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NUMBER REPRESENTATION IN YOUNG CHILDREN:
THE UNDERSTANDING OF QUANTITATIVE RELATIONS AMONG NUMERALS

by

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Abstract

NUMBER REPRESENTATION IN PRESCHOOL CHILDREN:
THE UNDERSTANDING OF QUANTITATIVE RELATIONS AMONG NUMERALS

by
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Adviser: Professor Geoffrey Saxe

The present research assessed preschool children's knowledge of cardinal number and ability to make comparisons of numerical values. Two studies were conducted. In the first study 15 3-year-old and 15 4-1/2-year-old children were asked to judge whether a series of additions, subtractions and spatial relocations changed the given value of a set of objects into another given value. It was hypothesized that children's performance on this task would be affected by set size and the direction of the requested change in numerical value. In the second study 15 3-year-old and 15 4-1/2-year-old children were asked to transform sets of a given value into another given value.

The findings of Study 1 indicated that children made more correct judgments on the spatial relocation tasks when set size was small. Children did not consistently base their judgments on either the length or the density of the array. It was also found that most of the children confused change in spatial extent with change in numerical value. Children did not make more correct judgments on this task when the numerical value was increasing rather than decreasing. On the addition and subtraction trials children also made more correct responses on the small set size trials.

An analysis of the strategies children used in Study 2 found 3 levels of strategy use which reflected unquantified, partially quantified and fully quantified conceptions of the numeration system. The findings of Study 2 indicated that children used strategies that could be characterized according to different levels of quantittative knowledge. Neither children's accuracy at producing the new set nor their use of strategies were affected by set size or the direction of numerical change.

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CHAPTER I

Introduction

Models of the development of numerical thinking in children take several different positions regarding children's capacity to represent and reason with cardinal numbers. One theoretical approach, best exemplified by Piaget's classic work (1952), suggests that children construct the concept of number over several years and before the age of 6 or 7 their ability to use counting and numbers is very limited. Piaget was interested in tracing the construction in thought of such logical relations as class and order which he saw as the basis of the number concept. The principal method in Piaget's research was the number conservation task in which he demonstrated that young children confuse spatial extent and number, and fail to understand the invariant numerical relationship between equivalent sets. Piaget saw the existence of those logical operations evidenced in the ability to conserve number as the hallmark of the number concept, and held that the preconserving child's use of counting and numeration is of a rote nature with little quantitative meaning.

More recently some investigators have taken another approach to the study of numerical thinking by focusing on the child's knowledge and use of counting. These researchers assume that preconserving children do use counting in ways that are quantitatively meaningful. Some of these researchers, most notably Gelman (Gelman, 1972;

Gelman & Gallistel, 1978) take the position that from infancy children have certain basic logical capacities that allow them to reason about number, and that these capacities are applied to the early use of counting. Other investigators take a weaker position and hold that while children can use the numeration system in some ways that are quantitatively meaningful, the ability to use counting in some contexts does not imply that the various skills and concepts regarding number are organized as a coherent whole. These recent investigations of counting have attempted to establish models of how children learn the sequence of number words (Fuson & Richards, 1980; Fuson, Richards & Briars, 1982), how they master the rules and procedures of counting (e.g., Gelman & Gallistel, 1978; Schaeffer, Eggleston & Scott, 1974), and how counting is used as a tool for reasoning about number (e.g., Saxe, 1977). In addition, several models have been proposed regarding the role counting might play in establishing such concepts as number conservation (Klahr & Wallace, 1976; Gelman & Gallistel, 1978; Saxe, 1979b), and how the structure of number word language reflects the child's logical thinking about number (Beilin, 1975).

Although these recent studies have documented that children can use counting in ways that show an understanding of certain basic principles (e.g., using a single number for each object, counting each object once and only once), questions still remain regarding the precise nature of this understanding. A central issue is the way in which children coordinate what they know about numerical operations such as addition and subtraction with what they know about using the numeration system to represent cardinal number. This problem is of interest because it involves the intersection of what the child knows about the logic of number operations and the language of number. For instance do young children who are able to count know that the numerical value of a set does not change unless elements are added or subtracted? Or, do preschoolers know that a 6 is greater than a 4, and that to get from 4 to 6 one must add a specified number of objects? These questions concern the relationship between children's knowledge of numbers and counting and their understanding of the effect of certain physical operations (addition and subtraction) on number. The present research focuses on this interaction and, in particular, how young children's knowledge of cardinal number is related to concepts about addition, subtraction and identity operations.

Since Piaget's work is the foundation for most later research, the following review of research will first describe Piaget's theoretical position on the relationship of counting to the development of the concept of number. Following the discussion of Piaget's research more recent research on counting will be reviewed.

Piagetian research on number concepts

Piaget (1952) demonstrated that children's understanding of number undergoes a change during development. He argued that children progress through a series of stages, each stage representing a qualitative shift in children's understanding of order and correspondence relations The behavior that Piaget used as evidence for this model of development was children's performance on a variety of numerical tasks, most notably the number conservation task. In this task children were presented with a linear array of objects and asked to construct a set numerically equal to that array. Children at the first stage merely matched the endpoints of the two arrays, paying no heed to the actual number in them. Children at the second stage matched the elements one-to-one, but believed that the equality relationship between the sets did not hold if the items in one row were spread apart or bunched together relative to the other row. At the third stage children understood that the relationship of one-to-one correspondence between the sets was lasting and that the equality between the sets was conserved despite the spatial transformation. Piaget argued that conservation of number requires the coordination of length and density. Piaget (1952; 1968) suggested that young children could not differentiate transformations that change only the spatial extent of a set of elements from those operations (addition and subtraction) that change number. He found that children will most often consider longer rows to have more because they use a "going beyond" strategy based on the relative spatial extent of the arrays to judge physical quantity. However, findings from other studies (Gelman, 1972) indicate that children do make this differentiation, and that many young children do not judge that a spatial change implies change in number.

In Piaget's view counting plays a very limited role in the development of the number concept prior to the achievement of conservation and consequently he did not conduct systematic research on the character of the preconserver's counting ability. The preconserver's use of counting is considered in a sense to be words empty of quantitative meaning. It is not until the child can understand and coordinate relations of order and correspondence that truly meaningful representation of number comes about. In Piaget's language the emphasis is principally on the operative structure of thought. It is this operative structure which determines the child's understanding and use of symbolic forms as in numeration.

Although Piaget's studies demonstrated that young children do not conserve the numerical equivalence between sets, several researchers have observed that during a conservation task some nonconservers understand that the <u>numerical value</u> of the sets do not change. For example, some children show the ability to correctly infer the number of objects in the collection that has been spatially transformed in a conservation task after that array has been screened by counting the objects in the other collection. This aibility has been called "quotite" conservation (Greco, 1962; Inhelder, Sinclair & Bovet, 1974; Morf, 1962) or "numeral invariance" (Saxe, Cohen & Rindskopf, 1980). For children at this level of understanding the activity of counting leads to the invariance of the child's numerical schemes (Klein, 1984).

The question arises, then, what quantitative meanings do numbers have for these children? Something of the quantitative significance of numerals is understood yet at the same time children seem to lack the understanding that identical numerals must represent the same amounts, that one "5" can't be more or less than another "5".

In light of these findings some attempt has been made by Piagetians to describe the place of counting in the construction of the number concept. According to Inhelder, Sinclair and Bovet (1974) counting allows the child to attribute a certain measure of individuality to the elements and to preserve their qualitative identity despite a change in position. Enumeration is also described by Inhelder et al. as facilitating the child's understanding of the serial aspect of the natural number system. Yet counting, though it may facilitate the acquisition of the number concept, is in no way sufficient or even central to its construction. "Indeed, it is most unlikely that counting or measuring will lead to the acquisition of a logical structure when this logical structure is itself necessary to know what to count or measure" (p. 36). The problem of how children's learning of the numeration system is determined by and interacts with the development of logical concepts such as conservation is of some interest in the recent research on number.

Research on children's counting

Recently several researchers have advanced the thesis that preschool children use counting in ways that imply some understanding of number. In contrast to Piaget, these investigators assume that preschooler's use of counting reflects some underlying understanding of certain logical principles, and this understanding allows children to reason effectively about number. Research on numeration can be organized around the following skills and concepts: (a) knowledge of the rules of counting (b) the use of counting in addition and subtraction (c) the acquisition of the counting sequence.

Knowledge of the rules or principles of counting

Some investigators have claimed that children's knowledge about counting reflects several logical capacities. The most prominent of these investigators, Gelman (Gelman & Gallistel, 1978) suggests that young children show an understanding of several counting principles. These counting principles include the following: (a) abstraction principle (any collection of entities can be a set of countables), (b) stable order principle (the words used in counting must be produced in a fashion that is stable from trial to trial), (c) one-to-one principle (each item in an array must receive one and only one counting word tag),(d) order irrelevant principle (the order in which a set is counted is irrelevant to producing an accurate count), and (e) cardinal principle (the last numeral in a count represents the number of objects in that set). Gelman and Gallistel claim that these principles form a scheme that guides and structures children's counting behavior. Once children acquire these principles they have the competence to use counting in a systematic way to represent and reason about number. Inaccuracies in the use of counting are attributed to errors in execution, not to deficiencies in children's conceptualization of how counting functions as a system of representation.

Gelman and Gallistel state that the counting principles become coordinated by the end of the fourth year and that this understanding reflects some concept of cardinal number. However their criteria for what constitutes an understanding of cardinal number differs substantially from Piaget's. Piaget's definition of cardinal number involves the application of the one-to-one principle in the comparison of more than one set. Piaget held that in order for a child to understand that two sets are numerically equivalent the child must understand the relationship of one-to-one correspondence between the sets. Gelman and Gallistel put forth a different view: "We do not agree that the recognition that sets can be placed in one-to-one correspondence marks the emergence of the initial appreciation of numerical equivalence... We find that the preschool child's judgment of whether two sets are numerically equal ordinarily rests on whether they yield the same cardinal numeron when counted" (p. 228).

Much of the evidence supporting this position comes from a series of "magic" studies in which children were asked to judge the relative number of objects in two arrays, one of which had been surrepetitiously changed. In these studies children judged an array of _n_ to still contain _n_ after the length, color and identity of the items had been altered. Further evidence from this study that children could reason about cardinal number was that when children were asked to undo the changes in the array they often constructed two sets that had the same numerosity.

Gelman and Gallistel found further evidence of children understanding cardinal number in observations they made of children's counting. In these studies they defined the cardinal principle as the understanding that the last number in a count is an index of the cardinal value of that set, and took as evidence for this understanding any of the following criteria: (1) repeating the last word of a count immediately following the count (2) emphasizing the last word of a count (3) repeating the final word of a count upon being asked for a second count. Using these criteria Gelman and Gallistel concluded that the cardinal principle is present even in 2-1/2 -year-olds, the youngest children they had studied.

Though the behavior described in this operational definition does suggest that preschool children have some understanding of the symbolic use of number, more evidence regarding children's knowledge of the number system is needed before claiming that the child has a fully developed understanding of cardinal number. In fact Gelman and Gallistel state as much: "We do not assume that counting is the only way individuals might arrive at a representation of the cardinal numerosity of a set. Nor do we assume that a child who uses the counting procedure to establish such a representation has a full appreciation of all the properties of cardinal number" (p.94). For instance, in addition to being able to count a set to determine its numerical value, reasoning about cardinal number must include the knowledge that a numerical value does not change unless elements are added or subtracted. Also, understanding cardinal value must include knowledge about the relations between numbers. Does a child know that 6 is greater than 4, and that to get from 4 to 6 one would have to add a specifed number of objects? Although Gelman and Gallistel do not claim that the children in their study have a full concept of cardinal number they do imply that they have a level of understanding that allows them to do quite a bit of reasoning about number. They make a distinction between numerical reasoning that involves representations of specific numerosities and algebraic reasoning that involves relations between unspecified numerosities. The prealgebraic child "can count one set and get the number four,... can count another set and also get the number four... He believes that displacing one set does not destroy its equivalence to the other because the numerosity of the displaced array will still be represented by the number four" (p.229). Thus Gelman and Gallistel suggest that the deficits that have been attributed to children's understanding of cardinal number are due for the most part to their inablity to make algebraic comparisons.

Gelman's model suggests that the counting principles serve as a scheme to assist the child in developing an understanding of cardinal number. There is also evidence that children derive the meanings of cardinal numbers outside of the context of counting. Wagner and Walters (1980) found that before understanding many children use the numbers "two" and "three" to mean either "many" or to represent the physical quantity of "twoness" or "threeness." This often occurs at age two, a full year before these children used counting to represent cardinal number. They also found that children altered the way they employed the counting principles to be congruent with some other estimate of the magnitude of a set. For instance, children often "fixed" their counts of two sets in a conservation task to accord with their belief that one array was more numerous.

To account for these findings Wagner and Walters made a distinction between the "semantic" use of numerals (evaluations of magnitude) and the "procedural" use of numerals (counting). They suggest that these two aspects of numeration develop separately and that developmental models of the symbolic use of number words must take into account both early emerging non-counting strategies for representing quantity, and the use of counting to produce representations of cardinal number. There may be moments in development when these two aspects of numeration are separate. For instance, it is possible that a child who has not mastered the procedures of counting does understand that two distinct numerals represent different magnitudes, and that this magnitude difference has a certain direction.

Further evidence of the indirect relationship between children's knowledge of the procedures of counting, and the symbolic use of numbers to represent cardinal value is also provided by Saxe (1977) who showed that the ability to count accurately does not necessarily lead children to use counting in numerical problem solving. Saxe investigated children's use of counting as a "notational system" for number: the use of counting to extract numerical information (i.e., determine the number of elements in a set), compare the number of objects in two or more sets, and reproduce a set (producing a new set of elements having the same number as a given set). Saxe found that preschool children use counting to compare and reproduce sets of objects and he documented developmental changes in the ways in which children use counting to mediate these comparisons and reproductions under various conditions. He found that very young children, though capable of counting, often did not use counting as an effective aid to compare or reproduce arrays. For example, when asked to numerically compare two arrays children of this age sometimes counted the arrays but based their judments on spatial extent. At transitional levels children tended to sometimes judge by spatial extent and sometimes by counting. It was not until age six (yet prior to conservation of number), that children used their counting as a tool to compare number. Thus, although children may know the proper sequence of counting words, and be able to use counting to determine the number of elements in a set, they may remain unable to use counting as an effective reasoning tool.

The use of counting in addition and subtraction

Another approach to understanding how children integrate their knowledge about numerical operations with their knowledge about the numeration system is to look at how children use counting to solve addition and subtraction problems. Children have been shown to use two basic types of counting strategies or algorithms for addition: "counting all" and "counting-on" (Carpenter & Moser, 1982; Fuson 1982; Fuson & Mierkiewicz, 1980). Counting all involves counting the elements in the sets that are to be added as a single count starting with "one." For example, a child who "counts all" when asked to add 4 + 3 would count the first 4 elements (i.e.," 1,2,3,4" and then add on 3 objects counting "5,6,7" as adding the 3 objects). Counting-on, a more advanced strategy, involves starting with the cardinal number of one set and continuing the count until the second set is exhausted. Using this strategy the child would count "5,6,7" as adding to the 4 objects.

It has been found that younger children are limited to using the counting-all algorithm and only later are able to use counting-on. This finding may reflect a developmental progression in childrens' understanding of the quantitative relations among numbers. It is as if for young children, the cardinal value arrived at through counting is conceived of as referring only to the immediate context and is not necessarily related to other cardinal numbers in some systematic way. Young children may have the notion that the last word in their count represents the cardinal value of that particular set at that particular time, but that the cardinal value may not be understood in a way that allows comparison with other cardinal values. Hence a child who cannot use the counting-on strategy does not always conceive of a number as part of a system in which all of the numbers are related along a scale of magnitude. This child must begin to count from "1" because the numbers take on this relationship only in the immediate context of counting.

Another method for looking at children's understanding of the correspondence between numbers and addition and subtraction operations is offered by Ginsburg and Russell (1981). This study assessed whether preschool and kindergarten children were able to understand that adding one item to or subtracting one item from a set increases or decreases that set by one to the next number (forward or backward) in the counting sequence. They called this understanding the "unit rule." In a task derived from Baroody (1979) children were told that there were a given number of objects, N, and were told to fix the set so there were N + 1. Using only small sets of objects (less than 5) Ginsburg and Russell found that over half of the preschoolers and almost 90% of the kindergartners were successful at this task.

In the same study Ginsburg and Russell also assessed children's ability to use the addition and subtraction of elements to transform one cardinal value into another value in constructing a set two or more larger (or smaller) than a given set. They posited that this task imposed a "cognitive strain" on the child's problem solving ability as compared to the previous task. In this task a puppet which was regularlv given a certain number of objects "each day" was then given a different number. The child was asked to fix the array so the puppet would have the same as before. The original set always contained 4 items and the number the child needed to add or subtract was either 2 or 3. Ginsburg and Russell found that preschool children understood which operation to perform nearly all of the time and performed it accurately most of the time.

This study suggests that children do have some understanding of the correspondence between relationships among numbers and the physical operations that relate these numbers into a system. Ginsburg and Russell limited their study to small sets. A question remains as to whether this understanding of number relations is limited to small sets or whether children generalize their understanding to larger sets as well.

Acquisition of the counting sequence

Fuson and her coworkers (Fuson, 1982; Fuson & Richards, 1980; Fuson, Richards & Briars, 1982) have looked at children's initial acquisition of the standard counting word sequence. They have characterized children's learning of the number word sequence as a progression from an unordered or "spew" stream of words, to an idiosyncratic list stable over counts, to a conventional list in which the relations among words have been consolidated into an "automatized chain."

Fuson and Richards (1980) have characterized five levels of "number word production." The first level, "spew," refers to a recitation of words with no predictable pattern within or between recitations. This level is usually very transitory. The child soon develops "stable strings" in which the order of the words, though not conventional, remains stable over counts. The stable string is soon replaced by the "standard string", number words in their conventional order. Once the standard list of number words is learned, the child begins to establish the order relations between contiguous words, arriving at a level of number word production called the "chain". The final level called the "automatized chain" or "represented chain" is described as being "a representational and abbreviated counting which can occur in the absence of real objects; acts of counting can be related and can themselves be counted". At this level of number word production the child's understanding of sequence allows the use of counting as a tool in numerical operations such as addition. For instance the child may perform an addition by starting with the number of elements in one set and "count-on' the number of items in a second set (e.g., the addition "4 + 2" is performed by counting up 2 from 4, "4, 5, 6"). However a child is not at a given level at a given time for the entire string of number words. Rather, different parts of the string are at different levels at any given time during development.

According to Fuson and Richards (1980), after acquiring a portion of the standard sequence the child begins to derive the ordered relations among the words. Thus, the child acquires the ability to start counting at an arbitrary point in the sequence, and relations such as the "immediate successor" (e.g., 7 comes immediately after 6), the "successor" (e.g., when counting, 7 comes after 5) and "between" becomes easier to deduce.

Fuson et al. (1982) also present findings indicating that both the ability to produce a backward number word sequence beginning from an arbitrary number, and the ability to start at and stop at arbitrary words in a backward sequence are more difficult than the analogous tasks in the forward direction. They suggest that children must gain a degree of mastery over the forward sequence before relations among words in the backward sequence become meaningful. Fuson et al.'s analysis focuses on children's recitation of number words in the absence of objects (what they call "number word production" as opposed to "counting"). However, Fuson et al. explain that the quantitative meanings actually called up with the utterance of a particular number word may depend heavily upon the context and function of the child's counting; for instance, whether the child is counting simply to produce a cardinal value, to compare two sets, to reproduce a given number of objects, etc. In order to learn more about how children understand these quantitative relations in different contexts Fuson and Richards compared what they called children's "alphabetic" or "sequence" knowledge of the counting words with their "cardinal" knowledge. Fuson and Richards operationally defined "sequence context" as the ability to judge which number comes before or after another in the sequence ("comes later/earlier than") and "cardinal context" as the ability to judge which of two numbers is more or less ("is bigger/smaller than"). This distinction is similar to Wagner and Walters (1980) procedural/semantic distinction that was discussed previously. Fuson and Richards suggest that because "sequence" number word contexts and "cardinal" contexts are separate for children, it is possible that they mentally process the order relations on sequence words differently from those on cardinal words. For instance, children might use some kind of "run through" to make sequence judgments and use a direct magnitude comparison for cardinal judgments; or they may use one type of process to answer questions on the other type.

Towards understanding developmental changes in children's conceptions of quantitative relations between numerals

The research that has just been reviewed indicates that very young children have some ability to represent and reason about cardinal number. However, investigators differ on the nature and extent of this understanding, with the differing theoretical positions exemplified by Piaget and Gelman. Piaget held that children's ability to reason about number is very limited until they grasp the concept of cardinal number, which he defined in terms of one-to-one correspondence relations between sets of objects. Gelman, on the other hand, holds that Piaget's criteria are too strict and that children can reason effectively about cardinal number without possessing a definition of number that rests on one-to-one correspondence relations. She maintains that very young children are able to reason about cardinal number, especially if the cardinal value of a set is expressed in terms of specific values, for instance when children count sets of objects.

The present study addresses several issues regarding children's understanding of cardinal number. It was thought that by asking children to judge what operation was required to go from one given cardinal value to another given value, one could learn something about children's notions of how cardinal numbers are related in a system of numeration. Two studies were conducted. In the first study preschool children were presented with sets of objects of a given cardinal value. They were asked to judge whether certain transformations performed by the interviewer on these objects (expansion, contraction, addition and subtraction) were adequate to change that set into another given cardinal value. In the second study children were asked to change a set of a given value into a set containing a different given value.

The aim of these studies was to determine whether preschool children are able to reason about the operations which relate one cardinal value to another. Gelman's position suggests that young children should be able to reason effectively about number in a context such as the present one in which a single set of objects is represented by a particular value. A more conservative view such as Piaget's suggests that these children's understanding of cardinal number is not yet integrated with their understanding of operations and therefore the kind of reasoning required in these tasks would be difficult for children of this age.

The purpose of Study 1 was to determine whether children of this age understand that the difference between two numerical values can be related by a particular operation. Study 1 consisted of two types of tasks. In one task children judged whether the numerical value of a set of objects could be changed by lengthening or shortening the spatial extent of the set ("spatial transformation task"). In this task the interviewer asked a puppet to change a set containing a given number of objects (the "original set") into a set containing another given number, 2 larger or 2 smaller than the original. For instance, the interviewer asked the puppet to fix a bunch of 4 things so it had 6 things. The puppet responded by either extending the length of the array or by decreasing its length without adding or subtracting elements. This was done in full sight of the child. The interviewer then asked the child if the puppet's response was right or wrong. These spatial transformations were performed on both small sets and large sets. The relationship between original set and goal set was also varied: from larger number to smaller number, and from smaller number to larger number (e.g., fixing a set of 6 into a set of 4, versus fixing a set of 2 into a set of 4).

In the second type of task the same children made judgments about changes in the numerical value of a set when objects were added to or subtracted from the set ("addition-subtraction task"). For example, the puppet was asked to change a set of 4 objects into a set of 6 objects. The puppet either added objects to or subtracted objects from that set and the child was asked to judge if the puppet was correct in what he did.

Piaget's model suggests that most preschool children would not have the capacity to make accurate judgments regarding the required operations. Children at this age would likely be at a phase in which they have little or no understanding of the quantitative relations among the words. For instance if, as Piaget claims, children of this age consistently confuse change in spatial extent with change in number they would be likely to judge that to change the cardinal value of a row of objects from 5 to 7 one could just extend the row without adding or subtracting objects.

Gelman's model, on the other hand, suggests that children's capacity to reason about numbers includes the understanding that spatial rearrangement doesn't change number, especially since the tasks involve operations on only a single set of objects and no judgments about equivalence between sets is called for. Children should also know the relative values of the number and they should know that to get from a smaller number to a larger number one would need to add objects and to go from a larger number to a smaller number one would have to subtract objects. Children should be able to use the specific cardinal values to make accurate judgments about the effects of addition, subtraction and identity operations.

In addition to evaluating the above theoretical positions the study also provides data on the influence of set size and the direction of numerical change on children's judgments. Several investigations have indicated that young children are first able to apply numerical reasoning on small sets early in development and we expected to confirm these findings in the present task. It was expected that if children could reason effectively about these problems it would be at small set size. Research also suggests that children's ability to use cardinal number may be at first closely tied to the act of counting and that the tasks used in this study may well be easier when the direction of change is increasing because that relationship reflects the child's usual experience of counting.

