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City University of New York PH.D. 1980

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UNITS IN PARAMETERIZED P-ADATROPIC NUMBER FIELDS

by

FARLEY MAWYER

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, the City University of New York.

1980

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This manuscript has been read and accepted for Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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Abstract

UNITS IN PARAMETERIZED P-ADATROPIC NUMBER FIELDS

by

Farley Mawyer

Adviser: Professor Harvey Cohn

Let p be a prime. Let $f(x)=x^n + a_{n-1} x^{n-1} +\ldots+ a_1 x + a_0, a_i \in Z$, $0 \leq i < n$. The polynomial f is p-adatropic if there are n + 1 consecutive integers, c_i, such that $|f(c_i)|$ is a power of p for each i. This paper will attempt to expand on the work of H. Cohn on the calculation of units in fields generated by p-adatropic polynomials. The problem to be discussed here can be divided into three major parts. They are:

1. Find all p-adatropic polynomials of degree < 5 which are given in terms of one or more parameters.

2. Let f(x) be an irreducible p-adatropic polynomial and let $f(\theta) = 0$. Let $k = Q(\theta)$. Using the fact that p-adatropic polynomials give many powers of p at values of x which differ by small integers, together with the fact that $f(m) = N_{k/Q}(m-\theta)$, we factor the n + 1 consecutive ideals, $(\theta-m_i)$, where $|f(m_i)|$ is a power of p. By looking at these ideals as elements of a vector space over Q, we are led to several units.

3. In the cases where we have r independent units, where r is the Dirichlet rank, we prove their independence by showing that the regulator is non-vanishing. This is done, using the computer as a guide, by finding bounds for the real roots of the parameterized defining polynomial and hence bounds for the conjugates of the units themselves.

ACKNOWLEDGEMENTS

I would like to thank my advisor, Professor Harvey Cohn for suggesting a topic which seemed to be designed especially for me. In our discussions he was always encouraging and his guidance is greatly appreciated.

I must thank all my friends and family for the support and encouragement they have given me through all these years. Notable among these have been John Hwang, Joel Levine and Ron Strauss. A special thank you to my parents for all they have done.

Mention should be made of Professor Ralph Kopperman of CCNY for his inspiring method of teaching. I would like to thank Sol Weintraub of Queens College for the marvelous way he taught my first course in Number Theory. I owe numerous thanks to my colleagues at Queens College. Most notable among these is Ken Kramer who introduced me to mathematical research. My thanks go also to the Chairman of the Mathematics Department at Queens College, Joseph Hershenov, who was always able to find a teaching position for me through the many years I spent as a graduate student. A special thank you to Minnie Green for the typing of this manuscript.

Finally, most special thanks go to my wife for her unwavering love, strength, understanding and most of all patience.

TABLE OF CONTENTS

Chapter I Introduction

We assume the reader to be familiar with the basic definitions from algebra which may be found in any standard textbook on the subject. We do feel obliged to mention that by a unit we shall mean an invertible algebraic integer. It is desirable for purposes of ideal factorization to find independent units parametrically. Because of its great difficulty we do not consider the more classical problem of finding fundamental units (see below). For an account of the history of the calculation of units the reader is referred to Zimmer [8].

We make the following definition:

Definition 1.1 Let p be a prime.
Let $f(x) = x_n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, $a_1 \in Z$, $0 \leq i < n$.
The polynomial f is said to be p-adatropic if there exist n + 1 consecutive rational integers, c_i, such that $|f(c_i)|$ is a power of p for each i.

From finite differencing we have the following:

Theorem 1.2 Let f(x) be a monic polynomial of degree n and let $x_0 \in R$.
Let $y_K = f(x_0+K)$, $K = 0, 1, \dots, n$

Then $\sum_{K=0}^{n} (-1)^K \binom{n}{K} y_K = (-1)^n n!$ (Equation 1.3)

Corollary 1.4 $\sum_{K=0}^{n} (-1)^K \binom{n}{K} K^n = (-1)^n n!$

Proof: Let $x_0 = 0$ and $f(x) = x^n$.

Corollary 1.5 Every p-adatropic polynomial has degree greater than or equal to p.

Proof: Obvious, because every term on the left side of equation 1.3 is divisible by p.

Of great importance is the following theorem of Dirichlet:

Theorem 1.6 Let $k = Q(\theta)$ be generated by an irreducible polynomial of degree n with r_1 real roots and r_2 pairs of complex conjugate roots. Let $r = r_1 + r_2 - 1$. Then every unit of the field k can be written as an unique product of the form:

$\Omega = \zeta\, \varepsilon_1^{t_1}\, \varepsilon_2^{t_2} \ldots \varepsilon_r^{t_r}$ $(t_1 \in z)$ where ζ is a root of unity belonging to k and each ε_1 is called a fundamental unit of the field.

If we have a system of r units, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r$ the regulator is defined (see [5]) as the following determinant:

$$R = \begin{vmatrix} \log|\varepsilon_1^{(1)}|, \ldots, & \log|\varepsilon_1^{(r_1)}|, & 2\log|\varepsilon_1^{(r_1+1)}|, \ldots, & 2\log|\varepsilon_1^{(r_1+r_2-1)}| \\ \cdot \quad \cdots & \cdot & \cdot \quad \cdots & \cdot \\ \cdot \quad \cdots & \cdot & \cdot \quad \cdots & \cdot \\ \cdot \quad \cdots & \cdot & \cdot \quad \cdots & \cdot \\ \log|\varepsilon_r^{(1)}|, \ldots, & \log|\varepsilon_r^{(r_1)}|, & 2\log|\varepsilon_r^{(r_1+1)}|, \ldots, & 2\log|\varepsilon_r^{(r_1+r_2-1)}| \end{vmatrix}$$

where $\varepsilon_i^{(j)}$ denotes the j^{th} conjugate of the unit ε_i

Also, as a matter of notation, if it is clear from context which polynomial is involved, Δ shall denote the discriminant of that polynomial. By a statement such as $(2) = 2_1\, 2_2$ we shall mean that the ideal (2) is the product of two ideal factors.

Theorem 1.7 In a field generated by a p-adatropic polynomial of degree p, the prime ideal (p) must split completely.

Proof: Obvious.

From corollary 1.5 we see that there are no linear p-adatropic polynomials. Furthermore, this result dictates that the only p-adatropic quadratic polynomials are those where $p = 2$. These 2-adatropic polynomials were studied extensively by H Cohn [3]. We summarize his results:

Let $v = (-1)^s\, 2^K$. The only parameterized family of 2-adatropic quadratic polynomials is the one given by $f(x) = x^2 + (1-v)\, x + v$. Let $f(\theta) = 0$. We factor the principal ideals:

$$(\theta+1) = 2_1^{K+1}$$

$$(\theta) = 2_2^{K}$$

$$(\theta-1) = 2_1$$

Cohn easily finds the unit $\varepsilon_0 = \frac{(\theta-1)^{K+1}}{(\theta+1)}$

Since the Dirichlet rank is 1, $\varepsilon_0 = \pm\, \varepsilon_1^{\,t}$ for the fundamental unit ε_1. Note that the discriminant of the polynomial is $(v-3)^2 - 8$ so our extensions are real except for $v = 2$ or 4.

Chapter II 2-adatropic cubics

§ 1 Overview

Here, equation 1.3 takes the form:

$$f(-1) - 3f(0) + 3f(1) - f(2) = -6.$$

This gives rise to the cubic equation

$$f(x)=x^3 + a\,x^2 + b\,x + c \quad \text{where}$$

$$c = f(0)$$

$$a = \frac{f(-1) + f(1)}{2} - f(0)$$

$$b = f(1) - 1 - a - c$$

Let $|v| = 2^K,\ K \in Z^+$.

In [4], Cohn listed all parameterized 2-adatropic cubic polynomials.

They are:

	f(-1)	f(0)	f(1)	f(2)
2A	v	4	2	v
2B	v	2	-v	-2v
2C	v	v	-2	-2v
2D	-2	v	v	4
2E	v	2	v	4v
2F	v	-v	-2	4v
2G	2	v	v	8

Table 2.1

We mention that 2C and 2G are merely different parameterizations of the same family of polynomials. This phenomenon occurs because here we have five consecutive powers of 2 as opposed to the four we require. We will now consider each case separately.

§ 2 2A

f(-1)	f(0)	f(1)	f(2)
v	4	2	v

is obtained from the polynomial

$$f(x) = x^3 + \frac{v-6}{2}\,x^2 - \frac{v}{2}\,x + 4$$

The discriminant, Δ, of this polynomial is given by

$$16 \quad \Delta = v^4 - 36 v^3 + 324 v^2 - 1728 v$$

So $\Delta < 0$ only for $v = 2, 4, 8, 16$ and in these cases the Dirichlet rank will be 1.

If $f(\theta) = 0$, we factor the ideals in $Q(\theta)$.

$$(\theta+1) = 2_1^{K}$$

$$(\theta) = 2_2 \, 2_3 \text{ if } K \neq 1, \quad (\theta) = 2_2^{\,2} \text{ if } K = 1$$

$$(\theta-1) = 2_1$$

We have the folloiwng units for $K \neq 1$:

$$\varepsilon_1 = \frac{(\theta-1)^K}{(\theta+1)}, \quad \varepsilon_2 = \frac{\theta(\theta-1)}{2}$$

Theorem 2.2 For those values of $v \neq -2$ where $\Delta > 0$, ε_1 and ε_2 are independent units.

Proof:

Case I: $v < 0$

We find

$$-1 < \theta_1 < 0, \quad 1 < \theta_2 < 2, \quad \frac{4-v}{2} < \theta_3 < \frac{6-v}{2}$$

$$0 < |\theta_1| < 1 \qquad 1 < |\theta_2| < 2$$

$$1 < |\theta_1 - 1| < 2 \qquad 0 < |\theta_2 - 1| < 1$$

$$0 < |\theta_1 + 1| < 1 \qquad 2 < |\theta_2 + 1| < 3$$

Thus,

$$|\varepsilon_1^{(1)}| = \frac{|\theta_1 - 1|^K}{|\theta_1 + 1|} > |\theta_1 - 1|^K > 1$$

$$|\varepsilon_2^{(1)}| = \frac{|\theta_1|\,|\theta_1 - 1|}{2} < \frac{|\theta_1 - 1|}{2} < 1$$

$$|\varepsilon_1^{(2)}| = \frac{|\theta_2-1|^K}{|\theta_2+1|} < \frac{1}{2} < 1$$

$$|\varepsilon_2^{(2)}| = \frac{|\theta_2||\theta_2-1|}{2} < 1$$

$$\therefore \quad R = \ln|\varepsilon_1^{(1)}| \cdot \ln|\varepsilon_2^{(2)}| - \ln|\varepsilon_1^{(2)}| \cdot \ln|\varepsilon_2^{(1)}| < 0$$

Case II: $v \geq 128$

$$\frac{8}{v} < \theta_1 < \frac{9}{v}, \quad \frac{v-5}{v} < \theta_2 < \frac{v-4}{v}, \quad \theta_3 < 0$$

$$K \ln\left|\frac{v-9}{v}\right| - \ln\left|\frac{v+9}{v}\right| < \ln|\varepsilon_1^{(1)}| < K \ln\left|\frac{v-8}{v}\right| - \ln\left|\frac{v+8}{v}\right|$$

$$K \ln\left|\frac{1}{2}\right| - \ln 2 < \ln|\varepsilon_1^{(1)}| < 0$$

$$-(K+1)\ln 2 < \ln|\varepsilon_1^{(1)}| < 0$$

$$\ln\left|\frac{8(v-9)}{2v^2}\right| < \ln|\varepsilon_2^{(1)}| < \ln\left|\frac{9(v-8)}{2v^2}\right|$$

$$\ln|\varepsilon_2^{(1)}| < \ln\left|\frac{8}{v}\right|$$

$$K \ln\left|\frac{4}{v}\right| - \ln\left|\frac{2v-4}{v}\right| < \ln|\varepsilon_1^{(2)}| < K \ln\left|\frac{5}{v}\right| - \ln\left|\frac{2v-5}{v}\right|$$

$$\ln|\varepsilon_1^{(2)}| < K \ln\left|\frac{8}{v}\right|$$

$$\ln|\varepsilon_1^{(2)}| < K(3-K)\ln 2$$

$$\ln\left|\frac{2(v-5)}{v^2}\right| < \ln|\varepsilon_2^{(2)}| < \ln\left|\frac{5(v-4)}{2v^2}\right|$$

$$-K \ln 2 < \ln|\varepsilon_2^{(2)}| < 0$$

$$R > [K(3-K)\ln 2][(3-K)\ln 2] - K(K+1)(\ln 2)^2$$

$$R > (\ln 2)^2 K(K^2-7K+8) > 0 \quad \text{since } K \geq 7.$$

Case III: $v = 32$ and $v = 64$ are easily verified by a hand caluclation.

