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ABELIAN GROUPS OF TWO AND THREE DIMENSIONAL SIMPLY TRANSITIVE AFFINE MOTIONS

by

HARVEY BRAVERMAN

A dissertation submitted to the Graduate
Faculty in Mathematics in partial fulfillment
of the requirements for the degree of Doctor
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Chapter 1

As the title of this dissertation indicates, we will discuss the abelian groups of two and three dimensional simply transitive affine motions. An abelian group of simply transitive affine motions will be designated AST-2 or AST-3 for two or three dimensions respectively.

We will begin in Chapter 2, section 1, with two definitions and two theorems that will establish the matrix form of an AST-2 or AST-3 group.

In section 2 we will show that there exists a parameter C in R such that for each C we have a representation of R^2 as an AST-2 group. Moreover, given two different values, C and C′, of the parameter can we find an affine motion δ and an isomorphism T such that (1) is a commutative diagram?

$$(1) \qquad \begin{tikzcd} R^2 \arrow[r, "\theta[C]"] \arrow[d, "T"] & (C) \arrow[d, "(\delta)"] \\ R^2 \arrow[r, "\theta[C']"] & (C') \end{tikzcd} \quad ,$$

where (C) is the AST-2 representation with parameter C and (δ) means conjugation by the affine motion δ. We say that two AST-2 representations are equivalent if they satisfy (1).

We will show the conditions on C and C′ for (1) to be satisfied. Our main result will be that the set of all AST-2 representations is divided into two equivalence classes under this equivalence relation.

In section 3 we will consider the representations of AST-3. We will show that representations of AST-3 are of two different types: M[A,B,C] and N[A,C,E] where in each type different representations will result from different sets of parameters. For each type of representation a procedure analagous to that used in section 2 will lead to the conclusion that the set of representations of type M[A,B,C] is divided into four equivalence classes under an equivalence relation analagous to (1) and that the set of representations of type N[A,C,E] is divided into two equivalence classes.

We conclude our discussion with section 4 in which we will show that one of the equivalence classes of type N[A,C,E] is an equivalence class of type M[A,B,C]. Thus, the total number of equivalence classes for AST-3 representations is five.

Chapter 2

§1.

Definition 1 An affine motion on R^n is a non-singular linear trnsformation followed by a translation by a fixed vector. For R^2 and R^3 an affine motion can be represented by the matrices

$$(1) \begin{bmatrix} a & b & \epsilon \\ e & f & \sigma \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and } (2) \begin{bmatrix} a & b & c & \epsilon \\ e & f & g & \sigma \\ h & i & j & \nu \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

respectively. The last column of the matrix represents the translation and the determinant of the matrix is non-zero. This implies that the set of all affine motions on R^n forms a group under matrix multiplication.

Definition 2 A group of matrices H is transitive on a vector space V if for any two vectors X and Y there exists a δ in H such that $\delta X = Y$. H is called simply transitive if δ is unique.

Theorem 1 Let H be an abelian group of affine motions, transitive on a finite dimensional vector space V. Then, relative to an appropriate basis for V, their matrix form is upper triangular with 1's down the main diagonal.

Proof: An affine motion on V has the matrix form

$$A = \left[\begin{array}{c|c} A' & k \\ \hline 0 & 1 \end{array}\right]$$

where A' is a non-singular linear transformation on V and k is a vector. We refer to A' as the linear part of A.

Let H' be the group of nxn matrices A' where the A' are the linear parts of the elements of H. The group H' is abelian. We apply to H' the results of Chapter IV, Section 9 of [5] and conclude that there exists a B' such that the transformation $(B')^{-1}A'B'$ has for each A' a block form

$$\begin{bmatrix} (\chi_1) & & \\ & \ddots & \\ & & (\chi_j) \end{bmatrix}$$

where the size of the blocks is independent of A'.

Furthermore, each χ_i is itself a block form

$$\chi_i = \begin{bmatrix} (\nu_1) & - & - & - \\ & \ddots & - & - \\ 0 & & \ddots & \\ & & & (\nu_t) \end{bmatrix}$$

where again the size of the blocks is independent of A'. Moreover, all the ν's in a given block are equal and have minimal polynomials which are irreducible.

We can define B as

$$B = \left[\begin{array}{c|c} B' & 0 \\ \hline 0 & 1 \end{array}\right]$$

. Thus, all the elements

$B^{-1}AB$ have the form

$$(1) \qquad \left[\begin{array}{cc|c} (\chi_1) & & \\ & & c \\ & (\chi_j) & \\ \hline & 0 & I \end{array}\right]$$

where A is in H. Because of transitivity the vectors c range over all of V.

We will prove that for each i, 1 is an eigenvalue of χ_i. Thus, we assume there exists an affine motion A_o in $B^{-1}HB$ such that there is at least one block χ_i which does not have an eigenvalue of 1. By a rearrangement of the blocks we let χ_1 be such a block and let its size be r. We will show that this assumption leads to a contradiction. This means that each χ_i has an eigenvalue of 1. Therefore, x-1 divides the minimum polynomial of the ν's which, as we remarked above, is an irreducible polynomial. Hence, the minimum polynomial is x-1. Therefore, each ν is the 1×1 matrix 1; this proves the theorem. We now proceed to the contradiction.

Consider $M^{-1}A_o M$ where

$$M = \left[\begin{array}{c|c} I & x \\ & \\ \hline 0 & 1 \end{array}\right]$$

. The first r entries in the last column are

(2) $[(\chi_1) - I_r] \begin{bmatrix} x_1 \\ \\ x_r \end{bmatrix} + \begin{bmatrix} c_1 \\ \\ c_r \end{bmatrix}$. Since by hypothesis $[(\chi_1) - I_r]$ is non-singular, we can find $x_1, - - -, x_r$ such that (2) is 0.

Consider the group $M^{-1}B^{-1}HBM$. This group is another abelian, transitive group. In this group there is a matrix of the form

$$C = \left[\begin{array}{ccc|c} (\theta_1) & & & 1 \\ & \ddots & & 0 \\ & & & \vdots \\ & & (\theta_j) & 0 \\ \hline & 0 & & 1 \end{array}\right]$$

. We calculate the first r entries of the last column of $CM^{-1}A_o M$. These are $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. But, the first r entries in the last column of $M^{-1}A_o MC$ are $(\chi_1) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. Since the group is abelian these must be equal. This implies (χ_1) has an eigenvalue of 1, a contradiction.

<u>Theorem 2</u> Let H be a group of AST-2 and J be a group of AST-3. Then, relative to an appropriate basis, the matrices of H are of the form

$$\begin{bmatrix} 1 & e(x,y) & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}, \text{ the } (x,y) \text{ range over all of } R^2,$$

and the matrices of J are of the form

$$\begin{bmatrix} 1 & f(x,y,z) & g(x,y,z) & z \\ 0 & 1 & h(x,y,z) & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ the } (x,y,z) \text{ range}$$

over all of R^3.

<u>Proof</u>: $\left[\begin{array}{c|c} A' & k \\ \hline 0 & 1 \end{array}\right]$ takes the origin to k. Simple transitivity means that there is a unique element in the group whose translation part is k. Hence A′ is determined by k. Furthermore every k in R^n shows up since the action is transitive.

According to Theorem 1, we may assume that the matrices are upper triangular with 1's appearing on the diagonal. The fact that A′ is determined by k is expressed via the functions e, f, g and h in the statement.

§2.

According to theorem 2 of section 1, the matrices of an AST-2 group have the form

$$\begin{bmatrix} 1 & e(x,y) & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}.$$

The fact that any two of the matrices commute leads to the result that the matrices of the group have the form

$$\begin{bmatrix} 1 & Cx & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}$$

where C is a parameter (Appendix 1, Theorem 1). We define a mapping $\theta[C]$: $R^2 \quad (C)$ by $\theta[C](x,w) = \exp F$ where $F = \begin{bmatrix} 0 & Cx & w \\ 0 & 0 & x \\ 0 & 0 & 0 \end{bmatrix}$ and $y = w + \frac{1}{2} Cx^2$.

Clearly, $\theta[C]$ is an isomorphism. Moreover, for any affine motion δ the mapping (δ): $(C) \quad \delta(C)\delta^{-1}$ is an isomorphism.

We wish to choose an affine motion δ so that $\delta(C)\delta^{-1}$ is an AST-2 group whose matrices are of the form

$$\begin{bmatrix} 1 & C'u & v \\ 0 & 1 & u \\ 0 & 0 & 1 \end{bmatrix}.$$

To accomplish this we choose $C' = \frac{aC}{f^2}$, $u = \frac{(e^2\epsilon - ae\sigma)}{D} Cx + ey + fx$ and $v = \frac{(ae\epsilon - a^2\sigma)}{D} Cx + ay + bx$ where $D = \det \delta \neq 0$ and we choose in δ, $e = 0$ if $C \neq 0$ (Appendix 1, theorem 2). Clearly, these definitions define an AST-2 group $(C') = \delta\,(C)\,\delta^{-1}$.