Hypotheses - Study 1

Since previous research has indicated that preschool children have a greater facility in reasoning about small sets (Cowan, 1979; Gelman, 1972; Winer, 1974; Zimiles, 1966) the following hypothesis was tested regarding the 3 and 4-1/2-year-old children in the present study:

1. Children will receive higher scores under the small set condition than under the large set condition on both the spatial task and the addition-subtraction task.

Studies also indicate that preschool children have more facility in using the counting sequence in the forward direction than in the backward direction (Fuson &Richards, 1980; Fuson et al., 1982). The following hypothesis was tested:

2. Children will receive higher scores under conditions in which the goal set is a larger number than the original set, than under conditions in which the goal set is smaller than the original.

The next set of hypotheses that were tested concern two positions that have been put forth regarding young children's use of length as the basis of making judgments about number. According to the work of Piaget and others, preschool children tend to judge that lengthening an array increases its number and shortening an array decreases its number. An opposing view is that even very young children have the concept of number invariance and, under favorable circumstances do not rely on length in making judgments about number. The following hypotheses were tested for the spatial task:

3. On those trials in which the numerical value of the goal set is greater than the original set, children will tend to judge that lengthening a row is correct rather than shortening a row.

4. On those trials in which the numerical value of the goal set is less than the original set, children will tend to judge that shortening a row is correct rather than lengthening a row.

5. The predicted effect of type of transformation will interact with age. Younger children for whom the numeral relations are not well established will be less likely to exhibit the predicted tendencies in Hypotheses 3 and 4. That is, younger children will not tend to associate increase in length with an increase in numerical value and a decrease in length with a decrease in numerical value.

Hypotheses - Study 2

The question addressed by the second study was how children's understanding of the relationship between number values and physical operations is reflected in their use of counting. While in study 1 children made judgments about the adequacy of certain transformations to change numerical values, in study 2 children were themselves allowed to make transformations on the sets. This permitted an analysis of children's use of different strategies, such as counting, in establishing physical representations of numerical differences.

In Study 2 a different sample of children was presented with a set of objects. They were told how many there were in the set and then asked to fix the set so it had a different given number of elements. Children were scored based on both their accuracy and on the strategies they used in solving the problem. Strategies were scored at 3 levels based on the degree to which they were "quantified". Regarding children's accuracy, we expected similar effects due to the set size and original-to-goal relation that we proposed in Study 1:

1. Children will receive higher accuracy scores on small set sizes than on large set sizes.

2. Children will receive higher accuracy scores on increasing value trials than on decreasing value trials.

It was also expected that these same effects would be seen on children's use of strategies:

3. Children will use more advanced level strategies on small sets than on large sets.

4. Children will use more advanced level strategies on increasing value trials than on decreasing value trials.

It was also expected that the types of strategies used by older and younger children would differ:

5. The older group of children will use higher level strategies than the younger group.

CHAPTER II

Study 1

Method

Design

The experimental design for this study was a 2 x 2 x 2 factorial design consisting of 1 between- and 2 within-subject factors. T'e 1 between-subject factor was age (3- and 4-1/2-year-olds). The 2 within-subject factors were set size (large and small), and the direction of transformation (smaller values to larger values, and larger values to smaller values).

Subjects

Thirty preschoolers, 15 each of 3-year-olds and 4-1/2-year-olds were selected from three nursery and elementary schools in the New York City area. The 3-year-olds group's ages ranged from 3-1 to 3-11, with a mean age of 3-4. The 4-1/2-year-old group's ages ranged from 4-6 to 4-11, with a mean age of 4-7. It was decided to select groups from 3- and 4-1/2 year populations because while it was desirable for there to be a range of ages, it was discovered through pilot testing that (1) almost all kindergarten children could judge all trials correctly on the spatial task and thus would ceiling out ot that task (2) many children who were younger than 3-years-old had trouble interpreting the verbal presentation of the task and were often unsure of the sequence of counting words above 5.

Materials

Materials included a puppet which fit over the interviewer's hand and color drawings of fruits and vegetables mounted on cards (4cm x 4cm). There were enough pictures of each type of fruit and vegetable to form 8 sets of 10 items.

Procedures

Prior to the administration of any tasks children were asked to count to 10. Any child unable to do so was not included in the study.

Children were introduced to a puppet. The interviewer then told the child that the puppet was going to help him make up some baskets of fruits and vegetables, and it would be the child's job to decide whether or not the puppet had done what the interviewer said. The task was introduced in the following way:

> Let's pretend that I need to make up these baskets of fruit so I can sell them at a store. Mr. Puppet here is going to help me get the right number of fruits in each basket. I'll give him some of these fruits in a basket and then I'll ask Mr. Puppet to fix them so they have as many as I need. If he makes the right number you should tell him that he is right. If he is wrong you should tell him that he is wrong.

The puppet was then presented with an array of pictures of fruits or vegetables and told, "In this basket I have x fruits. But I need y fruits. Can you fix this basket so it has y fruits?" The numbers x and y were 2 units apart, y being either 2 greater or 2 smaller than x. The puppet responded by either spreading or contracting the array, or by adding or subtracting 2 objects.

Each child received 32 trials in all: 8 trials each of spreading, contracting, adding and subtracting. Each set of 4 trials included 2 with small sets and 2 with large sets. In each of these sets of 4, 2 trials involved a request for an increase in value and 2 a request for a decrease in value. The trials were administered in one of eight randomized orders predetermined by the interviewer.

Under the small set size condition the puppet was asked to fix arrays of size 3 and 4 to sets 2 greater and 2 less (i.e., the transformations requested were 3 to 5, 4 to 6, 3 to 1, and 4 to 2). On the large set trials the transformations were from 5 to 7, 6 to 8, 7 to 5, and 8 to 6. On the spatial task trials the puppet performed a contraction and an expansion on the row for each of the requested transformations. For instance on one trial the puppet responded to a request to fix 3 into 5 by shortening the row, and on another trial by lengthening the row. After the puppet transformed an array it was immediately screened from view. Both the small/large numeral relation and the direction of spatial transformation (spreading/contracting) was counter-balanced.

In the addition-subtraction trials the puppet fixed the baskets by adding or subtracting objects rather than by lengthening or shortening arrays. The puppet fixed the sets by adding 2 objects, or subtracting 2 objects. Additions and subtractions occurred both on trials in which an increase in value was requested and on those trials in which a decrease in value was requested. On the addition and subtraction tasks the set of objects was screened from view before before objects were added or subtracted. Each child received a total of 16 addition-subtraction trials.

Scoring

Children were scored on the basis of their judgment for each of the 32 trials. A score of 0 was assigned to an incorrect judgment and a score of 1 to a correct judgment.

Results and Discussion - Study 1

The analyses are organized into three parts. The first set of analyses considers the influence of set size and change in numerical value on the accuracy of children's judgments. The next part presents analyses of patterns in children's error judgments. The final set of analyses compares children's performance on the spatial task with their performance on the addition-subtraction task.

The influence of set size and transformations on the accuracy of children's judgments

Spatial task. In order to understand the way in which set size and direction of numerical change influenced children's numerical evaluations, a comparison was made of children's mean scores on the spatial task under the 2 set size conditions (small and large) and the 2 numerical change conditions (increasing and decreasing). It was predicted that children would make more correct judgments in the small set size condition than in the large set size condition. It was also predicted that because children were more familiar with using counting in a forward direction and because their judgments about changes in numerical value are strongly linked to counting, children would make more correct judgments on those trials in which the change requested was from a smaller number to a larger number than when the change requested was from a larger number to a smaller number.

Table 1 contains children's mean scores on the spatial task as a function of set size, direction of required change (from smaller to larger or from larger to smaller number) and age. Because each child received every task condition the difference in the mean scores were analyzed using a profile analysis which takes into account repeated measures over trials[1]. A profile analysis testing for equal cell means indicated that the mean number of correct judgments in the small set condition for trials in which the original to goal relation was from larger to smaller was significantly larger than the mean for the smaller to larger condition, Hotelling $T^2 = 13.35$, $p < .05$. One explanation for the finding is that for the larger to smaller trials with small sets the puppet was being asked to fix the set so it has either 1 or 2 objects. Children may easily recognize that the transformation is not producing 1 or 2 objects and therefore are able to consistently make correct judgments.

<u>Addition-Subtraction</u> <u>Task</u>. The influence of set size and change in number on children's accuracy was also analyzed for the addition-subtraction task. As in the spatial task it was predicted that children would make more correct judgments when set size was small and when the numerical value was decreasing.

Table 1

Mean Number of Correct Judgments

On the Spatial Task

		Small Sets		Large Sets	
		Direction of Change in Numerical Value			
Age (Years)		Smaller to Larger	Larger to Smaller	Smaller to Larger	Larger to Smaller
3	Mean	2.80	3.73	2.80	2.87
	(SD)	(1.42)	(0.59)	(1.27)	(1.25)
4 1/2	Mean	3.20	3.87	3.00	3.20
	(SD)	(1.01)	(0.35)	(1.36)	(1.15)
Total	Mean	3.00	3.80	2.90	3.03
	(SD)	(1.23)	(0.48)	(1.30)	(1.19)

Note. Maximum score = 4.0.

Table 2 contains children's mean scores on the addition-subtraction task as a function of set size, direction of required change and age. A profile analysis testing for equal cell means indicated that as predicted the mean number of correct judgments in the small set condition was greater than in the large set condition for both the increasing value and decreasing value conditions, Hotelling $T^2 = 25.53, p<.05$ for the increasing condition; $T^2=80.64$, $p<.05$ for the decreasing condition, suggesting that small sets offer an advantageous context for judging numerical change.

The data in Table 2 also indicates that contrary to the hypothesis the mean number of correct judgments for both 3- and 4 1/2-year-olds is greater for those trials in which the value is decreasing rather than increasing. This is true for the small set condition only (Hotelling $T^2=22.90$). In other words, for small set sizes children made more correct judgments on those trials in which the interviewer requested a decrease in numerical value than on those trials in which the interviewer requested an increase in numerical value. This was the same finding as for the spatial task.

Table 2

Mean Number of Correct Judgments On

The Addition-Subtraction Task

Age (Years)		Small Sets: Smaller to Larger	Small Sets: Larger to Smaller	Large Sets: Smaller to Larger	Large Sets: Larger to Smaller
		Direction of Change in Numeral Value			
3	Mean	2.93	3.73	2.00	2.00
	(SD)	(0.70)	(0.59)	(0.93)	(0.93)
4 1/2	Mean	3.20	3.87	1.87	2.40
	(SD)	(0.67)	(0.35)	(1.25)	(1.06)
Total	Mean	3.07	3.80	1.93	2.20
	(SD)	(0.69)	(0.48)	(1.08)	(1.00)

Note. Maximum score = 4.0

Patterns in children's inaccurate judgments

Spatial Task. Most of the children in this study did not have a firm understanding that the numerical value of an array could not be changed by merely changing its spatial extent. Of the 30 subjects, only 11 gave all correct judgments on the spatial task trials; of the 15 3-year-olds only 5 judged correctly on each trial, and of the 15 4-year-olds, only 6 judged correctly on each trial. Thus there are clearly a number of children, both 3- and 4 1/2-year-olds, who confused spatial change with change in number.

In order to discover whether using large sets made the task more difficult an analysis of the number of errors that the 3- and 4 1/2-year-olds made on small and large set trials was performed. Table 3 contains the frequency distribution of children's incorrect judgments on the spatial task as a function of age and set size. It should be noted that for small sets very few children of either age made more than 2 errors over the 8 trials. However, for larger sets the number of errors greatly increased for the younger children with almost half of the 3-year-olds making 3 or more errors. To confirm that set size did influence

Table 3

Frequency Distribution of Children's Performances on the Spatial Task

Age (Years)	Number Incorrect		
	0	1 to 2	3 to 8
3			
Small sets	7	5	3
Large sets	6	2	7
4 1/2			
Small sets	7	6	2
Large sets	8	3	4

Note. Maximum number wrong = 8

children's tendency to make errors on the spatial task, contingency tables were formed by establishing 4 categories of children: those having 2 or less errors on both the large and small set trials, those having more than 2 errors on both small and large set trials, those having 2 or less errors on the small set trials and more than 2 errors on the large set trials, and those children having more than 2 errors on the small set trials and less than 2 errors on the large set trials. Chi-square analyses for both 3 and 4-1/2-year- olds indicated that children did tend to make less incorrect answers on small sets, X^2 $(1)=4.0$, $p<.05$, for both groups.

Children's judgments on the spatial task supplied evidence for testing the validity of Piaget's position that children make systematic judgments regarding quantity based on spatial extent. The data were analyzed to test both possibilities presented by Piaget: that some young children base their judgments about numerical change on length, and that other children base their judgments about numerical change on density. It should be pointed out that since children were making judgments about whether a change in numerical value can be affected by a spatial change, and not whether quantity per se could be changed, this analysis is only a partial or indirect test of Piaget's position. In other words the present task assessed children's concept of whether the <u>numeral</u> could change with a spatial relocation, not if the number of objects could change. It was hypothesised that there are differences between the way 3- and 4 1/2-year-olds understand the numeration system's significance in representing cardinal value and hence they interpret change in number due to spatial transformations differently.

It was assumed that if Piaget's position is correct and preconserving children use length as an index of number, those children not yet passing the spatial task (judging on all 16 trials that the number did not change) would think that lengthening a row of objects increases its number and therefore its numerical value, and that shortening the row decreases its number and numerical value. However, it was also hypothesised that there would be a difference in performance between 3- and 4 1/2-year-olds. It was expected that the 3-year-olds would not have a firm understanding of the correspondence between differences in numbers and differences in quantity. Though 3-year-olds might tend to think that the quantity increased by lengthening the array and decreased by shortening the array, they would not associate change in quantity with change in numerical value (e.g., from 4 to 6), and therefore would not show the same consistency of judgments based on length as would the 4-1/2-year-olds.

Table 4 presents data relevant to these hypotheses. The table contains the number of children falling into each of 4 categories formed by multiplying children's performance on two types of trials. The categories were formed by cross-classifying the number of times a child judged the transformation to be correct on trials in which the transformation was consistent with length as the index of number ("length based trials"), versus the number of times the child judged the transformation to be correct on trials in which the transformation was consistent with crowding as the index of number ("crowding based trials"). Length based trials were defined as those in which either the array was lengthened by the puppet in response to a request to increase the numerical value (e.g., "Make this 3 into 5"), or the array was shortened in response to a request to decrease the numerical value (e.g., "Make this 6 into 4"). Crowding based trials were defined as those in which either the array was made less dense by the puppet in response to a request to decrease the numerical value, or the array was made more dense in response to a request to increase the value. Thus children were assigned to 1 of 4 categories: low number of yes answers (0 to 2) on both length based and crowding based trials (Category I), low number of yes answers on length based trials and a high number of yes answers (3 to 8) on crowding based trials (Category II); high number of yes answers on length based trials and a low number of yes answers on crowding based trials (Category III); high number of yes answers on both length based and crowding based trials (Category IV).

Table 4

Children's Use of Lengthening and Crowding

As Criteria for Number Judgments

Age (Years)		Categories of Response (Number of yes answers)			
		I	II	III	IV
	Length based trials	(0-2)	(0-2)	(3-8)	(3-8)
	Crowding based trials	(0-2)	(3-8)	(0-2)	(3-8)
3		9	0	1	5
4 1/2		10	1	2	2
Total		19	1	3	7

Children whose judgments of change in numerical value were based on length would be in category III ,high number of "yes" answers on length based trials and low number of "yes" answers on crowding based trials. Children tending to use density as the basis of their response would fall into response category II : a high number of 'yes' responses on crowding based trials and low number on length based trials.

Table 4 indicates that children do not consistently base their judgments of numerical change on either length or density. There is only one 3-year-old and two 4-1/2-year-olds in category III. Thus neither 3- nor 4-1/2-year-olds are more likely to say that a transformation that is consistent with length as the index of number is correct. There is also no tendency for children to give more 'yes' answers on trials that are consistent with density as the basis of judgment. No 3-year-olds and only one 4 1/2-year-old fall into this category. These findings can be interpreted in two ways. It could be that most of the children were giving random responses. That is, they were not consistently using either length or density as an index of number and therefore saw no reason to judge that one or the other type of transformation accomplished the required change in numerical value. The other possibility is that children did think that quantity was either increasing or decreasing as a result of a particular type of spatial change but they did not have any way of associating this with the change in numerical value. For instance, some children might have consistently thought that spreading out an array increased its quantity, but they did not know whether the change in numerical value requested involved an increase or decrease in number, and therefore did not know whether to judge the transformation as correct or incorrect.

Addition-subtraction task. The previous analysis of children's performance on the spatial task indicated that children of this age do not always grasp that an addition or subtraction is needed to change numerical value. It was thought that an analysis of the patterns of children's judgments on the addition-subtraction task would give some indication of what children know about numeration and its relation to addition and subtraction operations. An analysis of the addition-subtraction task trials was performed to determine if children were more likely to give correct judgments on trials in which the correct transformation was performed versus those trials in which an incorrect transformation was performed. It seemed likely that children would feel more sure about identifying a wrong transformation as being incorrect (which would be scored as a correct judgment) than they would be about identifying a right transformation as being correct (also scored as a correct judgment). Though children would know enough about the numeration system to realize that a correct transformation was in the right direction (e.g., adding to get a set of higher numeral value) they would not have the strategies for determining if it were precisely correct. An incorrect transformation, on the other hand, would be seen by children as clearly wrong without the presence of the same kind of doubt. Table 5 presents the data relevant to this hypothesis. A profile analysis of these data revealed that there is a main effect in the direction hypothesized. Collapsing across age groups, the mean number of correct judgments on trials using incorrect transformations was greater than the mean number of correct judgments on the trials in which correct transformations were used, but only for small sets, Hotelling $T^2=60.58$, $p<.05$. Thus children do seem to be clearer about when an addition or subtraction transformation is wrong than when it is right. Why this is not true for larger sets may be due to the fact that children are generally less accurate on their judgments regarding larger sets, though there is a tendency for performance to be better on the incorrect transformations for large set sizes as well.

<u>Comparison of children's performance on the spatial task with Performance on the addition-subtraction task</u>

The data were also analyzed to determine whether children understood that numerical values remain invariant after a spatial relocation (numeral invariance) before they develop the basic notions of addition and subtraction that were tested in this study. One might think this to be the order of acquisition since those children not having quotite

Table 5

Mean Number of Correct Judgments on Correct Transformation Trials and Incorrect Transformation Trials in The Addition-Subtraction Task

Age (Years)		Small Sets: Correct	Small Sets: Incorrect	Large Sets: Correct	Large Sets: Incorrect
		Type of Transformation			
3	Mean	2.80	3.87	1.47	2.53
	(SD)	(0.78)	(0.35)	(1.41)	(1.30)
4 1/2	Mean	3.07	4.00	1.93	2.33
	(SD)	(0.70)	(0.00)	(0.96)	(1.35)
Total	Mean	2.93	3.93	1.70	2.43
	(SD)	(0.74)	(0.25)	(1.21)	(1.31)

Note. Maximum score = 4.0

do not seem to understand (at least in the present task context) that in order for numerical value to change elements must be added or subtracted - a basic addition and subtraction concept. If a child does not understand that to go from one number to another an addition or subtraction is required it seems probable that the child would also be uncertain about the required operation on at least some of the addition-subtraction trials.

Table 6 is a cross-tabulation of children's scores on the spatial task with their scores on the addition-subtraction task. If children tended to grasp numeral invariance before concepts of addition and subtraction one would expect to find a large percentage of the children with high spatial scores and low addition-subtraction scores. This is not the case. The data in this table indicate that most subjects scored at above the chance level of correct answers on both tasks. There was a group of 4 subjects who scored at below the chance level on the spatial task and better than chance on the addition-subtraction task. This suggests that some children grasp certain notions about addition and subtraction before they grasp numeral invariance, a finding which is contrary to the hypothesis stated above.

The data were also analyzed to determine if those children who fully grasped the concept of numeral invariance (correct judgments on all the spatial task trials) performed differently on the addition-subtraction task than those who did not grasp numeral invariance. The mean number of correct judgments on the addition- subtraction task for those children who scored perfectly on the spatial task was 11.9 (SD=1.83), while the mean score of those children who did not receive perfect scores on the spatial task was 11.47 (SD=1.23), not a significant difference. An analysis was made to determine whether children who did not grasp numeral invariance made errors that differed from those errors made by children who did have this understanding (perfect score on the spatial task). The data were first analyzed to determine whether these two groups of children differed in their knowledge of what type of operation was required on the addition-subtraction task.

When examining children's errors the interpretation to be given to a child's wrong judgment on a correct transformation was somewhat ambiguous. For instance, a "no" judgment on a 4-to-6-adding trial (interviewer asks puppet to change set of 4 into a set of 6 and puppet adds objects) could mean that the child thought the type of operation was wrong, or it could mean that the child just thought that the wrong number of objects were added or subtracted. Therefore, it was decided to look only at incorrect judgments on trials in which the wrong (inverse) operation was made. If a child judged that a wrong transformation (for instance, 7-to-5-adding) was correct he or she had to be saying that the wrong operation was correct (i.e., that one could add objects to get from a set of 7 to a set of 5).

Table 6

Cross-tabulation of Number of Correct Answers on The Spatial Task With the Addition-Subtraction Task

Spatial Task Scores	Addition-Subtraction Task Scores 0 to 2	3 to 8	9 to 14	15 to 16
0 to 2	0	0	0	0
3 to 8	0	0	4	0
9 to 14	0	1	12	0
15 to 16	0	0	10	3

Table 7 contains the number of children giving "yes" answers on trials with wrong transformations - transformations in which the operation was opposite to that needed for the requested change in value. Eight of the 16 trials involved operations which were contrary to the required operation. On these trials the interviewer either added objects when a subtraction was required or subtracted objects when an addition was required. These data are shown for those children who scored perfectly on the spatial task ("Numeral Invariance") versus those who did not ("No Numeral Invariance"). Since children could give "yes" responses on more than one of these trials the cells in the table are not mutually independent.

Looking at the performance on the addition-subtraction task of those 11 children who grasped numeral invariance several observations can be made. On small sets there were no numeral invariance children who accepted the opposite operation as being correct, and even for large sets only a few children made such errors. On both the 7-to-5-adding and the 8-to-6-adding transformations only 1 child judged that they were correct. The 7-to-9-subtracting and 8-to-10-subtracting transformation proved more difficult with 3 children judging them correct. Thus, knowledge that the numerical value cannot change without adding or subtracting did not always imply knowledge of what kind of operation was appropriate. It seems that these children had not sorted out the relationships among these larger values.

Table 7

Number of Numeral Invariance

and No Numeral Invariance Children

Making "Yes" Judgments on

Trials with Wrong Transformations

	Type of Transformation and the Original and Goal Values							
	Small Sets				Large Sets			
	Adding		Subtracting		Adding		Subtracting	
	3-1	4-2	3-5	4-6	7-5	8-6	7-9	8-10
Numeral Invariance	0	0	0	0	1	1	3	3
No Numeral Invariance	0	1	0	1	7	9	10	13

Note. The mean age of the Numeral invariance children was 46.9 months (SD=4.0) and the mean age of the No Numeral Invariance children was 47.5 months (SD=4.2). There were 11 children who were in the Numeral Invariance category and 19 in the No Numeral Invariance category. Since children could respond "yes" to more than one of the above trials the cells of the table are not mutually independent.

The performance of the 19 children who did not exhibit numeral invariance also revealed several interesting findings: (1) Very few children judged that the small set transformations were correct. No child judged that the change from 3 to 2 could be accomplished by adding and only 1 child judged that the change from 4 to 2 could be accomplished by adding; no children judged that the change from 3 to 5 could be accomplished by subtracting and only 1 child judged that the change from 4 to 6 could be accomplished by subtracting. (2) On the large set trials many children made the erroneous judgments that one could go from a larger numerical value to a smaller numerical value by adding, or that one could go from a smaller numerical value to a larger numerical value by subtracting. Of the 19 "non-quotite" subjects 7 judged the 7-to-5 transformation could be accomplished by adding; 9 judged the 8-to-6 transformation could be accomplished by adding; 10 judged the 7-to-9 change could be accomplished by subtracting; and 13 judged the 8-to-10 change could be accomplished by subtracting. These data suggest that there exists some uncertainty for many of the non-quotite children regarding the quantity relationships between numbers, even those numbers less than 10.

