left(\begin{matrix} r;a_p+n:\lambda z \\ s;\rho_t+n \end{matrix}\right)$$

$$= E_q(r;a_p:s;\rho_t:z\lambda[1-(1-\lambda)]_{-1}). \qquad (1)$$

The left hand side is equal to

$$\sum_{n=0}^{\infty} \frac{(1-\lambda)_n q^{1/2n(n-1)}}{(q)_n(\lambda z)^n}\cdot\frac{1}{2i}\int_C \frac{\Pi G(a_p+n-\xi)(\lambda z)^{\xi}}{G(1-\xi)\,\Pi G(\rho_t+n-\xi)\,\sin\pi\xi}\,d\xi\;.$$

Replacing ξ by $\xi+n$ and changing the order of summation and integration, we get

$$\frac{1}{2i}\int_C \frac{\Pi G(\alpha_p-\xi)z^{\xi}\lambda^{\xi}}{G(1-\xi)\,\Pi G(\rho_t-\xi)\,\sin\pi\xi}\,d\xi\;\sum_{n=0}^{\infty}\frac{(1-\lambda)_n q^{-n\xi}(q^{\xi})_n}{(q)_n}\;.$$

Now the series in the integrand can be written in the form

$$[1-(1-\lambda)]_{-\xi} .$$

Hence the result.

7. Still Another Expansion

We conclude this paper by deducing another infinite expansion of the type given in section 6 for the E_q-function with a basic binomial factor as its argument. We now prove

$$\sum_{n=0}^{\infty} \frac{(-\lambda)^n q^{1/2n(n-1)}}{(q)_n} E_q(r;\alpha_p;\rho_1-n,\rho_2,\ldots,\rho_s:z)$$

$$= [1-\lambda]_{\rho_1-1} E_q(r;\alpha_p;s;\rho_t:z[1-\lambda q^{\rho_1-1}]_{-1}). \tag{1}$$

To prove the above relation we substitute the integral 1.6(5) for the E_q-function on the left and as before, we get

$$\frac{1}{2\pi i} \int_C \frac{\Pi G(\alpha_p-\xi)\pi z^{\xi}}{G(1-\xi) \prod_{t=2}^{s} G(\rho_t-\xi) \sin\pi\xi} d\xi \sum_{n=0}^{\infty} \frac{(-\lambda)^n q^{1/2n(n-1)}}{(q)_n G(\rho_t-n-\xi)}$$

$$= \frac{1}{2\pi i} \int_C \frac{\Pi G(\alpha_p-\xi)\pi z^{\xi}}{G(1-\xi) \prod_{t=1}^{s} G(\rho_t-\xi) \sin\pi\xi} [1-\lambda]_{\rho_1-1-\xi}$$

$$= [1-\lambda]_{\rho_1-1} E_q(r;\alpha_p:s;\rho_t:z[1-\lambda q^{\rho_1-1}]_{-1}).$$

8. Conditions for Convergence

In order to investigate the convergence conditions for 2.2(5), 2.4(1), 2.5(1), 2.6(1), and 2.7(1), we first determine the order of E_q-functions for large values of the parameters. For the sake of simplicity, we take all the parameters and the variable z to be real and positive. The variable z is to be replaced by $|z|$ whenever it is negative. We shall find the estimates for the following E_q-functions for large positive values of n:

(i) $E_q(r;a_p+n:s;b_t+n:|z|)$,

(ii) $E_q(r;a_p+n:b_1+2n,b_2+n,\ldots,b_s+n:|z|q^n)$,

(iii) $E_q(a_1+n,a_2,\ldots,a_r,s;b_t:|z|q)$,

(iv) $E_q(r;a_p:b_1-n,b_2,\ldots,b_s:|z|)$.

Lemma 1. For $r \leq s$

$$E_q(r;a_p+n:s;b_t+n:|z|) \leq \frac{(a_1)_n,\ldots,(a_r)_n}{(b_1)_n,\ldots,(b_s)_n} E_q(r;a_p:s,b_t:|z|),$$

for large positive values of n and $b_p < a_p$, $(p=1,2,\ldots,r)$.

Proof. For $r \leq s$

$$E_q(r;a_p+n:s;b_t+n:|z|)$$

$$= \frac{G(a_1+n),\ldots,G(a_r+n)}{G(b_1+n),\ldots,G(b_s+n)}\, {}_r\phi_s\begin{bmatrix}a_1+n,\ldots,a_r+n, |1/z| \\ b_1+n,\ldots,b_s+n\end{bmatrix} \quad (1)$$

Let us consider the two series

$${}_r\phi_s\begin{bmatrix}a_1+1,a_2+1,\ldots,a_r+1;|1/z| \\ b_1+1,b_2+1,\ldots,b_s+1\end{bmatrix}$$

and

$$ {}_r\phi_s \begin{bmatrix} a_1,a_2,\ldots,a_r; |1/z| \\ b_1,b_2,\ldots,b_s \end{bmatrix}. $$

If R_m is the ratio of the coefficients of $|1/z|^m$ in the above two series, we have

$$ R_m = \prod_{p,q}^{r,s} \frac{(1-q^{a_p+m})(1-q^{b_p})}{(1-q^{a_p})(1-q^{b_p+m})} \leq 1, $$

for all positive m, if $b_p < a_p$ $(p=1,2,\ldots,r)$.

Hence,

$$ {}_r\phi_s \begin{bmatrix} a_1+1,a_2+1,\ldots,a_r+1; |1/z| \\ b_1+1,b_2+1,\ldots,b_s+1 \end{bmatrix} $$

$$ \leq {}_r\phi_s \begin{bmatrix} a_1,a_2,\ldots,a_r; |1/z| \\ b_1,b_2,\ldots,b_s \end{bmatrix}. \qquad (2) $$

Repeated application of 2.8(2) gives

$$ {}_r\phi_s \begin{bmatrix} a_1+n,a_2+n,\ldots,a_r+n; |1/z| \\ b_1+n,b_2+n,\ldots,b_s+n \end{bmatrix} $$

$$ \leq {}_r\phi_s \begin{bmatrix} a_1,a_2,\ldots,a_r; |1/z| \\ b_1,b_2,\ldots,b_s \end{bmatrix}. \qquad (3) $$

Applying 2.8(3) to 2.8(1), with $r \leq s$, we have

$$ E_q(r;a_p+n:s,b_t+n:|z|) $$

$$ \leq \frac{(a_1)_n(a_2)_n,\ldots,(a_r)_n}{(b_1)_n(b_2)_n,\ldots,(b_s)_n} \; E_q(r;a_p:s,b_t:|z|), $$

For large positive values of n and $b_p < a_p$ $(p=1,2,\ldots,r)$.

Lemma 2. For $r \leq s$

$$E_q(a_1+n, a_2+n, \ldots, a_r+n : b_1+2n, b_2+n, \ldots, b_s+n; |z|)$$

$$\leq \frac{(a_1)_n (a_2)_n, \ldots, (a_r)_n}{(b_1)_n (b_2)_n, \ldots, (b_s)_n} \; E_q(r; a_p : s; b_s+n : |z|) \tag{4}$$

for large positive values of n.

Proof. For $r \leq s$,

$$E_q(a_1+n, a_2+n, \ldots, a_r+n : b_1+2n, b_2+n, \ldots, b_s+n : |z|)$$

$$= \frac{G(a_1+n), \ldots, G(a_r+n)}{G(b_1+2n)G(b_2+n), \ldots, G(b_s+n)} \; {}_r\phi_s \left[\begin{matrix} a_1+n, a_2+n, \ldots, a_r+n; |1/z| \\ b_1+2n, b_2+n, \ldots, b_s+n \end{matrix}\right]. \tag{5}$$

Let us consider the two series,

$${}_r\phi_s \left[\begin{matrix} a_1, a_2, \ldots, a_r; |1/z| \\ b_1+1, b_2, \ldots, b_r \end{matrix}\right]$$

and

$${}_r\phi_s \left[\begin{matrix} a_1, a_2, \ldots, a_r; |1/z| \\ b_1, b_2, \ldots, b_s \end{matrix}\right].$$

If R_m be the ratio of the coefficients of $|1/z|^m$ in the above two series, we have

$$R_m = \frac{(1-q^{b_1})}{(1-q^{b_1+m})} \leq 1, \text{ for all } m \geq 0.$$

This gives

$$ {}_r\phi_s \begin{bmatrix} a_1, a_2, \ldots, a_r; |1/z| \\ b_1+1, b_2, \ldots, b_s \end{bmatrix} $$

$$ \leq {}_r\phi_s \begin{bmatrix} a_1, a_2, \ldots, a_r; |1/z| \\ b_1, b_2, \ldots, b_s \end{bmatrix} . \qquad (6) $$

Repeated application of 2.8(6) gives

$$ {}_r\phi_s \begin{bmatrix} a_1, a_2, \ldots, a_r; |1/z| \\ b_1+n, b_2, \ldots, b_s \end{bmatrix} $$

$$ \leq {}_r\phi_s \begin{bmatrix} a_1, a_2, \ldots, a_r; |1/z| \\ b_1, b_2, \ldots, b_s \end{bmatrix} . \qquad (7) $$

Repeated use of 2.8(7) and application of the result of Lemma 1 gives

$$ {}_r\phi_s \begin{bmatrix} a_1+n, a_2+n, \ldots, a_r+n; |1/z| \\ b_1+2n, b_2+n, \ldots, b_s+n \end{bmatrix} $$

$$ \leq {}_r\phi_s \begin{bmatrix} a_1, a_2, \ldots, a_r; |1/z| \\ b_1, b_2, \ldots, b_s \end{bmatrix} . \qquad (8) $$

Using 2.8(8), 2.8(5) gives that for $r \leq s$, $b_p < a_p$ $(p=1,2,\ldots,r)$, we have

$$ E_q(a_1+n, a_2+n, \ldots, a_r+n : b_2+n, \ldots, b_s+n : |z|) $$
$$ \leq \frac{(a_1)_n (a_2)_n, \ldots, (a_r)_n}{(b_1)_{2n} (b_2)_n, \ldots, (b_s)_n} \; E_q(r; a_p : s, b_t : |z|) . $$

Lemma 3. For r = s+1,

$$E_q(a_1+n,a_2,\ldots,a_r;s;b_t:|z|q^n)$$

$$\leq \frac{G(a_1+n)G(a_2-a_1-n),\ldots,G(a_{s+1}-a_1-n)}{G(1)G(b_1-a_1-n),\ldots,G(b_s-a_1-n)}$$

$$\times \prod_{m=0}^{\infty} \frac{(1+|z|^{-1}q^{a_1+m})(1+|z|q^{1-a_1+m})}{(1+|z|^{-1}q^{m-n})(1+|z|q^{1+m+n})}$$

$$\times {}_{s+1}\phi_s\left[\begin{matrix}a_1,1+a_1-b_1,\ldots,1+a_1-b_s;|z|Q\\ 1+a_1-a_2,\ldots,1+a_1-a_{s+1}\end{matrix}\right]$$

$$+ \frac{G(a_2)G(a_1+n-a_2)G(a_3-a_2),\ldots,G(a_{s+1}-a_2)}{G(1)G(b_1-a_2)G(b_2-a_2),\ldots,G(b_s-a_2)}$$

$$\times \prod_{m=0}^{\infty} \frac{(1+|z|^{-1}q^{a_2+m-n})(1+|z|q^{1-a_2+n+m})}{(1+|z|^{-1}q^{m-n})(1+|z|q^{1+m+n})}$$

$$\times {}_{s+1}\phi_s\left[\begin{matrix}a_2,1+a_2-b_1,\ldots,1+a_2-b_s;|z|Q'\\ 1+a_2-a_1,1+a_2-a_3,\ldots,1+a_2-a_{s+1}\end{matrix}\right] + \text{idem}(a_2;a_3,\ldots,a_{s+1}), \qquad (9)$$

where $Q = q^{\Sigma b_s - \Sigma a_s + 1}$ and $Q' = q^{\Sigma b_{s+1} - \Sigma a_{s+1}}$,

for large positive values of n.

For r > s+1, the order of E_q-function is still smaller.

Proof. For r = s+1,

$$E_q(a_1+n,a_2,\ldots,a_{s+1}:s;b_t:|z|q^n)$$

$$= \frac{G(a_1+n)G(a_2-a_1-n),\ldots,G(a_{s+1}-a_1-n)}{G(1)G(b_1-a_1-n),\ldots,G(b_s-a_1-n)}$$

$$\times \prod_{m=0}^{\infty} \frac{(1+|z|^{-1}q^{a_1+m})(1+|z|a^{1-a_1+m})}{(1+|z|^{-1}q^{m-n})(1+|z|q^{1+m+n})}$$

$$\times {}_{s+1}\phi_s\begin{pmatrix} a_1+n, 1+a_1+n-b_1,\ldots,1+a_1+n-b_s:|z|Q \\ 1+a_1+n-a_2,\ldots,1+a_1+n-a_{s+1} \end{pmatrix}$$

$$+ \frac{G(a_2)G(a_1+n-a_2),\ldots,G(a_{s+1}-a_2)}{G(1)G(b_1-a_2),\ldots,G(b_s-a_2)}$$

$$\times {}_{s+1}\phi_s\begin{pmatrix} a_2, 1+a_2-b_1,\ldots,1+a_2-b_s:|z|Q' \\ 1+a_2-a_1-n, 1+a_2-a_3,\ldots,1+a_2-a_{s+1} \end{pmatrix}$$

$$+ \text{idem}(a_2,a_3,\ldots,a_{s+1}), \tag{10}$$

where $Q = q^{\Sigma b_s - \Sigma a_s + 1}$ and $Q' = q^{\Sigma b_{s+1} - \Sigma a_{s+1}}$. We have as in Lemma 1 that

$${}_{s+1}\phi_s\begin{pmatrix} A_1+n, A_2+n,\ldots,A_{s+1}+n;|z|Q \\ B_1+n, B_2+n,\ldots,B_s+n \end{pmatrix}$$

$$\leq {}_{s+1}\phi_s\begin{pmatrix} A_1, A_2,\ldots,A_{s+1};|z|Q \\ B_1, B_2,\ldots,B_s \end{pmatrix}. \tag{11}$$

Also considering the two series

$${}_{s+1}\phi_s\begin{pmatrix} A_1, A_2,\ldots,A_{s+1};|z|Q' \\ B_1-1, B_2,\ldots,B_s \end{pmatrix}$$

and

$$ {}_{s+1}\phi_s\begin{pmatrix} A_1,A_2,\ldots,A_{s+1};|z|Q' \\ B_1,B_2,\ldots,B_s \end{pmatrix}, $$

if R_m be the ratio of the coefficients of $|z|^m Q'^m$ in the above two series, we have

$$ R_m = \frac{(1-q^{B_1+m})}{(1-q^{B_1-1})} \, . $$

$(1-q^{B_1-1})\ R_m \le 1$, for all $m \ge 0$.