Moreover, the mapping $\theta[C']: R^2 \rightarrow (C')$ defined by $\theta[C'](u,p) = \exp G$ where

$$G = \begin{bmatrix} 0 & C'u & p \\ 0 & 0 & u \\ 0 & 0 & 0 \end{bmatrix}$$

and $v = p + \frac{1}{2}C'u^2$ is an isomorphism.

We now define an isomorphism $T: R^2 \rightarrow R^2$ as $T = \theta^{-1}[C'](\delta)\theta[C]$.

Theorem 1 If (C) and (C′) are two AST-2 representations, then there exists a δ and a T such that the diagram

$$(1) \qquad \begin{tikzcd} R^2 \arrow[r, "\theta{[C]}"] \arrow[d, "T"] & (C) \arrow[d, "(\delta)"] \\ R^2 \arrow[r, "\theta{[C']}"] & (C') \end{tikzcd}$$

is a commutative diagram under the following conditions:

1. $C = 0$ if and only if $C' = 0$
2. If $C \neq 0$ we choose a δ such that $e = 0$ and a and f are chosen such that $\frac{a}{f^2} = \frac{C'}{C}$. (Appendix 1, theorem 2).

With these conditions we define u, v, θ and T as before.

The above theorem shows that the set of all AST-2 representations is divided into two equivalence classes by the equivalence relation (1).

If we require in $\delta(C)\delta^{-1} = (C')$ that $u = x$ and $v = y$, then it follows from Appendix 1, theorem 3 that $C = C'$ and that δ is in (C). Hence, $\delta(C)\delta^{-1} = (C)$. Thus, if we require $u = x$ and $v = y$ we find each AST-2 representation is its own equivalence class.

§3.

Part a

According to theorem 2, § 1 the matrices of an AST-3 group have the form

$$\begin{bmatrix} 1 & f(x,y,z) & g(x,y,z) & z \\ 0 & 1 & h(x,y,z) & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

We will first consider the case of $E = 0$. The fact that any two of the matrices commute will lead to the result (Appendix 2, section 1) that such matrices are of the form

$$M[A,B,C] = \begin{bmatrix} 1 & Ax+By & Cx+Ay & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

We define a mapping $\theta[A,B,C]\colon R^3 \longrightarrow M[A,B,C]$ as $\theta[A,B,C](x,y,r) = \exp H$ where

$$H = \begin{bmatrix} 0 & Ax+By & Cx+Ay & r \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $z = r + \frac{1}{2}[(Ax+By)y + (Cx+Ay)x]$. The above shows that $\theta[A,B,C]$ is an isomorphism. Moreover, for any affine motion δ, the mapping (δ): $(M[A,B,C]) \longrightarrow \delta(M[A,B,C])\delta^{-1}$ is an isomorphism.

We wish to choose an affine motion δ so that $\delta(M[A,B,C])\delta^{-1}$ is an AST-3 group with matrices of the form

$$\begin{bmatrix} 1 & A'u+B'v & C'u+A'v & w \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

To accomplish this we choose, if A,B,C are not all zero, $e = h = 0$ in δ and

$$A' = \frac{a}{(fj-gi)^2}\,[Ajf - Cif - Bjg + Agi],$$

$$B' = \frac{a}{(fj-gi)^2}\,[Bj^2 + Ci^2 - 2Aij],$$

$$C' = \frac{a}{(fj-gi)^2}\,[Cf^2 + Bg^2 - 2Agf],$$

$$w = \frac{a(g\nu - j\sigma)(Ax + By) + a(i\sigma - f\nu)(Cx + Ay)}{fj - gi}$$

$$+ az + by + cx,$$

$$v = fy + gx \text{ and}$$

$$u = iy + jx.$$

If $A = B = C = 0$ we choose $A' = B' = C' = 0$, $w = az + by + cx$, $v = ez + fy + gx$, and $u = hz + iy + jx$. (Appendix 2, section 2). Clearly, these definitions define an AST-3 group $(M[A',B',C']) = \delta(M[A,B,C])\delta^{-1}$. Moreover, the mapping $\theta[A',B',C']: R^3 \longrightarrow (M[A',B',C'])$ defined by $\theta[A',B',C'](u,v,p) = \exp I$ where $I =$

$$\begin{bmatrix} 0 & A'u + B'v & C'u + A'v & p \\ 0 & 0 & 0 & v \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $w = p + \frac{1}{2}[(A'u + B'v)v + (C'u + A'v)u]$ is an isomorphism.

We now define an isomorphism $T: R^3 \longrightarrow R^3$ as $T = \theta^{-1}[A',B',C'](\delta)\theta[A,B,C]$.

<u>Theorem 1</u> If $(M[A,B,C])$ and $(M[A',B',C'])$ are two AST-3 representations, then there exists an isomorphism T and an affine motion δ such that the diagram

$$\begin{tikzcd} R^3 \arrow[r, "{\theta[A,B,C]}"] \arrow[d, "T"] & (M[A,B,C]) \arrow[d, "(\delta)"] \\ R^3 \arrow[r, "{\theta[A',B',C']}"] & (M[A',B',C']) \end{tikzcd} \tag{1}$$

is a commutative diagram under the following conditions:

1. $A = B = C = 0$ if and only if $A' = B' = C' = 0$.
2. If not all A,B,C are zero then
   a) $A^2 = BC$ if and only if $(A')^2 = B'C'$
   b) $A^2 < BC$ if and only if $(A')^2 < B'C'$
   c) $A^2 > BC$ if and only if $(A')^2 > B'C'$,

where u,v,w, θ and T are to be defined as before.

The derivation of these conditions and the existance of a δ appear in Appendix 2, section 4.

The above conditions show that the set of all AST-3 representations of type $M[A,B,C]$ is divided into four equivalence classes under the equivalence relation (1).

If we require $u = x$, $v = y$ and $w = z$ in $\delta(M[A,B,C])\delta^{-1} = (M[A',B',C'])$ then we conclude from Appendix 2, section 5 that $A = A'$, $B = B'$, $C = C'$ and that δ is in $(M[A,B,C])$. Thus, each AST-3 representation is its own equivalence class if we require $u = x$, $v = y$ and $w = z$.

Part b

We now consider an AST-3 group whose matrices have the form

$$\begin{bmatrix} 1 & f(x,y,z) & g(x,y,z) & z \\ 0 & 1 & h(x,y,z) & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $h \neq 0$. The fact that any two of the matrices commute will lead to the result (Appendix 3, theorem 1) that such matrices are of the form

$$N[A,C,E] = \begin{bmatrix} 1 & Ax & Cx + Ay & z \\ 0 & 1 & Ex & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We define a mapping $\alpha\,[A,C,E]:R^3 \longrightarrow (N[A,C,E])$ as $\alpha\,[A,C,E](x,r,s) = \exp J$ where

$$J = \begin{bmatrix} 0 & Ax & Cx + Ar & s \\ 0 & 0 & Ex & r \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and where $y = r + \frac{1}{2}Ex^2$ and $z = s + Axr + \frac{1}{2}Cx^2 + \frac{AEx^3}{6}$.

The above shows that $\alpha[A,C,E]$ is an isomorphism. Moreover, if δ is any affine motion, the mapping (δ): $(N[A,C,E]) \longrightarrow \delta(N[A,C,E])\delta^{-1}$ is an isomorphism.

We wish to choose an affine motion δ so that $\delta(N[A,C,E])\delta^{-1}$ is an AST-3 group with matrices of the form

$$\begin{bmatrix} 1 & A'u & C'u+A'v & w \\ 0 & 1 & E'u & v \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

To accomplish this we choose; $i = h = 0$ and

1. If $A \neq 0$ we choose $e = 0$, $E' = \dfrac{fE}{j^2}$, $\dfrac{aA}{fj} = A'$ and

   $$C' = \frac{-2agA}{fj^2} + \frac{aC + bE}{j^2} + \frac{aAE\nu}{j^3}$$

2. If $A = 0$ we choose $A' = 0$, $E' = \dfrac{eC + fE}{j^2}$ and

   $$C' = \frac{aC + bE}{j^2}.$$

In each case we choose

$$w = \frac{a[a(g\nu - j\sigma) + e\,(j\epsilon - c\nu)]Ax}{D} - \frac{a\nu(Cx + Ay)}{j}$$

$$- \frac{b\nu Ex}{j} + az + by + cx,$$

$$v = \frac{e[a(g\nu - j\sigma) + e(j\epsilon - c\nu)]Ax}{D} - \frac{e\nu(Cx + Ay)}{j}$$

$$- \frac{f\nu Ex}{j} + ez + fy + gx \text{ and}$$

$u = jx$, where $D = \det \delta$ (Appendix 3, section 2). Clearly these definitions define an AST-3 group

$$(N[A',C',E']\ ) = \delta(N[A,C,E])\delta^{-1}.$$

Moreover, the mapping $\alpha[A',C',E']: R^3 \longrightarrow (N[A',C',E'])$ defined by $\alpha[A',C',E'](u,p,q) = \exp K$ where

$$K = \begin{bmatrix} 0 & A'u & C'u + A'p & q \\ 0 & 0 & E'u & p \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and where $v = p + \frac{1}{2}E'u^2$ and $w = q + A'up + \frac{1}{2}C'u^2 + \frac{A'E'u^3}{6}$

is an isomorphism.

We now define $T: R^3 \longrightarrow R^3$ as $T = \alpha^{-1}[A',C',E'](\delta)\alpha[A,C,E]$.

<u>Theorem 2</u> If $(N[A,C,E])$ and $(N[A',C',E'])$ are two AST-3 representations, then there exists an isomorphism T and an affine motion δ such that the diagram

$$\begin{tikzcd} R^3 \arrow[r, "{\alpha[A,C,E]}"] \arrow[d, "T"] & (N[A,C,E]) \arrow[d, "(\delta)"] \\ R^3 \arrow[r, "{\alpha[A',C',E']}"] & (N[A',C',E']) \end{tikzcd} \tag{2}$$

is a commutative diagram under the following conditions:

1. $A = 0$ if and only if $A' = 0$ and the elements of δ are chosen such that $eC + fE = E'j^2$, $aC + bE = C'j^2$.

2. If $A \neq 0$ then $A' \neq 0$ and the elements of δ are chosen such that $e = 0$, $fE = E'j^2$, $aA = fjA'$ and $C' = \frac{-2agA}{fj^2} + \frac{aC + bE}{j^2} + \frac{aAE\nu}{j^3}$, where u, v, w, α and T are defined as before.

The derivation of the above appears in Appendix 3, theorem 2. In each of the above we choose $i = h = 0$ in δ.

The above shows that the set of all AST-3 representations of type (N[A,C,E]) is divided into two equivalence classes by the equivalence relation (2).

Finally, if we require in $\delta(N[A,C,E])\delta^{-1} = (N[A',C',E'])$ that $u = x$, $v = y$ and $w = z$ then we conclude from Appendix 3, theorem 3 that $A = A'$, $C = C'$, $E = E'$ and δ is in (N[A,C,E]). Thus, each AST-3 representation is its own equivalence class if we require $u = x$, $v = y$ and $w = z$.

§4

Theorem 1 The conditions under which an affine motion δ satisfies (1),

$$(1) \qquad \delta \begin{bmatrix} 1 & Ax & Cx + Ay & z \\ 0 & 1 & Ex & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix} \delta^{-1} = \begin{bmatrix} 1 & A'u+B'v & C'u+A'v & w \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $E \neq 0$, are;

$$w = az + by + cx + \alpha\frac{(aC + bE)}{D}x,$$

$$v = ez + fy + gx + \alpha\frac{(eC + fE)}{D}x,$$

$$u = hz + iy + jx + \alpha\frac{(hC + iE)}{D}x$$

where $D = \det \delta \neq 0$ and $\alpha = - \begin{vmatrix} a & b & \varepsilon \\ e & f & \sigma \\ h & i & \nu \end{vmatrix}$ and;

1. If $A \neq 0$ (1) is not possible.
2. If $A = 0$ then $ei = fh$ and $(A')^2 = B'C'$ for A', B', C' not all zero.
3. If $A = 0$, $C \neq 0$ and $bh - ai \neq 0$ then
   a) $A' = B' = 0$, $C' \neq 0$ is not possible,
   b) $A' = C' = 0$, $B' \neq 0$ implies $e = f = 0$,
   $B'g^2 = aC + bE = \frac{E(bh - ai)}{h}$ and $C = \frac{-iE}{h}$.
   c) If A', B' and $C' \neq 0$, then $C' = \frac{-A'e}{h}$,
   $B' = \frac{-A'h}{e}$, $C = \frac{-iE}{h}$ and $(hg-ej)^2A' = eE(ai-bh) = hE(af-be)$.
4. If $A = 0$, $C \neq 0$ and $af - be \neq 0$, then
   a) $A' = C' = 0$, $B' \neq 0$ is not possible,
   b) $A' = B' = 0$, $C' \neq 0$ implies $i = h = 0$,
   $C'j^2 = aC + bE = \frac{E(be - af)}{e}$ and $C = \frac{-fE}{e}$.
   c) If A', B' and $C' \neq 0$, then $C' = \frac{-A'e}{h}$, $B' = \frac{-A'h}{e}$,
   $C = \frac{-iE}{h}$, $(hg - ej)^2A' = eE(ai - bh) = hE(af - be)$.
5. If $A = C = 0$, then $f = i = 0$ and
   a) If $bh - ai \neq 0$, then
      1. $A' = B' = 0$, $C' \neq 0$ is not possible.
      2. $A' = C' = 0$, $B' \neq 0$ implies $B'g^2 = bE$ and $e = 0$.

3. $A', B', C' \neq 0$ implies $C' = \frac{-A'e}{h}$, $B' = \frac{-A'h}{e}$

and $(hg - ej)^2A' = -ebhE$.

b) If $(af - be) \neq 0$, then

1. $A' = C' = 0$, $B' \neq 0$ is not possible

2. $A' = B' = 0$, $C' \neq 0$ implies $h = 0$ and

$C'j^2 = bE$.

3. $A', B', C' \neq 0$ implies $C' = \frac{-A'e}{h}$, $B' = \frac{-A'h}{e}$

and $(hg - ej)^2A' = -ebhE$.

This theorem shows that for $(A')^2 = B'C'$ and if not all A', B' and C' are zero, that the AST-3 representations $N[0, C, E]$ and $M[A', B', C']$ are equivalent. The equivalence relation being the commutative diagram

$$\begin{tikzcd}
R^3 \arrow[r, "{\alpha[0,C,E]}"] \arrow[d, "T"] & N[0,C,E] \arrow[d, "(\delta)"] \\
R^3 \arrow[r, "{\theta[A',B',C']}"] & M[A',B',C']
\end{tikzcd}$$

where $T = \theta^{-1}[A',B',C']\ (\delta)\ \alpha[0,C,E]$.

Therefore, for AST-3 representations there are exactly five equivalence classes.

<u>Theorem 2</u> There exists no affine motion δ such that (1) is true if we require $w = z$, $v = y$ and $u = x$.

The proofs of the above theorems are found in Appendix 4.

Appendix 1

§1

Theorem 1 If the matrices of an AST-2 group have the form

$$\begin{bmatrix} 1 & e(x,y) & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}$$

, then, for some constant C, $e(x,y) = Cx$.

Proof: Since the group is abelian, $x'e(x,y) = xe(x',y')$ for all values of x, x', y and y'. Let $x' = 1$, $y' = 0$ and $C = e(1,0)$. Hence, $e(x,y) = Cx$.

§2

Theorem 2

If (1):

$$\delta \begin{bmatrix} 1 & Cx & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \delta^{-1} = \begin{bmatrix} 1 & C'u & v \\ 0 & 1 & u \\ 0 & 0 & 1 \end{bmatrix},$$

where δ is the affine motion $\begin{bmatrix} a & b & \epsilon \\ e & f & \sigma \\ 0 & 0 & 1 \end{bmatrix}$, then

1. $$u = \frac{(e^2\epsilon - ae\sigma)Cx}{D} + ey + fx,$$

$$v = \frac{(ae\epsilon - a^2\sigma)Cx}{D} + ay + bx$$

where $D = \det \delta$.

(Note: the determinant of the matrix of the coefficients of x and y is $D \neq 0$ for both (2) and (3) below).

2. $C = 0$ if and only if $C' = 0$.

3. $C \neq 0$ implies $e = 0$ and $aC = f^2C'$.

Proof:

1. Follows immediately by multiplying (1) out and equating the elements of the third column with v and u respectively.
2. If $C = 0$ then from (1) $C'u = 0$. If $C' \neq 0$ then $e = f = 0$ and therefore $D = 0$, a contradiction. Thus, $C' = 0$. Conversely, if $C' = 0$, then in (1), $-e^2Cx = 0$ and $a^2Cx = 0$. If $C \neq 0$ then $a = 0$ and $e = 0$ contradicting $D \neq 0$.
3. If $C \neq 0$ then in (1) we have $e^2Cx = 0$. Therefore, $e = 0$. Also in (1), $\frac{a^2Cx}{D} = C'u$ yields $aC = f^2C'$.

Theorem 3 If in the above we require $u = x$ and $v = y$, then $C = C'$ and

1. If $C = 0$ then $a = f = 1$ and $b = e = 0$.
2. If $C \neq 0$ then $a = f = 1$, $e = 0$ and $b = \sigma C$.

In either case δ is of the same form as the matrix

$$\begin{bmatrix} 1 & Cx & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}.$$

Proof:

1. $C = 0$ if and only if $C' = 0$ follows from theorem 2. Moreover, the formulas for u and v imply $a = f = 1$ and $b = e = 0$.

2. If $C \neq 0$ then theorem 2 implies $e = 0$ and the formulas for u and v imply $a = f = 1$ and $b = \sigma C$. Moreover, $aC = f^2C'$ implies $C = C'$.

Appendix 2

§1

Theorem 1 If the matrices of an AST-3 group have the form $\begin{bmatrix} 1 & f(x,y,z) & g(x,y,z) & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix}$ for all (x,y,z) in R^3, then $f(x,y,z) = Ax + By$ and $g(x,y,z) = Cx + Ay$.

Proof: Since the group is abelian

(1) $yf(x',y',z') + xg(x',y',z') = y'f(x,y,z) + x'g(x,y,z)$ for all values of x,x',y,y',z and z'. Let $x'=0$, $y'=1$, $z'=0$, $B = f(0,1,0)$ and $A = g(0,1,0)$. Then (1) becomes $f(x,y,z) = Ax + By$. Let $x'=1$, $y'=0$, $z'=0$, $D = f(1,0,0)$ and $C = g(1,0,0)$. Then (1) becomes $g(x,y,z) = Cx + Dy$. If we replace f and g in (1) with these linear functions we obtain $y'Ax + x'Dy = yAx' + xDy'$. For $y' = 0$, $x'=1$ and $y=1$, we conclude $D = A$. Thus, $f(x,y,z) = Ax + By$ and $g(x,y,z) = Cx + Ay$.

§2

Theorem 2 If

$$(2):\delta \begin{bmatrix} 1 & Ax+By & Cx+Ay & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix} \delta^{-1} =$$

$$\begin{bmatrix} 1 & A'u+B'v & C'u+A'v & w \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{where } \delta = \begin{bmatrix} a & b & c & \varepsilon \\ e & f & g & \sigma \\ h & i & j & \nu \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and $D = \det \delta \neq 0$, then

1. If not all A,B,C are zero; $e = h = 0$,

   $$w = \frac{a(g\nu - j\sigma)(Ax + By) + a(i\sigma - f\nu)(Cx + Ay)}{fj - gi} +$$

   $az + by + cx$,

   $v = fy + gx$, $u = iy + jx$, (Note: the determinant of the matrix of the coefficients of x,y and z is $D \neq 0$).

   $$A' = \frac{a}{(fj-gi)^2}\ [Ajf - Cif - Bjg + Agi],$$

   $$B' = \frac{a}{(fj-gi)^2}\ [Bj^2 + Ci^2 - 2Aij],$$

   $$C' = \frac{a}{(fj-gi)^2}\ [Cf^2 + Bg^2 - 2Agf] \text{ and}$$

   $$(A')^2 - B'C' = (A^2 - BC)\ \frac{a^2}{(fj-gi)^2}$$

2. $A = B = C = 0$ if and only if $A' = B' = C' = 0$.

   In this case $w = az + by + cx$, $v = ez + fy + gx$ and $u = hz + iy + jx$. (Note: the determinant of the matrix of the coefficients of x,y and z is $D \neq 0$).

<u>Proof:</u>

1. In (2) the following equations result:

(3) $e[A\alpha + C\theta] = 0$, $e[B\alpha + A\theta] = 0$ where $(\alpha,\theta) = (aj - hc, bh - ai)$ or $(ec - ag, af - be)$ or $(hg - ej, ei - fh)$. A similar result is obtained by replacing e by h in (3).

If we assume $e \neq 0$ we obtain

(4) $A\alpha + C\theta = 0$, $B\alpha + A\theta = 0$.

a) If $A^2 - BC \neq 0$ (4) leads to a contradiction of $D \neq 0$. Thus, if $A^2 - BC \neq 0$ we obtain $e = 0$ and similarly $h = 0$.

b) If $A^2 = BC$ then (4) yields for the cases A,B,C not zero, $A = C = 0$ and $B \neq 0$, and $A = B = 0$ and $C \neq 0$ that $D = 0$, a contradiction. Thus, $e = h = 0$.

The equations for w, v and u result directly from (2) using the fact that $e = h = 0$ in δ.

Moreover, in (2) row 1, column 2 = $A'u + B'v$ and row 1, column 3 = $C'u + A'v$. This and the formulas for u and v imply the equality of the following coefficients of x and y:

(5) $$\frac{a}{fj-gi}(Aj - Ci) = A'j + B'g, \quad \frac{a}{fj-gi}(Bj-Ai) = A'i+B'f,$$

$$\frac{a}{fj-gi}(-Ag+Cf) = C'j + A'g, \quad \frac{a}{fj-gi}(-Bg+Af) = C'i+A'f.$$

The solution of the above equations yields the aforementioned equations for A', B' and C'. Moreover, those equations for A', B' and C' will result in the equation $(A')^2 - B'C' = (A^2 - BC)\dfrac{a^2}{(fj-gi)^2}$ being confirmed.

2. If $A = B = C = 0$ (2) yields the following results: $w = az + by + cx$, $v = ez + fy + gx$, $u = hz + iy + jx$, and $A'u + B'v = 0$, $C'u + A'v = 0$. These last two equations yield the following equations for the coefficients of x, y and z.

   (6) $A'h + B'e = 0$, $A'i + B'f = 0$, $A'j + B'g = 0$,
   $C'h + A'e = 0$, $C'i + A'f = 0$, $C'j + A'g = 0$.

   If we consider the cases a) $(A')^2 - B'C' \neq 0$ and b) $(A')^2 = B'C'$ (and the separate possibilities for (b) that $A', B', C' \neq 0$ or $A' = B' = 0$ and $C' \neq 0$ or $A' = C' = 0$ and $B' \neq 0$), we find that $D = 0$, a contradiction. Therefore, we conclude $A' = B' = C' = 0$.

   Conversely, if $A' = B' = C' = 0$ we will conclude $A = B = C = 0$. If not all of A, B, C are zero we obtain from part 1 of this theorem that $e = h = 0$ and $A^2 = BC$. For the separate possibilities; (a) $A,B,C \neq 0$, (b) $A = B = 0$ and $C \neq 0$, (c) $A = C = 0$ and $B \neq 0$ equations (5) lead to the result $D = 0$, a contradiction. Therefore, $A = B = C = 0$.

§3

Theorem 3 If

$$\delta \begin{bmatrix} 1 & Ax+By & Cx+Ay & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \delta^{-1} = \begin{bmatrix} 1 & A'u+B'v & C'u+A'v & w \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and if $A = A'$, $B = B'$ and $C = C'$, then the following is true:

1. If $A^2 - BC \neq 0$ and if we choose $a = fj - gi$, then $B(j - f) = 2Ai$, $C(f - j) = 2Ag$ and $-Bg = Ci$.
2. If $A^2 - BC \neq 0$ and if we choose $a = -(fj - gi)$, then $f = -j$ and $Ci - Bg = 2Aj$.
3. If $A^2 = BC$, then
    a) $A = 0$, $B = 0$, $C \neq 0$ implies $i = 0$ and $a = j^2$.
    b) $A = 0$, $C = 0$, $B \neq 0$ implies $g = 0$ and $a = f^2$.
    c) $A \neq 0$, $B \neq 0$, $C \neq 0$ implies
        1. $g = \dfrac{(\sqrt{a} - j)C}{A}$, $i = \dfrac{(\sqrt{a} - f)B}{A}$, $\sqrt{a} \neq f + j$, or
        2. $g = \dfrac{(\sqrt{a} - j)C}{A}$, $i = -\dfrac{(\sqrt{a} + f)B}{A}$, $\sqrt{a} \neq j - f$, or
        3. $g = -\dfrac{(\sqrt{a} + j)C}{A}$, $i = \dfrac{(\sqrt{a} - f)B}{A}$, $\sqrt{a} \neq f - j$, or

4. $g = \frac{-(\sqrt{a} + j)C}{A}$, $i = \frac{(\sqrt{a} + f)B}{A}$, $\sqrt{a} \neq -(f+j)$.