§ 3 2B

$f(-1)$	$f(0)$	$f(1)$	$f(2)$
v	2	$-v$	$-2v$

is obtained from the polynomial

$$f(x) = x^3 - 2x^2 - (v+1)\,x + 2$$

$$\Delta = 4(v^3 + 4v^2 + 23v + 9)$$

$$\Delta \geq 0 \text{ if } v \geq 0 \qquad (\text{Recall } |v| = 2^K)$$

We factor (for $K \neq 1$)

$$(\theta+1) = 2_2\, 2_3^{K-1}, \qquad (\theta-1) = 2_2^{K-1}\, 2_3$$

$$(\theta) = 2_1 \qquad , \qquad (\theta-2) = 2_1^{K+1}$$

We have the unit $\varepsilon_2 = \frac{(\theta)^{K+1}}{(\theta-2)}$ and we also find $\varepsilon_1^* = \frac{(\theta+1)(\theta-1)\theta^K}{2^K}$ but unfortunately we discover $\varepsilon_1^* = \pm\,\varepsilon_2$.

We may observe that when $v = t^2$ $(t > 0)$

$$f(1+t) = -\,2t$$

$$f(1-t) = 2t$$

and $\varepsilon_1 = \frac{(\theta-1-t)}{(\theta-1+t)}$

Theorem 2.3: For the field $Q(\theta)$ defined by the polynomial $f(x) = x^3 - 2x^2 - (t^2+1)\,x + 2$ where $t = 2^m$, ε_1 and ε_2 are independent units.

Proof:

$$-\,t < \theta_1 < 1 - t,\ t + 1 < \theta_2 < t + 2,\ 0 < \theta_3 < 1$$

$$|\varepsilon_1^{(1)}| > 2t > |$$

$$|\varepsilon_2^{(1)}| > (t-1)^{2m}/(t+2) > 1 \quad \text{if } m > 1$$

$$|\varepsilon_1^{(2)}| < \frac{1}{2t} < 1$$

$$|\varepsilon_2^{(2)}| > \frac{(t+1)^{2m+1}}{t} > 1$$

$\therefore$ $m > 1 \Longrightarrow R > 0$

If $m = 1$ we find $R \simeq 7.066$

§ 4 2D

$f(-1)$	$f(0)$	$f(1)$	$f(2)$
-2	v	v	4

where $f(x) = x^3 - \frac{v+2}{2}x^2 + \frac{v}{2}x + v$

and $16\,\Delta = 9v^4 - 28v^3 - 476v^2 + 64v.$

So $\Delta < 0$ iff $v = -4, -2, 2, 4, 8.$

Here (for $K \neq 1$), $(\theta+1) = 2_1$,

$$(\theta) = 2_2^{\,m}\, 2_3^{\,K-m} \qquad m \leq K$$

$$(\theta-1) = 2_1^{\,K}$$

$$(\theta-2) = 2_2\, 2_3$$

$$(\theta-\tfrac{v}{2}) = 2_2^{\,m}\, 2_3^{\,K-m}$$

$$\varepsilon_1 = \frac{(\theta+1)^K}{(\theta-1)}, \quad \varepsilon_2 = \frac{(\theta+1)(\theta-2)}{2}$$

Theorem 2.4: When $\Delta > 0$, the units ε_1 and ε_2 form an independent system.

Proof:

Case I: $v \geq 64$

$$\frac{1-v}{v} < \theta_1 < \frac{2-v}{v}, \quad \frac{2v+2}{v} < \theta_2 < \frac{2v+3}{v}, \quad \frac{v-1}{2} < \theta_3 < \frac{v}{2}$$

Therefore,

$$|\varepsilon_1^{(1)}| < 1, \qquad |\varepsilon_1^{(2)}| > 1$$

$$|\varepsilon_2^{(1)}| < 1, \qquad |\varepsilon_2^{(2)}| < 1$$

and $R = \ln |\varepsilon_1^{(1)}| \ln |\varepsilon_2^{(2)}| - \ln |\varepsilon_2^{(1)}| \ln |\varepsilon_1^{(2)}| > 0$

Case II: $v \leq -16$

$$\frac{-3}{2} < \theta_1 < -1,\ \frac{2v+3}{v} < \theta_2 < \frac{2v+2}{v},\ \frac{v}{2} < \theta_3 < \frac{v+1}{2}$$

We find,

$$|\varepsilon_1^{(1)}| < 1,\ |\varepsilon_2^{(1)}| < 1,\ |\varepsilon_2^{(2)}| < 1,\ |\varepsilon_1^{(2)}| > 1$$

and $R \neq 0$

Case III: $v = -8, 16, 32$

We see $|R| \simeq 7.827, 20.765, 48.085$, respectively.

§ 5 2E

f(-1)	f(0)	f(1)	f(2)
v	2	v	4v

are found from

$$f(x) = x^3 + (v-2)\, x^2 - x + 2 = (x+1)(x-1)(x-2) + v\, x^2$$

$$\Delta = -(8v^3 - 49v^2 + 136v - 36)$$

We see that for $|v| = 2^K$, v and Δ have the same sign.

We factor:

$$(\theta+1) = 2_2\, 2_3^{K-1}$$

$$(\theta) = 2_1$$

$$(\theta-1) = 2_2^{K-1}\, 2_3$$

$$(\theta-2) = 2_1^{K+2}$$

$$(\theta-2+v) = 2_1^{K}$$

We observe the units:

$$\varepsilon_1 = \frac{(\theta)^{K+2}}{(\theta-2)};\ \varepsilon_2 = \frac{(\theta^2-1)(\theta)^K}{2^K};\ \varepsilon_3 = \frac{(\theta)^K}{(\theta-2+v)}$$

and note with regret that

$\varepsilon_1 = \pm\, \varepsilon_2 = \pm\, \varepsilon_3$ and so when the Dirichlet rank is 2, we are at a loss to parametrically find a second independent unit.

§ 6 2F

$f(-1)$	$f(0)$	$f(1)$	$f(2)$
v	$-v$	-2	$4v$

are found from $f(x) = x^3 + \frac{3v-2}{2} x^2 - \frac{v+4}{2} x - v$

$16\Delta = 225 v^4 - 148 v^3 + 724 v^2 - 416 v + 576$

We factor: $(\theta+1) = 2_1{}^K$ $\qquad (\theta-1) = 2_1$

$(\theta) = 2_2{}^{K-1}\, 2_3$ $\qquad (\theta-2) = 2_2\, 2_3{}^{K+1}$

Our units are $\varepsilon_1 = \frac{(\theta-1)^K}{(\theta+1)}$ and $\varepsilon_2 = \frac{(\theta)^K (\theta-1)^{K^2-2} (\theta-2)^{K-2}}{2^{K^2-2}}$

Theorem 2.5: ε_1 and ε_2 are independent units.

Proof: Case I: $v \geq 4$

$$-\frac{2}{3} < \theta_1 < 0, \quad 1 < \theta_2 < \frac{3}{2}, \quad \theta_3 < -1$$

$$\therefore \quad |\varepsilon_1{}^{(1)}| > 1, \quad |\varepsilon_1{}^{(2)}| < 1$$

$$|\varepsilon_2{}^{(1)}| < \frac{5^{K^2-2}}{2^{K^2-4K+4}\, 3^{K^2+2K-4}} < 1, \quad |\varepsilon_2{}^{(2)}| < \frac{3^K}{2^{2K^2+K-4}} < 1$$

and $|R| > 0$

Case II: $v \leq -4$

$$-\frac{3}{4} < \theta_1 < \frac{-2}{3}, \quad 0 < \theta_2 < 1, \quad \theta_3 > 1$$

$$|\varepsilon_1{}^{(1)}| > 1 \qquad |\varepsilon_1{}^{(2)}| < 1$$

$$|\varepsilon_2{}^{(1)}| < 1 \qquad |\varepsilon_2{}^{(2)}| < 1$$

and $|R| > 0$

Case III: $v = 2, -2$

$R \simeq -.43023$ and $.14037$ respectively.

§ 7 2G

$f(-2)$	$f(-1)$	$f(0)$	$f(1)$	$f(2)$
$-2v$	2	v	v	8

come from $f(x) = x^3 - \frac{v-2}{2} x^2 + \frac{v-4}{2} x + v$.

$16 \quad \Delta = 9 v^4 - 140 v^3 + 244 v^2 - 1120 v + 576$

We see that $\Delta < 0$ only for $v = 2, 4, 8$

We factor:

$K \neq 1$	$K = 1$
$(\theta+1) = 2_1$	$(\theta+1) = 2_1$
$(\theta) = 2_2\, 2_3^{K-1}$	$(\theta) = 2_2$
$(\theta-1) = 2_1^{K}$	$(\theta-1) = 2_1$
$(\theta-2) = 2_2^{2}\, 2_3$	$(\theta-2) = 2_2^{3}$

Our units are $\varepsilon_1 = \frac{(\theta+1)^K}{(\theta-1)}$ and

(if $K \neq 1$) $\varepsilon_2 = \frac{(\theta+1)^{2K-3}(\theta)(\theta-2)^{K-2}}{2^{2K-3}}$

(if $K = 1$) $\varepsilon_2 = \frac{(\theta-2)}{(\theta)^3}$

Theorem 2.6: When $\Delta > 0$, ε_1 and ε_2 are independent.

Proof: Case I: $v \leq -16$

$$-1 < \theta_1 < -\frac{1}{2}\, ,\ \frac{7}{4} < \theta_2 < 2\, ,\ \theta_3 < -1$$

Here, $|\varepsilon_1^{(1)}| < 1$, $|\varepsilon_2^{(1)}| < 1$, $|\varepsilon_1^{(2)}| > 1$, $|\varepsilon_2^{(2)}| < 1$

and $|R| > 0$

Case II: $v \geq 128$

$$-\frac{3}{2} < \theta_1 < -1,\ 2 < \theta_2 < \frac{9}{4}\, ,\ \theta_3 > \frac{9}{4}$$

$|\varepsilon_1^{(1)}| < 1$, $|\varepsilon_2^{(1)}| < 1$, $|\varepsilon_1^{(2)}| > 1$ and $|\varepsilon_2^{(2)}| < 1$

So $|R| > 0$

Case III: $v = -8, -4, -2, 16, 32, 64$

We may find $R \neq 0$ by inspection.

Chapter III 3-adatropic cubics

§ 1 Overview

From the difference equation $f(-1) - 3f(0) + 3f(1) - f(2) = -6$ we may assume, without loss of generality, that $|f(-1)| = 3$. It can then be shown that the only parameterized families of 3-adatropic cubic polynomials are the following:

	f(1)	f(0)	f(1)	f(2)
3A	3	v	v	9
3B	3	3	v	3v
3C	3	v	-3	-3v
3D	-3	v	v	3

Table 3.1

We define K as $|v| = 3^K$, $K \in Z^+$.

We also note that the prime ideal (3) must split in $Q(\theta)$, where $f(\theta) = 0$.

We will now proceed, as in Chapter II, to examine each case separately. That is for each parameterized family we will find units algebraically and then, using the computer for guidance, bound them numerically. We will then manipulate these inequalities to show that the regulator is non-vanishing.

§ 2 3A

f(-1)	f(0)	f(1)	f(2)
3	v	v	9

where $f(x) = x^3 - \frac{v-3}{2} x^2 + \frac{v-5}{2} x + v$

and $16\,\Delta = 9\,v^4 - 168\,v^3 + 574\,v^2 - 2136\,v + 1225$.

So $\Delta < 0$ only when $v = 3$ or 9.

We factor: $(\theta+1) = 3_1 \qquad (\theta-1)\ 3_3^{\ K}$

$(\theta) = 3_2^{\ K} \qquad (\theta-2) = 3_1^{\ 2}$

We find the units $\varepsilon_1 = \frac{(\theta+1)^2}{(\theta-2)}$ and $\varepsilon_2 = \frac{(\theta+1)^K(\theta)(\theta-1)}{3^K}$

Theorem 3.2 When $\Delta > 0$, ε_1 and ε_2 are independent.

Proof:

Case I: $v \leq -81$

$$\frac{-v-1}{v} < \theta_1 < \frac{-v-2}{v}, \quad \frac{2v+6}{v} < \theta_2 < \frac{2v+5}{v}, \quad 1 < \theta_3 < 2$$

We discover that:

$$\frac{1}{3v^2} < \frac{1}{3v^2+v} < |\varepsilon_1^{(1)}| < \frac{4}{3v^2+2v} < \frac{1}{2v^2}$$

$$3^{-K(K+1)} < \frac{2(v+2)(v+1)}{(-v)^{K+3}} < |\varepsilon_2^{(1)}| < \frac{2^K(v+1)(2v+1)}{(-v)^{K+3}} < 3^{-(K-1)(K+1)}$$

$$\frac{-3(v+4)}{2} < \frac{(3v+6)^2}{-6v} < |\varepsilon_1^{(2)}| < \frac{(3v+5)^2}{-5v} < \frac{-9v}{5}$$

$$1 < \frac{|3v+6|^K(2v+6)(v+6)}{(-v)^{K-3}} < |\varepsilon_2^{(2)}| < \frac{|3v+5|^K(2v+5)(v+5)}{(-v)^{K+3}} < 2$$

and $R = \ln|\varepsilon_1^{(1)}| \cdot \ln|\varepsilon_2^{(2)}| - \ln|\varepsilon_1^{(2)}| \cdot \ln|\varepsilon_2^{(1)}|$

$$> \ln 3 \left[(K+1)(K-1)\ln\left|\frac{3(v+4)}{2}\right| - (2K+1)\ln 2\right] > 0$$

Case II: $v \geq 27$

$$\frac{-v-3}{v} < \theta_1 < \frac{-v-2}{v}, \quad \frac{2v+6}{v} < \theta_2 < \frac{2v+7}{v}, \quad \theta_3 > 3$$

It follows that

$$\frac{1}{v^2} < \frac{4}{3v^2+3v} < |\varepsilon_1^{(1)}| < \frac{9}{3v^2+2v} < 1$$

$$\frac{2^{K+1}(v+1)(v+2)}{v^{K+3}} < |\varepsilon_2^{(1)}| < \frac{(v+3)(2v+3)}{v^{K+2}} < \frac{3}{v^K} < 1$$

$$1 < v < \frac{(3v+6)^2}{7v} < |\varepsilon_1^{(2)}| < \frac{(3v+7)^2}{6v}$$

$$2 < \frac{2(v+2)^K(v+3)(v+6)}{v^{K+2}} < |\varepsilon_2^{(2)}| < \frac{(3v+7)^K(2v+7)(v+7)}{v^{K+3}} < v$$

$\therefore\quad R = \ln|\varepsilon_1^{(1)}| \cdot \ln|\varepsilon_2^{(2)}| - \ln|\varepsilon_1^{(2)}| \cdot \ln|\varepsilon_2^{(1)}| > K(\ln 3)^2[K^2-2K-1] > 0$

Case III: $v = -27, -9, -3$

$R \neq 0$ is seen by inspection.

§ 3 3B

$f(-1)$	$f(0)$	$f(1)$	$f(2)$
3	3	v	3v

are values of $f(x)=x^3 + \frac{v-3}{2} x^2 + \frac{v-5}{2} x + 3$

and $16\,\Delta = v^4 - 48\,v^3 + 646\,v^2 - 3216\,v + 1225.$

So $\Delta < 0$ for $v = 3, 9, 27$

We observe that 3 must split into principal factors here. In fact,

$$(\theta+1) = 3_1 \qquad (\theta-1) = 3_3^{\,K} \qquad (\theta+\tfrac{v-5}{2}) = 3_3$$

$$(\theta) = 3_2 \qquad (\theta-2) = 3_1^{\,K+1}$$

We have the units $\varepsilon_1 = \frac{(\theta+1)^{K+1}}{(\theta-2)}$ and $\varepsilon_2 = \frac{(\theta+1)^K(\theta)^K(\theta-1)}{3^K}$

Theorem 3.3 When $\Delta > 0$, ε_1 and ε_2 are independent.

Proof: Case I: $v \geq 243$

$$\frac{6-v}{v} < \theta_1 < \frac{7-v}{v}\,,\quad \frac{-7}{v} < \theta_2 < \frac{-6}{v}\,,\quad \theta_3 < -1$$

It can be shown that

$$|\varepsilon_1^{(1)}| < 3^{1+K-K^2} \qquad 3^{-K-2} < |\varepsilon_1^{(2)}| < 1$$

$$3^{-K^2} < |\varepsilon_2^{(1)}| < 1 \qquad |\varepsilon_2^{(2)}| < 3^{K-K^2}$$

So $R > (\ln 3)^2\, K[K^3-3K^2-2K+1] > 0$ since $K \geq 5$

Case II: $v \leq -81$

$$\frac{6-v}{v} < \theta_1 < \frac{5-v}{v}\,,\quad \frac{-5}{v} < \theta_2 < \frac{-6}{v}\,,\quad \theta_3 > 2$$

We may show

$|\varepsilon_1^{(1)}| < 3^{1+K-K^2}$ $\qquad$ $\frac{1}{3} < |\varepsilon_1^{(2)}| < 1$

$3^{-K^2} < |\varepsilon_2^{(1)}| < 1$ $\qquad$ $|\varepsilon_2^{(2)}| < 3^{2K-K^2}$

So $R > K[K^3-3K^2+2]\ (\ln 3)^2 > 0$ since $K \geq 4$

Case III: $v = 81, -3, -9, -27$

We find $R \neq 0$ by inspection.

§ 4 3C

$f(-1)$	$f(0)$	$f(1)$	$f(2)$
3	v	-3	-3v

come from $f(x) = x^3 - v\,x^2 - 4\,x + v$ and

$\Delta = 4\,v^4 + 61\,v^2 + 256 > 0$

Again, 3 must split into principal factors.

$(\theta+1) = 3_1$ $\qquad$ $(\theta-1) = 3_3$

$(\theta) = 3_2^{\ K}$ $\qquad$ $(\theta-2) = 3_1^{\ K+1}$

Our units are $\varepsilon_1 = \dfrac{(\theta+1)^{K+1}}{(\theta-2)}$ and $\varepsilon_2 = \dfrac{(\theta-1)^K(\theta+1)^K\,\theta}{3^K}$

Theorem 3.4 ε_1 and ε_2 are independent

Proof: Case I: $v \geq 27$

$$\frac{-v-2}{v} < \theta_1 < \frac{-v-1}{v}\,,\ \frac{v-2}{v} < \theta_2 < \frac{v-1}{v}\,,\ \theta_3 > 2.$$

We find,

$$|\varepsilon_1^{(1)}| < \frac{2^{K+1}}{v^K(3v+1)} < 1\,, \qquad |\varepsilon_1^{(2)}| > \frac{4^K}{3^K+2} > 1$$

$$|\varepsilon_2^{(1)}| < \frac{3}{v^{K-2}} < 1\,, \qquad |\varepsilon_2^{(2)}| < \left(\frac{4}{3^{K+1}}\right)^K < 1$$

Clearly, $R \neq 0$.

Case II: $v \leq -3$

$$-1 < \theta_1 < 0, \quad 1 < \theta_2 < \frac{3}{2}, \ \theta_3 < -1$$

$$|\varepsilon_1^{(1)}| < \frac{1}{2} \qquad |\varepsilon_1^{(2)}| > 2^{K+1}$$

$$|\varepsilon_2^{(1)}| < (\frac{2}{3})^K \qquad |\varepsilon_2^{(2)}| < (\frac{5}{12})^K \cdot \frac{3}{2} < 1$$

Again it is clear that $R \neq 0$.

Case III: $v = 3, 9$

Here we find $|R| \doteqdot 6.73$ and 38.68 respectively.

§ 5 3D

f(-1)	f(0)	f(1)	f(2)
-3	v	v	3

where $f(x) = x^3 - \frac{v+3}{2} x^2 + \frac{v+1}{2} x + v.$

$16 \Delta = 9 v^4 - 506 v^2 + 1$, $\Delta < 0$ iff $|v| = 3$

We factor: $(\theta+1) = 3_1 \qquad (\theta-1) = 3_3^K$

$(\theta) = 3_2^K \qquad (\theta-2) = 3_1$

Our units are $\varepsilon_1 = \frac{(\theta+1)}{(\theta-2)}$ and $\varepsilon_2 = \frac{(\theta)(\theta-1)(\theta+1)^K}{3^K}$

Theorem 3.5 When $|v| \neq 3$, ε_1 and ε_2 are independent.

Proof: Case I: $v \leq -27$

$$-2 < \theta_1 < -1, \quad 1 < \theta_2 < 2, \ \frac{v+1}{2} < \theta_3 < \frac{v+3}{2}$$

It can be shown that

$$|\varepsilon_1^{(1)}| < 1 \qquad |\varepsilon_1^{(2)}| < 1$$

$$|\varepsilon_2^{(1)}| < 1 \qquad |\varepsilon_2^{(2)}| > 1$$

$\therefore \ R \neq 0$

Case II: $v \geq 27$

$$\frac{1-v}{v} < \theta_1 < \frac{2-v}{v},\ \frac{2v+2}{v} < \theta_2 < \frac{2v+3}{v},\ \theta_3 > 3$$

It can be shown that

$$\frac{1}{3v-1} < |\varepsilon_1^{(1)}| < \frac{2}{3v-2}\ ,\ \frac{3v+2}{3} < |\varepsilon_1^{(2)}| < \frac{3v+3}{2}$$

$$(\frac{1}{v})^{K+1} < |\varepsilon_2^{(1)}| < (\frac{2}{v})^{K+1},\quad 2 < |\varepsilon_2^{(2)}| < 3$$

$$R > -\ln(3v-1) \cdot \ln 3 + (K+1)\ \ln(\frac{v}{2}) \cdot \ln(\frac{3v+2}{3})$$

$$> (K+1-\ln 3)\ \ln v - (\ln 3)^2 > 0$$

Case III: $|v| = 9$

We compute $|R| \approx 6.73$

Chapter IV 2-adatropic polynomials of degree 4

§ 1 Overview

When n = 4 the difference equation takes the form:

$f(-2) - 4\,f(-1) + 6\,f(0) - 4\,f(1) + f(2) = 24.$

When p = 2 we have so much flexibility in assigning parameters we are led to a rather large number of cases. We list them in Table 4.1 and consider them in greater detail afterward. We mention that Case 10 and Case 11 have been studied by H. Cohn (see [2],[3])

§ 2 The Table

In Table 4.1 we list all parameterized powers of 2 which satisfy the difference equation given at the beginning of the last section. We avoid the symmetry u, v, w, y, z <——> z, y, w, v, u but we remember to include u, v, w, y, z <——> u, y, w, v, z. We also avoid duplication which could occur if one family of polynomials parametrically gives more than 5 consecutive powers of 2. We mention that there are some parameterizations, such as 16v, v, 4, 2v, -4v, which give rise to polynomials with non-integer coefficients. These are also omitted from the table. In the table, 'type' refers to the degree of freedom we have for our parameter according to the following scheme:

P - only $v = (-2)^K$ allowed

M - only $v = -(-2)^K$ allowed

(blank) - both of the above v's may occur

If either symbol is followed by ; F it means that our field $K = Q(\theta)$ has a quadratic subfield.

Case	Type	f(-2)	f(-1)	f(0)	f(1)	f(2)
1	P	8	v	2	-v	4
2	M	16	v	2	-v	-4
3	P	4	-2	2	v	4v
4	M	-4	-4	2	v	4v
5		4	v	2	-2	4v
6		-4	v	2	-4	4v
7	P	32	v	-2	-v	4
8	P	4	-8	-2	v	4v
9		4	v	-2	-8	4v
10	F	4v	v	4	y	4y
11	F	4y	v	4	y	4v
12	F	y	v	4	-v	-y
13	P	32	v	-4	-v	16
14	M;F	64	v	-4	-v	-16
15	P	16	-8	-4	v	4v
16		16	v	-4	-8	4v
17	M	-16	-16	-4	v	4v
18		-16	v	-4	-16	4v
19	M	32	-4	-4	v	4v
20		32	v	-4	-4	4v
21	P	64	4	-4	v	4v
22	P	64	v	-4	4	4v
23	P	v	-4	-4	-8	-v
24	M	v	-8	-4	-4	-v
25	M	32	2	v	v	-2v
26	P	32	2	v	2v	2v
27	M	32	2	2v	v	-8v
28	M	32	2	-2v	v	16v
29	M	-16	2	8	v	4v
30	M	-64	2	16	v	4v
31	M	128	2	-16	v	4v
32	P	v	2	8	4	-v
33	P	v	2	16	16	-v
34	P	v	2	-16	-32	-v
35	P	16	-2	v	v	-2v
36	M	16	-2	v	2v	2v
37	P	16	-2	2v	v	-8v
38	P	16	-2	-2v	v	16v
39	P	-32	-2	8	v	4v
40	P	64	-2	-8	v	4v
41	M	v	-2	8	8	-v
42	M	v	-2	-8	-16	-v
43	P	-8	4	8	v	4v
44	M	8	-4	v	v	-2v
45	P	8	-4	v	2v	2v
46	M	8	-4	-2v	v	16v
47	M	8	-4	2v	v	-8v
48		32	2v	v	2	2v
49		32	v	v	2	-2v
50		32	v	2v	2	-8v

Case	Type	f(2)	f(-1)	f(0)	f(1)	f(2)
51		32	v	-2v	2	16v
52		-64	v	16	2	4v
53		128	v	-16	2	4v
54		-16	v	8	2	4v
55		16	v	v	-2	-2v
56		16	2v	v	-2	2v
57		16	v	2v	-2	-8v
58	F	16	v	-2v	-2	16v
59		-32	v	8	-2	4v
60		64	v	-8	-2	4v
61		-8	v	8	4	4v
62		8	v	v	-4	-2v
63		8	2v	v	-4	2v
64		8	v	2v	-4	-8v
65		8	v	-2v	-4	16v
66	M	8	8	8	v	4v
67		8	v	8	8	4v
68	P	8	v	8	-v	-32
69	P	8	v	-8	-v	64
70	M	8	-16	-8	v	4v
71		8	v	-8	-16	4v
72	P	8	v	2v	2v	16
73	M	8	2v	2v	v	16
74	P	8	v	-2v	-4v	16
75	M	8	-4v	-2v	v	16
76	P	-8	-8	v	v	-2v
77	M	-8	-8	v	2v	2v
78	P	-8	-8	2v	v	-8v
79	P	-8	-8	-2v	v	16v
80	F	-8	v	v	-8	-2v
81		-8	2v	v	-8	2v
82		-8	v	-2v	-8	16v
83	M	-8	v	8	-v	-16
84	P	-8	16	16	v	4v
85		-8	v	16	16	4v
86	M	-8	v	16	-v	-64
87	P	-8	-32	-16	v	4v
88		-8	v	-16	-32	4v
89	M	-8	v	-16	-v	128
90	M	-8	v	-2v	-4v	32
91	P	-8	-4v	-2v	v	32
92	P	-8	2v	2v	v	32
93		16v	2v	4	v	-4v
94		16v	v	4	v	-8v
95		-2v	v	4	-2v	-2v
96		-4v	v	4	-4v	-8v
97	M	2v	2	v	-8	-8v
98	P	2v	-8	v	2	-8v
99	P	2v	-2	v	-4	-8v
100	M	2v	-4	v	-2	-8v

Case	Type	f(-2)	f(-1)	f(0)	f(1)	f(2)
101	P	-2v	2	v	-8	-4v
102	M	-2v	-8	v	2	-4v
103	M	-2v	-2	v	-4	-4v
104	P	-2v	-4	v	-2	-4v

Table 4.1

(Parameterizations leading to 2-adatropic polynomials of degree 4.)

§ 3 The Cases

For each case listed above we shall give the coefficients of its 2-adatropic polynomial, $f(x) = x^4 + A x^3 + B x^2 + C x + D$.

We then give the coefficients of the corresponding cubic resolvent of f, $g(x) = x^3 + B x^2 + B_2 x + B_3$. These are given by (see [6])

$$B_2 = A C - 4D$$

$$B_3 = C^2 + (A^2 - 4B)D$$

If g(x) has a rational integer root, we may factor f(x) over a quadratic extension of Q.

We will mention units that may be found immediately but we do not strive for completeness at this time.

Case 1: 8 v 2 -v 4

$A = (v-1)/3,\ B = -3,\ C = -(4v-1)/3,\ D = 2$

$$B_2 = -\frac{4v^2-5v+73}{9},\quad B_3 = \frac{6v^2-4v+73}{3}$$

$$\varepsilon_1 = \frac{(\theta)^3}{(\theta+2)},\quad \varepsilon_2 = \frac{(\theta)^2}{(\theta-4)}$$

Case 2: 16 v 2 -v -4

$A = (v-5)/3,\ B = -3,\ C = -(4v-5)/3,\ D = 2$

$$B_2 = -\frac{4v^2-25v+97}{9},\quad B_3 = \frac{6v^2-20v+97}{3}$$

$$\varepsilon_1 = \frac{(\theta)^4}{(\theta+2)},\quad \varepsilon_2 = \frac{(\theta)^2}{(\theta-2)}$$

Case 3: 4 -2 2 v 4v

$A = (v-4)/6,\ B = (v-8)/2,\ C = (v+5)/3,\ D = 2$

$$B_2 = \frac{v^2+v-164}{18}, \quad B_3 = \frac{v^2-20v+214}{6}$$

$$\varepsilon_1 = \frac{(\theta)^2}{(\theta+2)}, \quad \varepsilon_2 = \frac{(\theta)^{K+2}}{(\theta-2)}, \quad \varepsilon_3 = \frac{(\theta+1)^K}{(\theta-1)}$$

Case 4: -4 -4 2 v 4v

$A = (v-2)/6,\ B = (v-10)/2,\ C = (v+7)/3,\ D = 2$

$$B_2 = \frac{v^2+5v-158}{18}, \quad B_3 = \frac{v^2-16v+274}{6}$$

$$\varepsilon_1 = \frac{(\theta)^2}{(\theta+2)}, \quad \varepsilon_2 = \frac{(\theta)^{K+2}}{(\theta-2)}$$

Case 5: 4 v 2 -2 4v

$A = v/2,\ B = (v-8)/2,\ C = -\ (v+1),\ D = 2$

$$B_2 = -\frac{v^2+v+16}{2}, \quad B_3 = \frac{3v^2-4v+66}{2}$$

$$\varepsilon_1 = \frac{(\theta)^2}{(\theta+2)}, \quad \varepsilon_2 = \frac{(\theta)^{K+2}}{(\theta-2)}$$

Case 6: -4 v 2 -4 4v

$A = (v+2)/2,\ B = (v-10)/2,\ C = -\ (v+3),\ D = 2$

$$B_2 = -\frac{v^2+5v+22}{2}, \quad B_3 = \frac{3v^2+8v+102}{2}$$

$$\varepsilon_1 = \frac{(\theta)^2}{(\theta+2)}, \quad \varepsilon_2 = \frac{(\theta)^{K+2}}{(\theta-2)}$$

Case 7: 32 v -2 -v 4

$A = (v-7)/3,\ B = 1,\ C = -\ (4v-7)/3,\ D = -\ 2$

$$B_2 = -\frac{4v^2-35v-23}{9}, \quad B_3 = \frac{14v^2-28v+23}{9}$$

$$\varepsilon_1 = \frac{(\theta)^5}{(\theta+2)}, \quad \varepsilon_2 = \frac{(\theta)^2}{(\theta-2)}$$

Case 8: 4 −8 −2 v 4v

$$A = (v-10)/6,\ B = (v-6)/2,\ C = (v+17)/3,\ D = -2$$

$$B_2 = \frac{v^2+7v-26}{18}, \quad B_3 = \frac{v^2+160v+46}{18}$$

$$\varepsilon_1 = \frac{(\theta)^2}{(\theta+2)}, \quad \varepsilon_2 = \frac{(\theta)^{K+2}}{(\theta-2)}$$

Case 9: 4 v −2 −8 4v

$$A = (v+2)/2,\ B = (v-6)/2,\ C = -(v+5),\ D = -2$$

$$B_2 = -\frac{v^2+7v-6}{2}, \quad B_3 = \frac{v^2+24v-2}{2}$$

$$\varepsilon_1 = \frac{(\theta)^2}{(\theta+2)}, \quad \varepsilon_2 = \frac{(\theta)^{K+2}}{(\theta-2)}$$

Case 10: 4v v 4 y 4y

$$f(x) = x^4 + \frac{y-v}{6}x^3 + (\frac{y+v}{2} - 5)x^2 + \frac{y-v}{3}x + 4$$

$$= x^4 + A\,x^3 + B\,x^2 + C\,x + D$$

$$f(x) = (x+4)(x^2 + (B-4)x + 2(A^2-2B))$$

let $\sigma = A^2 - 4\,(B-4)$

$$f(x) = [x^2 + \frac{A+\sqrt{\sigma}}{2}x + 2]\ [x^2 + \frac{A-\sqrt{\sigma}}{2}x + 2]$$

H. Cohn has studied the fields generated by these polynomials in some detail (see [3]). We merely mention this case for completeness.

Case 11: 4y v 4 y 4v

$$f(x) = x^4 + \frac{v-y}{2}x^3 + (\frac{v+y}{2} - 5)x^2 + (y-v)\,x + 4$$

$$= x^4 + A\,x^3 + B\,x^2 + C\,x + D$$

$$g(x) = (x-4)(x^2 + (B+4)\,x - 2(A^2-2B))$$

let $\sigma = A^2 - 4(B+4)$

$$f(x)=[x^2+\frac{A+\sqrt{\sigma}}{2}\,x-2]\;[x^2+\frac{A-\sqrt{\sigma}}{2}\,x-2]$$

This case was also done by H. Cohn (see [3]).

Case 12: $y \quad v \quad 4 \quad -v \quad -y$

$$f(x) = x^4+\frac{2v-y}{6}x^3-5x^2+\frac{y-8v}{6}x+4$$

$$= x^4+A\,x^3+B\,x^2+C\,x+D$$

$$g(x) = x^3-5x^2-(A^2+vA+16)\,x+(5A^2+2vA+v^2+80)$$

Case 13: $32 \quad v \quad -4 \quad -v \quad 16$

$A = (v-4)/3$, $B = 3$, $C = -(4v-4)/3$, $D = -4$

$$B_2=-\frac{4v^2-20v-128}{9}\;,\; B_3=\frac{4(v^2+32)}{3}$$

Case 14: $64 \quad v \quad -4 \quad -v \quad -16$

$A = (v-20)/3$, $B = 3$, $C = -(4v-20)/3$, $D = -4$

$B_2 = -4(v^2-25v+64)/9$, $B_3 = 4(v-8)(v+8)/3$

$g(x)$ has $-2(v-8)/3$ as a root

let $\sigma = (v^2-16v+100)/9$, $m = (v-8)/3$

$$f(x) = (x^2+\frac{A+\sqrt{\sigma}}{2}x+m+\sqrt{\sigma})(x^2+\frac{A-\sqrt{\sigma}}{2}x+m-\sqrt{\sigma})$$

$$\varepsilon_1 = \frac{m+\sqrt{\sigma}}{2}$$

Also if $(\theta) = 2_1^{\,2}$ or $\tilde{2}_1$

$$\varepsilon_2=\frac{(\theta)^3}{(\theta+2)} \quad \text{and} \quad \varepsilon_3=\frac{(\theta)^2}{(\theta-2)}$$

Case 15: $16 \quad -8 \quad -4 \quad v \quad 4v$

$A = (v-16)/6$, $B = (v-2)/2$, $C = (v+20)/3$, $D = -4$

$B_2 = (v^2+4v-32)/18$, $B_3 = 16v$

Case 16: 16 v −4 −8 4v

$A = v/2$, $B = (v-2)/2$, $C = -(v+4)$, $D = -4$

$B_2 = -(v^2+4v-32)/2$, $B_3 = 16v$

Case 17: −16 −16 −4 v 4v

$A = (v-8)/6$, $B = (v-10)/2$, $C = (v+28)/3$, $D = -4$

$B_2 = (v+16)(v+4)/18$, $B_3 = 16v$

Case 18: −16 v −4 −16 4v

$A = (v+8)/2$, $B = (v-10)/2$, $C = -(v+12)$, $D = -4$

$B_2 = -(v+4)(v+16)/2$, $B_3 = 16v$

Case 19: 32 −4 −4 v 4v

$A = (v-20)/6$, $B = (v+2)/2$, $C = (v+16)/3$, $D = -4$

$B_2 = (v-8)(v+4)/18$, $B_3 = 16v$

Case 20: 32 v −4 −4 4v

$A = (v-4)/2$, $B = (v+2)/2$, $C = -v$, $D = -4$

$B_2 = -(v-8)(v+4)/2$, $B_3 = 16v$

Case 21: 64 4 −4 v 4v

$A = (v-28)/6$, $B = (v+10)/2$, $C = (v+8)/3$, $D = -4$

$B_2 = (v-4)(v-16)/18$, $B_3 = 16v$

Case 22: 64 v −4 4 4v

$A = (v-12)/2$, $B = (v+10)/2$, $C = -(v-8)$, $D = -4$

$B_2 = -(v-4)(v-16)/2$, $B_3 = 16v$

Case 23: v −4 −4 −8 −v

$A = -(v-4)/6$, $B = -3$, $C = (v-16)/6$, $D = -4$

$B_2 = -(v^2-20v-512)/36$, $B_3 = -(v^2+512)/12$

Case 24: v -8 -4 -4 -v

$A = -(v+4)/6, \quad B = -3, \quad C = (v+16)/16, \quad D = -4$

$B_2 = -(v^2+20v-512)/36, \quad B_3 = -(v^2+512)/12$

Case 25: 32 2 v v -2v

$A = -(v+7)/3, \quad B = -v/2, \quad C = (5v+8)/6, \quad D = v$

$B_2 = -(5v^2+115v+56)/18, \quad B_3 = (4v^3+153v^2+276v+64)/36$

$\varepsilon_1 = (\theta+1)^K/(\theta-1)$

Case 26: 32 2 v 2v 2v

$A = -(v+14)/6, \quad B = 0, \quad C = (7v+8)/6, \quad D = v$

$$B_2 = -\frac{7v^2+250v+112}{36}, \quad B_3 = \frac{v^3+77v^2+308v+64}{36}$$

$\varepsilon_1 = (\theta+1)^{K+2}/(\theta-1)$

Case 27: 32 2 2v v -8v

$A = -(5v+14)/6, \quad B = -3v/2, \quad C = 4(v+1)/3, \quad D = 2v$

$$B_2 = -\frac{10v^2+110v+28}{9}, \quad B_3 = \frac{25v^3+388v^2+260v+32}{18}$$

$\varepsilon_1 = (\theta+1)^K/(\theta-1)$

Case 28: 32 2 -2v v 16v

$A = 7(v-2)/6, \quad B = 5v/2, \quad C = -2(v-2)/3, \quad D = -2v$

$B_2 = -(7v-2)(v-14)/9, \quad B_3 = -(49v^3-564v^2+228v-32)/18$

$\varepsilon_1 = (\theta+1)^K/(\theta-1)$

Case 29: -16 2 8 v 4v

$A = (v+10)/6, \quad B = (v-16)/2, \quad C = (v-8)/3, \quad D = 8$

$B_2 = (v^2+2v-656)/18, \quad B_3 = (v^2-40v+856)/3$

$\varepsilon_1 = (\theta+1)^K/(\theta-1)$

Case 30: -64 2 16 v 4v

$A = (v+34)/6, \quad B = (v-32)/2, \quad C = (v-20)/3, \quad D = 16$

$B_2 = (v^2+14v-1832)/18, \quad B_3 = (5v^2-56v+14240)/9$

$\varepsilon_1 = (\theta+1)^K/(\theta-1)$

Case 31: 128 2 -16 v 4v

$A = (v-62)/6, \quad B = (v+32)/2, \quad C = (v+28)/3, \quad D = -16$

$B_2 = (v^2-34v-584)/18, \quad B_3 = -(v^2-280v+1792)/3$

$\varepsilon_1 = (\theta+1)^K/(\theta-1)$

Case 32 v 2 8 4 -v

$A = -(v+2)/6, \quad B = -6, \quad C = (v+8)/6, \quad D = 8$

$B_2 = -(v^2+10v+1168)/36, \quad B_3 = (3v^2+16v+2336)/12$

$\varepsilon_1 = (\theta+1)^2/(\theta-1)$

Case 33: v 2 16 16 -v

$A = -(v+14)/6, \quad B = -8, \quad C = (v+56)/6, \quad D = 16$

$B_2 = -(v^2+70v+3088)/36, \quad B_3 = (17v^2+560v+24704)/36$

$\varepsilon_1 = (\theta+1)^4/(\theta-1)$

Case: 34 v 2 -16 -32 -v

$A = -(v-34)/6, \quad B = 0, \quad C = (v-136)/6, \quad D = -16$

$B_2 = -(v^2-170v+2320)/36, \quad B_3 = -v(5v-272)/12$

$\varepsilon_1 = (\theta+1)^5/(\theta-1)$

Case 35: 16 -2 v v -2v

$A = -(v+5)/3, \quad B = -(v+4)/2, \quad C = (5v+16)/6, \quad D = v$

$B_2 = -(5v^2+113v+80)/18, \quad B_3 = (4v^3+137v^2+548v + 256)/36$

$\varepsilon_1 = (\theta+1)^K/(\theta-1)$

Case 36: 16 −2 v 2v 2v

$A = -(v+10)/6, \quad B = -2, \quad C = (7v+16)/6, \quad D = v$

$B_2 = -(7v^2+230v+160)/36, \quad B_3 = (v^3+69v^2+612v+256)/36$

$\varepsilon_1 = (\theta+1)^{K+1}/(\theta-1)$

Case 37: 16 −2 2v v −8v

$A = -5(v+2)/6, \quad B = -(3v+4)/2, \quad C = 4(v+2)/3, \quad D = 2v$

$B_2 = -(10v^2+112v+40)/9, \quad B_3 = (25v^3+348v^2+516v+128)/18$

$\varepsilon_1 = (\theta+1)^{K}/(\theta-1)$

Case 38: 16 −2 −2v v 16v

$A = (7v-10)/6, \quad B = (5v-4)/2, \quad C = -(2v-8)/3, \quad D = -2v$

$B_2 = -(7v^2-110v+40)/9, \quad B_3 = -(49v^2-508v^2+452v-128)/18$

$\varepsilon_1 = (\theta+1)^{K}/(\theta-1)$

Case 39: −32 −2 8 v 4v

$A = (v+14)/6, \quad D = (v-20)/2, \quad C = (v-4)/3, \quad D = 8$

$B_2 = (v^2+10v-632)/18, \quad B_3 = (v^2-32v+1096)/3$

$\varepsilon_1 = (\theta+1)^{K}/(\theta-1)$

Case 40: 64 −2 −8 v 4v

$A = (v-34)/6, \quad B = (v+12)/2, \quad C = (v+20)/3, \quad D = -8$

$B_2 = (v^2-14v-104)/18, \quad B_3 = -(v^2-320v+184)/9$

$\varepsilon_1 = (\theta+1)^{K}/(\theta-1)$

Case 41: v −2 8 8 −v

$A = -(v+10)/6, B = -6, \quad C = (v+40)/6, \quad D = 8$

$B_2 = -(v^2+50v+1552)/36, \quad B_3 = (3v^2+80v+3104)/12$

$\varepsilon_1 = (\theta+1)^{3}/(\theta-1)$

Case 42: v -2 -8 -16 -v

$A = -(v-14)/6, \quad B = -2, \quad C = (v-56)/6, \quad D = -8$

$B_2 = -(v^2-70v-368)/36, \quad B_3 = -(7v^2-112v+736)/36$

$\varepsilon_1 = (\theta+1)^4/(\theta-1)$

Case 43: -8 4 8 v 4v

Is a special case of Case 12.

Case 44: 8 -4 v v -2v

$A = -(v+4)/3, \quad B = -(v+6)/2, \quad C = 5(v+4)/6, \quad D = v$

$B_2 = -(5v^2+112v+80)/18, \quad B_3 = (4v^3+129v^2+696v+400)/36$

Case 45: 8 -4 v 2v 2v

$A = -(v+8)/6, \quad B = -3, \quad C = (7v+20)/6, \quad D = v$

$B_2 = -(7v^2+220v+160)/36, \quad B_3 = (v^3+65v^2+776v+400)/36$

Case 46: 8 -4 -2v v 16v

$A = (7v-8)/6, \quad B = (5v-6)/2, \quad C = -(2v-10)/3, \quad D = -2v$

$B_2 = -(7v^2-115v+40)/9, \quad B_3 = -(49v^3-480v^2+576v-200)/18$

Case 47: 8 -4 2v v -8v

$A = -(5v+8)/6, \quad B = -(3v+6)/2, \quad C = (4v+10)/3, \quad D = 2v$

$B_2 = -(10v^2+113v+40)/9, \quad B_3 = (25v^3+328v^2+656v+200)/18$

Case 48: 32 2v v 2 2v

$A = (v-6)/2, \quad B = 0, \quad C = -(3v-8)/2, \quad D = v$

$B_2 = -(3v^2-10v+48)/4, \quad B_3 = (v^3-3v^2-12v+64)/4$

$\varepsilon_1 = (\theta-1)^{K+1}/(\theta+1)$

Case 49: 32 v v 2 -2v

$A = -3, \quad B = -v/2, \quad C = -(v-8)/2, \quad D = v$

$B_2 = -(5v+24)/2, \quad B_3 = (9v^2+20v+64)/4$

$\varepsilon_1 = (\theta-1)^K/(\theta+1)$

Case 50: 32 v 2v 2 -8v

$A = -(v+6)/2, \quad B = -3v/2, \quad C = 4, \quad D = 2v$

$B_2 = -(10v+12), \quad B_3 = (v^3+36v^2+36v+32)/2$

$\varepsilon_1 = (\theta-1)^K/(\theta+1)$

Case 51: 32 v -2v 2 16v

$A = 3(v-2)/2, \quad B = 5v/2, \quad C = -2(v-2), \quad D = -2v$

$B_2 = -(3v-2)(v-6), \quad B_3 = -(9v^3-84v^2+68v-32)/2$

$\varepsilon_1 = (\theta-1)^K/(\theta+1)$

Case 52: -64 v 16 2 4v

$A = (v+10)/2, \quad B = (v-32)/2, \quad C = -(v+4), \quad D = 16$

$B_2 = -(v^2+14v+168)/2, \quad B_3 = 5v^2 + 56v + 1440$

$\varepsilon_1 = (\theta-1)^K/(\theta+1)$

Case 53: 128 v -16 2 4v

$A = (v-22)/2, \quad B = (v+32)/2, \quad C = -(v-12), \quad D = -16$

$B_2 = -(v^2-34v+136)/2, \quad B_3 = -(3v^2-184v+768)$

$\varepsilon_1 = (\theta-1)^K/(\theta+1)$

Case 54: -16 v 8 2 4v

$A = (v+2)/2, \quad B = (v-16)/2, \quad C = -v, \quad D = 8$

$B_2 = -(v^2+2v+64)/2, \quad B_3 = 3v^2 - 8v + 264$

$\varepsilon_1 = (\theta-1)^K/(\theta+1)$

Case 55: 16 v v -2 -2v

$A = -1, \quad B = -(v+4)/2, \quad C = -v/2, \quad D = v$

$B_2 = -7v/2, \quad B_3 = 9v(v+4)/4$

$\varepsilon_1 = (\theta-1)^K/(\theta+1)$

Case 56: 16 2v v -2 2v

$A = (v-2)/2,\ B = -2,\ C = -3v/2,\ D = v$

$B_2 = -v(3v+10)/4,\ B_3 = v(v^2+5v+36)/4$

$\varepsilon_1 = (\theta-1)^{K+1}/(\theta+1)$

Case 57: 16 v 2v -2 -8v

$A = -(v+2)/2,\ B = -(3v+4)/2,\ C = 0,\ D = 2v$

$B_2 = -8v,\ B_3 = v(v^2+28v+36)/2$

$\varepsilon_1 = (\theta-1)^K/(\theta+1)$

Case 58: 16 v -2v -2 16v

$A = (3v-2)/2,\ B = (5v-4)/2,\ C = -2v,\ D = -2v$

$B_2 = -v(3v-10),\ B_3 = -3v(v-6)(3v-2)/2$

let $\sigma = (9v^2-4v+4)/4 > 0$

$$f(x) = [x^2 + \frac{A+\sqrt{\sigma}}{2} x + A + \sqrt{\sigma}][x^2 + \frac{A-\sqrt{\sigma}}{2} x + A - \sqrt{\sigma}]$$

$\varepsilon_1 = (\theta-1)^K/(\theta+1)$

Case 59: -32 v 8 -2 4v

$A = (v+6)/2,\ B = (v-20)/2,\ C = -(v+4),\ D = 8$

$B_2 = -(v^2+10v+88)/2,\ B_3 = 3v^2 + 16v + 408$

$\varepsilon_1 = (\theta-1)^K/(\theta+1)$

Case 60: 64 v -8 -2 4v

$A = (v-10)/2,\ B = (v+12)/2,\ C = -(v-4),\ D = -8$

$B_2 = -(v^2-14v-24)/2,\ B_3 = -(v^2-48v-8)$

$\varepsilon_1 = (\theta-1)^K/(\theta+1)$

Case 61: -8 v 8 4 4v

$A = v/2,\ B = (v-14)/2,\ C = -(v-2),\ D = 8$

$B_2 = -(v^2-2v+64)/2,\ B_3 = 3v^2 - 20v + 228$

Case 62: 8 v v -4 -2v

$A = 0, \quad B = -(v+6)/2, \quad C = -(v+4)/2, \quad D = v$

$B_2 = -4v, \quad B_3 = (9v^2+56v+16)/4$

Case 63: 8 2v v -4 2v

$A = v/2, \quad B = -3, \quad C = -(3v+4)/2, \quad D = v$

$B_2 = -v(3v+20)/4, \quad B_3 = (v^3+9v^2+72v+16)/4$

Case 64: 8 v 2v -4 -8v

$A = -v/2, \quad B = -3(v+2)/2, \quad C = -2, \quad D = 2v$

$B_2 = -7v, \quad B_3 = (v^3+24v^2+48v+8)/2$

Case 65: 8 v -2v -4 16v

$A = 3v/2, \quad B = (5v-6)/2, \quad C = -2(v+1), \quad D = -2v$

$B_2 = -v(3v-5), \quad B_3 = -(9v^3-48v^2+32v-8)/2$

Case 66: 8 8 8 v 4v

$A = (v+4)/6, \quad B = (v-10)/2, \; C = (v-14)/3, \quad D = 8$

$B_2 = (v^2-10v-632)/18, \quad B_3 = (v^2-52v+556)/3$

Case 67: 8 v 8 8 4v

$A = (v-4)/2, \quad B = (v-10)/2, \; C = -(v-6), \quad D = 8$

$B_2 = -(v^2-10v+88)/2, \quad B_3 = 3v^2 - 44v + 228$

Case 68: 8 v 8 -v -32

$A = (v-10)/3, \quad B = -9, \quad C = -(4v-10)/3, \quad D = 8$

$B_2 = -(4v^2-50v+388)/9, \quad B_3 = (8v^2-80v+1164)/3$

$\varepsilon_1 = (\theta-2)^3(\theta)^2/(\theta+2)^7$

Case 69: 8 v -8 -v 64

$A = (v+14)/3, \quad B = 7, \quad C = -(4v+14)/3, \quad D = -8$

$B_2 = -(4v^2+70v-92)/9, \quad B_3 = (8v^2-112v+644)/9$

Case 70: 8 −16 −8 v 4v

$A = (v-20)/6, \quad B = (v-2)/2, \quad C = (v+34)/3, \quad D = -8$

$B_2 = (v^2+14v-104)/18, \quad B_3 = -(v^2-292v-68)/9$

Case 71: 8 v −8 −16 4v

$A = (v+4)/2, \quad B = (v-2)/2, \quad C = -(v+10), \quad D = -8$

$B_2 = -(v^2+14v-24)/2, \quad B_3 = -(v^2-20v-36)$

Case 72: 8 v 2v 2v 16

$A = -(v-4)/6, \quad B = -(v+2)/2, \quad C = (2v-2)/3, \quad D = 2v$

$B_2 = -(v^2+67v+4)/9, \quad B_3 = (v+2)(v^2+70v+4)/18$

Case 73: 8 2v 2v v 16

$A = (v+4)/6, \quad B = -(v+2)/2, \quad C = -(2v+2)/3, \quad D = 2v$

$B_2 = -(v^2+77v+4)/9, \quad B_3 = (v+2)(v^2+86v+4)/18$

Case 74: 8 v −2v −4v 16

$A = (5v+4)/6, \quad B = (v-2)/2, \quad C = -(10v+2)/3, \quad D = -2v$

$B_2 = -(25v^2-47v+4)/9, \quad B_3 = -(25v^3-232v^2+80v-8)/18$

Case 75: 8 −4v −2v v 16

$A = -(5v-4)/6, \quad B = (v-2)/2, \quad C = (10v-2)/3, \quad D = -2v$

$B_2 = -(25v^2-97v+4)/9, \quad B_3 = -(25v^3-312v^2+240v-8)/18$

Case 76: −8 −8 v v −2v

$A = -(v+2)/3, \quad B = -(v+10)/2, \quad C = (5v+28)/6, \quad D = v$

$B_2 = -(5v^2+110v+56)/18, \quad B_3=(4v^3+113v^2+1016v+784)/36$

Case 77: −8 −8 v 2v 2v (32)

$A = -(v+4)/6, \quad B = -5, \quad C = 7(v+4)/6, \quad D = v$

$B_2 = -(7v^2+200v+112)/36, \quad B_3 = (v^3+57v^2+1128v+784)/36$

Here we have six consecutive powers of two parametrically and we are led to the units

$$\varepsilon_1 = \frac{(\theta-2)^{2K-3}(\theta)^{K-2}}{(\theta+2)^{K^2-K-1}}, \quad \varepsilon_2 = \frac{(\theta-3)^{2K-1}(\theta-1)^2}{(\theta+1)^{4K-1}}$$

Case 78: $-8 \quad -8 \quad 2v \quad v \quad -8v$

$A = -(5v+4)/6, \quad B = -(3v+10)/2, \quad C = (4v+14)/3, \quad D = 2v$

$B_2 = -(10v^2+115v+28)/9, \quad B_3 = (25v^3+288v^2+960v+392)/18$

Case 79: $-8 \quad -8 \quad -2v \quad v \quad 16v$

$A = (7v-4)/6, \quad B = (5v-10)/2, \quad C = -(2v-14)/3, \quad D = -2v$

$B_2 = -(14v^2-250v+56)/18, \quad B_3 = -(v-2)(49v^2-326v+196)/18$

Case 80: $(-2v) \quad -8 \quad v \quad v \quad -8 \quad -2v$

$A = 2, \quad B = -(v+10)/2, \quad C = -(v+12)/2, \quad D = v$

$B_2 = -(5v+12), \quad B_3 = 3(3v+4)(v+12)/4$

let $\lambda = (v+12)/2, \ \tau = [(v+4)^2+2^7]/4$

$g(\lambda) = 0$

$f(x) = [x^2 + x + \frac{-\lambda+\sqrt{\tau}}{2}][x^2 + x + \frac{-\lambda-\sqrt{\tau}}{2}]$

We note that we have a family of polynomials which always gives us powers of 2 at 6 consecutive integers. We are able to factor:

$$(\theta+3) = 2_{12}{}^{K}\, 2_{22}, \quad (\theta-1) = 2_{12}\, 2_{22}{}^{K-1}, \quad (\theta-1) = 2_{12}{}^{2}\, 2_{22}$$

$$(\theta+2) = 2_{11}{}^{2}\, 2_{21}, \quad (\theta) = 2_{11}\, 2_{21}{}^{K-1}, \quad (\theta-2) = 2_{11}{}^{K}\, 2_{21}$$

We find the quadratic unit $\varepsilon_3 = \dfrac{(\theta+2)^{K-1}(\theta-1)^{K-1}}{2^{K-2}(\theta+3)(\theta-2)}$ and

also the unit $\varepsilon_1 = \dfrac{(\theta+3)^{2K-3}(\theta+1)^{K-2}}{(\theta-1)^{K^2-K-1}}$. As an example we take

$K = 4$ and $\theta = \dfrac{-1+\sqrt{29+4\sqrt{33}}}{2}$.

$\varepsilon_3 = 23 + 4\sqrt{33}$ and

$\varepsilon_1 = 957776 + 169782\sqrt{33} - 130489\sqrt{29+4\sqrt{33}} - 23960\sqrt{33}\,\sqrt{29+4\sqrt{33}}$

Case 81: (-8v) -8 2v v -8 2v

$A = (v+4)/2,\quad B = -5,\quad C = -3(v+4)/2,\quad D = v$

$B_2 = -(3v^2+40v+48)/4,\quad B_3 = (v^3+17v^2+168v+144)/4$

Case 82: -8 v -2v -8 16v

$A = (3v+4)/2,\quad B = (5v-10)/2,\quad C = -2(v+3),\quad D = -2v$

$B_2 = -(3v^2+5v+12),\quad B_3 = -(v-2)(9v^2-6v+36)/2$

Case 83: -8 v 8 -v -16

$A = (v-2)/3,\quad B - 9,\quad C = -(4v-2)/3,\quad D = 8$

$B_2 = -(4v^2-10v+292)/9,\quad B_3 = 4(2v^2-4v+219)/3$

Case 84: -8 16 16 v 4v

$A = (v+20)/6,\quad B = (v-18)/2,\quad C = (v-34)/3,\quad D = 16$

$B_2 = (v^2-14v-1832)/18,\quad B_3 = (5v^2-196v+7940)/9$

Case 85: -8 v 16 16 4v

$A = (v-4)/2,\quad B = (v-18)/2,\quad C = -(v-10),\quad D = 16$

$B_2 = -(v^2-14v+168)/2,\quad B_3 = 5v^2 - 84v + 740$

Case 86: -8 v 16 -v -64

$A = (v-14)/3,\quad B = -17,\quad C = -(4v-14)/3,\quad D = 16$

$B_2 = -(4v^2-70v+772)/9,\quad B_3 = (32v^2-560v+13124)/9$

Case 87: -8 -32 -16 v 4v

$A = (v-28)/6,\quad B = (v-2)/2,\quad C = (v+62)/3,\quad D = -16$

$B_2 = (v^2+34v-584)/18,\quad B_3 = -(v^2-212v-44)/3$

Case 88: -8 v -16 -32 4v

$A = (v+12)/2, \quad B = (v-2)/2, \quad C = -(v+22), \quad D = -16$

$B_2 = -(v^2+34v+136)/2, \quad B_3 = -(3v^2+20v+156)$

Case 89: -8 v -16 -v 128

$A = (v+34)/3, \quad B = 15, \quad C = -(4v+34)/3, \quad D = -16$

$B_2 = -(4v^2+170v+580)/9, \quad B_3 = -(272v+2900)/3$

Case 90: -8 v -2v -4v 32

$A = (5v+20)/6, \quad B = (v-2)/2, \quad C = -(10v+10)/3, \quad D = -2v$

$B_2 = -(25v^2+53v+100)/9, \quad B_3 = -(25v^3-72v^2+144v-200)/18$

Case 91: -8 -4v -2v v 32

$A = -(5v+4)/6, \quad B = (v-2)/2, \quad C = 10(v-1)/3, \quad D = -2v$

$B_2 = -(25v^2-197v+100)/9, \quad B_3 = -(25v^3-472v^2+944v-200)/18$

Case 92: -8 2v 2v v 32

$A = (v+20)/6, \quad B = -(v+2)/2, \quad C = -(2v+10)/3, \quad D = 2v$

$B_2 = -(v^2+97v+100)/9, \quad B_3 = (v^3+120v^2+624v+200)/18$

Case 93: 16v 2v 4 v -4v

$A = -3v/2, \quad B = (3v-10)/2, \quad C = v, \quad D = 4$

$B_2 = -(3v^2+32)/2, \quad B_3 = 10v^2 - 24v + 80$

Case 94: 16v v 4 v -8v

$A = -2v, \quad B = v - 5, \quad C = 2v, \quad D = 4$

$B_2 = -4(v^2+4), \quad B_3=4[5v^2 - 4v + 20]/9$

Case 95: -2v v 4 -2v -2v

$A = v/2, \quad B = -(v+10)/2, \quad C = -2v, \quad D = 4$

$B_2 = -(v^2+16), \quad B_3 = 5v^2 + 8v + 80$

Case 96: $-4v \quad v \quad 4 \quad -4v \quad -8v$

$A = v/2, \quad B = -(3v+10)/2, \quad C = -3v, \quad D = 4$

$B_2 = -(3v^2+32)/2, \quad B_3 = 10v^2 + 24v + 80$

Case 97: $2v \quad 2 \quad v \quad -8 \quad -8v$

$A = -(5v-10)/6, \quad B = -(v+4), \quad C = (5v-40)/6, \quad D = v$

$B_2 = -(25v^2-106v+400)/46, \quad B_3 = (25v^3+69v^2+276v+1600)/36$

$\varepsilon_1 = (\theta+1)^3/(\theta-1)$

Case 98: $2v \quad -8 \quad v \quad 2 \quad -8v$

$A = -(5v+10)/6, \quad B = -(v+4), \quad C = (5v+40)/6, \quad D = v$

$B_2 = -(25v^2+394v+400)/36, \quad B_3 = (25v^3+269v^2+1076v+1600)/36$

$\varepsilon_1 = (\theta-1)^3/(\theta+1)$

Case 99: $2v \quad -2 \quad v \quad -4 \quad -8v$

$A = -(5v-2)/6, \quad B = -(v+4), \quad C = (5v-8)/6, \quad D = v$

$B_2 = -(25v^2+94v+16)/36, \quad B_3 = (25v^3+149v^2+500v+64)/36$

$\varepsilon_1 = (\theta+1)^2/(\theta-1)$

Case 100: $2v \quad -4 \quad v \quad -2 \quad -8v$

$A = -(5v+2)/6, \quad B = -(v+4), \quad C = (5v+8)/6, \quad D = v$

$B_2 = -(25v^2+194v+16)/36, \quad B_3 = (25v^3+189v^2+660v+64)/36$

$\varepsilon_1 = (\theta-1)^2/(\theta+1)$

Case 101: $-2v \quad 2 \quad v \quad -8 \quad -4v$

$A = -(v-10)/6, \quad B = -(v+4), \quad C = (v-40)/6, \quad D = v$

$B_2 = -(v^2+94v+400)/36, \quad B_3 = (v^3+125v^2+596v+1600)/36$

$\varepsilon_1 = (\theta+1)^3/(\theta-1)$

Case 102: $-2v \quad -8 \quad v \quad 2 \quad -4v$

$A = -(v+10)/6, \quad B = -(v+4), \quad C = (v+40)/6, \quad D = v$

$B_2 = -(v^2+194v+400)/36, \quad B_3 = (v^3+165v^2+756v+1600)/36$

$\varepsilon_1 = (\theta-1)^3/(\theta+1)$

Case 103: $-2v \quad -2 \quad v \quad -4 \quad -4v$

$A = -(v-2)/6, \quad B = -(v+4), \quad C = (v-8)/6, \quad D = v$

$B_2 = -(v^2+134v+16)/36, \quad B_3 = (v^3+141v^2+564v+64)/36$

$\varepsilon_1 = (\theta+1)^2/(\theta-1)$

Case 104: $-2v \quad -4 \quad v \quad -2 \quad -4v$

$A = -(v+2)/6, \quad B = -(v+4), \quad C = (v+8)/6, \quad D = v$

$B_2 = -(v^2+154v+16)/36, \quad B_3 = (v^3+149v^2+596v+64)/36$

$\varepsilon_1 = (\theta-1)^2/(\theta+1)$

Chapter V: 3-adatropic polynomials of degree 4.

§ 1 Overview

Our difference equation here, as in the last chapter, is $f(-2) - 4\,f(-1) + 6\,f(0) - 4\,f(1) + f(2) = 24$. Upon examining this equation we see that either $f(-2) + 6\,f(0) + f(2) = 0$ and $f(-1) + f(1) = -6$ or else $f(1) = -f(-1)$ and $f(-2) + 6\,f(0) + f(2) = 24$. We thus find that there are only the following 6 cases (as usual $|v| = 3^K$):

	f(-2)	f(-1)	f(0)	f(1)	f(2)
3A	-3v	-3	v	-3	-3v
3B	3	v	3	-v	3
3C	-3	v	3	-v	9
3D	-3	v	-9	-v	81
3E	-3	v	9	-v	-27
3F	-3v	-9	v	3	-3v

We will now proceed as in the last chapter to examine each case separately. Only in case 3B do we find 3 independent units.

§ 2 The six cases

3A: -3v -3 v -3 -3v

$$f(x) = x^4 - (v+4)\,x^2 + v$$

the resolvent, $g(x) = x^3 - (v+4)\,x^2 - 4v\,x + 4v(v+4)$

$$g(x) = (x^2-4v)(x-(v+4))$$

let $\tau = v^2 + 4v + 16$

We find that $\Delta = v[4\tau]^2$ so it is clear that 3 must ramify when K is odd. We are thus able to factor:

$$(\theta+2) = 3_1^{K+1}\ , \quad (\theta+1) = 3_2$$

$$(\theta-1) = 3_1\ , \quad (\theta-2) = 3_2^{K+1}$$

$(\theta) = \mathfrak{3}_3^{K}$, if 3 ramifies

$(\theta) = \mathfrak{3}_3^{K/2}\ \mathfrak{3}_4^{K/2}$, if 3 splits

$(\theta) = \tilde{\mathfrak{3}}_3^{\frac{K}{2}}$, if 3 neither splits nor ramifies

In all cases we have a quadratic unit, $\varepsilon_3 = \dfrac{(\theta+1)^K(\theta-1)^K(\theta)^2}{3^K}$

as well as the units $\varepsilon_1 = \dfrac{(\theta-1)^{K+1}}{(\theta+2)}$ and $\varepsilon_2 = \dfrac{(\theta+1)^{K+1}}{(\theta-2)}$

We see at once that $\varepsilon_1\ \varepsilon_2 = \pm\ \varepsilon_3$

A word about the roots of the polynomial. It is clear that if $v > 0$ all our roots are real and the Dirichlet rank is 3. In fact,

$$\theta_i = \frac{\pm\sqrt{v+4-2\sqrt{v}}\ \pm\sqrt{v+4+2\sqrt{v}}}{2}$$

and we see that for $v = 3^{2t}$, the Galois group of our extension is $C_2 \times C_2$. Rewriting θ_i as $\pm\sqrt{(v+4\ \pm\ \sqrt{\tau})/2}$ we easily see that when $v < 0$ we have exactly two real roots and hence, the Dirichlet rank is 2.

3B: 3 v 3 −v 3

$$f(x) = x^4 + \frac{v}{3}x^3 - 4\ x^2 - \frac{4v}{3}x + 3$$

$$g(x) = x^3 - 4\ x^2 - \frac{4(v^2+27)}{9}x + \frac{19v^2+432}{9}$$

We compute, $\Delta = \dfrac{256v^6-15435v^4-461376v^2+559872}{729}$

Since $3\,||\,\Delta$, 3 must ramify.

We factor the ideals:

$(\theta+2) = \mathfrak{3}_1$ $\qquad$ $(\theta-1) = \mathfrak{3}_1^{K}$

$(\theta+1) = \mathfrak{3}_2^{K}$ $\qquad$ $(\theta-2) = \mathfrak{3}_2$

$(\theta) = \mathfrak{3}_3$ $\qquad$ $(\theta+\frac{v}{3}) = \mathfrak{3}_3$ $\qquad$ $(|v|\neq 3)$

Note the 'bonus' ideal obtained because $f(-v/3) = 3$. We find the units $\varepsilon_1 = (\theta+2)^K/(\theta-1)$, $\varepsilon_2 = (\theta-2)^K/(\theta+1)$, $\varepsilon_3 = \frac{(\theta-2)(\theta+2)(\theta)^2}{3}$ and a 'bonus' unit $\varepsilon_4 = (\theta+\frac{v}{3})/(\theta)$.

Theorem 5.1 When $r = 3$ the units ε_1, ε_2, and ε_3 form an independent system.

Proof: We will outline the proof in the positive case for v sufficiently large. The negative case may be handled in an analogous manner and the exceptionally small values of v are easily checked by a computer.

Let $v \geq 81$

$$\frac{2}{v} < \theta_1 < \frac{3}{v}, \quad \frac{2v-2}{v} < \theta_2 < \frac{2v-1}{v}, \quad \frac{-2v-2}{v} < \theta_3 < \frac{-2v-1}{v}$$

$1<\lvert\varepsilon_1^{(1)}\rvert$	$1<\lvert\varepsilon_1^{(2)}\rvert<2^{2K+1}$	$\lvert\varepsilon_1^{(3)}\rvert<(2/v)^K<3^{K(1-K)}<1$
$1<\lvert\varepsilon_2^{(1)}\rvert<2^K$	$\lvert\varepsilon_2^{(2)}\rvert<(2/v)^K<3^{K(1-K)}<1$	$1<4^K<\lvert\varepsilon_2^{(3)}\rvert<5^K$
$3^{1-2K}<\lvert\varepsilon_3^{(1)}\rvert<3^{3-2K}<1$	$3^{-K}<\lvert\varepsilon_3^{(2)}\rvert<3^{3-K}$	$3^{1-K}<\lvert\varepsilon_3^{(3)}\rvert<1$

let $Q_{ij} = \ln \lvert\varepsilon_i^{(j)}\rvert$

then
$$R = \det \begin{vmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{vmatrix} = Q_{11}\,Q_1 - Q_{12}\,Q_2 + Q_{13}\,Q_3$$

where Q_K is the appropriate minor.

We see $Q_{11}, Q_1 > 0$.

It can be shown that

$$Q_{13}\,Q_3 - Q_{12}\,Q_2 > K^2\,(K-1)^2(2K-3)(\ln 3)^3 - K(2K+1)(2K-1)\ln 2\,\ln 3\,\ln 5$$
$$> K(\ln 3)^3\,[2K^4-7K^3+4K^2-3K+1] > 0$$

The last inequality holding because $K \geq 4$

$\therefore \quad R \neq 0.$

3C: $-3 \quad v \quad 3 \quad -v \quad 9$

$$f(x) = x^4 + \frac{v+3}{3} x^3 - 4x^2 - \frac{4v+3}{3} x + 3$$

$$g(x) = x^3 - 4x^2 - \frac{4v^2+15v+117}{9} x + \frac{19v^2+42v+468}{9}$$

We are able to factor some ideals:

$$(\theta+2) = 3_1, \ (\theta) = 3_3, \quad (\theta-1) = 3_1^{\ K}$$

$$(\theta-2) = 3_2^{\ 2} \text{ or } \tilde{3}_2 \text{ or } 3_2\, 3_4$$

where $\tilde{3}_2$ means an irreducible factor of 3 of degree 2, which is possible here only if $|v| = 3^{\text{even}}$.

Two units may be found:

$$\varepsilon_1 = \frac{(\theta+2)^K}{(\theta-1)}, \quad \varepsilon_2 = \frac{(\theta+2)(\theta-2)(\theta)}{3}$$

We point out that if $(\theta-2) \neq 3_2\, 3_4$, then $\frac{(\theta-2)^K}{(\theta+1)^2}$ will also be a unit. We do not make any claims to independence here.

3D: $-3 \quad v \quad -9 \quad -v \quad 81$

$$f(x) = x^4 + \frac{v+21}{3} x^3 + 8x^2 - \frac{4v+21}{3} x - 9$$

$$g(x) = x^3 + 8x^2 - \frac{4v^2+105v+117}{9} x + \frac{7v^2-210v-936}{9}$$

It is easily seen that $r = 2$ when $v = -27, -9$, and -3 and $r = 3$ otherwise. We are able to factor the ideals:

$$(\theta+2) = 3_1, \qquad (\theta-1) = 3_1^{\ K}$$

$$(\theta+1) = 3_3\, 3_4^{\ K-1}, \quad (\theta-2) = 3_3^{\ 3}\, 3_4$$

$$(\theta) = 3_2^{\ 2}$$

This factorization leads to the units

$$\varepsilon_1 = \frac{(\theta+2)^K}{(\theta-1)} \quad \text{and} \quad \varepsilon_2 = \frac{(\theta+2)^{6K-8}(\theta+1)^4(\theta-2)^{2K-4}(\theta)^{3K-4}}{3^{6K-8}}$$

3E: -3 v 9 -v -27

$$f(x) = x^4 + \frac{v-6}{3} x^3 - 10 x^2 - \frac{4v-6}{3} x + 9$$

$$g(x) = x^3 - 10 x^2 - \frac{4v^2-30v+360}{9} x + \frac{25v^2-156v+3600}{9}$$

For this parameterized family, the Dirichlet rank is always 3. We are able to factor the ideals:

$$(\theta+2) = 3_1 , \qquad (\theta-1) = 3_1^{K}$$

$$(\theta+1) = 3_3 \, 3_4^{K-1} , \qquad (\theta-2) = 3_3^{2} \, 3_4$$

$$(\theta) = 3_2^{2}$$

Our units here are $\varepsilon_1 = \frac{(\theta+2)^K}{(\theta-1)}$ and $\varepsilon_2 = \frac{(\theta+2)^{4K-6}(\theta+1)^2(\theta)^{2K-3}(\theta-2)^{2K-4}}{3^{4K-6}}$

3F: -3 v -9 v 3 -3 v

$$f(x) = x^4 - 2 x^3 - (v+4) x^2 + 8 x + v$$

$$g(x) = x^3 - (v+4) x^3 - 4(v+4) x + 4(v^2+5v+16)$$

We may compute $\Delta = 16v \; [v^4+9v^3+120v^2+541v+1536]$.

This implies that, at least when $|v| = 3^{\text{even}}$, 3 will ramify. We also note that f has four real zeros when v > 0 and two real zeros otherwise. When 3 ramifies we have the following factorization:

$$(\theta+2) = 3_1^{K+1} \qquad (\theta-1) = 3_1$$

$$(\theta+1) = 3_3^{2} \qquad (\theta-2) = 3_3^{K+1}$$

$$(\theta) = 3_2^{K}$$

This factorization gives us the units

$$\varepsilon_1 = \frac{(\theta-1)^{K+1}}{(\theta+2)} , \quad \varepsilon_2 = \frac{(\theta+1)^{K+1}}{(\theta-2)^2} , \quad \varepsilon_3 = \frac{(\theta+1)^K(\theta)(\theta-1)^K}{3^K}$$

but we note that $\varepsilon_1 \, \varepsilon_2 \, \varepsilon_3^{-1} = \frac{(\theta^2-1)|v|}{(\theta^2-4)(\theta^2-2\theta)} = \frac{(\theta^2-1)|v|}{(\theta^2-1)v} = \pm 1$

Chapter VI Afterword

We have studied all parameterized p-adatropic polynomials of degree n, where $n \leq 4$. Our success in finding units depends on our ability to factor ideals $(\theta - m)$ where $|f(m)| = p^K$. This factorization may be obvious, as when $n - p = 0$ where (p) must split into p ideal factors, but becomes much more difficult as $n - p$ increases.

Looking ahead to $n = 5$, we see that the difference equation is:

$$f(-2) - 5\,f(-1) + 10\,f(0) - 10\,f(1) + 5\,f(2) - f(3) = -120$$

We are able to spot the following parameterizations at a glance:

	f(-2)	f(-1)	f(0)	f(1)	f(2)	f(3)
	t	v	4	8	v	t
	t	16	w	w	8	t
p = 2:	8	v	w	w	v	128
	8	2w	w	y	2y	128
	8	-2y	w	y	-2w	128
p = 3:	t	27	w	w	3	t
	-9w	9	w	9	3	w
p = 5:	5	v	w	w	v	125

Table 6.1

(Some p-adatropic parameterizations of degree 5)

We would like to conclude with the following question:

Do parameterized p-adatropic number fields of degree n exist for every $p \leq n$ and if so, can something be said about the number of such parameterizations?

BIBLIOGRAPHY

[1] Borevich, Z.I. and Shafatevich, I.R. Number Theory, