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SOME ASPECTS OF STOCHASTIC ELECTRODYNAMICS CONNECTED WITH THE THERMAL EFFECTS OF ACCELERATION FOR HARMONIC OSCILLATOR DIPOLE SYSTEMS

by

Daniel C. Cole

A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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PREFACE

While an undergraduate, I became interested in a number of questions regarding the possibility of a deterministic description of nature. Is a determininistic description of nature incompatible with quantum theory? If such a theory could indeed exist, then wouldn't it necessarily be more fundamental than quantum theory? Would there be anything useful in such a description, or would such a theory simply entail the hypothesizing of additional "hidden variables" to quantum theory that only result in an entangled, complicated, and unverifiable description of nature?

Such questions have certainly been raised and discussed by many past researchers in physics. L. E. Ballentine's 1970 article entitled, "The Statistical Interpretation of Quantum Theory," provided me with a list of references to past research related to these questions. The early hidden variable model of D. Bohm fascinated me, as none of my early quantum mechanics courses ever mentioned that alternative theories were even at all possible. Other work that occupied my attention was research by J. S. Bell and work by E. Nelson.

Eventually, I came across the theory of stochastic electrodynamics, on which this thesis is based. The two main authors of this theory, whose work has influenced me, are T. W. Marshall and T. H. Boyer. I was particularly pleased to find the articles of these two researchers, for their detailed and sometimes lengthy calculations were usually carefully carried out, with few grandiose claims of revelations in physics except for the basic facts their papers set out to show. In such a speculative area as hidden variable theories, these articles were, indeed, a rare welcome.

The basic theory of stochastic electrodynamics impressed me greatly. Hypothesizing that an electromagnetic background radiation field, present even at zero temperature, must necessarily exist in order to maintain a stable equilibrium behavior for both classical charged particles and classical electromagnetic radiation, struck me as most appealing. Other features of this theory intrigued me. First, the demands of homogeneity, isotropy, and Lorentz invariance, limit the functional form of this backgound radiation spectrum. Second, the stochastic nature of this zero-point radiation has the same fundamentally deterministic character as does classical thermal electromagnetic radiation, as viewed by physicists before 1900. Finally, that such a simple hypothesis of a stochastic classical electromagnetic zero-point radiation field could qualitatively, at least, account for such perplexing phenomena as the stability of a classical hydrogen atom, particle diffraction and interference effects, and the tunneling of small particles through classically impenetrable potential barriers, seemed extremely beautiful to me.

Quantitatively, the phenomena just mentioned have not been described within the theory of stochastic electrodynamics, despite vigorous attempts by Boyer, Marshall, and others. Aside from the analysis of linear mechanical systems, detailed calculations remain to be carried out before such phenomena can be shown to be explained by stochastic electrodynamics, or, for that matter, to be shown to be incompatible with this theory.

After reading many of Professor Boyer's articles, I quickly realized how much I had to learn from someone with such deep physical insight into nature, as he clearly strikes me as possessing. I sincerely thank him for letting me work with him during the past two years, for openly sharing his thoughts and ideas with me, and in aiding me to improve my own research abilities. I greatly admire his ingenuity, conviction, and determination in carrying on the difficult research in stochastic electrodynamics, in such a way that is largely respected by the majority of physicists, despite their strong doubts that the theory of stochastic electrodynamics will lead to a correct description of nature.

Certainly this thesis does not pretend to answer any of the questions mentioned at the beginning of this Preface. Instead, these questions merely serve to indicate my own interests in choosing this thesis topic. However, I do

believe that the research described here contributes to the further investigation on the theory of stochastic electrodynamics. Moreover, the simpler conceptual framework of stochastic electrodynamics has enabled the calculations in this thesis to precede similar quantum electrodynamical calculations. Hence, at the very least, the calculations contained here (in particular, Part Two, on the thermal effects of acceleration for a spatially extended system) should aid quantum researchers in their studies of similar systems.

I greatly wish to thank my wife, Judy, for her support and understanding throughout my physics studies. She agreed to move with me from Vermont to New York City so that I might work with Professor Boyer. Only a non-city dweller can appreciate the courage such a move requires! She has nearly always listened to me when I have occasionally talked on about physics, rarely revealing that she may not understand what I might be saying. My only fear is that her own deep interest in psychology may explain what I perceive to be her apparent politeness!

I also wish to thank, in particular, my parents, as well as relatives and friends for their interest and support in my work. Lastly, I thank IBM for the financial support they have given me during the past two years to pursue my doctorate.

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PART TWO

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INTRODUCTION

Stochastic electrodynamics is a classical electron theory in which Planck's constant appears as a scale factor for what has been termed classical electromagnetic zero-point radiation. This theory of stochastic electrodynamics has been investigated by researchers to find its description of nature and its connection with quantum theory. For certain systems, its predictions have been shown to be equivalent to those of quantum theory; however, within these areas, stochastic electrodynamics often provides a simpler calculational scheme and a more transparent interpretation than quantum electrodynamics.

This thesis explores aspects of stochastic electrodynamics when its advantages are clear. Using this theory, new calculations are presented that are in the forefront of the current work on the thermal effects of acceleration through the vacuum. For the first time in either the classical or quantum literature, the thermal effects of acceleration are shown to hold for a particular spatially extended situation. Previous research had only treated point-like systems.

A topic that occupies a significant portion of this thesis is the van der Waals force between polarizable particles. In the past, stochastic electrodynamics has been shown to provide certain calculational advantages over quantum theory with regard to evaluating van der Waals forces. Here, these advantages are particularly evident, as the theory of stochastic electrodynamics provides a clear calculational scheme for evaluating the thermal-like van der Waals force between two accelerating oscillators.

This thesis is divided into three main parts. The problems analyzed here involve the behavior of classical harmonic dipole oscillators situated in homogeneous, isotropic random (Gaussian) classical electromagnetic radiation. Part One of this thesis finds the equation of motion for a small harmonic dipole oscillator uniformly accelerating through classical electromagnetic zero-point radiation. Part Two analyzes certain statistical properties of a pair of these oscillators accelerating through classical electromagnetic zero-point radiation and interacting with each other via emitted electromagnetic dipole radiation. In particular, the thermal-like van der Waals force between two accelerating oscillators is calculated here. Part Three relates the two-point field correlation functions of homogeneous, isotropic random classical electromagnetic radiation to the electromagnetic fields of a stationary fluctuating electric dipole. The relationships between these two quantities are then used to deduce van der Waals forces and other properties for unaccelerated oscillators immersed in random classical electromagnetic radiation.

An underlying element of commonality, other than stochastic electrodynamics, that exists between the problems on harmonic dipole oscillator systems treated in this thesis, is the subject of the thermal effects of acceleration through the vacuum. The first two parts of this thesis are contained entirely under this topic; however, the majority of the calculations presented in the last section are explicitly carried out only for nonaccelerating oscillator systems. Nevertheless, due to relationships found in Parts Two and Three for the random electromagnetic radiation field correlation functions, these calculations may be extended to accelerating oscillator systems. An outline presented in Part Three indicates how this extension may be accomplished.

In order to provide an appropriate setting for examining the problems in this thesis, a brief descriptive overview will now be given on the theory of stochastic electrodynamics. A current area of research in theoretical physics is involved with investigating this theory. Here, the interplay between classical electromagnetic radiation and classical charged particles has been reexamined in order to determine whether a significant portion of quantum phenomena may be understood in a natural way purely within the context of classical physics. Traditionally, of course, the problem of energy loss through emitted electromagnetic radiation by a classical charged particle orbiting an attractive center with an inverse law potential helped to persuade early quantum researchers that there must exist laws of nature that fall outside the domain of classical electron theory. A reanalysis (*1,2) of the equilibrium behavior for classical radiation and matter at zero and nonzero temperatures suggested, however, that an important element was missing in the early quantum researchers' work. This element consists of the hypothesis that classical electromagnetic zero-point radiation, present even at a temperature of absolute zero, needs to be taken into account in order to obtain an equilibrium behavior for classical electromagnetic radiation and classical charged particles that agrees with nature.

Hence, the theory of stochastic electrodynamics differs from traditional classical electron theory only in so far as electromagnetic zero-point radiation is taken into account. Mathematically, this random radiation enters the equations of motion as the homogeneous solution to Maxwell's equations. In traditional classical electrodynamics, this homogeneous solution is set equal to zero.

The electromagnetic zero-point radiation spectrum can be determined uniquely up to a scale factor by demanding that the spectrum be isotropic and homogeneous in space and that the expectation value for the electromagnetic energy density spectrum be Lorentz invariant. The random radiation scale factor is chosen in stochastic electrodynamics so as to provide the closest possible agreement between physical observation and theoretical prediction. It is at this point that Planck's constant enters the theory of stochastic electrodynamics by being the parameter that sets the scale of the zero-point radiation spectrum.

Thus, stochastic electrodynamics is essentially classical electron theory, but with stochastic behavior impressed upon charged particles through the presence of classical electromagnetic zero-point radiation. Determining the degree to which this theory corresponds to quantum theory and agrees with familiar and unfamiliar aspects of nature is the task that researchers in stochastic electrodynamics have set for themselves.

A number of interesting results have been obtained. Close connections have been established between quantum theory and stochastic electrodynamics for free electromagnetic fields (#3,4) and charged harmonic oscillators (#4,5). Other systems, such as the electric dipole rotator (#6) and a spinning magnetic dipole (#7), have also been found to exhibit properties closely connected with quantum theory. An understanding of these systems has led to purely classical descriptions for van der Waals forces (#8-17) and diamagnetic behavior (#5,4,18), as well as suggestive classical derivations (#1,2,19,20) of the blackbody radiation spectrum. Several reviews on stochastic electrodynamics exist which describe numerous other research efforts (#4,21-24).

Unfortunately, not all results obtained by the researchers in stochastic electrodynamics have been successful ones in terms of establishing a close connection with quantum theory. An attempt to deduce the structure of hydrogen using perturbation calculations within the framework of stochastic electrodynamics resulted in the prediction of an unstable atom (*25). Calculations of the frequency spectrum of scattered radiation from nonlinear systems seem to predict that the correct equilibrium radiation spectrum for classical nonlinear systems should be the Rayleigh-Jeans and not the Planck spectrum (*26,27,28). These last results apparently stand as contradictory arguments to the previously mentioned classical derivations of the Planck spectrum.

A thorough understanding of these negative results has not yet been obtained. Investigations into stochastic electrodynamics have been greatly impeded by the nonlinear stochastic differential equations that are encountered when trying to attack such problems as the hydrogen atom. The solutions of such equations are a largely unexplored area of applied mathematics. Approximation techniques commonly used in handling linear stochastic differential equations may result in totally erroneous results when treating their nonlinear counterparts (*29,30). Consequently, only linear systems have been treated with any substantial degree of confidence. The problems discussed in this thesis all deal with such linear mechanical systems.

Two areas of physics will now be discussed that are related to the problems analyzed within this thesis. Both subjects are areas that stochastic electrodynamics has been very successful in describing. The first has to do with van der Waals forces. For the situation of two neutral polarizable particles modeled by charged harmonic oscillators, the expectation value of the stochastic force acting between the particles at zero temperature has been calculated for all distances between the particles (#15). The potential function that arises from this calculation has been shown (#15) to agree exactly with calculations performed from the viewpoint of quantum electrodynamics (#31). Both sets of calculations were carried out nonrelativistically and both employed a dipole approximation for the sources. The resulting agreement seems most impressive in light of the fact that the two calculations agreed not just to lowest order, but to all orders in the fine structure constant. In this instance, the conceptual simplicity of stochastic electrodynamics afforded a distinct advantage to the viewpoint of quantum electrodynamics. Within the former framework, a subsequent calculation was carried out that readily generalized the earlier van der Waals force calculation to include the situation where a thermal radiation field was present in addition to the zero-point radiation field (#17). A calculation taking thermal radiation into account within quantum electrodynamics is considerably more complicated (#4). Hence, here is an example where stochastic electrodynamics provides an advantage over quantum electrodynamics by presenting a simpler calculational scheme.

A second successful application of stochastic electrodynamics, which is connected to the problems treated in this thesis, involves the prediction of thermal-like behavior of physical systems when subjected to a uniform relativistic acceleration through the so-called "vacuum". In quantum electrodynamics, the vacuum is taken to consist of the absence of all matter and of all photons; in stochastic electrodynamics, the vacuum also consists of the absence of matter, but radiation is assumed to be present in the form of classical electromagnetic zero-point radiation. Researchers in general relativity and quantum electrodynamics first predicted thermal behavior for quantum scalar fields (*32,33). When turning to examine the quantum electromagnetic field, a behavior that was not of a purely thermal nature was found to be predicted (*34). Similar results were found in the case of stochastic electrodynamics (*35).

Recently, a better understanding of these results for electromagnetic systems was gained when the behavior of an accelerated classical electric dipole oscillator was analyzed from within the context of stochastic electrodynamics (*36). Here, a purely thermal behavior was found for the accelerating oscillator. The key point in arriving at this deduction lay in considering the behavior of a physical system versus simply field correlation functions, as had been done previously in both the quantum and classical cases. The successful treatment of this problem may very well be the start of exciting work in stochastic electrodynamics and quantum electrodynamics involving the thermodynamics of electromagnetic systems suspended in gravitational fields. Thus, stochastic electrodynamics is helping to pave the way for a better understanding of the thermal effects of acceleration through the vacuum. Because of its relative conceptual and calculational simplicity, stochastic electrodynamics has preceded the quantum field theory calculations for some problems in this area.

Part One of this thesis extends the analysis of Ref. 36 by finding the equation of motion for a harmonic dipole oscillator, unrestricted in its direction of oscillation, that is uniformly accelerated through classical electromagnetic zero-point radiation (*37). Here, constraints are removed that were imposed in Ref. 36 upon the direction of oscillation for the accelerating oscillator. A Fermi-Walker transported coordinate system is introduced that significantly aids in the analysis of this accelerating system. The result is found that this accelerating harmonic dipole oscillator, without constraints upon its direction of oscillation, does indeed possess a thermal behavior.

Part Two considers two spatially separated harmonic dipole oscillators that are uniformly accelerated through classical electromagnetic zero-point radiation (*38). The equilibrium positions of the two oscillators are assumed to be constrained in such a way that they lie in a plane undergoing uniform acceleration along the direction perpendicular to the plane's surface. Using the Fermi-Walker transported coordinate system introduced in Part One, the equations of motion for this system of oscillators are deduced. Various statistical properties pertaining to this pair of oscillators are obtained, including the expectation value of the Lorentz force acting on one of the dipole oscillators. Thus, this problem combines two of the most successful applications of stochastic electrodynamics: namely, that of van der Waals forces and of thermal effects of acceleration.

Previous research on the thermal effects of acceleration, both for the quantum and classical points of view, have dealt only with point-like systems uniformly accelerating through the vacuum. Thus, the system considered in Part Two constitutes the first time that a spatially extended electromagnetic system has been examined for the thermal effects of acceleration. In particular, the force is calculated that an experimenter would measure between two closely separated dipole oscillators that are uniformly accelerated through the classical vacuum. If the thermal effects predicted for an accelerated point dipole system carry over to a spatially extended electromagnetic system, then this force between the accelerating oscillators should agree with the van der Waals force between two similarly constructed, but unaccelerated oscillators, that are situated in a thermal radiation bath. Under certain assumptions described in Part Two, such as a small laboratory approximation and a narrow linewidth approximation, this agreement is found to exist between these accelerated and unaccelerated-thermal spatially separated oscillator systems.

Identities are established in Part Two between the two-point field correlation functions obtained along trajectories described by uniform acceleration through classical electromagnetic zero-point radiation and the electromagnetic dipole fields of a uniformly accelerating fluctuating electric dipole. Without the validity of these identities, the calculations of Part Two could not have been carried through to establish the equivalence for van der Waals forces and certain statistical properties between an unaccelerated, thermally-bathed pair of dipole oscillators, and a similar pair of dipole oscillators accelerated through classical electromagnetic zero-point radiation.

Part Three establishes identities, similar to those just mentioned, between the two-point field correlation functions of homogeneous, isotropic random classical electromagnetic radiation, as evaluated at fixed spatial points in an inertial frame, and the electromagnetic fields of an unaccelerated fluctuating electric dipole (#39). These identities enable calculations to be carried out much more efficiently with regard to analysis on the behavior of electric dipoles immersed in homogeneous, isotropic random classical electromagnetic radiation. Moreover, the explicit use of these identities then enables these calculations to be extended to the corresponding case of a system of uniformly accelerated electric dipoles. These points are illustrated by a calculation in Part Three involving van der Waals forces for a system of N harmonic dipole oscillators. A second calculation in Part Three illustrates the importance of these identities in establishing that the presence of a harmonic dipole oscillator in homogeneous, isotropic random radiation does not alter the null value for the expectation value of the Poynting vector.

The work presented in this thesis does not extend stochastic electrodynamics beyond harmonic systems plus radiation. Instead, harmonic dipole oscillator systems are explicitly used here in order to obtain, in particular, a deeper physical understanding of electromagnetic systems accelerating through classical electromagnetic zero-point radiation. Undoubtedly, much of this work will be taken over into the literature of quantum fields in curved spacetime. The last section of this thesis contains material in addition to the subject of the thermal effects of acceleration through the vacuum. Here, a relationship is found for radiation field correlation functions that aid in calculations involving dipole oscillators, such as van der Waals forces between dipole harmonic oscillators.

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PART ONE

PROPERTIES OF A CLASSICAL CHARGED HARMONIC OSCILLATOR ACCELERATED THROUGH CLASSICAL ELECTROMAGNETIC ZERO-POINT RADIATION

I. INTRODUCTION

An observer uniformly accelerating through the vacuum of a scalar quantum field was found by Unruh and Davies (*1,2) to observe a Planckian spectrum of the scalar field characterized by the temperature of $T = \hbar a/2\pi ck$. Unfortunately, this beautifully simple relationship was not found for the correlation functions of the quantized electromagnetic field (*3).

Results analogous to those of quantum field theory were shown to exist within the context of classical theory for the situations of an observer uniformly accelerating through classical scalar zero-point radiation and through electromagnetic zero-point radiation (*4). These results were obtained by examining the correlation functions of the zero-point fields along a trajectory in space-time described by uniform acceleration. Hence, even here, the Planckian spectrum seen on acceleration in the scalar case did not carry over to the electromagnetic situation.

Recently, however, Boyer (*5) was able to recover Planck's spectrum within classical electromagnetism by considering the behavior of a charged harmonic oscillator uniformly accelerated through classical electromagnetic zero-point radiation. Thus, instead of simply examining the correlation functions of the electromagnetic field along a path described by uniform acceleration, the physical behavior of a uniformly accelerated electromagnetic system was analyzed. A key feature in solving the equation of motion of the oscillator was to retain all terms in the full relativistic radiation reaction expression except those that were negligible due to the assumed small size of the oscillator. From the solution of this equation and from the assumed statistical properties of the zero-point electromagnetic field, the second order moments were obtained for the displacement and velocity of the oscillating particle as seen by an observer uniformly accelerating with the system. These properties agreed exactly with those of a similar oscillator in an inertial frame but bathed in a Planckian classical electromagnetic radiation spectrum characterized by the Unruh-Davies temperature of $T = \hbar a/2\pi c k$.

The present article generalizes the classical analysis by removing one of the restrictions previously imposed on the oscillator system. In the model mentioned above (*5), physical constraints were assumed to exist which confined the oscillations of the charged particle to a plane perpendicular to the direction of uniform acceleration. Here, these constraints are removed. Oscillations may then occur along any spatial direction. The equation of motion for longitudinal oscillations, meaning oscillations along the direction of acceleration, is more complicated than the transverse case since there are additional terms due to relativistic effects. In order to ease calculational difficulties, a coordinate system that is Fermi-Walker transported along the trajectory of the equilibrium point of the oscillator is introduced. [This approach should also allow the study of an oscillator moving through electromagnetic zero-point radiation along other space-time trajectories of interest (#6).]

The equation obtained for the motion of an oscillator along the direction of acceleration of the system agrees with the equation governing motion perpendicular to the acceleration direction, except for a change in the expression for the oscillator frequency due to "red-shift effects". Indeed, the frequency of the oscillator in the Fermi-Walker coordinate system will in general be a function of the proper acceleration of the system, even for a transverse oscillator; only when the distance between "source and field point" is negligible compared to c^2/a, can one expect the dependence of the transverse oscillator frequency upon the proper acceleration to be removed. (Boyer's results apply in this particular limit.) In order to remove the frequency dependence upon acceleration for a longitudinal oscillator, one must also impose the restriction that $cT_{os} \ll c^2/a$, where T_{os} is the period of the oscillator. This condition makes negligible the red-shift effects mentioned previously. However, the limiting case of $cT_{os} \ll c^2/a$ is not an interesting situation in which to examine the thermal effects of acceleration, for this condition implies that the thermal energy associated with the acceleration is small compared to the zero-point energy of the oscillator. Thus, $cT_{os} \ll c^2/a$ implies that

$$kT = \frac{\hbar a}{2\pi c} \ll \frac{\hbar}{2\pi T_{os}} \approx \hbar\omega \, .$$

Therefore, in order to observe the thermal effects of acceleration in the longitudinal oscillations, it appears that one cannot impose this restriction.

Consequently, the frequency of the longitudinal oscillator in the accelerating coordinate system will depend upon the value of the proper acceleration of the system. As noted earlier, the same situation exists for the transverse oscillator when the "small source to field point limit" does not apply. This frequency dependence upon acceleration presents an additional complication that did not exist in Boyer's original transverse oscillator model. Nevertheless, the essential conclusion reached by Boyer in regard to the latter model will also hold for the longitudinal and the slightly more general transverse situations considered here, provided the change in frequency with acceleration is taken into account. Thus, let $\omega(a)$ be the frequency of the oscillator accelerating in classical zero-point radiation as seen by an observer moving with the equilibrium point of the oscillator. Let ω' be the frequency of an oscillator situated in an inertial frame with Planckian electromagnetic radiation characterized by $T = \hbar a / 2\pi c k$. If the two frequencies are selected so that $\omega(a) = \omega'$, then the statistical behavior of the oscillators, as observed in their respective coordinate systems, will be identical.

II. NEUTRAL OSCILLATOR IN HYPERBOLIC MOTION

The system that will be considered first is a neutral particle of rest mass m oscillating at the end of a massless spring, the equilibrium point of which moves with uniform proper acceleration $\vec{a}$. Assume that a constant force $\vec{f}_c = m\vec{a}$ exists so as to provide a uniform acceleration to the particle if the spring was not present. As described in an inertial frame I_* , the equation of motion for the displacement $\vec{x}_*(t)$ of the particle from the equilibrium point of the spring is given by

$$m\frac{d^2x_*^\mu}{d\tau^2} = F_{*sp}^\mu + F_{*c}^\mu \quad . \qquad (1)$$

The quantities F_{*sp}^μ and F_{*c}^μ denote the four-vector forces in the I_* frame associated with the spring and the three-vector force $\vec{f}_c$ causing the acceleration. The proper time of the particle is given by τ .

If the four-vector forces F_{*sp}^μ and F_{*c}^μ were written out in terms of x_*^μ and its derivatives, they would be fairly complicated and highly nonlinear functions of the latter quantities. Hence, it seems appropriate to attempt to transform the coordinates so as to obtain a differential equation that is more manageable. A coordinate system that seems a likely choice to make is one that is Fermi-Walker transported along the path of the equilibrium point of the spring, for in such a coordinate system, the oscillating particle's behavior is naturally described relative to the equilibrium position of the spring. It will be shown that if one chooses this coordinate system, and imposes a small oscillator restriction as measured in an inertial frame instantaneously at rest with respect to the equilibrium point of the spring, then one obtains a linear differential equation that can easily be solved.

The method for constructing a Fermi-Walker transported coordinate system is described in many standard textbooks on general relativity (*7) and will simply be summarized here in order to unify notation. Let the uniform acceleration $\vec{a}$ of the spring's equilibrium point be directed along the x axis of the I_* coordinate system. The position of this equilibrium point undergoing relativistic hyperbolic motion is described in the I_* system by (*8)

$$X_*^\mu = (ct_*\,;\, \vec{X}_*(t_*)) = \left(ct_*\,;\, \frac{c^2}{a}\left[1 + \left(\frac{at_*}{c}\right)^2\right]^{1/2}, 0, 0\right) , \qquad (2)$$

where it has been assumed for convenience that $X_* = \frac{c^2}{a}$ at $t_* = 0$. By making use of the relationships

$$\frac{d\vec{X}_*}{dt_*} = \frac{at_*\hat{x}}{\left[1 + \left(\frac{at_*}{c}\right)^2\right]^{1/2}}$$

and

$$\frac{dt_*}{d\tau_e} = \left[1 - \left|\frac{d\vec{X}_*}{dt_*}\right|^2\right]^{-1/2} ,$$

where τ_e is the proper time associated with the equilibrium point of the spring, X^μ_* can be expressed as

$$X^\mu_*(\tau_e) = \left(\frac{c^2}{a}\sinh\left(\frac{a\tau_e}{c}\right) ; \frac{c^2}{a}\cosh\left(\frac{a\tau_e}{c}\right) , 0 , 0 \right) \quad . \tag{3}$$

For convenience, the proper time τ_e has been chosen to equal zero when $t_* = 0$.

Using the above description, a coordinate system can be constructed that consists of four unit four-vectors $[e_\nu(\tau_e)]^\mu$ that are Fermi-Walker transported along the path of the equilibrium point of the spring (#9). Coordinates ξ^μ in the accelerated coordinate system can then be defined by $c\tau_e = \xi^0$ and the following conditions (#10):

$$x^\mu_* = \sum_{k=1}^{3} \xi^k [e_k]^\mu + X^\mu_* \quad , \tag{4}$$

or

$$\begin{aligned} ct_* &= \left(\xi^1 + \frac{c^2}{a}\right)\sinh\left(\frac{a\tau_e}{c}\right) \quad , \\ x_* &= \left(\xi^1 + \frac{c^2}{a}\right)\cosh\left(\frac{a\tau_e}{c}\right) \quad , \\ y_* &= \xi^2 \quad , \quad z_* = \xi^3 \quad . \end{aligned} \tag{5}$$

Two characteristics of the ξ^μ coordinates make them particularly useful in describing the accelerating oscillator system. First, ξ^0/c equals the proper time associated with the equilibrium point of the spring. Second, differences in the ξ^1, ξ^2, and ξ^3 coordinates are equal to the corresponding differences in the x, y, and z coordinates of an inertial frame instantaneously at rest with respect to the equilibrium point of the spring. Hence, lengths measured in the instantaneous rest system are equal to lengths expressed in terms of ξ^i, $i=1,2,3$. [This can immediately be deduced from Eqs. (7) below.]

The following set of inertial coordinate systems will now be introduced for future use. Let $I_{\tau_e'}$ be an inertial frame moving with speed $\frac{dx_*}{dt_*} = c\tanh\left(\frac{a\tau_e'}{c}\right)$ along the x axis of the I_* inertial frame. The I_* and $I_{\tau_e'}$ coordinate systems can be related by the Lorentz transformation

$$\begin{aligned} ct_{\tau_e'} &= ct_*\cosh\left(\frac{a\tau_e'}{c}\right) - x_*\sinh\left(\frac{a\tau_e'}{c}\right) \quad , \\ x_{\tau_e'} &= x_*\cosh\left(\frac{a\tau_e'}{c}\right) - ct_*\sinh\left(\frac{a\tau_e'}{c}\right) \quad , \\ y_{\tau_e'} &= y_* \qquad , \qquad z_{\tau_e'} = z_* \quad . \end{aligned} \tag{6}$$

From Eqs. (6), the I_* inertial frame is equivalent to the $I_{\tau_e'=0}$ system. Due to the above choice in origins of the I_* and $I_{\tau_e'}$ systems, when the equilibrium point of the spring at proper time τ_e' is given by

$$x_*^\mu = X_*^\mu(\tau_e') = \left(\frac{c^2}{a}\sinh\left(\frac{a\tau_e'}{c}\right) ; \frac{c^2}{a}\cosh\left(\frac{a\tau_e'}{c}\right) , 0 , 0\right)$$

then its position in the $I_{\tau_e'}$ inertial frame is described by $x_{\tau_e'}^\mu = \left(0 ; \frac{c^2}{a}, 0, 0\right)$. Finally, from Eqs. (5) and (6), one immediately obtains

$$ct_{\tau_e'} = \left(\xi^1 + \frac{c^2}{a}\right)\sinh\left(\frac{a}{c}(\tau_e - \tau_e')\right) \quad , \tag{7a}$$

$$x_{\tau_e'} = \left(\xi^1 + \frac{c^2}{a}\right)\cosh\left(\frac{a}{c}(\tau_e - \tau_e')\right) \quad , \tag{7b}$$

$$y_{\tau_e'} = \xi^2 \quad , \quad z_{\tau_e'} = \xi^3 \quad . \qquad (7c\ \&\ d)$$

The relationships given by Eqs. (5) will now be used to express Eq. (1) in terms of the ξ^μ coordinates of the accelerated coordinate system. By substituting Eqs. (5) into the identity $c^2 d\tau^2 = -dx_*^\mu dx_{*\mu}$, it can be shown that

$$\frac{d\tau_e}{d\tau} = \left[\left(1 + \frac{a\xi^1}{c^2} \right)^2 - \frac{1}{c^2} \frac{d\vec{\xi}}{d\tau_e} \cdot \frac{d\vec{\xi}}{d\tau_e} \right]^{-1/2} \quad . \qquad (8)$$

With the use of Eqs. (5) and (8), all derivatives of x_*^μ with respect to the proper time τ of the particle can be expressed in terms of ξ^i and the differentials of it with respect to τ_e. Hence, Eq. (1) can be rewritten in terms of the ξ^μ coordinates. Again, however, a very nonlinear differential equation will be obtained.

The assumption will now be made that the amplitude of the particle's oscillations about the equilibrium point can be made arbitrarily small (*11). Since one anticipates that the magnitude of $\frac{d^n\xi^i}{d\tau_e^n}$ will be approximately $A\omega^n$, where A is the amplitude of oscillation and ω is the frequency of the oscillator, then the restriction of small amplitude enables $\frac{d^n\xi^i}{d\tau_e^n}$ to be made arbitrarily small. It is under this assumption of small amplitude, where "small" refers to the amplitude measured by an inertial frame instantaneously at rest with respect to the accelerated equilibrium point, that the particle's equation of motion will be linearized. (Clearly the situation is far more complicated than this simple argument since important properties may be masked by linearizing inherently nonlinear differential equations. Fortunately, a harmonic potential is well suited for enforcing mathematical stability at small amplitudes. Consequently, one expects that the above argument holds in some sort of limit whereby the smaller the amplitude of oscillation, the closer the linearized equation approximates the actual one.)

Retaining terms to only first order in the ξ^i coordinates and their derivatives yields

$$\frac{d\tau_e}{d\tau} \approx 1 - \frac{a\xi^1}{c^2} \qquad (8')$$

$$\begin{aligned}
\frac{dx_*^0}{d\tau} &\approx \frac{d\xi^1}{d\tau_e}\sinh\left(\frac{a\tau_e}{c}\right) + c\cosh\left(\frac{a\tau_e}{c}\right) , \\
\frac{dx_*^1}{d\tau} &\approx \frac{d\xi^1}{d\tau_e}\cosh\left(\frac{a\tau_e}{c}\right) + c\sinh\left(\frac{a\tau_e}{c}\right) , \qquad (9) \\
\frac{dx_*^2}{d\tau} &\approx \frac{d\xi^2}{d\tau_e} , \qquad \frac{dx_*^3}{d\tau} \approx \frac{d\xi^3}{d\tau_e} ,
\end{aligned}$$

$$\begin{aligned}
\frac{d^2x_*^0}{d\tau^2} &\approx \left[\frac{d^2\xi^1}{d\tau_e^2} - \xi^1\left(\frac{a}{c}\right)^2 + a\right]\sinh\left(\frac{a\tau_e}{c}\right) + \left(\frac{d\xi^1}{d\tau_e}\frac{a}{c}\right)\cosh\left(\frac{a\tau_e}{c}\right) , \\
\frac{d^2x_*^1}{d\tau^2} &\approx \left[\frac{d^2\xi^1}{d\tau_e^2} - \xi^1\left(\frac{a}{c}\right)^2 + a\right]\cosh\left(\frac{a\tau_e}{c}\right) + \left(\frac{d\xi^1}{d\tau_e}\frac{a}{c}\right)\sinh\left(\frac{a\tau_e}{c}\right) , \qquad (10) \\
\frac{d^2x_*^2}{d\tau^2} &\approx \frac{d^2\xi^2}{d\tau_e^2} , \qquad \frac{d^2x_*^3}{d\tau^2} \approx \frac{d^2\xi^3}{d\tau_e^2} .
\end{aligned}$$

Equations (10) reexpress the left-hand side of Eq. (1) in terms of the ξ^μ coordinates. Rewriting the right side of Eq. (1) in terms of the accelerated coordinate system involves transforming the four-vector forces in an appropriate manner. Rather than directly making this transformation from the I_* frame to the Fermi-Walker transported coordinate system, the transformation will be made from I_* to the inertial frame $I_{\tau_e'}$ via the inverse of Eqs. (6). This allows one to utilize the familar connection between four-vector and three-vector forces in order to make use of the well-known expressions for three-vector forces in inertial frames. As will be shown, these expressions for F_*^μ can then be written in terms of the ξ^μ coordinates by choosing τ_e' to equal τ_e.

Using the expression $F_*^\mu = F_{\tau_e'}^\nu \frac{\partial x_*^\mu}{\partial x_{\tau_e'}^\nu}$ and the inverse of Eqs. (6) yields

$$
\begin{aligned}
F_*^0 &= F_{\tau_e'}^0 \cosh\left(\frac{a\tau_e'}{c}\right) + F_{\tau_e'}^1 \sinh\left(\frac{a\tau_e'}{c}\right) , \\
F_*^1 &= F_{\tau_e'}^0 \sinh\left(\frac{a\tau_e'}{c}\right) + F_{\tau_e'}^1 \cosh\left(\frac{a\tau_e'}{c}\right) , \qquad (11) \\
F_*^2 &= F_{\tau_e'}^2 \quad , \qquad F_*^3 = F_{\tau_e'}^3 .
\end{aligned}
$$

The four-vector forces $F_{\tau_e'}^\mu$ can then be written in terms of the three-vector forces $\vec{f}_{\tau_e'}$ via

$$
F_{\tau_e'}^\mu = \frac{dt_{\tau_e'}}{d\tau} \frac{d p_{\tau_e'}^\mu}{dt_{\tau_e'}} = \frac{dt_{\tau_e'}}{d\tau}\left(\vec{f}_{\tau_e'} \cdot \frac{d\vec{x}_{\tau_e'}}{dt_{\tau_e'}} \frac{1}{c} \; ; \; \vec{f}_{\tau_e'} \right) , \qquad (12)
$$

where $p_{\tau_e'}^\mu$ is the particle's four-momentum as measured in the $I_{\tau_e'}$ frame. The quantity

$$
\frac{dt_{\tau_e'}}{d\tau} = \left[1 - \frac{1}{c^2} \frac{d\vec{x}_{\tau_e'}}{dt_{\tau_e'}} \cdot \frac{d\vec{x}_{\tau_e'}}{dt_{\tau_e'}} \right]^{-1/2} \qquad (13)
$$

can be expressed in terms of the ξ^μ coordinates by using Eqs. (7). One obtains

$$\frac{dx^1_{\tau_e'}}{dt_{\tau_e'}} = \left[\frac{d\xi^1}{d\tau_e}\cosh\left(\frac{a}{c}(\tau_e - \tau_e')\right) + \left(c + \frac{a\xi^1}{c}\right)\sinh\left(\frac{a}{c}(\tau_e - \tau_e')\right)\right]\frac{1}{D}$$

$$\frac{dx^2_{\tau_e'}}{dt_{\tau_e'}} = \frac{d\xi^2}{d\tau_e}\frac{1}{D} \quad , \quad \frac{dx^3_{\tau_e'}}{dt_{\tau_e'}} = \frac{d\xi^3}{d\tau_e}\frac{1}{D} \quad , \tag{14}$$

where $$D = \frac{1}{c}\frac{d\xi^1}{d\tau_e}\sinh\left(\frac{a}{c}(\tau_e - \tau_e')\right) + \left(1 + \frac{a\xi^1}{c^2}\right)\cosh\left(\frac{a}{c}(\tau_e - \tau_e')\right).$$

The inertial reference frame $I_{\tau_e'}$ in which $F^{\mu}_{\tau_e'}$ is chosen to be evaluated is completely arbitrary since the transformation of Eqs. (11) applies to any value of τ_e'. This arbitrariness can be used to simplify the problem by choosing τ_e' to equal τ_e. This procedure is perfectly well defined, since from the transformations of Eqs. (6), the parameter τ_e' can be treated as a continuous variable to automatically select the inertial frame instantaneously at rest with respect to the spring's equilibrium point for all values of τ_e. Therefore, setting τ_e' equal to τ_e in Eqs. (11) is equivalent to evaluating the four-vector forces in the instantaneous rest frame of the spring's equilibrium point. In this frame, the harmonic restoring force is most simply expressed.

With the above condition, Eqs. (14) reduce to

$$\left.\frac{dx^i_{\tau_e'}}{dt_{\tau_e'}}\right|_{\tau_e'=\tau_e} = \frac{(d\xi^i/d\tau_e)}{(1 + a\xi^1/c^2)} \quad , \qquad i = 1, 2, 3 \quad . \tag{15}$$

From Eqs. (15), (13), and (12), the following results are obtained to first order in the ξ^i coordinates:

$$\left.\frac{dx^i_{\tau_e'}}{dt_{\tau_e'}}\right|_{\tau_e'=\tau_e} \approx \frac{d\xi^i}{d\tau_e} \quad , \qquad i = 1, 2, 3 \quad , \tag{16}$$

$$\left.\frac{dt_{\tau_e'}}{d\tau}\right|_{\tau_e'=\tau_e} \approx 1 \quad , \tag{17}$$

$$F^{\mu}_{\tau_e} \approx \left(\vec{f}_{\tau_e} \cdot \frac{d\vec{\xi}}{d\tau_e}\frac{1}{c} \; ; \quad \vec{f}_{\tau_e} \right) \quad . \tag{18}$$

The force $\vec{f}_{\tau_e c}$ has been assumed to equal $m\vec{a}$, as it is the force in the instantaneous rest frame of the spring's equilibrium point that provides the particle with a relativistic hyperbolic motion in the absence of the spring. The force $\vec{f}_{\tau_e sp}$ requires more discussion, however, as it is not so immediately dealt with.

If the equilibrium point of the oscillator is at rest in an inertial frame, one usually defines a harmonic oscillator restoring force as $\vec{f}_{sp}=-k\vec{x}$ for an isotropic oscillator and $f^i_{sp}=-k^i x^i$, $i=1,2,3$, for an anisotropic oscillator. For a nonrelativistic oscillator obeying the equation of motion

$$m\frac{d^2x^i}{dt^2} = -k^i_0 x^i \quad , \qquad i = 1, 2, 3 \quad , \tag{19}$$

this yields a simple harmonic motion of frequency $\omega^i_0=\left(\frac{k^i_0}{m}\right)^{1/2}$ along each of the three orthogonal spatial axes. Although one often thinks of a spring obeying Hook's law as constituting the physical example for Eq. (19), it is well known that the area of applicability of Eq. (19) extends to all stable systems of small amplitude governed by forces that are functions of position alone and are reexpressed in terms of the system's normal coordinates (*12). The force $-k^i x^i$ is then simply the first term in a Taylor's expansion of the forces acting on the particle about the point of equilibrium. [If one instead uses the relativistic expression for the momentum of the particle and writes

$$\frac{d}{dt}\left\{ m_o\left[1-\left|\frac{\vec{v}}{c}\right|^2\right]^{-1/2}\frac{dx^i}{dt}\right\} = -k^i x^i \quad , \qquad i = 1,2,3 \quad , \qquad (20)$$

then the motion no longer follows a simple harmonic behavior characterized by $A\sin(\omega t-\beta)$. By restricting the motion to small amplitudes, however, so that $v/c \approx \frac{\omega_o A}{c} \ll 1$, then Eq. (20) is again approximately described by Eq. (19) (*13,14).]

For an oscillator system in which the equilibrium point is undergoing uniform acceleration, the appropriate generalization to the force $-k^i x^i$ must be deduced. To a certain degree, this becomes strictly a matter of definition for what is meant by an "accelerating simple harmonic oscillator." In Appendix A and B of this article, an explicit physical model is examined to help motivate the form for the oscillator's restoring force that will be used in this section of

$$f^i_{\tau_e sp} = K\delta_{i1} - k^i \xi^i \quad , \qquad i = 1,2,3 \quad . \qquad (21)$$

Since $\vec{\xi} = \vec{x}_{\tau_e} - \hat{x}\frac{c^2}{a}$ in I_{τ_e} [$\tau_e' = \tau_e$ in Eqs. (7)], then Eq. (21) may be viewed as applicable to those systems where $f^i_{\tau_e sp}$ is a function of position and the appropriate conditions of small amplitude, stability, and normal coordinates are satisfied. The simplifying assumption was made that $f^2_{\tau_e sp}=0$ and

$f^3_{\tau_e sp} = 0$ when ξ^2 and ξ^3 equal zero. The constants K and k^i are assumed, in general, to be functions of the proper acceleration of the oscillator's equilibrium point. When $a = 0$, it will be assumed that $K = 0$ and k^i reduces to the value of the constant k^i_0 in Eq. (19) for an unaccelerated oscillator. The examples in the Appendices bring these points out more clearly. Finally, the presence of the nonzero value of K requires that the uniform acceleration of the equilibrium point be given by $ma = f_c + K$ instead of $ma = f_c$.

Combining the above expressions with Eq. (18) yields

$$F^{\nu}_{\tau_e} \approx \left((f_c + K)\frac{d\xi^1}{d\tau_e}\frac{1}{c} ;\quad f_c + K - k^1\xi^1 ,\ -k^2\xi^2 ,\ -k^3\xi^3 \right) . \qquad (22)$$

The terms $-\sum_{i=1}^{3} k^i \xi^i \frac{d\xi^i}{d\tau_e}\frac{1}{c}$ that would occur in $F^0_{\tau_e}$ have been ignored due to the small amplitude assumption. Now combining Eqs. (1), (10), (11) and (22) gives the result that

$$m\left\{\left[\frac{d^2\xi^1}{d\tau_e^2} - \xi^1\left(\frac{a}{c}\right)^2 + a\right]\sinh\left(\frac{a\tau_e}{c}\right) + \left[\frac{d\xi^1}{d\tau_e}\frac{a}{c}\right]\cosh\left(\frac{a\tau_e}{c}\right)\right\}$$
$$= (f_c + K)\frac{d\xi^1}{d\tau_e}\frac{1}{c}\cosh\left(\frac{a\tau_e}{c}\right) + (f_c + K - k^1\xi^1)\sinh\left(\frac{a\tau_e}{c}\right) , \qquad (23)$$

$$m\left\{\left[\frac{d^2\xi^1}{d\tau_e^2} - \xi^1\left(\frac{a}{c}\right)^2 + a\right]\cosh\left(\frac{a\tau_e}{c}\right) + \left[\frac{d\xi^1}{d\tau_e}\frac{a}{c}\right]\sinh\left(\frac{a\tau_e}{c}\right)\right\}$$
$$= (f_c + K)\frac{d\xi^1}{d\tau_e}\frac{1}{c}\sinh\left(\frac{a\tau_e}{c}\right) + (f_c + K - k^1\xi^1)\cosh\left(\frac{a\tau_e}{c}\right) , \qquad (24)$$

$$\frac{d^2\xi^2}{d\tau_e^2} = -\frac{k^2}{m}\xi^2 , \qquad (25) \qquad\qquad \frac{d^2\xi^3}{d\tau_e^2} = -\frac{k^2}{m}\xi^3 . \qquad (26)$$

Equations (23) and (24) are equivalent and can be rewritten as

$$\frac{d^2\xi^1}{d\tau_e^2} = -\left[\frac{k^1}{m} - \left(\frac{a}{c}\right)^2\right]\xi^1 \quad . \qquad (27)$$

Provided that $k^1 > \left(\frac{a}{c}\right)^2 m$, then the ξ^i coordinates follow simple harmonic motion of the form $A^i \sin(\omega^i t - \beta^i)$, with angular frequencies given by

$$\begin{aligned} \omega^1 &= \left(\frac{k^1}{m} - \left(\frac{a}{c}\right)^2\right)^{1/2} \quad , \\ \omega^2 &= \left(\frac{k^2}{m}\right)^{1/2} \quad , \qquad (28) \\ \omega^3 &= \left(\frac{k^3}{m}\right)^{1/2} \quad . \end{aligned}$$

The results of this section may cause some puzzlement over the origin of the change in frequency associated with the ξ^1 coordinate due to the $\left(\frac{a}{c}\right)^2$ term. The dimensionless quantity $\frac{a\xi^i}{c^2}$, which gave rise to this change in frequency [see the second term in the first bracket on the left-hand side of Eqs. (23) and (24)], is sometimes referred to as a red-shift effect (*15). In Eq. (8'), this term is the first order correction to the ratio between the rates of proper time of the oscillating particle and the equilibrium point.

In terms of proper acceleration and three-vector forces, the following explanation helps to understand the frequency change associated with the ξ^1 coordinate. In order for the particle to remain at a constant distance ξ^1 from the equilibrium point, one can show that its proper acceleration must be given by

$$a' = \frac{a}{\left(1 + \frac{a\xi^1}{c^2}\right)} \approx a - \xi^1\left(\frac{a}{c}\right)^2$$

for $\xi^1 \ll \frac{c^2}{a}$. Under this condition of constant value ξ^1, no oscillations will take place and both the particle and the equilibrium point will have the same instantaneous rest frame. Since

$$ma' = f_c + K - k^1\xi^1 \approx m\left(a - \xi^1\left(\frac{a}{c}\right)^2\right)$$

and one requires that $ma = f_c + K$, then k^1 must equal $m\left(\frac{a}{c}\right)^2$. This is precisely the limiting condition of oscillatory motion as predicted by Eq. (27), for only if $k^1 > m\left(\frac{a}{c}\right)^2$ will oscillations occur (*16,17).

III. CHARGED OSCILLATOR IN HYPERBOLIC MOTION

The results of the previous section are easily extended to the case of a particle with rest mass m and charge e. As before, the particle may be pictured as oscillating at the end of a massless spring, the equilibrium point of which undergoes uniform proper acceleration $a\hat{x}$. The force $f^i_{\tau_e sp}$ is again taken to be equal to $K\delta_{i1}-k^i\xi^i$. The constant force $\vec{f}_c$ may now be obtained from a constant electric field $\vec{E}_o$ in the x direction which satisfies $ma=eE_o+K$. The major change in the equation of motion arises from the relativistic radiation reaction terms due to the charged particle's motion. In the I_* frame, the Lorentz-Dirac equation that describes the charged particle's behavior is (*18)

$$m\frac{d^2x^\mu_*}{d\tau^2} = F^\mu_{*sp} + \frac{e}{c}\mathcal{F}^{\mu\nu}_*\frac{dx_{*\nu}}{d\tau} + \Gamma^\mu \qquad , \qquad (29)$$

where

$$\Gamma^\mu = \frac{2}{3}\frac{e^2}{c^3}\left[\frac{d^3x^\mu_*}{d\tau^3} - \frac{1}{c^2}\left(\frac{d^2x^\lambda_*}{d\tau^2}\frac{d^2x_{*\lambda}}{d\tau^2}\right)\frac{dx^\mu_*}{d\tau}\right] \quad . \qquad (30)$$

Here, F^μ_{*sp} is again the four-force associated with the spring, and $\mathcal{F}^{\mu\nu}_*$ is the electromagnetic field tensor due to the presence of the electric field $\vec{E}_o$.