Hence

$$ (1-q^{B_1-1})\ {}_{s+1}\phi_s\begin{pmatrix} A_1,A_2,\ldots,A_{s+1};|z|Q' \\ B_1-1,B_2,\ldots,B_s \end{pmatrix} $$

$$ \le {}_{s+1}\phi_s\begin{pmatrix} A_1,A_2,\ldots,A_{s+1};|z|Q' \\ B_1,B_2,\ldots,B_s \end{pmatrix}. \qquad (12) $$

Repeated application of 2.8(12) gives

$$ (q^{B_1-1})_n\ {}_{s+1}\phi_s\begin{pmatrix} A_1,A_2,\ldots,A_{s+1};|z|Q' \\ B_1-n,B_2,\ldots,B_s \end{pmatrix} $$

$$ \le {}_{s+1}\phi_s\begin{pmatrix} A_1,A_2,\ldots,A_{s+1};|z|Q' \\ B_1,B_2,\ldots,B_s \end{pmatrix} . \qquad (13) $$

Applying 2.8(11) and 2.8(13) to 2.8(10) with $r = s+1$, we have

$$ E_q(a_1+n,a_2,\ldots,a_{s+1}:b_t:|z|q^n) $$

$$ \le \frac{G(a_1+n)G(a_2-a_1-n),\ldots,G(a_{s+1}-a_1-n)}{G(1)G(b_1-a_1-n),\ldots,G(b_s-a_1-n)} $$

$$\times \prod_{m=0}^{\infty} \frac{(1+|z|^{-1}q^{a_1+m})(1+|z|q^{1-a_1+m})}{(1+|z|^{-1}q^{m-n})(1+|z|q^{1+m+n})}$$

$$\times {}_{s+1}\phi_s\begin{pmatrix} a_1, 1+a_1-b_1, \ldots, 1+a_1-b_s; |z|Q \\ 1+a_1-a_2, \ldots, 1+a_1-a_{s+1} \end{pmatrix}$$

$$+ \frac{G(a_2)G(a_1+n-a_2)G(a_3-a_2), \ldots, G(a_{s+1}-a_2)}{G(1)G(b_1-a_2), \ldots, G(b_s-a_2)}$$

$$\times \prod_{m=0}^{\infty} \frac{(1+|z|^{-1}q^{a_2+m-n})(1+|z|q^{1-a_2+n+m})}{(1+|z|^{-1}q^{m-n})(1+|z|q^{1+n+m})(q^{a_2-a_1})_n}$$

$$\times {}_{s+1}\phi_s\begin{pmatrix} a_2, 1+a_2-b_1, \ldots, 1+a_2-b_s; |z|Q' \\ 1+a_2-a_1, 1+a_2-a_3, \ldots, 1+a_2-a_{s+1} \end{pmatrix}$$

$$+ \text{idem}(a_2, a_3, \ldots, a_{s+1}),$$

where $Q = q^{\Sigma b_s - \Sigma a_s + 1}$ and $Q' = q^{\Sigma b_{s+1} - \Sigma a_{s+1}}$.

For $r > s+1$, because of the presence of the quadratic power, $q^{1/2m(m+i)}$ in the argument of the hypergeometric functions the order of each term is still smaller than in the previous case.

Lemma 4. For $r = s+1$, $|z| < 1$,

$$E_q(a_1, a_2, \ldots, a_{s+1} : b_1-n, b_2, \ldots, b_s : |z|)$$

$$\leq K \prod_{m=0}^{\infty} \frac{(1-q^{b_1-a_1-n+m})}{(z^{1+a_1-b_1})_n} q^{a(\lambda+n)} + \text{idem}(a_1, a_2, \ldots, a_r),$$

for large positive values of n, where K and a are suitable constants independent of n, and $\lambda = \Sigma b_s - \Sigma a_r + 1$. For $r > s+1$, the order of the E_q-function is still smaller.

<u>Proof.</u> For $r \geq s+1$

$$E_q(a_1,a_2,\ldots,a_r:b_1-n,b_2,\ldots,b_s:|z|)$$

$$= \frac{G(a_1)G(a_2-a_1),\ldots,G(a_r-a_1)}{G(1)G(b_1-a_1-n),\ldots,G(b_s-a_1)}$$

$$\times \prod_{m=0}^{\infty} \frac{(1+|z|^{-1}q^{a_1+m})(1+|z|q^{1-a_1+m})}{(1+|z|^{-1}q^m)(1+|z|q^{1+m})}$$

$$\times {}_{s+1}\phi_s\left(\begin{matrix} a_1, 1+a_1-b_1+n, 1+a_1-b_2,\ldots,1+a_1-b_s; |z|Q \\ 1+a_1-a_2,\ldots,1+a_1-a_r \end{matrix}\right)$$

$$+ \text{idem}(a_1,a_2,\ldots,a_r),$$

where $Q = q^{1/2(m+1)(r-s-1)}q^{\Sigma b_s - \Sigma a_r + a_1(r-s-a)+1-n}$. (14)

Let us first consider the case $r = s+1$ and consider the two series of the type

$${}_{s+1}\phi_s\left(\begin{matrix} A_1, A_2+1, A_3,\ldots,A_{s+1}; |z|Q \\ B_1, B_2, B_3,\ldots,B_s \end{matrix}\right)$$

and

$${}_{s+1}\phi_s\left(\begin{matrix} A_1, A_2, A_3,\ldots,A_{s+1}; |z|Q \\ B_1, B_2, B_3,\ldots,B_s \end{matrix}\right) \tag{15}$$

where $Q = q^{\alpha-n}$.

If R_m is the ratio of the coefficients of $|z|^m Q^m$ in the above two series, we have

$$R_m = \frac{(1-q^{A_2+m})}{(1-q^{A_s})}$$

$$(1-q^{A_s})\, R_m \leq 1 \text{ for all } m \geq 0.$$

Hence

$$(1-q^{A_s}) \;{}_{s+1}\phi_s \begin{pmatrix} A_1, A_2+1, A_3, \ldots, A_{s+1}; |z|Q \\ B_1, B_2, B_3, \ldots, B_s \end{pmatrix}$$

$$\leq {}_{s+1}\phi_s \begin{pmatrix} A_1, A_2, \ldots, A_{s+1}; |z|Q \\ B_1, B_2, \ldots, B_s \end{pmatrix} . \qquad (16)$$

Repeated application of 2.8(16) gives

$${}_{s+1}\phi_s \begin{pmatrix} A_1, A_2+n, A_3, \ldots, A_{s+1}; |z|Q \\ B_1, B_2, B_3, \ldots, B_s \end{pmatrix}$$

$$\leq {}_{s+1}\phi_s \begin{pmatrix} A_1, A_2, \ldots, A_{s+1}; |z|Q \\ B_1, B_2, \ldots, B_s \end{pmatrix} . \qquad (17)$$

Applying 2.8(17) to 2.8(14) with $r = s+1$, we have

$$E_q(a_1, a_2, \ldots, a_{s+1} : b_1-n, b_2, \ldots, b_s : z)$$

$$\leq K \prod_{m=0}^{\infty} \frac{(1-q^{b_1-a_1-n+m})}{(q^{1+a_1-b_1})_n} \;{}_{s+1}\phi_s \begin{pmatrix} a_1, 1+a_1-b_1, \ldots, 1+a_1-b_s; |z|Q \\ 1+a_1-a_2, \ldots, 1+a_1-a_{s+1} \end{pmatrix}$$

$$+ \text{idem}(a_1; a_2, \ldots, a_{s+1})$$

$$\sim K \prod_{m=0}^{\infty} \frac{(1-q^{b_1-a_1-n+m})}{(q^{1+a_1-b_1})_n} q^{a(\lambda+n)} + \text{idem}(a_1,a_2,\ldots,a_{s+1}),$$

where K and a are suitable constants independent of n. In deriving the last estimate we have used the asymptotic value of ${}_{s+1}\phi_s$ for large values of the argument (Hahn [1]). For $r > s+1$, because of the presence of the quadratic power $q^{1/2m(m+1)}$ in the argument of the hypergeometric functions, the order of each term is still smaller than in the previous case. Applying the above Lemmas we get the following conditions for convergence:

$r > s$ and $|\lambda q^{-a_1}| < 1$ for 2.2(15),

$r \leq s$ and $z \neq 0$ for 2.4(1),

$r \leq s$ and $z \neq 0$ for 2.5(1),

$r \leq s$ and $z \neq 0$ for 2.6(1),

$r \geq s$ and $|\lambda| < 1$ for 2.7(1).

It may be remarked that no attempt has been made to examine the conditions of convergence for values of r and s other than those mentioned above because the only importance of the expansions proved lies in their mere existence.

CHAPTER III

CERTAIN GENERAL EXPANSION OF BASIC GENERALIZED MACROBERT'S E-FUNCTIONS - II

1. Introduction

In this chapter I give a very general expansion involving therein the generalized E_q-function in terms of a series of another E_q-function and it has been shown that from this expansion a large variety of elegant results can be obtained. In our analysis we shall come across functions similar to the E_q-function but in which the base of all the terms is not the same. In particular, we shall deal with functions of the type, ($q' \equiv q^m$, m a positive integer, $r \leq s$)

$$\frac{G(\alpha_1),\ldots,G(\alpha_r)G_{q'}(k/m),\ldots,G_{q'}(k+m-1/m)}{G(\beta_1),\ldots,G(\beta_s)}$$

$$\times\ \Phi\begin{bmatrix} q^{\alpha_1},\ldots,q^{\alpha_r},q'^{k/m},\ldots,q'^{k+m-1/m};-1/z \\ q^{\beta_1},\ldots,q^{\beta_s} \end{bmatrix}$$

which we denote by

$$E_{q'}\begin{bmatrix} q^{\alpha_1},\ldots,q^{\alpha_r},q'^{k/m},\ldots,q'^{k+m-1/m};z \\ q^{\beta_1},\ldots,q^{\beta_s} \end{bmatrix}$$

$$= E_{q'}[r;\alpha_p:\Delta(m;k):s,\beta_r;z],$$

where the symbol $\Delta(m;\alpha)$ stands for m parameters of the type

$$q^{\alpha/m}, q^{\alpha+1/m}, \ldots, q^{\alpha+m-1/m} .$$

2. The Main Theorem

Subject to suitable conditions of convergence

$$\sum_{n=0}^{\infty} \frac{(-1)^{\varepsilon n}[(a_A)]_n [\mu z]^n q^{n\theta+\frac{1}{2}wn(n+1)}}{(q)_n [(b_B)]_n}$$

$$E_q \left[\begin{matrix} \Delta[m;(\alpha_S)+n], \Delta[m;(\beta_T)-n], \Delta[2m;(p_P)+n], \Delta[2m;(r_R)-n], (c_C) : zq^{\lambda} \\ \Delta[m;(\gamma_U)+n], \Delta[m;(\delta_V)-n], \Delta[2m;(e_E)+n], \Delta[2m;(f_F)-n]; (d_D) \end{matrix} \right]$$

$$\equiv \frac{1}{2\pi i} \int_C \frac{G[(c_C)-\xi] \prod_{u=0}^{m-1} G_q\left[\frac{(\alpha_S)+u}{m} - \xi\right] G_q\left[\frac{(\beta_T)+u}{m} - \xi\right] \prod_{u=0}^{2m-1} G_q\left[\frac{(p_P)+u}{2m} - \xi\right] G_q\left[\frac{(r_R)+u}{2m} - \xi\right]}{G[1-\xi] G[(d_D)-\xi] \prod_{u=0}^{m-1} G_q\left[\frac{(\gamma_U)+u}{m} - \xi\right] G_q\left[\frac{(\delta_V)+u}{m} - \xi\right] \prod_{u=0}^{2m-1} G_q\left[\frac{(e_E)+u}{2m} - \xi\right] G_q\left[\frac{(f_F)+u}{2m} - \xi\right]}$$

$$\times \frac{\pi}{\sin \pi\xi} {}_{A+M}\Phi_{B+N} \left[\begin{matrix} (a_A), (\alpha_S)-m\xi, 1-(\delta_V)+m\xi, (p_P)-2m\xi, 1-(f_F)+2m\xi; Q \\ (b_B), (\gamma_U)-m\xi, 1-(\beta_T)+m\xi, (e_E)-2m\xi, 1-(r_R)+2m\xi \end{matrix} \right] \times (zq^{\lambda})^{\xi} d\xi \tag{1}$$

where $Q = (-1)^{\varepsilon+T+R-F-V} q^{-1/2(n+1)(V+F-T-R-W)} q^{\theta-m\xi(V+2F-T-2R)}$

$$- \Sigma\beta_T - \Sigma r_R + \Sigma\delta_V + \Sigma f_F ,$$

and C denotes the contour given in 1.2(4) and the notation (a_A) stands for the sequence of A parameters $a_1, a_2, \ldots, a_A$. This notation will be followed throughout the present work. Also w is a positive integer.

Proof. To prove 3.2(1) we use the contour integral for the E_q-function on the left side and change the order of integration and summation, assumed to be justified by the absolute convergence of series of E_q'. The expression then on simplification reduces to 3.2(1).

Now, whenever the inner hypergeometric series in 3.2(1) can be summed up in terms of q-infinite products of the form.

$$G[(a_r)-m\xi]/G[(b_s)-m\xi],$$

one can evaluate the contour integral on the right hand side in terms of another E_q-function and hence obtain a sum of a series of E_q-function as another E_q-function. Exploiting this technique a large number of expansions can be obtained by specializing the parameters in our theorem. As illustration, I mention some of the typical results.