Summary

The first set of analyses concerned several hypotheses regarding the influence of two task variables, set size and the relative values of the original and goal set. The results indicated that as hypothesized children did make more correct judgments on the spatial transformation task for those trials in which set size was small. However, the hypothesis that children would make more correct judgments when the numerical value was increasing was not confirmed. In fact when set size was small children fared better when the value was decreasing. One explanation for this better performance is that the goal value in these trials was always 1 or 2. The spatial transformation was performed in full view thus allowing the child to make inferences about change in value based on a visual intuition. Since an array of 2 objects is easily identifiable it is possible that children were visually subitizing or rapidly counting the arrays allowing them to judge that the quantity left after the spatial relocation was not 2. It is also possible that children have a better understanding of cardinal number for smaller set values than for the larger values and this understanding includes invariance rules.

On the addition subtraction task children again made more correct judgments on the small set trials in which the original- to-goal relationship was decreasing. Since in this task the array was hidden from view there was no possibility of children visually subitizing the arrays after elements were added or subtracted. The only criterion for evaluating change in value was the type of transformation that they observed, suggesting that children's more accurate judgments on small sets was due to their greater understanding of the numerical relations for small sets.

The second set of analyses concerned the patterns of errors that children made on the two types of tasks. For the spatial task it was hypothesized that the pattern of children's judgments would suggest one of the following forms of reasoning: (1) spatial relocation did not change the numerical value (2) spatial relocation changed the value of the set in some way that was consistent with an evaluation based on either length or density (3) spatial relocation changed the numerical value but without any consistent criteria for judging which type of transformation would affect the value in which ways. It was hypothesized that many of the 3-year-olds would not have established the relationship between differences in numbers and quantitative change and therefore would be in this last group. Many 4-1/2-year-olds, however, would understand that changes in numerical values require changes in quantity, and perceiving the spatial changes as changes in quantity, would also think that the numerical value changed. The results did not bear out these hypotheses. Although the judgments of a majority of the children indicated that they thought that spatial relocations could change numerical value, they did not tend to use either length or density consistently as a criterion for making their judgments.

The fact that a large percentage of the subjects did think that spatial relocations could result in numerical change is consistent with many other studies on conservation but somewhat at odds with Saxe, Cohen and Rindskopf (1980) study in which "numeral invariance" was found even among preschoolers. It was somewhat surprising that there were not more children who conserved the numerical value over the transformation. Saxe et al. (1980) found that even though subjects did not conserve number on the classical Piagetian task, they did make judgments that indicated a knowledge of numerical invariance - that the numerical value of a set did not change with a spatial relocation. Apparently many of the subjects in this study, did not fully understand numeral invariance.

These findings also put into question Gelman and Gallistel's contention that children at this age can reason effectively about cardinal number when sets are given definite numerical values. In fact most of the children in the present study could not reason in such a way and gave many judgments that implied they did not understand invariance rules.

The final set of analyses were of children's performance on the spatial task versus the addition-subtraction task. Though not demonstrating a clear order of acquisition of the two concepts, the results did suggest that children with numeral invariance were more advanced than those children without numeral invariance in understanding addition and subtraction operations at large set sizes. The performance of both those children understanding invariance and those not understanding invariance indicates that for small sets children know what operation is needed to go from a larger to a smaller number and from a smaller to a larger number. However, at larger set sizes even those children who did grasp numerical invariance on the spatial task were not always clear as to the effects of addition and subtraction operations. Some of these children did not properly judge whether an addition or subtraction operation was required to achieve the goal value. Thus, though children may recognize that <u>some</u> operation is necessary in order to change numerical value they do not seem to be clear on the relative magnitues of the two numerical values. This finding is consistent with research by Morf (1962) who found that even some 5- to 7-year-olds did not understand the "connexity" of the series of whole numbers generated by the iterative addition of one unit. However, there are also research findings indicating that children in this same age range can make local inferences of "one more" for a particular number in the whole number series (Ginsburg & Russell, 1981; Walters, 1983).

Notes

1. A profile analysis assumes that 2 independent random samples of individuals have been administered a battery of tests with the responses to those tests being expressed in compatible units (Morrison,). Three questions are asked about the population frofiles of the samples:

1. Are the population mean profiles similar, in the sense that if graphed the line segments of adjacent tests are parallel.

2. If 2 population profiles are indeed parallel, are they also at the same level?

3. Again assuming parallelism, are the population means of the tests different?

Question 1 refers to the hypothesis of no response by group interaction, while question 2 addresses itself to the hypothesis of equal group effects. Because the tests for equal levels and response effects have no meaning if a group response interaction is present, question 1 has been accored priority among the tests. Question 3 is treated by a repeated-measures technique.
The profile analysis in this study was done using the EMLIN computer program.

CHAPTER III

Study 2

Method

Design

The experimental design was a 2 x 2 x 2 repeated measures factorial design consisting of 1 between- and 2 within- subject factors. The between subject factor was age with 2 levels (3- and 4-1/2-year-olds). The within-subject factors were set size (large and small) and direction of transformation (smaller numerals to larger numerals, and larger numerals to smaller numerals).

Subjects

Subjects were 30 preschool children, 15 each of 3- and 4 1/2 year-olds selected from middle class nursery and elementary schools in the New York area. The range of ages for the 3-year-olds was 3-3 to 3-11, with a mean age of 3-6. The range of age for the 4-year-olds was 4-6 to 4-11, with a mean age of 4-7.

Materials

Materials included a puppet which fit over the hand of the interviewer and color drawings of fruits and vegetables mounted on cards (4cm x 4cm). There were enough pictures of each type of fruit and vegetable to form 8 sets of 10 items.

Procedures

All children were asked to count to 10. Those unable to do so were not included in the study. Four children were eliminated from the study on the basis of this screening. Children were introduced to the task in the following way:

> I have some baskets full of fruits and vegetables.
> I need to deliver them to the store but
> the store owner wants a certain number in each basket and
> my baskets doesn't have the right number.
> I'll show you some baskets with some
> fruits and vegetables already in them.
> I'll tell you how many fruits or vegetables
> the store owner wants in each basket.
> I'd like you to fix them so they have the
> right number.

The interviewer then placed a set of one type of fruit or vegetable cards on a board upon which a basket was drawn. The interviewer told the child how many there were in the basket and how many he wanted there to be. For example: "Here is a basket of bananas. There are 3 bananas here.

The store owner wants 5 bananas in the basket. Can you fix the basket so there are 5?" The child was allowed to manipulate the materials however he or she wished. The child's judgments and comments were recorded by the interviewer on a coding sheet. Children were also given a pile of 8 pictures additional to those in the array in order to allow him or her to add objects to the basket.

Children received 4 trials using small set sizes and 4 trials using large set sizes. Under the small set size condition the child was asked to make the following transformations: 3 to 5, 5 to 3, 4 to 6, 6 to 4. In the large set size condition the child was asked to make these transformations: 7 to 9, 9 to 7, 8 to 10, 8 to 6.

Children's behaviors were recorded on a coding form that had been developed from observations of video tapes made during a pilot study. Each manipulation including the number of objects added or subtracted and any counting that took place was recorded as well as any comments that the child made. The child's performance on a trial was considered ended when the child verbally indicated that he or she was done, or when the child responded "yes" to the question, "So is this y (the goal value)?".

Children's accuracy was recorded as the difference between the number of pictured objects requested by the interviewer (the goal value) and the actual number of pictured objects the child had on the table.

Results and Discussion - Study 2

The purpose of the following analysis is to describe developmental differences in children's performance as revealed by their solutions to the various tasks. The analyses are divided into three parts. The first part concerns children's accuracy on the tasks. The goal of this analysis was twofold. First, to determine the effects of set size and the direction of transformation on children's accuracy. These analyses test hypotheses that children would be more accurate at establishing the goal value when sets were small, confirming previous research on the influence of set size, and when the required numerical change was from smaller values to larger values, a prediction arising from research which suggests that children's numerical reasoning at this age was still wedded to the forward direction of the counting sequence.

The second set of analyses is a description of the varieties of solution strategies that children produced. First, an overview of strategy types is presented. These strategies are described in terms of developmental differences in the degree to which children have quantified the numeration system. Using this taxonomy of children's strategies an analysis is presented of whether children at these ages differentiate change in numerical value from change in spatial extent. In study 1 it was found that the majority of children failed to make this differentiation when judging whether a spatial transformation changed number. The present study allowed us to observe whether children would spontaneously change the spatial extent of the arrays in an attempt to change the numerical value. A second analysis focused on the effect of set size and the relationship between the original and goal values on the use of strategies.

Finally, another analysis of strategy use is offered but this time focusing on children's "first move" in attempting a solution. Such an analysis offers a clearer idea of what children's intuitions are about the relationship among cardinal numbers. The analysis of these first attempt strategies parallels the analyses of the overall strategies, assesing the influence of set size and the direction of transformation.

<u>Children's Accuracy</u>

It was predicted that children would be more accurate in arriving at the correct goal number on the small set size trials than on the large set size trials, and children should be more accurate when the numerical value of the goal set was greater than the original set, than when the goal value was less than the original value. Children were scored for accuracy on each trial; accurate responses were counted as 1 and inaccurate responses as 0. The mean accuracy scores of 3- and 4 1/2-year-olds are presented in Table 8. Because children received all of the task conditions a profile analysis was performed. This analysis indicated that there was a significant main effect only for age with 4 1/2-year-olds receiving higher scores than 3-year-olds, $F(1,28) = 57.76$, $p < .05$). Clearly 3-year-olds are not very successful at producing accurate sets on this task as indicated by the range of scores in Table 8. Contrary to the hypotheses there were no significant differences between children's performance on small versus large set size trials nor on the tasks with increasing values versus tasks with decreasing values.

These findings regarding set size suggest that the ability to perform the task is not mastered gradually first for small sets and then for larger. This finding is contrary to the results of study 1 which suggested that children do master certain relations for small sets first. It may be that this task in which children are allowed to engage in manipulating the objects allows children to arrive at the accurate value despite the uncertainty they have about the relations between the larger numbers. They do not have to make judgments about transformations that they see happen rather quickly; instead they can engage in various trial and error strategies that can result in an accurate response.

Table 8

Mean Accuracy Scores

Age (Years)		Small Sets: Smaller to Larger	Small Sets: Larger to Smaller	Large Sets: Smaller to Larger	Large Sets: Larger to Smaller
3	Mean	.27	.53	.27	.33
	(SD)	(0.59)	(0.83)	(0.59)	(0.62)
4 1/2	Mean	1.60	1.93	1.40	1.53
	(SD)	(0.63)	(0.26)	(0.83)	(0.83)
Total	Mean	.93	1.23	.83	.93
	(SD)	(0.91)	(0.94)	(0.91)	(0.94)

Note. Maximum score = 2.0

Overview of Children's Solution Strategies

In order to determine if there was some developmental pattern in children's strategy use the typology of strategies described below was developed and each child's performance was coded for each task. The codes were devised in a pilot study in which children's performance on each trial was recorded on video tape. The experimenter and an observer trained in the use of the strategy codes viewed the tapes of 8 subjects and coded their performance. The interrater reliability on the 64 trials coded was 83%.

Strategies were grouped into 3 categories reflecting the level of children's quantitative understanding of the numeration system. Strategies at the first level described as "unquantified" reflected an understanding in which children did not relate the numbers to each other with any quantitative meaning. This level was characterized by either a random adding or subtracting of objects, counting the objects without attempting any transformation on the sets, or by spatially relocating the elements without adding or subtracting. At the second level, referred to as "partially quantified", children's strategies reflected the development of some quantitative understanding. Children understood general relations of magnitude between larger and smaller numbers, but they did not understand the unit increments between numbers. The strategies at this level were characterized by trial-and-error strategies in which the child generally used an operation that was in the right direction (i.e., adding to arrive at a higher value and subtracting to arrive at a lower value) but were not accurate in their attempts. Thus strategies at this level implied that children knew that certain numbers were greater or smaller than others but did not understand this relation in terms of unit differences. Children at the highest strategy level understood this unit relation. These strategies consisted of various types of counting-all or counting-on in which the child arrived at the accurate value for the set. Using this model each of the strategies was classified into one of the 3 levels as follows:

Pre-quantified Strategies

1. No attempt was made to change the array, or the array was only spatially rearranged.

2. Count, no action.
   Child counted the array and then stopped. Made no manipulation.

3. Action, no count.
   Child made an addition or subtraction and then stopped. The addition or subtraction was inaccurate.

4. Action with counting, stop. Operation was not accurate.

5. Trial and error. Lack of consistency between operations and goal.

5a. Objects were added or subtracted in a manner inconsistent with the original to goal relationship. In other words objects were subtracted in going from a lesser to a greater numerical value (original goal) or added in going from a from a greater to a lesser numerical value (original goal).

5b. Both consistent and inconsistent additions and subtractions took place within the trial.

5c. Additions or subtractions were inconsistent with the original-to-goal relations. Child counted those added or subtracted.

Partially quantified

6. Take away and start over. Child took cards away, then counted the cards as placing them out until the goal value was reached.

   6a. Child counted from "one" as objects were subtracted.

   6b. No counting of those objects taken away.

7. Child added all the available cards and then counted up to the goal value.

Child added all the available cards (i.e., whatever was in the "extra" pile that was given to the child in addition to the original array). Sometimes these cards are counted as they were put out. Next the child counted the new array until the goal value was reached in the count, then removed all extra cards.

8. Trial and error with counting-all. Child added or subtracted a few objects making a kind of approximation. Then the child counted the new array and adjusted the number. This count was usually a count of all the objects starting with "one" (i.e., a count-all after a manipulation). This trial and error strategy had several variations:

    8a. The addition or subtraction was consistent with the original to goal relationship. In other words, objects were added in going from a lesser to a greater numeral (original $<$ goal) and subtracted in going from a greater to a lesser numeral (original $>$ goal).

    8b. Child's addition or subtraction was consistent with the original-to-goal relationship, and the child counted those taken away or added.

<u>Quantitative strategies</u>

9. Counting-all strategy: Counting-all strategy is used on first attempt. This was accomplished in several ways.

9a. If the original was greater than the goal then the child counted up to the goal and removed extra objects.

9b. If the original was greater than the goal the child counted the entire array and then counted backwards as removing objects.

9c. If the original was less than the goal the child counted the entire array and then added on objects while continuing the count; e.g., in going from 4 to 6 the child counted the original 4 and then added 2 saying "5,6".

10. Counting-on strategy.

The counting-on strategy was used on the first attempt. This was accomplished in several ways:

10a. If the original was greater than the goal the child counted backwards from the original until arriving at the goal. The child took away one object with each numeral recited.

10b. If the original was less than the goal the child did not count the whole array but began with the original and added on objects as counting up from the original ("n,n+1,n+2,...").

Children's Use of Strategies.

The number of children using each of the strategies described above is presented in Table 9. The most prevalent strategy that children used was trial and error in which they added objects to or subtracted objects from the array and then proceeded to count in order to check their actions. Often children seemed to have little idea of whether to add or subtract objects on their initial attempts as well as after counting. Prequantified strategies 5a., 5b. and 5c. characterize children whose trial and error approach showed some inconsistency with the actual relations between the numbers. Partially quantified strategies 8a., and 8b. indicate those children who use this trial and error approach with some kind of consistency.

Children used several forms of counting-all or counting-on strategies. A counting-all strategy involved counting all of the objects present in the array and either adding on the extra objects needed if it was required or subtracting objects if there were too many. For instance, in going from a set of 4 to a set of 6 a child would count "1,2,3,4" and then seeing the count was short of the 6 needed, would add on two saying "5,6". If the goal set was smaller than the original set (e.g., the goal was to go from a set of 9 to a set of 7) the child would count up to the goal value, "1,2,3,4,5,6,7" and remove the 2 extra. The first strategy was used by 7 of the children, while the second of these strategies was also used by 7 children.

The most advanced strategy, "counting-on", was used by only 3 of the children and only once by each of them. In counting-on the child begins with the numerical value of the set and then begins to count-up or down from that point. For instance, in going from 4 to 6 the child would say "4" and then adding 2 would say "5,6". It seemed surprising, actually, that this strategy was observed so infrequently. Apparently children at this age are still very much wedded to the process of counting and cannot isolate a value from the string of numbers.

Children's use of spatial transformations as a means of achieving the goal value.

In Study 1 it was found that the majority of children confused changes in the spatial extent of the array with changes in number. The present study permitted an analysis of whether children would spontaneously change the spatial extent of an array in order to change the numerical value.

The frequency distribution presented in Table 9 shows that although there were several children who used strategies in which they did not subtract or add any objects, most children consistently used strategies in which their manipulations involved additions or subtractions. There were a number of children who used strategies in which they did not use counting at all. Strategy 3 in which children performed some action on the set and then stopped without counting was used by 22 children on at least one trial, and 4 of these children used such a strategy exclusively - they used no counting on any trials. Children who did use counting, however, did not always proceed to add or subtract objects to the set. Whatever meaning these children gave to their counting it was not used as a problem solving strategy. Only 2 of the children changed the spatial extent as a way of transforming the original set into a set of the goal value. One child used this type of strategy once, and another child used this strategy over all of the 8 trials. Thus when left to their own devices very few children spontaneously considered spatial change as a way to effect numerical change.

Table 9

Frequency of Overall Strategy Use

Strategy	Number of Trials on Which Strategy Was Used *	
	1 to 2	3 or More
Prequantified		
1. No attempt	1	1
2. Count,no action	2	4
3. Act,no count	11	11
4. Act with count,stop.	3	
5a.Trial-error,inconsit.	5	1
5b.Trial-error,consis and inc.	4	
Partially quantified		
6a.Take away	1	
6b.Take away - no count	1	
7. Add all	1	
8a.Trial-error,consis	12	8
8b.Trial-error,counts	2	
Fully quantified		
9a.Count-all,remove	5	2
9c.Count-all	5	2
10. Count-on	3	

Influence of set size and direction of change in value on strategy use.

Several predictions were made regarding the influence of age, set size and direction of change on strategy use. It was predicted that (1) 4-1/2-year-olds would use advanced strategies more frequently than 3-year-olds (2) both 3- and 4-1/2-year-olds would use more advanced strategies on those trials in which the set size was smaller than large (3) that both 3- and 4-1/2-year-olds would use more advanced strategies on trials in which the goal value was greater than the original value versus trials in which the goal value was smaller than the original value.

To test these hypotheses children were scored 0 for a prequantified strategy, 1 for a partially quantified strategy and 2 for a fully quantified strategy. Table 10 presents the mean overall strategy score for each of the 4 types of trials. A profile analysis found that there was a main effect only for age with 4 1/2-year-olds scoring higher than 3-year-olds, $F(1,28) = 15.04$, $p<.05$. One should note, however, that the scores for both 3 and 4 1/2-year-olds are quite low. In no condition did the mean score for 4-1/2-year-olds even reach 2.0, a score that would signify that children were on the average scoring 1.0 (partially quantified) on each of the 2 trials in that condition. The hypotheses regarding set size and direction of the original to goal value were not confirmed.

Table 10

Mean Overall Strategy Scores of 3- and 4 1/2-year olds.

		Small Sets		Large Sets	
Age		Smaller to Larger	Larger to Smaller	Smaller to Larger	Larger to Smaller
3	Mean	.60	.40	.33	.47
	(SD)	(1.12)	(0.74)	(0.72)	(0.83)
4 1/2	Mean	1.60	1.60	1.73	1.53
	(SD)	(1.12)	(1.30)	(1.34)	(1.36)
Total	Mean	1.10	1.00	1.03	1.00
	(SD)	(1.21)	(1.20)	(1.27)	(1.23)

Note. Maximum score = 4.0

Developmental shifts in children's strategies.

To gain a better picture of some of the developmental shifts in children's strategy use an analysis was made of changes in the prevalence of different strategies for the two age groups. Table 11 contains a frequency count of the highest level of strategy that 3- and 4-1/2-year-olds used over the 8 trials. Collapsing the data in Table 11 into those children whose highest strategy were not quantitified versus those whose highest strategy were fully quantified indicate that there were a significantly greater number of 4-1/2-year-olds using fully quantified strategies than there were 3-year-olds using fully quantified strategies, (1) =16.66, p .05. Though most of the 4-1/2-year-olds did use the fully quantified strategy at some time, the majority of children overall reached either prequantified or partially quantified strategies as their highest level.

Table 12 shows the number of children whose highest scores were at the prequantified, partially quantified and fully quantified level as a function of set size. A comparison across age groups of the distributions of children's strategy use suggest that set size had no effect on the highest level of strategy used.

Table 11

Frequency of Highest Strategy Use By Age

Age (Years)	Highest Strategy Used Prequantified	Partially Quantified	Fully Quantified
3	8	6	1
4 1/2	1	2	12

In order to determine if set size has an effect on strategy use contingency tables were created by cross multiplying set size by strategy level. The 4 categories of response were as follows: fully quantified on both small and large sets, not fully quantified on both small and large sets, not fully quantified on small sets and fully quantified on large sets, and fully quantified on small sets and not fully quantified on large sets. A series of chi-square analyses on these contingency tables showed no effect for set size. Referring to Table 12, it is interesting to note that even when set size is small only one 3-year-old used a fully-quantified strategy on any of the trials and that most 3-year-olds were at the prequantified level. It was only among the 4-1/2-year-olds that children began to use the fully quantified strategies with any regularity.

Table 13 contains frequency counts of children's highest level of strategy use as a function of age and the direction of numerical change. As with set size the pattern of children responding at the 3 levels is similar for both types of numerical change suggesting that the direction of transformation has little effect on children's use of strategies.

Table 12

Frequency of

Highest Strategy Use as a

Function of Set Size and Age

Age (Years)	Highest Strategy Used Pre-Quantified	Partially Quantified	Fully Quantified
3			
Small sets	9	5	1
Large sets	10	5	0
4 1/2			
Small sets	2	7	6
Large sets	2	5	8

To test this a series of contingency tables similar to those described in the previous analysis of set size was created by multiplying children's strategy level (fully quantified/not fully quantified) by the two directions of numerical change (small to large/ large to small). Again, a series of chi-square analyses of these tables showed no significant differences in children's strategy use due to set size.

First attempt Strategies

The previous analysis focused on children's overall strategy on each trial in changing the original value to the goal value. Children's strategies on their "first move" in modifying the original array were also analyzed in order to provide some measure of children's intuitive understanding of the operation relating the numerical values of the original and goal sets.

The following coding scheme was derived from the same observations used by the interviewer in recording the "overall" strategies described above. As with the overall strategies these codes were subclassified into 3 categories: prequantified, partially quantified, and fully quantified.

Table 13

Frequency of

Strategy Use as a

Function of Numerical Change and Age

Age (Years)	Highest Strategy Used		
	Pre-Quantified	Partially Quantified	Fully Quantified
3			
Small to Large	9	5	1
Large to Small	10	5	0
4 1/2			
Small to Large	1	6	8
Large to Small	2	6	7

Prequantified

1. No attempt to transform set, piling up objects or a spatial relocation without adding or subtracting.

2. Counted to determine the number in the set without acting on the set as in a count-all or count-on.

3. Added or subtracted objects inconsistent with the original-to-goal relationship. For example subtracting 1 when transformation was supposed to be from 6 to 8.

4. Counted objects that were added or subtracted,and the operation was inconsistent with goal. For example, the child subtracts 3 objects while counting them "1,2,3" when asked to make a larger set.

5. Added all the objects made available by the interviewer.

6. Subtracted all objects in the array.

Partially-quantified

7. Added objects to or subtracted objects from the array, the operation was consistent with the relation between the original and goal set but inaccurate.

8. Counted objects added or subtracted to the array, operation consistent with the goal.

Fully quantified

9. Count-all to goal. For example a child asked to make a set of 3 into a set of 5 counts the set "1,2,3" then adding more objects counts "4,5".

10. Count-on to goal. A child asked to make a set of 3 into a set of 5 would start counting with "3" and adding 2 say "4,5".

11. Added or subtracted objects and was accurate without counting.

Developmental shifts in children's first attempt strategies.

As mentioned before it was thought that the previous analysis of children's overall strategy scores, which were derived from a description of the child's performance over an entire trial, might have masked children's first intuitions of what operation was necessary to go from the original set to the goal set. In other words children may not have a good first intuition about whether an addition or subtraction was required but, over the course of a trial the child might have picked up information which would result in a strategy score that implied a more advanced understanding. Or, conversely, children may have had a good initial understanding of the relationship between the original value and the goal value but over the course of the trial became confused and used low level strategies. To determine what the nature of these first move strategies were children were scored 0, 1, and 2 on each trial for using prequantified, partially quantified and fully quantified first attempt strategies respectively. Table 14 contains children's mean first attempt strategy scores. A profile analysis of these scores found that there was a main effect for age with 4 1/2-year-olds scoring higher than 3-year-olds. Thus the older group certainly reflected a greater understanding of what operation was needed to arrive at the new set as evidenced by their first move towards a solution to the task. Set size did have some effect on performance. When children were asked to create a set of smaller value they were better able to do this where set size was small, Hotelling T^2 (1,28) = 16.00, $p < .05$. This finding differs from the analysis of both accuracy and the use of overall strategies. These analyses found that small sets did not result in children being more accurate nor in their using more advanced overall strategies. It appears that when small sets are used, however, children did have a somewhat better notion of what first move to make in solving the problem. It is also interesting that when children's mean scores on their first attempts are compared to the overall strategy scores (Table 10) the first attempt scores are generally higher. Again this suggests that children's first intuitions about the relations often reflect a more advanced level of quantitative thinking than is reflected in the series of trial and error attempts they make over the course of the trial. It may be that in coming to work out the problem children become confused and make mistakes that are reflected in their overall strategy scores.

As with the overall strategies, in order to get a better picture of the developmental shifts in children's strategy use an analysis was made of changes in the prevalence of different strategies for the 2 age groups. Table 15 contains the frequencies of children's highest strategy use. The distribution indicates that all but 1 of the 4-1/2-year-olds are scoring at the fully quantified level while only one-third of the 3-year-olds are scoring at this level. A chi-square analysis of scores collapsing over the prequantified and partially quantified strategies indicated that there was a significant difference between the number of fully quantified scorers across ages, X^2 (1) = 12.92, $p < .01$.

Tables 16 and 17 contain respectively the frequencies of children's highest first move strategy use as a function of the set size and the direction of numerical change (increasing value or decreasing value). There were a significantly greater number of children using fully quantified strategies for the smaller sets, as indicated by a chi-square analysis of 3-year-olds using non-quantified versus quantified strategies on small versus large sets, $X^2(1)=6.0$, $p < .05$. No significant differences were found in performance due to the direction of numerical relation.

Table 14

Mean First Move Strategy Scores as a Function of Set Size and Age

		Small Sets		Large Sets	
Age (Years)		Smaller to Larger	Larger to Smaller	Smaller to Larger	Larger to Smaller
3	Mean	1.07	1.00	0.53	0.33
	(SD)	(1.28)	(1.36)	(0.83)	(0.72)
4 1/2	Mean	2.87	3.47	2.53	2.27
	(SD)	(1.06)	(0.92)	(1.73)	(1.58)
Total	Mean	1.97	2.23	1.53	1.30
	(SD)	(1.47)	(1.70)	(1.68)	(1.56)

Note. Maximum score = 4.0

Table 15

Frequency of Highest First Move Strategy Use

As a Function of Age

Age (Years)	Highest Strategy Used		
	Pre-Quantified	Partially Quantified	Fully Quantified
3 years	4	6	5
4 1/2 years	0	1	14

Table 16 indicates that none of the 3-year-olds used a first move strategy that demonstrated a fully quantified understanding of the relationship between the original and goal set when the set size was large, though almost half did indicate some understanding of the direction of the change in value. The large majority of 4-1/2-year-olds did have an intuitive notion of what operation was needed to achieve the goal value as evidenced by the fact that 12 children in this age group used the fully quantified strategy at least once. These findings suggest that it is not until after 4 years of age that most children begin to grasp the connection between differences in number values and the operations that these differences imply. It is possible that children have a general understanding of the relations between the numbers but lack strategies that allow them to accurately produce the new value. The existence of this kind of "directional" understanding in 3-year-olds is supported by the fact that nearly half of the 3-year-olds use some partially quantified strategy on the large set trials.

Table 16

Frequency of

Highest Strategy Use as a

Function of Set Size and Age

	Highest Strategy Used		
Age (Years)	Pre-Quantified	Partially Quantified	Fully Quantified
3			
Small	4	6	5
Large	8	7	0
4 1/2			
Small	0	3	12
Large	1	2	12

Table 17

Frequency of

Highest Strategy Use as a

Function of Numerical Change and Age

	Highest Strategy Used		
Age (Years)	Pre-Quantified	Partially Quantified	Fully Quantified
3			
Increase	6	7	2
Decrease	7	4	4
4 1/2			
Increase	0	3	12
Decrease	0	3	12

Summary

This second study was designed to assess children's knowledge of the relations between numbers as reflected in their ability to change a set of a given value into a set of another given value when the children made the transformations.

The findings indicated that children's accuracy on this task was a function of age. Three-year-olds were not very accurate on the task with a majority of them arriving at solutions that were 2 or more off from the correct answer. If this task is accepted as a measure of children's ability to compare cardinal number, 3-year-olds seem to have a poor grasp of these relations. The hypotheses that were made regarding the influence of set size and the direction of numerical change were not supported. Working with small sets did not appear to make the task any easier nor was there a difference in children's success rate when the numerical change was increasing versus decreasing.

In addition to an analysis of children's accuracy, a description was given of the strategies that children used in attempting a solution. Strategies were classified as either prequantified, partially quantified, or fully quantified. Among the partially quantified strategies the most frequently used was trial-and-error, and the most sophisticated fully quantified strategy, counting-on, was rarely used even by 4-1/2-year-olds. It was also noted that very few children attempted to change the value of the set by merely rearranging the objects.

There were clear developmental shifts in children's strategies from age 3 to age 4-1/2. Four-and-one-half-year-olds tended to use higher level strategies. Even when sets were small most 3-year-olds used prequantified strategies. Neither set size nor the direction of numerical change affected children's use of strategies. However, when children's first moves at solutions were analyzed, set size did influence the level of children's strategy use. It was suggested that for small sets children do have accurate intuitions about the proper operation that is required but this intuition is masked when children are allowed full manipulations of the materials.

CHAPTER IV

General Discussion and Conclusions

The present study was designed to assess preschool children's knowledge of cardinal number. The focus of the study was on children's ability to make comparisons of numerical values. Children were asked to judge what numerical operations were needed to go from a set of a given value to a set of another given value. In the first study children were asked to judge whether certain additions, subtractions and spatial relocations carried out by the experimenter changed the numerical value of that set. In the second study children were asked to themselves transform a set containing a given number of elements into a set containing another given number.

By using these tasks the present study was able to assess children's ability to make cardinal number comparisons in terms of the operations relating numbers. Most previous studies of children's ability to make numerical comparisons have required children to judge the relative quantity in two sets of objects and usually without specifying to the child the numerical value of the arrays (e.g., Beilin, 1968; Siegler, 1981). In addition the design of the study provided information about certain factors which influence children's ability to make these judgments and comparisons. Set size was manipulated in order to determine if children could reason about small sets before they extended these capacities to larger sets. The required operation was manipulated in order to determine if children had more facility with using numeration in the forward direction than in the backward direction.

The study addressed the relative validity of two differing positions regarding what preschool children know about representing cardinal number. One view suggests that children know very little about the use of numeration and counting to represent and reason about number because they lack a basic understanding of the definition of cardinal number. This view takes the position that until children can define cardinal number in terms of one-to-one correspondences between sets they do not really understand cardinal value. The other position suggests that the capability to use counting to represent and reason about number is possessed by even preschool children. According to this view skills such as counting to determine the value of a given set and making accurate comparisons of numerical values must reflect some understanding of cardinal number.

The position taken in the present study is that a full understanding of cardinal number involves several different types of knowledge and skills. Though preschool children do have some understanding of cardinal number and possess skills that allow them to reason about number, this understanding is limited, particularly in the extent to which children understand the relationship between numerical operations and differences in cardinal number. The present study was designed to discover whether children of this age (1) know that the numerical value of a set does not change unless an addition or subtraction occurs (numeral invariance) and (2) can determine those changes in numerical value that require an addition and those that require a subtraction. In addition the study provided data regarding the order in which these two skills are acquired. The analysis of children's performance on these numerical problems also offered some description of developmental shifts in children's use of problem solving strategies. Following is a discussion of the findings as they relate to each of these issues, as well as a discussion of children's performance in terms of recent information processing models.

Knowledge of Numeral Invariance

Study 1 provided data regarding children's ability to distinguish operations that changed numerical value from those that did not. Previous research findings beginning with Piaget's well known method for studying number conservation have found that children younger than 5 or 6-years-old can often be led, through misleading perceptual cues, into making judgments about quantity based on spatial extent rather than actual number. Although there is research showing that for small sets even very young children don't have such confusions (Gelman, 1978), most studies (e.g., Beilin, 1968; Siegler, 1981), in which many combinations of transformations, set sizes and equalities were used, have found that the youngest children tended to base their judgments on spatial extent and only later base their judgments on the type of transformation. Piaget (1968) claimed that the reason for this finding is that evaluation by length is based on "an ordinal quantification which is already a concept, for moroe complex and general than the experiment on number alone would lead us to suppose" (p.976).

The spatial transformation task used in Study 1 of the present research was different from the previous methods in two ways. First, children were asked to judge whether the numerical value of a set was changed to another given value as the result of certain transformations, rather than being asked to judge whether the quantity per se was changed as in most previous studies. Children's knowledge of the symbolic meaning of the words used in counting and not their knowledge about numerical relations more generally was of interest. Secondly, children's judgments regarding numerical change were in reference to the change in numerical value of a single set rather than a comparison of two sets. It was thought that presenting children with only one set might limit the perceptual stumbling blocks involved in comparing two sets, as well as simplifying the cognitive demands of the task. Klein (1984) has made a useful distinction between three types of problems that have been used to investigate children's arithmetic abilities. The first type of task requires only directional reasoning about addition and subtraction operations. That is, whether adding yields more and subtracting less. The second type of study requires an integration of directional reasoning with the initial quantitative relationship between two sets. A third type of study requires reasoning about reciprocal relations between joint addition and subtraction operations. The present study simplified the task facing the child to judgments about the change in numerical value of only a single set of objects. The goal was to look at children's knowledge of directional relations but in terms of numerical values rather than judgments of more or less. Previous theory and research has suggested that when children first begin to gain the capacity to quantify number their logic can be cahracterized as functions (Klein, 1984; Piaget, 1968; Piaget, Grize, Szeminska & Vinh Bang, 1977). In this type of thinking every action is a series of ordered movements which come to some end (in the sense of a certain distance). It is these ordered movements that dominate the young child's considerations when evaluating length and when making other quantifications such as number. The task in Study 1 can be considered as a method of determining to what extent these one-way functions are mapped onto children's understanding and use of the numeration system.

The findings of the present research support previous findings that most 3-and 4-1/2-year-olds judge that spatial relocations can change numerical values. Most children in

Study 1 were willing to judge that a set of objects of a given value could be changed into another given value merely by extending or shortening the length of the array. It was also found that children were best able to judge that the array's value was not changed when the set size was small. The finding regarding set size is consisitent with several studies that have demonstrated that for small set sizes even very young children can conserve number (Cowan, 1979; Gelman, 1972; Winer, 1974; Zimiles, 1966).

The responses children gave on the spatial transformation task were not consistent, however, with some previous investigations which found that children consistently base their numerical evaluation on length or density (Beilin 1968; Piaget, 1952,1968; Siegler, 1981). One possible explanation for this was that with only one row of objects being transformed as in the present study the visual cues for making the comparison were not as salient as in the studies in which two arrays were inspected and compared. That is there were not two rows of objects presented simultaneously that the child could visually compare.

Study 2 also provided data relevant to the issue of whether children confuse operations that change only spatial extent with those that change number. In that study, in which children were allowed to themselves transform the sets from one given value to another, very few children attempted to change the sets by merely lengthening, shortening or otherwise manipulating the row without adding or subtracting objects. This is in contrast to Study 1 in which most of the children were willing to judge that merely changing the length of the array was sufficient to change its number. It appears that in a context that calls for children to themselves act on the set as in Study 2, children do not tend to spontaneously change the spatial extent, but rather tend to think that some type of addition or subtraction must occur. This finding suggests that the types of confusions between change in length and change in number found in many previous studies is not general, in the sense that the confusion extends to all task contexts. Even at age 3 the meaning of the counting sequence is linked to changes in the number of discrete objects. The children in this study had an understanding of addition and subtraction that belied their failure in certain conservation tasks. It could be said that their ability to <u>perform</u> the task is at a higher level than their <u>competence</u> to judge the effect of similar operations in a different context. However, this is not to say that they always knew which of these two operations was required.

<u>Knowledge about addition and subtraction</u>

That many children confused spatial relocation with change in number in Study 1 does not necessarily imply that these children do not understand anything about how differences in numerical values are related to the operations of addition and subtraction. A child could think, wrongly, that a spatial relocation of objects in an array results in a change in value, yet might also consistently judge that adding increases value and subtracting decreases value. Data regarding this issue come from both Study 1 and Study 2. In the first study, children were better able to judge that an operation opposite to the required operation was incorrect, than they were able to judge that the proper operation was accurate. In other words for the addition-subtraction trials children made more correct judgments on those trials in which the wrong type of transformation was performed, than on those trials in which the correct transformation occurred. These results suggest that before children have effective strategies for determining the precise result of a given transformation (i.e., what the numerical value is after the addition or subtraction transformation) they do have notions about whether the operation of addition or subtraction is required.

The analysis of children's first move strategies in Study 2 also provided some evidence regarding children's ability to determine the correct operation. The majority of 3-year-olds used strategies that suggested they did not understand this relationship. This was true at both small and large set sizes. Four-year-olds, on the other hand, were much more likely to use first attempt strategies that indicated they were aware of the proper operation needed to establish the goal set.

However, children's understanding of these numerical relations is clearly limited even at 4-1/2-years. The fact that there were very few children using the counting-on strategy, in which they could begin counting with a number in order to add on objects rather than beginning a count with 1, suggests that children this age have not fully grasped the significance of the sequence of numbers. Fuson (1982) has outlined several different, though related meanings of the last word in a count and claimed that "the confluence of these different meanings permits the counting-on procedure to function" (p. 68). For example, to know how many objects are in a set the child must connect the counting meaning of the last word with the cardinal meaning of that word. It may be that most of the children in this study had not yet made this connection between the two meanings. In Fuson and Richard's (1980) terms the preschoolers in the present study have not fully "decomposed" the relations between the numbers such that a numerical value clearly represents to them a summation of objects.

Order of acquisition

The findings just cited indicate that preschool children tended to distinguish when an addition was required and when a subtraction was required. This was especially true in the case of small sets. The question then arises whether children come to understand the effect of addition and subtraction operations on numerical value before or after they understand that numbers do not vary unless objects are added or subtracted (numeral invariance).

Study 1 provides evidence regarding the order in which children acquire these two concepts: (1) that numerical value is invariant over spatial relocations (numeral invariance), and (2) that addition operations increase value and subtraction operations decrease value. In comparing children's performance on the spatial relocation trials with their performance on the addition-subtraction trials it was found that children who understood numeral invariance tended to make more correct judgments on the addition-subtraction trials than did children who did not grasp numeral invariance. There were very few children who performed well on the addition-subtraction tasks and poorly on the spatial tasks. This finding does suggest that children come to the understanding of numeral invariance before understanding certain addition and subtraction operations. However, even those children who did not yet grasp numeral invariance seldom judged on small sets that operations <u>opposite</u> to the correct operation were correct. When they did judge wrongly, their mistakes tended to occur at large set sizes. This is consistent with the findings from Study 1 that children's ability to judge the effects of transformations is greater for small sets. Previous research (Beilin, 1968; Brush, 1978; Cooper et al., 1978; Klein, 1984; Gelman,

1972) has found that even 3-year-olds understand that adding increases number (makes a set have more than it did before the operation) and subtracting decreases number (makes a set have less than it did before the operation). Therefore the present findings suggest that the deficit in children's knowledge lies not in their understanding of the effect of the operations on quantity, but rather their lack of understanding of relations of magnitude between the larger numbers. In other words, it is not that children don't understand that addition increases quantity and subtraction decreases quantity, but rather that children aren't clear on what the quantity relations are among the numbers. Children of this age are not sure that numbers further along the sequence represent a greater quantity than one preceding it. Not knowing this they are not clear on whether an addition or a subtraction is required to go from say a 6 to an 8. Children do better with small numbers possibility because they can easily percieve the difference between a 2 and 3 or a 4 and a 5 just by the visual patterns of arrays. this is not true of differences between the larger numbers where a quick inspection does not allow one to easily differentiate a 7 from a 9, for instance. This finding is consistent with previous research by Fuson (Fuson, 1982; Fuson & Richards, 1980) and Siegler and Robinson (1982). Fuson's work suggests that the process of building the order relations between numbers occurs over an extended period of time and is known for the smaller numbers before the larger numbers. Siegler and Robinson present a model of preschoolers knowledge of number which suggests that though children may very early on place numbers in psychological categories which are differentiated as small and large, their knowledge of the relations between particular numbers may be quite tentative and unstable, and may vary depending on the particualar situations in which they are called upon to exhibit this knowledge.

Strategy Use

Strategies were grouped into 3 categories reflecting the level of children's quantitative understanding of the numeration system. The first level of understanding was characterized as unquantified with children not having much of a sense of the numbers being related to one another in a quantitative scheme. Strategies at this level reflected an understanding in which children did not relate the numbers to each other with any quantitiative meaning. This level was characterized by either a random adding or subtracting of objects, counting the objects without attempting any transformation on the sets or by spatially relocating the elements without adding or subtracting. In the second stage children's understanding seemed to be partially quantified in that they understood some of the relations of magnitude between the numbers but could not quantify differences in terms of units. Children understood general relations of magnitude between larger and smaller numbers, but they did not understand the unit increments between numbers. That is, they knew that certain numbers were greater or smaller than others but they did not understand this relation in terms of unit differences. Finally children use strategies at a fully quantified level in which differences between numbers can be arrived at in a precise manner. Children at this level seem to understand that each increment in the sequence of counting words represents one unit and that the units between all adjacent numbers are equal. Whether this fully quantified strategy truly reflects an understanding of units is open to question and is a problem for future research. There is some research suggesting, for example that preschool children do not have a good grasp on the unit concept in measures of length and mass (Inhelder et al., 1974). The extent to which children of 3- or 4-years-old truly understand the concept of an arithmetic unit is open to question and is a problem for future research. There is some research suggesting, for example, that most preschool children do not have a grasp of the unit concept in measures of length and mass (Inhelder et al., 1974).

The fact that children at this age are still unsure of the operations required to establish the requested numerical value challenges those models of number development which suggest that children have a well developed understanding of cardinal number before entering school. The present data indicate rather that many children are still in the process of working out these relations especially with larger sets and do not fully understand the structure of counting as a way of representing number.

Children's problem solving as information processing

The preceding analysis of children's strategies has suggested that children's performance on the task was indicative of their knowledge about the structure of the numeration system. It was demonstrated that preschool children often do not understand what operations are required to achieve a given change in numerical value. In this view changes in children's strategies and in the accuracy of their strategy use is due to a growth in their understanding of the relations between the words used in counting.

Another way to explain children's performance on these tasks would be to consider changes in performance as reflecting changes in their conceptualization of word problems, rather than changes in their understanding of the numerical relations among numbers, or changes in their ability to do certain kinds of arithmetic operations. In this view the growth in children's ability to solve word problems results from changes in their ability to represent the problem as it is expressed verbally.

Several attempts have been made at classifiying different types of word problems (Carpenter & Moser, 1981; Riley, Greeno & Heller, 1983; Vergnaud, 1981). Generally in these schemes problems differ on two dimensions: their semantic structure and on the identity of the unknown quantity. Differences in the semantic structure of the problems places different demands on children's conceptual knowledge about increases, decreases, combinations and comparisons. For example one can make a distinction between problems that involve <u>actions</u> such as making a change on one set or equalizing two sets, and problems that involve <u>static relations</u> such as counting or comparing two sets (Riley et al., 1983). The second basic way problems differ, the identity of the unknown quantity, refers to what is given in the problem and what is to be found. Applying these distinctions to the present study, the semantic structure of the problem required children to <u>transform</u> a given set (i.e., perform some action) in which the unknown quantity was the <u>number of objects that were to be added or subtracted</u>.

The way in which children apply their conceptual understanding to various problem types has also been subjected to different theoretical analyses. For example, Riley et al. have delineated three categories of knowledge necessary for solving arithmetic word problems. First children must employ <u>problem schemata</u> for understanding the semantic relations presented in the problem. Next, certain <u>action schemata</u> must be used for representing the child's knowledge about the actions involved in the solution. Finally, the child must have some strategic knowledge for planning solutions. Using this model, the greater accuracy of 4-1/2-year-olds in solving the problems in the present study is due to their better ability to represent problem information rather than their having a better understanding of the actions required to solve the problem. Thus with age children refine such action schemas as making a set of a given number, putting in objects, taking out objects, counting-all, etc.

An attempt at describing a developmental model of children's progress would need to take into account both the problem schemata and the action schemata that are required to relate the problem statement to the actions needed to solve the problem. For example, Riley et al. (1983) have proposed a set of models which they have used to describe the kinds of performance that are typical of young children on arithmetic word problems. These problem schemata range in complexity from simple schemes which allow the child to merely represent a quantitative relation to more complex schemes for effecting change, combining and comparing operations.

This type of analysis of children's problem solving has some general implications about the conclusions that can be drawn from the present study. Because certain children failed to show an understanding of a concept (for example, that one must add objects in order to go from a smaller number to a larger number) should not be taken as firm evidence that they lack understanding of the concept; there may be other tasks in which they show that they understand the concept quite well. Yet it is also wrong to assert that a schema for solving a problem is understood if evidence can be found for that understanding in some limited task domain. In order to gain a better understanding of children's true understanding of the numeration system one would need to use multiple tasks that would get at their understanding from several angles.

Conclusions

The findings of the present study suggest that previous research that has demonstrated preschool children's ability to count and to solve certain problems using counting may in fact conceal a great deal of uncertainty in children's understanding of the quantitative relations between the numbers. It was demonstrated that preschool children do not know in all contexts that the numerical representation of a set cannot be changed unless additions and subtractions occur. It was also documented that there was a lack of understanding, especially with large sets (i.e., larger than 5) regarding the type of operations required to establish the goal set.

An examination of children's strategy use showed that preschool children are still unsure of the operations that relate numerical values and that even 4-1/2-year-olds sometimes use strategies that can be classified as prequantified. Although preschool children certainly have some idea of how counting is used to represent cardinal number their concept of cardinal number is far from fully developed.

The findings of the present research suggest that procedural knowledge of the numeration system or (knowing how to count) does not imply full understanding of the quantitative relations that the numeration system represents. This is not to say that procedural skills do not in some way underlie understanding. Resnick (1982) has suggested that children do in fact invent certain arithmetic calculation procedures and that these inventions which reflect an understanding of number, "can come about only when procedures become well enough established that their results can be inspected and compared" (p. 149). The precise way in which counting interacts with other schemes to produce a full understanding of number remains a problem for future research.

Implications for education. The findings of the present research indicate several limitations in children's ability to understand certain basic numerical relations. It seems that a mature understanding of cardinal number involves several interrelated concepts. One should bear this point in mind in developing curriculum for young children. Teachers and curriculum planners would do well to approach the teaching of concepts from many angles, thus encouraging children to integrate what may be very separate skills into an integrated understanding of counting as a system for representing number.

The findings that these young children seem to perform better in Study 2 in which they were able to actively manipulate the objects suggests that teachers may want to assess children's abilities in contexts that allow them this type of activity and that they should be cautious in making evaluations too quickly based on a single type of performance.

Several cautions that must be stated regarding such applications in general. First, it should be pointed out that the capacities that one observes in children under certain experimental conditions may not be observed in more typical school situations. In most experiments children get an unusual amount of support and guidance, while it is the goal of most educational objectives to have the child acquire the ability to apply principles independently. Another issue in applying research to education is the question of readiness. For example, one might draw the conclusion from some of Piaget's work that conservation of number should be a precondition for teaching arithmetic. At least one researcher (Gelman, 1980) has suggested, however, that conservation may mark the onset of a higher order ability - reasoning about undetermined values, a kind of "algebraic" understanding of number. If this is true, it is not clear in what sense conservation could be considered as a precondition for teaching about number. Although the preschool children in the present study clearly lack certain basic arithmetical concepts, this is not to say that they aren't ready to attempt to learn certain basic addition and subtraction tasks. It may well be that it is through these trial and error attempts that children come to fully understand these basic numerical operatons.

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