Most of the results from the previous section can be used to evaluate the new terms that have been introduced in the equation of motion. Using Eqs. (8') and (10) and again invoking the small amplitude approximation yields

$$\frac{d^3x_*^0}{d\tau^3} \approx \frac{d^3\xi^1}{d\tau_e^3}\sinh\left(\frac{a\tau_e}{c}\right) + \left[2\left\{\frac{d^2\xi^1}{d\tau_e^2}\frac{a}{c} - \xi^1\left(\frac{a}{c}\right)^3\right\} + \frac{a^2}{c}\right]\cosh\left(\frac{a\tau_e}{c}\right) \quad ,$$
$$\frac{d^3x_*^1}{d\tau^3} \approx \frac{d^3\xi^1}{d\tau_e^3}\cosh\left(\frac{a\tau_e}{c}\right) + \left[2\left\{\frac{d^2\xi^1}{d\tau_e^2}\frac{a}{c} - \xi^1\left(\frac{a}{c}\right)^3\right\} + \frac{a^2}{c}\right]\sinh\left(\frac{a\tau_e}{c}\right) \quad , \tag{31}$$
$$\frac{d^3x_*^2}{d\tau^3} \approx \frac{d^3\xi^2}{d\tau_e^3} \quad , \qquad \frac{d^3x_*^3}{d\tau^3} \approx \frac{d^3\xi^3}{d\tau_e^3} \quad .$$

From Eqs. (10),

$$\frac{d^2x_*^\lambda}{d\tau^2}\frac{d^2x_{*\lambda}}{d\tau^2} \approx a^2 + 2a\left\{\frac{d^2\xi^1}{d\tau_e^2} - \xi^1\left(\frac{a}{c}\right)^2\right\} \quad . \tag{32}$$

Substituting Eqs. (9), (31) and (32) into Eq. (30) results in the expressions

$$\begin{aligned}\Gamma^0 \approx \frac{2}{3}\frac{e^2}{c^3}\Bigg(&\frac{d^3\xi^1}{d\tau_e^3}\sinh\left(\frac{a\tau_e}{c}\right) + \left[2\left\{\frac{d^2\xi^1}{d\tau_e^2}\frac{a}{c} - \xi^1\left(\frac{a}{c}\right)^3\right\} + \frac{a^2}{c}\right]\cosh\left(\frac{a\tau_e}{c}\right) \\ &- \frac{1}{c^2}\left[a^2 + 2a\left\{\frac{d^2\xi^1}{d\tau_e^2} - \xi^1\left(\frac{a}{c}\right)^2\right\}\right]\left[\frac{d\xi^1}{d\tau_e}\sinh\left(\frac{a\tau_e}{c}\right) + c\cosh\left(\frac{a\tau_e}{c}\right)\right]\Bigg) \\ \approx \frac{2}{3}\frac{e^2}{c^3}&\left[\frac{d^3\xi^1}{d\tau_e^3} - \left(\frac{a}{c}\right)^2\frac{d\xi^1}{d\tau_e}\right]\sinh\left(\frac{a\tau_e}{c}\right) \quad ,\end{aligned} \tag{33}$$

$$\Gamma^1 \approx \frac{2}{3}\frac{e^2}{c^3}\left[\frac{d^3\xi^1}{d\tau_e^3} - \left(\frac{a}{c}\right)^2\frac{d\xi^1}{d\tau_e}\right]\cosh\left(\frac{a\tau_e}{c}\right) \quad , \tag{34}$$

$$\Gamma^i \approx \frac{2}{3}\frac{e^2}{c^3}\left[\frac{d^3\xi^i}{d\tau_e^3} - \left(\frac{a}{c}\right)^2\frac{d\xi^i}{d\tau_e}\right] \quad , \qquad i = 2, 3 \quad . \tag{35}$$

Finally, the term $\frac{e}{c}\mathcal{F}_*^{\mu\nu}\frac{dx_{*\nu}}{d\tau}$ can be expressed in the ξ^μ coordinates by using the explicit expression for $\mathcal{F}_*^{\mu\nu}$ (*19) and Eqs. (9). As should be expected, the same results are obtained as in the previous section with f_c replaced by eE_0 , since I_* and I_{τ_e} see the same electric field along the x direction.

After substituting the above quantities into the

Lorentz-Dirac equation and observing that the $\mu=0$ and $\mu=1$ equations again yield the same result to first order in ξ^i, one obtains the following equations of motion

$$\frac{d^2\xi^i}{d\tau_e^2} = -(\omega^i)^2\xi^i + \Gamma\left[\frac{d^3\xi^i}{d\tau_e^3} - \left(\frac{a}{c}\right)^2\frac{d\xi^i}{d\tau_e}\right] \quad , \quad i = 1, 2, 3 , \qquad (36)$$

where $\Gamma = \frac{2}{3}\frac{e^2}{mc^3}$ and ω^i is again given by Eq. (28).

IV. CHARGED OSCILLATOR UNIFORMLY ACCELERATED IN CLASSICAL ELECTROMAGNETIC ZERO-POINT RADIATION

The situation considered next is the oscillating charged particle of Sec. III, again undergoing relativistic hyperbolic motion, but now in the presence of what has been termed classical electromagnetic zero-point radiation (*20). The Lorentz-Dirac equation is now altered simply by including the zero-point radiation fields $\vec{E}_*^{zp}(\vec{x}_*, t_*)$ and $\vec{B}_*^{zp}(\vec{x}_*, t_*)$ in addition to the $\vec{E}_0$ field in the electromagnetic field tensor $\mathcal{F}_*^{\mu\nu}$. Loosely phrased, these additional fields now act as a driving force to the simple harmonic oscillator of Sec. II with damping terms of Sec. III. Hence, an equilibrium behavior of the particle's motion is obtained, since the energy radiated by the particle's oscillations must be supplied by the work done by the zero-point fields in maintaining the particle's oscillations.

Using Eqs. (9) and the appropriate expression for $\mathcal{F}_*^{\mu\nu}$, yields

$$\frac{e}{c}\mathcal{F}_*^{\mu\nu}\frac{dx_{*\nu}}{d\tau} \approx$$

$$\frac{e}{c}\begin{bmatrix} (E_{*x}^{zp} + E_0)\left(\frac{d\xi^1}{d\tau_e}\cosh\left(\frac{a\tau_e}{c}\right) + c\sinh\left(\frac{a\tau_e}{c}\right)\right) + E_{*y}^{zp}\frac{d\xi^2}{d\tau_e} + E_{*z}^{zp}\frac{d\xi^3}{d\tau_e} \\ (E_{*x}^{zp} + E_0)\left(\frac{d\xi^1}{d\tau_e}\sinh\left(\frac{a\tau_e}{c}\right) + c\cosh\left(\frac{a\tau_e}{c}\right)\right) + B_{*z}^{zp}\frac{d\xi^2}{d\tau_e} - B_{*y}^{zp}\frac{d\xi^3}{d\tau_e} \\ E_{*y}^{zp}\left(\frac{d\xi^1}{d\tau_e}\sinh\left(\frac{a\tau_e}{c}\right) + c\cosh\left(\frac{a\tau_e}{c}\right)\right) - B_{*z}^{zp}\left(\frac{d\xi^1}{d\tau_e}\cosh\left(\frac{a\tau_e}{c}\right) + c\sinh\left(\frac{a\tau_e}{c}\right)\right) + B_{*x}^{zp}\frac{d\xi^3}{d\tau_e} \\ E_{*z}^{zp}\left(\frac{d\xi^1}{d\tau_e}\sinh\left(\frac{a\tau_e}{c}\right) + c\cosh\left(\frac{a\tau_e}{c}\right)\right) + B_{*y}^{zp}\left(\frac{d\xi^1}{d\tau_e}\cosh\left(\frac{a\tau_e}{c}\right) + c\sinh\left(\frac{a\tau_e}{c}\right)\right) - B_{*x}^{zp}\frac{d\xi^2}{d\tau_e} \end{bmatrix} . \quad (37)$$

Additional complicated terms due to the zero-point fields now appear in the equations of motion for the charged particle. Solving the resulting differential equations without imposing certain limits would indeed be very difficult. However, the small oscillator limit cannot so arbitrarily be imposed as it was in the previous cases considered. Now the fluctuating zero-point fields will be the determining factor in the size of the amplitude of oscillation. Hence, even the use of Eqs. (9) in obtaining Eqs. (37) must be reexamined.

The following reasoning is intended to provide some rationale for the approximations that will be made subsequently. The amplitude of the frequency component of the zero-point fields near the resonant frequency of the oscillator is anticipated to be the main contributing factor to the amplitude of the oscillator. Let the former quantities be denoted by $\vec{E}^{zp}(\omega^i)$ and $\vec{B}^{zp}(\omega^i)$. If $\vec{E}^{zp}(\omega^i)$ and $\vec{B}^{zp}(\omega^i)$ are sufficiently small enough, then one would expect that ξ^i and any derivative $\frac{d^n\xi^i}{d\tau_e^n}$ would roughly be of first order in these quantities. Consequently, any single power of ξ^i or its derivatives and any single power of the fields $\vec{E}_*^{zp}$ and $\vec{B}_*^{zp}$ will be treated as first order quantities in $\vec{E}^{zp}(\omega^i)$ and $\vec{B}^{zp}(\omega^i)$. Terms of the form $\frac{e}{c}B_*^{zp}\frac{d\xi}{d\tau_e}$, $\frac{e}{c}E_*^{zp}\frac{d\xi}{d\tau_e}$, $-k\xi\frac{d\xi}{d\tau_e}$, and $cE_0\frac{d\xi}{d\tau_e}\frac{d\xi}{d\tau_e}\frac{1}{c^2}$ will be considered of second order in $\vec{E}^{zp}(\omega^i)$ and $\vec{B}^{zp}(\omega^i)$. The quantity $\frac{e}{c}\mathcal{F}_*^{\mu\nu}\frac{dx_{*\nu}}{d\tau}$ can then be linearized; the earlier linearization steps followed in Secs. II and III will also hold (*21).

Three equations of motion similar to Eq. (36) are now obtained which include the effects of the zero-point fields,

$$\frac{d^2\xi^i}{d\tau_e^2} = -(\omega^i)^2\xi^i + \Gamma\left[\frac{d^3\xi^i}{d\tau_e^3} - \left(\frac{a}{c}\right)^2\frac{d\xi^i}{d\tau_e}\right] + \frac{e}{m}E^{zp}_{\tau_e i}(\vec{\xi},\tau_e) \ , \quad i = 1,2,3 \ , \tag{38}$$

where

$$\begin{aligned} E^{zp}_{\tau_e 1}(\vec{\xi},\tau_e) &= E^{zp}_{*x}(\vec{\xi},\tau_e) \ , \\ E^{zp}_{\tau_e 2}(\vec{\xi},\tau_e) &= \cosh\left(\frac{a\tau_e}{c}\right)E^{zp}_{*y}(\vec{\xi},\tau_e) - \sinh\left(\frac{a\tau_e}{c}\right)B^{zp}_{*z}(\vec{\xi},\tau_e) \ , \\ E^{zp}_{\tau_e 3}(\vec{\xi},\tau_e) &= \cosh\left(\frac{a\tau_e}{c}\right)E^{zp}_{*z}(\vec{\xi},\tau_e) + \sinh\left(\frac{a\tau_e}{c}\right)B^{zp}_{*y}(\vec{\xi},\tau_e) \ . \end{aligned} \tag{39}$$

The quantities $\vec{E}^{zp}_{\tau_e i}$ are the electric fields measured in the inertial frame I_{τ_e} along the three orthogonal coordinate space axes. The quantities $(\vec{\xi},\tau_e)$ in the arguments of the fields represent the space-time position of the particle at which to evaluate the fields. Using a dipole approximation for the fields, the arguments $\vec{\xi}$ are then set equal to zero.

Hence, when the small oscillator assumption is made, all three directions of motion are described by the differential equation expressed by Eq. (38). This turns out to be true despite the additional complicating terms that must be taken into account for oscillations occurring in a direction parallel to that of the uniform acceleration $\vec{a}$.

The above linear stochastic differential equations may now be solved in order to determine the statistical properties imposed upon the oscillating particle by the fluctuating zero-point fields. Fortunately, this work has already been carried out in Ref. 5, where the behavior of the particle was investigated under the restriction that oscillations were confined to the directions perpendicular to $\vec{a}$. By comparing Eq. (14) of Ref. 5 to Eq. (38) of the present article, it can immediately be seen that they are of the same form when the dipole approximation in the fields is made. The only difference is that in the present paper, ω^i is recognized to be, in general, a function of a , the uniform acceleration of the oscillator's equilibrium point. Of course, this difference in no way effects the method of solution. Hence, the conclusion of Ref. 5 can be immediately applied here, with a slight change in interpretation.

V. CONCLUSION

The results of the previous section lead to the following conclusion. Consider an oscillating charged particle uniformly accelerated through classical electromagnetic zero-point radiation. Let $\omega^i(a)$, for $i=1,2,3$, be the natural frequency of the motion of the particle along each of the spatial axes of the Fermi-Walker transported coordinate system introduced in Sec. II. Now consider a second oscillating charged particle at rest in an inertial frame and bathed in classical electromagnetic zero-point radiation plus Planckian electromagnetic radiation. Let the latter spectrum be characterized by the Unruh-Davies temperature of $T=\hbar a/2\pi ck$. Let this oscillator have a natural frequency $\omega'^i=\omega^i(a)$ along each of the three spatial axes in the inertial frame. One can then conclude that the statistical properties for these two oscillators will be identical, as observed in their respective coordinate systems.

APPENDIX A: MODEL OF A TRANSVERSE ACCELERATED OSCILLATOR

The following simple model is presented in order to motivate the form assumed for the spring three-vector force given by Eq. (21). A transverse oscillator model is considered here; the following section considers the analogous longitudinal oscillator model. From these examples, it should be apparent how one could construct more general stationary charge distributions in the Fermi-Walker transported coordinate system, with symmetry axes along the $\hat{x}$, $\hat{y}$, and $\hat{z}$ directions, and that result in the validity of Eq. (21) in the limit of small amplitude A.

In order to create a force which depends linearly upon the oscillating particle's displacement from equilibrium, two charged particles will be placed and held fixed in the positions $\vec{\xi}_{\pm}=(0,\pm\ell,0)$ of the accelerating coordinate system. Consequently, they will possess the same proper acceleration $\vec{a}$ as does the equilibrium point $\vec{\xi}=(0,0,0)$. Assume that these two particles each have a charge q of the same sign as the charge e of the oscillating particle. Let the latter particle be constrained so that its position is described by $\vec{\xi}=(0,\xi^2(\tau_\xi),0)$. By restricting the amplitude of oscillation A to be much smaller than the length ℓ, the force of the outer two charges on the center particle can be expanded in a Taylor series in ξ^2. For $\frac{A}{\ell} \ll 1$, this force is adequately approximated by retaining only the first-order term in ξ^2, thereby yielding the desired model for a simple harmonic oscillator restoring force. Calculating the value of the proportionality constant k^2 will then determine the dependence of $\omega^2 = \left(\frac{k^2}{m}\right)^{1/2}$ upon the proper acceleration of the oscillator.

If one does the above calculation for an unaccelerated oscillator, it is found that $k_0^2 = 4eq/\ell^3$. Proceeding to the situation of an accelerated system, one must use the standard expression for the retarded electric field of a point charge given by (#22)

$$\vec{E} = q\left[\frac{(\hat{n}-\vec{\beta})}{\gamma^2(1-\vec{\beta}\cdot\hat{n})^3R^2}\right]_{ret} + \frac{q}{c}\left[\frac{\hat{n}\otimes\{(\hat{n}-\vec{\beta})\otimes\dot{\vec{\beta}}\}}{(1-\vec{\beta}\cdot\hat{n})^3R}\right]_{ret} . \tag{40}$$

In order to conform with the analysis of Sec. II, the force on the oscillating particle should be evaluated along the y axis of an inertial frame I_{τ_e}, instantaneously at rest with respect to the equilibrium point of the oscillator. From Eq. (40), the quantity of interest is

$$E_{y\pm} = \left[\frac{q\,n_y}{R^2(1-\beta n_x)^3}\left\{(1-\beta^2) + R\,n_x\dot{\beta}\right\}\right]_{\substack{\pm \\ ret}} , \tag{41}$$

where the $\pm$ signs indicate the field due to the charge at $\xi^2 = \pm\ell$.

In the limit of $\frac{A}{\ell} \ll 1$, $F_{\tau_e y} = e(E_{y+} + E_{y-})$ can be expanded in terms of ξ^2 to yield

$$F_{r_e y}(\xi^2) \approx 2e \left.\frac{dE_{y+}}{d\xi^2}\right|_{\vec{\xi}=0} \xi^2 = -k^2\xi^2 \quad , \qquad (42)$$

where the symmetry of the model has been used to set

$$\left.(E_{y+} + E_{y-})\right|_{\vec{\xi}=0} = 0 \quad \text{and}$$

$$\left.\frac{dE_{y+}}{d\xi^2}\right|_{\vec{\xi}=0} = \left.\frac{dE_{y-}}{d\xi^2}\right|_{\vec{\xi}=0} \quad .$$

Through rather lengthy calculations, one can now obtain an expression for k^2. This involves calculating the retarded time t_{r+} associated with the charge at $\vec{\xi}=(0, \ell, 0)$, expressing all quantities in the expression for E_{y+} in terms of t_{r+}, expanding t_{r+} to first order in ξ^2, and finally propagating those first-order terms to Eq. (42).

Clearly, despite the simplicity of this transverse accelerated oscillator model, k^2 will be an extremely complicated function of the proper acceleration a. As one might expect, only in a particular limit, namely where $\ell \ll \frac{c^2}{a}$, will k^2 reduce to the value k_0^2 in Eq. (19). Of course, in most cases of interest, this limit is easily satisfied. As discussed in the Introduction, however, the thermal effects of acceleration for an oscillator are only expected to be observable when cT_{os} is not small compared with c^2/a. Hence, in order to see thermal effects and yet have $k^2 \approx k_0^2$, then $\ell \ll c^2/a$ and $cT_{os} \gtrsim \frac{c^2}{a}$ must both be satisfied. (As indicated elsewhere, the conclusion of this article does not depend upon satisfying the condition $k^2 = k_0^2$. The present discussion simply examines from a classical point of view some of the subtleties involved with the dependence of k^2 upon a.)

With regard to the present model, one way to satisfy the above conditions is by letting $q \rightarrow 0$ as $\ell \rightarrow 0$ in such a way that $4qe/\ell^3$ remains of constant value k_o^2. Of course, $\xi^2 \ll \ell$ must remain satisfied and k_o^2 chosen such that $cT_{os} \gtrsim \frac{c^2}{a}$. Using the relationships

$$t_{r\pm} = -\frac{(\ell \mp \xi^2)}{c}\left[1 + \left(\frac{a(\ell \mp \xi^2)}{2c}\right)^2\right]^{1/2} \quad , \qquad R_{r\pm} = -ct_{r\pm} \quad ,$$

$$\beta_{r\pm} = \frac{at_{r\pm}/c}{\left[1 + \left(\frac{at_{r\pm}}{c}\right)^2\right]^{1/2}} \quad , \qquad \dot{\beta}_{r\pm} = \frac{a/c}{\left[1 + \left(\frac{at_{r\pm}}{c}\right)^2\right]^{3/2}} \quad , \tag{43}$$

$$n_{xr\pm} = \frac{\frac{c^2}{a} - \frac{c^2}{a}\left[1 + \left(\frac{at_{r\pm}}{c}\right)^2\right]^{1/2}}{R_{r\pm}} \quad , \qquad n_{yr\pm} = \left(\frac{\mp\ell + \xi^2}{R_{r\pm}}\right) \quad .$$

and following the operations mentioned earlier, one can then show that

$$k^2 = \frac{4qe}{\ell^3}\left(1 + O\left(\frac{a\ell}{c^2}\right)\right)$$

for $\quad \frac{a\ell}{c^2} \ll 1 \quad .$

APPENDIX B: MODEL OF A LONGITUDINAL ACCELERATED OSCILLATOR

A simple model for longitudinal oscillations, analogous to the example in Appendix A, will be briefly examined here. Let two particles of charge q be held fixed at the positions $\vec{\xi}_{\pm} = (\pm \ell, 0, 0)$ of the accelerating coordinate system. Consequently, they will possess proper accelerations $a_{\pm} = a/(1 \pm a\ell/c^2)$. Assume that the oscillating particle of charge e is constrained so its motion is described by $\vec{\xi} = (\xi^1(\tau_e), 0, 0)$. In conformity with the analysis of Sec. II, the force on the oscillating particle should be evaluated in the instantaneous rest frame of the equilibrium point.

For an unaccelerated system, $k_0^1 = 4qe/\ell^3$. In the case where $a \neq 0$, one must again use Eq. (40) in order to obtain the force on the oscillating particle. One obtains

$$F_{\tau_e x} = eq\left\{ -\left[\frac{1-\beta_{r+}}{1+\beta_{r+}}\right]\frac{1}{R_{r+}^2} + \left[\frac{1+\beta_{r-}}{1-\beta_{r-}}\right]\frac{1}{R_{r-}^2} \right\} , \tag{44}$$

where the $\pm$ signs again indicate the respective quantities associated with the source particles at $\xi^1 = \pm \ell$. In order to evaluate Eq. (44), the following expressions are needed:

$$t_{r\pm} = \mp \frac{1}{2c(\xi^1 + c^2/a)} \left\{ \left(\frac{c^2}{a_\pm}\right)^2 - \left(\xi^1 + \frac{c^2}{a}\right)^2 \right\} \quad ,$$

$$R_{r\pm} = -c\,t_{r\pm} \quad , \qquad \beta_{r\pm} = \frac{a_\pm t_{r\pm}/c}{\left[1 + (a_\pm t_{r\pm}/c)^2\right]^{1/2}} \quad . \tag{45}$$

Substituting Eqs. (45) into Eq. (44) and expanding $F_{\gamma_e x}$ to first order in ξ^1 results in an expression of the form $F_{\gamma_e x} = K - k^1 \xi^1$. When $a \neq 0$, then $K \neq 0$. If $\ell \ll \frac{c^2}{a}$ and $\frac{4qe}{\ell^3} = k_o^1$, then

$$K \approx \frac{-2eq}{\ell^2}\left(\frac{a\ell}{c^2}\right) \approx 0$$

and

$$k^1 = \frac{4qe}{\ell^3}\left\{1 + O\left(\frac{a\ell}{c^2}\right)\right\} \quad .$$

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6) Other trajectories that might be of interest are discussed by J. R. Letaw in Phys. Rev. D 23, 1709 (1981).

7) In particular, see C. W. Misner, K. Thorne and J. A. Wheeler, Gravitation (W. H. Freeman and Company, San Francisco, 1973), Secs. 6.4-6.6, pp. 169-174.

8) See, for example, the discussion on hyperbolic motion in Sec. 6.2 of Ref. 7, or the discussion by F. Rohrlich in Classical Charged Particles (Addison-Wesley, Reading, Mass., 1965), Sec. 6-11, pp. 169-172. Also, see Refs. 4 and 5.

9) See Eq. (6.6) in Ref. 7.

10) See Eqs. (6.16) and (6.17) in Ref. 7.

11) See the discussion of the small oscillator assumption on p. 1091 of Ref. 5.

12) See the discussion on small oscillations of stable systems by H. Goldstein, Classical Mechanics, 2nd ed. (Addison-Wesley, Reading, Mass., 1980), Chap. 6, p. 243.

13) See pp. 324 and 325 of Ref. 12. Also, a related simple problem is given by W. Rindler, Essential Relativity: Special, General, and Cosmolgical, 2nd ed. (Springer-Verlag, New York, 1977), prob. 5.14, p. 263. Using Rindler's relativistic expressions, one can explicitly show that v/c->0 as the amplitude->0.

14) Rohrlich chooses to give another definition for a

"relativistic oscillator" in Sec. 6-14 of Ref. 8. In the limit of small amplitude, his example also reduces to the nonrelativistic situation of Eq. (19).

15) See Ref. 7, p. 1008 and the related discussion on p. 393.

16) See Rindler's discussion on pp. 50 and 51 of Ref. 13. Also, see Ref. 7, problems 37.4, 6.6, and 6.7.

17) If a force field other than $\vec{f}_{r_e c} = m\vec{a}$ could be constructed that appeared as a stationary force field in the instantaneous rest frame of the spring's equilibrium point, then the frequency shift indicated by Eq. (28) could be altered and even made to vanish. The case of vanishing frequency shift would occur if $\vec{f}_{r_e c}$ was replaced by $f_{r_e c} = mc^2\hat{x}/(\frac{c^2}{a}+\xi^1) \approx ma\hat{x}(1 - a\xi^1/c^2)$, which corresponds to a force that falls off inversely with distance in the $\hat{x}$ direction (see Rindler's comments on p. 51 of Ref. 13).

18) See, for example, Rohrlich's text in Ref. 8, Eq. (6-57).

19) See Rohrlich's text in Ref. 8, Eqs. (4-47) and (4-48).

20) See the discussion and references on stochastic electrodynamics given in Refs. 4 and 5.

21) Obviously, the argument presented here is extremely rough. However, a rigorous justification of the approximations made in obtaining Eqs. (38) would be very difficult to give, if not impossible, without a better understanding of the effects of the high-frequency components of the zero-point fields upon the charged particle's behavior. Unfortunately, such difficulties plaugue most of the integals that occur in stochastic electrodynamics. Presumably, some sort of physical mechanism exists so that the high frequency contributions may be ignored.

22) See, for example, J.D.Jackson, Classical Electrodynamics, 2nd ed. (Wiley, New York, 1975), p. 657, Eq. (14.14).

PART TWO

THERMAL EFFECTS OF ACCELERATION FOR A SPATIALLY EXTENDED ELECTROMAGNETIC SYSTEM IN CLASSICAL ELECTROMAGNETIC ZERO-POINT RADIATION: TRANSVERSELY POSITIONED CLASSICAL OSCILLATORS

I. INTRODUCTION

Recently, the behavior has been analyzed for certain classical electromagnetic dipole systems that are relativistically, uniformly accelerated through classical electromagnetic zero-point radiation (*1,2,3). Under the assumption of a narrow linewidth approximation, the statistical properties of these accelerated point-like electromagnetic systems have been found to agree with the corresponding statistical properties of similarly constructed, but unaccelerated systems that are bathed in a classical electromagnetic Planckian radiation spectrum. This agreement occurs when the temperature T that characterizes the Planckian radiation spectrum is related to the acceleration a of the accelerated systems by the Unruh-Davies relationship of $T = \hbar a/2\pi c k$ (*4,5).

The above results are generalized in this article from the consideration of point-like electromagnetic systems to a spatially extended electromagnetic system that is relativistically, uniformly accelerated through classical electromagnetic zero-point radiation. Assuming that the equivalency, at least in some sense, between the accelerated and unaccelerated-thermal electromagnetic dipole systems is not merely accidental, but is indicative of a deeper physical relationship, then one should expect that the same sort of equivalency occurs for a spatially extended electromagnetic system. Under certain specified conditions examined in this article, this result is precisely what is found. Hence, for the first time in either the classical or quantum literature, the thermal effects of acceleration are shown here to hold for a particular spatially extended situation.

The spatially extended electromagnetic system considered in this article consists of two classical charged simple harmonic oscillators, oriented such that their equilibrium positions lie in a plane perpendicular to the direction of acceleration. No restrictions are placed upon the direction of oscillations for the two oscillators. These two spatially separated systems interact with each other via the electromagnetic radiation that each one emits. Hence, the statistical behavior of these two oscillators are correlated because of this interaction and because the two oscillators are being driven at different points in space by correlated values of classical electromagnetic zero-point fields. If the thermal properties of acceleration carry over to a spatially extended electromagnetic system, then the correlated statistical properties of this pair of accelerated charged oscillators must agree with the corresponding properties for a similar pair of unaccelerated charged oscillators that are bathed in a thermal radiation spectrum characterized by the Unruh-Davies temperature.

The theoretical basis that will be used here for analyzing the above electromagnetic system is that of classical electrodynamics with classical electromagnetic zero-point radiation, which has often been termed stochastic electrodynamics. (See references 6, 7, and 8 for reviews on this field of research.) The van der Waals force between two nonrelativistic classical charged harmonic oscillators, taken in the electric dipole limit, has been previously calculated within the context of stochastic electrodynamics, for the situation where the oscillators are situated in classical electromagnetic zero-point radiation (#9). The result was found to agree with the corresponding result of quantum electrodynamics to all orders in the fine structure constant. This calculation of the van der Waals force was generalized, within the domain of stochastic electrodynamics, to include the situation where the two oscillators were situated in a thermal plus zero-point classical electromagnetic radiation spectrum (#10).

From the standpoint of stochastic electrodynamics, the van der Waals force is simply the expectation value of the total Lorentz force acting on one of the charged oscillators. Hence, the results of the calculations of Ref. 10 may be compared to the expectation value of the force between the two accelerated oscillators considered in this article, thereby presenting a starting point for a comparison of the statistical properties of the accelerated and unaccelerated-thermal pair of charged harmonic oscillators. These calculations are carried out for a special oscillator system with restricted oscillatory motion in Secs. IIIB and IIIC and for the general oscillator system with unrestricted oscillatory motion in Sec. IVA.

Additional statistical properties for the accelerated oscillator system are considered in Secs. IIID for the special oscillator system and in Sec. IVB for the general oscillator system. These properties consist of the correlation functions of the coordinate positions of each oscillator, as well as correlation functions of higher time derivatives of each oscillator's coordinate position. In the process of examining these statistical properties for the pair of oscillators, a somewhat deeper analysis is presented on the statistical properties for a single accelerated charged oscillator.

Certain assumptions and approximations will be made in the analysis presented here on the properties of the pair of accelerated oscillators. The small oscillator assumption will be imposed (see Refs. 1 and 3), which enables the equations of motion to be linearized in the appropriate Fermi-Walker transported coordinate system (see Ref. 3). The radiation reaction damping constant of $\Gamma = \frac{2}{3}(e^2/mc^3)$ will be taken to be small compared to the other time constants of the system, thereby enabling the narrow linewidth approximation to be employed when evaluating integral expressions for the expectation values of certain stochastic quantities. This approximation was also used when analysing the statistical properties of the dipole systems considered in Refs.

1, 2, and 3.

Two additional approximations will be made here that did not enter into the analysis of the electromagnetic dipole systems treated in Refs. 1, 2, and 3. Both of these additional assumptions involve the spatial separation of the two oscillators being considered here. For this reason, these additional assumptions did not appear in the work of Refs. 1, 2, and 3, which treated only the behavior of single accelerated electromagnetic dipole systems.

First, a "small laboratory" condition will be imposed of

$$R \ll \frac{c^2}{a} \quad , \qquad (1)$$

where R is the distance of separation between the equilibrium points of the two oscillators. If one demands that the approximate distance of $\frac{1}{2}a\left(\frac{R}{c}\right)^2$ that a light ray would travel along the direction of acceleration from one oscillator to another be much less than the distance c^2/a to the event horizon, then one arrives at the condition of Eq. (1). Under this condition, R is much larger than the approximate distance of $\frac{1}{2}a\left(\frac{R}{c}\right)^2$ that that one of the oscillators will accelerate in time (R/c). Consequently, a light ray propagating from one oscillator to the other will make a small angle to the plane perpendicular to the direction of acceleration of the system. Thus, the condition of Eq. (1) reduces the physical distinguishability between the accelerated and unaccelerated-thermal pair of charged oscillators interacting via the electromagnetic radiation emitted from each oscillator.

The second approximation that will be imposed here, which also had no role in tho analysis of previous single point-like electromagnetic systems, consists of the condition

$$\frac{\omega_0 R}{c} \ll 1 \quad , \tag{2}$$

where ω_0 is the resonant frequency of the accelerated oscillators. This condition is traditionally termed the unretarded van der Waals condition. Possibly, the general results of this article hold when this condition is relaxed; this possibility will not be examined here, however. Hence, the equivalence in certain statistical properties between the accelerated and unaccelerated-thermal oscillator systems will only be established in this article when the condition of Eq. (2) applies.

Experimentally, the unretarded van der Waals condition of Eq. (2) is of interest because it describes the region in which one would be most likely to physically discern the thermal effects of acceleration for the extended system considered here. Roughly speaking, one would expect these thermal-like properties to be discernable from the zero-point motion of each oscillator when

$$\hbar\omega_0 \lesssim kT = \frac{\hbar a}{2\pi c} \quad . \tag{3}$$

Combining Eqs. (1) and (3) results in Eq. (2): namely, the unretarded van der Waals condition.

Thus, there are four primary conditions under which the accelerated and unaccelerated-thermal spatially extended oscillator systems will be examined in this article. These conditions consist of the small laboratory approximation of Eq. (1), the unretarted van der Waals condition of Eq. (2), the small oscillator assumption, and the narrow linewidth approximation.

The outline of this article is as follows. Section II describes in more detail the geometrical configuration of the accelerated pair of oscillators. The Fermi-Walker transported coordinate system is briefly introduced here; Ref. 3 should be referred to for further details. Using the work of Ref. 3, the equations of motion for the pair of accelerated oscillators are readily deduced in Sec. II. Lengthy calculations that involve the electromagnetic fields of an electric dipole undergoing relativistic uniform acceleration are contained in Appendix A so as not to sidetrack the main discussion presented in this article. Section III treats the special case in which each oscillator is confined to oscillations perpendicular to the plane defined by the direction of acceleration and the axis connecting the two oscillators. Section IV treats the general case where no restrictions are imposed upon the direction of oscillations. Appendices B, C, and D present additional calculations that were felt to be too lengthy for the main discussion of this article. In particular, however, it should be noted that as a result of the calculations in Appendices A and C, the electromagnetic fields radiated by a uniformly accelerating classical electric dipole are shown to be intimately related to the correlation functions of the electromagnetic fields of the classical electromagnetic zero-point radiation spectrum. It is precisely because of these relationships that agreement is obtained between the various expressions examined here for the accelerated and unaccelerated-thermal oscillator systems.

II. DESCRIPTION OF ACCELERATING SYSTEM

The accelerated classical charged harmonic oscillators considered here are assumed to have equilibrium positions following relativistic hyperbolic trajectories in space-time described by the constant proper acceleration $\vec{a}$. The equilibrium positions of the two oscillators are assumed to be situated such that they lie in a plane perpendicular to the direction of acceleration. Consequently, both oscillator equilibrium positions possess the same instantaneous inertial rest frame.

As was done in Refs. 1 and 3, a set of inertial reference frames I_{τ_e} will be introduced here such that the I_{τ_e} frame constitutes the instantaneous rest frame at proper time τ_e for the equilibrium positions of the two oscillators. (The inertial frame $I_{\tau_e=0}$ will be denoted by I_*.) Let these inertial frames be situated such that their x axes, or i=1 axes, lie along the direction of acceleration and their y axes, or i=2 axes, lie along the direction of separation between the two oscillators.

Let the two oscillators be denoted by the labels A or B. Let oscillator A have an i=2 and i=3 coordinate value of +R/2 and 0, respectively, within each inertial frame I_{τ_e} ; let oscillator B have corresponding coordinate values of

−R/2 and 0. As described in Ref. 3, the equilibrium position of each oscillator may be described in the I_{τ_e} inertial frame at proper time τ_e' by

$$X^{\mu}_{\binom{A}{B}\tau_e}(\tau_e') = \left(cT_{\binom{A}{B}\tau_e}(\tau_e') \, ; \, \vec{X}_{\binom{A}{B}\tau_e}(\tau_e') \right)$$
$$= \left(\frac{c^2}{a}\sinh\left(\frac{a}{c}(\tau_e'-\tau_e)\right) ; \, \frac{c^2}{a}\cosh\left(\frac{a}{c}(\tau_e'-\tau_e)\right), \pm\frac{R}{2}, 0 \right). \tag{4}$$

In Ref. 3, a Fermi-Walker transported coordinate system was introduced to aid in the description of a uniformly accelerated oscillator. This coordinate system may be described by the coordinates $\xi^{\mu}=(c\tau_e \, ; \vec{\xi})$, where τ_e is again the proper time of the equilibrium position of the accelerated oscillators. The coordinates ξ^{μ} are related to the coordinates $x^{\mu}_{\tau_e'}$ of an inertial frame $I_{\tau_e'}$ by

$$x^0_{\tau_e'} = ct_{\tau_e'} = \left(\xi_1 + \frac{c^2}{a}\right)\sinh\left(\frac{a}{c}(\tau_e'-\tau_e)\right) \quad , \tag{5a}$$

$$x^1_{\tau_e'} = x_{\tau_e'} = \left(\xi_1 + \frac{c^2}{a}\right)\cosh\left(\frac{a}{c}(\tau_e'-\tau_e)\right) \quad , \tag{5b}$$

$$x^2_{\tau_e'} = y_{\tau_e'} = \xi_2 \quad , \quad x^3_{\tau_e'} = z_{\tau_e'} = \xi_3 \quad , \tag{5c \& d}$$

where $x^i_{\tau_e'} = x_{\tau_e' i}$ and $\xi^i = \xi_i$, for i=1,2,3.

In order to examine the behavior of two oscillators accelerating through classical electromagnetic zero-point radiation, the effects of the fields of each oscillator acting upon the other oscillator must be taken into account. Hence, the equations of motion that must be solved for the combined system will consist of a set of coupled stochastic differential equations. These equations can readily be deduced from the work of Ref. 3, in which the introduction of the Fermi-Walker transported coordinate system and the assumption of the small oscillator approximation sufficiently simplified the equations of motion for a single accelerating oscillator that the equations of motion could easily be solved. The same method will be applied here in the case of two accelerating oscillators.

The equations of motion for two accelerating oscillators may be obtained by treating the electromagnetic field tensor $\mathcal{F}_*^{\mu\nu}$ that occurs in Eq. (37) of Ref. 3 as being due to the constant electric field $\hat{x}E_0$, the zero-point fields $\vec{E}_*^{zp}$ and $\vec{B}_*^{zp}$, and the dipole fields $\vec{E}_*^{D}$ and $\vec{B}_*^{D}$ arising from the action of one oscillator upon the other. (As discussed in Ref. 3, the constant field $\hat{x}E_0$ provides the mechanism for a relativistic uniform acceleration of a charged particle in the absence of other forces.) Following the argument presented in Sec. IV of Ref. 3, the quantity $\frac{e}{c}\mathcal{F}_*^{\mu\nu}\frac{dx_{*\nu}}{d\tau}$ will be linearized with repect to the ξ_i coordinates of the Fermi-Walker transported coordinate system under the assumption of the validity of the small oscillator approximation.

The model assumed for the classical charged harmonic oscillator of Ref. 3 will also be taken for this article. This model consisted of a classical charged particle of rest mass m and charge +e that was attracted to a uniformly accelerating equilibrium point by a simple harmonic potential, as measured in the instantaneous rest frame of the equilibrium point. [See Eq. (21) of Ref. 3, along with the discussion that accompanies this equation, for further description of the simple harmonic potential.]

This oscillator model will be further elaborated upon in this article in order to obtain an accelerating system that approximates, in the small oscillator limit, a pure electric dipole in the instantaneous rest frame of the equilibrium point of the oscillator. More specifically, a continuous negative charge distribution with net charge $-e$ will be assumed to surround the equilibrium point of the oscillator, as described in the instantaneous rest frame of the equilibrium point. This charge distribution will be taken to be stationary in the Fermi-Walker transported coordinate system and to possess axes of symmetry along the $i=1,2,3$ coordinate axes. The oscillating particle of charge $+e$ will be assumed to oscillate inside this "rigid" cloud distribution of negative charge, where the word "rigid" is used here to denote the fact that all points of the charge distribution possess the same instantaneous inertial rest frame throughout the full evolution of their trajectories. For sufficiently small amplitudes of oscillations, this rigid cloud of negative charge may be constructed in such a way as to be the source of the simple harmonic oscillator potential acting on the oscillating particle. (See the discussion in appendices A and B of Ref. 3 on specific simple harmonic oscillator models.) Moreover, for sufficiently small amplitudes of oscillation and for a sufficiently small volume of negative charge, the total oscillator system approximates an electric dipole in the instantaneous rest frame of the oscillator's equilibrium point.

The i^{th} component of this electric dipole in the rest frame of the oscillator's equilibrium point is given by $e\xi_{Li}(\tau_e)$, where $\xi_{Li}(\tau_e)$ is the distance along the rest frame's i^{th} axis from the oscillator's equilibrium point to the oscillating particle. Here, "L" is a label that takes on the values A or B, depending on which of the two accelerating oscillators is being discussed. The argument τ_e is again the proper time of the equilibrium point of the accelerating oscillator. (It should be noted that a distinction is being made here between ξ_i and ξ_{Li}. The coordinate along the i^{th} spatial axis in the Fermi-Walker transported coordinate system is given by ξ_i. The difference in coordinate values along the i^{th} axis of the Fermi-Walker transported coordinate system, between the L labeled oscillating particle and its equilibrium point, is denoted by ξ_{Li}.)

The dipole fields $\vec{E}^{DL}$ and $\vec{B}^{DL}$ mentioned earlier, where now the label L has been added in order to distinguish the dipole fields arising from the A or B accelerating oscillator, can be shown to depend linearly upon the quantities ξ_{Li}, $\frac{d\xi_{Li}}{d\tau_e}$, and $\frac{d^2\xi_{Li}}{d\tau_e^2}$. Ignoring second order terms in the coordinates will then mean dropping products of the dipole fields of one of the oscillators and the coordinates of the other oscillator. Equations (38) of Ref. 3 may then be readily generalized to the following set of equations for i=1,2,3:

$$\frac{d^2\xi_{Ai}}{d\tau_e^2} = -(\omega_i)^2\xi_{Ai} + \Gamma\left\{\frac{d^3\xi_{Ai}}{d\tau_e^3} - \left(\frac{a}{c}\right)^2\frac{d\xi_{Ai}}{d\tau_e}\right\} + \frac{e}{m}\left\{E^{ZP}_{\tau_e i}(\vec{R}_A,\tau_e) + E^{DB}_{\tau_e i}(\vec{R}_A,\tau_e)\right\} \quad , \tag{6a}$$

$$\frac{d^2\xi_{Bi}}{d\tau_e^2} = -(\omega_i)^2\xi_{Bi} + \Gamma\left\{\frac{d^3\xi_{Bi}}{d\tau_e^3} - \left(\frac{a}{c}\right)^2\frac{d\xi_{Bi}}{d\tau_e}\right\} + \frac{e}{m}\left\{E^{ZP}_{\tau_e i}(\vec{R}_B,\tau_e) + E^{DA}_{\tau_e i}(\vec{R}_B,\tau_e)\right\} \quad . \tag{6b}$$

Here, $\vec{R}_A$ and $\vec{R}_B$ represent the $\vec{\xi}$ vector positions (i.e., ξ_i, for i=1,2,3) of the equilibrium positions of oscillator A and B, respectively. Hence, for the configuration of the oscillator system considered here, $\vec{R}_A = +\hat{y}\frac{R}{2}$ and $\vec{R}_B = -\hat{y}\frac{R}{2}$. The arguments $(\vec{R}_A,\tau_e)$ and $(\vec{R}_B,\tau_e)$ are used to indicate that the fields are to be evaluated at proper time τ_e along the trajectory of the equilibrium point of the indicated oscillator, A or B. (Hence, a dipole approximation has been made in evaluating these fields.) The quantity $E^{ZP}_{\tau_e i}$ denotes the zero-point electric fields, while $E^{DA}_{\tau_e i}$ and $E^{DB}_{\tau_e i}$ represent the electric dipole fields of the A and B oscillators, respectively. The subscript τ_e occurring in these three fields is used here to indicate that the fields are to be evaluated in the inertial frame I_{τ_e} that is instantaneously at rest with respect to the equilibrium points of the oscillators at proper time τ_e .

The fields $\vec{E}_{\tau_e}$ and $\vec{B}_{\tau_e}$ in an inertial frame I_{τ_e} are readily expressed in terms of the fields $\vec{E}_{\tau_e'}$ and $\vec{B}_{\tau_e'}$ of another inertial frame $I_{\tau_e'}$ using the usual Lorentz transformation. Writing this transformation out in vector components yields:

$$\vec{E}_{\tau_e} = \hat{x}E_{\tau_e'1} + \hat{y}\gamma_{(\tau_e-\tau_e')}\left(E_{\tau_e'2} - \beta_{(\tau_e-\tau_e')}B_{\tau_e'3}\right) + \hat{z}\gamma_{(\tau_e-\tau_e')}\left(E_{\tau_e'3} + \beta_{(\tau_e-\tau_e')}B_{\tau_e'2}\right) , \tag{7}$$

$$\vec{B}_{\tau_e} = \hat{x}B_{\tau_e'1} + \hat{y}\gamma_{(\tau_e-\tau_e')}\left(B_{\tau_e'2} + \beta_{(\tau_e-\tau_e')}E_{\tau_e'3}\right) + \hat{z}\gamma_{(\tau_e-\tau_e')}\left(B_{\tau_e'3} - \beta_{(\tau_e-\tau_e')}E_{\tau_e'2}\right) , \tag{8}$$

where $\gamma_{(\tau_e-\tau_e')} = \cosh\left(\frac{a}{c}(\tau_e-\tau_e')\right)$ and $\beta_{(\tau_e-\tau_e')} = \tanh\left(\frac{a}{c}(\tau_e-\tau_e')\right)$.

In order to solve Eqs. (6a) and (6b), expressions for the dipole fields $E^{DB}_{\tau_e i}(\vec{R}_A,\tau_e)$ and $E^{DA}_{\tau_e i}(\vec{R}_B,\tau_e)$ must be obtained. Unfortunately, the calculation involved in determining these fields is quite lengthy; consequently, this work is deferred to Appendix A in order not to sidetrack from the main discussion presented here. Instead, the main result of this calculation will simply be summarized here before proceeding with the process of solving Eqs. (6a) and (6b).

Let L be the label of a time-dependent electric dipole undergoing relativistic, uniform acceleration. (Here, L will take on the values A or B, as the model of the accelerating charged oscillator described earlier will be treated in the limit where the model is a pure electric dipole in its instantaneous rest frame.) Let the spatial position of this electric dipole in the Fermi-Walker transported coordinate system be given by $\vec{\xi} = \hat{y}R_L$. From Appendix A, the electromagnetic fields at the coordinate position $\xi^\mu = (c\tau_e;\, 0, R, 0)$ due to the "L" labeled accelerating electric dipole is given by

$$E^{DL}_{\tau_e i}(\hat{y}R, \tau_e) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{E}^{DL}_i(\hat{y}R, \Omega)\exp(-i\Omega\tau_e)\,d\Omega \quad , \tag{9}$$

$$B^{DL}_{\tau_e i}(\hat{y}R, \tau_e) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widetilde{B}^{DL}_i(\hat{y}R, \Omega)\exp(-i\Omega\tau_e)d\Omega \quad , \tag{10}$$

where,
$$\widetilde{E}^{DL}_i(\hat{y}R,\Omega) = -\frac{m}{e^2}\sum_{j=1}^{3} n_{ij}(\hat{x}a, \hat{y}(R-R_L),\Omega)(e\widetilde{\xi}_{Lj}(\Omega)) \quad , \tag{11}$$

$$\widetilde{B}^{DL}_i(\hat{y}R,\Omega) = -\frac{m}{e^2}\sum_{j=1}^{3} \rho_{ij}(\hat{x}a \, ; \hat{y}(R-R_L),\Omega)(e\widetilde{\xi}_{Lj}(\Omega)) \quad , \tag{12}$$

and $e\widetilde{\xi}_{Lj}(\Omega)$ is the Fourier transform of the j^{th} component of the electric-dipole as measured in its instantaneous rest frame (*11). More specifically,

$$\left[e\,\xi_{Li}(\tau_e)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[e\,\xi_{Li}(\Omega)\right]\exp(-i\Omega\tau_e)d\Omega \quad . \tag{13}$$

The quantities n_{ij} and ρ_{ij} that occur in Eqs. (11) and (12) are rather complicated functions of the acceleration a, coordinate difference $(R-R_L)$, and frequency Ω ; the explicit functional forms are given in Eqs. (A39)-(A46) of Appendix A.

In order to solve Eqs. (6a) and (6b), the Fourier transform of the zero-point electric fields that occur in Eqs. (6a) and (6b) must be introduced. Hence, let

$$E^{zp}_{\tau_e i}(\hat{y}R,\tau_e) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} E^{zp}_i(\hat{y}R,\Omega)\exp(-i\Omega\tau_e)d\Omega \quad . \tag{14}$$

The argument that appears here of $(\hat{y}R,\tau_e)$ again represents the ξ^{μ} coordinate position at which to evaluate the electric field.

Now, Eqs. (9), (11), and (14) may be substituted into Eqs. (6a) and (6b) to yield the following set of equations:

$$C_i(a,\Omega)\tilde{\xi}_{Ai}(\Omega) + \sum_{j=1}^{3} \eta_{ij}(\hat{x}a + \hat{y}R,\Omega)\tilde{\xi}_{Bi}(\Omega) = \frac{e}{m}\tilde{E}_i^{zp}(+\hat{y}\frac{R}{2},\Omega) \quad , \tag{15a}$$

$$C_i(a,\Omega)\tilde{\xi}_{Bi}(\Omega) + \sum_{j=1}^{3} \eta_{ij}(\hat{x}a - \hat{y}R,\Omega)\tilde{\xi}_{Ai}(\Omega) = \frac{e}{m}\tilde{E}_i^{zp}(-\hat{y}\frac{R}{2},\Omega) \quad , \tag{15b}$$

where
$$C_i(a,\Omega) = -\Omega^2 + (\omega_i)^2 - i\Gamma(\Omega^3 + \Omega(\tfrac{a}{c})^2) \quad . \tag{16}$$

Equations (15a) and (15b) may be simplified somewhat by noting from Eqs. (A39)–(A43) that $\eta_{ii}(\hat{x}a,+\hat{y}R,\Omega) = \eta_{ii}(\hat{x}a,-\hat{y}R,\Omega)$, $\eta_{ij}(\hat{x}a,+\hat{y}R,\Omega) = -\eta_{ij}(\hat{x}a,-\hat{y}R,\Omega)$ for $i \neq j$, $\eta_{12} = -\eta_{21}$, and $\eta_{i3} = \eta_{3i} = 0$ for $i \neq 3$. Using conventional matrix methods, Eqs. (15a) and (15b) can then be solved for $\tilde{\xi}_{Ai}(\Omega)$ and $\tilde{\xi}_{Bi}(\Omega)$ in terms of $\tilde{E}_i^{zp}(+\hat{y}\frac{R}{2},\Omega)$ and $\tilde{E}_i^{zp}(-\hat{y}\frac{R}{2},\Omega)$. After taking the inverse Fourier transform of $\tilde{\xi}_{Ai}(\Omega)$ and $\tilde{\xi}_{Bi}(\Omega)$, one then obtains the solutions to $\xi_{Ai}(\tau_e)$ and $\xi_{Bi}(\tau_e)$.

After imposing the conditions of Eqs. (1) and (2), these solutions will be used in Secs. III and IV to deduce certain statistical properties of the accelerating oscillator system. In particular, the expectation value will be computed for the component of the Lorentz force along the $\hat{y}$ direction that acts on one of the accelerating charged oscillators, as given in the instantaneous rest frame of the oscillator's equilibrium point.

III. SPECIAL CASE OF TWO ACCELERATING DIPOLE OSCILLATORS

A. Description of Special Case

The full calculations involved in obtaining the statistical properties of the accelerating charged oscillators are rather long. Consequently, it seems appropriate to examine the simplest case possible that illustrates the main physics of the system before proceeding with the most general case. Fortunately, a special case does exist that will serve this purpose; in addition, the solution of this problem will be shown to aid in solving the more general oscillator case that will be discussed in Sec. IV.

This special case arises by examining Eqs. (15a) and (15b) for i=3. As shown in Appendix A [see Eq. (A43)], $n_{13} = n_{31} = n_{23} = n_{32} = 0$. Hence, oscillations in the $\hat{x}$ and $\hat{y}$ directions of the two accelerating charged oscillators do not give rise to electric dipole fields in the $\hat{z}$ direction at the other oscillator, as expressed in the instantaneous rest frame of the two oscillators. Consequently, the i=3 set of equations obtained from Eqs. (15a) and (15b) are not coupled to the corresponding i=1,2 set of equations. Hence, the special case of two accelerating electric dipole oscillators with nonzero electric dipole moments only along the $\hat{z}$ direction may be safely studied for the main underlying physics of the accelerating system.

Equations (15) are easily solved for the case of i=3. One obtains

$$\tilde{\mathcal{E}}_{A3}(\Omega) = \frac{e}{2m}\left\{ \tilde{E}_3^{ZP}(+\hat{y}\tfrac{R}{2},\Omega)\left[\frac{1}{C_3^a - \Omega_{33}^a} + \frac{1}{C_3^a + \Omega_{33}^a}\right] + \tilde{E}_3^{ZP}(-\hat{y}\tfrac{R}{2},\Omega)\left[\frac{-1}{C_3^a - \Omega_{33}^a} + \frac{1}{C_3^a + \Omega_{33}^a}\right]\right\} , \quad (17a)$$

$$\tilde{\mathcal{E}}_{B3}(\Omega) = \frac{e}{2m}\left\{ \tilde{E}_3^{ZP}(+\hat{y}\tfrac{R}{2},\Omega)\left[\frac{-1}{C_3^a - \Omega_{33}^a} + \frac{1}{C_3^a + \Omega_{33}^a}\right] + \tilde{E}_3^{ZP}(-\hat{y}\tfrac{R}{2},\Omega)\left[\frac{1}{C_3^a - \Omega_{33}^a} + \frac{1}{C_3^a + \Omega_{33}^a}\right]\right\} . \quad (17b)$$

Here, $C_3(a,\Omega)$ and $\Omega_{33}(\hat{x}a,+\hat{y}R,\Omega)$ have been abbreviated by C_3^a and Ω_{33}^a, where the dependence on acceleration is still explicitly indicated in order to later aid in making comparisons with equations describing the unaccelerated situation.

The remainder of Sect. III uses Eqs. (17a) and (17b) to determine certain statistical properties for the pair of accelerated oscillators. Section IIIB imposes a narrow linewidth approximation, the small oscillator assumption, and the conditions of Eqs. (1) and (2), in order to determine the expectation value of the component of the Lorentz force in the $\hat{y}$ direction acting upon one of the oscillators. Consequently, the integrals encountered in calculating this force are readily evaluated by employing a resonant approximation. The value for the Lorentz force of the accelerated oscillators is then found to agree with the unretarded van der Waals force for the corresponding unaccelerated-thermal oscillator system. This resonant approximation is reexamined in Sec. IIIC in order to show

how this correspondence arises between the accelerated and unaccelerated-thermal oscillator systems. Section IIID then turns to an examination of other statistical properties of the accelerated pair of charged harmonic oscillators.

B. Expectation Value of Lorentz Force in Resonant Case

Let $\vec{F}_{\tau_e}(t_{\tau_e})$ be the Lorentz force on a uniformly accelerating charged oscillator at time t_{τ_e} in the I_{τ_e} inertial frame. According to the construction of the set of inertial reference frames I_{τ_e} in Ref. 3, when $t_{\tau_e}=0$, then the I_{τ_e} inertial frame is instantaneously at rest with respect to the equilibrium point of the oscillator. Hence, $\vec{F}_{\tau_e}(t_{\tau_e}=0)$ represents the force in the rest frame of the equilibrium point of the oscillator at proper time τ_e of the equilibrium point.

Appendix B presents a simple derivation of the Lorentz force on a particular model of a charged oscillator that moves in some arbitrary trajectory through space-time. The expression for the force in Appendix B is given to first order in the amplitude of the oscillator. This is equivalent to treating the charged oscillator as being a pure electric dipole in the inertial rest frame of the equilibrium point of the oscillator.

Equation (B4) may be immediately applied here to obtain the appropriate expression for the Lorentz force on a uniformly accelerating oscillator, when the oscillator is taken in the electric dipole limit. Let $\vec{X}_{\tau_e}(t_{\tau_e})$ and $\vec{X}_{\tau_e}(t_{\tau_e})+\Delta\vec{x}_{\tau_e}(t_{\tau_e})$ be the respective positions of the equilibrium point and the oscillating particle at time t_{τ_e} in the I_{τ_e} inertial frame. One then obtains from Eq. (B4) that

$$F_{\tau_e i}(t_{\tau_e}=0) = e\sum_{j=1}^{3} \Delta x_{\tau_e j}(0)\frac{\partial}{\partial x_{\tau_e i}} E_{\tau_e j}(\vec{x}_{\tau_e}, t_{\tau_e})\Big|_{\vec{X}_{\tau_e}(0),\,0}$$

$$+\frac{e}{c}\frac{d}{dt_{\tau_e}}\Big(\vec{\Delta x}_{\tau_e}(t_{\tau_e})\otimes\vec{B}_{\tau_e}(\vec{X}(t_{\tau_e}), t_{\tau_e})\Big)_i\Big|_{t_{\tau_e}=0} \quad , \qquad (18)$$

where, in the case of the pair of accelerating oscillators being considered here, $\vec{E}_{\tau_e}$ and $\vec{B}_{\tau_e}$ are the total electric and magnetic fields due to the zero-point radiation fields and the electromagnetic fields of the opposite accelerating oscillator.

The expectation value of the i=2 component of Eq. (18) will be calculated in this article, as this is the force component along the direction of separation of the two oscillators. As will be shown quite generally in Sec. IV, where no restrictions will be placed on the direction in which oscillations may occur, the expectation value of the second term of Eq. (18) is of order $(aR/c^2)^2$ less than the first term. Hence, the second term may be neglected in the small laboratory approximation of Eq. (1). For the situation being considered in the present section, where oscillations are confined to the $\hat{z}$ direction, the second term of Eq. (18) is exactly equal to zero, as will be shown shortly.

The zero-point electromagnetic fields will be specified in the I_* inertial frame, which, as mentioned earlier, is equivalent to the $I_{\tau_e=0}$ coordinate system. Using the Lorentz transformation of Eqs. (7) and (8), the zero-point fields may then be obtained in all I_{τ_e} inertial frames. The functional form that will be used for the zero-point fields will consist of the following relationships, which have been frequently used in performing calculations in stochastic electrodynamics (*12):

$$\vec{E}_*^{zp}(\vec{x}_*, t_*) = \sum_{\lambda=1}^{2} \int d^3k\, \mathcal{h}(\omega)\hat{\epsilon}(\vec{k},\lambda)\cos\left(\vec{k}\cdot\vec{x}_* - \omega t_* + \Theta(\vec{k},\lambda)\right) \quad , \tag{19}$$

$$\vec{B}_*^{zp}(\vec{x}_*, t_*) = \sum_{\lambda=1}^{2} \int d^3k\, \mathcal{h}(\omega)\left(\hat{k}\otimes\hat{\epsilon}(\vec{k},\lambda)\right)\cos\left(\vec{k}\cdot\vec{x}_* - \omega t_* + \Theta(\vec{k},\lambda)\right) \; . \tag{20}$$

Identities that the polarization vectors $\hat{\epsilon}(\vec{k},\lambda)$ satisfy and that will be used in later calculations are

$$\hat{\epsilon}(\vec{k},\lambda)\cdot\hat{\epsilon}(\vec{k},\lambda') = \delta_{\lambda,\lambda'} \quad , \quad \vec{k}\cdot\hat{\epsilon}(\vec{k},\lambda) = 0 \quad , \qquad (21\,a\,\&\,b)$$

$$\begin{aligned}\sum_{\lambda=1}^{2} \epsilon_i(\vec{k},\lambda)\epsilon_j(\vec{k},\lambda) &= \sum_{\lambda=1}^{2} \left(\hat{k}\otimes\hat{\epsilon}(\vec{k},\lambda)\right)_i \left(\hat{k}\otimes\hat{\epsilon}(\vec{k},\lambda)\right)_j \\ &= \delta_{ij} - \frac{k_i k_j}{k^2} \quad ,\end{aligned} \tag{22}$$

$$\sum_{\lambda=1}^{2} \epsilon_i(\vec{k},\lambda)\left(\hat{k}\otimes\hat{\epsilon}(\vec{k},\lambda)\right)_j = \sum_{\ell} \epsilon_{ij\ell}\frac{k_\ell}{k} \quad . \tag{23}$$

The frequency ω in Eqs. (19) and (20) satisfies $\omega = c|\vec{k}|$. The phase angle $\Theta(\vec{k},\lambda)$ is a random variable, independently distributed for each $\vec{k}$ and λ, that ranges between 0 and 2π with uniform probability density. The function $\mathcal{h}(\omega)$ is given by

$$\mathcal{h}^2(\omega) = \frac{\hbar\omega}{2\pi^2} \quad , \tag{24}$$

so as to insure the Lorentz invariance of the stochastic properties of the zero-point fields and to obtain agreement with nature in regard to quantum mechanical phenomena (see Refs. 6 and 12).

Using Eq. (18), the expectation value of the i=2 component of the Lorentz force on oscillator A will now be calculated. (By symmetry, the value of this component must be of equal magnitude and opposite sign to the similar quantity calculated for oscillator B.) According to the construction of the Fermi-Walker transported coordinate system in Ref. 3, the quantity $\Delta x_{\tau_e j}(0)$ that occurs in the first term of Eq. (18) may be replaced by $\xi_{Aj}(\tau_e)$. From Eq. (5c), $\xi_2 = x_{\tau_e 2}$; hence, the operator $\frac{\partial}{\partial x_{\tau_e 2}}$ may be replaced by $\frac{\partial}{\partial \xi_2}$. Let the quantity $\vec{F}_{\tau_e}(t_{\tau_e})$ be relabeled by $\vec{F}_{\tau_e}(\tau_e')$, where the inertial coordinate time t_{τ_e} of the equilibrium point is related to the proper time τ_e' by

$$t_{\tau_e}(\tau_e') = \frac{c}{a} \sinh\left(\frac{a}{c}(\tau_e' - \tau_e)\right) \quad . \tag{25}$$

From Eq. (25),

$$\left.\frac{d\tau_e'}{dt_{\tau_e}}\right|_{\tau_e'=\tau_e} = \left.\frac{1}{\cosh\left(\frac{a}{c}(\tau_e'-\tau_e)\right)}\right|_{\tau_e'=\tau_e} = 1 \quad . \tag{26}$$

For the special case being considered in this section of

$$\vec{\Delta x}_{A\tau_e}(t_{\tau_e}(\tau_e')) = \hat{z}\,\xi_{A3}(\tau_e') \quad \text{and} \quad \vec{\Delta x}_{B\tau_e}(t_{\tau_e}(\tau_e')) = \hat{z}\,\xi_{B3}(\tau_e') \quad , \tag{27a \& b}$$

the expectation value of the i=2 force component of Eq. (18) is given by

$$\langle F_{A\tau_e 2}(\tau_e)\rangle = e\left\langle \xi_{A3}(\tau_e)\frac{\partial}{\partial\xi_2}E^{zp}_{\tau_e 3}(\vec{\xi},\tau_e)\Big|_{\vec{\xi}=\hat{y}\frac{R}{2}}\right\rangle + e\left\langle \xi_{A3}(\tau_e)\frac{\partial}{\partial\xi_2}E^{DB}_{\tau_e 3}(\vec{\xi},\tau_e)\right\rangle\Big|_{\vec{\xi}=\hat{y}\frac{R}{2}} \tag{28}$$
$$+\frac{e}{c}\frac{d}{d\tau_e'}\left\langle \xi_{A3}(\tau_e')\left[B^{zp}_{\tau_e 1}(\hat{y}\tfrac{R}{2},\tau_e')+B^{DB}_{\tau_e 1}(\hat{y}\tfrac{R}{2},\tau_e')\right]\right\rangle\Big|_{\tau_e'=\tau_e}.$$

The arguments in the fields have been relabeled here in terms of the ξ^μ coordinates.

From Eqs. (13), (17), and the inverse of (14), the first term of Eq. (28) becomes

$$e\left\langle \xi_{A3}(\tau_e)\frac{\partial}{\partial\xi_2}E^{zp}_{\tau_e 3}(\vec{\xi},\tau_e)\Big|_{\vec{\xi}=\hat{y}\frac{R}{2}}\right\rangle = \frac{e}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\Omega'\exp(-i\Omega'\tau_e)\left\langle \tilde{\xi}^3_A(\Omega')\frac{\partial}{\partial\xi_2}E^{zp}_{\tau_e 3}(\vec{\xi},\tau_e)\Big|\right\rangle_{\vec{\xi}=\hat{y}\frac{R}{2}}$$
$$=\left(\frac{e^2}{2m}\right)\frac{1}{2\pi}\int_{-\infty}^{\infty}d\Omega'\exp(-i\Omega'\tau_e)\int_{-\infty}^{\infty}d\tau_e'\exp(i\Omega'\tau_e')\,\times$$
$$\times\left\{\left[\frac{1}{C_3^{a'}-n_{33}^{a'}}+\frac{1}{C_3^{a'}+n_{33}^{a'}}\right]\left\langle E^{zp}_{\tau_e' 3}(\hat{y}\tfrac{R}{2},\tau_e')\frac{\partial}{\partial\xi_2}E^{zp}_{\tau_e 3}(\vec{\xi},\tau_e)\Big|_{\vec{\xi}=\hat{y}\frac{R}{2}}\right\rangle\right. \tag{29}$$
$$\left.+\left[\frac{-1}{C_3^{a'}-n_{33}^{a'}}+\frac{1}{C_3^{a'}+n_{33}^{a'}}\right]\left\langle E^{zp}_{\tau_e' 3}(-\hat{y}\tfrac{R}{2},\tau_e')\frac{\partial}{\partial\xi_2}E^{zp}_{\tau_e 3}(\vec{\xi},\tau_e)\Big|_{\vec{\xi}=\hat{y}\frac{R}{2}}\right\rangle\right\}.$$

In order to carry this calculation further, the two quantities involving the expectation values in Eq. (29) must be evaluated. The following quantity is therefore of interest:

$$\left\langle E^{zp}_{\tau_e i}(\hat{y}R,\tau_e)\frac{\partial}{\partial\xi_2}E^{zp}_{\tau_e' j}(\vec{\xi},\tau_e')\Big|_{\vec{\xi}=\hat{y}R'}\right\rangle = \frac{\partial}{\partial\xi_2}\left\langle E^{zp}_{\tau_e i}(\hat{y}R,\tau_e)E^{zp}_{\tau_e' j}(\hat{y}\xi_2,\tau_e')\right\rangle\Big|_{\xi_2=R'}. \tag{30}$$

The correlation function to the right of the $\frac{\partial}{\partial\xi_2}$ operator on the right side of Eq. (30) is shown in Appendix C to be equal to

$$\left\langle E^{zp}_{\tau_e i}(\hat{y}R,\tau_e)E^{zp}_{\tau_e' j}(\hat{y}R',\tau_e')\right\rangle = \int_0^\infty f^{zp}_{ij}(\hat{x}a,\hat{y}(R'-R),\Omega)\cos(\Omega(\tau_e'-\tau_e))\,d\Omega\,, \tag{C1}$$

where f^{zp}_{ij} is a rather complicated function of a, R and

Ω , and is intimately related to the quantity η_{ij} that occurs in the expression for the electric dipole fields of Eq. (11). More will be said about this relationship later. The appropriate relationship between Eq. (30) and the functions f_{ij}^{zp} in Eq. (C1) can be easily deduced; this relationship is given in Eq. (C57).

From Eqs. (29), (C57), and (C60), one obtains

$$e\left\langle \xi_{A3}(\tau_e)\frac{\partial}{\partial\xi_2}E^{zp}_{\tau_e 3}(\vec{\xi},\tau_e)\Big|_{\vec{\xi}=\hat{y}\frac{R}{2}}\right\rangle = \left(\frac{e^2}{2m}\right)\frac{1}{2\pi}\int_0^\infty d\Omega' \exp(-i\Omega'\tau_e)\int_{-\infty}^{\infty}d\tau_e' \exp(i\Omega'\tau_e') \times$$
$$\times\left[\frac{-1}{C_3^{a'}-\eta_{33}^{a'}}+\frac{1}{C_3^{a'}+\eta_{33}^{a'}}\right]\int_0^\infty d\Omega\frac{\partial}{\partial R}f_{33}^{zp}(\hat{x}a,\hat{y}R,\Omega)\cos(\Omega(\tau_e'-\tau_e))$$
$$=\left(\frac{e^2}{2m}\right)\frac{1}{2\pi}\int_0^\infty d\Omega\frac{\partial}{\partial R}f_{33}^{zp}(\hat{x}a,\hat{y}R,\Omega)\int_{-\infty}^{\infty}d\Omega' \exp(-i\Omega'\tau_e)\times$$
$$\times\left[\frac{-1}{C_3^{a'}-\eta_{33}^{a'}}+\frac{1}{C_3^{a'}+\eta_{33}^{a'}}\right]\frac{2\pi}{2}\left\{\delta(\Omega'+\Omega)e^{-i\Omega\tau_e}+\delta(\Omega'-\Omega)e^{i\Omega\tau_e}\right\} . \quad (31)$$

From Eq. (16), it can be seen that

$$C_i(a,-\Omega) = C_i^*(a,\Omega) \quad . \quad (32)$$

This same property is demonstrated for η_{ij}^a in Appendix A [see Eq. (A47)]. Hence, from Eqs. (31), (32), and (A47), one has that

$$e\left\langle \xi_{A3}(\tau_e)\frac{\partial}{\partial\xi_2}E^{zp}_{\tau_e 3}(\vec{\xi},\tau_e)\Big|_{\vec{\xi}=\hat{y}\frac{R}{2}}\right\rangle = \frac{e^2}{2m}\int_0^\infty d\Omega\left\{-\frac{R_e(C_3^a-\eta_{33}^a)}{|C_3^a-\eta_{33}^a|^2}+\frac{R_e(C_3^a+\eta_{33}^a)}{|C_3^a+\eta_{33}^a|^2}\right\}\frac{\partial}{\partial R}f_{33}^{zp}(\hat{x}a,\hat{y}R,\Omega) . \quad (33)$$

The second term in Eq. (28) may be calculated in a similar manner. From Eqs. (11) and (13), one obtains

$$
\begin{aligned}
& e\left\langle \xi_{A3}(\tau_e)\frac{\partial}{\partial\xi_2}E^{DB}_{\tau_e 3}(\vec{\xi},\tau_e)\Big|_{\vec{\xi}=\hat{y}\frac{R}{2}}\right\rangle \\
&= \frac{e}{2\pi}\int_{-\infty}^{\infty} d\Omega' \exp(-i\Omega'\tau_e)\int_{-\infty}^{\infty} d\Omega'' \exp(-i\Omega''\tau_e)\left(-\frac{m}{e}\right)\left[\frac{\partial}{\partial R}\eta_{33}(\hat{x}a,\hat{y}R,\Omega'')\right]\left\langle \tilde{\xi}_{A3}(\Omega')\tilde{\xi}_{B3}(\Omega'')\right\rangle . \qquad (34)
\end{aligned}
$$

From Eqs. (17a) and (17b) and the inverse of Eq. (14),

$$
\begin{aligned}
& \left\langle \tilde{\xi}_{A3}(\Omega')\tilde{\xi}_{B3}(\Omega'')\right\rangle \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau_e' \exp(i\Omega\tau_e')\int_{-\infty}^{\infty} d\tau_e'' \exp(i\Omega\tau_e'')\left(\frac{e}{2m}\right)^2 \times \\
&\quad \times \Bigg\langle \left\{ E_3^{zp}\left(\hat{y}\frac{R}{2},\tau_e'\right)\left[\frac{1}{C_3^{a'}-\eta_{33}^{a'}} + \frac{1}{C_3^{a'}+\eta_{33}^{a'}}\right] + E_3^{zp}\left(-\hat{y}\frac{R}{2},\tau_e'\right)\left[\frac{-1}{C_3^{a'}-\eta_{33}^{a'}} + \frac{1}{C_3^{a'}+\eta_{33}^{a'}}\right]\right\} \times \qquad (35) \\
&\qquad \times \left\{ E_3^{zp}\left(\hat{y}\frac{R}{2},\tau_e''\right)\left[\frac{-1}{C_3^{a''}-\eta_{33}^{a''}} + \frac{1}{C_3^{a''}+\eta_{33}^{a''}}\right] + E_3^{zp}\left(-\hat{y}\frac{R}{2},\tau_e''\right)\left[\frac{1}{C_3^{a''}-\eta_{33}^{a''}} + \frac{1}{C_3^{a''}+\eta_{33}^{a''}}\right]\right\} \Bigg\rangle .
\end{aligned}
$$

This expression can be evaluated with the use of Eq. (C1) and the symmetry relationship of Eq. (C56). The algebraic identity consisting of

$$
(\pm A_-' + A_+')(-A_-'' + A_+'') + (\mp A_-' + A_+')(A_-'' + A_+'') = 2(\mp A_-' A_-'' + A_+' A_+'') \quad , \qquad (36)
$$

where $A_\pm = 1/[C_3^a \pm \eta_{33}^a]$, will be found to be helpful in obtaining the relationship

$$
\begin{aligned}
& \left\langle \tilde{\xi}_{A3}(\Omega')\tilde{\xi}_{B3}(\Omega'')\right\rangle \\
&= \left(\frac{e}{2m}\right)^2 2\pi\int_0^{\infty} d\Omega\left[\delta(\Omega'-\Omega)\delta(\Omega''+\Omega) + \delta(\Omega'+\Omega)\delta(\Omega''-\Omega)\right] \times \\
&\quad \times \Bigg\{ f_{33}^{zp}(\hat{x}a,0,\Omega)\left[-\frac{1}{(C_3^{a'}-\eta_{33}^{a'})}\frac{1}{(C_3^{a''}-\eta_{33}^{a''})} + \frac{1}{(C_3^{a'}+\eta_{33}^{a'})}\frac{1}{(C_3^{a''}+\eta_{33}^{a''})}\right] \qquad (37) \\
&\qquad + f_{33}^{zp}(\hat{x}a,\hat{y}R,\Omega)\left[\frac{1}{(C_3^{a'}-\eta_{33}^{a'})}\frac{1}{(C_3^{a''}-\eta_{33}^{a''})} + \frac{1}{(C_3^{a'}+\eta_{33}^{a'})}\frac{1}{(C_3^{a''}+\eta_{33}^{a''})}\right]\Bigg\} .
\end{aligned}
$$

After substituting Eq. (37) into Eq. (34) and using Eqs. (32) and (A47), one obtains the relationship of

$$e\left\langle \xi_{A3}(\tau_e)\frac{\partial}{\partial \xi_2} E^{DB}_{\tau_e 3}(\vec{\xi},\tau_e)\Big|_{\vec{\xi}=\hat{y}\frac{R}{2}}\right\rangle \tag{38}$$
$$= \frac{-e^2}{2m}\int_0^\infty d\Omega \left\{ -\frac{\left(f^{zp}_{33}(\hat{x}a,0,\Omega)-f^{zp}_{33}(\hat{x}a,\hat{y}R,\Omega)\right)}{|C^a_3-\eta^a_{33}|^2} + \frac{\left(f^{zp}_{33}(\hat{x}a,0,\Omega)+f^{zp}_{33}(\hat{x}a,\hat{y}R,\Omega)\right)}{|C^a_3+\eta^a_{33}|^2}\right\}\frac{\partial \mathrm{Re}\,\eta^a_{33}}{\partial R} .$$

The third term in Eq. (28), which came from the second term of Eq. (18), is easily shown to equal zero. From Eq. (8), $B_{\tau_e 1}(\vec{\xi},\tau_e') = B_{\tau_e' 1}(\vec{\xi},\tau_e')$. This means that all quantities inside the expectation value signs in the third term of Eq. (28) depend only upon τ_e'. Using Eqs. (17), (A38), (C1), and (C2), explicit calculations can then be carried out to demonstrate that the expectation value of this quantity is a number that is totally independent of the value of τ_e'. Alternatively, this demonstration follows more generally from the physical demand that the act of accelerating through the zero-point fields must yield statistical properties that are stationary in the proper time τ_e. Using either argument, however, the net result is that the third term of Eq. (28) is exactly equal to zero.

The integrals that appear in Eqs. (33) and (38) will now be evaluated under the assumption that the damping time $\Gamma = \frac{2}{3}(e^2/mc^3)$ is much smaller than the other time constants that describe the oscillator's behavior, such as the period of the oscillator at resonance (for an electron, $\Gamma \approx 6\times10^{-24}$ seconds). Under this condition, which will be made more specific shortly, the quantities $1/|C^a_3 \pm \eta^a_{33}|^2$, which occur in the integrands in Eqs. (33) and (38), behave as sharply peaked functions near the point $\Omega = \omega_3$. This resonant property of $1/|C_3 \pm \eta^a_{33}|^2$ can then be exploited to aid in evaluating the integrals in Eqs. (33) and (38).

Insight into the resonant behavior of $1/|C_3^a \pm n_{33}^a|^2$ can be gained by first examining the less complicated quantity of

$$\frac{1}{|C_i(a,\Omega)|^2} = \frac{1}{\left[\left(-\Omega^2 + (\omega_i)^2\right)^2 + \Gamma^2\Omega^6\left(1 + \left(\frac{a/c}{\Omega}\right)^2\right)^2\right]} \quad . \tag{39}$$

This quantity occurs in calculations involving the acceleration of a single oscillator; more will be said about this situation in Sect. IIID. When Γ is sufficiently small that

$$\Gamma\omega_i\left(1 + \left(\frac{a/c}{\omega_i}\right)^2\right) << 1 \quad , \tag{40}$$

then $1/|C_i^a|^2$ becomes a sharply peaked function of Ω about the point ω_i. Equation (39) is then well approximated by

$$\frac{1}{|C_i(a,\Omega)|^2} \approx \frac{1}{(\Omega - \omega_i)^2(2\omega_i)^2 + \Gamma^2(\omega_i)^6\left(1 + \left(\frac{a}{c\omega_i}\right)^2\right)^2} \quad . \tag{41}$$

The resonance full width at half-maximum of the peak described by this function is given by

$$\Delta_{Si}^a = \Gamma(\omega_i)^2\left(1 + \left(\frac{a/c}{\omega_i}\right)^2\right) \quad , \tag{42}$$

where the superscript a and subscript S indicate the single accelerated oscillator situation.

Turning now to examine the resonant properties of

$$\frac{1}{\left|C_3^a \pm \Pi_{33}^a\right|^2} = \frac{1}{\left(-\Omega^2 + (\omega_3)^2 \pm Re\,\Pi_{33}^a\right)^2 \pm \left(Im(C_3^a \pm \Pi_{33}^a)\right)^2} \quad , \tag{43}$$

it immediately becomes evident that the functional dependence of $\Pi_{33}(\hat{x}a, \hat{y}R, \Omega)$ upon Ω must first be analyzed. As explained in the Introduction, if a physical equivalence of some sort does indeed exist between the accelerated pair of charged oscillators and the unaccelerated, but thermally-bathed pair of charged oscillators, then the likely situation to observe such an equivalence is when the unretarded van der Waals condition applies of $\frac{\omega_i R}{c} \ll 1$. When $\left(\frac{aR}{c^2}\right) \ll 1$ and $\frac{\Omega R}{c} \ll 1$, the appropriate functional form for Π_{33}^a can readily be deduced from Eq. (A41). The quantity $\exp(i\Omega\Delta T_-)$ must be expanded in a Taylor series in ΔT_-, with terms up to the third power of ΔT_- retained. The identities

$$\lim_{R\to 0}\left(\frac{1 - \frac{c\Delta T_-}{R_-}}{R_-^2}\right) = \frac{1}{6}\left(\frac{a}{c^2}\right)^2 \tag{44}$$

and

$$\lim_{R\to 0}\left(\frac{c\Delta T_-}{R_-}\right) = 1 \quad , \tag{45}$$

which follow from

$$\Delta T_- = \frac{c}{a}\sinh^{-1}\left(\frac{aR_-}{c^2}\right) \approx \frac{c}{a}\left\{\frac{aR_-}{c^2} - \frac{1}{6}\left(\frac{aR_-}{c^2}\right)^3 + \ldots\right\} \tag{46}$$

and $R_- = R\left(1 + 2(aR/2c^2)^2\right)^{1/2}$ [see Eqs. (A19), (A21), and (A22)], are then useful in establishing that

$$\Pi_{33}(\hat{x}a, \hat{y}R, \Omega) \approx \left\{\frac{3}{2}\Gamma\left(\frac{c}{R}\right)^3 - i\Gamma\Omega^3\left(1 + \left(\frac{a}{c\Omega}\right)^2\right)\right\} \times$$

$$\times \left\{1 + \mathcal{O}\left(\frac{aR}{c^2}\right) + \mathcal{O}\left(\frac{\Omega R}{c}\right)\right\} \quad , \tag{47}$$

when $\Omega R/c \ll 1$ and $aR/c^2 \ll 1$. Hence, when these two conditions apply, then $Re\, \eta^a_{33}$ is essentially independent of both frequency Ω and acceleration a , while $Im\, \eta^a_{33}$ retains a dependence on both of these quantities. The next order terms in $Re\, \eta^a_{33}$ and $Im\, \eta^a_{33}$ vary as $\frac{1}{R^2}$ and R^1, respectively. As may be seen when i=j=3, Eqs. (47) and (16) agree with the following result, which may be established with a bit of work from Eqs. (44), (45) and (A39)-(A43):

$$\lim_{R\to 0} Im\, \eta_{ij}(\hat{x}a, \hat{y}R, \Omega) = +\delta_{ij}\, Im\, C_i(a,\Omega) = -\delta_{ij}\Gamma\left[\Omega^3 + \Omega\left(\frac{a}{c}\right)^2\right] \quad . \tag{48}$$

The $a=0$ value for η^a_{33} will be abbreviated by η_{33}; from Eq. (A41), this quantity is given by

$$\eta_{33}(\hat{x}a, \hat{y}R, \Omega)\bigg|_{a=0} = -\frac{e^2}{m}K^3\left[\frac{1}{KR} + \frac{i}{(KR)^2} - \frac{1}{(KR)^3}\right]\exp\left(\frac{i\Omega R}{c}\right) \quad . \tag{49}$$

When $\frac{\Omega R}{c} \ll 1$ and $\frac{aR}{c^2} \ll 1$, then from Eqs. (47) and (49),

$$Re\, \eta^a_{33} \approx Re\, \eta_{33} \approx \frac{3}{2}\Gamma(c/R)^3 \quad . \tag{50}$$

When the conditions expressed by Eqs. (40), (1) and (2) hold, with ω_0 replaced by ω_i in Eq. (2), then from Eqs. (43) and (47), $1/|C^a_3 \pm \eta^a_{33}|^2$ attains a maximum as a function of Ω at approximately the point $\Omega = \omega_{3\pm}$, where

$$\omega_{3\pm} = \left[(\omega_3)^2 \pm Re\, \eta^a_{33}\bigg|_{\omega_{3\pm}}\right]^{1/2} \approx \left[(\omega_3)^2 \pm \frac{3}{2}\frac{\Gamma}{(R/c)^3}\right]^{1/2} \quad . \tag{51}$$

By making Γ sufficiently small enough, Eq. (43) describes a sharply peaked function about $\Omega = \omega_{3\pm}$ that may be approximated by replacing all quantities that do not involve $\Omega - \omega_{3\pm}$ by their value at $\omega_{3\pm}$:

$$\frac{1}{|C_3^a \pm n_{33}^a|^2} \approx \frac{1}{(\Omega - \omega_{3\pm})^2 (2\omega_{3\pm})^2 + \left(\mathrm{Im}(C_3^a \pm n_{33}^a)\big|_{\omega_{3\pm}}\right)^2} \quad . \tag{52}$$

Here, $\mathrm{Im}\, n_{33}^a|_{\omega_{3\pm}}$ may be approximated by the imaginary part of Eq. (47), evaluated at $\Omega = \omega_{3\pm}$, where the first order terms of $O(\frac{aR}{c^2})$ and $O(\omega_{3-}R/c)$ should be retained in the case of ω_{3-} in order not to result in a zero value for $\mathrm{Im}(C_3^a - n_{33}^a)\big|_{\omega_{3-}}$. [Although the actual value of these first order terms are readily obtained, they are not needed here, as $\mathrm{Im}(C_3^a \pm n_{33}^a)$ will be shown to cancel out in later calculations.]

The resonance full width at half-maximum of the peak described by Eq. (52) is given by

$$\begin{aligned} \Delta_{D3\pm}^a &= \frac{1}{\omega_{3\pm}} \left| \mathrm{Im}(C_3^a \pm n_{33}^a) \right|_{\omega_{3\pm}} \\ &\le \frac{1}{\omega_{3\pm}} \left\{ |\mathrm{Im}\, C_3^a| + |\mathrm{Im}\, n_{33}^a| \right\} \Big|_{\omega_{3\pm}} \approx 2\Gamma(\omega_3)^2 \left(1 + \left(\frac{a}{c\omega_3}\right)^2\right) , \end{aligned} \tag{53}$$

where terms of order $O(\frac{aR}{c^2})$ and $O(\frac{\omega_3 R}{c^2})$ have been ignored in the last step [see Eqs. (1), (2), and (47)]. The subscript D in $\Delta_{D3\pm}^a$ is used here to indicate the double accelerated oscillator situation. From Eq. (53), the width of the peak becomes narrower as Γ becomes smaller; likewise, since $\mathrm{Im}(C_3^a \pm n_{33}^a)$ is proportional to Γ, the maximum of Eq. (52) increases as Γ decreases in value. Assuming that Γ is sufficiently small that Eq. (40) is valid, one obtains from Eqs. (40) and (53) that

$$\frac{\Delta^{a}_{D3\pm}}{\omega_3} \ll 1 \quad . \tag{54}$$

Using the resonant property that Eq. (43) acquires for small values of Γ, the integrals occurring in Eqs. (33) and (38) will now be evaluated. The factor of $\exp(i\Omega\Delta\tau_-)$ that occurs in the expression for Π^{a}_{33} [and also in $f^{zp}_{ij}(\hat{x}a,\hat{y}R,\Omega)$, as will be shown shortly] varies very little over the width of the peak of Eq. (43), since from Eqs. (1), (46), (A19), and (A21), $\Delta\tau_- \approx R/c$; hence, from Eqs. (2) and (54),

$$\Delta^{a}_{D3\pm}\Delta\tau_- \approx \left(\frac{\Delta^{a}_{D3\pm}}{\omega_3}\right)\left(\frac{\omega_3 R}{c}\right) \ll 1 \quad . \tag{55}$$

Likewise, by applying Eqs. (54) and (55), one can readily show that all other quantities in Eqs. (33) and (38) vary very little over the width of the resonant peak.

The steps in Sect. IIIC of Ref. 10 will now be followed quite closely. Using the resonant approximation, all quantities in the integrands of Eqs. (38) that do not involve $\Omega-\omega_{3\pm}$ will be replaced by their values at $\omega_{3\pm}$. The lower limit of integration will be replaced by $-\infty$, as the additional contribution to the integrals is negligible compared to the resonant contribution. Under these approximations, Eq. (38) becomes

$$
\begin{aligned}
&e\left\langle \xi_{A3}(\tau_e)\frac{\partial}{\partial\xi_2}E^{DB}_{\tau_e 3}(\vec{\xi},\tau_e)\Big|_{\vec{\xi}=\hat{y}R/2}\right\rangle \\
&= -\frac{e^2}{2m}\Bigg\{-\left(f^{zp}_{33}(\hat{x}a,0,\omega_{3-})-f^{zp}_{33}(\hat{x}a,\hat{y}R,\omega_{3-})\right)\frac{\partial \mathrm{Re}\,\Pi_{33}}{\partial R}\Big|_{\omega_{3-}}\int_{-\infty}^{\infty}\frac{d\Omega}{(\Omega-\omega_{3-})^2(2\omega_{3-})^2+\left(\mathrm{Im}(C^{a}_{3}-\Pi^{a}_{33})\big|_{\omega_{3-}}\right)^2} \\
&\qquad +\left(f^{zp}_{33}(\hat{x}a,0,\omega_{3+})+f^{zp}_{33}(\hat{x}a,\hat{y}R,\omega_{3+})\right)\frac{\partial}{\partial R}\mathrm{Re}\,\Pi_{33}\Big|_{\omega_{3+}}\int_{-\infty}^{\infty}\frac{d\Omega}{(\Omega-\omega_{3+})^2(2\omega_{3+})^2+\left(\mathrm{Im}(C^{a}_{3}+\Pi^{a}_{33})\big|_{\omega_{3+}}\right)^2}\Bigg\}
\end{aligned}
$$

$$\approx -\frac{e^2}{2m}\,\frac{\pi}{2}\left\{ -\left[\frac{f_{33}^{2P}(\hat{x}a,0,\omega_{3-}) - f_{33}^{2P}(\hat{x}a,\hat{y}R,\omega_{3-})}{\omega_{3-}\left|\,\mathrm{Im}(C_3^a - n_{33}^a)\big|_{\omega_{3-}}\right|}\right]\frac{\partial}{\partial R}\,\mathrm{Re}\,n_{33}\Big|_{\omega_{3-}} + \left[\frac{f_{33}^{2P}(\hat{x}a,0,\omega_{3+}) + f_{33}^{2P}(\hat{x}a,\hat{y}R,\omega_{3+})}{\omega_{3+}\left|\,\mathrm{Im}(C_3^a + n_{33}^a)\right|_{\omega_{3+}}}\right]\frac{\partial}{\partial R}\,\mathrm{Re}\,n_{33}\Big|_{\omega_{3+}} \right\} \quad , \quad (56)$$

where $\mathrm{Re}\,n_{33}^a$ was replaced by its approximate value given by Eq. (50). The integral of

$$\int_{-\infty}^{\infty} \frac{dx}{4A^2x^2+B^2} = \frac{\pi}{2|A\cdot B|} \qquad (57)$$

was used in obtaining Eq. (56).

If a similar approximation procedure is followed for Eq. (33) by replacing all quantities in the integrands that do not involve $\Omega-\omega_{3\pm}$ by their values at $\omega_{3\pm}$, then one finds that the numerators of the integrands contain quantities $(-\Omega+\omega_{3\pm})$ due to the terms $\mathrm{Re}(C_3^a \pm n_{33}^a)$. The presence of these terms in the integrand essentially negate the resonant effect of the denominator, thereby resulting in integrals that are negligible in magnitude when compared with the magnitude of the integrals in Eq. (38). Hence, the approximation will be made that the first term of Eq. (28) contributes negligibly to the expectation value of the force; therefore, the latter quantity is approximately equal to the second term in Eq. (28).

As shown in Eqs. (C3) and (C5), the quantities $f_{ij}^{2P}(\hat{x}a,\hat{y}R,\Omega)$, which arise in the cosine expansion of the correlation functions of Eq. (C1), obey a very interesting relationship to the terms $n_{ij}(\hat{x}a,\hat{y}R,\Omega)$ that appear in the dipole fields of Eq. (11). With the use of Eqs. (C3) and

(C5), along with Eq. (48), one can show that

$$\left[\frac{f_{33}^{ZP}(\hat{x}a,0,\Omega) \pm f_{33}^{ZP}(\hat{x}a,\hat{y}R,\Omega)}{\left|Im(C_3^a \pm n_{33}^a)\right|}\right] = + \left.\frac{\hbar\coth\left(\frac{\hbar\Omega}{2kT}\right)}{\frac{e^2}{m}\pi}\right|_{T=\frac{\hbar a}{2\pi ck}} . \tag{58}$$

Neglecting the first term of Eq. (28) and substituting Eqs. (56) and (58) into (28), yields

$$\langle F_{A\tau_e 2}(\tau_e)\rangle = -\frac{\hbar}{4}\left\{-\left[\frac{1}{\Omega}\coth\left(\frac{\hbar\Omega}{2kT}\right)\frac{\partial}{\partial R}Re\,n_{33}\right]\Bigg|_{\omega_{3+}} + \left[\frac{1}{\Omega}\coth\left(\frac{\hbar\Omega}{2kT}\right)\frac{\partial}{\partial R}Re\,n_{33}\right]\Bigg|_{\omega_{3-}}\right\}\Bigg|_{T=\frac{\hbar a}{2\pi ck}} . \tag{59}$$

Each of the two terms in Eq. (59) may be expanded in $(\omega_{3\pm}-\omega_3)$ about ω_3 by using Eq. (51). Assuming that the radiation damping constant Γ is sufficiently small to satisfy

$$\Gamma \ll \left(\frac{\omega_3 R}{c}\right)^2 \frac{R}{c} , \tag{60}$$

then $\omega_{3\pm}$ may be approximated by

$$\omega_{3\pm} \approx \omega_3 \pm \frac{1}{2}\left(\frac{3}{2}\frac{\Gamma}{(R/c)^3}\right)\frac{1}{\omega_3} - \frac{1}{8}\left(\frac{3}{2}\frac{\Gamma}{(R/c)^3}\right)^2\frac{1}{(\omega_3)^3} . \tag{61}$$

Hence, from Eqs. (59) and (61), one can then show that

$$\langle F_{A\tau_e 2}(\tau_e)\rangle \approx -\frac{\partial}{\partial R}\left\{\frac{\hbar}{8\omega_3}\left[Re\,n_{33}\right]^2\frac{\partial}{\partial\Omega}\left[\frac{1}{\Omega}\coth\left(\frac{\hbar\Omega}{2kT}\right)\Bigg|_{T=\frac{\hbar a}{2\pi ck}}\right]\Bigg|_{\Omega=\omega_3}\right\} , \tag{62}$$

Equation (62) agrees exactly with the force expression of Eq. (38) in Ref. 10, where the force was calculated along the direction of separation between two classical charged oscillators that were bathed in classical electromagnetic Planckian plus zero-point radiation. Hence, in regard to the expectation value of the force between two charged oscillators, an equivalence has been demonstrated for the special uniformly accelerated dipole oscillator system discussed in this section and a similar unaccelerated oscillator system held fixed in an inertial frame, but bathed with zero-point plus thermal electromagnetic radiation characterized by the temperature of $T = \hbar a / 2\pi c k$.

C. Expectation Value of Force: General Expression

In the previous section, a resonant approximation was used to obtain an expression for the expectation value of the Lorentz force acting on one of the accelerating oscillators, as expressed in the instantaneous rest frame of the equilibrium point of the oscillator. The question naturally arises as to what the expression would be for this force if the resonant approximation was not used. Such a calculation has already been carried out for the case of two unaccelerated classical charged oscillators that are bathed in classical electromagnetic thermal plus zero-point radiation [see Ref. 10, Eqs. (8) and (9)]. This calculation applies for the case of nonrelativistic classical charged oscillators taken in the electric dipole limit; the result is valid for all values of the separation distance R and for all orders in the fine structure constant. A corresponding calculation has been carried out within the context of quantum electrodynamics when the temperature T equals zero (*13); the results for these two sets of calculations have been shown to agree exactly (see Ref. 9).

In this section, a result analogous to the above mentioned exact expression for the van der Waals force of the unaccelerated-thermal oscillator case will be obtained for the accelerated oscillator case within the context of stochastic electrodynamics. This force expression will be derived by using the small oscillator approximation in solving the equations of motion and by then taking the electric dipole limit when calculating the force on an oscillator. As yet, similar calculations involving the interaction energy between two dipole oscillators uniformly accelerating through the quantum electrodynamical vacuum have not been carried out by researchers in quantum electrodynamics. A connection, described in Ref. 7, between the calculations performed in stochastic electrodynamics and in quantum electrodynamics, may prove helpful to researchers in carrying these calculations over to quantum electrodynamics.

By obtaining the above mentioned exact force expression for the accelerated oscillator system, a comparison can then be made to the analogous van der Waals force expression in the unaccelerated-thermal oscillator system. As will be shown, these two expressions differ from each other. By employing the approximations of the previous section, the two expressions are then brought into agreement. The calculations of this section illustrate how this agreement is obtained.

From Eqs. (28), (33), (38), (48), and (C3), and noting that the third term of Eq. (28) is exactly equal to zero, one obtains

$$
\begin{aligned}
&\langle F_{A\tau_e 2}(\tau_e)\rangle \\
&= -\pi\int_0^\infty \frac{d\Omega}{\Omega}\, h_T^2(\Omega)\Big|_{T=\frac{\hbar a}{2\pi c k}} \left\{ \frac{\mathrm{Re}(C_3^a+n_{33}^a)\frac{\partial}{\partial R}\mathrm{Im}(C_3^a+n_{33}^a)-\mathrm{Im}(C_3^a+n_{33}^a)\frac{\partial}{\partial R}\mathrm{Re}(C_3^a+n_{33}^a)}{|C_3^a+n_{33}^a|^2} \right. \\
&\qquad\qquad \left. +\frac{\mathrm{Re}(C_3^a-n_{33}^a)\frac{\partial}{\partial R}\mathrm{Im}(C_3^a-n_{33}^a)-\mathrm{Im}(C_3^a-n_{33}^a)\frac{\partial}{\partial R}(C_3^a-n_{33}^a)}{|C_3^a-n_{33}^a|^2} \right\} . \qquad (63)
\end{aligned}
$$

Using the algebraic relationship of

$$\frac{\mathrm{Re}(C_3^a \pm n_{33}^a)\frac{\partial}{\partial R}\mathrm{Im}(C_3^a \pm n_{33}^a) - \mathrm{Im}(C_3^a \pm n_{33}^a)\frac{\partial}{\partial R}\mathrm{Re}(C_3^a \pm n_{33}^a)}{|C_3^a \pm n_{33}^a|^2}$$

$$= \mathrm{Im}\left(\frac{1}{(C_3^a \pm n_{33}^a)}\frac{\partial}{\partial R}(C_3^a \pm n_{33}^a)\right) = \frac{\partial}{\partial R}\mathrm{Im}\ln\left(1 \pm \frac{n_{33}^a}{C_3^a}\right) , \quad (64)$$

the following result can be obtained from Eqs. (63), (64), and (C5):

$$\langle F_{A\tau_e 2}(\tau_e)\rangle = -\frac{\partial}{\partial R}U(\hat{x}a, \hat{y}R) \quad , \qquad (65)$$

where $$U(\hat{x}a,\hat{y}R) = \frac{\hbar}{2\pi}\int_0^\infty d\Omega \coth\left(\frac{\hbar\Omega}{2kT}\right)\Bigg|_{T=\frac{\hbar a}{2\pi c k}} \mathrm{Im}\ln\left(1 - \left(\frac{n_{33}^a}{C_3^a}\right)^2\right) \quad . \qquad (66)$$

In the unaccelerated case, the function analogous to $U(\hat{x}a,\hat{y}R)$ (see Refs. 9 and 10) is the quantity found within quantum electrodynamics for the energy of interaction between the dipole oscillators.

Equations (8) and (9) of Ref. 10 constitute the analogous expressions to Eqs. (65) and (66) for the expectation value of the force between two unaccelerated oscillators bathed in thermal plus zero-point electromagnetic radiation. [In order to apply Eqs. (9) and (10) of Ref. 10 to the special case considered here, where oscillations are restricted to the $\hat{z}$ direction, the factor $(1 - n_x^2/c^2)(1 - n_y^2/c^2)$ that appears in the natural logarithm function should be replaced by a factor of unity.] As one will note, although the two force expressions for the accelerated and unaccelerated-thermal oscillator situations differ from each other, they are in direct correspondence with each other; full agreement between the two expressions may be obtained by replacing the quantities of η^a_{33} and C^a_3 in Eq. (66) by their a->0 counterparts of Eq. (49) and [see Eq. (16)]

$$C_3(a,\Omega)\Big|_{a=0} = -\Omega^2 + (\omega_3)^2 - i\Gamma\Omega^3 \quad . \qquad (67)$$

Thus, there exists a direct correspondence, but distinct difference, between the force expressions of the accelerated and the unaccelerated-thermal situations. The main reason for the existance of the direct relationship between the two expressions is due to the identities of Eqs. (C3) and (C5), which relates the η^a_{ij} function that appears in the expression for the electric dipole fields to the cosine transform of the correlation function of the zero-point electric fields. From these two equations comes the $\coth(\pi c/a)$ factor, which is the distinguishing factor that relates the accelerating situation to the unaccelerated-thermal case, where the temperature is related to the acceleration by $T=\hbar a/2\pi c k$. Section IIID shows that Eq. (C3) is the main property that enables other direct relationships to be established other than simply the expectation value of the force.

In regard to the difference between the two force expressions, which arises from the difference in the expressions for C^a_3 and η^a_{33} when $a=2\pi c k T/\hbar$ versus $a=0$, one might wonder what were the key steps of Sect. IIIB that enabled the force expressions for the two situations to be not only directly related, but also equivalent in the case of the unretarded van der Waals condition of Eq. (2). The key steps turn out to be the small laboratory condition of Eq. (1), the near field condition of Eq. (2), and the small damping constant condition given by Eq. (40).

The form of Eq. (63) for the force expressions is particularly useful for illustrating the means by which the two force expressions become approximately equal when these three conditions are applied (*14). Under the latter conditions, the two terms in the integrand of Eq. (63) that contain the quantities

$$\frac{\mathrm{Re}(C_3^a \pm \Omega_{33}^a)\frac{\partial}{\partial R}\mathrm{Im}(C_3^a \pm \Omega_{33}^a)}{|C_3^a \pm \Omega_{33}^a|^2}$$

are of negligible contribution to the integral, because $\mathrm{Re}(C_3^a \pm \Omega_{33}^a) = 0$ when $\Omega = \omega_{3\pm}$. Since $\frac{\partial}{\partial R}\mathrm{Re}\,C_3^a \simeq 0$, the remaining two terms in Eq. (63) consist of integrals that are evaluated in the following way when the resonant approximation is used:

$$\pi\int_0^\infty \frac{d\Omega}{\Omega}\, h_T^2(\Omega)\Big|_{T=\frac{\hbar a}{2\pi c k}} \frac{\mathrm{Im}(C_3^a \pm \Omega_{33}^a)\frac{\partial}{\partial R}\mathrm{Re}\,\Omega_{33}^a}{|C_3^a \pm \Omega_{33}^a|^2}$$

$$\approx \pi\left[\frac{h_T^2(\Omega)}{\Omega}\Big|_{T=\frac{\hbar a}{2\pi c k}} \mathrm{Im}(C_3^a \pm \Omega_{33}^a)\frac{\partial}{\partial R}\mathrm{Re}\,\Omega_{33}^a\right]_{\Omega=\omega_{3\pm}} \int_{-\infty}^{\infty} \frac{d\Omega}{(\Omega-\omega_{3\pm})^2(2\omega_{3\pm})^2 + \left(\mathrm{Im}(C_3^a \pm \Omega_{33}^a)\big|_{\Omega=\omega_{3\pm}}\right)^2}$$

$$= \pi\left[\frac{h_T^2(\Omega)}{\Omega}\Big|_{T=\frac{\hbar a}{2\pi c k}} \mathrm{Im}(C_3^a \pm \Omega_{33}^a)\frac{\partial}{\partial R}\mathrm{Re}\,\Omega_{33}^a\right]_{\Omega=\omega_{3+}} \frac{\pi}{2\omega_{3\pm}\,|\mathrm{Im}(C_3^a \pm \Omega_{33}^a)|_{\Omega=\omega_{3+}}} \quad . \qquad (68)$$

Remembering that the unaccelerated-thermal case has the exact same integral as Eq. (68), but with Ω_{33}^a and C_3^a replaced by their $a \to 0$ counterparts, then the canceling of $\mathrm{Im}(C_3^a \pm \Omega_{33}^a)$ in the numerator and denominator of Eq. (68)

removes this distinguishing factor from the two force expressions. Moreover, from Eq. (50), $Re\,\Pi_{33}^{a}$ and $Re\,\Pi_{33}$ are both approximately equal to $\frac{3}{2}\Gamma(c/R)^3$ when $\frac{aR}{c^2} \ll 1$ and $\frac{\omega_{3\pm}R}{c} \ll 1$. Hence, under these conditions, the two force expressions become not only directly related, but also equivalent.

Thus, the above analysis traces through the steps by which the use of the conditions of Eqs. (1) and (40) make the two force expressions equivalent in the case examined here of the unretarded van der Waals situation given by Eq. (2). In Sect. IIID, other properties of the two oscillator system will be shown to be equivalent by following similar steps to those outlined above.

D. Other Properties of Special System

In the previous section, the expression for the expectation value of the force between two charged oscillators was obtained, both for the accelerated and unaccelerated-thermal systems. Properties other than the force should be examined in order to determine to what extent the accelerated and unaccelerated-thermal systems possess identical statistical properties, as observed in their respective coordinate systems. Other properties that one might want to examine are the correlation functions of the position and the time derivatives of the position for the two oscillating particles of the two accelerating oscillators.

Up until now, the only property that has been examined for the accelerating oscillator system has been the force between the two oscillators. One reason for having concentrated on this property is that the force between two polarizable particles is, undoubtably, an easier quantity to obtain experimentally than is the correlation function in position between two spatially separated oscillating particles. The second reason is that Ref. 10 already contains the analysis of the van der Waals force between two charged oscillators in a thermal bath. The existance of this work then allowed a direct comparison to be made between the force expressions for the unaccelerated-thermal case treated in Ref. 10 and the accelerated situation discussed in this article.

Before proceeding with an examination of additional statistical properties for a pair of accelerating oscillators, a brief review will be given of the statistical properties of a single accelerating oscillator. This review will serve to show how the relationships derived in Appendix C enable the single accelerating oscillator case to agree in certain statistical properties with the unaccelerated-thermal single oscillator situation. In the process of giving this brief review, a simple generalization will be made to the original work by Boyer in Ref. 1, by examining the correlation function $\langle \xi_i(\tau_{e_0})\xi_j(\tau_{e_0}+\tau_e)\rangle$ versus the less general case of $\tau_e = 0$.

The appropriate equation of motion for the single accelerating oscillator system is given by Eq. (38) of Ref. (3), which is equivalent to Eq. (6a) or (6b) of this article when $\vec{E}^{D}_{\tau_e} = 0$. The solution to this equation is given by

$$\xi_i(\tau_e) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e}{m}\,\frac{\tilde{E}_i^{zp}(0,\Omega)}{C_i(a,\Omega)}\exp(-i\Omega\tau_e)\,d\Omega\,. \qquad (69)$$

The positional correlation function is then given by

$$\langle \xi_i(\tau_{e_0})\xi_j(\tau_{e_0}+\tau_e)\rangle$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\Omega'\exp(-i\Omega'\tau_{e_0})\int_{-\infty}^{\infty} d\Omega''\exp\left(-i\Omega''(\tau_{e_0}+\tau_e)\right)\left(\frac{e}{m}\right)^2 \frac{\langle \tilde{E}_i^{zp}(0,\Omega')\tilde{E}_j^{zp}(0,\Omega'')\rangle}{C_i(a,\Omega')C_j(a,\Omega'')}\,. \qquad (70)$$

Using the inverse of Eq. (14) along with Eqs. (C1), (C3), and (48), yields

$$\langle \tilde{E}_i^{zp}(0,\Omega')\tilde{E}_j^{zp}(0,\Omega'')\rangle =$$

$$= -\delta_{ij}\frac{(2\pi)^2}{2}\frac{m}{e^2}\int_0^\infty \frac{d\Omega}{\Omega}\, h_T^2(\Omega)\Big|_{T=\frac{\hbar a}{2\pi ck}} \mathrm{Im}\, C_i(a,\Omega)\left[\delta(\Omega'-\Omega)\delta(\Omega''+\Omega)+\delta(\Omega'+\Omega)\delta(\Omega''-\Omega)\right] \quad . \tag{71}$$

Substituting Eq. (71) into Eq. (70) gives the result of

$$\langle \xi_i(\tau_{e0})\,\xi_j(\tau_{e0}+\tau_e)\rangle = -\delta_{ij}\frac{2\pi}{m}\int_0^\infty \frac{d\Omega}{\Omega}\, h_T^2(\Omega)\Big|_{T=\frac{\hbar a}{2\pi ck}} \frac{\mathrm{Im}\, C_i(a,\Omega)}{|C_i(a,\Omega)|^2}\cos(\Omega\tau_e) \quad . \tag{72}$$

Similar steps to the above are followed when analyzing the behavior of an unaccelerated oscillator in a thermal radiation bath characterized by $T=\hbar a/2\pi ck$ (#15). The coordinates τ_e and ξ_i are then replaced by the coordinates in the inertial frame at rest with respect to the equilibrium point of the oscillator. The factor of $C_i(a,\Omega)$ in the denominator of Eq. (69) must be replaced by its a->0 counterpart; consequently, this change must be propagated to the denominators in Eqs. (70) and (72). In place of $\tilde{E}_i^{zp}(0,\Omega)$ in Eqs. (69)-(71), one must use the corresponding Fourier transform, at a stationary point in an inertial frame, for the electric field of a zero-point plus thermal radiation spectrum. This electric field is given by the expression of Eq. (19), but with $h^2(\omega)=\frac{\hbar\omega}{2\pi^2}$ replaced by $h_T^2(\omega) = \frac{\hbar\omega}{2\pi^2}\coth(\hbar\omega/2kT)$. In calculating the appropriate correlation function corresponding to Eq. (71), one then finds that the $h_T^2(\Omega)$ factor on the right-hand side of Eq. (71) is correct, but the factor of $\mathrm{Im}\,C_i(a,\Omega)$ must be replaced by $\mathrm{Im}\,C_i(0,\Omega)$. Equation (72) then follows, with $C_i(a,\Omega)$ replaced by $C_i(0,\Omega)$ on the right-hand side.

The resonant factor of $1/|C_i(a,\Omega)|^2$ that occurs in Eq. (72) was discussed in Sect. IIIC. The damping constant Γ will be assumed to be small enough that Eq. (40) is valid. The approximate width $\Delta^a_{\xi i}$ of the peak described by $\frac{1}{|C_i^a|^2}$ is then given by Eq. (42). When Γ is taken to be sufficiently small to satisfy

$$|\Delta^a_{\xi i}\tau_e| = \Gamma(\omega_i)^2\left(1+\left(\frac{a/c}{\omega_i}\right)^2\right)|\tau_e| \ll 1 \qquad , \tag{73}$$

then the factor of $\cos(\Omega\tau_e)$ may be approximated as being constant over the width of the resonant peak. When Eq. (73) is not satisfied, then the variation of the $\cos(\Omega\tau_e)$ factor over the width of the resonant peak will tend to wash out the resonant contribution, thereby invalidating the resonant approximation and decreasing the absolute value of $\langle\xi_i(\tau_{eo})\xi_i(\tau_{eo}+\tau_e)\rangle$. [For atomic processes, with $2\pi\omega_i$ of the order of 10^{-16} sec. and $\Gamma\approx 6\text{x}10^{-24}$ sec. for an electron, then the condition of Eq. (73) is satisfied when τ_e is less than about 10^3 times the period of the oscillator.] Employing the resonant approximation results in a factor of $\frac{1}{|\text{Im}\,C_i(a,\omega_i)|} = \frac{-1}{\text{Im}\,C_i(a,\omega_i)}$, which cancels out the factor in Eq. (72) that distinguishes the accelerated from the unaccelerated-thermal situation. Hence, under the condition of Eq. (73),

$$\langle\xi_i(\tau_{eo})\xi_j(\tau_{eo}+\tau_e)\rangle \approx \frac{\delta_{ij}\hbar}{2m\omega_i}\coth\left(\frac{\hbar\omega_i}{2kT}\right)\Bigg|_{T=\frac{\hbar a}{2\pi ck}}\cos(\omega_i\tau_e) \qquad , \tag{74}$$

which applies for both the accelerated and unaccelerated-thermal situations.

Correlation functions of the time derivatives of ξ_i can be calculated from Eqs. (70) and (71). For example,

$$\left\langle \xi_i(\tau_{e0}) \frac{d}{d\tau_e}\xi_j\Big|_{\tau_{e0}+\tau_e} \right\rangle = +\delta_{ij}\frac{2\pi}{m}\int_0^\infty d\Omega\, h_T^2(\Omega)\Big|_{T=\frac{\hbar a}{2\pi c k}} \frac{\mathrm{Im}\,C_i(a,\Omega)}{|C_i(a,\Omega)|^2}\sin(\Omega\tau_e) \quad , \tag{75}$$

$$\left\langle \frac{d\xi_i}{d\tau_e}\Big|_{\tau_{e0}} \frac{d\xi_j}{d\tau_e}\Big|_{\tau_{e0}+\tau_e} \right\rangle = -\delta_{ij}\frac{2\pi}{m}\int_0^\infty d\Omega\cdot\Omega\, h_T^2(\Omega)\Big|_{T=\frac{\hbar a}{2\pi c k}} \frac{\mathrm{Im}\,C_i(a,\Omega)}{|C_i(a,\Omega)|^2}\cos(\Omega\tau_e) \quad , \tag{76}$$

or, more generally,

$$\left\langle \frac{d^m\xi_i}{d\tau_e^m}\Big|_{\tau_{e0}} \frac{d^n\xi_j}{d\tau_e^n}\Big|_{(\tau_{e0}+\tau_e)} \right\rangle$$

$$= -\delta_{ij}\frac{2\pi}{m}\int_0^\infty d\Omega\cdot\Omega^{(m+n-1)} h_T^2(\Omega)\Big|_{T=\frac{\hbar a}{2\pi c k}} \frac{\mathrm{Im}\,C_i(a,\Omega)}{|C_i(a,\Omega)|^2} \begin{Bmatrix} (-1)^{\left(\frac{n-m+1}{2}\right)}\sin(\Omega\tau_e) \\ (-1)^{\left(\frac{n-m}{2}\right)}\cos(\Omega\tau_e) \end{Bmatrix} \begin{matrix} [(m+n)\ \text{odd}] \\ [(m+n)\ \text{even}] \end{matrix} \ . \tag{77}$$

For large values of Ω, the integrand of Eq. (77) behaves like $\Omega^{(m+n-3)}$ [see Eqs. (16) and (C5)]. When $0 \le (m+n) < 2$, the integral is perfectly well defined. When $2 \le (m+n)$ and $\tau_e \ne 0$, a finite integral may be obtained using the method of Appendix C, by inserting a factor of $\exp(-\epsilon\Omega)$, where $\epsilon > 0$, and then taking the limit of $\epsilon \to 0$ after performing the integration. When $\tau_e = 0$, this method will not work, and an infinite quantity will result. Presumably, this high frequency divergence would not arise if the oscillator's equation of motion was treated in a fully relativistic manner; in order to prove this, however, one would need to go beyond the small oscillator approximation discussed in Refs. 1 and 3 and deal with the full, nonlinear stochastic differential equations of motion.

If the limit of e->0 is taken before the limit of ϵ ->0, then a delta function arises from the resonant denominator, which results in a finite quantity even in the case of $\tau_e = 0$. More specifically (*16),

$$\lim_{\Gamma \to 0} \frac{\Gamma}{|C_i(a,\Omega)|^2} = \frac{\pi\left\{\delta(\Omega-\omega_i)+\delta(\Omega+\omega_i)\right\}}{2(\omega_i)^4\left(1+\left(\frac{a/c}{\omega_i}\right)^2\right)} \quad . \tag{78}$$

Following this limit procedure in evaluating Eq. (77) gives

$$\lim_{e\to 0}\left\langle \left.\frac{d^m\xi_i}{d\tau_e^m}\right|_{\tau_{e0}} \left.\frac{d^n\xi_j}{d\tau_e^n}\right|_{\tau_{e0}+\tau_e}\right\rangle$$

$$= \delta_{ij}\left[\frac{\hbar}{2m\omega_i}\right](\omega_i)^{(m+n)} \left.\coth\left(\frac{\hbar\omega_i}{2kT}\right)\right|_{T=\frac{\hbar a}{2\pi ck}} \left\{\begin{matrix} (-1)^{\left(\frac{n-m+1}{2}\right)}\sin(\omega_i\tau_e) \\ (-1)^{\left(\frac{n-m}{2}\right)}\cos(\omega_i\tau_e) \end{matrix}\right\} \begin{matrix} [(m+n)\ \text{odd}] \\ [(m+n)\ \text{even}]\ , \end{matrix} \tag{79}$$

where the $\epsilon \to 0$ limit is not indicated, but understood to be taken after letting e->0. Taking the limit of e->0 in stochastic electrodynamics has been termed "random mechanics" by Boyer in Refs. 6 and 7, and has been shown to have a close connection with quantum mechanics.

As in the case of Eq. (74), the above result of Eq. (79) holds in the e->0 limit for both the accelerated and the unaccelerated-thermal single oscillator systems. The factor that multiplies the delta functions in Eq. (78) cancels the factor of $\mathrm{Im}\, C_i(a,\Omega)$ that occurs in Eq. (77), thereby removing the distinguishing difference between the expressions for the accelerated and unaccelerated-thermal systems.

Other statistical quantities that should be considered for a single accelerating oscillator involve higher-order momente of ξ_i and its time derivatives; namely,

$$\left\langle \frac{d^{n_1}\xi_{i_1}}{d\tau_e^{n_1}}\bigg|_{\tau_{e1}} \frac{d^{n_2}\xi_{i_2}}{d\tau_e^{n_2}}\bigg|_{\tau_{e2}} \cdots \frac{d^{n_m}\xi_{i_m}}{d\tau_e^{n_m}}\bigg|_{\tau_{em}} \right\rangle \quad .$$

Properly treating the random phases involved in this higher-order moment results in this quantity being zero for an odd-order moment and being equal to a sum of products of the two-point correlation functions in Eq. (79) for an even-order moment (#17). Thus, for example,

$$\langle a_1 a_2 a_3 a_4 \rangle = \langle a_1 a_2\rangle\langle a_3 a_4\rangle + \langle a_1 a_3\rangle\langle a_2 a_4\rangle + \langle a_1 a_4\rangle\langle a_2 a_3\rangle \quad , \qquad (80)$$

where $\quad a_\ell = \left.\dfrac{d^{n_\ell}\xi_{i_\ell}}{d\tau_e^{n_\ell}}\right|_{\tau_{e\ell}} \quad .$

Hence, the connection between these higher-order moments for the accelerated and unaccelerated-thermal systems follows the same rules described above for the second-order moments.

Using the above discussion of a single accelerating oscillator as a guide, the examination of the statistical properties of a spatially separated pair of accelerated oscillators will now be continued. The comparison of this system with its unaccelerated-thermal counterpart is simplified by restricting attention to the special oscillator model considered here in Sect. III, where oscillations are confined to the $\hat{z}$ direction. For this special system, a direct correspondence exists between the expressions for various physical properties of the accelerated and unaccelerated-thermal situations. This was first demonstrated in Sect. IIIC for the case of the force expression of Eq. (65);

simply by replacing the functions of η^{a}_{33} and C^{a}_{3} by their a->0 limits resulted in the appropriate force expression for the unaccelerated-thermal oscillator system. A similar correspondence exists for other properties of the two oscillator system, as will be shown shortly. Again, as occurred for the force expression, when the conditions of Eqs. (1) and (40) are imposed upon the unretarded accelerating oscillator system, then the expectation value of these other properties become equivalent for the accelerated and unaccelerated-thermal situations.

When the more general oscillator model is considered in Sect. IV, where oscillations will no longer be confined to the $\hat{z}$ direction, the direct correspondence between expressions for the accelerated and unaccelerated-thermal situations will no longer apply. Instead, additional complicated terms of order $O\left(\frac{aR}{c^2}\right)$ will arise in the expression for the properties of the accelerated oscillator system, thereby complicating the analysis of the system. Of course, when the condition of Eq. (1) is imposed, then the direct correspondence again applies between expressions for the accelerated and unaccelerated-thermal systems. Likewise, when the damping constant is made sufficiently small, where the specific condition will be given shortly, then in the case of the unretarded van der Waals situation, the expectation values of these expressions again become equivalent for the two systems.

Using Eqs. (17a), (17b), (C1), (C3), (C56), and (48), the following results can be obtained:

$$\langle \tilde{\xi}_{A3}(\Omega')\tilde{\xi}_{A3}(\Omega'')\rangle = \langle \tilde{\xi}_{B3}(\Omega')\tilde{\xi}_{B3}(\Omega'')\rangle$$
$$= -\frac{\pi^2}{m}\int_0^\infty \frac{d\Omega}{\Omega} h_T^2(\Omega)\Big|_{T=\frac{\hbar a}{2\pi c k}} \left\{ \delta(\Omega'-\Omega)\delta(\Omega''+\Omega) + \delta(\Omega'+\Omega)\delta(\Omega''-\Omega) \right\} \times \tag{81}$$
$$\times \left\{ \frac{\mathrm{Im}(C_3^a - \eta_{33}^a)}{|C_3^a - \eta_{33}^a|^2} + \frac{\mathrm{Im}(C_3^a + \eta_{33}^a)}{|C_3^a + \eta_{33}^a|^2} \right\} ,$$

$$\langle \tilde{\xi}_{A3}(\Omega')\tilde{\xi}_{B3}(\Omega'')\rangle = -\frac{\pi^2}{m}\int_0^\infty \frac{d\Omega}{\Omega} h_T^2(\Omega)\Big|_{T=\frac{\hbar a}{2\pi c k}} \left\{ \delta(\Omega'-\Omega)\delta(\Omega''+\Omega) + \delta(\Omega'+\Omega)\delta(\Omega''-\Omega) \right\} \times$$
$$\times \left\{ \frac{-\mathrm{Im}(C_3^a - \eta_{33}^a)}{|C_3^a - \eta_{33}^a|^2} + \frac{\mathrm{Im}(C_3^a + \eta_{33}^a)}{|C_3^a + \eta_{33}^a|^2} \right\} . \tag{82}$$

From these two expressions, the following moments can be obtained:

$$\left\langle \frac{d^m\xi_{A3}}{d\tau_e^m}\bigg|_{\tau_{eo}} \frac{d^n\xi_{A3}}{d\tau_e^n}\bigg|_{(\tau_{eo}+\tau_e)} \right\rangle = \left\langle \frac{d^m\xi_{B3}}{d\tau_e^m}\bigg|_{\tau_{eo}} \frac{d^n\xi_{B3}}{d\tau_e^n}\bigg|_{(\tau_{eo}+\tau_e)} \right\rangle$$
$$= -\frac{\pi}{m}\int_0^\infty d\Omega\cdot\Omega^{(m+n-1)} h_T^2(\Omega)\Big|_{T=\frac{\hbar a}{2\pi c k}} \left\{ \frac{\mathrm{Im}(C_3^a - \eta_{33}^a)}{|C_3^a - \eta_{33}^a|^2} + \frac{\mathrm{Im}(C_3^a + \eta_{33}^a)}{|C_3^a + \eta_{33}^a|^2} \right\} \times \tag{83}$$
$$\times \begin{Bmatrix} (-1)^{\left(\frac{n-m+1}{2}\right)} \sin(\Omega\tau_e) \\ (-1)^{\left(\frac{n-m}{2}\right)} \cos(\Omega\tau_e) \end{Bmatrix} \quad \begin{matrix} [(m+n)\ \text{odd}] \\ [(m+n)\ \text{even}] \end{matrix} ,$$

$$\left\langle \frac{d^m\xi_{A3}}{d\tau_e^m}\bigg|_{\tau_{eo}} \frac{d^n\xi_{B3}}{d\tau_e^n}\bigg|_{\tau_{eo}+\tau_e} \right\rangle = -\frac{\pi}{m}\int_0^\infty d\Omega\cdot\Omega^{(m+n-1)} h_T^2(\Omega)\Big|_{T=\frac{\hbar a}{2\pi c k}} \left\{ \frac{-\mathrm{Im}(C_3^a - \eta_{33}^a)}{|C_3^a - \eta_{33}^a|^2} + \frac{\mathrm{Im}(C_3^a + \eta_{33}^a)}{|C_3^a + \eta_{33}^a|^2} \right\} \times \tag{84}$$
$$\times \begin{Bmatrix} (-1)^{\left(\frac{n-m+1}{2}\right)} \sin(\Omega\tau_e) \\ (-1)^{\left(\frac{n-m}{2}\right)} \cos(\Omega\tau_e) \end{Bmatrix} \quad \begin{matrix} [(m+n)\ \text{odd}] \\ [(m+n)\ \text{even}] \end{matrix} .$$

Equations (83) and (84) can be shown to apply in the unaccelerated-thermal situation when C_3^a and η_{33}^a are replaced by their a->0 limits and when, of course, the ξ^μ coordinates are replaced by the coordinates of the inertial system at rest with respect to the equilibrium point of the oscillators. This statement may be proven easily by direct calculation. In the process of verifying this statement, the following points may be helpful to note. First, Eqs. (83) and (84) were obtained by using Eqs. (17a) and

(17b) and by applying the electric field correlation function formulae of Appendix C. Second, Eqs. (17a) and (17b) can readily be shown to be the appropriate oscillator coordinate expressions in the unaccelerated-thermal situation when C_3^a and n_{33}^a are replaced by their a->0 limits and when the quantities of $\tilde{E}_3^{zp}(\pm \S R, \Omega)$ are replaced by the corresponding Fourier transforms for the electric field of a zero-point plus thermal radiation spectrum. Third, the correlation functions of the thermal plus zero-point electromagnetic fields in an inertial frame possess the same distinguishing factor of $\coth(\hbar\omega_3/2kT)$ as do the correlation functions of zero-point electromagnetic fields in a nonrotating, uniformly accelerating coordinate system, when $T = \hbar a/2\pi c k$. If the factor of $\coth\left(\frac{\hbar\omega_3}{2kT}\right)\Big|_{T=\frac{\hbar a}{2\pi c k}}$ is retained in the correlation functions for the accelerated system, but the quantity of n_{ij}^a is replaced by its a->0 limit [see Eqs. (C1) and (C3)], then one can show that the corresponding unaccelerated-thermal field correlation functions are obtained. The net result of these points is that Eqs. (83) and (84) apply for the unaccelerated-thermal situation, when C_3^a and n_{33}^a are replaced by their a->0 limits.

Paralleling the single oscillator case, then when the conditions hold of Eq. (40), $0 \leq (m+n) < 2$, and

$$|\Delta^a_{D3\pm}\tau_e| \lesssim 2\Gamma(\omega_3)^2\left(1 + \left(\frac{a/c}{\omega_3}\right)^2\right)|\tau_e| \ll 1 \quad , \qquad (85)$$

the resonant approximation used in the force calculations for the unretarded van der Waals situation may again be employed here. Assuming that Eq. (60) is valid, so that Eq. (61) applies, then for $0 \leq (m+n) < 2$,

$$
\begin{aligned}
&\left\langle \left.\frac{d^m \xi_{A3}}{d\tau_e^m}\right|_{\tau_{e0}} \left.\frac{d^n \xi_{A3}}{d\tau_e^n}\right|_{\tau_{e0}+\tau_e} \right\rangle = \left\langle \left.\frac{d^m \xi_{B3}}{d\tau_e^m}\right|_{\tau_{e0}} \left.\frac{d^n \xi_{B3}}{d\tau_e^n}\right|_{\tau_{e0}+\tau_e} \right\rangle \\
&\quad \approx \frac{\hbar}{2m}\begin{pmatrix} (-1)^{\left(\frac{n-m+1}{2}\right)} \\ (-1)^{\left(\frac{n-m}{2}\right)} \end{pmatrix} \Bigg\{ \omega_3^{(m+n-1)} \coth\left(\frac{\hbar\omega_3}{2kT}\right)\Bigg|_{T=\frac{\hbar a}{2\pi c k}} \begin{pmatrix} \sin(\omega_3 \tau_e) \\ \cos(\omega_3 \tau_e) \end{pmatrix} \\
&\qquad + \frac{1}{8}\left(\frac{3\Gamma}{2\left(\frac{R}{c}\right)^3 \omega_3}\right)^2 \left(\left[-\frac{1}{\omega_3}\frac{\partial}{\partial\Omega} + \frac{\partial^2}{\partial\Omega^2} \right] \left[\Omega^{(m+n-1)} \coth\left(\frac{\hbar\Omega}{2kT}\right)\Bigg|_{T=\frac{\hbar a}{2\pi c k}} \begin{Bmatrix} \sin(\Omega\tau_e) \\ \cos(\Omega\tau_e) \end{Bmatrix} \right] \right)\Bigg|_{\Omega=\omega_3} \Bigg\} \begin{matrix} [(m+n) \text{ odd}] \\ [(m+n) \text{ even}] \end{matrix} ,
\end{aligned} \tag{86}
$$

$$
\begin{aligned}
&\left\langle \left.\frac{d^m \xi_{A3}}{d\tau_e^m}\right|_{\tau_{e0}} \left.\frac{d^n \xi_{B3}}{d\tau_e^n}\right|_{\tau_{e0}+\tau_e} \right\rangle \\
&\quad \approx \frac{\hbar}{2m}\begin{pmatrix} (-1)^{\left(\frac{n-m+1}{2}\right)} \\ (-1)^{\left(\frac{n-m}{2}\right)} \end{pmatrix} \left(\frac{3}{2}\frac{\Gamma}{\left(\frac{R}{c}\right)^3 \omega_3}\right) \frac{\partial}{\partial\Omega}\left[\Omega^{(m+n-1)} \coth\left(\frac{\hbar\Omega}{2kT}\right)\Bigg|_{T=\frac{\hbar a}{2\pi c k}} \begin{Bmatrix} \sin(\Omega\tau_e) \\ \cos(\Omega\tau_e) \end{Bmatrix} \right]\Bigg|_{\Omega=\omega_3} \begin{matrix} [(m+n) \text{ odd}] \\ [(m+n) \text{ even}] \end{matrix} ,
\end{aligned} \tag{87}
$$

where the above results are given to lowest order in Γ . The second term in Eq. (86) gives the first order correction term to the corresponding single oscillator expression of Eq. (79).

When the limit of e->0 is taken, as was done for the single accelerating oscillator, then the restriction of (m+n)<2 is removed. Equation (86) reduces to the single accelerating oscillator case of Eq. (79), and a value of zero is obtained for Eq. (87). The latter result corresponds to the quantum mechanical case of two uncharged oscillators separated by a distance R that is large compared to the approximate size of each oscillator, so that their wave functions do not overlap.

Equations (86) and (87) apply for both the accelerated and unaccelerated-thermal systems. In Eqs. (83) and (84), the distinguishing factors in the expressions for the two situations were the factors of $\mathrm{Im}(C_3^a \pm \eta_{33}^a)/|C_3^a \pm \eta_{33}^a|^2$ and their a->0 counterparts. When the resonant approximation is made in the case of the unretarded van der Waals situation, as was done in arriving at Eqs. (86) and (87), then this distinction between the expressions for the two situations becomes negligible due to a cancellation of the distinguishing factors. The same sort of cancellation occurred in the single oscillator situation when proceeding from Eq. (77) to Eq. (79). Moreover, since higher-order moments of the time derivatives of ξ_{A3} and ξ_{B3} can be expressed in terms of the two-point correlation functions of Eqs. (86) and (87), then these expressions for the higher-order moments also agree between the two situations. Hence, these results extend the domain of equivalence between the accelerated and unaccelerated-thermal situations, which was found to occur for the case of a single oscillator, to the case of a spatially extended system consisting of two oscillators.

IV. GENERAL CASE OF TWO ACCELERATING DIPOLE OSCILLATORS

A. Expectation Value of Force

The situation will now be investigated where all restrictions are removed as to the direction in which oscillations are allowed to occur. As noted in Sec. IIIA, the i=3 equation of motion is uncoupled from the i=1,2 equations of motion. Hence, the i=3 solution to the equation of motion for the special oscillator model considered in Sec. III also applies to the more general oscillator model considered in this section; this solution is given by Eqs. (17a) and (17b).

Turning now to the i=1,2 equations of motion given by Eqs. (15a) and (15b), a convenient symmetric matrix form may be readily obtained by examining the symmetry properties of $\eta_{ij}(\hat{x}a,\hat{y}R,\Omega)$. Using Eqs. (A39)–(A43), the i=1,2 equations of motion may be written as

$$\begin{bmatrix} C_1^a & 0 & \eta_{11}^a & \eta_{12}^a \\ 0 & C_2^a & -\eta_{12}^a & \eta_{22}^a \\ \eta_{11}^a & -\eta_{12}^a & C_1^a & 0 \\ \eta_{12}^a & \eta_{22}^a & 0 & C_2^a \end{bmatrix} \begin{bmatrix} \tilde{\xi}_{A1}(\Omega) \\ \tilde{\xi}_{A2}(\Omega) \\ \tilde{\xi}_{B1}(\Omega) \\ \tilde{\xi}_{B2}(\Omega) \end{bmatrix} = \frac{e}{m} \begin{bmatrix} \tilde{E}_1^{zp}(+\hat{y}\frac{R}{2},\Omega) \\ \tilde{E}_2^{zp}(+\hat{y}\frac{R}{2},\Omega) \\ \tilde{E}_1^{zp}(-\hat{y}\frac{R}{2},\Omega) \\ \tilde{E}_2^{zp}(-\hat{y}\frac{R}{2},\Omega) \end{bmatrix} \quad . \qquad (88)$$

In conformity with the notation used previously for

$\eta_{33}(\hat{x}a,+\hat{y}R,\Omega)$, $\eta_{ij}(\hat{x}a,+\hat{y}R,\Omega)$ has been abbreviated in Eq. (88) by η^a_{ij}.

The above set of equations may be readily solved simply by obtaining the inverse of the matrix on the left. The Lorentz force acting between the two oscillators may then be calculated in the same manner as in Sec. III. Similar results to those of Sec. III are found, except that now there exist additional terms of order $O\left(\frac{aR}{c^2}\right)$. The source of these additional terms arise from the components of the matrix in Eq. (88) that couple the i=1 and i=2 set of equations. The matrix component that serves to couple these equations consists of the quantity $\eta_{12}(a\hat{x},+\hat{y}R,\Omega)$. As can be seen from Eqs. (A39)-(A42), η^a_{12} is of order $\left(\frac{aR}{c^2}\right)$ times the magnitude of η^a_{11}, η^a_{22}, and η^a_{33}.

Assuming the condition of Eq. (1) is satisfied, then the η^a_{12} terms in the matrix of Eq. (88) may be ignored. This effectively decouples the i=1,2 set of equations and allows the form of the solution for i=3 to be applied here also, where the error incurred in making this approximation consists of terms of order $O(aR/c^2)$. Hence, for i=1,2,3,

$$\tilde{\xi}_{Ai}(\Omega)=\frac{e}{2m}\left\{1+(1-\delta_{i3})O\left(\frac{aR}{c^2}\right)\right\}\times$$
$$\times\left\{\tilde{E}^{zp}_i\left(\hat{y}\frac{R}{2},\Omega\right)\left[\frac{1}{C^a_i-\eta^a_{ii}}+\frac{1}{C^a_i+\eta^a_{ii}}\right]+\tilde{E}^{zp}_i\left(-\hat{y}\frac{R}{2},\Omega\right)\left[\frac{-1}{C^a_i-\eta^a_{ii}}+\frac{1}{C^a_i+\eta^a_{ii}}\right]\right\},\quad (89a)$$

$$\tilde{\xi}_{Bi}(\Omega)=\frac{e}{2m}\left\{1+(1-\delta_{i3})O\left(\frac{aR}{c^2}\right)\right\}\times$$
$$\times\left\{\tilde{E}^{zp}_i\left(\hat{y}\frac{R}{2},\Omega\right)\left[\frac{-1}{C^a_i-\eta^a_{ii}}+\frac{1}{C^a_i+\eta^a_{ii}}\right]+\tilde{E}^{zp}_i\left(-\hat{y}\frac{R}{2},\Omega\right)\left[\frac{+1}{C^a_i-\eta^a_{ii}}+\frac{1}{C^a_i+\eta^a_{ii}}\right]\right\}.\quad (89b)$$

Physically, this decoupling of the i=1,2 set of equations occurs when the distance $\alpha R_- = (aR/2c^2)R_-$ that an oscillator accelerates in time R_-/c (see Appendix A) is small compared to R_-; this condition is equivalent to Eq. (1). As mentioned in the Introduction, when this condition applies, then the angle is very small that a light ray would make to the y axis when propagating from one oscillator to the other. For small angles, the following quantity then becomes negligible: namely, the electromagnetic force acting on one of the oscillating particles in the $\hat{x}$ ($\hat{y}$) direction due to the oscillations in the $\hat{y}$ ($\hat{x}$) direction of the other oscillator. These two sets of oscillations then become independent, which results in the anticipated effective decoupling of the i=1,2 set of equations.

Using Eqs. (18), (89a), and (89b), the expectation value of the i=2 component of the Lorentz force acting on oscillator A can now be calculated. The first term of Eq. (18) can be easily expressed in terms of the ξ^μ coordinates in the same way that the first two terms of Eq. (28) were obtained; the only difference is that a sum from j=1 to j=3 must now be included. Hence,

$$\langle F_{A\tau_e 2}(\tau_e)\rangle = e\sum_{j=1}^{3}\left\langle \xi_{Aj}(\tau_e)\frac{\partial}{\partial\xi_2}E^{ZP}_{\tau_e j}(\vec{\xi},\tau_e)\Big|_{\vec{\xi}=\hat{y}\frac{R}{2}}\right\rangle$$
$$+\sum_{j=1}^{3}\left\langle \xi_{Aj}(\tau_e)\frac{\partial}{\partial\xi_2}E^{DB}_{\tau_e j}(\vec{\xi},\tau_e)\Big|_{\vec{\xi}=\hat{y}\frac{R}{2}}\right\rangle + \langle A_2\rangle \quad , \qquad (90)$$

where A_2 is equal to the i=2 component of the second term of Eq. (18), evaluated for oscillator A.

The first two terms of Eq. (90) can be evaluated in the same way that their corresponding terms in Eq. (28) were evaluated. The third term of Eq. (28) requires a fair amount of additional work, however. The end result of this calculation is relatively uninteresting, however, as $\langle A_2 \rangle$ is at most of order $(aR/c^2)^2$ times $\langle F_{A\tau_e 2}(\tau_e)\rangle$, when the resonant approximation is used in the case of the unretarded van der Waals condition. Consequently, in order not to sidetrack the discussion of the main contribution to $\langle F_{A\tau_e 2}(\tau_e)\rangle$, the calculation of $\langle A_2 \rangle$ is presented in Appendix D.

If the $O(aR/c^2)$ terms in Eqs. (89a) and (89b) are ignored, then the first term of Eq. (90) is of the same form as Eq. (33), but now there exists a subscript j that must be summed from j=1 to j=3. In order to obtain the second term of Eq. (90), Eqs. (9) and (11) must be used. Since the $i \neq j$ terms of Ω^a_{ij} are approximately (aR/c^2) times the size of Ω^a_{ii}, these off diagonal terms will be ignored here also. Again dropping the $O\left(\frac{aR}{c^2}\right)$ terms in Eqs. (89a) and (89b), then the second term in Eq. (90) is of the same form as Eq. (38); again, a subscript j must be included that is summed from j=1 to j=3.

Following the same steps as led from Eq. (63) to Eqs. (65) and (66), then yields

$$\langle F_{A\tau_e 2}(\tau_e)\rangle \approx -\frac{\partial}{\partial R} U(\hat{x}a, \hat{y}R) \quad , \qquad (91)$$

$$U(\hat{x}a,\hat{y}R) = \frac{\hbar}{2\pi}\int_0^\infty d\Omega \coth\left(\frac{\hbar\Omega}{2kT}\right)\Bigg|_{T=\frac{\hbar a}{2\pi c k}} \operatorname{Im} \ln\left[\left(1-\left(\frac{\Omega^a_{11}}{C^a_1}\right)^2\right)\left(1-\left(\frac{\Omega^a_{22}}{C^a_2}\right)^2\right)\left(1-\left(\frac{\Omega^a_{33}}{C^a_3}\right)^2\right)\right] \quad . \qquad (92)$$

Besides the additional terms of order $\mathcal{O}\left(\frac{aR}{c^2}\right)$ that have been dropped, Eqs. (91) and (92) do not agree with their unaccelerated-thermal counterparts due to the n_{ii}^a and C_i^a terms [see Eqs. (8) and (9) of Ref. 10].

Agreement is again obtained when the resonant approximation is made in the case of the unretarded van der Waals situation, where the condition for this situation is now given by

$$\frac{\omega_i R}{c} \ll 1 \quad , \quad \text{for} \quad i = 1,2,3 \quad . \qquad (93)$$

Use of the resonant approximation enables Eqs. (91) and (92) to be written in a form similar to that of Eq. (62), but, of course, with a sum included over the required j index. When $aR/c^2 \ll 1$ and $\Omega R/c \ll 1$, then from Eqs. (A39)-(A40), (44), and (45),

$$n_{11}^a = n_{33}^a = \left[+\tfrac{3}{2}\Gamma\left(\tfrac{c}{R}\right)^3 - i\Gamma\Omega^3\left(1+\left(\tfrac{a/c}{\Omega}\right)^2\right)\right]\times\left[1+\mathcal{O}\left(\tfrac{aR}{c^2}\right)+\mathcal{O}\left(\tfrac{\Omega R}{c^2}\right)\right] \quad , \qquad (94)$$

$$n_{22}^a = \left[-3\Gamma\left(\tfrac{c}{R}\right)^3 - i\Gamma\Omega^3\left(1+\left(\tfrac{a/c}{\Omega}\right)^2\right)\right]\times\left[1+\mathcal{O}\left(\tfrac{aR}{c^2}\right)+\mathcal{O}\left(\tfrac{\Omega R}{c^2}\right)\right] \quad . \qquad (95)$$

The real parts of these quantities agree with their a=0 counterparts. If Eq. (40) is assumed to hold not just for j=3, but also for j=1 and j=2, then, in conjunction with Eq. (93), equations similar to Eqs. (54) and (55) may be obtained for j=1,2,3. Generalizing Eq. (60) to hold also for j=1,2,3 , combining this equation with the generalized forms of Eqs. (54) and (55), and employing the resonant approximation used in Sect. III, yields

$$\langle F_{A\tau_e 2}(\tau_e)\rangle \approx -\sum_{j=1}^{3} \frac{\partial}{\partial R}\left\{ \frac{\hbar}{8}\frac{(\mathrm{Re}\, n_{jj})^2}{\omega_i}\frac{\partial}{\partial \Omega}\left[\frac{1}{\Omega}\coth\left(\frac{\hbar\Omega}{2kT}\right)\Big|_{T=\frac{\hbar a}{2\pi ck}}\right]\Big|_{\Omega=\omega_i}\right\}$$

where $\mathrm{Re}\, n_{ii}$ is given by the real part of Eqs. (94) and (95). This expression agrees precisely with the corresponding unaccelerated-thermal force expression [see Eq. (39) of Ref. (10)].

B. Other Statistical Properties of General System

Most of the work required to obtain the force expression in Sec. IVA had already been done in Sec. III; the only further complication occurring in Sec. IVA that was not present in Sec. IIIB was in the additional terms of order $\mathcal{O}\left(\frac{aR}{c^2}\right)$. Taking these terms to be of negligible contribution, however, then reduced the treatment of Sec. IVA to essentially that of Sec. IIIB.

The same situation occurs in this section. The results of Sec. IIID will be drawn heavily upon in order to obtain the correlation functions for the general oscillator model considered here in Sec. IV. When terms of order $\mathcal{O}\left(\frac{aR}{c^2}\right)$ are ignored, then agreement is again obtained between these correlation functions and those of the unaccelerated-thermal system.

Using Eqs. (89a), (89b), (C1), and (C3), and recognizing that $\eta_{ij} \approx \left(\frac{aR}{c^2}\right)\eta_{ii}$ for $i \neq j$, enables Eqs. (81) and (82) to be generalized into the following compact notation:

$$
\begin{aligned}
\left\langle \tilde{\xi}_{\binom{A}{A}i}(\Omega')\tilde{\xi}_{\binom{A}{B}j}(\Omega'')\right\rangle &= \left\langle \tilde{\xi}_{\binom{B}{B}i}(\Omega')\tilde{\xi}_{\binom{B}{A}j}(\Omega'')\right\rangle \\
&= -\frac{\pi^2}{m}\left(\delta_{ij} + \mathcal{O}\left(\frac{aR}{c^2}\right)(1-\delta_{i3})(1-\delta_{j3})\right)\int_0^\infty \frac{d\Omega}{\Omega}\, h_T^2(\Omega)\Big|_{T=\frac{\hbar a}{2\pi c^2}} \times \\
&\quad \times \left\{\delta(\Omega'-\Omega)\delta(\Omega''+\Omega) + \delta(\Omega'+\Omega)\delta(\Omega''-\Omega)\right\}\left\{\pm\frac{\mathrm{Im}(C_i^a-\eta_{ii}^a)}{|C_i^a-\eta_{ii}^a|^2} + \frac{\mathrm{Im}(C_i^a+\eta_{ii}^a)}{|C_i^a+\eta_{ii}^a|^2}\right\} \, . \qquad (97)
\end{aligned}
$$

As can be shown by explicit calculation, the exact expression for the unaccelerated-thermal situation is given by dropping the $\mathcal{O}\left(\frac{aR}{c^2}\right)$ terms in Eq. (97) and changing the C_i^a and η_{ii}^a terms to their a->0 values.

The moments of Eqs. (83) and (84) may be generalized in the following way by using Eq. (97):

$$\left\langle \frac{d^m}{d\tau_e^m}\xi_{\binom{A}{A}i}\Big|_{\tau_{e0}} \frac{d^n}{d\tau_e^n}\xi_{\binom{A}{B}j}\Big|_{\tau_{e0}+\tau_e} \right\rangle = \left\langle \frac{d^m}{d\tau_e^m}\xi_{\binom{B}{A}i}\Big|_{\tau_{e0}} \frac{d^n}{d\tau_e^n}\xi_{\binom{B}{B}j}\Big|_{\tau_{e0}+\tau_e} \right\rangle$$
$$= -\left(\delta_{ij}+O\left(\frac{aR}{c^2}\right)(1-\delta_{i3})(1-\delta_{j3})\right)\frac{\pi}{m}\int_0^\infty d\Omega\cdot\Omega^{(m+n-1)} h_T^2(\Omega)\Big|_{T=\frac{\hbar a}{2\pi ck}} \times \begin{Bmatrix} (-1)^{\left(\frac{n-m+1}{2}\right)}\sin(\Omega\tau_e) \\ (-1)^{\left(\frac{n-m}{2}\right)}\cos(\Omega\tau_e) \end{Bmatrix} \times$$
$$\times \left\{ \pm\frac{\mathrm{Im}(C_i^a-\eta_{ii}^a)}{|C_i^a-\eta_{ii}^a|^2} + \frac{\mathrm{Im}(C_i^a+\eta_{ii}^a)}{|C_i^a+\eta_{ii}^a|^2} \right\} \quad \begin{matrix} [(m+n)\ \text{odd}] \\ [(m+n)\ \text{even}] \end{matrix} \tag{98}$$

Again, the analogous expression for the unaccelerated-thermal situation may be obtained by following the steps just mentioned at the end of the last paragraph.

Parallelling the steps of Sect. IIID, then when $0 \le (m+n) < 2$ and the conditions of Eq. (40) and

$$|\Delta^a_{Di\pm}\tau_e| \le 2\,\Gamma(\omega_i)^2\left(1+\left(\frac{a/c}{\omega_i}\right)^2\right)|\tau_e| \ll 1 \tag{99}$$

are valid for i=1,2,3 , the resonant approximation for the unretarded van der Waals situation may again be applied here. Under these conditions, Eqs. (86) and (87) may be generalized to

$$\left\langle \frac{d^m\xi_{Ai}}{d\tau_e^m}\Big|_{\tau_{e0}} \frac{d^n\xi_{Aj}}{d\tau_e^n}\Big|_{(\tau_{e0}+\tau_e)} \right\rangle = \left\langle \frac{d^m\xi_{Bi}}{d\tau_e^m}\Big|_{\tau_{e0}} \frac{d^n\xi_{Bj}}{d\tau_e^n}\Big|_{(\tau_{e0}+\tau_e)} \right\rangle \tag{100}$$
$$\approx \frac{\hbar}{2m}\left\{ \omega_i^{(m+n-1)}\coth\left(\frac{\hbar\omega_i}{2kT}\right)\Big|_{T=\frac{\hbar a}{2\pi ck}} \begin{Bmatrix}\sin(\omega_i\tau_e)\\ \cos(\omega_i\tau_e)\end{Bmatrix} + \frac{1}{8}\left(\frac{\mathrm{Re}\,\eta_{ii}}{\omega_i}\right)^2\left(\left[-\frac{1}{\omega_i}\frac{\partial}{\partial\Omega}+\frac{\partial^2}{\partial\Omega^2}\right]\left[\Omega^{(m+n-1)}\coth\left(\frac{\hbar\Omega}{2kT}\right)\Big|_{T=\frac{\hbar a}{2\pi ck}}\begin{Bmatrix}\sin(\Omega\tau_e)\\ \cos(\Omega\tau_e)\end{Bmatrix}\right]\right)\Big|_{\Omega=\omega_3}\right\} \times$$
$$\times\left(\delta_{ij}+O\left(\frac{aR}{c^2}\right)(1-\delta_{i3})(1-\delta_{j3})\right)\begin{pmatrix}(-1)^{\left(\frac{n-m+1}{2}\right)}\\ (-1)^{\left(\frac{n-m}{2}\right)}\end{pmatrix} \quad \begin{matrix}[(m+n)\ \text{odd}]\\ [(m+n)\ \text{even}]\end{matrix}$$

$$\left\langle \frac{d^m\xi_{Ai}}{d\tau_e^m}\Big|_{\tau_{e0}} \frac{d^n\xi_{Bj}}{d\tau_e^n}\Big|_{(\tau_{e0}+\tau_e)} \right\rangle \approx \frac{\hbar}{2m}\left(\frac{\mathrm{Re}\,\eta_{ii}}{\omega_i}\right)\frac{\partial}{\partial\Omega}\left[\Omega^{(m+n-1)}\coth\left(\frac{\hbar\Omega}{2kT}\right)\Big|_{T=\frac{\hbar a}{2\pi ck}}\begin{Bmatrix}\sin(\Omega\tau_e)\\ \cos(\Omega\tau_e)\end{Bmatrix}\right]\Big|_{\Omega=\omega_3} \times \tag{101}$$
$$\times\left(\delta_{ij}+O\left(\frac{aR}{c^2}\right)(1-\delta_{i3})(1-\delta_{j3})\right)\begin{pmatrix}(-1)^{\left(\frac{n-m+1}{2}\right)}\\ (-1)^{\left(\frac{n-m}{2}\right)}\end{pmatrix} \quad \begin{matrix}[(m+n)\ \text{odd}]\\ [(m+n)\ \text{even}]\end{matrix}$$

where $\mathcal{R}e\, n_{ii}$ is again given by the real part of Eqs. (94) and (95). Along with Eq. (93), the generalized forms of Eqs. (40), (60), and (85), valid for i=1,2,3 , were assumed in arriving at Eqs. (100) and (101).

Equations (100) and (101) apply for both the accelerated and unaccelerated-thermal situations. These expressions are given to lowest order in the damping constant Γ . As discussed in Sec. IIID, when the limit of e->0 is taken, then the restriction of (m+n)<2 is removed. Equations (100) and (101) then reduce to the case of a single oscillator that is noninteracting with any other oscillators.

V. CLOSING REMARKS

During recent years, a close connection in physical behavior has been established between point-like electromagnetic systems undergoing relativistic hyperbolic motion through electromagnetic zero-point radiation, and similar electromagnetic systems, held fixed in an inertial frame, but bathed in thermal electromagnetic radiation characterized by the temperature $T=\hbar a/2\pi ck$. This connection between these accelerated and unaccelerated-thermal electromagnetic systems consists of the agreement in their stochastic properties, when the former accelerated systems are described in a Fermi-Walker transported coordinate reference frame and the latter unaccelerated-thermal systems are described in their inertial rest frame.

The calculations of this article demonstrate that the connection just mentioned between accelerated and unaccelerated-thermal electromagnetic point-like systems also applies to the spatially extended electromagnetic system considered here: namely, two spatially separated charged simple harmonic oscillators, each taken in the electric dipole limit. Using the narrow linewidth approximation, the small oscillator assumption, the unretarded van der Waals condition, and a small laboratory condition, a number of stochastic properties were shown to agree between the accelerated and unaccelerated-thermal situations for such a pair of oscillators. The expectation value of the component of the Lorentz force along the axis separating the two accelerating oscillators was calculated and found to agree with the van der Waals force of a similar unaccelerated, but thermally situated pair of oscillators. Also, all combinations were analyzed for the n-point correlation functions of the derivatives of each oscillator's position; all such correlation functions for the accelerated oscillator system agreed with the corresponding correlation function of the unaccelerated-thermal oscillator system.

A set of exact relationships, namely Eqs. (C1)-(C4), were obtained in Appendices A and C; these identities relate the fields of an accelerating electric dipole to the correlation functions of classical electromagnetic zero-point fields. These relationships are what enabled the connections to be made between the accelerated and unaccelerated-thermal systems studied in this article. The complexity of these identities, as evidenced by the calculations of Appendices A and C, are such that, with little doubt, these relationships would not have been discovered had there not existed the physically motivating concept that the connection found between accelerated and unaccelerated-thermal point-like electromagnetic systems should also apply to spatially extended electromagnetic systems.

APPENDIX A: FIELDS OF UNIFORMLY ACCELERATING CHARGED OSCILLATOR, TAKEN IN THE ELECTRIC DIPOLE LIMIT

This appendix contains a calculation of the electric and magnetic fields due to a uniformly accelerating oscillator; these fields will be evaluated at the position of the second accelerating oscillator being considered in this article. The oscillator model described in Sec. II will be assumed. Hence, each oscillator will be taken to consist of an oscillating charge +e that moves inside a small region of negative charge distribution, with net charge -e. The latter distribution will be assumed to be centered about the oscillator's equilibrium point. At points far away from this small region of negative charge, the electric and magnetic fields arising from this negative charge distribution may be taken to be that of a single charge -e that is located at the oscillator's equilibrium point.

A fairly simple method will be described for calculating the electric and magnetic fields of one of the oscillators, where the fields are expressed in the I_{τ_e} frame and evaluated at the space-time point $\xi^{\mu}=(c\tau_e;\vec{\xi})$. The fields will first be found in an inertial frame $I_{\tau_e-\Delta\tau_-}$, where $\Delta\tau_-$ is defined to be the difference in proper time of the oscillator's equilibrium point for a light signal to travel from the equilibrium point of the oscillator to the space-time point $(c\tau_e ; \vec{\xi})$. A simple Lorentz transformation of the fields to the rest frame I_{τ_e} of the other oscillator will then yield the fields that will be used in Eqs. (6a) and (6b) of this article. As will be shown shortly, this indirect method of calculation significantly reduces the algebraic complications that a direct calculation would involve.

According to the above construction, $\tau_e - \Delta\tau_-$ equals the proper time of the oscillator's equilibrium point for the start of the light signal just mentioned. Consequently, the retarded inertial coordinate time for the starting time of this light signal is given by $t_{\tau_e - \Delta\tau_-} = 0$ [see Eq. (5a), with $\xi_1 = 0$]. Hence, due to the original definition for the $I_{\tau_e'}$ inertial rest frames, the oscillator giving rise to the electromagnetic fields at $(c\tau_e ; \vec{\xi})$ has its equilibrium point at rest in the $I_{\tau_e - \Delta\tau_-}$ inertial frame at proper time $t_{\tau_e - \Delta\tau_-} = 0$.

For notational purposes, let all quantities in the $I_{\tau_e - \Delta\tau_-}$ frame be indicated by a prime. Let the space-time point $x^{\mu}_{\tau_e - \Delta\tau_-} = (ct' ; \vec{x}')$ be given in the Fermi-Walker transported coordinate system by $\xi^{\mu} = (c\tau_e ; \vec{\xi})$. The expression for the electric field of an accelerating charged oscillator in the $I_{\tau_e - \Delta\tau_-}$ frame, evaluated at the space-time point $(c\tau_e ; \vec{\xi})$, is then given by:

$$\vec{E}_{\tau_e-\Delta\tau_-}(\vec{\xi},\tau_e)$$
$$= e\left\{\frac{(\vec{R}_+' - R_+'\vec{\beta}_+')(1-\beta_+'^2)}{(R_+' - \vec{\beta}_+'\cdot\vec{R}_+')^3}\right\}_{t_{r+}'} + \frac{e}{c}\left\{\frac{\vec{R}_+'\otimes\{(\vec{R}_+' - R_+'\vec{\beta}_+')\otimes\dot{\vec{\beta}}_+'\}}{(R_+' - \vec{\beta}_+'\cdot\vec{R}_+')^3}\right\}_{t_{r+}'}$$
$$-e\left\{\frac{(\vec{R}_-' - R_-'\vec{\beta}_-')(1-\beta_-'^2)}{(R_-' - \vec{\beta}_-'\cdot\vec{R}_-')^3}\right\}_{t_{r-}'} - \frac{e}{c}\left\{\frac{\vec{R}_-'\otimes\{(\vec{R}_-' - R_-'\vec{\beta}_-')\otimes\dot{\vec{\beta}}_-'\}}{(R_-' - \vec{\beta}_-'\cdot\vec{R}_-')^3}\right\}_{t_{r-}'} . \quad (A1)$$

Here, t_{r-}' equals the retarded time associated with the negative charge, while t_{r+}' is the analogous retarded time associated with the oscillating positive charge. The other quantities in Eq. (A1) follow the form of conventional usage (#18); their exact functional forms will be specified shortly.

Let $\overrightarrow{\Delta E}'$ be equal to the top two terms of Eq. (A1) minus the same two terms evaluated at the retarded time t_{r-}' rather than t_{r+}'. Then, Eq. (A1) can be expressed as the four terms of Eq. (A1), all evaluated at t_{r-}', plus $\Delta\vec{E}'$. The latter term will be evaluated later by using a Taylor's series expansion.

Let $\vec{X}'$ and $\vec{X}' + \overrightarrow{\Delta x}'$ be the vector positions of the negative charge and the oscillating charge, respectively, as expressed in the $I_{\tau_e-\Delta\tau_-}$ frame. The vector position at which the fields are to be evaluated is given by $\vec{x}' = (x', y', z')$. Let the symbol $\Big|_{r-}$ be a shortened notation for evaluating all quantities at t_{r-}'. The following notation should then be fairly obvious:

$$\left(\vec{R}_-'\right)_{\tau_-} = \left(\vec{x}'-\vec{X}'\right)_{\tau_-} \quad , \tag{A2}$$

$$\left(\vec{R}_+'\right)_{\tau_-} = \left(\vec{x}'-\vec{X}'-\vec{\Delta x}'\right)_{\tau_-} = \left(\vec{R}_-'-\vec{\Delta x}'\right)_{\tau_-} \quad , \tag{A3}$$

$$\left(\vec{\beta}_-'\right)_{\tau_-} = \left(\frac{1}{c}\frac{d\vec{X}'}{dt'}\right)_{\tau_-} = \left(\frac{(at'/c)\hat{x}}{\left[1+\left(\frac{at'}{c}\right)^2\right]^{1/2}}\right)_{t'=0} = 0 \quad , \tag{A4}$$

$$\left(\vec{\beta}_+'\right)_{\tau_-} = \frac{1}{c}\left(\frac{d}{dt'}\left(\vec{X}'+\vec{\Delta x}'\right)\right)_{\tau_-} = \left(\frac{1}{c}\dot{\vec{\Delta x}}'\right)_{\tau_-} \quad , \tag{A5}$$

$$\left(\dot{\vec{\beta}}_-'\right)_{\tau_-} = \left(\frac{1}{c}\frac{d^2\vec{X}'}{dt'^2}\right)_{\tau_-} = \frac{(a/c)\hat{x}}{\left[1+\left(\frac{at'}{c}\right)^2\right]^{3/2}}\Bigg|_{t'=0} = \frac{a}{c}\hat{x} \quad , \tag{A6}$$

$$\left(\dot{\vec{\beta}}_+'\right)_{\tau_-} = \frac{1}{c}\left(\frac{d^2}{dt'^2}\left(\vec{X}'+\vec{\Delta x}'\right)\right)_{\tau_-} = \frac{a}{c}\hat{x} + \left(\frac{1}{c}\ddot{\vec{\Delta x}}'\right)_{\tau_-} \quad . \tag{A7}$$

The relationships used in Eqs. (A4) and (A6) follow from Eq. (2) of Ref. (3). As described earlier, the value of t'_{τ_-} is given by $t'_{\tau_-} = 0$. Hence, $\left(\vec{\beta}_-'\right)_{\tau_-} = 0$, which was the main reason for choosing $I_{\tau_e-\Delta\tau_-}$ in which to first evaluate the dipole fields.

All of the terms in Eq. (A1) that are evaluated at $\tau_e-\Delta\tau_-$ may be expressed in terms of the quantities in Eqs. (A2) through (A7). In keeping with the small oscillator assumption, all quantities in Eq. (A1) will be evaluated only to first order in $\vec{\Delta x}'$, $\dot{\vec{\Delta x}}'$, and $\ddot{\vec{\Delta x}}'$. The following expressions are then obtained:

$$\left(R_+'\right)_{\tau_-} = \left|\vec{x}'-\vec{X}'-\vec{\Delta x}'\right|_{\tau_-} \approx \left(R_-' - \frac{\vec{\Delta x}'\cdot\vec{R}_-'}{R_-'}\right)_{\tau_-} \quad , \tag{A8}$$

$$\left(R_+' - \vec{\beta}_+'\cdot\vec{R}_+'\right)_{\tau_-} \approx \left(R_-' - \frac{\vec{\Delta x}'\cdot\vec{R}_-'}{R_-'} - \frac{\dot{\vec{\Delta x}}'}{c}\cdot\vec{R}_-'\right)_{\tau_-} \quad , \tag{A9}$$

$$\left(1-(\beta_+')^2\right)_{\tau_-} = 1 - \left(\frac{\dot{\vec{\Delta x}}'}{c}\right)^2_{\tau_-} \approx 1 \quad , \tag{A10}$$

$$\left(R_+' - \vec{\beta}_+'\cdot\vec{R}_+'\right)^{-3}_{\tau_-} \approx \left(\frac{1}{R_-'^3}\left\{1 + 3\frac{\vec{\Delta x}'\cdot\vec{R}_-'}{R_-'^2} + \frac{3\dot{\vec{\Delta x}}'\cdot\vec{R}_-'}{cR_-'}\right\}\right)_{\tau_-} \quad , \tag{A11}$$

$$\left(\left(\vec{R}_+' - R_+'\vec{\beta}_+'\right)\otimes\dot{\vec{\beta}}_+'\right)_{\tau_-} \approx \left(\left(\vec{R}_-' - \vec{\Delta x}' - R_-'\frac{\dot{\vec{\Delta x}}'}{c}\right)\otimes\dot{\vec{\beta}}_-'\right)_{\tau_-} + \left(\vec{R}_-'\otimes\frac{\ddot{\vec{\Delta x}}'}{c}\right)_{\tau_-} \quad . \tag{A12}$$

Other quantities may be linearized in the same way. Hence, Eq. (A1) becomes:

$$\vec{E}_{\tau_e-\Delta\tau_-}(\vec{\xi},\tau_e) \tag{A13}$$

$$\approx e\left\{\frac{3(\overline{\Delta x}'\cdot\overline{R}'_-)\overline{R}'_-}{R'^5_-}+\frac{3(\dot{\overline{\Delta x}}'\cdot\overline{R}'_-)}{cR'^4_-}\overline{R}'_- - \frac{\overline{\Delta x}'}{R'^3_-} - \frac{\dot{\overline{\Delta x}}'}{c}\frac{1}{R'^2_-}\right\}_{r_-}$$

$$+\frac{e}{c}\left\{\left(\overline{R}'_-\otimes(\overline{R}'_-\otimes\frac{\ddot{\overline{\Delta x}}'}{c}) - \overline{\Delta x}'\otimes(\overline{R}'_-\otimes\dot{\vec{\beta}}'_-) - \overline{R}'_-\otimes(\overline{\Delta x}'\otimes\dot{\vec{\beta}}'_-) - \overline{R}'_-\otimes(R'_-\frac{\dot{\overline{\Delta x}}'}{c}\otimes\dot{\vec{\beta}}'_-)\right.\right.$$

$$\left.\left.+(\overline{R}'_-\otimes\{\overline{R}'_-\otimes\dot{\vec{\beta}}'_-\})\frac{3\overline{\Delta x}'\cdot\overline{R}'_-}{R'^2_-}+(\overline{R}'_-\otimes(\overline{R}'_-\otimes\dot{\vec{\beta}}'_-))\frac{3\dot{\overline{\Delta x}}'\cdot\overline{R}'_-}{R'_-c}\right)\left(\frac{1}{R'^3_-}\right)\right\}_{r_-} + \overline{\Delta E}'.$$

In order to evaluate $\overline{\Delta E}'$, the difference in the retarded times must be found to lowest order in $\Delta x_i'$. Assuming that $z' = (Z')_{r_-}$, then one can show from a simple geometrical picture that

$$c(t'_{r_+}-t'_{r_-}) \approx +\left[\Delta y'\Big|_{r_-}\left(\frac{y'-Y'}{R'_-}\right)_{r_-} + \Delta x'\Big|_{r_-}\left(\frac{x'-X'}{R'_-}\right)_{r_-}\right], \tag{A14}$$

where $\vec{X}'=(X',Y',Z')$, $\overline{\Delta x}'=(\Delta x',\Delta y',\Delta z')$, and $\vec{x}'=(x',y',z')$.

Using a Taylor's expansion of the top two terms of Eq. (A1) in terms of the difference in the retarded times of Eq. (A14), yields

$$\overline{\Delta E}' \approx \left(e\left[\frac{-\dot{\vec{\beta}}'_-}{R'^2_-}+\frac{3\overline{R}'_-(\dot{\vec{\beta}}'_-\cdot\overline{R}'_-)}{R'^4_-}\right]_{r_-}\right.$$

$$\left.+\frac{e}{c}\left[\frac{\overline{R}'_-\otimes(\overline{R}'_-\otimes\ddot{\vec{\beta}}'_-)}{R'^3_-}\,\frac{3\dot{\vec{\beta}}'_-\cdot\overline{R}'_-}{R'_-}\right]_{r_-}\right)(t'_{r_+}-t'_{r_-}), \tag{A15}$$

where $\overline{\Delta E}'$ is given to first order in the $\Delta x_i'$ coordinates. Equation (A15) was obtained simply by differentiating the top two terms of Eq. (A1) by t', evaluating this quantity at the retarded time t'_{r_-}, and then multiplying this expression by $(t'_{r_+}-t'_{r_-})$. Since the latter factor was already of first-order in the $\Delta x_i'$ coordinates, then this fact enabled the dependence of the differentiated top two terms of Eq.

(A1) upon the $\Delta x_i'$ coordinates to be ignored. Moreover, the first-order dependence of Eq. (A14) upon the $\Delta x_i'$ coordinates allowed the Taylor's expansion to be carried out only to the first order in the time dependence. Finally, the following expressions were used in deducing Eq. (A15):

$$\left(\frac{d\vec{R}_-'}{dt'}\right)_{r_-} = \left(\vec{\beta}_-'\right)_{r_-} = 0 \quad , \qquad (A16)$$

$$\left(\ddot{\vec{\beta}}_-'\right)_{r_-} = \left.\frac{-3\hat{x}\left(\frac{a}{c}\right)^3 t'}{\left[1+\left(\frac{at'}{c}\right)^2\right]^{3/2}}\right|_{t'=0} = 0 \quad . \qquad (A17)$$

Equations (A13), (A14), and (A15) will now be used to obtain the electric field at the space-time point $x^{\mu}_{\tau_e-\Delta\tau} = (ct';\vec{x}') = \left(ct';\ \hat{y}(R_L \pm R) + \hat{x}\frac{c^2}{a}\left[1+\left(\frac{at'}{c}\right)^2\right]^{1/2}\right)$, or, equivalently, $\xi^{\mu} = (c\tau_e;\ \hat{y}(R_L \pm R))$; the electric field will be taken to be due to an accelerated charged oscillator with equilibrium point located at $\vec{\xi} = \hat{y}R_L$. Let the quantities $(R_-')_{r_-}$ and $(\vec{R}_-')_{r_-}$ be abbreviated by R_- and $\vec{R}_-$, respectively. The following simple relationship then holds:

$$(R_-)^2 = (t' - t_{r_-}')^2 c^2$$

$$= R^2 + \left(\frac{c^2}{a}\left[1+\left(\frac{at'}{c}\right)^2\right]^{1/2} - \frac{c^2}{a}\left[1+\left(\frac{at_{r_-}'}{c}\right)^2\right]^{1/2}\right)^2 \quad . \qquad (A18)$$

Substituting in $t_{r_-}' = 0$ and then solving for t' enables one to deduce that

$$R_- = ct' = R(1+\alpha^2)^{1/2} \quad , \qquad (A19)$$

$$\vec{R}_- = \hat{x}\alpha R \pm \hat{y}R \quad , \qquad (A20)$$

$$\alpha = \frac{aR}{2c^2} \quad , \qquad (A21)$$

$$\Delta\tau_- = \frac{c}{a}\sinh^{-1}\left(\frac{aR_-}{c^2}\right) \quad . \qquad (A22)$$

Combining Eqs. (A13)–(A15) and (A19)–(A22) then yields the following, where a superscript DL has been added to indicate dipole fields due to oscillator L:

$$\begin{aligned}
&\vec{E}^{DL}_{\tau_e-\Delta\tau_-}\left(\hat{y}(R_L \pm R), \tau_e\right) \\
&\quad = \hat{x}e\left\{ -\Delta x'_L\left(\frac{R^2}{R'^5_-}\right)(1+6\alpha^2+8\alpha^4) - \frac{\Delta\dot{x}'_L}{c}\left(\frac{R^2}{R'^4_-}\right)(1+4\alpha^2) - \frac{\Delta\ddot{x}'_L}{c^2}\left(\frac{R^2}{R'^3_-}\right)\right. \\
&\qquad\qquad \left. \mp \Delta y'_L\left(\frac{R^2}{R'^5_-}\right)\alpha(1+4\alpha^2) \mp \frac{\Delta\dot{y}'_L}{c}\left(\frac{R^2}{R'^4_-}\right)\alpha(1-2\alpha^2) \pm \frac{\Delta\ddot{y}'_L}{c^2}\left(\frac{R^2}{R'^3_-}\right)\alpha \right\}_{r_-} \\
&\qquad + \hat{y}e\left\{ \pm\Delta x'_L\left(\frac{R^2}{R'^5_-}\right)\alpha(1+10\alpha^2+12\alpha^4) \pm \frac{\Delta\dot{x}'_L}{c}\left(\frac{R^2}{R'^4_-}\right)\alpha\, 3(1+2\alpha^2) \pm \frac{\Delta\ddot{x}'_L}{c^2}\left(\frac{R^2}{R'^3_-}\right)\alpha \right. \\
&\qquad\qquad \left. + \Delta y'_L\left(\frac{R^2}{R'^5_-}\right)(2+9\alpha^2+10\alpha^4) + \frac{\Delta\dot{y}'_L}{c}\left(\frac{R^2}{R'^4_-}\right)(2+3\alpha^2-2\alpha^4) - \frac{\Delta\ddot{y}'_L}{c^2}\left(\frac{R^2}{R'^3_-}\right)\alpha^2 \right\}_{r_-} \\
&\qquad + \hat{z}e\left\{ -\Delta z'_L\frac{1}{R'^3_-}(1+2\alpha^2) - \frac{\Delta\dot{z}'_L}{c}\frac{1}{R'^2_-}(1+2\alpha^2) - \frac{\Delta\ddot{z}'_L}{c^2}\frac{1}{R'_-} \right\}_{r_-} \quad . \qquad (A23)
\end{aligned}$$

The magnetic field that is due to the accelerating charged oscillator must also be obtained. The magnetic field in the inertial frame is easily obtained by using

$$\vec{B}^{DL}_{\tau_e-\Delta\tau_-}\left(\hat{y}(R_L \pm R), \tau_e\right) = \left(\hat{n}'_+ \otimes \vec{E}'_+\right)_{r_+} + \left(\hat{n}'_- \otimes \vec{E}'_-\right)_{r_-} \quad , \qquad (A24)$$

where $(\vec{E}'_+)_{r_+}$ consists of the first two terms of Eq. (A1), $(\vec{E}'_-)_{r_-}$ consists of the second two terms, and

$$(\hat{n}'_-)_{r_-} = (\vec{R}_-/R_-) \quad , \qquad (A25)$$

$$\begin{aligned}
(\hat{n}'_+)_{r_+} &= \left(\frac{\vec{R}'_- - \Delta\vec{x}'_L}{|\vec{R}'_- - \Delta\vec{x}'_L|}\right)_{r_+} \approx \left(\frac{\vec{R}'_- - \Delta\vec{x}'_L}{|\vec{R}'_- - \Delta\vec{x}'_L|}\right)_{r_-} \\
&\approx \left\{ \frac{\vec{R}'_-}{R'_-} - \frac{\Delta\vec{x}'_L}{R'_-} + \frac{\vec{R}'_-(\Delta\vec{x}'_L\cdot\vec{R}'_-)}{(R'_-)^3} \right\}_{t_{r_-}} \quad . \qquad (A26)
\end{aligned}$$

The first approximation sign in Eq. (A26) follows from making a Taylor's expansion in $(t_{r_+} - t_{r_-})$, using Eq. (A16), and retaining terms only to first order in the $\Delta x'_{Li}$ coordinates. Hence, from Eq. (A16), the second term in the Taylor's expansion is identically equal to zero. Due to Eq. (A14), higher-order terms in the Taylor's expansion are not linear in the $\Delta x'_{Li}$ coordinates; hence, only the lowest-order term in the expansion need be retained.

Let $\vec{E}^{DL'}$ be an abreviated form for the electric field in Eq. (A23). From Eqs. (A1) and (A23), $(\vec{E}'_+)_{r_+} \approx \vec{E}^{DL'} - (\vec{E}'_-)_{r_-}$, where the field $\vec{E}^{DL'}$ is given by Eq. (A23) to first order in the $\Delta\vec{x}'$ coordinates. Hence, to this same degree of approximation,

$$\vec{B}^{DL}_{\tau_e - \Delta\tau_-}\left(\hat{y}(\mathcal{R}_L \pm R), \tau_e\right) \approx \left\{ \frac{\vec{R}'_- \otimes \vec{E}^{DL'}}{R'_-} + \frac{\Delta\vec{x}'_L \otimes \vec{E}'_-}{R'_-} - \frac{(\vec{R}'_- \otimes \vec{E}'_-)(\Delta\vec{x}'_L \cdot \vec{R}'_-)}{(R'_-)^3} \right\}_{r_-} , \qquad (A27)$$

where $(\vec{E}'_-)_{r_-}$ is given by

$$(\vec{E}'_-)_{r_-} = \hat{x}\frac{eR\alpha}{R^3_-} \mp \hat{y}\frac{eR}{R^3_-}(1+2\alpha^2) \quad . \qquad (A28)$$

From Eqs. (A23), (A27), and (A28), the following expression may be obtained:

$$\begin{aligned}\vec{B}^{DL}_{\tau_e - \Delta\tau_-}\left(\hat{y}(\mathcal{R}_L \pm R), \tau_e\right) = {} & \hat{x}e\left\{ \mp \frac{\Delta\dot{z}'_L}{c}\frac{R}{R'^3_-}(1+2\alpha^2) \mp \frac{\Delta\ddot{z}'_L}{c^2}\frac{R}{R'^2_-} \right\}_{r_-} \\ & + \hat{y}e\left\{ +\Delta z'_L \frac{R}{R'^4_-} 2\alpha(1+\alpha^2) + \frac{\Delta\dot{z}'_L}{c}\frac{R}{R'^3_-}\alpha(1+2\alpha^2) + \frac{\Delta\ddot{z}'_L}{c^2}\frac{R}{R'^2_-}\alpha \right\}_{r_-} \qquad (A29) \\ & + \hat{z}e\left\{ \pm \Delta x'_L\left(\frac{R}{R'^4_-}\right)(6\alpha^2+12\alpha^4) \pm \frac{\Delta\dot{x}'_L}{c}\frac{R}{R'^3_-}(1+6\alpha^2) \pm \frac{\Delta\ddot{x}'_L}{c^2}\frac{R}{R'^2_-} \right. \\ & \qquad \left. + \Delta y'_L \frac{R}{R'^4_-}(4\alpha+10\alpha^3) + \frac{\Delta\dot{y}'_L}{c}\frac{R\alpha}{R'^3_-}(3-2\alpha^2) - \frac{\Delta\ddot{y}'_L}{c^2}\frac{R}{R'^2_-}\alpha \right\}_{r_-} .\end{aligned}$$

Using Eqs. (A23), (A29), and (A22), the electric and magnetic fields in the I_{τ_e} frame may be obtained via the Lorentz transformation

$$\vec{E}^{DL}_{\tau_e} = \hat{x}E_x^{DL'} + \hat{y}\gamma_{\Delta\tau_-}(E_y^{DL'} - \beta_{\Delta\tau_-}B_z^{DL'}) + \hat{z}\gamma_{\Delta\tau_-}(E_z^{DL'} + \beta_{\Delta\tau_-}B_y^{DL'}) \quad , \qquad (A30)$$

$$\vec{B}^{DL}_{\tau_e} = \hat{x}B_x^{DL'} + \hat{y}\gamma_{\Delta\tau_-}(B_y^{DL'} + \beta_{\Delta\tau_-}E_z^{DL'}) + \hat{z}\gamma_{\Delta\tau_-}(B_z^{DL'} - \beta_{\Delta\tau_-}E_y^{DL'}) \quad , \qquad (A31)$$

where
$$\gamma_{\Delta\tau_-} = \cosh\left(\frac{a\Delta\tau_-}{c}\right) = (1+2\alpha^2) \quad , \qquad (A32)$$

$$\beta_{\Delta\tau_-}\gamma_{\Delta\tau_-} = \sinh\left(\frac{a\Delta\tau_-}{c}\right) = \frac{aR_-}{c^2} = 2\alpha(1+\alpha^2)^{1/2} \quad . \qquad (A33)$$

The quantities of $\vec{\Delta x}_L'$, $\dot{\vec{\Delta x}}_L'$, and $\ddot{\vec{\Delta x}}_L'$, all evaluated at $t_{\tau_e-\Delta\tau_-} = t_{\tau_-} = 0$, that will occur in Eqs. (A30) and (A31) after substituting in Eqs. (A23) and (A29), may be expressed in terms of the ξ^μ coordinates. From the discussion presented in Appendix D, [see, in particular, Eqs. (D6) and (D8)], one can show that

$$(\vec{\Delta x}_{\tau_e-\Delta\tau_-})_L\Big|_{t_{\tau_e-\Delta\tau_-}=0} = \vec{\xi}_L\Big|_{\tau_e-\Delta\tau_-} \quad , \qquad (A34)$$

$$\frac{d(\vec{\Delta x}_{\tau_e-\Delta\tau_-})_L}{d\,t_{\tau_e-\Delta\tau_-}}\Bigg|_{t_{\tau_e-\Delta\tau_-}=0} = \frac{d\vec{\xi}_L}{d\tau_e}\Bigg|_{\tau_e-\Delta\tau_-} \quad , \qquad (A35)$$

$$\frac{d^2(\vec{\Delta x}_{\tau_e-\Delta\tau_-})_L}{d\,t^2_{\tau_e-\Delta\tau_-}}\Bigg|_{t_{\tau_e-\Delta\tau_-}=0} = \hat{x}\left(\frac{d^2\xi_{L1}}{d\tau_e^2} - \left(\frac{a}{c}\right)^2\xi_{L1}\right)_{\tau_e-\Delta\tau_-} + \hat{y}\,\frac{d^2\xi_{L2}}{d\tau_e^2}\Bigg|_{\tau_e-\Delta\tau_-} + \hat{z}\,\frac{d^2\xi_{L3}}{d\tau_e^2}\Bigg|_{\tau_e-\Delta\tau_-} \quad . \qquad (A36)$$

Combining Eqs. (A23) and (A29)-(A36), along with Eq. (13), yields the following results (#19):

$$E^{DL}_{\tau_e i}(\hat{y}(R_L \pm R), \tau_e) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\Omega\, \exp(-i\Omega\tau_e)\left\{-\frac{m}{e^2}\sum_{j=1}^{3} n_{ij}(\hat{x}a, \pm\hat{y}R, \Omega)\left[e\tilde{\xi}_{Lj}(\Omega)\right]\right\} \quad , \qquad (A37)$$

$$B^{DL}_{\tau_e i}\left(\hat{y}(R_L \pm R), \tau_e\right)$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\Omega \exp(-i\Omega\tau_e)\left\{-\frac{m}{e^2}\sum_{j=1}^{3}\rho_{ij}(\hat{x}a, \pm\hat{y}R, \Omega)\left[e\tilde{\xi}_{Lj}(\Omega)\right]\right\} \quad , \qquad (A38)$$

where

$$\eta_{11}(\hat{x}a, \pm\hat{y}R, \Omega)$$
$$= -\frac{e^2}{m}K^3\left[\frac{R^2}{R_-^2}\right]\left[\frac{1}{KR_-} + \frac{i(1+4\alpha^2)}{(KR_-)^2} - \frac{(1+2\alpha^2+4\alpha^4)}{(KR_-)^3}\right]\exp(i\Omega\Delta\tau_-) \quad , \qquad (A39)$$

$$\eta_{22}(\hat{x}a, \pm\hat{y}R, \Omega)$$
$$= -\frac{e^2}{m}K^3\left[\frac{R^2}{R_-^2}\right]\left[-\frac{\alpha^2}{KR_-} - \frac{i(2+\alpha^2+2\alpha^4)}{(KR_-)^2} + \frac{(2+5\alpha^2)}{(KR_-)^3}\right]\exp(i\Omega\Delta\tau_-) \quad , \qquad (A40)$$

$$\eta_{33}(\hat{x}a, \pm\hat{y}R, \Omega) = -\frac{e^2}{m}K^3\left[\frac{1}{KR_-} + \frac{i(1+2\alpha^2)}{(KR_-)^2} - \frac{1}{(KR_-)^3}\right]\exp(i\Omega\Delta\tau_-) \quad , \qquad (A41)$$

$$\eta_{12}(\hat{x}a, \pm\hat{y}R, \Omega) = -\eta_{21}(\hat{x}a, \pm\hat{y}R, \Omega)$$
$$= -\frac{e^2}{m}K^3\left[\pm\frac{R^2}{R_-^2}\right]\alpha\left[-\frac{1}{KR_-} + \frac{i(1-2\alpha^2)}{(KR_-)^2} - \frac{(1+4\alpha^2)}{(KR_-)^3}\right]\exp(i\Omega\Delta\tau_-) \quad , \qquad (A42)$$

$$\eta_{i3}(\hat{x}a, \pm\hat{y}R, \Omega) = \eta_{3i}(\hat{x}a, \pm\hat{y}R, \Omega) = 0 \quad , \quad \text{for } i \neq 3 \quad , \qquad (A43)$$

and

$$\rho_{13}(\hat{x}a, \pm\hat{y}R, \Omega) = -\rho_{31}(\hat{x}a, \pm\hat{y}R, \Omega)$$
$$= -\frac{e^2}{m}K^3\left[\pm\frac{R}{R_-}\right]\left[\frac{1}{KR_-} + \frac{i(1+2\alpha^2)}{(KR_-)^2}\right]\exp(i\Omega\Delta\tau_-) \quad , \qquad (A44)$$

$$\rho_{23}(\hat{x}a, \pm\hat{y}R, \Omega) = +\rho_{32}(\hat{x}a, \pm\hat{y}R, \Omega)$$
$$= -\frac{e^2}{m}K^3\left[\frac{R}{R_-}\right]\alpha\left[\frac{1}{KR_-} + \frac{i(1+2\alpha^2)}{(KR_-)^2}\right]\exp(i\Omega\Delta\tau_-) \quad , \qquad (A45)$$

$$\rho^a_{11} = \rho^a_{22} = \rho^a_{33} = \rho^a_{12} = \rho^a_{21} = 0 \quad . \qquad (A46)$$

In the above expressions, $K = \frac{\Omega}{c}$ **;** R_- **and** α **are given by Eqs. (A19) and (A21), respectively. Thus, as may be seen from above,**

$$\eta_{ij}(\hat{x}a, \hat{y}R, -\Omega) = \eta^*_{ij}(\hat{x}a, \hat{y}R, \Omega) \quad , \qquad (A47)$$

$$\rho_{ij}(\hat{x}a, \hat{y}R, -\Omega) = \rho^*_{ij}(\hat{x}a, \hat{y}R, \Omega) \quad . \qquad (A48)$$

APPENDIX B: FORCE ON CHARGED OSCILLATOR, TAKEN IN THE ELECTRIC DIPOLE LIMIT

As was done in Appendix A, the model assumed here for a charged oscillator will be a +e positive point charge, with mass m, that oscillates inside a small distribution of negative charge, with net charge −e. This charge distribution will be assumed to be constructed in such a way that all points of the negative charge distribution possess the same instantaneous inertial rest frame. For the purposes of this section, the trajectory of the charged oscillator will not be restricted to that of uniform acceleration.

Let $\vec{X}(t)$ be the trajectory of the center of the negative charge distribution, as expressed in some arbitrary inertial frame. Let $\vec{X}(t) + \Delta\vec{x}(t)$ be the position of the oscillating positive charge. When $|\Delta\vec{x}|$ and the volume of negative charge are made infinitesimally small, then the Lorentz force on this system, due to electric and magnetic fields $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$, is approximately given by

$$\vec{F} = e\left\{ \vec{E}(\vec{X}+\Delta\vec{x},t) + \frac{1}{c}\left(\frac{d}{dt}(\vec{X}+\Delta\vec{x})\right)\otimes\vec{B}(\vec{X}+\Delta\vec{x},t)\right\} - e\left\{\vec{E}(\vec{X},t) + \frac{1}{c}\left(\frac{d\vec{X}}{dt}\right)\otimes\vec{B}(\vec{X},t)\right\}$$

$$\approx e\left[(\Delta\vec{x}\cdot\nabla)\vec{E}\Big|_{\vec{X},t} + \frac{1}{c}\left(\frac{d\Delta\vec{x}}{dt}\right)\otimes\vec{B}(\vec{X},t) + \frac{1}{c}\left(\frac{d\vec{X}}{dt}\right)\otimes\left\{(\Delta\vec{x}\cdot\nabla)\vec{B}\Big|_{\vec{X},t}\right\}\right] \quad . \tag{B1}$$

In the instantaneous inertial rest frame of the charge distribution, $\frac{d\vec{X}}{dt} = 0$, so the third term in Eq. (B1) drops out. In the limit where $|\vec{\Delta x}| \to 0$, $e \to \infty$, and the volume of the negative charge distribution goes to zero in such a way that the total charge distribution is described by a finite value for $e\vec{\Delta x} = \vec{p}$, then in the special case of the instantaneous inertial rest frame, Eq. (B1) becomes of the same form as the familiar expression

$$\vec{F} = (\vec{p}\cdot\nabla)\vec{E} + \frac{1}{c}\dot{\vec{p}}\otimes\vec{B} \quad , \qquad\qquad (B2)$$

for the Lorentz force on a stationary electric dipole.

The second term in Eq. (B1) can be rewritten as

$$\frac{1}{c}\left(\frac{d\vec{\Delta x}}{dt}\right)\otimes\vec{B}\left(\vec{X}(t),t\right)$$
$$= \frac{1}{c}\frac{d}{dt}\left\{\vec{\Delta x}\otimes\vec{B}(\vec{X},t)\right\} - \frac{1}{c}\vec{\Delta x}\otimes\left\{\left(\frac{d\vec{X}}{dt}\cdot\nabla\right)\vec{B}\Big|_{\vec{X},t}\right\} + \vec{\Delta x}\otimes\left(\nabla\otimes\vec{E}\Big|_{\vec{X},t}\right) \quad , \qquad (B3)$$

Combining Eq. (B3) with Eq. (B1) yields

$$F_i(t) = e\sum_{j=1}^{3}\Delta x_j\frac{\partial}{\partial x_i}E_j\Big|_{\vec{X}(t),t} + \frac{e}{c}\frac{d}{dt}\left(\vec{\Delta x}\otimes\vec{B}(\vec{X},t)\right)_i$$
$$-\frac{e}{c}\left(\vec{\Delta x}\otimes\left\{\left(\frac{d\vec{X}}{dt}\cdot\nabla\right)\vec{B}\Big|_{\vec{X}(t),t}\right\}\right)_i + \frac{e}{c}\left(\frac{d\vec{X}}{dt}\otimes\left\{(\vec{\Delta x}\cdot\nabla)\vec{B}\Big|_{\vec{X}(t),t}\right\}\right)_i \quad . \qquad (B4)$$

In the instantaneous inertial rest frame of the negative charge distribution, the last two terms equal zero.

APPENDIX C: CORRELATION FUNCTIONS OF FIELDS

In this section, the following relationships will be shown to be valid:

$$\langle E^{zp}_{i\tau_{e0}}(\hat{y}R_0,\tau_{e0})E^{zp}_{j(\tau_{e0}+\tau_e)}(\hat{y}(R_0+R),\tau_{e0}+\tau_e)\rangle$$
$$= \langle B^{zp}_{i\tau_{e0}}(\hat{y}R_0,\tau_{e0})B^{zp}_{j(\tau_{e0}+\tau_e)}(\hat{y}(R_0+R),\tau_{e0}+\tau_e)\rangle$$
$$= \int_0^\infty f^{zp}_{ij}(\hat{x}a,\hat{y}R,\Omega)\cos(\Omega\tau_e)\,d\Omega \quad , \qquad (C1)$$

$$\langle B^{zp}_{i\tau_{e0}}(\hat{y}R_0,\tau_{e0})E^{zp}_{j(\tau_{e0}+\tau_e)}(\hat{y}(R_0+R),\tau_{e0}+\tau_e)\rangle$$
$$= \int_0^\infty g^{zp}_{ij}(\hat{x}a,\hat{y}R,\Omega)\sin(\Omega\tau_e)\,d\Omega \quad , \qquad (C2)$$

where f^{zp}_{ij} and g^{zp}_{ij} are related to the functions n^a_{ij} and ρ^a_{ij} [see Eqs. (A37)-(A46)] for an accelerated electric dipole oscillator by (*20, 21)

$$f^{zp}_{ij}(\hat{x}a,\hat{y}R,\Omega) = \left.\frac{-2\pi\hbar^2_T(\Omega)}{\frac{e^2}{m}\Omega}\right|_{T=\frac{\hbar a}{2\pi ck}} \mathrm{Im}\left(n_{ij}(\hat{x}a,\hat{y}R,\Omega)\right) \quad , \qquad (C3)$$

$$g^{zp}_{ij}(\hat{x}a,\hat{y}R,\Omega) = \left.-\frac{2\pi\hbar^2_T(\Omega)}{\frac{e^2}{m}\Omega}\right|_{T=\frac{\hbar a}{2\pi ck}} \mathrm{Re}\left(\rho_{ji}(\hat{x}a,\hat{y}R,\Omega)\right) \quad , \qquad (C4)$$

$$\hbar^2_T(\Omega) = \frac{\hbar\Omega}{2\pi^2}\coth\left(\frac{\hbar\Omega}{2kT}\right) = \frac{1}{\pi^2}\left\{\frac{\hbar\Omega}{2} + \frac{\hbar\Omega}{\exp(\hbar\Omega/2kT)-1}\right\} \quad . \qquad (C5)$$

It should be noted that the i,j indices are reversed in order on the left and right sides of Eq. (C4), but they occur in the same order on both sides of Eq. (C3). The definition given for $\hbar^2_T(\Omega)$ in Eq. (C5) generalizes the function $\hbar^2(\Omega)$ in Eq. (24) to the case of the thermal plus zero-point spectrum, in which the temperature of the stochastic electromagnetic radiation field may not necessarily equal zero. More specifically, when $T \neq 0$, then $h_T(\Omega)$ is used in place of $h(\Omega)$ in the expressions for the electric and magnetic fields of Eqs. (19) and (20) (#22). When $T \to 0$, then $h_T(\Omega) \to h(\Omega)$. From Eq. (C5), the factor below that appears in Eqs. (C3) and (C4) may be written more simply as

$$\left.\frac{2\pi h_T^2(\Omega)}{\frac{e^2}{m}\Omega}\right|_{T=\frac{\hbar a}{2\pi c k}} = \frac{\hbar \coth\left(\frac{\pi c \Omega}{a}\right)}{\left(\frac{e^2}{m}\right)\pi} \quad . \qquad (C6)$$

The demonstration of the validity of Eqs. (C1)-(C4) will be given by explicitly calculating the expectation value of the quantities on the left side of Eqs. (C1) and (C2), and then showing that the expressions that result from these calculations are related to n_{ij}^a and ρ_{ij}^a via Eqs. (C1)-(C4). Unfortunately, the algebraic manipulations involved in this demonstration are fairly lengthy.

In order to establish the validity of Eqs. (C1) and (C3), a set of identities will first be proven; these identities will considerably shorten the proofs that follow. Let

$$\beta = \frac{a}{2c} \quad , \qquad \alpha = \frac{aR}{2c^2} \quad , \qquad b = \frac{\sinh(\beta\tau_e)}{\alpha} \quad . \qquad (C7a,b,c)$$

From Eqs. (A19), (A21), and (A22), $\Delta\tau_-$ may be rewritten as

$$\Delta\tau_- = \frac{1}{2\beta}\sinh^{-1}\left(2\alpha(1+\alpha^2)^{1/2}\right) \quad . \qquad (C8)$$

Let I_c be defined as the integral listed below. From this definition, three useful identities can be established that involve I_c, $\left[\frac{dI_c}{d\Delta\tau_-}\right]$, and $\left[-\frac{d^2 I_c}{d\Delta\tau_-^2}\right]$:

$$I_c \equiv \int_0^\infty \coth\left(\frac{\pi\Omega}{2\beta}\right)\sin(\Omega\Delta\tau_-)\cos(\Omega\tau_e)d\Omega = \frac{\beta}{\alpha}(1+\alpha^2)^{1/2}\left(\frac{1}{1-b^2}\right) , \qquad (C9)$$

$$\left[\frac{dI_c}{d\Delta\tau_-}\right] = \int_0^\infty \coth\left(\frac{\pi\Omega}{2\beta}\right)\cos(\Omega\Delta\tau_-)\cos(\Omega\tau_e)\Omega\cdot d\Omega$$

$$= -\left(\frac{\beta}{\alpha}\right)^2\left\{\frac{1}{(1-b^2)} + 2(1+\alpha^2)\frac{b^2}{(1-b^2)^2}\right\} , \qquad (C10)$$

$$\left[-\frac{d^2 I_c}{d\Delta\tau_-^2}\right] = \int_0^\infty \coth\left(\frac{\pi\Omega}{2\beta}\right)\sin(\Omega\Delta\tau_-)\cos(\Omega\tau_e)\Omega^2\cdot d\Omega$$

$$= -2\left(\frac{\beta}{\alpha}\right)^3(1+\alpha^2)^{1/2}\left\{\frac{1}{(1-b^2)} + (2\alpha^2+5)\frac{b^2}{(1-b^2)^2} + 4(1+\alpha^2)\frac{b^4}{(1-b^2)^3}\right\}. \qquad (C11)$$

The first identity above can be demonstrated by reexpressing I_c as

$$I_c = \int_0^\infty \coth\left(\frac{\pi\Omega}{2\beta}\right)\frac{1}{2}\left\{\sin\left(\Omega(\tau_e+\Delta\tau_-)\right) - \sin\left(\Omega(\tau_e-\Delta\tau_-)\right)\right\}d\Omega , \qquad (C12)$$

and then utilizing the following relationship which may be obtained from standard integral tables (#23):

$$\int_0^\infty \coth(b'x)\sin(a'x)dx = \frac{\pi}{2b'}\coth\left(\frac{\pi a'}{2b'}\right) . \qquad (C13)$$

The expression obtained for I_c, namely,

$$I_c = \frac{\beta}{2}\left\{\coth\left(\beta(\tau_e+\Delta\tau_-)\right) - \coth\left(\beta(\tau_e-\Delta\tau_-)\right)\right\} , \qquad (C14)$$

must then be differentiated with respect to $\Delta\tau_-$ in order to obtain corresponding expressions for $\left[\frac{dI_c}{d\Delta\tau_-}\right]$ and $\left[\frac{d^2 I_c}{d\Delta\tau_-^2}\right]$. Applying the hyperbolic trigonometric sum of angles formulae, substituting in Eqs. (C7c) and (C8), and combining terms, will then result in the identities of Eqs. (C9)-(C11).

Three more relationships will be established before returning to an examination of Eqs. (C1) and (C3). These relationships are:

$$\int_0^\infty dw \cos(wb)\sin(w) = \frac{1}{(1-b^2)} \quad , \qquad (C15)$$

$$\int_0^\infty dw\cdot w\cos(wb)\cos(w) = -\left[\frac{1}{(1-b^2)} + \frac{2b^2}{(1-b^2)^2}\right] \quad , \qquad (C16)$$

$$\int_0^\infty dw\cdot w^2\cos(wb)\sin(w) = -2\left[\frac{1}{(1-b^2)} + \frac{5b^2}{(1-b^2)^2} + \frac{4b^4}{(1-b^2)^3}\right] \quad . \qquad (C17)$$

Equation (C15) may be obtained by using a temporary cutoff and then setting $b'=b$ and $c'=1$ in the following integral:

$$\int_0^\infty dw \cos(wb')\sin(wc') = \lim_{\epsilon\to 0_+}\int_0^\infty dw\, e^{-\epsilon w}\cos(wb')\sin(wc')$$

$$= \lim_{\epsilon\to 0_+} \mathrm{Im}\int_0^\infty dw\, e^{-\epsilon w}\frac{1}{2}\left[e^{iw(c'+b')} + e^{iw(c'-b')}\right] = \frac{c'}{(c'^2-b'^2)} \quad . \qquad (C18)$$

Equations (C16) and (C17) may then be obtained by repeatedly differentiating Eqs. (C18) with respect to c' and then setting $c'=1$ and $b'=b$.

Finally, Eqs. (C9)-(C11) and Eqs. (C15)-(C17) may be combined as follows:

$$\int_0^\infty dw \cos(wb)\sin(w) = \frac{\alpha}{\beta}\frac{1}{(1+\alpha^2)^{1/2}} I_c \quad , \qquad (C19)$$

$$\int_0^\infty dw\cdot w\cos(wb)\cos(w) = -\frac{\alpha}{\beta}\frac{\alpha^2}{(1+\alpha^2)^{3/2}} I_c + \frac{\alpha^2}{\beta^2}\frac{1}{(1+\alpha^2)}\frac{dI_c}{d\Delta T_-} \quad , \qquad (C20)$$

$$\int_0^\infty dw\cdot w^2\cos(wb)\sin(w) = \frac{\alpha}{\beta}\frac{(\alpha^2-2\alpha^4)}{(1+\alpha^2)^{5/2}} + \frac{3\left(\frac{\alpha}{\beta}\right)^2\alpha^2}{(1+\alpha^2)^2}\frac{dI_c}{d\Delta T_-} + \left(\frac{\alpha}{\beta}\right)^3\frac{1}{(1+\alpha^2)^{3/2}}\left[-\frac{d^2 I_c}{d\Delta T_-^2}\right] . \qquad (C21)$$

Turning back to Eqs. (C1) and (C3), consider first the correlation function for the electric fields when i=j=1. (The second part of Eq. (C1), which involves the correlation functions of the magnetic fields, may be proven in nearly exactly the same way as what follows for the electric fields, simply by using Eqs. (8) and (20) in place of Eqs. (7) and (19) and by following the steps outlined below.) The initial part of the following calculations proceed along the lines of Ref. 24 and so will simply be quickly summarized here in order to unify notation. Let the $I_{\tau_e=0}$ inertial frame be again denoted by I_*. From Eqs. (7) and (19), the following expression may be obtained:

$$
\begin{aligned}
&\left\langle E^{zp}_{\tau_{eo}1}(\hat{y}R_o, \tau_{eo})\, E^{zp}_{(\tau_{eo}+\tau_e)1}(\hat{y}(R_o \pm R), \tau_{eo}+\tau_e)\right\rangle \\
&\quad = \left\langle E^{zp}_{*1}(\hat{y}R_o, \tau_{eo})\, E^{zp}_{*1}(\hat{y}(R_o \pm R), \tau_{eo}+\tau_e)\right\rangle \\
&\quad = \sum_{\lambda'}\sum_{\lambda''}\int d^3k' \int d^3k''\, h(\Omega')h(\Omega'')\, \epsilon_x'\, \epsilon_x'' \times \\
&\qquad\quad \times \Big\langle \cos\Big[k_1'\frac{c^2}{a}\cosh\Big(\frac{a\tau_{eo}}{c}\Big) + k_2' R_o - k'\frac{c^2}{a}\sinh\Big(\frac{a\tau_{eo}}{c}\Big) + \theta'\Big] \times \\
&\qquad\qquad \times \cos\Big[k_1''\frac{c^2}{a}\cosh\Big(\frac{a}{c}(\tau_{eo}+\tau_e)\Big) + k_2''(R_o \pm R) - k''\frac{c^2}{a}\sinh\Big(\frac{a}{c}(\tau_{eo}+\tau_e)\Big) + \theta''\Big]\Big\rangle .
\end{aligned}
\qquad (C22)
$$

Use of the identity

$$
\left\langle \cos(A+\theta(\vec{k}',\lambda'))\cos(B+\theta(\vec{k}'',\lambda''))\right\rangle = \frac{1}{2}\delta_{\lambda',\lambda''}\,\delta^3(\vec{k}''-\vec{k}')\cos(B-A) \quad , \qquad (C23)
$$

then results in the expression

$$
\left\langle E^{zp}_{\tau_{eo}1}(\ldots)\, E^{zp}_{(\tau_{eo}+\tau_e)1}(\ldots)\right\rangle \qquad (C24)
$$

$$
=\sum_{\lambda'}\int_{1} \left\langle E^{zp}_{\tau_{eo}1}(\ldots)\, E^{zp}_{(\tau_{eo}+\tau_e)1}(\ldots)\right\rangle \qquad (C24)
$$

$$
=\sum_{\lambda'}\int d^3k'\, h^2(\Omega')(\epsilon_x')^2\frac{1}{2}\cos\Big(k_1'\frac{c^2}{a}\Big[\cosh\Big(\frac{a}{c}(\tau_{eo}+\tau_e)\Big)-\cosh\Big(\frac{a}{c}\tau_{eo}\Big)\Big] \pm k_2' R - \frac{k'c^2}{a}\Big[\sinh\Big(\frac{a}{c}(\tau_{eo}+\tau_e)\Big)-\sinh\Big(\frac{a}{c}\tau_{eo}\Big)\Big]\Big),
$$

where the left side of the equation has been abbreviated for the moment.

Let $\sigma = \tau_{eo} + \tau_e/2$. Let K and $\vec{K}$ be defined by the following Lorentz transformation:

$$K' = \gamma_\sigma (K + \beta_\sigma K_1) \quad , \qquad (C25a)$$

$$K_1' = \gamma_\sigma (K_1 + \beta_\sigma K) \quad , \qquad (C25b)$$

$$K_2' = K_2 \quad , \quad K_3' = K_3 \quad , \qquad (C25c \,\&\, d)$$

where $\gamma_\sigma = \cosh(\frac{a\sigma}{c})$ and $\beta_\sigma = \tanh(\frac{a\sigma}{c})$. One can then verify the following relationship:

$$K_1' \frac{c^2}{a}\left[\cosh\left(\frac{a}{c}(\tau_{eo}+\tau_e)\right) - \cosh\left(\frac{a\tau_{eo}}{c}\right)\right] - K'\frac{c^2}{a}\left[\sinh\left(\frac{a}{c}(\tau_{eo}+\tau_e)\right) - \sinh\left(\frac{a\tau_{eo}}{c}\right)\right]$$
$$= -\frac{2c^2}{a} K \sinh\left(\frac{a\tau_e}{2c}\right) \quad . \qquad (C26)$$

Using Eqs. (22), (24), and (C25a)-(C25d), then yields the results of

$$d^3K' \mathcal{h}^2(\Omega') \frac{1}{K'^2} = d^3K \mathcal{h}^2(\Omega) \frac{1}{K^2} \quad , \qquad (C27)$$

$$\sum_\lambda \epsilon_x'^2 = 1 - \frac{K_1'^2}{K'^2} = \frac{K^2 - K_1^2}{K'^2} \quad . \qquad (C28)$$

From Eqs. (C24) and (C26)-(C28), one can then show that

$$\left\langle E^{zp}_{\tau_{eo}1}(\hat{y}R_o, \tau_{eo}) E^{zp}_{(\tau_{eo}+\tau_e)1}(\hat{y}(R_o \pm R), \tau_{eo}+\tau_e)\right\rangle$$
$$= \frac{1}{2}\int d^3K \mathcal{h}^2(\Omega)\left(1 - \frac{K_1^2}{K^2}\right)\cos(K_2 R)\cos\left(-\frac{2c^2}{a}K\sinh\left(\frac{a\tau_e}{c}\right) \pm K_2 R\right) \quad . \qquad (C29)$$

Hence, the correlation function of Eq. (C29) is independent of τ_{e0} and R_0 , as one would expect.

The second cosine term in Eq. (C29) may be expanded into two terms using the cosine sum of angles formula. One of these terms results in an integrand that is odd in k_2 , thereby vanishing upon angular integration. Choosing the i=2 axis as the polar axis, performing all angular integrations, letting $w = kR$, and using Eqs. (C7a)-(C7c), yields the expression of

$$\left\langle E^{zp}_{\tau_{e0}1}(\hat{y}R_0,\tau_{e0})\,E^{zp}_{(\tau_{e0}+\tau_e)1}\left(\hat{y}(R_0 \pm R),\tau_{e0}+\tau_e\right)\right\rangle$$

$$= \frac{\hbar c}{\pi R^4}\int_0^\infty dw\cos(wb)\left[w^2\sin w + w\cos w - \sin w\right] \quad . \qquad (C30)$$

Now the identities of Eqs. (C19)-(C21) may be employed. Substituting these in and collecting terms results in

$$\left\langle E^{zp}_{\tau_{e0}1}(\hat{y}R_0,\tau_{e0})\,E^{zp}_{(\tau_{e0}+\tau_e)1}\left(\hat{y}(R_0 \pm R),\tau_{e0}+\tau_e\right)\right\rangle$$

$$= \frac{\hbar c}{\pi R^4}\left\{\left[-\frac{d^2 I_c}{d\Delta\tau_-^2}\right]\left(\frac{\alpha}{\beta}\right)^3\frac{1}{(1+\alpha^2)^{3/2}} + \frac{dI_c}{d\Delta\tau_-}\left(\frac{\alpha}{\beta}\right)^2\left[\frac{3\alpha^2}{(1+\alpha^2)^2}+\frac{1}{(1+\alpha^2)}\right]\right.$$

$$\left. + I_c\frac{\alpha}{\beta}\left[\frac{\alpha^2-2\alpha^4}{(1+\alpha^2)^{5/2}} - \frac{\alpha^2}{(1+\alpha^2)^{3/2}} - \frac{1}{(1+\alpha^2)^{1/2}}\right]\right\}$$

$$= \int_0^\infty d\Omega\cos(\Omega\tau_e)\left[\frac{-\hbar}{\frac{e^2}{m}\pi}\coth\left(\frac{\pi c\Omega}{a}\right)\right]\times$$

$$\times\, \mathrm{Im}\left\{-\frac{e^2}{m}k^3\frac{R^2}{R_-^2}\left[\frac{1}{kR_-} + i\frac{(1+4\alpha^2)}{(kR_-)^2} - \frac{(1+2\alpha^2+4\alpha^4)}{(kR_-)^3}\right]\exp(i\Omega\Delta\tau_-)\right\} \quad . \qquad (C31)$$

The last line above results from the use of Eqs. (C9)-(C11), (A19), (C7a), and (C7b). By comparing Eq. (C31) with the expressions of Eqs. (C6) and (A39), one can now see that Eqs. (C1) and (C3) have indeed been verified for the i=j=1 situation.

The other cases may be verified in a similar way. Consider the case of i=j=3. The first line below follows from Eq. (7). The second line may be obtained from Eqs. (19), (20), (C23), and (C26). Finally, the third line may be obtained by using Eqs. (22), (23), and (C27), and then combining a few terms.

$$
\begin{aligned}
&\left\langle E^{zp}_{\tau_{e0}3}(\hat{y}R_0,\tau_{e0})\,E^{zp}_{(\tau_{e0}+\tau_e)3}\big(\hat{y}(R_0\pm R),\tau_{e0}+\tau_e\big)\right\rangle \qquad\qquad (C32)\\
&\quad=\Big\langle \gamma_{\tau_{e0}}\left[E_{*3}(\hat{y}R_0,\tau_{e0})+\beta_{\tau_{e0}}B_{*2}(\hat{y}R_0,\tau_{e0})\right]\\
&\qquad\qquad\times\gamma_{(\tau_{e0}+\tau_e)}\left[E_{*3}\big(\hat{y}(R_0\pm R),\tau_{e0}+\tau_e\big)+\beta_{(\tau_{e0}+\tau_e)}B_{*2}\big(\hat{y}(R_0\pm R),\tau_{e0}+\tau_e\big)\right]\Big\rangle\\
&\quad=\sum_{\lambda'}\int d^3k'\,h^2(\Omega')\,\frac{1}{2}\cos\!\left(-\frac{2c^2}{a}k\sinh\!\left(\frac{a\tau_e}{2c}\right)\pm k_2R\right)\gamma_{\tau_{e0}}\gamma_{(\tau_{e0}+\tau_e)}\,\times\\
&\qquad\qquad\times\left\{\epsilon_3'^2+\beta_{(\tau_{e0}+\tau_e)}\epsilon_3'(\hat{k}'\otimes\hat{\epsilon}')_2+\beta_{\tau_{e0}}\epsilon_3'(\hat{k}'\otimes\hat{\epsilon}')_2+\beta_{\tau_{e0}}\beta_{(\tau_{e0}+\tau_e)}\left((\hat{k}'\otimes\hat{\epsilon}')_2\right)^2\right\}\\
&\quad=\frac{1}{2}\int d^3k\,\frac{h^2(\Omega)}{k^2}\cos\!\left(-\frac{2c^2}{a}k\sinh\!\left(\frac{a\tau_e}{2c}\right)\pm k_2R\right)\gamma_{\tau_{e0}}\gamma_{(\tau_{e0}+\tau_e)}\,\times\\
&\qquad\qquad\times\left\{k_2'^2\left(1-\beta_{\tau_{e0}}\beta_{(\tau_{e0}+\tau_e)}\right)+\left[k_1'^2-k'k_1'\left(\beta_{\tau_{e0}}+\beta_{(\tau_{e0}+\tau_e)}\right)+k'^2\beta_{\tau_{e0}}\beta_{(\tau_{e0}+\tau_e)}\right]\right\}.
\end{aligned}
$$

As may be verified by using Eqs. (C25a) and (C25b),

$$
\gamma_{\tau_{e0}}\gamma_{(\tau_{e0}+\tau_e)}\left\{k_1'^2-k'k_1'\left(\beta_{\tau_{e0}}+\beta_{(\tau_{e0}+\tau_e)}\right)+k'^2\beta_{\tau_{e0}}\beta_{(\tau_{e0}+\tau_e)}\right\}
=k_1^2\left(1+\sinh^2\!\left(\frac{a\tau_e}{2c}\right)\right)-k^2\sinh^2\!\left(\frac{a\tau_e}{2c}\right) \qquad (C33)
$$

Likewise, one may show that

$$
\gamma_{\tau_{e0}}\gamma_{(\tau_{e0}+\tau_e)}\left(1-\beta_{\tau_{e0}}\beta_{(\tau_{e0}+\tau_e)}\right)=1+2\sinh^2\!\left(\frac{a\tau_e}{2c}\right) \qquad (C34)
$$

Combining these results with Eq. (C32) demonstrates that the i=j=3 correlation functions are independent of R_0 and τ_{e0}.

After angular integrations are performed and $w=kR$ and Eqs. (C7a)-(C7c) are substituted into Eq. (C32), then Eq.

(C32) can be shown to be equal to $(\mathrm{I}_{33}+\mathrm{II}_{33})$, where

$$\mathrm{I}_{33} = \frac{\hbar c}{\pi R^4}\int_0^\infty dw\cos(bw)\left(w^2\sin w + w\cos w - \sin w\right) \quad , \tag{C35}$$

$$\mathrm{II}_{33} = \frac{\hbar c}{\pi R^4}(\alpha^2 b^2)\int_0^\infty dw\cos(bw)\left(w^2\sin w + 3w\cos w - 3\sin w\right) \quad . \tag{C36}$$

Comparing Eqs. (C35) and (C30), shows that the two expressions are identical, so I_{33} may immediately be set equal to Eq. (C31). Equation (C36) may be reexpressed by again using a temporary cutoff in the following way:

$$\mathrm{II}_{33} = -\frac{\hbar c\alpha^2}{\pi R^4}\int_0^\infty dw\left[\frac{d^2}{dw^2}\cos(bw)\right]\left[w^2\sin w + 3w\cos w - 3\sin w\right]$$

$$= \frac{-\hbar c\alpha^2}{\pi R^4}\lim_{\epsilon\to 0_+}\int_0^\infty dw\, e^{-\epsilon w}\left[\frac{d^2}{dw^2}\cos(bw)\right]\left[w^2\sin w + 3w\cos w - 3\sin w\right] \quad . \tag{C36'}$$

Integrating by parts twice can then be shown to yield

$$\mathrm{II}_{33} = \frac{\hbar c\alpha^2}{\pi R^4}\int_0^\infty dw\cos(bw)\left[w^2\sin w - w\cos w + \sin w\right] \quad , \tag{C37}$$

since taking the limit of e->0 after performing the integration by parts results in the vanishing of all additional terms (#25).

Substituting Eqs. (C19)-(C21) into Eq. (C37), using the result of (C31) for (C35), and combining terms, yields

$$\left\langle E^{ZP}_{\tau_{e0}3}(\hat{y}R_0,\tau_{e0})E^{ZP}_{(\tau_{e0}+\tau_e)3}(\hat{y}(R_0\pm R),\tau_{e0}+\tau_e)\right\rangle$$

$$= \frac{\hbar c}{\pi R^4}\left\{\left[-\frac{d^2 I_c}{d\Delta\tau_-^2}\right]\left(\frac{\alpha}{\beta}\right)^3\frac{1}{(1+\alpha^2)^{1/2}} + \frac{dI_c}{d\Delta\tau_-}\left(\frac{\alpha}{\beta}\right)^2\frac{(1+2\alpha^2)}{(1+\alpha^2)} - I_c\frac{\alpha}{\beta}\frac{1}{(1+\alpha^2)^{3/2}}\right\}$$

$$= \int_0^\infty d\Omega\cos(\Omega\tau_e)\left[\frac{-\hbar}{\frac{e^2}{m}\Omega}\coth\left(\frac{\pi c\Omega}{a}\right)\right]\times$$

$$\times\ \mathrm{Im}\left\{-\frac{e^2}{m}K^3\left[\frac{1}{KR_-} + \frac{i}{(KR_-)^2}(1+2\alpha^2) - \frac{1}{(KR_-)^3}\right]\exp(i\Omega\Delta\tau_-)\right\} \quad . \tag{C38}$$

Comparing Eq. (C38) with the expressions of Eqs. (C6) and (A41) shows that Eqs. (C1) and (C3) are indeed valid for the i=j=3 situation.

Consider now the i=j=2 case. Following the same steps indicated in Eq. (C32) results in nearly an identical expression to the last line of Eq. (C32), but with the $(k_2')^2$ term that appears there replaced by $(k_3')^2$. Using Eqs. (C33) and (C34) again and performing angular integrations yields the result that the i=j=2 correlation function equals $(\mathrm{I}_{22}+\mathrm{II}_{22})$, where $\mathrm{II}_{22}=-\mathrm{II}_{33}$ and

$$\mathrm{I}_{22} = \frac{2\hbar c}{\pi R^4}\int_0^\infty dw \cos(wb)\left[-w\cos w + \sin w\right] \quad . \qquad (C39)$$

Using Eqs. (C19)-(C21) and (C38) and combining terms, results in

$$\begin{aligned}
&\left\langle E^{zp}_{\tau_{eo}2}(\hat{y}R_o,\tau_{eo})\,E^{zp}_{(\tau_{eo}+\tau_e)2}(\hat{y}(R_o\pm R),\tau_{eo}+\tau_e)\right\rangle \\
&= \frac{\hbar c}{\pi R^4}\left\{-\left[-\frac{d^2 I_c}{d\Delta\tau_-^2}\right]\left(\frac{\alpha}{\beta}\right)^3\frac{\alpha^2}{(1+\alpha^2)^{3/2}} - \frac{dI_c}{d\Delta\tau_-}\left(\frac{\alpha}{\beta}\right)^2\frac{(2+\alpha^2+2\alpha^4)}{(1+\alpha^2)^2} + I_c\frac{\alpha}{\beta}\frac{(2+5\alpha^2)}{(1+\alpha^2)^{5/2}}\right\} \\
&= \int_0^\infty d\Omega\cos(\Omega\tau_e)\left[\frac{-\hbar}{\frac{e^2}{m}\Omega}\coth\left(\frac{\pi c\Omega}{a}\right)\right] \times \qquad (C40) \\
&\qquad \times\, \mathrm{Im}\left\{-\frac{e^2}{m}K^3\left(\frac{R}{R_-}\right)^2\left[\frac{-\alpha^2}{KR_-} - \frac{i(2+\alpha^2+2\alpha^4)}{(KR_-)^2} + \frac{(2+5\alpha^2)}{(KR_-)^3}\right]\right\} \quad .
\end{aligned}$$

Hence, Eqs. (C1) and (C3) have been verified for the i=j=2 case, as may be seen by comparing Eq. (C40) with Eqs. (C6) and (A40).

Consider now the i=1, j=2 situation for Eqs. (C1) and (C3). As may be readily verified, the i=2, j=1 case differs from the former case by simply a minus sign. The same steps followed in earlier calulations will result in the first line indicated below. This expression may be rewritten by the means of the second line. A single integration by parts then results in the third line.

$$\langle E^{2P}_{\tau_{e0}1}(\hat{y}R_o,\tau_{e0})E^{2P}_{(\tau_{e0}+\tau_e)2}(\hat{y}(R_o\pm R),\tau_{e0}+\tau_e)\rangle$$
$$=\pm\frac{\hbar c\alpha b}{\pi R^4}\int_0^\infty dw\,\sin(wb)\left[w^2\cos w-w\sin w\right]$$
$$=\pm\frac{\hbar c\alpha}{\pi R^4}\lim_{\epsilon\to 0_+}\int_0^\infty dw\,e^{-\epsilon w}\left[-\frac{d}{dw}\cos(wb)\right]\left[w^2\cos w-w\sin w\right]$$
$$=\mp\frac{\hbar c\alpha}{\pi R^4}\int_0^\infty dw\cos(wb)\left(w^2\sin w-w\cos w+\sin w\right)\quad . \qquad (C41)$$

Substituting Eqs. (C19)-(C21), collecting terms, and then comparing the expressions with Eqs. (C6) and (A42), enables one to verify Eqs. (C1) and (C3) for i=1, j=2.

The remaining correlation functions of Eq. (C1) consist of the situations where $i\neq j$ and either i=3 or j=3. Following the steps employed for the earlier correlation functions enables one to show that the expressions for these examples have integrands that are odd in K_y. Hence, these correlation functions are identically equal to zero, which agrees with the result of Eq. (A43).

Turning now to the verification of Eqs. (C2) and (C4), a number of additional identities will be introduced in order to aid in this task. In analogy with the definition for I_c, let I_s be defined as the integral listed below. Three identities can then be established that involve I_s, $\left[-\frac{dI_s}{d\Delta\tau_-}\right]$, and $\left[-\frac{d^2I_s}{d\Delta\tau_-^2}\right]$:

$$I_s \equiv \int_0^\infty \coth\left(\frac{\pi\Omega}{2\beta}\right)\cos(\Omega\Delta\tau_-)\sin(\Omega\tau_e)\,d\Omega = -\left(\frac{\beta}{\alpha}\right)\frac{b\cosh(\beta\tau_e)}{(1-b^2)} \quad , \qquad (C42)$$

$$\left[-\frac{dI_s}{d\Delta\tau_-}\right] = \int_0^\infty \coth\left(\frac{\pi\Omega}{2\beta}\right)\sin(\Omega\Delta\tau_-)\sin(\Omega\tau_e)\Omega\,d\Omega$$
$$= -2\left(\frac{\beta}{\alpha}\right)^2(1+\alpha^2)^{1/2}\left\{\frac{1}{(1-b^2)} + \frac{b^2}{(1-b^2)^2}\right\}b\cosh(\beta\tau_e) \quad , \qquad (C43)$$

$$\left[-\frac{d^2I_s}{d\Delta\tau_-^2}\right] = \int_0^\infty \coth\left(\frac{\pi\Omega}{2\beta}\right)\cos(\Omega\Delta\tau_-)\sin(\Omega\tau_e)\Omega^2\,d\Omega$$
$$= 2\left(\frac{\beta}{\alpha}\right)^2\left\{\frac{(3+2\alpha^2)}{(1-b^2)} + \frac{(7+6\alpha^2)}{(1-b^2)^2}b^2 + \frac{4(1+\alpha^2)b^4}{(1-b^2)^3}\right\}b\cosh(\beta\tau_e) \quad . \qquad (C44)$$

The first identity may be obtained by reexpressing I_s in the same manner that I_c was rewritten in Eq. (C12), and then applying Eq. (C13). These steps result in

$$I_s = \frac{\beta}{2}\left\{\coth\left(\beta(\tau_e+\Delta\tau_-)\right) + \coth\left(\beta(\tau_e-\Delta\tau_-)\right)\right\} \quad . \qquad (C45)$$

Substituting Eqs. (C7c) and (C8) into Eq. (C45) then yields Eq. (C42). The first and second differentiations of Eq. (C45) with respect to $\Delta\tau_-$, followed by appropriate substitutions of Eqs. (C7c) and (C8), will then yield Eqs. (C43) and (C44).

In analogy with Eqs. (C19)-(C21), three more useful relationships are:

$$\int_0^\infty dw \sin(wb)\cos(w) = \frac{-b}{(1-b^2)} \quad , \qquad (C46)$$

$$\int_0^\infty dw\cdot w\sin(wb)\sin(w) = -2b\left\{\frac{1}{(1-b^2)} + \frac{b^2}{(1-b^2)^2}\right\} \quad , \qquad (C47)$$

$$\int_0^\infty dw\cdot w^2\sin(wb)\cos(w) = 2b\left\{\frac{3}{(1-b^2)} + \frac{7b^2}{(1-b^2)^2} + \frac{4b^4}{(1-b^2)^3}\right\} \quad . \qquad (C48)$$

Equation (C46) may be obtained by setting $b'=1$ and $c'=b$ in Eq. (C18). Equations (C47) and (C48) may be obtained by repeated differentiations with respect to b' and then setting $b'=1$ and $c'=b$.

Combining Eqs. (C42)-(C44) and (C46)-(C48) yields:

$$\cosh(\beta\tau_e)\int_0^\infty dw \sin(wb)\cos(w) = \frac{\alpha}{\beta} I_s \qquad , \qquad (C49)$$

$$\cosh(\beta\tau_e)\int_0^\infty dw \cdot w\sin(wb)\sin(w) = \left(\frac{\alpha}{\beta}\right)^2 \frac{1}{(1+\alpha^2)^{1/2}}\left[-\frac{dI_s}{d\Delta\tau_-}\right] \qquad , \qquad (C50)$$

$$\cosh(\beta\tau_e)\int_0^\infty dw \cdot w^2\sin(wb)\cos(w)$$
$$= \frac{-\alpha^2\left(\frac{\alpha}{\beta}\right)^2}{(1+\alpha^2)^{3/2}}\left[-\frac{dI_s}{d\Delta\tau_-}\right] + \frac{\left(\frac{\alpha}{\beta}\right)^3}{(1+\alpha^2)}\left[-\frac{d^2 I_s}{d\Delta\tau_-^2}\right] \qquad . \qquad (C51)$$

Turning back to Eqs. (C2) and (C4), consider first the correlation function for $i=1$, $j=3$. Following the same steps used in the evaluation of the correlation functions previously considered, along with the identity

$$\gamma_{(\tau_{e_0}+\tau_e)}\left(K' - \beta_{(\tau_{e_0}+\tau_e)} K_1'\right) = \cosh\left(\frac{a\tau_e}{2c}\right) K - \sinh\left(\frac{a\tau_e}{2c}\right) K_1 \qquad , \qquad (C52)$$

where $\vec{K}'$ is related to $\vec{K}$ via Eqs. (C25a)-(C25d), one can obtain the result of

$$\left\langle B^{zp}_{\tau_{e_0}1}(\hat{y}R_0, \tau_{e_0}) E^{zp}_{(\tau_{e_0}+\tau_e)3}\left(\hat{y}(R_0 \pm R), \tau_{e_0}+\tau_e\right)\right\rangle$$
$$= \mp\frac{\hbar c}{\pi R^4}\cosh(\beta\tau_e)\int_0^\infty dw \sin(wb)\left\{w^2\cos w - w\sin w\right\} \qquad . \qquad (C53)$$

With the use of Eqs. (C50) and (C51), Eqs. (C2) and (C4) can then be verified for $i=1, j=3$ [see Eq. (A44)]. Repeating the same set of operations for the case of $i=3, j=1$ reveals that a minus sign is introduced into the expression of the former correlation function; this result also agrees with Eq. (A44).

Now consider the correlation function of Eq. (C2) when i=2,j=3. Using Eqs. (C25a) and (C25b), the following relationship may be established:

$$\gamma_{(\tau_{e0}+\tau_e)}\gamma_{\tau_e}\left[K'K_1'\left(1+\beta_{\tau_{e0}}\beta_{(\tau_{e0}+\tau_e)}\right)-\beta_{(\tau_{e0}+\tau_e)}K'^2-\beta_{\tau_{e0}}K_1'^2\right]$$

$$= K_1 K + (K_1^2-K^2)\cosh\left(\frac{a\tau_e}{2c}\right)\sinh\left(\frac{a\tau_e}{2c}\right) \quad . \qquad (C54)$$

Following previous steps and using Eq. (C54) results in the expression of

$$\left\langle B^{zp}_{\tau_{e0}2}(\hat{y}R_0,\tau_{e0})E^{zp}_{(\tau_{e0}+\tau_e)3}(\hat{y}(R_0\pm R),\tau_{e0}+\tau_e)\right\rangle$$

$$= -\frac{\hbar c}{\pi R^4}\cosh\left(\frac{a\tau_e}{2c}\right)\alpha b\int_0^\infty dw\cos(wb)\left[w^2\sin w+3w\cos w-3\sin w\right]$$

$$= +\frac{\hbar c}{\pi R^4}\alpha\cosh\left(\frac{a\tau_e}{2c}\right)\int_0^\infty dw\sin(wb)\left[w^2\cos w-w\sin w\right] \quad , \qquad (C55)$$

where the second line above was obtained by integrating by parts once and using a temporary cutoff. Using Eqs. (C49)-(C51) in Eq. (C55) then enables Eqs. (C2) and (C4) to be verified when i=2,j=3 [see Eq. (A45)]. Repeating the same set of operations for the case of i=3,j=2 results in an identical expression for the corresponding correlation function; this result agrees with Eq. (A45).

All the other correlation functions of Eq. (C2) may be shown to equal zero, either because of Eq. (23) in the i=j=1 case, or because the integrands are odd in K_3 , thereby resulting in a value of zero when the angular integrations are carried out. This result agrees with Eq. (A46). Hence, the verification of Eqs. (C2) and (C4) has been completed.

From Eqs. (C1)–(C4) and (A39)–(A46), certain symmetry relationships may be readily deduced for $f_{ij}^{zp}(\hat{x}a,\hat{y}R,\Omega)$ and $g_{ij}^{zp}(\hat{x}a,\hat{y}R,\Omega)$. A symmetry property that was used in Sec. IIIB consists of

$$f_{ii}^{zp}(\hat{x}a,+\hat{y}R,\Omega) = f_{ii}^{zp}(\hat{x}a,-\hat{y}R,\Omega) \quad , \qquad (C56)$$

which follows immediately from Eqs. (C1), (C3), and (A39)–(A41). Likewise, when the i,j indices of Eq. (C1) are not equal to each other, then from Eqs. (A42) and (A43), f_{ij}^{zp} is an odd function of R. With regard to τ_e, the correlation functions of Eqs. (C1) and (C2) may immediately be deduced to be even and odd functions of τ_e, respectively, because of the cosine and sine expansions.

Two final relationships, namely, Eqs. (C57) and (C60), will be derived in this section, as these identities are required for the calculations of Sect. IIIB. As will be shown,

$$\left\langle E_{\tau_e' i}^{zp}(\hat{y}R',\tau_e')\frac{\partial}{\partial\xi_2}E_{\tau_e'' j}^{zp}(\vec{\xi},\tau_e'')\Big|_{\vec{\xi}=\hat{y}R''}\right\rangle$$
$$= \int_0^\infty d\Omega \cos\left(\Omega(\tau_e''-\tau_e')\right)\frac{\partial}{\partial\Delta R}f_{ij}^{zp}(\hat{x}a,\hat{y}\Delta R,\Omega)\Big|_{\Delta R=R''-R'} \quad . \qquad (C57)$$

By using Eq. (C1), Eq. (C57) can be verified almost immediately; nevertheless, the following demonstration will be given, as it is useful in establishing the validity of Eq. (C60). First, consider the i=j=1 case. Using the identity of

$$\left\langle \cos\left(A+\theta(\vec{K}',\lambda')\right)\sin\left(B+\theta(\vec{K}'',\lambda'')\right)\right\rangle$$

$$= \frac{1}{2}\,\delta_{\lambda',\lambda''}\,\delta^3(\vec{K}''-\vec{K}')\sin(B-A) \quad , \qquad (C58)$$

along with Eqs. (C25a)–(C25d) and (C26), then the following steps may readily be followed:

$$\left\langle E^{zp}_{\tau_e' 1}(\hat{y}R',\tau_e')\frac{\partial}{\partial\xi_2}E^{zp}_{\tau_e'' 1}(\vec{\xi},\tau_e'')\Big|_{\vec{\xi}=\hat{y}R''}\right\rangle = \sum_{\lambda'}\sum_{\lambda''}\int d^3K'\int d^3K''\,h(\Omega')h(\Omega'')\epsilon_x'\epsilon_x'' \times$$

$$\times\left\langle \cos\left(K_1'\frac{c^2}{a}\cosh\left(\frac{a\tau_e'}{c}\right)+K_2'R'-K_1'\frac{c^2}{a}\sinh\left(\frac{a\tau_e'}{c}\right)+\theta'\right)\times\right.$$

$$\left.\times\frac{\partial}{\partial\xi_2}\cos\left(K_1''\left(\frac{c^2}{a}+\xi_1\right)\cosh\left(\frac{a\tau_e''}{c}\right)+K_2''\xi_2+K_3''\xi_3-K''\left(\frac{c^2}{a}+\xi_1\right)\sinh\left(\frac{a\tau_e''}{c}\right)+\theta''\right)\Big|_{\vec{\xi}=\hat{y}R''}\right\rangle$$

$$= -\frac{1}{2}\sum_{\lambda'}\int d^3K'\,h^2(\Omega')\epsilon_x'^2 K_2\sin\left(-\frac{2c^2}{a}K\sinh\left(\frac{a\tau_e}{2c}\right)+K_2(R''-R')\right)$$

$$= \frac{\partial}{\partial\Delta R}\sum_{\lambda'}\int d^3K'\,h^2(\Omega')\epsilon_x'^2\cos\left(-\frac{2c^2}{a}K\sinh\left(\frac{a\tau_e}{2c}\right)+K_2\Delta R\right)\Big|_{\Delta R=R''-R'} \quad . \qquad (C59)$$

From the earlier calculations involving the i=j=1 case of Eq. (C1), one can see that Eq. (C59) falls under the form of Eq. (C57) for i=j=1. The other situations of Eq. (C57), where i and j do not necessarily equal one, may be proven in a similar way, since the same set of operations above would be followed in all these other cases. The only change in the expressions for the other cases would be that the factor of $(\epsilon_x')^2$ in the last two lines of Eq. (C59) would be replaced by some other factor, which certainly does not change the final relationship one would arrive at of Eq. (C57).

As may be seen in the second to last line of Eq. (C59), an odd factor of K_2 appears in the integrand. As long as i=j, this may be shown to be a common feature for the correlation functions of Eq. (C57). Hence, when i=j and $R''=R'$, then one may prove that

$$\left\langle E^{zp}_{\tau_e' i}(\hat{g}R', \tau_e') \frac{\partial}{\partial \xi_2} E^{zp}_{\tau_e'' i}(\vec{\xi}, \tau_e'')\Big|_{\vec{\xi} = \hat{g}R''} \right\rangle = 0 \quad \text{for} \quad R'' = R', \qquad (C60)$$

since under these conditions, an integrand that is odd in k_2 results, thereby yielding a value of zero after angular integrations are performed.

APPENDIX D: EVALUATION OF <A_2> IN EQ. (90)

The quantity A_2 in Eq. (90) arises from the second term of Eq. (18). By reexpressing A_2 in terms of the ξ^μ coordinates, then the task of obtaining the expectation value of A_2 becomes much easier. Hence, this procedure will now be followed.

Of the three vector components of $\vec{\Delta x}_{\tau_e}(t_{\tau_e})$ that appear in the second term of Eq. (18), $\Delta y_{\tau_e}(t_{\tau_e})$ and $\Delta z_{\tau_e}(t_{\tau_e})$ may be immediately replaced by $\xi_{A2}(\tau_e')$ and $\xi_{A3}(\tau_e'')$. The relationship between t_{τ_e} and τ_e' is given by

$$ct_{\tau_e}(\tau_e') = \left(\xi_{A1}(\tau_e') + \frac{c^2}{a}\right)\sinh\left(\frac{a}{c}(\tau_e' - \tau_e)\right) \quad . \qquad (D1)$$

The quantity $\Delta x_{\tau_e}(t_{\tau_e})$ is equal to the x position of the oscillating particle minus the x position of the equilibrium point of the oscillator, where both quantities are evaluated at time $t_{\tau_e}(\tau_e')$. The x position of the A oscillating particle is given by

$$x_{A\tau_e}(t_{\tau_e}) = \left(\xi_{A1}(\tau_e') + \frac{c^2}{a}\right)\cosh\left(\frac{a}{c}(\tau_e' - \tau_e)\right) \quad , \qquad (D2)$$

where τ_e' is again related to t_{τ_e} by Eq. (D1). The x position of the equilibrium point of the A oscillator at time

$t_{\tau_e} = T_{\tau_e}$ is given by

$$X_{A\tau_e}(\tau_e'') = \frac{c^2}{a}\cosh\left(\frac{a}{c}(\tau_e''-\tau_e)\right) \quad , \qquad (D3)$$

where τ_e'' is related to the time coordinate T_{τ_e} of the equilibrium point by

$$cT_{\tau_e}(\tau_e'') = \frac{c^2}{a}\sinh\left(\frac{a}{c}(\tau_e''-\tau_e)\right) \quad . \qquad (D4)$$

As illustrated in Fig. 1, the value of $T_{\tau_e}(\tau_e'')$ in Eq. (D4) should be set equal to the value of $t_{\tau_e}(\tau_e')$ in Eq. (D1). Hence, τ_e' and τ_e'' are related to each other by

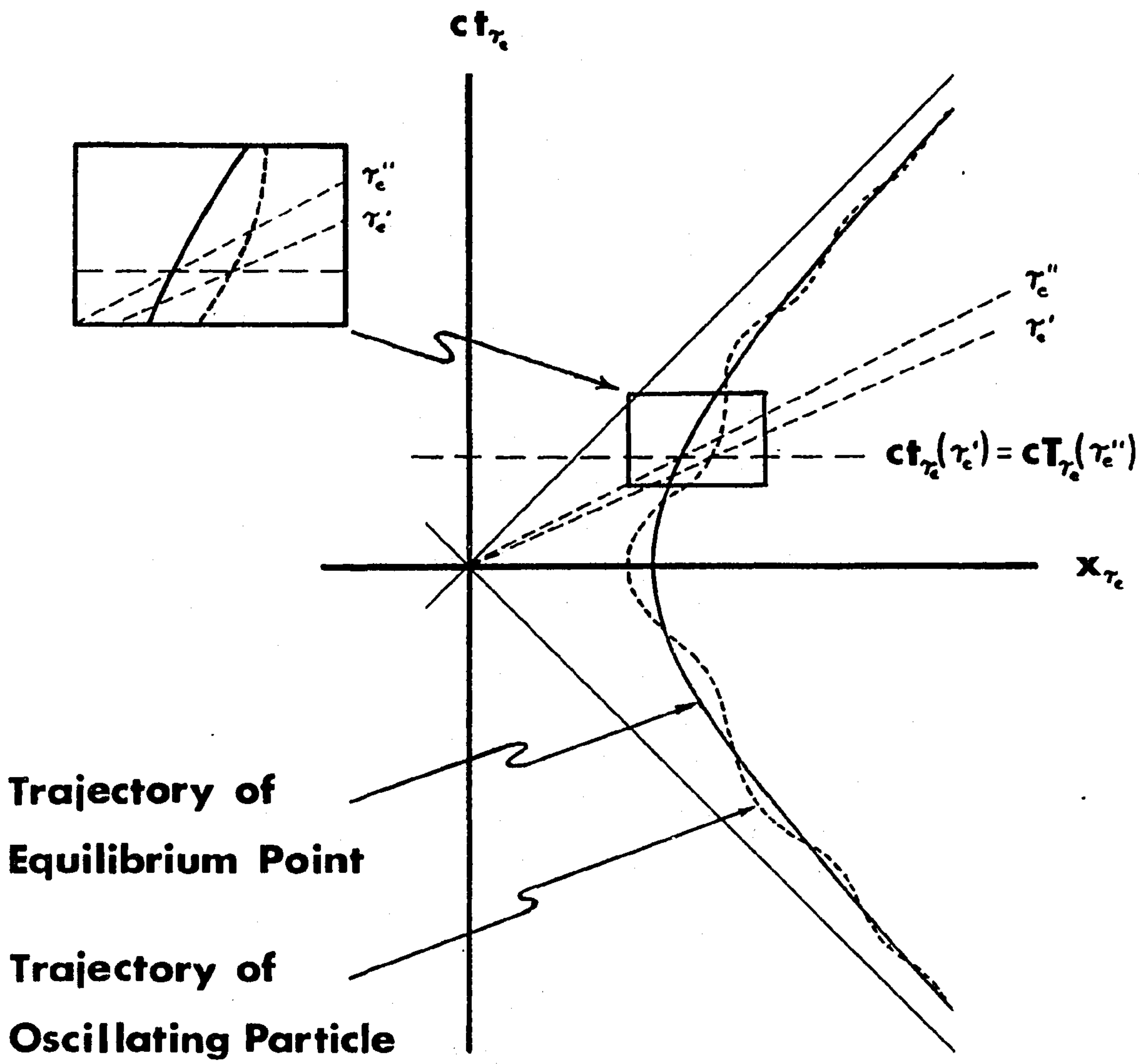
$$\frac{c^2}{a}\sinh\left(\frac{a}{c}(\tau_e''-\tau_e')\right) = \left(\xi_{A1}(\tau_e') + \frac{c^2}{a}\right)\sinh\left(\frac{a}{c}(\tau_e'-\tau_e)\right) \quad . \qquad (D5)$$

From Eqs. (D3) and (D5), $X_{A\tau_e}$ can be reexpressed in terms of the value of τ_e' that occurs in Eq. (D1). Hence,

$$\begin{aligned} \Delta x_{A\tau_e}(t_{\tau_e}) &= \left(x_{A\tau_e} - X_{A\tau_e}\right)\Big|_{t_{\tau_e}} \\ &= \left(\xi_{A1}(\tau_e') + \frac{c^2}{a}\right)\cosh\left(\frac{a}{c}(\tau_e'-\tau_e)\right) \\ &\qquad - \frac{c^2}{a}\left[1 + \left(1 + \frac{a\xi_{A1}(\tau_e')}{c^2}\right)^2\sinh^2\left(\frac{a}{c}(\tau_e'-\tau_e)\right)\right]^{1/2} \\ &\approx \frac{\xi_{A1}(\tau_e')}{\cosh\left(\frac{a}{c}(\tau_e'-\tau_e)\right)}\left(1 + O\left(\frac{a\xi_{A1}}{c^2}\right)\right) \quad , \qquad (D6) \end{aligned}$$

where the last line above follows from the previous line by the assumption that $\left(\frac{a\xi_{A1}}{c^2}\right) \ll 1$ and by then appropriately expanding the quantity under the square root sign. Thus, the three components of $\overrightarrow{\Delta x}_{\tau_e}(t_{\tau_e})$ have now been reexpressed

FIGURE 1


ct_{τ_e}
τ_e''
τ_e'
τ_e''
τ_e'
$ct_{\tau_e}(\tau_e') = cT_{\tau_e}(\tau_e'')$
x_{τ_e}
Trajectory of Equilibrium Point
Trajectory of Oscillating Particle

in terms of the coordinates $\xi_{Ai}(\tau_e')$.

The distinction between τ_e' of Eq. (D1) and τ_e' of Eq. (D4), where τ_e' and τ_e'' are related to each other by Eq. (D5), may now be ignored when reexpressing the remaining quantities in the second term of Eq. (18) in terms of the ξ_{Ai} coordinates. The reason for this is that τ_e' and τ_e'' differ from each other only by terms of order $O\left(\frac{a\xi_{A1}}{c^2}\right)$, and $\Delta\vec{X}_{A\tau_e}$ has already been shown to be of first order in $\vec{\xi}_A$.

Consequently, from Eq. (D1), the following time derivative occurring in Eq. (18) may be written as

$$\frac{d}{dt_{\tau_e}}\left(\qquad \right)\Bigg|_{t_{\tau_e}=0} = \frac{d\tau_e'}{dt_{\tau_e}}\Bigg|_{\tau_e'=\tau_e} \frac{d}{d\tau_e'}\left(\qquad \right)\Bigg|_{\tau_e'=\tau_e} , \qquad (D7)$$

where
$$\frac{d\tau_e'}{dt_{\tau_e}}\Bigg|_{\tau_e'=\tau_e} = \frac{1}{\left(1+\frac{a\xi_{A1}(\tau_e)}{c^2}\right)} = 1 + O\left(\frac{a\xi_{A1}}{c^2}\right) . \qquad (D8)$$

Following the above comment, the second term of Eq. (D8) may then be ignored.

Making the substitutions that have been indicated so far in the expression for A_2 from Eq. (18), then yields

$$A_2 = \frac{e}{c}\left\{1 + O\left(\frac{a\xi_{A1}}{c^2}\right)\right\} \times$$
$$\times \frac{d}{d\tau_e'}\left(\xi_{A3}(\tau_e')B_{\tau_e 1}\left(\frac{R}{2}\hat{y},\tau_e'\right) - \frac{\xi_{A1}(\tau_e')}{\cosh\left(\frac{a}{c}(\tau_e'-\tau_e)\right)} B_{\tau_e 3}\left(\hat{y}\frac{R}{2},\tau_e'\right)\right)\Bigg|_{\tau_e'=\tau_e} , \qquad (D9)$$

where the arguments in the fields have again been relabeled by the ξ^μ coordinates, as was done in the transition from Eq. (18) to Eq. (28). The field $\vec{B}_{\tau_e}(\hat{y}\frac{R}{2},\tau_e')$ in Eq. (D9) represents the sum of the zero-point magnetic field plus the magnetic field due to oscillator B, where both are evaluated in the I_{τ_e} frame at proper time τ_e' along the trajectory of the equilibrium point of the A oscillator.

The fields $B_{\tau_e i}$ in Eq. (D9) may be reexpressed in terms of the fields of the $I_{\tau_e'}$ frame via the Lorentz transformations of Eqs. (7) and (8). Making these substitutions, differentiating all the obvious terms of $\cosh\left(\frac{a}{c}(\tau_e'-\tau_e)\right)$ and $\sinh\left(\frac{a}{c}(\tau_e'-\tau_e)\right)$, and then evaluating these terms at $\tau_e'=\tau_e$, yields

$$A_2 \approx \frac{e}{c}\frac{d}{d\tau_e'}\left(\xi_{A3}(\tau_e')B_{\tau_e'1}(\hat{y}\tfrac{R}{2},\tau_e') - \xi_{A1}(\tau_e')B_{\tau_e'3}(\hat{y}\tfrac{R}{2},\tau_e')\right)\Big|_{\tau_e'=\tau_e}$$

$$+\left[\frac{a\,\xi_{A1}(\tau_e)}{c^2}\right]e\,E_{\tau_e 2}(\hat{y}\tfrac{R}{2},\tau_e) \qquad , \qquad\qquad (D10)$$

where the terms of order $\mathcal{O}(a\xi_{A1}/c^2)$ from Eq. (D9) have been ignored.

When the expectation value of the expression for A_2 in Eq. (D10) is taken, then the same argument employed in Sec. IIIB on the zero value of the third term of Eq. (28) may again be used here in conjunction with the first two terms of Eq. (D10) [see the brief discussion immediately following Eq. (38)]. Consequently,

$$\langle A_2\rangle \approx \frac{ea}{c^2}\left\langle \xi_{A1}(\tau_e)E^{ZP}_{\tau_e 2}(\hat{y}\tfrac{R}{2},\tau_e)\right\rangle$$

$$+\frac{ea}{c^2}\left\langle \xi_{A1}(\tau_e)E^{DB}_{\tau_e 2}(\hat{y}\tfrac{R}{2},\tau_e)\right\rangle \quad . \qquad\qquad (D11)$$

The above two terms may be evaluated by following the steps of Sec. IIIB and by making use of the required relationships in Appendices A and C. In keeping with the aim of this article, the value of these two terms are of interest here when the resonant approximation is employed in the case of the unretarded van der Waal's situation.

If the terms of order $\mathcal{O}(aR/c^2)$ in Eqs. (89a) and (89b) are dropped as an initial approximation, then the first term of Eq. (D11) can be shown to be negligible compared to the second one. This demonstration follows exactly the same argument that was used to show Eq. (33) was negligible compared to Eq. (38) when the resonant approximation was employed in the unretarded van der Waals situation.

Under these same conditions, the evaluation of the second term of Eq. (D11) also proceeds along steps presented in Sec. IIIB. Again, if terms of order $\mathcal{O}(aR/c^2)$ in Eq. (89a) and (89b) are ignored as an initial approximation, then one can show that the second term of Eq. (D11) is approximately $(aR/c^2)^2$ times the value of the second term of Eq. (90), when the latter is evaluated under similar conditions. The factor of $(aR/c^2)^2$ arises essentially from the ratio of

$$\frac{\frac{a}{c^2} \mathcal{R}_e \, n^a_{21}}{\frac{\partial}{\partial R} \mathcal{R}_e \, n^a_{ii}} \approx \frac{\frac{a}{c^2}\left(\frac{aR}{c^2}\right) \mathcal{R}_e \, n_{ii}}{\frac{1}{R} \mathcal{R}_e \, n_{ii}} = \left(\frac{aR}{c^2}\right)^2 .$$

If the initial approximation of ignoring terms of order $\mathcal{O}\left(\frac{aR}{c^2}\right)$ is reexamined and these terms are retained, then one can trace through the calculations indicated above to see that the contribution of these terms will also yield a final result characterized by the factor of $\left(\frac{aR}{c^2}\right)^2$.

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9) T. H. Boyer, Phys. Rev. A <u>7</u>, 1832 (1973).

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11) The factor of $-m/e^2$ in Eqs. (11) and (12) is introduced here to reduce the number of constants that would otherwise be carried along in equations to be solved shortly. Strictly speaking, of course, the electromagnetic fields of an electric dipole should only depend upon the electric dipole and not upon the factor $-m/e^2$, which involves a mass m and charge e that are extraneous to the electric dipole itself. This apparent difficulty is immediately resolved upon examining Eqs. (A39)–(A45), which define n_{ij} and ρ_{ij} ; here, an extra factor of $-e^2/m$ was introduced that cancels with the factor of $-m/e^2$ in Eqs. (11) and (12). By defining the quantities $n^D_{ij} = -\frac{m}{e^2} n_{ij}$ and $\rho^D_{ij} = -\frac{m}{e^2} \rho_{ij}$, which are independent of $-m/e^2$, then Eqs. (11) and (12) may be rewritten without the factor of $-m/e^2$ appearing.

12) Boyer first used this functional form for the zero-point fields in his initial paper on stochastic electrodynamics: T. H. Boyer, Phys. Rev. <u>186</u>, 1304 (1969).

13) M. J. Renne, Physica <u>53</u>, 193 (1971).

14) Since Eqs. (63) and (65)-(66) are equivalent, then Eq. (63) also holds in the case of the unaccelerated-thermal case when η^{a}_{jj} and ζ^{a}_{j} are replaced by their a->0 counterparts of Eqs. (49) and (67). See Eq. (14) of Ref. 10.

15) See, for example, Ref. 7.

16) See the short discussion accompanying Eq. (36) in the article by T. H. Boyer in Phys. Rev. A <u>18</u>, 1228 (1978).

17) Reference 7 describes how to handle higher order quantities involving the random phases. Also see, for example, T. H. Boyer, Phys. Rev. D <u>13</u>, 2832 (1976), Eqs. (56) and (57).

18) See, for example, J. D. Jackson, <u>Classical Electrodynamics</u>, 2nd ed. (Wiley, New York, 1975), p. 657, Eq. (14.14).

19) As explained in Ref. (11), a factor of $-\frac{m}{e^2}$ was introduced in Eqs. (A37) and (A38) in order to simplify other calculations throughout this article. This factor cancels with the factor of $-e^2/m$ in Eqs. (A39)-(A46), which define η_{ij} and ρ_{ij} . Hence, the electric dipole fields of Eqs. (A37) and (A38) are, in fact, independent of this factor.

20) Again, as explained in Refs. (11) and (19), the factor of $-e^2/m$ appears in Eqs. (C3) and (C4) only because of the way in which η_{ij} and ρ_{ij} were defined. Substituting in the quantities $\eta^{D}_{ij} = -\frac{m}{e^2}\eta_{ij}$ and $\rho^{D}_{ij} = -\frac{m}{e^2}\rho_{ij}$, which are independent of $-m/e^2$, will eliminate this factor from Eqs. (C3) and (C4).

21) Similar relationships between the two-point field correlation functions and the electromagnetic fields of a nonaccelerating electric dipole are discussed elsewhere: see, D. C. Cole, "Correlation Functions for Homogeneous, Isotropic Random Classical Electromagnetic Radiation and the Electromagnetic Fields of a Fluctuating Classical Electric Dipole," to be published in Phys. Rev. A. This article appears as Part III in this thesis.

22) See Eqs. (1), (2), and (91) of Ref. 7.

23) I. S. Gradshteyn and I. M. Ryzhik, <u>Tables of Integrals, Series, and Products</u> (Academic, New York, 1980), p. 504, No 8. A small algebraic manipulation results in Eq. (C13).

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25) Although not explicitly stated in Eqs. (C15)-(C17), (C35), (C36), and (C37), the meaning of these integrals should be remembered to be given by the operations of first inserting an effective cutoff into the integral, then evaluating the integral, and finally taking an appropriate limit to remove the cutoff.

PART THREE

CORRELATION FUNCTIONS FOR HOMOGENEOUS, ISOTROPIC RANDOM CLASSICAL ELECTROMAGNETIC RADIATION AND THE ELECTROMAGNETIC FIELDS OF A FLUCTUATING CLASSICAL ELECTRIC DIPOLE

I. INTRODUCTION

A number of calculations have been performed within the context of classical electrodynamics that agree with the results of quantum electrodynamics. This agreement occurs provided a nonzero homogeneous solution to Maxwell's equations is assumed to exist in the form of random electromagnetic radiation that is present even when the temperature of the radiation equals zero. Assuming the stochastic properties of this classical electromagnetic zero-point radiation to be that of a Gaussian process in the fields, then the demands of isotropy, homogeneity, and Lorentz invariance result in the functional form of the radiation's spectrum being uniquely specified up to a multiplicative constant. Comparison with experiment then yields the numerical value for this multiplicative constant, which is found to agree with Planck's constant. Hence, it is in this manner that Planck's constant enters into this classical electrodynamical theory. The name frequently given to this classical theory is stochastic electrodynamics. (For reviews on this field of research, see Refs. 1, 2, and 3.)

The electrodynamical system that has received the most attention within stochastic electrodynamics has been the charged harmonic oscillator. The equation of motion for this system is a linear stochastic differential equation; hence, the steady state solution is readily obtained by the use of Fourier transforms. Although the physical interpretation for the behavior of such a system is markedly different from that of quantum electrodynamics, most of the physically observable statistical properties of this system agree between the two theories of stochastic and quantum electrodynamics. Perhaps the most remarkable agreement has been found in the case of the van der Waals force between two nonrelativistic charged harmonic oscillators, each taken in the electric dipole limit; the force expressions calculated within both theories at temperature T=0 agree for all distances between the two oscillators and to all orders in the fine structure constant (\#4).

Other electrodynamical systems, such as the classical hydrogen atom, have not been so successfully tackled within stochastic electrodynamics; in most cases, the results obtained for these systems have not agreed with physical observation. Such systems are described by nonlinear stochastic differential equations. It remains unclear whether the basic theory of stochastic electrodynamics is an incorrect description of nature or whether the difficult mathematics of these nonlinear systems have simply not been solved with sufficient accuracy.

At this point in time, perhaps the most appropriate viewpoint of the theory of stochastic electrodynamics is that, at the very least, it offers alternative means for calculating certain quantities within quantum electrodynamics. In some instances, as in the case of van der Waals forces (#4,5) or of the thermal effects of electromagnetic dipole systems accelerating through the so-called vacuum (#6,7,8,9), there exist calculational advantages of stochastic electrodynamics to the more traditional calculational methods of quantum electrodynamics.

In Section II of this article, the two-point correlation function of the electromagnetic radiation fields in stochastic electrodynamics are evaluated at fixed spatial points within an inertial reference frame. This calculation has certainly been done before; here, however, the correlation functions are shown to be expressible in terms of the electromagnetic fields radiated by a fluctuating electric dipole (#10). The use of this functional form for the correlation functions simplifies many of the calculations that have been performed previously within the context of stochastic electrodynamics; in particular, calculations involving the charged harmonic oscillator, taken in the electric dipole limit, become much more efficient and tractable. The calculations of Secs. III and IV illustrate this point.

Recently, the correlation functions of the classical electromagnetic zero-point fields were calculated along spatially separated trajectories described by relativistic uniform acceleration (#9). These correlation functions were shown to be related to the radiated electromagnetic fields of a uniformly accelerated electric dipole. Hence, this relationship may be viewed as a generalization of the unaccelerated case discussed in the present article. Thus, there exists a second advantage, besides greater simplicity, for recasting previous calculations that have been done in stochastic electrodynamics in such a way that the functional form of the correlation functions in Sec. II are utilized: namely, in many cases, without too much additional work, the calculations for nonaccelerating electromagnetic systems may then be readily generalized to the corresponding situation of electromagnetic systems uniformly accelerating through classical electromagnetic zero-point radiation. Such is the case for the calculations presented in Secs. III and IV.

In Sec. III, the expressions obtained for the correlation functions in Sec. II are used to compute the van der Waals force acting on a single harmonic dipole oscillator that is a member of an arbitrary configuration of N oscillators. This calculation for N oscillators generalizes the work of Refs. 4 and 5 for two oscillators. The case where the temperature of the electromagnetic radiation equals zero has previously been carried out for an N-oscillator system in quantum electrodynamics (see Ref. 11); the results found in the present article, via the means of stochastic electrodynamics, agree exactly with the results of this particular work.

By using the functional form found in Sec. II for the correlation functions of the electromagnetic radiation fields, the force calculation of the N-oscillator system in Sec. II may be generalized to the case of a system of N transversely positioned oscillators that are uniformly accelerated through classical electromagnetic zero-point radiation. This generalization may be carried out by combining the work of Ref. 9, which calculates the expectation value of the force between a pair of transversely positioned accelerating charged oscillators, along with the calculation of Sec. III, which gives the expectation value of the force acting on one charged oscillator of an N-oscillator nonaccelerating system.

Section IV of this article repeats a calculation presented in Appendix B of Ref. 1 that proves the expectation value of the Poynting vector equals zero at a point in space near a stationary harmonic dipole oscillator. The difference between the proof given in Ref. 1 and the proof presented here, however, is that the proof of the present article uses the functional form obtained in Sec. II for the correlation functions of the classical electromagnetic radiation fields. Explicitly using this functional form demonstrates that the reason the proof can be carried out is precisely because of the relationships found in Sec. II between the correlation functions of electromagnetic radiation and the electromagnetic fields of a fluctuating electric dipole. The original proof did not identify this fact. Moreover, the explicit use of these relationships then enables this proof to be extended to the case of a single accelerating charged harmonic oscillator. This generalization is briefly discussed in Sec. IV and sketched more fully in an appendix.

II. CORRELATION FUNCTIONS OF CLASSICAL ELECTROMAGNETIC RADIATION FIELDS

The functional form for the electromagnetic radiation fields that is often used for performing explicit calculations within stochastic electrodynamics consist of the following expressions (*12):

$$\vec{E}^{in}(\vec{x},t) = \sum_{\lambda=1}^{2} \int d^3k\, \mathcal{h}_{in}(\omega)\hat{\epsilon}(\vec{k},\lambda)\cos\left(\vec{k}\cdot\vec{x}-\omega t+\theta(\vec{k},\lambda)\right) \quad , \qquad (1)$$

$$\vec{B}^{in}(\vec{x},t) = \sum_{\lambda=1}^{2} \int d^3k\, \mathcal{h}_{in}(\omega)\left(\hat{k}\otimes\hat{\epsilon}(\vec{k},\lambda)\right)\cos\left(\vec{k}\cdot\vec{x}-\omega t+\theta(\vec{k},\lambda)\right) \quad . \qquad (2)$$

Thus, the radiation fields are expressed here as a sum of plane waves; hence, they satisfy Maxwell's equations in free space. The phase angle $\theta(\vec{k},\lambda)$ is treated here as a random variable that takes on values between 0 and 2π with uniform probability density. For each value of $\vec{k}$ and λ, $\theta(\vec{k},\lambda)$ is independently distributed. The polarization vectors satisfy the relationships of

$$\hat{\epsilon}(\vec{k},\lambda)\cdot\hat{\epsilon}(\vec{k},\lambda') = \delta_{\lambda\lambda'} \quad , \qquad (3)$$

$$\vec{k}\cdot\hat{\epsilon}(\vec{k},\lambda) = 0 \quad , \qquad (4)$$

which can be used to show that

$$\sum_{\lambda=1}^{2} \epsilon_i(\vec{k},\lambda)\epsilon_j(\vec{k},\lambda) = \sum_{\lambda=1}^{2} \left(\hat{k}\otimes\hat{\epsilon}(\vec{k},\lambda)\right)_i \left(\hat{k}\otimes\hat{\epsilon}(\vec{k},\lambda)\right)_j = \delta_{ij} - \frac{k_i k_j}{k^2} \quad , \tag{5}$$

$$\sum_{\lambda=1}^{2} \epsilon_i(\vec{k},\lambda)\left(\hat{k}\otimes\hat{\epsilon}(\vec{k},\lambda)\right)_j = \sum_{\ell=1}^{3} \epsilon_{ij\ell}\frac{k_\ell}{k} \quad . \tag{6}$$

The frequency ω in Eqs. (1) and (2) is defined by $\omega = c|\vec{k}|$. When the electromagnetic fields of Eqs. (1) and (2) belong to a thermal radiation field described by the temperature T, then the quantity $h_{in}(\omega)$ will be denoted by $h_T(\omega)$, where

$$h_T^2(\omega) = \frac{\hbar\omega}{2\pi^2}\coth\left(\frac{\hbar\omega}{2kT}\right) = \frac{1}{\pi^2}\left\{\frac{\hbar\omega}{2} + \frac{\hbar\omega}{\left[\exp\left(\frac{\hbar\omega}{kT}\right)-1\right]}\right\} \quad . \tag{7}$$

When T=0, then Eqs. (1) and (2) constitute the classical electromagnetic zero-point fields. The quantity $h_{T=0}(\omega)$ will be abbreviated by $h(\omega)$. From Eq. (7),

$$h_{T=0}^2(\omega) = h^2(\omega) = \frac{\hbar\omega}{2\pi^2} \quad . \tag{8}$$

The two-point correlation functions of the fields in Eqs. (1) and (2) will now be evaluated. Using the probability distribution described earlier for the random variables $\theta(\vec{k},\lambda)$, one can show that

$$\left\langle \cos\left(A+\theta(\vec{k}',\lambda')\right)\cos\left(B+\theta(\vec{k}'',\lambda'')\right)\right\rangle$$

$$= \frac{1}{2}\delta_{\lambda',\lambda''}\,\delta^3(\vec{k}''-\vec{k}')\cos(B-A) \quad , \tag{9}$$

where the brackets < > are used here to indicate that the expectation value is to be taken for the quantity within the brackets. Using Eqs. (1), (2), (5), and (9), the following two-point correlation functions may be expressed by:

$$\langle E_i^{in}(\vec{x}_1,t_1)E_j^{in}(\vec{x}_2,t_2)\rangle = \langle B_i^{in}(\vec{x}_1,t_1)B_j^{in}(\vec{x}_2,t_2)\rangle$$

$$= \frac{1}{2}\int d^3k'\,\mathcal{A}_{in}^2(\omega')\left[\delta_{ij} - \frac{k_i'k_j'}{k'^2}\right]\cos\left(\vec{k}'\cdot(\vec{x}_2-\vec{x}_1)-\omega'(t_2-t_1)\right) \quad . \qquad (10)$$

The rotation matrix [O] will now be introduced such that

$$\vec{R} = (\vec{x}_2-\vec{x}_1) = [O]\,\hat{z}R \quad , \qquad (11)$$

where $R = |\vec{x}_2-\vec{x}_1|$. Let θ and ϕ be the polar and azimuthal angles of $(\vec{x}_2-\vec{x}_1)$. The transformation transformation of Eq. (11) can be accomplished as shown in Fig. 1, which results in an explicit form for [O] of

$$[O] = \begin{bmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \qquad (12)$$

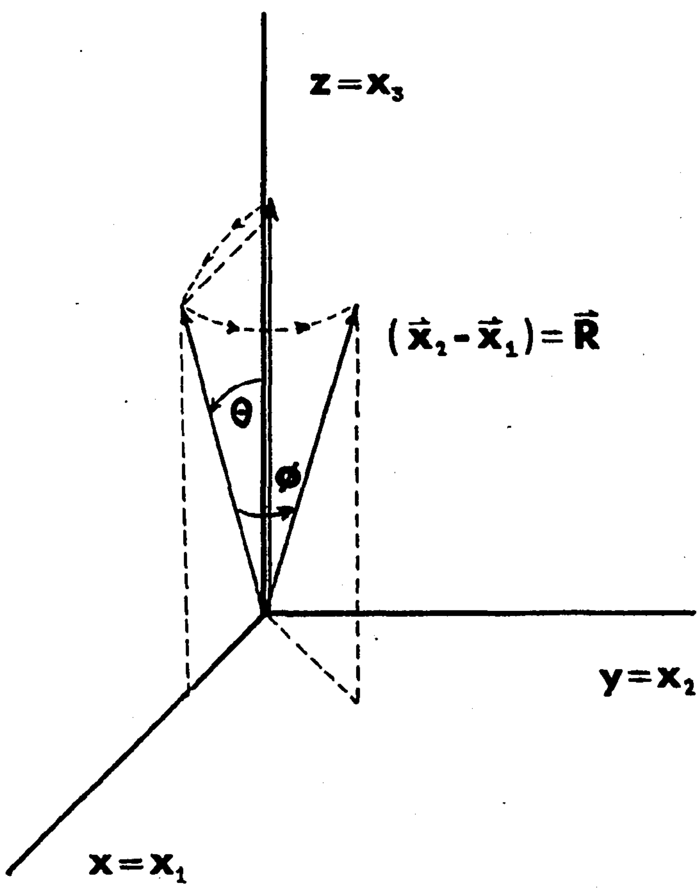
Substituting $\vec{k}' = [O]\vec{k}$ and Eq. (11) into Eq. (10), then yields

$$\langle E_i^{in}(\vec{x}_1,t_1)E_j^{in}(\vec{x}_2,t_2)\rangle$$

$$= \frac{1}{2}\int d^3k\,\mathcal{A}_{in}^2(\omega)\left[\delta_{ij} - \sum_{m,n=1}^{3} O_{im}O_{jm}\frac{k_ik_j}{k^2}\right]\cos\left(k_3R-\omega(t_2-t_1)\right) \quad . \qquad (13)$$

The cosine term in Eq. (13) can be expanded into two terms using the cosine sum of angles formula. Only

FIGURE 1



$\cos(K_3 R)\cos(\omega(t_2-t_1))$ will remain, since $\sin(K_3 R)\sin(\omega(t_2-t_1))$ results in an integrand odd in K_3. Moreover, the second term in brackets in Eq. (13) results in a nonzero contribution only when m=n; otherwise, the integrand is odd either in K_1, K_2, or K_3. Performing angular integrations, substituting in $\sum_{m=1}^{2} \Theta_{im}\Theta_{jm} = \delta_{ij} - \Theta_{i3}\Theta_{j3}$, and realizing from Eq. (12) that $\Theta_{i3} = \frac{R_i}{R}$, yields

$$\langle E_i^{in}(\vec{x}_1,t_1)E_j^{in}(\vec{x}_2,t_2)\rangle = \langle B_i^{in}(\vec{x}_1,t_1)B_j^{in}(\vec{x}_2,t_2)\rangle$$

$$= \int_0^\infty d\omega \cos\left(\omega(t_2-t_1)\right) f_{ij}^{in}(\vec{x}_2-\vec{x}_1,\omega) \quad , (14)$$

where $$f_{ij}^{in}(\vec{R},\omega) = \frac{2\pi \mathcal{R}_{in}^2(\omega)}{\omega} \mathrm{Im}\left(n_{ij}^D(\vec{R},\omega)\right) \quad , (15)$$

$$n_{ij}^D(\vec{R},\omega) = K^3\left[\frac{(\delta_{ij}-R_iR_j/R^2)}{KR} + i\frac{(\delta_{ij}-3R_iR_j/R^2)}{(KR)^2} - \frac{(\delta_{ij}-3R_iR_j/R^2)}{(KR)^3}\right]e^{iKR} . \quad (16)$$

The quantity $n_{ij}^D(\vec{R},\omega)$ is intimately related to the electric field $\vec{E}^{D\alpha}(\vec{R},t)$ of a fluctuating electric dipole $\vec{p}_\alpha(t)$. Here, the superscript D on n_{ij}^D and $\vec{E}^{D\alpha}$ stands for "dipole"; the index α on $\vec{E}^{D\alpha}$ and $\vec{p}_\alpha$ will be used to label a particular electric dipole under discussion. Let $\vec{R}_\alpha$ represent the position of $\vec{p}_\alpha(t)$. From the formulae of standard textbooks (*13), one can readily show that

$$E_i^{D\alpha}(\vec{R},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega \exp(-i\omega t)\left\{\sum_{j=1}^{3} n_{ij}^D(\vec{R}-\vec{R}_\alpha,\omega)\tilde{p}_{\alpha j}(\omega)\right\} \quad , \quad (17)$$

where $$p_{\alpha i}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega \exp(-i\omega t)\tilde{p}_{\alpha i}(\omega) \quad . \quad (18)$$

A similar relationship can be obtained between the magnetic field of a fluctuating electric dipole and the two-point correlation function of the radiation fields listed below. The first line in Eq. (19) follows from Eqs. (1), (2), (6), and (9). The second line may be obtained by again introducing the rotation matrix [O], using the cosine sum of angles formula, and recognizing that $K_m \cos(K_3 R)$ results in an integrand that is odd in K_m.

$$
\begin{aligned}
&\langle B_i^{in}(\vec{x}_1,t_1)E_j^{in}(\vec{x}_2,t_2)\rangle \\
&\quad = \frac{1}{2}\int d^3K' \mathit{h}_{in}^2(\omega')\sum_{\ell=1}^{3}\epsilon_{ji\ell}\frac{K'_\ell}{K'}\cos\left(\vec{K}'\cdot(\vec{x}_2-\vec{x}_1)-\omega'(t_2-t_1)\right) \\
&\quad = \frac{1}{2}\int d^3K \mathit{h}_{in}^2(\omega)\sum_{\ell=1}^{3}\epsilon_{ji\ell}\sum_{m=1}^{3}O_{\ell m}\frac{K_m}{K}\sin(K_3R)\sin\left(\omega(t_2-t_1)\right) \quad .
\end{aligned}
\tag{19}
$$

Only the m=3 term in Eq. (19) yields an integrand that is even in K_3. After performing angular integrations, one obtains

$$
\langle B_i^{in}(\vec{x}_1,t_1)E_j^{in}(\vec{x}_2,t_2)\rangle = \int_0^\infty d\omega \sin\left(\omega(t_2-t_1)\right) g_{ij}^{in}(\vec{x}_2-\vec{x}_1,\omega) \quad , \tag{20}
$$

where
$$
g_{ij}^{in}(\vec{R},\omega) = \frac{2\pi \mathit{h}_{in}^2(\omega)}{\omega}\,\mathrm{Re}\left(\rho_{ji}^{D}(\vec{R},\omega)\right) \quad , \tag{21}
$$

$$
\rho_{ij}^{D}(\vec{R},\omega) = -K^3\sum_{\ell=1}^{3}\epsilon_{ij\ell}\frac{R_\ell}{R}\left[\frac{1}{KR}+\frac{i}{(KR)^2}\right]e^{iKR} \quad . \tag{22}
$$

It should be noted that the order of the i,j indices are reversed on the right and left sides of Eq. (21). This convention then agrees with the results found in Ref. 9, where the two-point field correlation functions were evaluated in a plane uniformly accelerating through classical electromagnetic zero-point radiation.

The quantity $\rho^{D}_{ij}(\vec{R},\omega)$ is intimately related to the magnetic field $\vec{B}^{D\alpha}(\vec{R},t)$ of a fluctuating electric dipole $\vec{p}_{\alpha}(t)$. Again, from the formulae of standard textbooks (*13), one can readily show that

$$B^{D\alpha}_{i}(\vec{R},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega \exp(-i\omega t)\left\{\sum_{j=1}^{3}\rho^{D}_{ij}(\vec{R}-\vec{R}_{\alpha},\omega)\,\tilde{p}_{\alpha j}(\omega)\right\} \quad , \qquad (23)$$

where $\tilde{p}_{\alpha j}(\omega)$ and ρ^{D}_{ij} are given in Eqs. (18) and (22).

Thus, from Eqs. (14)-(15) and (20)-(21), the two-point correlation functions of the electromagnetic radiation fields in Eqs. (1) and (2) have been related to the functions η^{D}_{ij} and ρ^{D}_{ij} that appear in the expressions of Eqs. (17) and (23) for the electric and magnetic fields of a fluctuating electric dipole. These relationships will be found useful in performing calculations in stochastic electrodynamics that involve the forced stochastic behavior of an electric dipole due to the electromagnetic radiation fields of Eqs. (1) and (2). Sections III and IV of the present article contain such calculations.

Various symmetry properties can be identified for the two-point correlation functions of Eqs. (14) and (20). Both correlation functions depend only upon the difference in time (t_2-t_1) and the difference in spatial position $(\vec{x}_2-\vec{x}_1)$. Because of the cosine and sine expansions in Eqs. (14) and (20), the correlation functions of Eqs. (14) and (20) are even and odd functions of (t_2-t_1), respectively. Other properties of the correlation functions that may readily be deduced from earlier equations are:

$$f_{ij}^{in}(\vec{R},\omega) = f_{ij}^{in}(-\vec{R},\omega) \quad , \qquad (24)$$

$$f_{ij}^{in}(\bar{R},\omega) = f_{ji}^{in}(\bar{R},\omega) \quad , \qquad (25)$$

$$g_{ij}^{in}(\bar{R},\omega) = -g_{ij}^{in}(-\bar{R},\omega) \quad , \qquad (26)$$

$$g_{ij}^{in}(\bar{R},\omega) = -g_{ji}^{in}(\bar{R},\omega) \quad . \qquad (27)$$

Similar relationships to those of Eqs. (14), (15), (17), (18), (20), (21), and (23) have recently been obtained between the correlation functions of the classical electromagnetic zero-point radiation fields, as calculated along trajectories described by relativistic uniform acceleration, and the electromagnetic fields of a uniformly accelerated electric dipole (*9). The configuration assumed for this calculation consisted of two points located in a plane, where the plane followed a trajectory of uniform acceleration along the normal to the plane. Because of the correspondence between the functional forms of the field correlation functions found in the present section and those found for an accelerating system, many of the calculations performed in an unaccelerated-thermal system can be carried over to a system uniformly accelerating through classical electromagnetic zero-point radiation. This point will be mentioned again briefly in Sec. III and illustrated a bit more clearly in Sec. IV. (Reading Ref. 9 should greatly clarify this point.)

Due to a connection discussed in Ref. (2) between the electromagnetic radiation field correlation functions in stochastic electrodynamics and the corresponding expectation value of electric and magnetic field operators in quantum electrodynamics, the relationships of Eqs. (14) and (20) may be immediately carried over to quantum electrodynamics. Let

$$[\underline{a},\underline{b}]_+ = \underline{a}\,\underline{b} + \underline{b}\,\underline{a} \quad , \qquad (28)$$

where $\underline{a}$ and $\underline{b}$ are underlined to denote quantum mechanical operators. Then, from Ref. 2 and Eqs. (14) and (21),

$$\langle 0|\tfrac{1}{2}[\underline{E}_i(\vec{x}_1,t_1),\underline{E}_j(\vec{x}_2,t_2)]_+|0\rangle = \langle 0|\tfrac{1}{2}[\underline{B}_i(\vec{x}_1,t_1),\underline{B}_j(\vec{x}_2,t_2)]_+|0\rangle$$
$$= \int_0^\infty d\omega \cos\big(\omega(t_2-t_1)\big)\, f_{ij}^{zp}(\vec{x}_2-\vec{x}_1,\omega) \ , \qquad (29)$$

$$\langle 0|\tfrac{1}{2}[\underline{B}_i(\vec{x}_1,t_1),\underline{E}_j(\vec{x}_2,t_2)]_+|0\rangle$$
$$= \int_0^\infty d\omega \sin\big(\omega(t_2-t_1)\big)\, g_{ij}^{zp}(\vec{x}_2-\vec{x}_1,\omega) \ . \qquad (30)$$

Here, $\underline{E}_i(\vec{x},t)$ and $\underline{B}_i(\vec{x},t)$ are the electric and magnetic field operators in quantum electrodynamics; Eqs. (29) and (30) represent the vacuum expectation value of the symmetrized products of these operators. The functions f_{ij}^{zp} and g_{ij}^{zp} in Eqs. (29) and (30) are given by Eqs. (15), (16), (21), and (22), with $\mathcal{h}^2_{in}(\omega)$ replaced by the function of Eq. (8) that is appropriate for the zero-point radiation field situation. Equations (29) and (30) may be generalized to the situation of a thermal radiation spectrum by replacing the vacuum state $|0\rangle$ on the left sides of Eqs. (29) and (30) by the appropriate incoherent superposition of photon states at temperature T; in correspondence with this change, the function $h_{in}(\omega)$, which occurs in the expressions for f_{ij}^{in} and g_{ij}^{in} of Eqs. (15) and (21), should be replaced by the thermal expression of Eq. (7). (Referring to Ref. 2 should clarify these points.)

III. RETARDED VAN DER WAALS FORCE FOR A SYSTEM OF N CLASSICAL HARMONIC OSCILLATORS

In this section, an arbitrary configuration of N charged classical harmonic oscillators will be considered, where each oscillator will be taken in the electric dipole limit. A thermal plus zero-point electromagnetic radiation field will be assumed to exist, corresponding to the choice of $h_T(\omega)$ for $h_{in}(\omega)$ in Eq. (7). This radiation field provides the mechanism for the forced steady state behavior of each oscillator. All oscillators interact with each other via the electromagnetic radiation they emit due to their forced harmonic motion.

The expectation value of the Lorentz force on one of the oscillators will be calculated in this section. From the viewpoint of stochastic electrodynamics, this quantity is simply the van der Waals force. In the process of carrying this calculation through, frequent use will be made of the relationships found in Sec. II of the present article.

The model chosen for each charged harmonic oscillator will be that of a point mass m with charge +e that oscillates under the action of a simple harmonic potential. A convenient model for providing the mechanism for such a potential consists of a spherical uniform distribution of charge, with net value $-e$. If a $+e$ point charge is located within this sphere at a position $\vec{x}_\alpha(t)$ from the center of this charge distribution, then the particle will experience a force proportional to the displacement $\vec{x}_\alpha(t)$. For a sufficiently small spherical volume of charge distribution and for sufficiently small amplitudes of oscillation of the point particle, the net charge configuration then approximates an electric dipole of value $+e\vec{x}_\alpha(t)$.

Under the small oscillator approximation (see Refs. 6 and 8), the equation of motion for one of N oscillators is given by

$$m\ddot{x}_{\alpha i} = -m\omega_0^2 x_{\alpha i} + m\Gamma \dddot{x}_{\alpha i} + eE_i^T(\vec{R}_\alpha, t) + e\sum_{\beta\neq\alpha} E_i^{D\beta}(\vec{R}_\alpha, t) \quad . \tag{31}$$

where $i=1,2,3$ and $\alpha = 1,2,\ldots,N$. Here, the index α serves as a label to distinguish each of the N oscillators. The quantity $\Gamma = \frac{2}{3}\frac{e^2}{mc^3}$ is the radiation reaction damping constant. The force constant of the harmonic potential is denoted by $m\omega_0^2$. The equilibrium position of the α^{th} oscillator is given by $\vec{R}_\alpha$ and the displacement from equilibrium by $\vec{x}_\alpha$. The electric field $\vec{E}^T$ stands for the field of Eq. (1), where $h_{in}(\omega)$ is replaced by $h_T(\omega)$ in Eq. (7). Finally, $\vec{E}^{D\beta}$ represents the electric dipole field of the β^{th} oscillator; hence, the expression for this quantity is given by Eqs. (16)–(18), where $p_{\alpha i}(t)$ in Eq. (18) is approximated by $ex_{\alpha i}(t)$.

In order to solve the linear stochastic differential set of equations indicated by Eq. (31), the following Fourier transforms will be introduced:

$$x_{\alpha i}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega \exp(-i\omega t)\tilde{x}_{\alpha i}(\omega) \quad , \qquad (32)$$

$$E_i^T(\vec{R},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega \exp(-i\omega t)\tilde{E}_i^T(\vec{R},\omega) \quad . \qquad (33)$$

Also, let

$$n_{ij}(\vec{R},\omega) = -\frac{e^2}{m} n_{ij}^D(\vec{R},\omega) \quad , \qquad (34)$$

$$\rho_{ij}(\vec{R},\omega) = -\frac{e^2}{m} \rho_{ij}^D(\vec{R},\omega) \quad , \qquad (35)$$

Replacing $\tilde{P}_{\alpha i}(\omega)$ in Eq. (17) by $e\tilde{x}_{\alpha i}(\omega)$, then from Eqs. (17) and (31)-(34), one obtains

$$C(\omega)\tilde{x}_{\alpha i}(\omega) + \sum_{\beta\neq\alpha}\sum_{j=1}^{3} n_{ij}(\vec{R}_\alpha - \vec{R}_\beta, \omega)\tilde{x}_{\beta i}(\omega) = \frac{e}{m}\tilde{E}_i^T(\vec{R}_\alpha,\omega) \quad , \qquad (36)$$

where $$C(\omega) = -\omega^2 + \omega_o^2 - i\Gamma\omega^3 \quad . \qquad (37)$$

Equation (36) can be expressed in the form of

$$\sum_{\beta=1}^{N}\sum_{j=1}^{3} M_{\alpha i;\beta j}(\omega)\,\tilde{x}_{\beta j}(\omega) = \frac{e}{m}\frac{\tilde{E}_i^T(\vec{R}_\alpha,\omega)}{C(\omega)} \quad , \qquad (38)$$

where $$M_{\alpha i;\beta j}(\omega) = \left\{ \delta_{\alpha\beta}\delta_{ij} + (1-\delta_{\alpha\beta})\frac{n_{ij}(\vec{R}_\alpha - \vec{R}_\beta, \omega)}{C(\omega)} \right\} \quad . \qquad (39)$$

From Eqs. (32) and (38),

$$X_{\alpha i}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \exp(-i\omega t) \left\{ \frac{e}{m} \sum_{\beta=1}^{N} \sum_{j=1}^{3} \left(M^{-1}(\omega) \right)_{\alpha i;\beta j} \frac{\tilde{E}_i^T(\vec{R}_\beta, \omega)}{C(\omega)} \right\} \quad . \tag{40}$$

Using this solution for the displacement of the α^{th} oscillating particle, the expectation value of the Lorentz force on the α^{th} oscillator can be obtained. Again approximating each oscillator as an electric dipole, the Lorentz force on the oscillator is given by

$$F_{\alpha i}(t) = \left(e\vec{x}_\alpha(t) \cdot \nabla \right) E_i(\vec{x},t) \Big|_{\vec{x}=\vec{R}_\alpha} + \left(\frac{e\dot{\vec{x}}_\alpha}{c} \otimes \vec{B}(\vec{R}_\alpha,t) \right)_i \quad , \tag{41}$$

where $\vec{E}$ and $\vec{B}$ represent the total electric and magnetic fields due to the radiation fields of Eqs. (1) and (2) and the dipole fields of all the other (N-1) oscillators. When taking the expectation value of Eq. (41), the relationship

$$\left\langle \left(\dot{\vec{x}}_\alpha \otimes \vec{B}(\vec{R}_\alpha, t) \right) \right\rangle = \left\langle \frac{d}{dt} \left(\vec{x}_\alpha(t) \otimes \vec{B}(\vec{R}_\alpha, t) \right) \right\rangle + c \left\langle \vec{x}_\alpha(t) \otimes \left(\nabla \otimes \vec{E}(\vec{x},t) \Big|_{\vec{x}=\vec{R}_\alpha} \right\rangle \tag{42}$$

is helpful, since the first term on the right-hand side of Eq. (42) equals zero, as will be proven shortly. Hence, from Eqs. (41) and (42),

$$\langle F_{\alpha i}^T(t) \rangle = \left\langle \sum_{j=1}^{3} e x_{\alpha j}(t) \frac{\partial}{\partial R_{\alpha i}} E_j^T(\vec{R}_\alpha, t) \right\rangle + \left\langle \sum_{j=1}^{3} e x_{\alpha j}(t) \frac{\partial}{\partial R_{\alpha i}} \left(\sum_{\beta \neq \alpha} E_j^{D\beta}(\vec{R}_\alpha, t) \right) \right\rangle \quad , \tag{43}$$

where a superscript T has been added in $\langle F_{\alpha i}^T(t) \rangle$ in order to designate the thermal situation at temperature T.

In order to prove that the first term of Eq. (42) equals zero, the operation of taking the time derivative should be interchanged with taking the expectation value. From Eqs. (14), (20), (23), (33), and (40), one can then show that

$$\left\langle \vec{x}_\alpha(t) \otimes \left[\vec{B}^{T}(\vec{R}_\alpha, t) + \sum_{\beta \neq \alpha} \vec{B}^{D\beta}(\vec{R}_\alpha, t) \right] \right\rangle$$

is independent of time. Alternatively, one may present a more general argument by physically demanding that the behavior of the set of oscillators constitute a process that is stationary in time. The expectation value of two quantities connected to the behavior of the oscillators must then depend only upon the difference in the time arguments of the two quantities.

In order to evaluate the first quantity on the right-hand side of Eq. (43), the consideration of the following quantity will prove helpful:

$$\left\langle \tilde{x}_{\alpha j}(\omega') \frac{\partial}{\partial R_{\alpha i}} \tilde{E}_j^{T}(\vec{R}_\alpha, \omega'') \right\rangle$$
$$= \sum_{\beta=1}^{N} \sum_{\ell=1}^{3} \left(M^{-1}(\omega) \right)_{\alpha j;\beta\ell} \frac{e}{mC(\omega)} \left\langle \tilde{E}_\ell^{T}(\vec{R}_\beta, \omega') \frac{\partial}{\partial R_{\alpha i}} \tilde{E}_j^{T}(\vec{R}_\alpha, \omega'') \right\rangle \quad . \qquad (44)$$

From the inverse of Eq. (33),

$$\left\langle \tilde{E}_\ell^{T}(\vec{R}_\beta, \omega') \frac{\partial}{\partial R_{\alpha i}} \tilde{E}_j^{T}(\vec{R}_\alpha, \omega'') \right\rangle$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt' \exp(i\omega' t') \int_{-\infty}^{\infty} dt'' \exp(i\omega'' t'') \left\langle E_\ell^{T}(\vec{R}_\beta, t') \frac{\partial}{\partial R_{\alpha i}} E_j^{T}(\vec{R}_\alpha, t'') \right\rangle \quad . \qquad (45)$$

As will be proven shortly,

$$\left\langle E_\ell^T(\bar{R}_\beta,t')\frac{\partial}{\partial R_{\alpha i}}E_j^T(\bar{R}_\alpha,t'')\right\rangle$$
$$=\int_0^\infty d\omega\cos(\omega(t''-t'))\frac{\partial}{\partial R_{\alpha i}}\left\{(1-\delta_{\alpha\beta})f_{\ell j}^T(\vec{R}_\alpha-\bar{R}_\beta,\omega)\right\} \qquad . \qquad (46)$$

where $f_{\ell j}^T$ is given by Eq. (15), with $h_{in}(\omega)$ replaced by $h_T(\omega)$. From Eqs. (45) and (46),

$$\left\langle E_\ell^T(\bar{R}_\beta,\omega')\frac{\partial}{\partial R_{\alpha i}}E_j^T(\bar{R}_\alpha,\omega'')\right\rangle$$
$$=\pi\int_0^\infty d\omega\left[\delta(\omega'-\omega)\delta(\omega''+\omega)+\delta(\omega'+\omega)\delta(\omega''-\omega)\right]\frac{\partial}{\partial R_{\alpha i}}\left\{(1-\delta_{\alpha\beta})f_{\ell j}^T(\vec{R}_\alpha-\bar{R}_\beta,\omega)\right\} \quad . \quad (47)$$

From Eqs. (16), (34), (37), and (39),

$$C(-\omega)=C^*(\omega) \qquad , \qquad (48)$$

$$n_{ij}^D(\bar{R},-\omega)=n_{ij}^{D*}(\bar{R},\omega) \qquad , \qquad n_{ij}(\bar{R},-\omega)=n_{ij}^*(\bar{R},\omega) \qquad , \qquad (49a\ \&\ b)$$

$$M_{\alpha i;\beta j}(-\omega)=M_{\alpha i;\beta j}^*(\omega) \qquad . \qquad (50)$$

From Eqs. (44), (47), (48), and (50), one can then show that the first term on the right side of Eq. (43) is given by

$$\left\langle\sum_{j=1}^3 ex_{\alpha j}(t)\frac{\partial}{\partial R_{\alpha i}}E_j^T(\bar{R}_\alpha,t)\right\rangle$$
$$=\frac{e^2}{m}\sum_{\beta=1}^N\sum_{j,\ell=1}^3\int_0^\infty d\omega\,\mathrm{Re}\left\{\frac{(M^{-1}(\omega))_{\alpha j;\beta\ell}}{C(\omega)}\right\}\frac{\partial}{\partial R_{\alpha i}}\left\{(1-\delta_{\alpha\beta})f_{\ell j}^T(\bar{R}_\alpha-\bar{R}_\beta,\omega)\right\} . \qquad (51)$$

This expression may be put into a more convenient form by the use of the following relationship:

$$\frac{\partial}{\partial R_{\alpha i}}\left\{(1-\delta_{\alpha\beta}) f_{\ell j}^{T}(\bar{R}_\alpha - \bar{R}_\beta, \omega)\right\}$$
$$= -\frac{2\pi \hbar_T^2(\omega)}{\frac{e^2}{m}\omega}\frac{\partial}{\partial R_{\alpha i}} \mathrm{Im}\left\{\delta_{\alpha\beta}\delta_{\ell j} C(\omega) + (1-\delta_{\alpha\beta}) n_{\ell j}(\bar{R}_\alpha - \bar{R}_\beta, \omega)\right\}$$
$$= -\frac{2\pi \hbar_T^2(\omega)}{\frac{e^2}{m}\omega}\frac{\partial}{\partial R_{\alpha i}} \mathrm{Im}\left\{C(\omega) M_{\alpha\ell;\beta j}(\omega)\right\} \quad , \qquad (52)$$

which can be verified by using Eqs. (15), (34), and (39). Substituting Eq. (52) into Eq. (51) then yields

$$\left\langle \sum_{j=1}^{3} e x_{\alpha j}(t) \frac{\partial}{\partial R_{\alpha i}} E_j^T(\bar{R}_\alpha, t)\right\rangle$$
$$= -2\pi \sum_{\beta=1}^{N} \sum_{j,\ell=1}^{3} \int_0^\infty \frac{d\omega}{\omega} \hbar_T^2(\omega) \mathrm{Re}\left\{\frac{M^{-1}(\omega)_{\alpha j;\beta\ell}}{C(\omega)}\right\} \mathrm{Im}\left\{C(\omega)\frac{\partial}{\partial R_{\alpha i}} M_{\alpha\ell;\beta j}(\omega)\right\} \quad . \qquad (53)$$

The missing proof of Eq. (46) will now be given. First, consider the case where $\alpha \neq \beta$. Combining the obvious relationship of

$$\left\langle E_\ell^T(\bar{R}_\beta, t')\frac{\partial}{\partial R_{\alpha i}} E_j^T(\bar{R}_\alpha, t'')\right\rangle = \frac{\partial}{\partial R_{\alpha i}}\left\langle E_\ell^T(\bar{R}_\beta, t') E_j^T(\bar{R}_\alpha, t'')\right\rangle \qquad (54)$$

with Eq. (14), one can easily verify Eq. (46) when $\alpha \neq \beta$. When $\alpha = \beta$, then the following identity must be used:

$$\left\langle \cos\left(A + \Theta(\bar{k}', \lambda')\right) \sin\left(B + \Theta(\bar{k}'', \lambda'')\right)\right\rangle = \frac{1}{2}\delta_{\lambda'\lambda''}\delta^3(\bar{k}'' - \bar{k}')\sin(B - A) \quad . \qquad (55)$$

From Eqs. (1), (55), and (5), Eq. (46) becomes, for $\alpha = \beta$,

$$\left\langle E_\ell^T(\bar{R}_\alpha, t')\frac{\partial}{\partial R_{\alpha i}} E_j^T(\bar{R}_\alpha, t'')\right\rangle$$
$$= \frac{1}{2}\int d^3k\, \hbar^2(\omega)\left(\delta_{\ell j} - \frac{k_\ell k_j}{k^2}\right)(-k_i)\sin\left(0 - \omega(t_2 - t_1)\right) \quad . \qquad (56)$$

By inspection, the integrand of Eq. (56) must be odd in either K_1, K_2, or K_3, thereby resulting in the integral being identically equal to zero when $\alpha=\beta$. Hence, Eq. (46) has been verified.

The second term of Eq. (43) will now be evaluated. From Eq. (34) and the Fourier transforms of Eqs. (17) and (40),

$$\left\langle \tilde{x}_{\alpha j}(\omega')\frac{\partial}{\partial R_{\alpha i}}\tilde{E}_j^{D\beta}(\bar{R}_\alpha,\omega'')\right\rangle$$
$$= -\frac{e}{m}\sum_{\gamma,\delta=1}^{N}\sum_{\ell,m,n=1}^{3}\frac{\left(M^{-1}(\omega')\right)_{\alpha j;\gamma\ell}}{C(\omega')}\left[\frac{\partial}{\partial R_{\alpha i}}\eta_{jm}(\bar{R}_\alpha-\bar{R}_\beta,\omega'')\right]\frac{\left(M^{-1}(\omega'')\right)_{\beta m;\delta n}}{C(\omega'')}\left\langle \tilde{E}_\ell^T(\bar{R}_\gamma,\omega')\tilde{E}_n^T(\bar{R}_\delta,\omega'')\right\rangle . \tag{57}$$

From the inverse of Eq. (33), along with Eq. (14),

$$\left\langle \tilde{E}_\ell^T(\bar{R}_\gamma,\omega')\tilde{E}_n^T(\bar{R}_\delta,\omega'')\right\rangle$$
$$= \pi\int_0^\infty d\omega\left[\delta(\omega'-\omega)\delta(\omega''+\omega)+\delta(\omega'+\omega)\delta(\omega''-\omega)\right] f_{\ell n}^T(\bar{R}_\delta-\bar{R}_\gamma,\omega) . \tag{58}$$

With the use of Eqs. (57), (58), (48), (49b), and (50), the second term of Eq. (43) can be expressed as

$$\left\langle \sum_{j=1}^{N} e x_{\alpha j}(t)\frac{\partial}{\partial R_{\alpha i}}\left(\sum_{\beta\neq\alpha}E_j^{D\beta}(\bar{R}_\alpha,t)\right)\right\rangle$$
$$= -\frac{e^2}{m}\sum_{\beta,\gamma,\delta=1}^{N}\sum_{j,\ell,m,n=1}^{3}\int_0^\infty d\omega\, f_{\ell n}^T(\bar{R}_\delta-\bar{R}_\gamma,\omega)\frac{\mathrm{Re}\left\{\left(M^{-1}(\omega)\right)^*_{\alpha j;\gamma\ell}\left(M^{-1}(\omega)\right)_{\beta m;\delta n}\frac{\partial}{\partial R_{\alpha i}}\left[(1-\delta_{\alpha\beta})\eta_{jm}(\bar{R}_\alpha-\bar{R}_\beta,\omega)\right]\right\}}{|C(\omega)|^2} . \tag{59}$$

Here, it should be noted that a factor of $(1-\delta_{\alpha\beta})$ has been included on the right-hand side of Eq. (59), thereby enabling the index β to be summed from 1 to N without restriction.

Two substitutions can be made to simplify Eq. (59).

First, from Eq. (39),

$$\frac{\partial}{\partial R_{\alpha i}}\left[(1-\delta_{\alpha\beta})\eta_{jm}(\vec{R}_\alpha-\vec{R}_\beta,\omega)\right]=\frac{\partial}{\partial R_{\alpha i}}\left[C(\omega)\underset{\alpha j;\beta m}{M}(\omega)\right] \quad . \qquad (60)$$

Second, from Eqs. (15) and (34),

$$f^{T}_{\ell n}(\bar{R}_\delta-\bar{R}_\gamma,\omega)=\frac{-2\pi h^2_T(\omega)}{\frac{e^2}{m}\omega}\,\mathrm{Im}\,\eta_{\ell n}(\bar{R}_\delta-\bar{R}_\gamma,\omega) \quad . \qquad (61)$$

This second substitution may be improved upon. By using $\sin(kR)\approx kR-\frac{1}{6}(kR)^3$ and $\cos(kR)\approx 1-\frac{1}{2}(kR)^2$ for $kR \ll 1$, one may verify from Eqs. (16), (34), and (37) that

$$\lim_{R\to 0}\,\mathrm{Im}\,\eta_{\ell n}(\hat{R}R,\omega)=-\delta_{\ell n}\frac{2}{3}\frac{e^2}{mc^3}\omega^3=\delta_{\ell n}\,\mathrm{Im}\,C(\omega) \quad . \qquad (62)$$

Consequently, from Eqs. (62) and (39),

$$\begin{aligned}\mathrm{Im}\,\eta_{\ell n}(\bar{R}_\delta-\bar{R}_\gamma,\omega)&=\mathrm{Im}\left[\delta_{\delta\gamma}\,\delta_{\ell n}C(\omega)+(1-\delta_{\delta,\gamma})\eta_{\ell n}(\vec{R}_\delta-\vec{R}_\gamma,\omega)\right]\\&=\mathrm{Im}\left[C(\omega)\underset{\delta\ell;\gamma n}{M}(\omega)\right] \quad . \qquad (63)\end{aligned}$$

Substituting Eqs. (60), (61), and (63) into Eq. (59) yields

$$\begin{aligned}&\left\langle\sum_{j=1}^{N}ex_{\alpha j}(t)\frac{\partial}{\partial R_{\alpha i}}\Big(\sum_{\beta\neq\alpha}E_j^{\flat\beta}(\vec{R}_\alpha,t)\Big)\right\rangle \qquad (64)\\&=2\pi\sum_{\beta,\gamma,\delta=1}^{N}\;\sum_{j,\ell,m,n=1}^{3}\int_0^\infty\frac{d\omega}{\omega}h^2_T(\omega)\frac{\mathrm{Im}\left\{C(\omega)\underset{\delta\ell;\gamma n}{M}(\omega)\right\}}{|C(\omega)|^2}\,\mathrm{Re}\left\{\left(\underset{\alpha j;\gamma\ell}{M^{-1}}(\omega)\right)^{*}\left(\underset{\beta m;\delta n}{M^{-1}}(\omega)\right)C(\omega)\frac{\partial}{\partial R_{\alpha i}}\underset{\alpha j;\beta m}{M}(\omega)\right\} \quad .\end{aligned}$$

A fair amount of algebraic manipulations must now be employed in order to bring Eq. (64) into a form that is compatible with Eq. (53). First, the imaginary and real terms below may be expanded as shown:

$$\sum_{\gamma,\delta=1}^{N} \sum_{\ell,n=1}^{3} \mathrm{Im}\{C M_{\delta\ell;\gamma n}\} \mathrm{Re}\{(M^{-1})^*_{\alpha j;\gamma\ell} (M^{-1})_{\beta m;\delta n} C \frac{\partial}{\partial R_{\alpha i}} M_{\alpha j;\beta m}\}$$

$$= \frac{1}{4i}\Big(C\big(\frac{\partial}{\partial R_{\alpha i}} M_{\alpha j;\beta m}\big)\Big[\sum_{\gamma,\ell} C (M^{-1})^*_{\alpha j;\gamma\ell} \sum_{\delta,n} \{(M^{-1})_{\beta m;\delta n} M_{\delta\ell;\gamma n}\} \tag{65}$$

$$- \sum_{\delta,n} C^*(M^{-1})_{\beta m;\delta n} \sum_{\gamma,\ell} \{(M^{-1})^*_{\alpha j;\gamma\ell} M^*_{\delta\ell;\gamma n}\}\Big]$$

$$+ C^*\big(\frac{\partial}{\partial R_{\alpha i}} M^*_{\alpha j;\beta m}\big)\Big[\sum_{\delta,n} C (M^{-1})^*_{\beta m;\delta n} \sum_{\gamma,\ell} \{(M^{-1})_{\alpha j;\gamma\ell} M_{\delta\ell;\gamma n}\}$$

$$- \sum_{\gamma,\ell} C^*(M^{-1})_{\alpha j;\gamma\ell} \sum_{\delta,n} \{(M^{-1})^*_{\beta m;\delta n} M^*_{\delta\ell;\gamma n}\}\Big]\Big).$$

From Eqs. (39), (34), and (16), the following symmetry rules for $M_{\alpha i;\beta j}$ may readily be verified:

$$M_{\alpha i;\beta j} = M_{\beta i;\alpha j} \quad , \tag{66}$$

$$M_{\alpha i;\beta j} = M_{\alpha j;\beta i} \quad . \tag{67}$$

By using Eqs. (66) and (67), a pair of Kronecker deltas may be obtained from each of the terms enclosed in curley brackets in Eq. (65). For example,

$$\sum_{\delta,n} \{ (M^{-1})_{\beta m;\delta n} M_{\delta\ell;\gamma n} \} = \sum_{\delta,n} \{ (M^{-1})_{\beta m;\delta n} M_{\delta n;\gamma\ell} \} = \delta_{\beta\gamma} \delta_{m\ell} \quad . \tag{68}$$

Both terms in the rectangular brackets of Eq. (65) may then be shown to be equal to

$$-2i\,\mathrm{Im}\big(C^*(M^{-1})_{\alpha j;\beta m}\big) \quad .$$

Combining terms enables Eq. (65) to be expressed as

$$\sum_{\gamma,\delta=1}^{N}\sum_{\ell,n=1}^{3} \mathrm{Im}\left\{C M_{\delta\ell;\gamma n}\right\} \mathrm{Re}\left\{(M^{-1})^{*}_{\alpha j;\gamma\ell} (M^{-1})_{\beta m;\delta n} C \frac{\partial}{\partial R_{\alpha i}} M_{\alpha j;\beta m}\right\}$$
$$= -\mathrm{Im}\left(C^{*}(M^{-1})_{\alpha j;\beta m}\right) \mathrm{Re}\left(C \frac{\partial}{\partial R_{\alpha i}} M_{\alpha j;\beta m}\right) . \qquad (69)$$

By now substituting Eq. (69) into Eq. (64), using the simple relationship of

$$\frac{\mathrm{Im}\left(C^{*}(M^{-1})_{\alpha j;\beta m}\right)}{|C|^{2}} = \mathrm{Im}\left(\frac{(M^{-1})_{\alpha j;\beta m}}{C}\right) , \qquad (70)$$

and relabeling the m dummy index as ℓ, finally yields

$$\left\langle \sum_{j=1}^{N} e x_{\alpha j}(t) \frac{\partial}{\partial R_{\alpha i}} \left(\sum_{\beta\neq\alpha} E_{j}^{D\beta}(\bar{R}_{\alpha},t)\right)\right\rangle$$
$$= -2\pi \sum_{\beta=1}^{N} \sum_{j,\ell=1}^{3} \int_{0}^{\infty} \frac{d\omega}{\omega} \hbar_{T}^{2}(\omega)\, \mathrm{Im}\left(\frac{(M^{-1})_{\alpha j;\beta\ell}}{C}\right) \mathrm{Re}\left(C \frac{\partial}{\partial R_{\alpha i}} M_{\alpha j;\beta\ell}\right) . \qquad (71)$$

If Eq. (67) is used to switch the ℓ, j indices of $M_{\alpha\ell;\beta j}$ in Eq. (53), then it can immediately be seen that Eq. (53) and (71) are of the same form. Combining Eqs. (43), (53), and (71), then yields

$$\langle F_{\alpha i}^{T}(t)\rangle = -2\pi \sum_{\beta=1}^{N} \sum_{j,\ell=1}^{3} \int_{0}^{\infty} \frac{d\omega}{\omega} \hbar_{T}^{2}(\omega)\, \mathrm{Im}\left((M^{-1})_{\alpha j;\beta\ell} \frac{\partial}{\partial R_{\alpha i}} M_{\alpha j;\beta\ell}\right) . \qquad (72)$$

Let the determinant of [M] be denoted by $|M|$; let $\Delta_{\alpha j;\beta\ell}$ be the cofactor of the matrix element $M_{\alpha j;\beta\ell}$. Since $|M|$ may be expressed as

$$|M| = \sum_{\beta=1}^{N} \sum_{\ell=1}^{3} M_{\alpha j;\beta\ell} \Delta_{\alpha j;\beta\ell} , \qquad (73)$$

then the inverse matrix elements may be written as

$$(M^{-1})_{\alpha j;\beta \ell} = \frac{\Delta_{\alpha j;\beta \ell}}{|M|} = \frac{1}{|M|}\frac{\partial}{\partial M_{\alpha j;\beta \ell}}|M| = \frac{\partial}{\partial M_{\alpha j;\beta \ell}} \ln|M| \quad . \tag{74}$$

Consequently,

$$\langle F^{T}_{\alpha i}(t)\rangle = -2\pi\int_0^\infty \frac{d\omega}{\omega} h^2_T(\omega)\,\mathrm{Im}\left[\sum_{\beta=1}^{N}\sum_{j,\ell=1}^{3}\left(\frac{\partial \ln|M|}{\partial M_{\alpha j;\beta \ell}}\right)\left(\frac{\partial M_{\alpha j;\beta \ell}}{\partial R_{\alpha i}}\right)\right] \quad . \tag{75}$$

Using Eq. (66) to write Eq. (75) more symmetrically,

$$\langle F^{T}_{\alpha i}(t)\rangle = -\pi\int_0^\infty \frac{d\omega}{\omega} h^2_T(\omega)\,\mathrm{Im}\left[\sum_{\beta}\sum_{j,\ell}\left(\frac{\partial \ln|M|}{\partial M_{\alpha j;\beta \ell}}\right)\left(\frac{\partial M_{\alpha j;\beta \ell}}{\partial R_{\alpha i}}\right) + \sum_{\beta}\sum_{j,\ell}\left(\frac{\partial \ln|M|}{\partial M_{\beta j;\alpha \ell}}\right)\left(\frac{\partial M_{\beta j;\alpha \ell}}{\partial R_{\alpha i}}\right)\right] \quad . \tag{76}$$

This expression for $\langle F^{T}_{\alpha i}(t)\rangle$ may be simplified significantly by considering the quantity within the square brackets. From Eq. (39), the quantity

$$\frac{\partial}{\partial R_{\alpha i}} M_{\beta j;\gamma \ell}$$

is nonzero only when $\beta=\alpha$ and $\gamma\neq\alpha$, or when $\beta\neq\alpha$ and $\gamma=\alpha$. By inspection of Eq. (76), all such nonzero contributions of this quantity have already been included within the square brackets. Hence, one may simply include the remaining terms within the indicated summation, since the remaining terms equal zero. More specifically,

$$\langle F^{T}_{\alpha i}(t)\rangle = -\pi\int_0^\infty \frac{d\omega}{\omega} h^2_T(\omega)\,\mathrm{Im}\left[\sum_{\gamma,\beta=1}^{N}\sum_{j,\ell=1}^{3}\left(\frac{\partial \ln|M|}{\partial M_{\beta j;\gamma \ell}}\right)\left(\frac{\partial M_{\beta j;\gamma \ell}}{\partial R_{\alpha i}}\right)\right] \quad . \tag{77}$$

Hence, as may be seen from Eq. (77), $\langle F^{T}_{\alpha i}(t)\rangle$ may be written as

$$\langle F^{T}_{\alpha i}(t)\rangle = -\frac{\partial}{\partial R_{\alpha i}} U^{T} \quad , \tag{78}$$

where $$U^{T} = \pi\int_0^\infty \frac{d\omega}{\omega}\, h^2_T(\omega)\, \mathrm{Im}\left[\ln\left|M(\omega)\right|\right] \quad . \tag{79}$$

Equations (78) and (79) generalize the van der Waals expressions of Eqs. (8) and (9) in Ref. 5. The latter result dealt with the expectation value of the component of the Lorentz force along the axis separating two electric dipole oscillators situated in thermal plus zero-point electromagnetic radiation. From Eqs. (16), (34), and (39), Eq. (79) may be readily shown to reduce to Eq. (9) of Ref. 5 when N=2 and $\vec{R}=\hat{z}R$. [The configuration of $\vec{R}=\hat{z}R$ was chosen in Refs. 4 and 5. From Eqs. (16) and (34), $\eta_{ij}(\hat{z}R,\omega) = \delta_{ij}\eta_i$, where η_i is given in Eqs. (19) and (20) of Ref. 5.]

When the temperature T equals zero, then Eq. (79) may be compared to the result of Eq. (18) in Ref. 11 that was obtained via the means of quantum electrodynamics. When T=0, then Eq. (8) must be used in Eq. (79). From the line following Eq. (13) of Ref. 11 and the comment $\overleftrightarrow{G}_{\alpha\alpha}(z)=0$ at the top of page 202, one can deduce that $\overleftrightarrow{G}_{\alpha\beta}$ may be written as

$$\begin{aligned} \underset{\alpha i;\beta j}{G(\omega)} &= (1-\delta_{\alpha\beta})\left(\nabla_{\alpha i}\nabla_{\beta j}-\frac{\omega^2}{c^2}\right)\left(\frac{\exp\left(\frac{i\omega}{c}|\vec{R}_\alpha-\vec{R}_\beta|\right)}{|\vec{R}_\alpha-\vec{R}_\beta|}\right) \\ &= \frac{m}{e^2}(1-\delta_{\alpha\beta})\eta_{ij}(\vec{R}_\alpha-\vec{R}_\beta,\omega) \quad . \end{aligned} \tag{80}$$

From Eq. (16) of Ref. 11, $\tilde{\alpha}(\omega)=\frac{e^2}{m}\frac{1}{C(\omega)}$. By now following the steps in the first part of Sect. IVB of Ref. 4, from Eq. (82) to Eq. (87), the above result of Eq. (80) can be shown to be equivalent to Eq. (18) of Ref. 11.

The calculations in this section may be generalized to the situation of N oscillators located in a plane undergoing uniform acceleration through classical electromagnetic zero-point radiation. The equations that allow this generalization to be made consist of the relationships between the two-point radiation field correlation functions and the fields of a fluctuating electric dipole found in Eqs. (14), (15), (17), (20), (21), and (23) of the present article, and in Eqs. (A37), (A38), and (C1)-(C4) of Ref. 9. Although this generalization will not be carried out here, a careful reading of the present section and of Ref. 9 will indicate how this generalization may be accomplished.

IV. EXPECTATION VALUE OF POYNTING VECTOR IN PRESENCE OF AN ELECTRIC DIPOLE OSCILLATOR

If a classical charged harmonic oscillator is bathed in classical electromagnetic radiation, then the oscillator will be forced into a steady state motion by the radiation. Consequently, the charged oscillator will emit electromagnetic radiation of its own. Consider the case where the oscillator is taken in the electric dipole limit. Let the statistical properties of the electromagnetic radiation causing the oscillator's forced motion be isotropic and homogeneous in space. Under these conditions, one can show that the expectation value of the Poynting vector, due to the total electromagnetic radiation, is exactly equal to zero. This is precisely the situation that occurs when the dipole oscillator is not present. Hence, the presence of the oscillator does not alter the basic flow pattern of electromagnetic radiation.

The proof of the above statement has been given previously in Appendix B of Ref. 1. This proof will be reconstructed in the present section of this article in such a way as to explicitly use the relationships found in Sec. II. As will be seen, what enables the proof to be carried out are precisely these relationships between the two-point correlation functions of classical electromagnetic radiation fields and the electromagnetic fields radiated by a fluctuating electric dipole. This fact was not identified in the original proof. At the end of this section, the means for extending this proof to the case of a uniformly accelerating oscillator will be given.

Let a single oscillator be situated at the origin of a Cartesian coordinate system in an inertial reference frame. The model assumed for the oscillator will be the same as that of Sec. III. Let the background electromagnetic radiation be described by Eqs. (1) and (2). As may readily be deduced from Eq. (31), with $\vec{E}^{D\beta}$ omitted on the right-hand side, along with Eqs. (32), (33), and (37), the motion of a single oscillator is given by

$$x_i(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \exp(-i\omega t) \left\{ \frac{e}{m} \frac{\tilde{E}_i^{in}(0,\omega)}{C(\omega)} \right\} \quad . \qquad (81)$$

The expectation value of the Poynting vector due to the total electromagnetic radiation at a location $\vec{R}$ and time t is given by

$$\begin{aligned} \langle S_k(\vec{R},t)\rangle &= \left\langle \frac{c}{4\pi}\left(\vec{E}_{total}(\vec{R},t) \otimes \vec{B}_{total}(\vec{R},t)\right)_k \right\rangle \\ &= \frac{c}{4\pi} \sum_{i,j=1}^{3} \epsilon_{ijk} \Big\{ \langle E_i^{in}(\vec{R},t) B_j^{in}(\vec{R},t)\rangle + \langle E_i^{in}(\vec{R},t) B_j^{D}(\vec{R},t)\rangle \\ &\qquad + \langle E_i^{D}(\vec{R},t) B_j^{in}(\vec{R},t)\rangle + \langle E_i^{D}(\vec{R},t) B_j^{D}(\vec{R},t)\rangle \quad . \end{aligned} \qquad (82)$$

The four terms in Eq. (82) can readily be evaluated by the use of Eq. (81) and the relationships in Sec. II. Using the same method as in Eq. (62), one can show from Eqs. (22) and (35) that

$$\lim_{R\to 0} \mathrm{Re}\left[\rho_{ij}(R\hat{R},\omega)\right] = 0 \ . \qquad (83)$$

From Eq. (83), the result follows that the correlation function of Eq. (20) equals zero when $\vec{x}_2 = \vec{x}_1$; consequently, the first term on the right of Eq. (82) equals zero. More specifically, from Eqs. (20), (21), (35), and (83),

$$\langle E_i^{in}(\bar{R},t) B_j^{in}(\bar{R},t)\rangle = 0 \ . \qquad (84)$$

The second term on the right of Eq. (82) may be evaluated by reexpressing B_j^D by Eqs. (23), (81), and (33) and then using Eqs. (14), (15), (34), and (48). From Eqs. (22) and (35),

$$\rho_{ij}^{D}(\bar{R},-\omega) = \rho_{ij}^{D*}(\bar{R},\omega) \quad , \qquad (85a)$$

$$\rho_{ij}(\bar{R},-\omega) = \rho_{ij}^{*}(\bar{R},\omega) \quad . \qquad (85b)$$

Combining the equations mentioned above,

$$\langle E_i^{in}(\bar{R},t) B_j^{D}(\bar{R},t)\rangle \qquad (86)$$

$$= \frac{\pi}{2i(e^2/m)} \sum_{\ell=1}^{3} \int_0^\infty \frac{d\omega}{\omega} \frac{k_{in}^2(\omega)}{|C(\omega)|^2} \Big\{ C^*(\omega)\eta_{\ell i}(\bar{R},\omega)\rho_{j\ell}(\bar{R},\omega) + C(\omega)\eta_{\ell i}(\bar{R},\omega)\rho_{j\ell}^*(\bar{R},\omega)$$

$$- C^*(\omega)\eta_{\ell i}^*(\bar{R},\omega)\rho_{j\ell}(\bar{R},\omega) - C(\omega)\eta_{\ell i}^*(\bar{R},\omega)\rho_{j\ell}^*(\bar{R},\omega)\Big\} .$$

Using virtually identical treatments, one may obtain the following expressions for the third and fourth terms of Eq. (82):

$$\langle E_i^D(\bar{R},t)B_j^{in}(\bar{R},t)\rangle \tag{87}$$
$$= \frac{\pi}{2i(e^2/m)}\sum_{\ell=1}^{3}\int_0^\infty \frac{d\omega}{\omega}\frac{h_{in}^2(\omega)}{|C(\omega)|^2}\Big\{C(\omega)n_{i\ell}^*(\bar{R},\omega)\rho_{\ell j}(-\bar{R},\omega)-C^*(\omega)n_{i\ell}(\bar{R},\omega)\rho_{\ell j}(-\bar{R},\omega)$$
$$+C(\omega)n_{i\ell}^*(\bar{R},\omega)\rho_{\ell j}^*(-\bar{R},\omega)-C^*(\omega)n_{i\ell}(\bar{R},\omega)\rho_{\ell j}^*(-\bar{R},\omega)\Big\},$$

$$\langle E_i^D(\bar{R},t)B_j^D(\bar{R},t)\rangle \tag{88}$$
$$= \frac{\pi}{2i(e^2/m)}\sum_{\ell=1}^{3}\int_0^\infty \frac{d\omega}{\omega}\frac{h_{in}^2(\omega)}{|C(\omega)|^2}\Big\{-C(\omega)n_{i\ell}(\bar{R},\omega)\rho_{j\ell}^*(\bar{R},\omega)+C^*(\omega)n_{i\ell}(\bar{R},\omega)\rho_{j\ell}^*(\bar{R},\omega)$$
$$-C(\omega)n_{i\ell}^*(\bar{R},\omega)\rho_{j\ell}(\bar{R},\omega)+C^*(\omega)n_{i\ell}^*(\bar{R},\omega)\rho_{j\ell}(\bar{R},\omega)\Big\}.$$

Hence, all four terms of Eq. (82) have been expressed in terms of the functions n_{ij} and ρ_{ij} that appear in the expressions for the electric and magnetic fields of a fluctuating electric dipole. This was accomplished by using the relationships of Eqs. (14), (15), (20), and (21) between the two-point correlation functions of classical electromagnetic radiation fields and the electromagnetic fields radiated by an electric dipole. At this point, the symmetries of

$$\rho_{ij}(-\bar{R},\omega) = -\rho_{ij}(\bar{R},\omega) \quad , \tag{89}$$

$$n_{ij}(\bar{R},\omega) = n_{ji}(\bar{R},\omega) \quad , \tag{90}$$

$$\rho_{ij}(\bar{R},\omega) = -\rho_{ji}(\bar{R},\omega) \quad , \tag{91}$$

which are readily deduced from Eqs. (16), (22), (34), and (35), may be introduced to show that all terms immediately cancel upon combining Eqs. (82), (84), and (86)-(88). Hence, $\langle S_k^{in}(\bar{R},t)\rangle = 0$.

Thus ends the proof of this section of the fact that the presence of an electric dipole oscillator does not alter $\langle S_k^{in}(\vec{R},t)\rangle$ from its zero value. The important assumptions used in this proof were that the linear dipole oscillator was stationary in an inertial frame and bathed with homogeneous, isotropic electromagnetic radiation described by Eqs. (1) and (2).

Without too much difficu..., however, this proof may be extended to the case of a dipole oscillator uniformly accelerating through classical electromagnetic radiation. This analogous situation requires that $\langle S_k^{in}(\vec{R},t)\rangle$ be evaluated in the instantaneous inertial rest frame of the accelerating oscillator. Instead of using the relationships in Sec. II of the present article, the analogous relationships of Appendices A and C of Ref. 9 must be employed. The steps of the proof given in this section for an unaccelerated oscillator may then be followed up through Eq. (88). The symmetries of Eqs. (89)-(91) may not be employed, however, as these do not carry over to the acceleration case.

A sketch of this calculation is given in the appendix of this article. Let $\vec{a}$ denote the proper acceleration of the oscillator; let R denote the distance from the oscillator to the point at which the Poynting vector is evaluated, taken along a perpendicular to the acceleration. Provided that a small laboratory condition is imposed such that terms of order $O(\frac{aR}{c^2})$ are ignored, then a null value is obtained for the expectation value of the Poynting vector in the instantaneous rest frame of the accelerating oscillator. This result agrees with the exact value of zero that is obtained for the expectation value of the Poynting vector when the oscillator is not present, as given in a nonrotating coordinate system uniformly accelerating through classical electromagnetic zero-point radiation. Hence, for terms up to order $\mathcal{O}(\frac{aR}{c^2})$, the presence of an oscillator within a plane uniformly accelerating through classical electromagnetic zero-point radiation does not alter the expectation value of the radiated electromagnetic momentum within this accelerating plane.

V. CLOSING REMARKS

Relationships were derived in Sec. II between the two-point field correlation functions for homogeneous and isotropic random (Gaussian) classical electromagnetic radiation and the electromagnetic fields of a classical fluctuating electric dipole. Section III explicitly used these relationships in order to obtain the van der Waals force on an oscillator surrounded by (N-1) other dipole oscillators, all of which were bathed in classical electromagnetic zero-point plus thermal radiation. Section IV used the relationships of Sec. II to show that the expectation value of the Poynting vector in the presence of an oscillator, bathed with homogeneous and isotropic classical electromagnetic radiation, is unaltered from its null value that occurs when the oscillator is not present. Brief discussions were also given in Secs. III and IV and the Appendix as to how the calculations of Secs. III and IV could be extended to the case of a small laboratory uniformly accelerating through classical electromagnetic radiation. What enables this extension to be made is the generalization found in Ref. 9: namely, when the classical electromagnetic zero-point field two-point correlation functions are evaluated along a uniformly accelerating trajectory, then they are related to the electromagnetic fields of a uniformly accelerating electric dipole in the same way that occurs for the relationships of Sec. II in the unaccelerated case.

APPENDIX

Section IV presented a proof that the expectation value of the Poynting vector, evaluated in the presence of isotropic and homogeneous random (Gaussian) classical electromagnetic radiation, is unchanged from its null value when a harmonic dipole oscillator is also present. Generalizing the relationships of Sec. II to the relationships found in Ref. 9, this proof may be extended to the case of an oscillator uniformly accelerating through classical electromagnetic zero-point radiation. In order to make this extension, a fair degree of familiarity is required of Ref. 9. Consequently, the calculation sketched below assumes that Ref. 9 is readily accessible to the reader.

In keeping with the work of Ref. 9, let $\hat{x}$ be along the direction of acceleration. The corresponding relationships to Eqs. (16) and (22) of the present article were evaluated in Ref. 9 for the case where the vector position $\vec{R}$ [here, $\vec{R}$ corresponds to the argument of η_{ij} and ρ_{ij} in Eqs. (16) and (22)] was contained in the plane that was accelerating along with the oscillator and oriented such as to be perpendicular to the $\hat{x}$ direction [see Eqs. (A39)-(A46) of Ref. 9]. The Fermi- Walker transported coordinate system used in Refs. 8 and 9 was constructed so as to have one coordinate axis along the $\hat{x}$ direction and two orthogonal coordinate axes lying in this accelerating plane and parallel to the $\hat{y}$ and $\hat{z}$ directions. Let $\vec{\rho}$ indicate a vector in this coordinate system; let $\vec{\rho}=\hat{y}R$ be a point in the accelerating plane at a distance R along the y axis from the accelerating oscillator. It is at this point that the Poynting vector will be evaluated.

Equations (82)–(88) still hold for this accelerating system when $\vec{R}$ is replaced by $\vec{\rho}=\hat{y}R$, t is replaced by the proper time τ_e of the accelerating oscillator's equilibrium point, all fields are evaluated in the instantaneous inertial rest frame of the oscillator's equilibrium point, $\mathcal{h}^2_{in}(\omega)$ is replaced by Eq. (7) for $T=\hbar a/2\pi ck$, and the functions C, n_{ij}, and ρ_{ij} are replaced by their appropriate generalizations of C^a, n^a_{ij}, and ρ^a_{ij} that occur for the accelerated siyuation (#14). The latter functions are given in Eqs. (16) and (A39)–(A46) of Ref. 9. Combining these functions with Eqs. (82)–(88) then yields

$$\langle S^{ZP}_{\tau_e K}(\hat{y}R,\tau_e)\rangle = \frac{c}{(e^2/m)}\int_0^\infty \frac{d\omega}{\omega}\mathcal{h}^2_T(\omega)\Big|_{T=\frac{\hbar a}{2\pi ck}} \mathrm{Im}(n^a_{12})\left[\mathrm{Re}\left(\frac{\rho^a_{32}}{C^a}\right)\delta_{K2}+\mathrm{Re}\left(\frac{\rho^a_{31}}{C^a}\right)\delta_{K1}\right] \quad . \qquad (92)$$

The subscript τ_e on $S^{ZP}_{\tau_e K}$ indicates that the Poynting vector is to be evaluated in the inertial reference frame instantaneously at rest with respect to the accelerating oscillator at proper time τ_e. The superscript ZP on $S^{ZP}_{\tau_e K}$ indicates that the background radiation, through which the trajectory of uniform acceleration takes place, consists of classical electromagnetic zero-point radiation.

As will be noticed in deducing Eq. (92) from Eqs. (82)-(88), a large number of the terms cancel and drop out; nevertheless, Eq. (92) does not equal zero exactly, as was the situation in the unaccelerated case (§15). As shown in Ref. 9, however, when $i \neq j$, then the magnitude of n_{ij}^{a} is approximately $\left(\frac{aR}{c^2}\right)$ times the magnitude of n_{ii}^{a}. Thus, when the small laboratory condition of

$$\frac{aR}{c^2} \ll 1 \qquad (93)$$

is considered, then the result of Eq. (92) shows that the only terms that remain after combining Eqs. (82)-(88) are terms that are first-order in $\left(\frac{aR}{c^2}\right)$, or higher. All terms of zeroth-order in $\left(\frac{aR}{c^2}\right)$ cancel precisely.

It should be noted that restricting attention to terms of zeroth-order in $\left(\frac{aR}{c^2}\right)$ does not mean that $\left\langle S^{zp}_{\tau_e k}(\mathcal{G}R,\tau_e)\right\rangle$ has simply been expanded in a power series in the acceleration and only the zeroth-order term in the acceleration examined. Such a case would be quite trivial, indeed, since then the null result for $\left\langle S^{zp}_{\tau_e k}(\mathcal{G}R,\tau_e)\right\rangle$, obtained to zeroth-order in the acceleration, would simply be a restatement of the unaccelerated result in Sec. IV. On the contrary, the zeroth-order terms in $\left(\frac{aR}{c^2}\right)$ for the four expressions corresponding to Eqs. (84) and (86)-(88) in the acceleration case, do depend upon the acceleration; in particular, they depend upon the spectral function

$$\mathscr{h}^2_T(\omega)\Big|_{T=\frac{\hbar a}{2\pi c k}} = \frac{\hbar\omega}{2\pi^2}\coth\left(\frac{\pi c\omega}{a}\right) \quad . \qquad (94)$$

In addition, these four expressions depend upon the functions C^a, η^a_{ij}, and ρ^a_{ij}, which contribute additional dependency upon acceleration even after dropping terms of order $\mathcal{O}\left(\frac{aR}{c^2}\right)$. Adding these four expressions together in order to form $\langle S^{zp}_{\tau_e k}(\mathfrak{g}R,\tau_e)\rangle$, then yields a null value for $\langle S^{zp}_{\tau_e k}(\mathfrak{g}R,\tau_e)\rangle$, when terms of order $\mathcal{O}\left(\frac{aR}{c^2}\right)$ are ignored.

References 6, 8, and 9 analyzed the equivalency that exists in certain physical properties between a system of classical dipole oscillators in a thermal radiation bath and a similar set of oscillators uniformly accelerating through classical electromagnetic zero-point radiation. As first noted in Ref. 6, the stochastic behavior of a single accelerating oscillator agrees with the behavior of an unaccelerated oscillator bathed in classical electromagnetic thermal radiation characterized by the spectral function of Eq. (94). The calculation outlined above for the expectation value of the total Poynting vector, when a single oscillator is present, shows another property that has a correspondence between the accelerated and unaccelerated-thermal single oscillator situations. In this case, the narrow linewidth approximation used in Refs. 6, 8, and 9 was not required; only the small laboratory condition was needed. Here, the expectation value of the Poynting vector was evaluated in the instantaneous rest frame of the oscillator and in the plane that included the oscillator and that was perpendicular to the direction of acceleration. This quantity was shown to equal zero, provided that terms of order $\mathcal{O}\left(\frac{aR}{c^2}\right)$ were ignored, thereby agreeing with the null effect upon the expectation value of radiated electromagnetic momentum when a harmonic dipole oscillator is included within a thermal radiation bath.

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10) Since the functional form for the electric (magnetic) field of an electric dipole agrees with the functional form for the magnetic (electric) field of a magnetic dipole, then the two-point field correlation functions of this article could just as easily have been related to the fields of a magnetic dipole.

11) M. J. Renne, Physica 53, 193 (1971).

12) See Refs. 1 and 2 for the motivation for using these expressions. Also see T. H. Boyer, Phys. Rev. 186, 1304 (1969).

13) See, for example, J. D. Jackson, Classical Electrodynamics, 2nd ed. (Wiley, New York, 1975), p. 395, Eq. (9.18).

14) As was done in Ref. 9, the notation of n_{ij}^{a} and p_{ij}^{a} stands for the functions $n_{ij}(\hat{x}a, +\hat{y}R, \omega)$ and $p_{ij}(\hat{x}a, +\hat{y}R, \omega)$, where R is assumed to be positive here.

15) The expectation value of the Poynting vector in the Fermi-Walker transported coordinate system may be readily shown to be exactly equal to zero when the oscillator is not present. This may be deduced from the fact that $\langle E^{zp}_{\tau_e i}(\vec{\xi},\tau_e) B^{zp}_{\tau_e j}(\vec{\xi},\tau_e)\rangle = 0$, where $\vec{\xi}$ is a point in the Fermi-Walker transported coordinate system.