3. Special Cases of 3.2(1)

Taking $B = E = F = P = R = T = V = W = 0$, $A = S = U = \mu z = 1$, $\varepsilon = 2$, $\theta = \rho - \beta - \alpha$; $\alpha_1 = \alpha$, $a_1 = \beta$, $\gamma_1 = \rho$ and summing the inner ${}_2\Phi_1$ in the integrand by the basic analogue of Gauss's Theorem (1.6(2)), we get, that

$$\sum_{r=0}^{\infty} \frac{(q^{\beta})_r q^{r(\rho-\beta-\alpha)}}{(q)_r} E_q'\left[\begin{matrix}\Delta(m;\alpha+r),(c_C):zq^{\lambda}\\ \Delta(m;\rho+r),(d_D)\end{matrix}\right]$$

$$= \frac{G(\rho-\beta-\alpha)}{G(\rho-\alpha)} E_q'\left[\begin{matrix}\Delta(m;\alpha),(c_C):zq^{\lambda}\\ \Delta(m,\rho-\beta),(d_D)\end{matrix}\right].$$

(ii) Taking $A = B = E = F = P = R = T = U = V = W = \lambda = 0$; $\mu z = S = 1$, $\varepsilon = 2$, $\theta = k$, $\alpha_1 = \alpha$ and summing the inner ${}_1\Phi_0$, we get

$$\sum_{n=0}^{\infty} \frac{q^{kn}}{(q)_n} \; E_{q'}\left[\begin{matrix}\Delta(m;\alpha+n),(c_C):zq^{\lambda}\\ (d_D)\end{matrix}\right]$$

$$= G(k)\; E_q\left[\begin{matrix}(c_C),\Delta[m;\alpha]:zq^{\lambda}\\ (d_D),\Delta[m;k+\alpha]\end{matrix}\right] .$$

(iii) Taking $F = P = R = T = U = V = \lambda = w = 0$, $A = B = \mu z = E = \theta = 1$, $\varepsilon = S = 2$, $a_1 = q^{-r}$, $b_1 = q^{s}$, $e_1 = 1 + \alpha + \beta - \delta - \gamma$, $\alpha_1 = \alpha$, $\alpha_2 = \beta$ and summing the inner ${}_3\Phi_2$ (1.6(4)) in the integrand on the right, we get

$$\sum_{n=0}^{r} \frac{(q^{-r})_n q^{n}}{(q)_n (q^{\delta})_n} \; E_{q'}\left[\begin{matrix}(c_C),\Delta[m;\alpha+n],\Delta[m;\beta+n]:zq^{\lambda}\\ (d_D):\Delta[2m,1+\alpha+\beta-\delta-r+n]\end{matrix}\right]$$

$$= \frac{q^{(1-\delta)r}+(-1)^{r} q^{r/2(r+1)}}{(q^{\delta})_r} \, E_{q'}\left[\begin{matrix}(c_C),\Delta[m,\alpha]:zq^{\lambda}\\ (d_D),\Delta[m;1+\alpha-\delta-\gamma],\Delta[m,1+\beta-\delta-\gamma]\end{matrix}\right].$$

(iv) Taking $E = F = P = R = w = \lambda = 0$; $S = T = U = V = \varepsilon = 2$; $\mu z = 1$, $A = 4$, $B = 3$, $a_1 = q^{\alpha}$, $a_2 = q^{1+1/2\alpha}$; $a_3 = -q^{1+1/2\alpha}$; $a_4 = q^{-p}$, (p is a positive integer); $b_1 = q^{1/2\alpha}$, $b_2 = -q^{1/2\alpha}$, $b_3 = q^{1+\alpha+p}$, $\alpha_1 = \beta$; $\alpha_2 = \gamma$, $\beta_1 = \beta-\alpha$; $\beta_2 = \gamma-\alpha$, $\gamma_1 = 1+\alpha-\delta$; $\gamma_2 = 1+\alpha-\eta$; $\delta_1 = 1-\delta$; $\delta_2 = 1-\eta$; $\theta = 2\alpha - \beta - \delta - \eta - \gamma + 3$ and summing the inner well-poised ${}_8\Phi_7$ (1.6(5)) in the integrand, we get

$$\sum_{n=0}^{p} \frac{(q^{\alpha})_n (q^{1+1/2\alpha})_n (-q^{1+1/2\alpha})_n (q^{-p})_n q^{n(2\alpha-\beta-\delta-\eta-\gamma+3)}}{(q)_n (q^{1/2\alpha})_n (-q^{1/2\alpha})_n (q^{1+\alpha+p})_n}$$

$$\times E_{q'}\left[\begin{matrix}(c_C),\Delta[m,\gamma+n],\Delta[m,\beta+n],\Delta[m,\beta-\alpha-n],\Delta[m,\gamma-\alpha-n]:zq^{\lambda}\\ (d_D),\Delta[m,1+\alpha-\delta+n],\Delta[m,1+\alpha-\eta+n],\Delta[m;1-\delta-n],\Delta[m;1-\eta-n]\end{matrix}\right]$$

$$= \frac{(-1)^{+p} q^{-p/2(p+1)} q^{\alpha p} (q^{1+\alpha})_p (q^{1+\alpha-\gamma-\eta})_p (q^{1+\gamma-\beta-\eta})_p}{(q^{1+\alpha-\beta-r-\eta-\delta})_p}$$

$$\times E_{q'}\left[\begin{matrix}(c_C),\Delta[m,\beta],\Delta[m;\gamma],\Delta[m,\beta-\alpha],\Delta[m,\gamma-\alpha],\Delta[m,\beta-\alpha-p],\Delta[m,\gamma-\alpha-p]:zq^{\lambda}\\(d_D),\Delta[m,1+\alpha-\delta],\Delta[m,1+\alpha-\eta],\Delta[m;1-\delta],\Delta[m;1-\eta],\Delta[2m;\lambda-\alpha+\gamma-p]\end{matrix}\right].$$

(v) Taking $E = F = P = R = S = T = U = w = 0$; $\varepsilon = 2$, $A = B = 0$, $V = 2$, $\delta_1 = \rho$; $\delta_2 = \sigma$ and summing the inner ${}_2\Phi_1$ by the q-analogue of Gauss's Theorem (1.6(4)) and on integration we get

$$\sum_{n=0}^{\infty} \frac{q^{n(n+\alpha)}}{(q)_n (q^{\alpha})_n} E_{q'}\left[\begin{matrix}(c_C):zq^{\lambda}\\(d_D);\Delta(m;\rho-n),\Delta(m;\sigma-n)\end{matrix}\right]$$

$$= G(\alpha)\, E_{q'}\left[\begin{matrix}\Delta(2m;\alpha+\rho+\sigma-2),(c_C):zq^{\lambda}\\\Delta(m;\alpha),\Delta(m;\sigma),\Delta(m;\alpha+\rho-1),\Delta(m;\alpha+\sigma-1),(d_D)\end{matrix}\right].$$

(vi) Taking $E = F = P = R = S = T = 0$; $\mu z = 1$, $\theta = (k-2)$; $A = U = V = w = 3$; $B = 2$; $a_1 = q^k$; $a_2 = q^{1+1/2k}$, $a_3 = -q^{1+1/2k}$, $b_1 = q^{1/2k}$; $b_2 = -q^{1/2k}$; $\gamma_1 = \alpha$, $\gamma_2 = \beta$; $\gamma_3 = \gamma$; $\delta_1 = \alpha-k$; $\delta_2 = \beta-k$; $\delta_3 = \gamma-k$, we get, on summing the inner well-poised ${}_6\Phi_5$ (1.6(6)), we get on integration

$$\sum_{n=0}^{\infty} \frac{(q^k)_n (q^{1+1/2k})_n (-q^{1+1/2k})_n q^{n(k-2)+3n/2(n+1)}}{(q)_n (q^{1/2k})_n (-q^{1/2k})_n}$$

$$\times E_{q'}\left[\begin{matrix}(c_C):zq^{\lambda}\\(d_D),\Delta(m;\alpha+n),\Delta(m,\beta+n),\Delta(m;\gamma+n),\infty(m,\alpha-k-n),\Delta(m;\beta-k-n),\Delta(m,\gamma-k-n)\end{matrix}\right]$$

$$= \frac{1}{G(1+k)} E_{q'}\left[\begin{matrix}(c_C),\Delta(3m;\alpha+\beta+\gamma-2k-2):zq^{\lambda}\\(d_D),\Delta(m,\alpha-k),\Delta(m;\beta-k),\Delta(m;\gamma-k),\Delta(m,\beta+\gamma-k-1),\Delta(2m;\alpha+\beta-1-1),\\\Delta(2m;\alpha+\gamma-k-1)\end{matrix}\right].$$

One could, similarly, give other special cases of the main theorem.

4. Convergence Conditions

A typical series that occurs in the previous section is

$$I = E_q\left[\begin{matrix}(a_A)+n;(c_C)-n;(e_E);z\\(b_B)+n;(d_D)-n;(f_F)\end{matrix}\right]$$

$$= \frac{1}{2\pi i}\int_C \prod_{s=0}^{\infty} \frac{(1-q^{1-x+s})(1-q^{(b_B)+n+s-x})(1-q^{(d_D)-n+s-x})(1-q^{(f_F)-x+s})}{(1-q^{(a_A)+n-x+s})(1-q^{(c_C)-n+s-x})(1-q^{(e_E)-x+s})} \times \frac{\pi z^x dx}{\sin\pi x}.$$

For large values of n, if (a_A), (b_B), (c_C), and (d_D) are positive

$$I \sim \frac{1}{2\pi i}\int_C \prod_{s=0}^{\infty} \frac{(1-q^{1-x+s})(1-q^{(f_F)-x+s})}{(1-q^{(e_E)-x+s})} \frac{\pi z^x dx}{\sin\pi x} \prod_{s=1}^{n} \frac{(1-q^{x+s-(d_D)})}{(1-q^{x+s-(c_C)})} \cdot q^{(C-D)(x+s)}.$$

If $C > D$, then for $n \to \infty$

$$I \sim \frac{1}{2\pi i}\int_C \prod_{s=0}^{\infty} \frac{(1-q^{1-x+s})(1-q^{(f_F)-x+s})(1-q^{x+s-(d_D)})}{(1-q^{(e_E)-x+s})(1-q^{x+s-(c_C)})} \cdot \frac{\pi z^x dx}{\sin\pi x},$$

which is independent of n and is a bounded function of z.

The above estimation, although only a rough estimate, nevertheless, serves to show that the infinite series involved in the preceeding sections converge.

CHAPTER IV

INTEGRATION INVOLVING THE q-ANALOGUE OF GENERALIZED MACROBERT'S E-FUNCTION WITH RESPECT TO THEIR PARAMETERS - I

1. Introduction

In this chapter we evaluate a very general integral involving E_q-functions, where the integration has been performed with respect to parameters. The main integral is evaluated in section 2 in the form of a covering theorem for evaluation of simpler integrals.

2. The Main Integral

The integral to be evaluated, with suitable restriction on the parameters, is

$$I = \frac{1}{2i} \int_C \frac{G[(a_A)-\xi]Z^{\xi}}{G[(b_B)-\xi]G[1-\xi] \sin\pi\xi} E_q \begin{bmatrix} (\alpha_P)-\xi, (c_C) : Zq^{\theta} \\ (\beta_Q)-\xi, (d_D) \end{bmatrix} d\xi, \qquad (1)$$

the path of integration being the same as in 1.4(1), and $P \geq Q$,
$|\{\arg(z) - w_2 w_1^{-1} \log|z|\}| < \pi$, $|q| < 1$.
The E_q-functions are further assumed to satisfy a relation of the type

$$E_q[(a_A):(b_B):Z]\, E_q[(c_C):(d_D):Zq^{\theta}]$$
$$= \Sigma_j K_j E_q \begin{bmatrix} (l_L)_j : Z^{m_j} q^{\phi} \\ (r_R)_j \end{bmatrix} Z^{S_j}, \qquad (2)$$

where Σ_j is a finite summation, ϕ is a positive integer and the symbol $(g_G)_j$; $(J=1,2,\ldots)$ stands for the sequence $(g_1)_j, (g_2)_j, \ldots, (g_G)_j$.