Proof:

1. The first part of theorem 2 of this appendix yields $a^2 = (fj - gi)^2$. If we choose $a = fj - gi$, then (5) of theorem 2, with $A = A'$, $B = B'$ and $C = C'$, leads to the desired result.
2. If in the above we choose $a = -(fj - gi)$, then (5) of theorem 2, with $A = A'$, $B = B'$ and $C = C'$, leads to the desired result.
3. Parts (a) and (b) are obtained by direct substitution into (5) of theorem 2 with $A = A'$, $B = B'$ and $C = C'$.

   In order to prove part (c) we notice that (5) of theorem 2 with $A = A'$, $B = B'$ and $C = C'$ yields $aAj = (Aj^2 + Bjg)f + (aC - Agj - Bg^2)i$, $-aAg = (-aC + Cj^2 + Ajg)f + (-Cjg - Ag^2)i$. If we assume the determinant of the matrix of the coefficients of i and f is non-zero we are led to $D = \det \delta = 0$, a contradiction. Similarly, (5) of theorem 2 yields $-aAi = (-aB + Afi + Bf^2)j + (-Ai^2 - Bfi)g$, $aAf = (Cfi + Af^2)j + (aB - Ci^2 - Afi)g$ and we must conclude the determinant of the matrix of the coefficients of j and g is zero.

   The fact that the above two determinants are zero yields the equations $a(Ag + Cj)^2 - a^2C^2 = 0$ and $a(Ai + Bf)^2 - a^2B^2 = 0$. These imply a is positive and the various possibilities for the roots of these equations yields the desired results for part (c).

Moreover, the relationships of each part of the above theorem, when directly substituted into the equations for A', B' and C' of theorem 2, part 1, yield $A' = A$, $B' = B$ and $C' = C$.

§4

We repeat here some of the results of theorem 2, namely if we wish to find a δ such that

$$(7) \quad \delta \begin{bmatrix} 1 & Ax+By & Cx+Ay & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix} \delta^{-1} = \begin{bmatrix} 1 & A'u+B'v & C'u+A'v & w \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\delta = \begin{bmatrix} a & b & c & \epsilon \\ e & f & g & \sigma \\ h & i & j & \nu \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $D = \det \delta \neq 0$, then

1. If A, B, C are not all zero then $e = h = 0$ and;

$$(8) \quad A' = \frac{a}{(fj-gi)^2} [Ajf - Cif - Bjg + Agi],$$

$$(9)\ B' = \frac{a}{(fj-gi)^2}\ [Bj^2 + Ci^2 - 2Aij],$$

$$(10)\ C' = \frac{a}{(fj-gi)^2}\ [Cf^2 + Bg^2 - 2Agf]\ \text{and}$$

$$(11)\ (A')^2 - B'C' = (A^2 - BC)\frac{a^2}{(fj-gi)^2}\ .$$

2. $A = B = C = 0$ if and only if $A' = B' = C' = 0$.

The equations for u, v and w of theorem 2 imply that for any u, v and w an x, y and z can be found to satisfy them, and conversely. Equation (11) and part 2 above imply that if (7) is to be satisfied then

a) $A^2 = BC$ if and only if $(A')^2 = B'C'$.

b) $A^2 - BC<0$ if and only if $(A')^2 - B'C'<0$.

c) $A^2 - BC>0$ if and only if $(A')^2 - B'C'>0$.

d) $A = B = C = 0$ if and only if $A' = B' = C' = 0$.

The above can be viewed as the surface $x^2 = yz$ dividing three dimensional Euclidean space into four regions:

a) The surface itself (excluding the origin).

b) The "inside" of the surface.

c) The "outside" of the surface.

d) The origin.

It remains to be shown that given any two sets of parameters (A,B,C) and (A_1,B_1,C_1) in any one of the four regions cited above we can find a δ such that (7) is satisfied, where u, v and w are related to x, y and z as in theorem 2. The existance of such a δ implies that the set of AST-3 representations of the type M[A,B,C] is divided into four equivalence classes under the commutative diagram equivalence relation.

<u>Theorem 4</u> If (A,B,C) and (A_1,B_1,C_1) are any two points on the surface $x^2 = yz$ (excluding the origin), then there exists an affine motion δ such that (8), (9) and (10) are satisfied.

<u>Proof:</u> The point (A,B,C) can be taken into the point (A_1,B_1,C_1) by use of some or all of the following:

a) $(A,\frac{A^2}{C},C) \longrightarrow (A', \frac{(A')^2}{C'}, C')$ where

A,A',C and $C' \neq 0$ and C and C′ have the same sign.

b) $(0,0,C) \longrightarrow (A', \frac{(A')^2}{C'}, C')$ where C,

A′ and $C' \neq 0$ and C and C′ have the same sign.

c) $(0,C,0) \longrightarrow (A', \frac{(A')^2}{C'}, C')$ where A′,

C and $C' \neq 0$ and C and C′ have the same sign.

d) $(C,C,C) \longrightarrow (C', C', C')$ where C and C′ $\neq 0$ and C and C′ are of opposite sign.

It is geometrically evident that the point (A,B,C) can be taken into the point (A_1,B_1,C_1) by use of a composition of the above maps or their inverses. The δ that is used to satisfy (8), (9) and (10) is the product of the various

"δ's" used in the individual maps, the order of the factors being the same as the order of the mappings in the composition.

It remains then to show that for each of the above maps an affine motion δ can be found which satisfies (8), (9) and (10).

1. The mapping (a) is obtained by choosing in δ;

   $a = f = j = 1,\ e = h = 0,$

   $$i = \frac{-A'C \pm |A| \sqrt{C'C}}{C'C} \quad \text{and} \quad g = \frac{-C' \pm \operatorname{sgn}A \sqrt{C'C}}{A'} .$$

   The choice of either the positive or negative root in both i and g depends upon which sign satisfies the condition $|A| \sqrt{C'C} \neq \pm (C'A + CA')$.

   The derivation of the above is obtained by choosing a, f and j = 1 to eliminate the i and g of theorem 3, part 3c1. e = h = 0 comes from theorem 2. The above formulas for i and g are obtained by direct substitution of the points of the mapping into (8), (9) and (10). The inequality condition is the condition that $1 - gi \neq 0$ which is required if δ is to be an affine motion.

2. The mapping (b) is obtained by choosing in δ;

   $a = f = j = 1,\ e = h = 0,\ \ i = \frac{-A'}{C'}$ and

$g = \dfrac{-C' \pm \operatorname{sgn}A' \sqrt{C'C}}{A'}$ where one can choose either the positive or negative root. The derivation of this result is obtained in the same manner as the derivation of the mapping (a).

3. The mapping (c) is obtained by choosing in δ; $a = f = j = 1$, $e = h = 0$, $i = \dfrac{-A' \pm \operatorname{sgn}C' \sqrt{C'C}}{C'}$ and $g = \dfrac{-C'}{A'}$ where one can choose either the positive or negative root. The derivation of this result is obtained in the same manner as the derivation of the mapping (a).

4. The mapping (d) is obtained by choosing in δ; $a = -1$, $f = j = 1$, $e = h = 0$ and $i = g = -1 - \sqrt{\dfrac{-C}{C'}}$. The derivation of this result is obtained in essentially the same manner as the derivation of the mapping (a). The choice of $a = -1$ is because C and C' are of opposite sign.

<u>Theorem 5</u> If (A,B,C) and $A_1,B_1,C_1)$ are any two points on the "inside" of the surface $x^2 = yz$, then there exists an affine motion δ such that (8), (9), (10) and (11) are satisfied.

Proof: The point (A,B,C) can be taken into the point (A_1,B_1,C_1) by use of some or all of the following:

(a) $(A,\frac{A^2}{D},C) \longrightarrow (A',\frac{(A')^2}{D},C)$ where $|D| < |C|$, D, A, A' and $C \neq 0$ and D and C have the same sign.

(b) $(0,\frac{A^2}{D},C) \longrightarrow (A,\frac{A^2}{D},C)$ where $|D| < |C|$, A, C, and $D \neq 0$ and D and C are of the same sign.

(c) $(D, D, C) \longrightarrow (D', D', C')$ where D, C, D', and $C' \neq 0$, $|D| < |C|$, $|D'| < |C'|$, D and C are of the same sign and D' and C' are of the same sign.

It is geometrically evident that the point (A,B,C) can be taken into the point (A_1,B_1,C_1) by use of a composition of the above maps or their inverses. The δ that is used to satisfy (8), (9), (10) and (11) is the product of the various "δ's" used in the individual maps, the order of the factors being the same as the order of the mappings in the composition.

It remains then to show that for each of the above maps an affine motion δ can be found which satisfies (8), (9), (10) and (11).

1. The mapping (a) is obtained by choosing in δ;

$f = j = 1$, $e = h = 0$, $i = \frac{A-A'}{2D}$, $g = \frac{-C(A-A')}{2AA'}$ and

$$a = \frac{A'}{A} + \frac{C}{4A^2 D}(A-A')^2.$$

The derivation of this result is obtained by choosing $f = j = 1$ to eliminate the i and g of theorem 3, part 1. $e = h = 0$ comes from theorem 2. The above formulas for a, i and g are obtained by substituting the points of the mapping in (11), and then choosing $a = \frac{A'}{A}(1-gi)$ and using this in equations (8), (9), and (10) to find i and g.

2. The mapping (b) is obtained by choosing in δ;

$f = j = 1$, $e = h = 0$, $i = \frac{A}{D}\left[-1 + \sqrt{\frac{C-D}{C}}\right]$,

$g = \frac{C}{A}\left[-1 + \sqrt{\frac{C-D}{C}}\right]$ and $a = 2\left[1 - \frac{C}{D} + \frac{C}{D}\sqrt{\frac{C-D}{C}}\right]\sqrt{\frac{C-D}{C}}$.

The derivation of this result is obtained in the same manner as the derivation for the mapping (a).

3. The mapping (c) is obtained by choosing in δ;

$f = j = 1$, $e = h = 0$ and

(a) If $RD + D' \neq 0$, where $R = \sqrt{\frac{(D')^2 - D'C'}{D^2 - DC}}$,

$i = \frac{RD - D'}{RD + D'}$, $g = \frac{RC - C'}{RD + D'}$ and

$$a = \left[1 - \frac{(RD - D')(RC - C')}{(RD + D')^2}\right] R$$

(b) If $RD + D' = 0$ then $RC + C' = 0$ and $i = 0$, $g = 0$ and $a = -R$ are chosen.

The derivation of part (a) of the above result is obtained in the same manner as the derivation for the mapping (a). For part (b) above if $RD + D' = 0$, the definition of R implies $RC + C' = 0$. Analysis of equations (8), (9), (10) and (11) reveal the above choices of i, g and a will yield the desired mapping.

Theorem 6 If (A,B,C) and (A_1,B_1,C_1) are any two points on the "outside" of the surface $x^2 = yz$, then there exists an affine motion δ such that (8), (9), (10) and (11) are satisfied.

Proof: The point (A,B,C) can be taken into the point (A_1,B_1,C_1) by use of some or all of the following:

(a) $(A, \frac{A^2}{D}, C) \longrightarrow (A', \frac{(A')^2}{D}, C)$ where $|D| > |C|$ if D and C are of the same sign and A, A′, D and $C \neq 0$.

(b) $(A, \frac{A^2}{D}, 0) \longrightarrow (A', \frac{(A')^2}{D}, 0)$ where A, A′ and $D \neq 0$.

(c) $(0, \frac{A^2}{D}, C) \longrightarrow (A, \frac{A^2}{D}, C)$ where A, C and $D \neq 0$ and C and D are of opposite sign.

(d) $(A, 0, C) \longrightarrow (A, \frac{A^2}{D}, C)$ where A and $D \neq 0$

and $|D| > |C|$ if C and D are of the same sign.

(e) $(D, D, C) \longrightarrow (D', D', C')$ where $|D| > |C|$ if D and C have the same sign, $|D'| > |C'|$ if D' and C' have the same sign and D, D', C and $C' \neq 0$.

(f) $(D, D, C) \longrightarrow (D', D', 0)$ where $|D| > |C|$ if D and C are of the same sign and $D, D' C \neq 0$.

It is geometrically evident that the point (A,B,C) can be taken into the point (A_1,B_1,C_1) by use of a composition of the above maps or their inverses. The δ that is used to satisfy (8), (9), (10) and (11) is the product of the various "δ's" used in the individual maps, the order of the factors being the same as the order of the mappings in the composition.

It remains then to show that for each of the above maps an affine motion δ can be found which satisfies (8), (9), (10) and (11).

1. The mapping (a) is obtained by choosing in δ;

$f = j = 1, e = h = 0,$

(12) $i = \frac{A - A'}{2D}$, (13) $g = \frac{-C(A - A')}{2AA'}$ and

(14) $a = \frac{A'}{A} + \frac{C}{4A^2D}(A - A')^2$ providing

(15) $\left[\frac{A}{A'}\right]^2 - 2\frac{A}{A'} + 4\frac{A}{A'}\frac{D}{C} + 1 \neq 0$. If (15) fails then the mapping (a) is obtained by choosing in δ; $e = h = 0$, $f = 1 - \frac{C(A+A')}{2DA'}$, $j = 1 - \frac{C(A+A')}{2DA}$,

$g = \frac{C}{A}\left[1 - \frac{(A+A')}{2A'}\right]$, $i = \frac{A}{D}\left[1 - \frac{(A+A')}{2A}\right]$ and

$a = \left[\frac{C}{D} - 1\right]\left[\frac{-A'}{A} + \frac{C}{4A^2D}(A+A')^2\right]$.

The derivation of the above result (excluding the exception mentioned above) is the same as that for the mapping (a) of theorem 5. To derive the results for the exception we use the composition of the mappings

$(A, \frac{A^2}{D}, C) \longrightarrow (-A, \frac{A^2}{D}, C)$ and $(-A, \frac{A^2}{D}, C) \longrightarrow (A', \frac{(A')^2}{D}, C)$

where A, A', D and $C \neq 0$ and $|D| > |C|$ if D and C are of the same sign.

For the mapping $(A, \frac{A^2}{D}, C) \longrightarrow (-A, \frac{A^2}{D}, C)$,

the matrix is obtained by choosing $f = j = 1$, $e = h = 0$, $i = \frac{A}{D}$, $g = \frac{C}{A}$ and $a = \left[\frac{C}{D} - 1\right]$ where i, g and a were found by substituting the above points into formulas (12) - (14). One notes that (15) is satisfied for this mapping.

For the mapping $(-A, \frac{A^2}{D}, C) \longrightarrow (A', \frac{(A')^2}{D}, C)$

the matrix is obtained by choosing $f = j = 1$, $e = h = 0$,

$i = -\frac{(A+A')}{2D}$, $g = -\frac{C(A+A')}{2AA'}$ and $a = \frac{-A'}{A} + \frac{C}{4A^2D}(A+A')^2$,

where i, g and a were found from substituting the above points into formulas (12) - (14). One notes that (15) is satisfied for this case.

The matrix δ of the exception is the product of the matrices obtained in the above two mappings.

2. The mapping (b) is obtained by choosing in δ;

$f = j = 1$, $e = h = 0$, $i = \frac{A-A'}{2D}$, $g = 0$ and $a = \frac{A'}{A}$.

The derivation of the above result is obtained by letting $C = 0$ in the mapping (a).

3. The mapping (c) is obtained by choosing in δ; $f = j = 1$, $e = h = 0$, $i = \frac{A}{D}\left[-1 + \sqrt{\frac{C-D}{C}}\right]$, $g = \frac{C}{A}\left[-1 + \sqrt{\frac{C-D}{C}}\right]$ and $a = 2\left[1 - \frac{C}{D} + \frac{C}{D}\sqrt{\frac{C-D}{C}}\right]\sqrt{\frac{C-D}{C}}$.

The derivation of the above result is the same as that for the mapping (b) of theorem 5.

4. The mapping (d) is obtained by choosing in δ;

$f = j = 1$, $e = h = 0$, $i = \frac{A}{C}(R-1)$, $g = \frac{-D}{A}(R-1)^2$ and $a = \left[1 + \frac{D}{C}(R-1)^3\right]R$ where $R = \sqrt{\frac{D-C}{D}}$, providing $C \neq 0$ or $C \neq -8D$. If $C = -8D$ the mapping (d) is obtained by choosing in δ; $f = j = 1$, $e = h = 0$,

$i = \frac{A}{2D}$, $g = \frac{-16D}{A}$ and $a = -27$. If $C = 0$ the mapping (d) is obtained by choosing in δ; $f = j = 1$, $g = 0$, $i = \frac{-A}{2D}$, $a = 1$ and $e = h = 0$.

The derivation of the above result is obtained by direct substitution of the points of the mapping into equations (8) - (11) and choosing $a = (1-gi)\sqrt{\frac{D-C}{D}}$. $f = j = 1$ was chosen in order to eliminate the i and g of theorem 3, part 1, and $e = h = 0$ comes from theorem 2. The derivation of the results for the exception $C = -8D$ is done in the same manner as the above where we choose $a = -3(1-gi)$. The derivation of the results for the exception $C = 0$ is done in the same manner as the above where we choose $a = 1-gi$.

5. The mapping (e) is obtained by choosing in δ;

   $f = j = 1$, $e = h = 0$ and

   a) If $RD + D' \neq 0$ and $-(D'C + C'D + 2D'D) \neq 2R(D^2 - DC)$ where $R = \sqrt{\frac{(D')^2 - D'C'}{D^2 - DC}}$ then $i = \frac{RD-D'}{RD+D'}$, $g = \frac{RC-C'}{RD+D'}$ and $a = \left[1 - \frac{(RD-D')(RC-C')}{(RD+D')^2}\right]R$.

   b) If $RD+D' = 0$ then $RC+C' = 0$ and we choose $i = g = 0$ and $a = -R$.

   c) If $RD-D' \neq 0$ and $-(D'C + C'D + 2DD') = 2R(D^2-DC)$

we choose $i = \dfrac{RD+D'}{RD-D'}$, $g = \dfrac{RC+C'}{RD-D'}$ and

$$a = -\left[1 - \frac{(RD+D')(RC+C')}{(RD-D')^2}\right] R.$$

d) If $RD-D' = 0$ then $RC-C' = 0$ and we choose

$i = g = 0$ and $a = R$.

The derivation of the above is obtained in the same manner as that of the mapping (c) of theorem 5.

6. The mapping (f) is obtained by choosing in δ;

$f = j = 1$, $e = h = 0$, $i = \dfrac{RD-D'}{RD+D'}$, $g = \dfrac{RC}{RD+D'}$, and

$a = \left[1 - \dfrac{(RD-D')(RC)}{(RD+D')^2}\right] R$ providing $C \neq -8D$. If

$C = -8D$ we choose $f = j = 1$, $e = h = 0$, $i = -2$, $g = 4$

and $a = \dfrac{-3D'}{D}$.

The derivation of the above is carried out by substituting $C' = 0$ in 5a above. For the exception $C = -8D$ we substitute $C' = 0$ and $C = -8D$ in part 5c above.

<u>§5</u>

Theorem 7 If we require an affine motion δ such that

$$\delta \begin{bmatrix} 1 & Ax+By & Cx+Ay & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix} \delta^{-1} = \begin{bmatrix} 1 & A'u+B'v & C'u+A'v & w \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and $u = x$, $v = y$ and $w = z$, then

1. If $A = B = C = 0$ we obtain $a = f = j = 1$, $e = g = h = i = b = c = 0$ and $A' = B' = C' = 0$. This implies δ is in $(M[A,B,C])$.
2. If not all A,B,C are zero we obtain $a = f = j = 1$, $e = g = h = i = 0$, $b = \sigma B + \nu A$ and $c = \sigma A + \nu C$. Moreover, $A = A'$, $B = B'$ and $C = C'$. This implies δ is in $(M[A,B,C])$.

Proof:

1. From theorem 2 we obtain $A' = B' = C' = 0$ and $w = z = az + by + cx$, $v = y = ez + fy + gx$ and $u = x = hz + iy + jx$. Thus, $a = f = j = 1$ and $e = g = h = i = b = c = 0$.
2. From theorem 2, $e = h = 0$ and $u = x = iy + jx$, $v = y = fy + gx$. Thus, $f = j = 1$ and $i = g = 0$. Moreover $w = z = -a\sigma(Ax + By) - a\nu(Cx + Ay) + az + by + cx$. This implies, $a = 1$, $b = \sigma B + \nu A$ and $c = \sigma A + \nu C$. Moreover, the equations for A', B', and C' on page 22 imply $A = A'$, $B = B'$ and $C = C'$.

Appendix 3

§1

Theorem 1 If the matrices of an AST-3 group have the form

$$\begin{bmatrix} 1 & f(x,y,z) & g(x,y,z) & z \\ 0 & 1 & h(x,y,z) & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and $h \not\equiv 0$, then $h(x,y,z) = Ex$, $f(x,y,z) = Ax$ and $g(x,y,z) = Cx + Ay$.

Proof: Since the group is abelian

(1) $x'h(x,y,z) = xh(x',y',z')$

(2) $h(x,y,z)f(x',y'z') = h(x',y'z')f(x,y,z)$

(3) $y'f(x,y,z) + x'g(x,y,z) = yf(x',y',z') + xg(x',y',z')$.

Let $x' = 1$, $y' = 0$, $z' = 0$, $h(1,0,0) = E$, $f(1,0,0) = A$ and $g(1,0,0) = C$. Then (1) becomes $h(x,y,z) = Ex$, (2) becomes $f(x,y,z) = Ax$ and (3) becomes $g(x,y,z) = Cx + Ay$.

§2

Theorem 2 If

$$(3)\quad \delta \begin{bmatrix} 1 & Ax & Cx+Ay & z \\ 0 & 1 & Ex & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix} \delta^{-1} = \begin{bmatrix} 1 & A'u & C'u+A'v & w \\ 0 & 1 & E'u & v \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\delta = \begin{bmatrix} a & b & c & \varepsilon \\ e & f & g & \sigma \\ h & i & j & \nu \\ 0 & 0 & 0 & 1 \end{bmatrix}$

and $D = \det \delta \neq 0$, then

1. $i = h = 0$
2. $$w = \frac{a[a(g\nu - j\sigma) + e(j\varepsilon - c\nu)]Ax}{D} - \frac{a\nu(Cx+Ay)}{j} - \frac{b\nu Ex}{j} + az + by + cx,$$

   $$v = \frac{e[a(g\nu - j\sigma) + e(j\varepsilon - c\nu)]Ax}{D} - \frac{e\nu(Cx+Ay)}{j} - \frac{f\nu Ex}{j} + ez + fy + gx \text{ and}$$

   $u = jx$ (Note: the determinant of the matrix of the coefficients of x, y and z is $D \neq 0$).
3. If $A \neq 0$, then $e = 0$ and $fE = E'j^2$, $aA = fjA'$ and

   $$C' = \frac{-2agA}{fj^2} + \frac{aC+bE}{j^2} + \frac{aAE\nu}{j^3}$$
4. $A = 0$ if and only if $A' = 0$ and $eC + fE = E'j^2$ and $aC + bE = C'j^2$.

Proof:

1. In (3) the element in row 2 column 3 equals E' times the element in row 3 column 4. This yields h, the coefficient of z, is zero. Moreover, row 3

column 1 = 0 and row 3 column 2 = 0. This implies the coefficients, i^2eE and i^2aE, of x equal zero. Thus, i = 0.

2. In (3) we find the elements of the fourth column are the desired results.

3. In (3), using i = h = 0, row 1 column 1 must equal 1 and row 2 column 1 must equal zero. Thus, the following coefficients of x are zero; aejA = 0 and $e^2jA = 0$. This implies e = 0. From (3) we also require row 2 column 3 to equal E′u; row 1 column 2 to equal A′u and row 1 column 3 to equal C′u+A′v, where u and v are as in part 2 with i = h = e = 0. We equate the coefficients of x and y and obtain the desired results.

4. If A = 0 the relations amongst the entries of (3) cited above, with i = h = 0, and the formulas for u and v imply that A′ = 0, $eC + fE = E'j^2$ and $aC + bE = C'j^2$.

If A′ = 0 and A ≠ 0 we find from $e^2jA = 0$ that e = 0. From row 1 column 2 equals A′u we find, with i = h = 0, that a = 0. This contradicts D ≠ 0. Thus, A = 0.

It is clear from theorem 2 that the existance of an affine motion δ implies the set of AST-3 representations of the form N[A,C,E] is divided into two equivalence classes under the commutative diagram equivalence relation.

§3

Theorem 3 If in (3) we require $w = z$, $v = y$ and $u = x$, then $e = h = i = 0$, $a = f = j = 1$, $b = \nu A$, $g = \nu E$, $c = \sigma A + \nu C$, $A = A'$, $C = C'$ and $E = E'$. From the above it is clear that δ is contained in N[A,B,C].

Proof: From theorem 2 parts 1 and 2 we obtain $e = h = i = 0$, $a = f = j = 1$, $b = \nu A$, $c = \sigma A + \nu C$ and $g = \nu E$. From the relationships for row 1 column 2 $= A'x$, row 2 column 3 $= E'x$ and row 1 column 3 $= C'x + A'y$ we obtain $A = A'$, $C = C'$ and $E = E'$.

Appendix 4

Theorem 1 The conditions under which an affine motion δ satisfies (1),

$$(1):\delta\begin{bmatrix} 1 & Ax & Cx+Ay & z \\ 0 & 1 & Ex & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix}\delta^{-1} = \begin{bmatrix} 1 & A'u+B'v & C'u+A'v & w \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $E \neq 0$, are; $w = az + by + cx + \alpha\frac{(aC+bE)}{D}x$,

$$v = ez + fy + gx + \alpha\frac{(eC+fE)}{D}x \text{ and}$$

$$u = hz + iy + jx + \alpha\frac{(hC+iE)}{D}x \text{ where}$$

$D = \det \delta \neq 0$ and $\alpha = -\begin{vmatrix} a & b & \epsilon \\ e & f & \sigma \\ h & i & \nu \end{vmatrix}$, and;

1. If $A \neq 0$ (1) is not possible.
2. If $A = 0$ then $ei = fh$ and $(A')^2 = B'C'$ for A', B', C' not all zero.
3. If $A = 0$, $C \neq 0$ and $bh - ai \neq 0$ then
   a) $A' = B' = 0$, $C' \neq 0$ is not possible.
   b) $A' = C' = 0$, $B' \neq 0$ implies $e = f = 0$,
      $B'g^2 = aC + bE = E\frac{(bh-ai)}{h}$ and $C = \frac{-iE}{h}$.

c) If A', B' and $C' \neq 0$ then $C' = \frac{-A'e}{h}$, $B' = \frac{-A'h}{e}$,

$C = \frac{-iE}{h}$ and $(hg-ej)^2A' = eE(ai-bh) = hE(af-be)$.

4. If $A = 0$, $C \neq 0$ and $af - be \neq 0$, then

a) $A' = C' = 0$, $B' \neq 0$ is not possible.

b) $A' = B' = 0$, $C' \neq 0$ implies $i = h = 0$,

$C'j^2 = aC + bE = \frac{E(be-af)}{e}$ and $C = \frac{-fE}{e}$.

c) If A',B' and $C' \neq 0$, then $C' = \frac{-A'e}{h}$, $B' = \frac{-A'h}{e}$,

$C = \frac{-iE}{h}$, $(hg-ej)^2A' = eE(ai-bh) = hE(af-be)$.

5. If $A = C = 0$, then $f = i = 0$ and

a) If $bh-ai \neq 0$, then

1. $A' = B' = 0$, $C' \neq 0$ is not possible.

2. $A' = C' = 0$, $B' \neq 0$ implies $B'g^2 = bE$ and $e = 0$.

3. A',B',$C' \neq 0$ implies $C' = \frac{-A'e}{h}$, $B' = \frac{-A'h}{e}$ and

$(hg-ej)^2A' = -ebhE$.

b) If $af-be \neq 0$ then

1. $A' = C' = 0$, $B' \neq 0$ is not possible.

2. $A' = B' = 0$, $C' \neq 0$ implies $h = 0$ and $C'j^2 = bE$.

3. A',B',$C' \neq 0$ implies $C' = \frac{-A'e}{h}$, $B' = \frac{-A'h}{e}$

and $(hg-ej)^2A' = -ebhE$.

Proof:

The equations for w, v and u follow from multiplying (1) out and equating the results for the fourth column with w, v and u , using the fact from below that $A = 0$.

1. In (1) the fact that certain entries are to be 0 or 1 yield the following coefficients of y:

(2) $e(ei-fh)A = 0$, $e(bh-ai)A = 0$ and $e(af-be)A = 0$.

If $A \neq 0$ and we assume $e \neq 0$ we will contradict $D \neq 0$. Thus, $e = 0$. We also have a set of equations similar to (2) except the factor e is replaced by h. These yield $h = 0$. With $e = h = 0$ we have for the zero or one entries of (1) the following coefficients of x:

(3) $-aifE = 0$, $-ai^2E = 0$ and $af^2E = 0$.

If $a \neq 0$, then $f = i = 0$. This implies $D = 0$, a contradiction. Thus, $a = 0$. However, $a = e = h = 0$ implies $D = 0$, a contradiction. Therefore, $A \neq 0$ is not possible.

2. If $A = 0$ we obtain from (1) the following coefficients of x:

(4) $(aC+bE)(ei-fh) = 0$, $(eC+fE)(ei-fh) = 0$ and $(hC+iE)(ei-fh) = 0$.

If $ei-fh \neq 0$ and either $C = 0$ or $C \neq 0$ we contradict $D \neq 0$. Therefore, $ei = fh$. Moreover, in (1) we have $A'u + B'v$ = row 1 column 2 and $C'u + A'v$ = row 1 column 3. Since $A = 0$ we have the coefficients of y and z in the above equations are zero. This implies

(5) $A'h + B'e = 0$, $C'h + A'e = 0$,

$A'i + B'f = 0$, $C'i + A'f = 0$.

If $(A')^2 - B'C' \neq 0$ we contradict $D \neq 0$. Thus, $(A')^2 = B'C'$.

If $A' = B' = C' = 0$ then row 1 column 2 and row 1 column 3 of (1) are zero. This yields the first two equations below. Moreover, from (1) the other entries that are 0 or 1 yield the last four equations below for the coefficients of x.

(6) $(aC+bE)(bh-ai) = 0$, $(aC+bE)(af-be) = 0$,

$(eC+fE)(bh-ai) = 0$, $(hC+iE)(bh-ai) = 0$,

$(eC+fE)(af-be) = 0$ and $(hC+iE)(af-be) = 0$.

The above equations and the fact that $ei = fh$ imply $D = 0$, a contradiction. Thus, A',B' and C' are not all zero.

3. $A = 0$, $C \neq 0$ and $bh - ai \neq 0$

a) If $A' = B' = 0$, $C' \neq 0$ then $A'u + B'v = 0$.

This implies $(aC+bE)(bh-ai) = 0$. Moreover, the last four equations of (6) are true. However, this contradicts $D \neq 0$. Therefore, $A' = B' = 0$ and $C' \neq 0$ is not possible.

b) If $A' = C' = 0$ and $B' \neq 0$ then from (5) $A'h+B'e = 0$ and $A'i+B'f = 0$. This implies $e = f = 0$. Moreover, from (6), since $(bh-ai) \neq 0$, we have $eC+fE = 0$ and $hC+iE = 0$. Also, row 1 column 2 $= A'u+B'v$. Equating the coefficients of x yields $(aC+bE)(bh-ai) = B'[(eC+fE)\alpha+Dg]$. These equations yield the desired results.

c) If A', B' and $C' \neq 0$ we have from (6) $hC+iE = 0$ and $eC+fE = 0$. This implies $D = \frac{(hg-ej)(aC+bE)}{E}$. The equations (5) and the above imply the desired results for C, C' and B'.

We equate the coefficients of x from $C'u+A'v =$ row 1 column 3 of (1) and obtain $(aC+bE)(af-be) = D(C'j+A'g)$. This implies

$$\frac{(hg-ej)(aC+bE)}{E}\left[-\frac{A'ej}{h}+A'g\right] = (aC+bE)(af-be).$$

Thus, $(hg-ej)^2A' = hE(af-be) = eE(ai-bh)$.

4. The proof for each part is essentially the same as

for part 3 above.

5. If $A = C = 0$ we have from (1) for the entries that are 0 or 1 the following coefficients of x:

(7) $fE(bh-ai) = 0$, $iE(bh-ai) = 0$,

$fE(af-be) = 0$ and $iE(af-be) = 0$.

This implies $i = f = 0$. Parts (a) and (b) follow respectively from parts 3 and 4 above.

Theorem 2 There exists no affine motion δ such that (1) is true if we require $w = z$, $v = y$ and $u = x$.

Proof: In theorem 1 above if $w = z$, $v = y$ and $u = x$ we obtain $a = 1$, $b = 0$, $c = \nu C$, $e = 0$, $f = 1$, $g = \nu E$, $h = 0$, $i = 0$ and $j = 1$. Moreover, from the entries that are 0 or 1 we obtain $e(ec-ag)A + e(af-be)C + fE(af-be) = 0$ as a coefficient of x. However, the above results for e, a and f imply $E = 0$, a contradiction.

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AUTOBIOGRAPHICAL STATEMENT

Harvey Braverman was born in Brooklyn, New York on November 12, 1940. He was raised in Brooklyn and completed all his preliminary education there. He has received a B.S. and an M.A. from Brooklyn College of the City University of New York. Since 1965 he has been a student at the Graduate Center of the City University of New York. Since 1964 he has been a member of the instructional staff of New York City Community College and currently holds the rank of Assistant Professor and coordinator of Mathematics at the Voorhees Campus of New York City Community College. He is married, has one child and lives in New Jersey.