The following known formulae due to N. Agarwal [1] will be used in evaluating the integral 4.2(1):

$$\frac{G(1)}{(1-q)} \int_0^1 E_q(q\lambda)\lambda^{k-1} \; E_q\left[\begin{matrix} r;\alpha_p:z/\lambda^m \\ s;\beta_Q \end{matrix}\right] d(q\lambda)$$

$$= \frac{G(k)}{G_{q'}(k/m),\ldots,G_{q'}((k+m-1/m))} E_{q'}[r+m;\alpha_p,\Delta(m,k):s,\beta_Q:z], \tag{3}$$

where m is a positive integer and Rl(k) > 0, and also if m is a positive integer, Rl(β) > Rl(α) > 0,

$$\frac{G(1)}{(1-q)} \int_0^1 \lambda^{\alpha-1}(1-q\lambda)_{\beta-\alpha-1} \; E_q\left[\begin{matrix} r;\alpha_p:z/\lambda^m \\ s,\beta_Q \end{matrix}\right] d(q\lambda)$$

$$= \frac{G(\beta-\alpha)G(\alpha)G_{q'}(\beta/m),\ldots,G_{q'}(\beta+m-1/m)}{G(\beta)G_{q'}(\alpha/m),\ldots,G_{q'}((\alpha+m-1/m))}$$

$$\times E_{q'}\left[\begin{matrix} r;\alpha_p:\Delta(m;\alpha):z \\ s;\beta_Q:\Delta(m,\beta) \end{matrix}\right] . \tag{4}$$

3. <u>Evaluation of 4.2(1)</u>

Using 4.2(3) and 4.2(4) with m=1 in 4.2(1), we get that

$$I = \frac{1}{2i} \int_C \frac{G[(a_A)-\xi]z^\xi}{G[1-\xi]G[(b_B)-\xi] \sin\pi\xi} \left\{ \prod_{n=1}^{Q} G[\beta_n-\alpha_n] \right\}^{-1}$$

$$\times \prod_{n=1}^{Q} \frac{G(1)}{(1-q)} \int_0^1 \lambda_n^{\alpha_n-\xi-1} (1-q\lambda_n)_{\beta_n-\alpha_n-1}$$

$$\times \prod_{n=Q+1}^{P} \left\{ \frac{G(1)}{(1-q)} \int_0^1 \lambda_n^{\alpha_n-\xi-1} \; E_q(q\lambda_n) \; E_q\left[\begin{matrix} (c_C):zq^\theta/\lambda_1,\lambda_2,\ldots,\lambda_p \\ (d_D) \end{matrix}\right] d\lambda_n \right\} d\xi . \tag{1}$$

Changing the order of integration, whenever it is justified, such that the λ_n-integral becomes the inner most integral, we get that

$$I = \left\{\prod_{n=1}^{Q} G[\beta_n-\alpha_n]\right\}^{-1} \prod_{n=1}^{Q} \frac{G(1)}{(1-q)} \int_0^1 \lambda_n^{\alpha_n-1}(1-q\lambda_n)_{\beta_n-\alpha_n-1}\,d(q\lambda_n)$$

$$\times \prod_{n=Q+1}^{P} \left\{\frac{G(1)}{(1-q)} \int_0^1 E_q(q\lambda_n)\lambda_n^{\alpha_n-1} \quad E_q\begin{bmatrix}(c_C):zq^{\theta}/\lambda_1\lambda_2,\ldots,\lambda_p \\ (d_D)\end{bmatrix}\right.$$

$$\times E_q\begin{bmatrix}(a_A):z/\lambda_1\lambda_2,\ldots,\lambda_p \\ (b_B)\end{bmatrix} d(q\lambda_n). \tag{2}$$

Now using the assumed relation 4.2(2) between E_q-functions and changing the order of summation and integration, we get that,

$$\left\{\prod_{n=1}^{Q} G[\beta_n-\alpha_n]\right\}^{-1} \prod_{n=1}^{Q} \frac{G(1)}{(1-q)} \Sigma_j k_j z^{s_j} \int_0^1 \lambda_n^{\alpha_n-s_j-1} (1-q\lambda_n)_{\beta_n-\alpha_n-1}\,d(q\lambda_n)$$

$$\times \prod_{n=Q+1}^{P} \left\{\frac{G(1)}{(1-q)} \int_0^1 E_q(q\lambda_n)\lambda_n^{\alpha_n-s_j-1} \quad E_q\begin{bmatrix}(1_L)_j:z^{m_j}q^{\phi}/(\lambda_1\lambda_2,\ldots,\lambda_p)^{m_j} \\ (r_R)_j\end{bmatrix}\right. d(q\lambda_n). \tag{3}$$

Once again, using 4.2(3) and 4.2(4) to evaluate the λ-integrals, we get

$$I = \Sigma_j k_j z^{s_j} \frac{G[(\alpha_p-s_j)]G_{q'}[\Delta(m_j;(\beta_Q)-s_j]}{G[(\beta_Q)-s_j]G_{q'}[\Delta(m_j;(\alpha_p)-s_j]}$$

$$\times E_{q'}\begin{bmatrix}(1_L)_j;\Delta[m_j;(\alpha_p)-s_j]:z^{m_j}q^{\phi} \\ (r_R)_j;\Delta[m_j;(\beta_Q)-s_j]\end{bmatrix} \tag{4}$$

where $Rl(\beta_n) > Rl(\alpha_n) > s_j$, $n=1,2,\ldots,Q$; $Rl(\alpha_n) > s_j$, $n=Q+1,\ldots,P_j$; $j=1,2,\ldots,n$.

4. Certain Known Cases of 4.2(2) and Integrals

In this section, I first cite certain known cases in which an identity of the type 4.2(2) holds.

$$\text{(i)}\quad \prod_{n=0}^{\infty}\left(\frac{1-zq^{\alpha+\beta-\gamma+n}}{1-zq^{n}}\right) {}_2\phi_1\left(\begin{matrix} q^{\gamma-\alpha}, q^{\gamma-\beta}; zq^{\alpha+\beta-\gamma} \\ q^{\gamma} \end{matrix}\right) = {}_2\phi_1\left(\begin{matrix} q^{\alpha}, q^{\beta}; z \\ q^{\gamma} \end{matrix}\right),$$

which is a basic analogue of Euler's identity [Tract]

$$\text{(ii)}\quad {}_2\phi_1\left(\begin{matrix} q^{\alpha}, q^{\beta}; z \\ q^{\alpha+\beta+1/2} \end{matrix}\right) \; {}_2\phi_1\left(\begin{matrix} q^{\alpha}, q^{\beta}; zq^{-1/2} \\ q^{\alpha+\beta-1/2} \end{matrix}\right)$$

$$= {}_2\phi_1\left(\begin{matrix} q^{\alpha+1/2}, q^{\beta+1/2}; zq^{1/2} \\ q^{\alpha+\beta+1/2} \end{matrix}\right) \; {}_2\phi_1\left(\begin{matrix} q^{\alpha-1/2}, q^{\beta-1/2}; z \\ q^{\alpha+\beta-1/2} \end{matrix}\right)$$

$$= {}_4\phi_3\left(\begin{matrix} x^{2\alpha}, x^{2\beta}, x^{\alpha+\beta}, -x^{\alpha+\beta}; zx^{-1} \\ x^{2\alpha+2\beta-1}, x^{\alpha+\beta+1/2}, -x^{\alpha+\beta+1/2} \end{matrix}\right)$$

$$\text{(iii)}\quad {}_2\phi_1\left(\begin{matrix} q^{\alpha+1/2}, q^{\beta}, zq^{-1/2} \\ q^{\alpha+\beta} \end{matrix}\right) \; {}_2\phi_1\left(\begin{matrix} q^{\alpha-1/2}, q^{\beta}; z \\ q^{\alpha+\beta} \end{matrix}\right)$$

$$= {}_4\phi_3\left(\begin{matrix} x^{2\alpha}, x^{2\beta}, x^{\alpha+\beta+1/2}, -x^{\alpha+\beta-1/2}; zx^{-1} \\ x^{2\alpha+2\beta-1}, x^{\alpha+\beta}, -x^{\alpha+\beta} \end{matrix}\right)$$

$$\text{(iv)}\quad {}_2\phi_1\left(\begin{matrix} q^{\alpha},q^{\alpha+1/2};zq^{-1/2} \\ q^{2\alpha} \end{matrix}\right) \quad {}_2\phi_1\left(\begin{matrix} q^{\beta},q^{\beta-1/2};z \\ q^{2\beta} \end{matrix}\right)$$

$$= {}_4\phi_3\left(\begin{matrix} x^{\alpha+\beta-1/2},-x^{\alpha+\beta-1/2},x^{\alpha+\beta},-x^{\alpha+\beta};zx^{-1} \\ x^{2\alpha-2\beta-1},-x^{2\alpha},-x^{2\beta} \end{matrix}\right)$$

where $x \equiv q^{1/2}$.

The relation (ii), (iii), and (iv) are the ones from which the basic analogue of Cayley Orr identities were deduced by V. Singh [1]. Lastly,

$$\text{(v)}\quad {}_2\phi_1\left(\begin{matrix} q^{\alpha},q^{\beta};zq^{1/2} \\ q^{\alpha+\beta+1/2} \end{matrix}\right) \quad {}_2\phi_1\left(\begin{matrix} q^{\alpha},q^{\beta};z \\ z^{\alpha+\beta+1/2} \end{matrix}\right)$$

$$= {}_4\phi_3\left(\begin{matrix} x^{2\alpha},x^{2\beta},x^{\alpha+\beta},-x^{\alpha+\beta};z \\ x^{2\alpha+2\beta},x^{\alpha+\beta+1/2},-x^{\alpha+\beta+1/2} \end{matrix}\right),$$

is the basic analogue of Clausen's identity given by Singh [1].

Now using the identities (i)-(v) in 4.3(4) one can evaluate certain new integrals. I mention below evaluation of some such integrals.

In 4.2(1) and 4.3(4) taking $B = S_1 = 0$; $A = D = j = m_1 = R = 1$; $C = L = 2$; $\vartheta = \gamma - \alpha - \beta$; $\phi = 0$, $a_1 = \alpha + \beta - \gamma$; $C_1 = \gamma - \alpha$; $C_2 = \gamma - \beta$; $d = \gamma$; $\ell_1 = \alpha$; $\ell_2 = \beta$; $r_1 = \gamma$;

$$K_1 = \frac{G(\alpha+\beta-\gamma)G(\gamma-\alpha)G(\gamma-\beta)}{G(\alpha)G(\beta)},$$

we find that 4.2(2) is satisfied by virtue of 4.4(1) and hence we get, that

$$\frac{1}{2i}\int_C \frac{G(\alpha+\beta-\gamma-\xi)z^{\xi}}{G(1-\xi)\ \sin\pi\xi}\ E_q\left[\begin{matrix}\gamma-\alpha,\gamma-\beta,(\alpha_p)-\xi;zq^{\gamma-\alpha-\beta}\\ \gamma,(\beta_Q)-\xi\end{matrix}\right]\ d\xi$$

$$=\frac{G(\alpha+\beta-\gamma)G(\gamma-\alpha)G(\gamma-\beta)}{G(\alpha)G(\beta)}\ E_q\left[\begin{matrix}\alpha,\beta,(\alpha_p);z\\ \gamma,(\beta_Q)\end{matrix}\right] \quad , \qquad (1)$$

provided that $Rl(\beta_n) > Rl(\alpha_n) > 0$, $n=1,2,\ldots,Q$; $Rl(\alpha_n) > 0$, $n=Q+1,\ldots,P$, and $|\{\arg(z) - w_2w_1^{-1}\log|z|\}| < \pi$, $|q| < 1$;

(ii) Next in 4.2(1) and 4.3(4), take $S_1 = \phi = 0$; $B = D = j = m = 1$; $A = C = 2$; $R = 3$; $L = 4$; $\theta = -1/2$; $a_1 = c_1 = q^{\alpha}$; $a_2 = c_2 = q^{\beta}$; $b_1 = d_1 = q^{\alpha+\beta+1/2}$; $\ell_1 = x^{2\alpha}$; $\ell_2 = x^{2\beta}$; $\ell_3 = x^{\alpha+\beta}$; $\ell_4 = -x^{\alpha+\beta}$; $r_1 = x^{2\alpha+2\beta}$; $r_2 = x^{\alpha+\beta+1/2}$; $r_3 = -x^{\alpha+\beta+1/2}$;

$$K_1 = \left[\frac{G(\alpha)G(\beta)}{G(\alpha+\beta+1/2)}\right]^2 \frac{G(x^{2\alpha+2\beta})G(x^{\alpha+\beta+1/2})G(-x^{\alpha+\beta+1/2})}{G(x^{2\alpha})G(x^{2\beta})G(x^{\alpha+\beta})G(-x^{\alpha+\beta})} \ .$$

Then 4.2(2) is satisfied by virtue of 4.4(v) and we obtain

$$\frac{1}{2i}\int_C \frac{G(\alpha-\xi)G(\beta-\xi)z^{\xi}}{G(1-\xi)G(\alpha+\lambda+1/2-\xi)\ \sin\pi\xi}\ E_q\left[\begin{matrix}\alpha,\beta,(\alpha_p)-\xi;zq^{-1/2}\\ \alpha+\beta+1/2,(\beta_Q)-\xi\end{matrix}\right]\ d\xi$$

$$=\left[\frac{G(\alpha)G(\beta)}{G(\alpha+\beta+1/2)}\right]^2 \frac{G(x^{2\alpha+2\beta})G(x^{\alpha+\beta+1/2})G(-x^{\alpha+\beta+1/2})}{G(x^{2\alpha})G(x^{2\beta})G(x^{\alpha+\beta})G(-x^{\alpha+\beta})}$$

$$\times\ E_q\left[\begin{matrix}x^{2\alpha},x^{2\beta},x^{\alpha+\beta},-x^{\alpha+\beta},(\alpha_p);z\\ x^{2\alpha+2\beta},x^{\alpha+\beta+1/2},-x^{\alpha+\beta+1/2},(\beta_Q)\end{matrix}\right] \quad , \qquad (2)$$

provided $Rl(\beta_n) > Rl(\alpha_n) > 0$, $n=1,2,\ldots,Q$; $Rl(\alpha_n) > 0$, $n=Q+1,\ldots,P$,

$|\{\arg(z) - w_2 w_1^{-1} \log|z|\}| < \pi$, for $r \leq s+1$, and x stands for $q^{1/2}$ and the terms x^A in the E_q-symbol are on the base x.

5. Certain Summation of Bilateral Series

In this section we discuss a bilateral expansion of a general class of series involving products of two E_q-functions. The class of series which can (under suitable assumptions) be summed up is of the type:

$$S = \sum_{n=-\infty}^{\infty} \frac{[(a_A)]_n q^{n\lambda_1+\lambda_2 n(n+1)}}{[(b_B)]_n} E_q\left[\begin{matrix}(c_C),(g_G)+n,(l_L)-n;zq^{\theta}\\(e_E),(j_J)+n,(p_P)-n\end{matrix}\right]$$

$$\times E_q\left[\begin{matrix}(d_D),(h_H)+n;(m_M)-n;z\\(f_F),(k_K)+n,(i_I)-n\end{matrix}\right] \quad . \tag{1}$$

Substituting the Barnes type of contour integral for the E_q-functions and changing the order of integration and summation, provided we are justified in doing so, we get that

$$S = \frac{1}{2i}\int_C \frac{G[(c_C)-\xi]G[(g_G)-\xi]G[(l_L)-\xi](zq^{\theta})^{\xi}}{G[1-\xi]G[(e_E)-\xi]G(j_J)-\xi]G[p_P)-\xi]\ \sin\pi\xi}\, d\xi$$

$$\times \frac{1}{2i}\int_C \frac{G[(d_D)-\xi]G[(h_H)-\xi]G(m_M)-\xi]z^{\xi}}{G[1-\xi]G[(f_F)-\xi]G[(k_K)-\xi]G[(i_I)-\xi]\ \sin\pi\xi}$$

$$\times {}_{A+G+P+H+Q}\psi_{B+J+K+L+M}\left[\begin{matrix}(a_A),(g_G)-\xi,1-(p_P)+\xi,(h_H)-\xi,1-(i_I)+\xi:Q\\(b_B),(j_J)-\xi,1-(l_L)+\xi,(k_K)-\xi,1-(m_M)+\xi\end{matrix}\right] d\xi, \tag{2}$$

where ${}_p\psi_p\left[\begin{matrix}(a_p):z\\(b_p)\end{matrix}\right]$ is a basic bilateral hypergeometric series

$$= \sum_{n=-\infty}^{\infty} \frac{[(a_p)]_n}{[(b_p)]_n} z^n, \quad \left|\frac{b_1 b_2,\ldots,b_p}{a_1 a_2,\ldots,a_p}\right| < |z| < 1 \;;$$

z may sometimes be a function of n also and $Q = (-)\; L-P+M-I_q+\Sigma p_P+\Sigma I_I-\Sigma 1_L-\Sigma m_M+\xi(L-P)+\zeta(M-I)+1/2(n+1)(L-P+M-I+\lambda_2)$.

Now assuming that the inner basic bilateral hypergeometric series is such that it can be summed as

$$\Pi\left[\begin{matrix}(r_R),(t_T)-\xi,(v_V)-\xi,(x_X)-\xi-\zeta;\\(s_S);(u_U)-\xi,(w_W)-\xi,(y_Y)-\xi-\zeta\end{matrix}\right], \tag{3}$$

then once again using the contour integral for the E_q-function, we get that:

$$S = \Pi\left[\begin{matrix}(r_R);\\(s_S)\end{matrix}\right] \frac{1}{2i}\int_{C_1} \frac{G[(c_C)-\xi]G[(g_C)-\xi]G[(1_L)-\xi]G[(v_V)-\xi]}{G[1-\xi]G[(e_E)-\xi]G[(j_J)-\xi]G[(p_P)-\xi]G[(w_W)-\xi]\;\sin\pi\xi}$$

$$\times E_q\left[\begin{matrix}(d_D),(h_H),(m_M),(t_T),(x_X)-\xi;z\\(f_F),(k_K),(i_I),(u_U),(y_U)-\xi\end{matrix}\right] (zq^{\theta})^{\xi}d\xi \;. \tag{4}$$

Assuming further that a relation of the type

$$E_q\left[\begin{matrix}(c_C),(g_G),(1_L),(v_V);zq^{\theta}\\(e_E),(j_J),(p_P),(w_W)\end{matrix}\right] \quad E_q\left[\begin{matrix}(d_D),(h_H),(m_M),(t_T);z\\(f_F),(k_K),(i_I),(u_U)\end{matrix}\right]$$

$$= \Sigma_j \gamma_j z^{\rho_j}\; E_q\left[\begin{matrix}(\delta)_j;q^{\phi}z^{\sigma_j}\\(\Sigma)_j\end{matrix}\right] \tag{5}$$

exists, we can evaluate 4.5(4) by the help of 4.3(4). In fact we get that

$$S = \Pi\begin{bmatrix}(r_R);\\(s_S)\end{bmatrix} \Sigma_j\gamma_j z^{\rho_j} \frac{G[(x_X)-\rho_j]G_q,[\Delta(\sigma_j,(y_Y)-\rho_j)]}{G_q[(y_Y)-\rho_j]G_q,[\Delta(\sigma_j,(x_X)-\rho_j)]}$$

$$\times E_q\begin{bmatrix}(\delta)_j,\Delta[\sigma_j,(x_X)-\rho_j];z^{\sigma_j}q^{\phi}\\(\Sigma)_j,\Delta[\sigma_j,(y_Y)-\rho_j]\end{bmatrix}. \qquad (6)$$

As an illustration of 4.5(6) we mention the following case by taking particular values, when the general theorem holds. Taking $E = G = L = M = H = 0$, $F = J = K = P = I = 1$; $C = 3$; $A = B = D = 4$; $\theta = \gamma - \alpha - \beta$; $\lambda_1 = 2a - b - c$; $a_1 = q^{1+1/2a}$; $a_2 = -q^{1+1/2a}$; $a_3 = q^b$; $a_4 = q^c$; $b_1 = q^{1/2a}$; $b_2 = -q^{1/2a}$; $b_3 = q^{1+a-b}$; $b_4 = q^{1+a-c}$; $c_1 = \alpha + \beta - \gamma$; $c_2 = a + e - c$; $c_3 = a + e - b$; $d_1 = \gamma - \alpha$; $d_2 = \gamma - \beta$; $d_3 = a + h - c$; $d_4 = a + h - b$; $f_1 = \gamma$; $j_1 = a + e$; $k_1 = a + h$; $p_1 = e$; $i = h$; $\lambda_2 = 1$

$$S = \sum_{n=-\infty}^{+\infty} \frac{(q^{1+1/2a})_n(-q^{1+1/2a})_n(q^b)_n(q^c)_n q^{n(2a-b-c+n(n+1)}}{(q^{1/2a})_n(-q^{1/2a})_n(q^{1+a-b})_n(q^{1+a-c})_n}$$

$$\times E_q\begin{bmatrix}\alpha+\beta-\gamma,a+e-c,a+e-b;zq^{\gamma-\alpha-\beta}\\e-n,a+e+n\end{bmatrix}$$

$$\times E_q\begin{bmatrix}\gamma-\alpha,\gamma-\beta,a+h-c,a+h-b;z\\\gamma,h-n,a+h+n\end{bmatrix}. \qquad (1)$$

Substituting the Barnes integral for the E_q-function and changing the order of summation and integration, which is justified for Rl(2a+h-c+e-b) > 0, further using Bailey's summation theorem for a well-poised ${}_6\psi_6$ [1.6(6)] for the inner basic bilateral hypergeometric series, we get on simplification

$$S = \frac{G(1-b)G(1-e)G(1+a-c)G(1+a-b)G(\alpha+\beta-\gamma)G(\gamma-\alpha)G(\gamma-\beta)}{G(a)G(1-a)G(1+a)G(1+a-b-c)G(\alpha)G(\beta)}$$

$$\times \quad F_q\left[\begin{matrix}\alpha,\beta,2a-b-c+e+h;z\\ \gamma,a+e+h-1\end{matrix}\right].$$

The conditions of convergence of the various infinite series expansions obtained in the previous sections can be dealt with by means of the estiamtes of the type discussed in Section 3.4.

CHAPTER V

INTEGRATION INVOLVING THE q-ANALOGUE OF GENERALIZED MACROBERT'S E_q-FUNCTION WITH RESPECT TO THEIR PARAMETERS - II

1. Introduction

In this chapter we discuss the integration of MacRobert's E_q-function with respect to parameters, where the contour integrals are of the Slater type [3]. A very general basic integral has been evaluated and then certain interesting special cases are discussed.

2. The Main Integral

We now proceed to integrate the integral

$$I = \frac{t}{2\pi i}\int_{-i\pi/t}^{i\pi/t} \Pi\begin{bmatrix}(g)+s,(h)-s\\ (j)+s,(k)-s,1+a_1-b_r-s\end{bmatrix} E_q[r+1;a_p+s;r,b_1+s,b_2+s,\ldots,b_r+s,\ b_r+2s;-z]ds \tag{1}$$

where $0 < q < 1$, $q = e^{-t}$, $t > 0$. (Appendix IV (3)).

To evaluate 5.2(1) we write the series definition (1.3(7)) of the E_q-function in 5.2(1) and use the result of Slater [3], namely, let

$$\tau(q^s) = \Pi\begin{bmatrix}(a)q^s,(b)q^{-s},(g)q^s,(h)q^{-s},zq^s,q^{1-s}/z;\\ (c)q^s,(d)q^{-s},(j)q^s,(k)q^{-s};\end{bmatrix},$$

$$I = \frac{t}{2\pi i}\int_{-i\pi/t}^{i\pi/t} \pi(q^s)\ {}_{C+D+E}\Phi_{A+B+F}\begin{bmatrix}(c)q^s,(d)q^{-s},(e);\\ (a)q^s,(b)q^{-s},(f);\end{bmatrix} x\ ds, \tag{2}$$

$$\sum_C = \sum_{\nu=1}^{C} \Pi(1/c_\nu) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} ((d)c_\nu;q)_{2m+n}(qc_\nu/(a);q)_n((k)c_\nu;q)_{m+n}$$

$$\frac{(qc_\nu/(g);q)_{m+n}((e);q)_m x^m z^{m+n} \alpha_1^m \alpha_2^{m+n}}{((b)c_\nu;q)_{2m+n}(qc_\nu/(c)';q)_n((h)c_\nu;q)_{m+n}(qc_\nu/(j),q)_{m+n}((f);q)_m(q;q)_m(q;q)_n}$$

$$+ \sum_{\nu=1}^{J} \Pi(1/j_\nu) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} ((c)/(j_\nu);q)_{m-n}((d)j_\nu;q)_{m+n}(qj_\nu/(g);q)_n$$

$$\frac{((k)j_\nu;q)_n((e);q)_m x^m \beta^n}{((a)/j_\nu;q)_{m-n}((b)j_\nu;q)_{m+n}(gj_\nu/(j)';q)_n((h)j_\nu;q)_n((f);q)_m(q;q)_m(q;q)_n} \tag{3}$$

where $\alpha_1 \equiv (-c_\nu q^{1/2n+1/2})^{C-A} \dfrac{a_1 a_2 \cdot \cdots \cdot a_A}{c_1 c_2 \cdot \cdots \cdot c_C}$,

$\alpha_2 \equiv (-c_\nu q^{1/2m+1/2n+1/2})^{J-G-1} \dfrac{g_1 g_2 \cdot \cdots \cdot g_G}{j_1 j_2 \cdot \cdots \cdot j_J}$,

and

$$\beta \equiv (-j_\nu z q^{1/2n+1/2})^{J-G-1} \frac{g_1 g_2 \cdot \cdots \cdot g_G}{j_1 j_2 \cdot \cdots \cdot j_J};$$

$$\sum_D = \sum_{\nu=1}^{D} \Pi(d_\mu) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty}$$

$$\times \frac{((c)d_\mu;q)_{2m+n}(qd_\mu/(b);q)_n((j)d_\mu;q)_{m+n}(qd_\mu/(h);q)_{m+n}((e);q)_m x^n \gamma_1^n, \gamma_2^{m+n}}{(a)d_\mu;q)_{2m+n}(qd_\mu/(d)';q)_n((g)d_\mu,q)_{m+n}(qd_\mu/(k);q)_{m+n}((f);q)_m(q;q)_m(q;q)_n}$$

$$+ \sum_{\mu=1}^{K} \Pi(k_\mu) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty}$$

$$\times \frac{((c)k_\mu;q)_{m+n}((b)/k_\mu;q)_{m+n}((j)k_\mu;q)_n(qk_\mu/(h);q)_n((e);q)_m x^m \delta^n}{((a)k_\mu;q)_{m+n}((d)/k_\mu;q)_{m+n}((g)k_\mu;q)_n(qk_\mu/(k)';q)_n(f);q)_m(q;q)_m(q;q)_n} ,$$

(4)

where $\nu_1 \equiv (-d_\mu q^{1/2n+1/2})^{D-B} \dfrac{d_1 d_2 \cdot \ldots \cdot d_D}{b_1 b_2 \cdot \ldots \cdot b_B}$,

$$\nu_2 \equiv (-d_\mu q^{1/2m+1/2n+1/2})^{K-H} \frac{h_1 h_2 \cdot \ldots \cdot h_H}{k_1 k_2 \cdot \ldots \cdot k_K} ,$$

and $\quad \delta \equiv (-k_\mu q^{1/2n+1/2})^{K-H-1} \dfrac{h_1 h_2 \cdot \ldots \cdot h_H}{k_1 k_2 \cdot \ldots \cdot k_K}$.

Then, when $|q| < 1$, $t > 0$ and $R(x) > 0$, we have

(i) $I = \Sigma_D$ if (a) $D > B$ or $D = B$ and $R\ell(b_1 b_2 \ldots b_B / d_1 d_2 \cdot \ldots \cdot d_D) \geq 0$
and (b) $K > H + 1$ or $K = H + 1$ and $R\ell(h_1 h_2, \ldots, h_H / k_1 k_2, \ldots, k_K) \geq 0$
and $I \sim \Sigma_D$ if (c) $D < B$ or (d) $K < H + 1$,

(ii) $I = \Sigma_C$ if (a) $C > A$ or $C = A$ and $R\ell(a_1 a_2 \cdot \ldots \cdot a_A / c_1 c_2 \cdot \ldots \cdot c_C) \geq 0$
and (b) $J > G + 1$ or $J = G + 1$ and $R\ell(g_1 g_2 \cdot \ldots \cdot g_G / j_1 j_2 \cdot \ldots \cdot j_J) \geq 0$
and $I \sim \Sigma_C$ if (c) $C < A$ or (d) $J < G + 1$.

Writing the series definition of E_q-function in 5.2(1), we get, for

$$A(a_1) = \Pi \begin{bmatrix} q, b_1 - a_1, \ldots, b_{r-1} - a_1 \\ a_2 - a_1, \ldots, a_{r+1} - a_1 \end{bmatrix} \times \frac{1}{\Pi(1 - z^{-1} q^n)(1 - zq^{1+n})} ,$$

$$I = \left\{ A(a_1)\,\frac{t}{2\pi i}\int_{-i\pi/t}^{i\pi/t} \Pi \begin{bmatrix} (g)+s,\ (h)-s,\ b_r-a_1+s,\ z^{-1}q^{a_1+s},\ zq^{1-a_1-s} \\ (j)+s,\ (k)-s,\ a_1+s,\ 1+a_1-b_r-s \end{bmatrix} \times \right.$$

$$ {}_{r+1}\Phi_r \begin{bmatrix} a_1+s, 1+a_1-b_1, \ldots, 1+a_1-b_{r-1}, 1+a_1-b_r-s; zq^{1+\Sigma b_r-\Sigma a_{r+1}} \\ 1+a_1-a_2, 1+a_1-a_3, \ldots, 1+a_1-a_{r+1} \end{bmatrix}$$

$$\left. + \text{ idem } (a_1, a_2, \ldots, a_{r+1}) \right\} ds. \qquad (5)$$

Now, to evaluate 5.2(5), we use 5.2(2) for $C = 1$, $D = 1$, $A = 0$, $B = 0$, $E = r-1$, $F = r$, $x = zq^{1+\Sigma b_r-\Sigma a_{r+1}}$, $z = z^{-1}q^{a_1}$, to get that, if

$$\sum_1 = A(a_1) \left\{ \prod_0^{\infty}(1-q^{1-a_1+n}) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (q^{1+2a_1-b_r})_{2m+n} (q^{(k)+a_1})_{m+n} (q^{1+a_1-(g)})_{m+n} \right.$$

$$\frac{(q^{1+2a_1-b_r})_{m+n} (q^{1+a_1-b_1})_m (q^{1+a_1-b_2})_m \cdots (q^{1+a_1-b_{r-1}})_m (z^{-1}q^{a_1})^{m+n} \alpha_1^{\,n} \alpha_2^{\,m+n}\; q^{m(1+\Sigma b_r-\Sigma a_{r+1})}}{(q)_m (q)_n (q^{(h)+a_1})_{m+n} (q^{1+a_1-(j)})_{m+n} (q^{1+a_1-a_2})_m (q^{1+a_1-a_3})_m \cdots (q^{1+a_1-a_{r+1}})}$$

$$+ \sum_{\nu=1}^{J} \Pi(1-q^{1-j_\nu+n}) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (q^{a_1-j_\nu})_{m-n} (q^{1+a_1-b_r-j_\nu})_{m+n} (q^{1+j_\nu-(j)})_n (q^{1+j_\nu-b_r+a_1})_n$$

$$\frac{(q^{(k)+j_\nu})_n (q^{1+a_1-b_1})_m (q^{1+a_1-b_2})_m \cdots (q^{1+a_1-b_{r-1}})_m \beta^n z^m q^{m(1+\Sigma b_r-\Sigma a_{r+1})}}{(q^{1+j_\nu-(j')})_n (q^{(b)+j_\nu})_n (q^{1+a_1-a_2})_m (q^{1+a_1-a_3})_m \cdots (q^{1+a_1-a_{r+1}})_m (q)_m (q)_n}$$

$$\left. + \text{ idem } (a_1, a_2, \ldots, a_{r+1}) \right\} \qquad (6)$$

where $\alpha_1 = -q^{1/2(n+1)}$, $\alpha_2 = (-a_1 q^{1/2(m+n+1)})^{J-G-2} q^{\Sigma(g)-\Sigma(j)+b_r-a_1}$

$$\beta = [-j_\nu z^{-1} q^{a_1+1/2(n+1)}]^{J-G-2} q^{\Sigma(g)-\Sigma(j)+b_r-a_1}$$

$$\sum_2 = A(a_1)\Bigg\{\prod_0^\infty (1-q^{1+a_1-b_r+n}) \sum_{m=0}^\infty \sum_{n=0}^\infty (q^{1+2a_1-b_r})_{2m+n} (q^{(j)+1+a_1-b_r})_{m+n}$$

$$\frac{(q^{2+a_1-b_r-(h)})_{m+n} (q^{1+a_1-b_1})_m (q^{1+a_1-b_2})_m \cdots (q^{1+a_1-b_{r-1}})_m z^m q^{m(1+\Sigma b_r-\Sigma a_{r+1})} \gamma_1^m \gamma_2^{m+n}}{(q)_m (q)_n (q^{1+a_1-b_r+(g)})_{m+n} (q)_{m+n} (q^{2+a_1-b_r-(k)})_{m+n} (q^{1+a_1-a_2})_m \cdots (q^{1+a_1-a_{r+1}})_m}$$

$$+ \sum_{\mu=1}^K \Pi(1-q^{k_\mu+n}) \sum_{m=0}^\infty \sum_{n=0}^\infty (q^{a_1+k_\mu})_{m+n} (q^{k_\mu+(j)})_n (q^{1+k_\mu-(h)})_n (q^{1+a_1-b_1})_m \cdots$$

$$\frac{(q^{1+a_1-b_{r-1}})_m z^m q^{(1+\Sigma b_r-\Sigma a_{r+1})} \delta^n}{(q)_m (q)_n (q^{1+a_1-b_r-k_\mu})_{m+n} (q^{(g)+k_\mu})_n (q^{b_r-a_1-k_\mu})}$$

$$\times (q^{1+k_\mu-k'})_n (q^{1+a_1-a_2})_m (q^{1+a_1-a_3})_m \cdots (q^{1+a_1-a_{r+1}})_m + \text{idem}(a_1 a_2, \ldots, a_{r+1}) \tag{7}$$

where $\gamma_1 = -q^{1+a_1-b_r+1/2(n+1)}$

$$\gamma_2 = (-q^{1+a_1-b_r+1/2(m+n+1)})^{K-H-1} \frac{h_1 h_2 \cdots\cdots h_H}{k_1 k_2 \cdots\cdots k_K}$$

and $\delta = (-k_\mu q^{1/2(n+1)})^{K-H-1} \dfrac{h_1 h_2 \cdots\cdots h_H}{k_1 k_2 \cdots\cdots k_K}$

for $|q| < 1$, $t > 0$, $R\ell(zq^{1+\Sigma b_r - \Sigma a_{r+1}}) > 0$, we have

(i) $I = \Sigma_2$ if $K > H + 1$ or $K = H + 1$ and $R\ell\left(\frac{h_1 h_2 \cdots h_H}{k_1 k_2 \cdots k_K}\right) \geq 0$

(ii) $I \sim \Sigma_2$ if $K < H + 1$

(iii) $I = \Sigma_1$ if $J > G + 2$ or $J = G + 2$ and $R\ell\left(\frac{g_1 g_2 \cdots g_G b_r}{j_1 j_2 \cdots j_J a_1}\right) \geq 0$

(iv) $I \sim \Sigma_1$ if $J < G + 2$.

Special Cases

(I) For $r = 0$ in 5.2(5), 5.2(6), and 5.2(7), we get that if

$$I_1 = \frac{t}{2\pi i} \int_{-i\pi/t}^{i\pi/t} \pi\begin{bmatrix}(g)+s, (h)-s; \\ (j)+s, (k)-s;\end{bmatrix} E_q[a_1+s;-;-z]ds$$

$$\sum_1 = \Pi(q)\left\{\prod_o^\infty (1-q^{1-a_1+n}) \sum_{m=0}^\infty \sum_{n=0}^\infty (q^{(k)+a_1})_{m+n} (q^{1+a_1-(g)})_{m+n} (z^{-1} q^{a_1})^{m+n}\right.$$

$$\times \frac{\alpha_1^n \alpha_2^{m+n} z^m q^{m(1-a_1)}}{(q^{(h)+a_1})_{m+n} (q^{1+a_1-(j)})_{m+n} (q)_m (q)_n}$$

$$+ \frac{\sum_{\nu=1}^J \prod_o^\infty [1-q^{1-j_\nu+n}] \sum_{m=0}^\infty \sum_{n=0}^\infty (q^{a_1-j_\nu})_{m-n} (q^{1+j_\nu-(j)})_n (q^{(k)+j_\nu})_n \beta^m z^m q^{m(1-a_1)}}{(q^{1+j_\nu-(j)'})_n (q^{(h)+j_\nu})_n (q)_m (q)_n}$$

(8)

where

$$\alpha \equiv -q^{1/2(n+1)}, \ \alpha_2 = [-a_1 q^{1/2(m+n+1)}]^{J-G-1} q^{\Sigma(g)-\Sigma(j)}$$

$$\beta \equiv [-j_\nu z^{-1} q^{a_1+1/2(n+1)}]^{J-G-1} q^{\Sigma(g)-\Sigma(j_\nu)}$$

$$\sum_2 = \Pi[q] \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{z^m q^{m(1-a_1)} \gamma_1^{\,m} \gamma_2^{\,m+n}}{(q)_m (q)_n} \right.$$

$$+ \frac{\sum_{\mu=1}^{k} \prod_{o}^{\infty} (1-q^{k_\mu+n}) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (q^{a_1+k_\mu})_{m+n} (q^{k_\mu+(j)})_n (q^{1+k_\mu-(h)})_n z^m q^{m(1-a_1)} \delta^n}{(q)_m (q)_n (q^{(g)+k_\mu})_n (q^{1+k_\mu-k'})_n}, \tag{9}$$

where

$$\gamma_1 \equiv -q^{1+a_1+1/2(n+1)}, \ \gamma_2 = (-q^{1+a_1+1/2(M+N+1)})^{K-H-1} \frac{h_1 h_2 \cdots\cdots h_H}{k_1 k_2 \cdots\cdots k_K}$$

$$\delta \equiv (-k_\mu q^{1/2(n+1)})^{K-H-1} \frac{h_1 h_2 \cdots\cdots h_H}{k_1 k_2 \cdots\cdots k_K}$$

for $|q| < 1$, $R\ell(zq^{1-a_1}) > 0$, we have

(i) $I_1 = \Sigma_2$ if $K > H + 1$ or $K = H + 1$ and $R\ell\left(\frac{h_1 h_2 \cdots\cdots h_H}{k_1 k_2 \cdots\cdots k_K}\right) \geq 0$

$\sim \Sigma_2$ if $K < H + 1$.

(ii) $I_1 = \Sigma_1$ if $J > G + 1$ or $J = G + 1$, $R\ell\left(\frac{g_1 g_2 \cdots\cdots g_G}{j_1 j_2 \cdots\cdots j_J}\right) \geq 0$

$\sim \Sigma_1$ if $J < G + 1$.

(II) In (I) taking $J = G + 1$, we get,

$I_1 = \Sigma_1$, where

$$\Sigma_1 = \Pi(q)\left\{\Pi[q^{1+a_1}]\sum_{n=0}^{\infty}\frac{(q^{(k)+a_1})_n(q^{1+a_1-(g)})_n(z^{-1}q^{a_1})^n(-1)^n q^{1/2(n+1)}}{(q^{(h)+a_1})_n(q^{1+a_1-(j_{G+1})})_n(q)_n}\right.$$

$$\times {}_{K+G}\Phi_{H+G+1}\left[\begin{matrix}q^{(k)+a_1+n}, q^{1+a_1-(g)+n}, q^{1+\Sigma(g)-\Sigma(j_{G+1})}\\ q^{(h)+a_1+n}, q^{1+a_1-(j_{G+1})+n}\end{matrix}\right]$$

$$+\sum_{\nu=1}^{G+1}\Pi[1-(j_\nu), zq^{1-\Sigma(g)-\Sigma(j_{G+1})-j_\nu}; zq^{1-a_1-\Sigma(j)-\Sigma(j_{G+1})}]$$

$$\times {}_{G+K+1}\Phi_{G+H+1}\left[\begin{matrix}q^{1+j_\nu-(g)}, q^{(k)+j_\nu}, 1/zq^{\Sigma(g)-\Sigma(j_{G+1})+j_\nu}, zq^{1-\Sigma(g)-\Sigma(j_{G+1})-a_1}\\ q^{1+j_\nu-(g)}, q^{(h)+j_\nu}, q^{1-a_1+j_\nu}\end{matrix}\right] \tag{10}$$

(III) Again taking $K = H + 1$ in (I), we get

$$I_1 = \Sigma_2 = \Pi(q)\sum_{n=0}^{\infty}\frac{1}{(q)_n\Pi(zq^{3/2+1/2n})}$$

$$+\frac{\prod_{r=0}^{\infty}(zq^{1+k_\mu})}{\prod_{r=0}^{\infty}(zq^{1-a_1})}\;\frac{\sum_{\mu=1}^{H+1}\Pi(1-q^{k_\mu+n})\sum_{n=0}^{\infty}(q^{a_1+k_\mu})_n(q^{k_\mu+(j)})_n(q^{1+k_\mu-(h)})\delta^n}{(q)_n(q^{(g)+k_\mu})_n(q^{1+k_\mu-k'})_n} \tag{11}$$

where $\delta \equiv \dfrac{h_1h_2\cdots\cdots h_H}{k_1k_2\cdots\cdots k_{H+1}}$.

(IV) Next, for $r = 1$ in 5.2(5, 6, and 7), we have

$$I_2 = \frac{t}{2\pi i}\int_{-i\pi/t}^{i\pi/t} \Pi\begin{bmatrix}(g)+s,(h)-s\\ (j)+s,(k)-s,1+a_1-b_1-s\end{bmatrix} \times E_q[a_1+s,a_2+s,b_1+2s;-z]ds \tag{12}$$

where $0 < q < 1$, $q = e^{-t}$, $t > 0$,

$$\sum_1 = \Pi\begin{bmatrix}q,b_1-a_1;\\ a_2-a_1,\end{bmatrix}\frac{1}{\Pi(1-z^{-1}q^n)(1-zq^{1+n})}\Bigg\{\Pi(1-q^{1-a_1+n})$$

$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(q^{1+2a_1-b_1})_{2m+n}(q^{(k)+a_1})_{m+n}(q^{1+a_1-(g)})_{m+n}$$

$$\times\frac{(q^{1+2a_1-b_1})_{m+n}(+z^{-1}q^{a_1})^{m+n}\alpha_1^{\,n}\alpha_2^{\,m+n}z^m q^{m(1+b_1-a_2-a_1)}}{(q^{(h)+a_1})_{m+n}(q^{1+a_1-(j)})_{m+n}(q^{1+a_1-a_2})_m(q)_m(q)_n}$$

$$+\sum_{\nu=1}^{J}\Pi(1-q^{1-j_\nu+n})\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(q^{a_1-j_\nu})_{m-n}(q^{1+a_1-b_1+j_\nu})_{m+n}(q^{1+j_\nu-(g)})_n$$

$$\times\frac{(q^{1+j_\nu-b_1+a_1})_n(q^{(k)+j_\nu})_n q^{m(1+b_1-a_2-a_1)}\beta^m z^m}{(q^{1+j_\nu-(j)'})_m(q^{(h)+j_\nu})_n(q^{1+a_1-a_2})_n(q)_m(q)_n} + \text{idem}(a_1;a_2)\Bigg\} \tag{13}$$

where $\alpha_1 \equiv -1^{1/2(n+1)}$, $\alpha_2 \equiv (-a_1q^{1/2(m+n+1)})^{J-G-2}q^{\Sigma(g)-\Sigma(j)+b_1-a_1}$

and $\beta \equiv [-j_\nu z^{-1}q^{a_1+1/2(m+1)}]^{J-G-2}q^{\Sigma(g)-\Sigma(j)+b_1-a_1}$

and

$$\sum_2 = \Pi\left[1, \begin{matrix} b_1-a_1, \\ a_2-a_1, \end{matrix}\right] \frac{1}{\Pi(1-z^{-1}q^n)(1-zq^{1+n})}$$

$$\left\{ \prod_o^{\infty}(1-q^{1+a_1-b_1+n}) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(q^{1+2a_1-b_1})_{2m+n}(q^{(j)+1+a_1-b_1})_{m+n}(q^{2+a_1-b_1-(h)})_{m+n} \right.$$

$$\frac{z^m q^{m(1+b_1-a_2-a_1)}\gamma_1^m \gamma_2^{m+n}}{(q^{1+a_1-b_1+(g)})_{m+n}(q)_m(q)_n(q)_{m+n}(q^{2+a_1-b_1-(k)})_{m+n}(q^{1+a_1-a_2})_m}$$

$$+ \frac{\sum_{\mu=1}^{K} \Pi(1-q^{k_\mu+n}) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(q^{a_1+k_\mu})_{m+n}(q^{k_\mu+(j)})_n(q^{1+k_\mu-(h)})_n z^m q^{m(1+b_1-a_2-a_1)}\delta^n}{(q^{1+a_1-b_1-k_\mu})_{m+n}(q^{(g)+k_\mu})_n(q^{b_1-a_1+k_\mu})_n(q^{1+k_\mu-k'})_n(q^{1+a_1-a_2})_n(q)_m(q)_n}$$

$$\left. + \text{idem}\ (a_1;a_2)\quad . \right\} \qquad (14)$$

where $\gamma_1 = -q^{1+a_1-b_1+1/2(n+1)}$

$$\gamma_2 = (-q^{1+a_1-b_1+1/2(m+n+1)})^{K-H-1}\ \frac{h_1h_2\cdots\cdots h_H}{k_1k_2\cdots\cdots k_K}$$

for $|q| < 1,\ t > 0,\ R\ell(zq^{1+b_1-a_2-a_1}) > 0.$

Then $I_2 = \Sigma_2$, if $K > H + 1$ or $K = H + 1$ and $R\ell\ \dfrac{(h_1h_2\cdots\cdots h_H)}{(k_1k_2\cdots\cdots k_K)} \geq 0.$

(ii) $I_2 \sim \Sigma_2$ if $K < H + 1$

(iii) $I_2 = \Sigma_1$ if $J > G + 2$ or $J = G + 2$, $R\ell\ \dfrac{(g_1g_2\cdots\cdots g_G b_1)}{(j_1j_2\cdots\cdots j_J a_1)} \geq 0$

(iv) $I_2 \sim \Sigma_1$ if $J < G + 2$.

Further, taking $J = G + 2$, in 5.2(12) we get

$$I_2 = \Pi\begin{bmatrix} q, b_1-a_1, \\ a_2-a_1, \end{bmatrix} \frac{1}{\Pi(1-z^{-1}q^n)(1-zq^{1+n})} \Big\{ \Pi(1-q^{1-a_1+n})$$

$$\frac{\sum_{n=0}^{\infty} (q^{1+2a_1-b_1})_n (q^{(k)+a_1})_n (q^{1+a_1-(g)})_n (q^{1+2a_1-b_1})_n (z^{-1}q^{a_1})^n \alpha_1^n \alpha_2^n}{(q)_n (q^{(h)+a_1})_n (q^{1+a_1-(g_{G+2})})_n}$$

$$\times {}_{G+K+J+1}\Phi_{H+G+3}\begin{pmatrix} q^{1+a_1-(g)+m}, q^{1+2a_1-b+2m}, q^{(k)+a_1+m}, q^{1+a_1 (j)+m}, q^{1+2a_1-b_1+m}; Q \\ q^{(h)+a_1+m}, q^{1+a_1-(j_{G+2})}, q^{1+a_1-a_2} \end{pmatrix}$$

$$+ \sum_{\nu=1}^{G+2} (1-q^{1-j_\nu+m}) \Big\{ \sum_{n=0}^{\infty} (q^{1+a_1-b_1-j_\nu})_n (q^{1+j_\nu-b_1+a_1})_n (q^{k+j_\nu})_n$$

$$\frac{(-1)^n q^{n/2(n+1)} q^{-n(a_1-j_\nu)} (q^{1+j_\nu-(g)})_n}{(q^{1+a_1-j_\nu-m})_m (q^{1+a_1-a_2})_n (z^{(h)+(j_\nu)})_n (q)_n}$$

$${}_2\Phi_{J-1}\begin{bmatrix} q^{1+a_1-b_1+m}, q^{a_1-j_\nu}; Q_1 \\ q^{1+j_\nu-(j')} \end{bmatrix} + \text{idem } (a_1; a_2) \Big\}$$

where $Q_1 = zq^{(b_1-a_2-a_1)}$, $Q = \alpha_2 zq^{(1+b_1-a_2-a_1)}$, $\alpha_1 = -q^{1/2(n+1)}$, $\alpha_2 = q^{\Sigma(g)-\Sigma(j_{G+2})+b_1-a_1}$, and $\beta = q^{\Sigma(g)-\Sigma(j_{G+2})+b_1-a_1}$.

One could further take particular values for J and K to get the values of more simple integrals.

APPENDIX I

SYMBOLIC INDEX

$(a)_n \equiv a(a+1)(a+2)\ldots(a+n-1)$, $n \geq 1$.

$(a)_0 = 1$

$$1 + \frac{ab}{c}\frac{z}{1} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \ldots$$

$$= {}_2F_1[a,b;c;z]$$

or

$${}_2F_1[a,b;c;z] = \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n\, n!} z^n$$

$$E[a,b;c;-\tfrac{1}{z}] = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)z}\, 2F_1[a,b;c;z]$$

$$(a,q)_n = (1-a)(1-aq)(1-aq^2)\ \ldots\ (1-aq^{n-1})$$

$$(a,q)_0 = 1$$

$$(a,q)_{-n} = \frac{(-a)^{-n}\, q^{n/2(n+1)}}{(q/a;q)_n}$$

$${}_{r+1}\Phi_r\begin{bmatrix} a_1, a_2, \ldots\ a_{r+1}; z \\ b_1, b_2, \ldots\ b_r \end{bmatrix} = \sum_{n=1}^{\infty} \frac{(a_1)_n, \ldots\ (a_{r+1})_n}{(q)_n(b_1)_n\ \ldots\ (b_r)_n}\, z^n$$

$${}_1\Phi_1[a;b;z] = \sum_{n=1}^{\infty} \frac{(a)_n}{(q)_n(b)_n}\, z^n q^{n/2(n-1)} \qquad |z| < 1$$

$$E_q(x) = \prod_{n=1}^{\infty}(1-xq^n) = \sum_{n=1}^{\infty}\frac{(-1)^n q^{n/2(n-1)}}{(q)_n}\, x^n ,$$

$$(x+y)_\alpha \equiv x^\alpha(1+y/x)_\alpha = \prod_{n=0}^{\infty}\frac{(1+y/x\ q^n)}{(1+y/x\ q^{\alpha+n})}$$

$$\frac{1}{(1+x)_\alpha} = {}_1\Phi_0(\alpha;-x) \quad \text{for} \quad |x| < 1$$

$$G(\alpha) = \left\{\prod_{n=0}^{\infty}(1-q^{\alpha+n})\right\}^{-1}$$

$$\mathop{S}_0^x f(y)d(qy) = x(1-q)\sum_{i=0}^{\infty} q^i\ f(q^i x)$$

-AI.2-

$$\int_x^{\infty} f(y)\, d(qy) = x(1-q) \sum_{j=1}^{j=\infty} q^j f(q^j)$$

$$\int_0^{\infty} f(y)\, d(qy) = (1-q) \sum_{j=-\infty}^{j=\infty} q^j f(q^j)$$

$$E_q(\alpha,\beta: :z) = \frac{G(\alpha)}{(1-q)} \int_0^1 E_q(q\lambda)\lambda^{\beta-1}\ {}_1\Phi_0(\alpha; \frac{-\lambda}{z})\ d(q\lambda); \qquad z \neq 0.$$

$$E_q(r;a_p:s;b_t:z) = \frac{G(a_1)\ \ldots\ G(a_r)}{G(b_1)\ \ldots\ G(b_s)}\ {}_r\Phi_s \left[\begin{matrix} a_1,\ldots,a_r; \frac{1}{z} \\ b_1,\ldots,b_s \end{matrix}\right]$$ Also

if $f(x) = x^{\alpha} \sum_{r=-\infty}^{\infty} a_r x^r$,

then $f[x+h] = \sum_{r=-\infty}^{\infty} a_r [x+h]_{a+r}$ where $[x+y]_{\alpha} = x^{\alpha}[1+y/x]_{\alpha} =$

$$\prod_{n=0}^{\infty} \frac{(1+y/x\ q^n)}{(1+y/x\ q^{\alpha+n})},$$

and $$\frac{G(\alpha_1)\ldots G(\alpha_r) G_{q'}(k/m),\ldots,a_{q'}(k+m-1/m)}{G(\beta_1),\ldots,G(\beta_s)}$$

$$\times\ \Phi \left[\begin{matrix} q^{\alpha_1},\ldots,q^{\alpha_r},\ q'^{k/m},\ldots,q'^{k+m-1/m}; z \\ q^{\beta_1},\ldots,q^{\beta_s} \end{matrix}\right]$$

which we denote by

$$E_{q'}\left[\begin{matrix} q^{\alpha_1},\ldots,q^{\alpha_r},\ q'^{k/m},\ldots,q'^{k+m-1/m}; z \\ q^{\beta_1},\ldots,q^{\beta_s} \end{matrix}\right]$$

where the symbol $\Delta(m;\alpha)$ stands for m parameters of the type

$$q'^{\alpha/m},\ q'^{\frac{\alpha+1}{m}},\ldots,q'^{\frac{\alpha+m-1}{m}}.$$

In the notation for ordinary hypergeometric function, the symbol

$(a)_n \equiv a(a+1)(a+2 \ldots (a+n-1)$

$(a_1)_0 = 1.$

Since many authors have used the same symbol $(a)_n$ for $(a,q)_n$.

In order to avoid any ambiguity in the present work $(a)_n$ may be read as $(a,q)_n$.

APPENDIX II

Summation Theorems for Ordinary Hypergeometric Series

THE BINOMIAL THEOREM: $_1F_0[a;;z] = (1-z)^{-a}$.

SAALSCHUTZ'S THEOREM: $_3F_2[a,b,-n;c;d;1] = \dfrac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}$ provided that $c + d = a + b - n + 1$.

GAUSS'S THEOREM: $_2F_1[a,b;c;1] = \Gamma\begin{bmatrix}c, c-a-b\\ c-a, c-b\end{bmatrix}$ or when $b = -n$ Vandermonde's theorem, $_2F_1[a,-n;c;1] = \dfrac{(c-a)_n}{(c)_n}$.

DIXON'S THEOREM: $_3F_2\begin{bmatrix}a, & b, & c & 1\\ & 1+a-b & 1+a-c & \end{bmatrix}$

$$= \Gamma\begin{bmatrix}1+\frac{1}{2}a, 1+a-b, 1+a-c, 1+\frac{1}{2}a-b-c\\ 1+a, 1+\frac{1}{2}a-b, 1+\frac{1}{2}a-c, 1+a-b-c\end{bmatrix}$$

or if $c = -n$

$$_3F_2\begin{bmatrix}a, & b, & -n & ; 1\\ & 1+a-b & 1+a+n & \end{bmatrix}$$

$$= \frac{(1+a)_n(1+\frac{1}{2}a-b)_n}{(1+\frac{1}{2}a)_n(1+a-b)_n}$$

$$_5F_4\begin{bmatrix}a, & 1+\frac{1}{2}a, & b, & c, & c; & 1\\ & \frac{1}{2}a & 1+a-b, & 1+a-c, & 1+a-d; & \end{bmatrix}$$

$$= \Gamma\begin{bmatrix}1+a-b, 1+a-c, 1+a-d, 1+a-b-c-d\\ 1+a \quad, 1+a-b-c, 1+a-b-d, 1+a-c-d\end{bmatrix}$$

or if $d = -n$

$$_5F_4\begin{bmatrix}a, & 1+\frac{1}{2}a, & b, & c, & -n; & 1\\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a+n & \end{bmatrix}$$

$$= \frac{(1+a)_n(1+a-b-c)_n}{(1+a-b)_n(1+a-c)_n} .$$

DOUGALL'S THEOREM: $_7F_6\begin{bmatrix}a, & 1+\frac{1}{2}a, & b, & c, & d, & c, & -n; & 1\\ & \frac{1}{2}a & 1+a-b & 1+a-c, & 1+a-d, & 1+a-e, & 1+a+n & \end{bmatrix}$

$$= \frac{(1+a)_n(1+a-b-c)_n(1+a-b-d)_n(1+a-c-d)_n}{(1+a-b)_n(1+a-c)_n(1+a-d)_n(1+a-b-c-d)_n}$$

provided that $1+2a = b+c+d+e-n$.

$$ {}_4F_3\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c; & -1 \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c & \end{matrix}\right] = \Gamma\left[\begin{matrix} 1+a-b, & 1+a-c \\ 1+a & 1+a-c \end{matrix}\right] $$

or if $c = -n$

$$ {}_4F_3\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & -n; & -1 \\ & \frac{1}{2}a, & 1+a-b, & 1+a+n & \end{matrix}\right] = \frac{(1+a)_n}{(1+a-b)_n} $$

$$ {}_5F_4\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c, & d; & 1 \\ & \frac{1}{2}a & 1+a-b, & 1+a-c, & 1+a-d & \end{matrix}\right] $$

$$ = \Gamma\left[\begin{matrix} 1+a-b, 1+a-c, 1+a-d, 1+a-b-c-d \\ 1+a \quad , 1+a-b-c, 1+a-b-d, 1+a-c-d \end{matrix}\right] $$

or if $d = -n$

$$ {}_5F_4\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c, & -n; & 1 \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a+n & \end{matrix}\right] = \frac{(1+a)_n(1+a-b-c)_n}{(1+\frac{1}{2}a)_n(1+a-b)_n} \quad . $$

APPENDIX III

SUMMATION THEOREMS FOR BASIC SERIES

(1) GAUSS'S ANALOGUE: $_2\Phi_1[a,b;c;c/ab] = \Pi\begin{bmatrix} c/a, c/b \\ c, \quad c/ab \end{bmatrix}$ or when $b = q^{-n}$

$$_2\Phi_1[a,q^{-n};c;cq^n/a] = \frac{(c/a)_n}{(c)_n}$$

(2) SAALSCHUTZ'S ANALOGUE: $_3\Phi_2[a,b,q^{-n};c,d;q] = \dfrac{(c/a)_n(c/b)_n}{(c)_n(c/ab)_n}$

provided that $cd = abq^{1-n}$.

(3) DIXON'S ANALOGUE: $_6\Phi_5\begin{bmatrix} a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & d; & aq/bcd \\ & \sqrt{a}, & \sqrt{a}, & aq/b, & aq/c, & aq/d \end{bmatrix}$

$$= \Pi\begin{bmatrix} aq, & aq/cd, & aq/bd, & aq/bc \\ aq/b & aq/c & aq/d & aq/bcd \end{bmatrix} .$$

(4) $_6\Phi_5\begin{bmatrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & q^{-N}; \\ & \sqrt{a}, & \sqrt{a}, & aq/b, & aq/c, & aq^{1+N}/bc \end{bmatrix}$

$$= \frac{(aq)_N(\sqrt{aq/b})_N}{(aq/b)_N(aq/c)_N}$$

(5) HEINE'S THEOREM: $_1\Phi_0[a;z] = \Pi[az;\ q]$

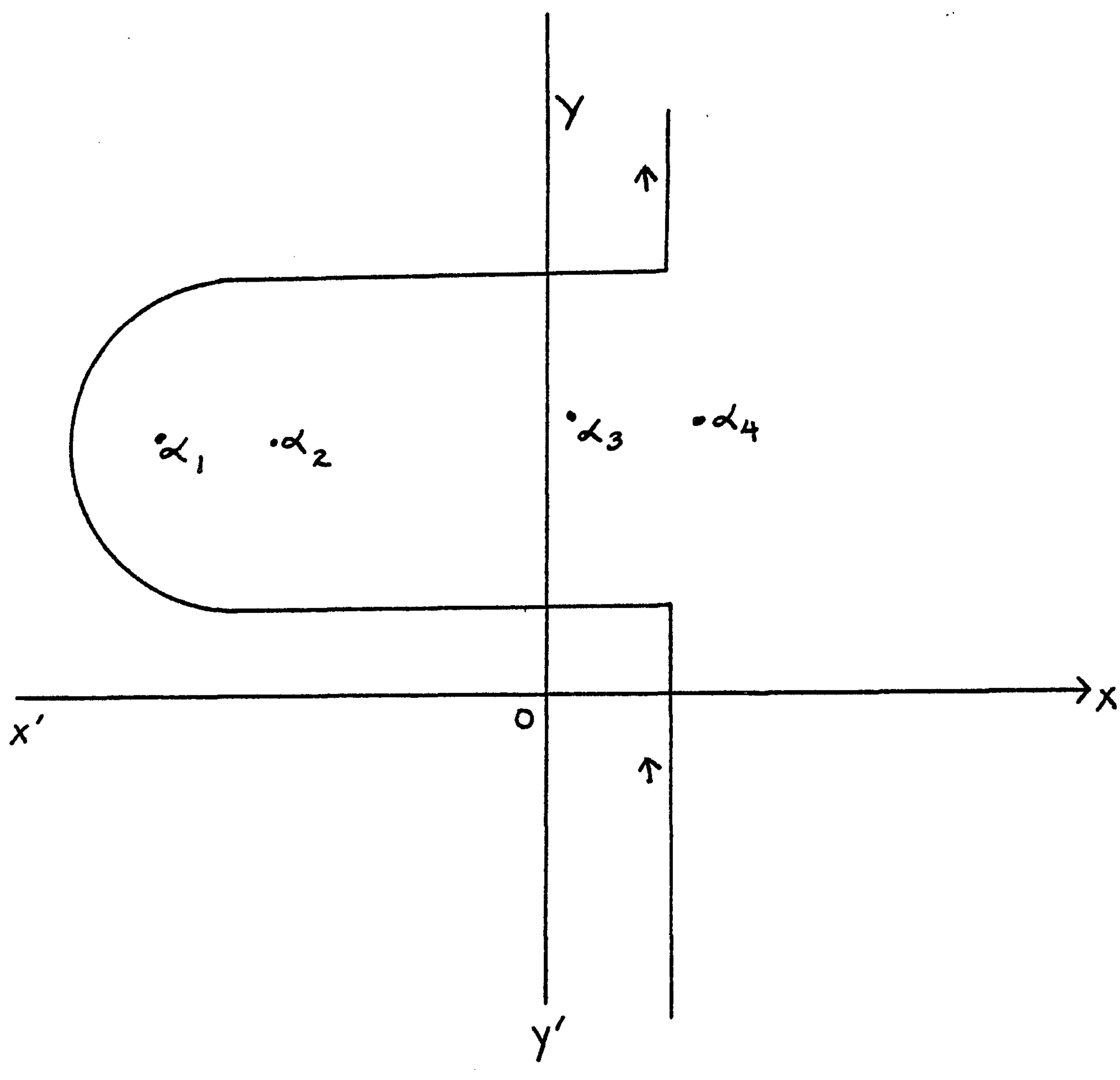
(6) THE BASIC ANALOGUE OF DONGALL'S THEOREM:

$$_8\tilde{\Phi}_7\begin{bmatrix} a & q\sqrt{a} & -q\sqrt{a} & b & c & d & e & q^{-N} & q \\ & \sqrt{a} & -\sqrt{a} & aq/b & aq/c & aq/d & aq/e & q^{N+1} \end{bmatrix}$$

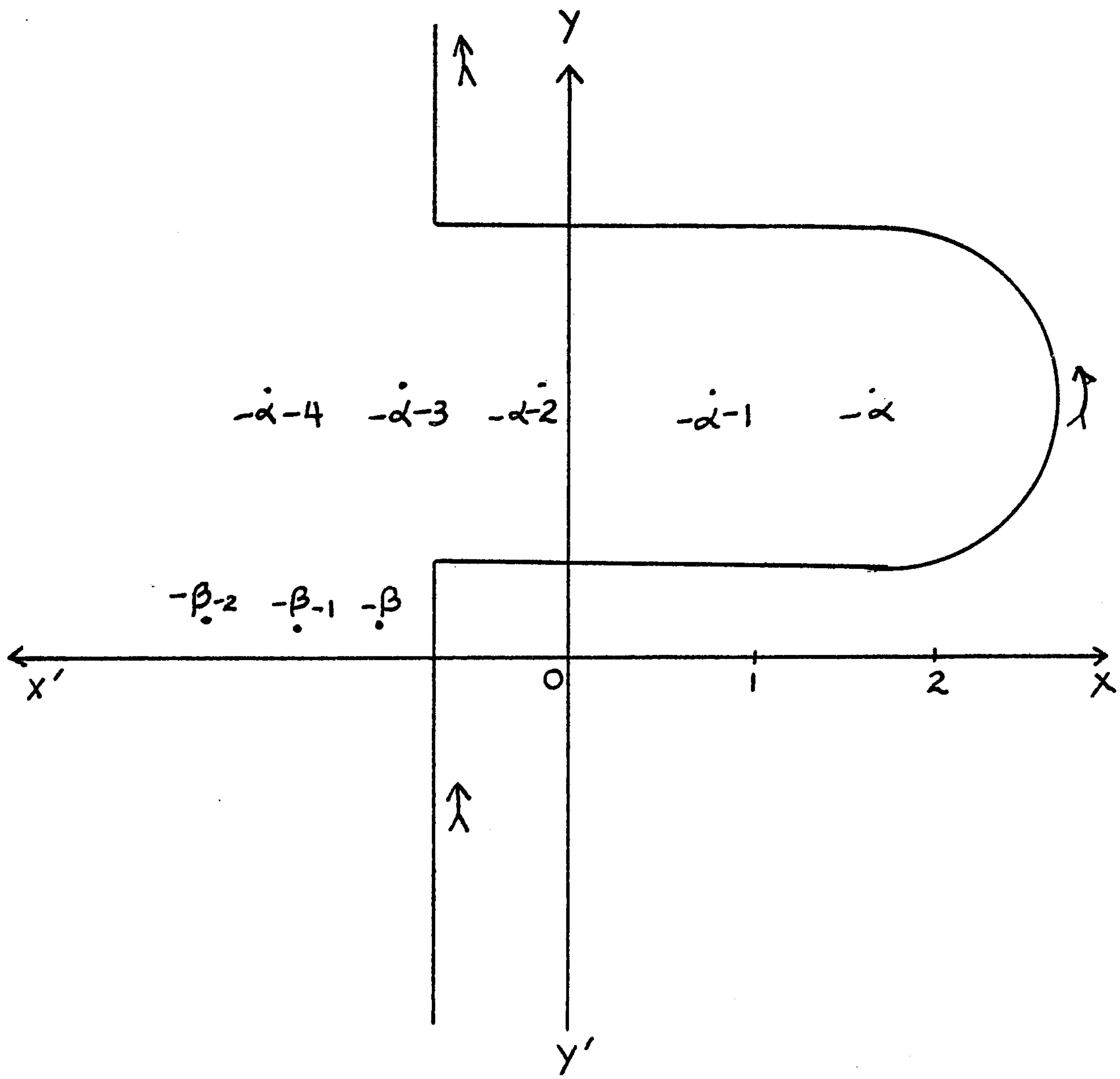
$$= \frac{(aq;N)(aq/cd;N)(aq/bd;N)(aq/bc;N)}{(aq/b,N)(aq/e,N)(aq/d;N)(aq/bcd,N)}$$

provided that $bcde = a^2q^{N+1}$, and N is a positive integer.

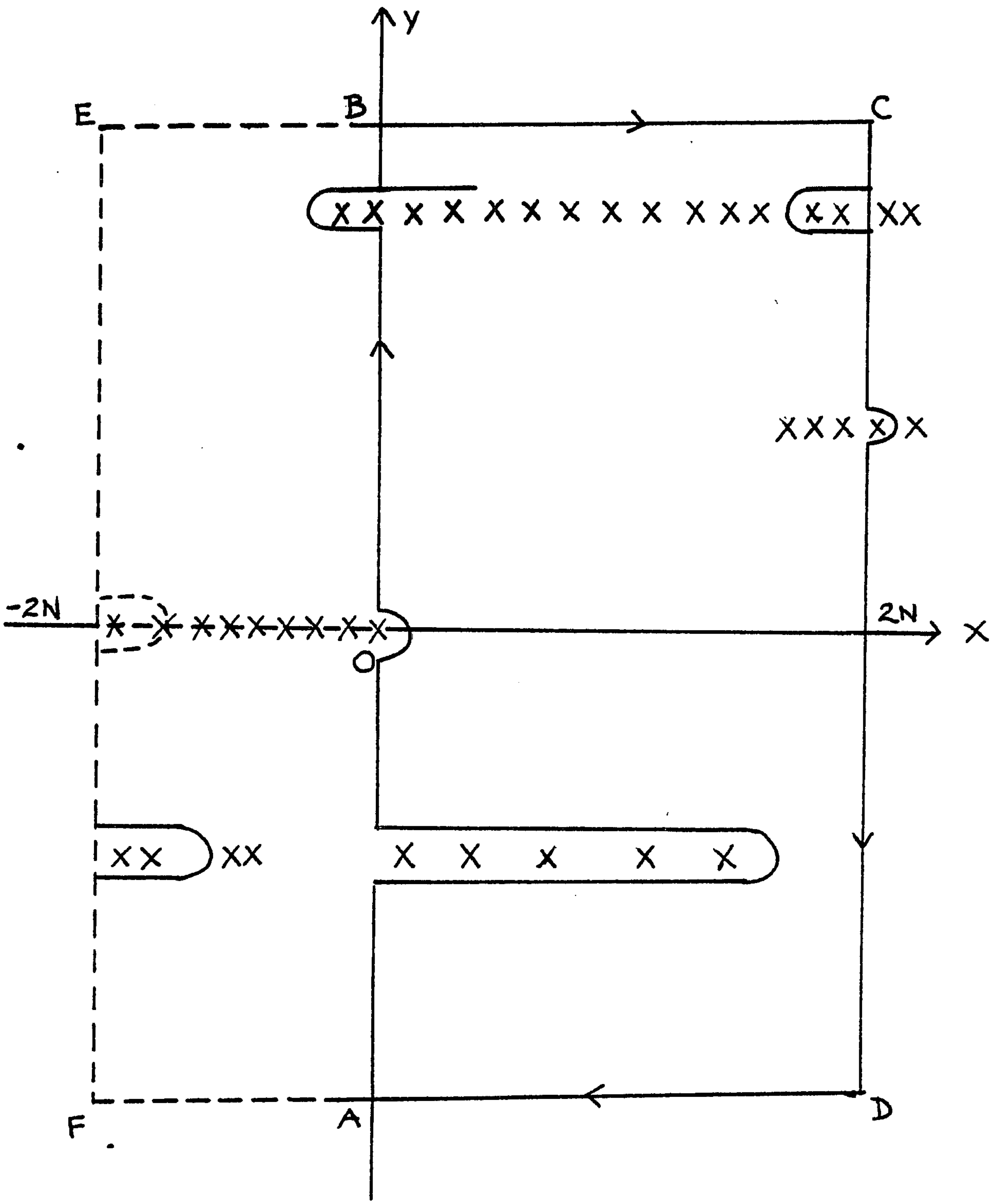
APPENDIX IV-1



APPENDIX IV.2



APPENDIX IV.3



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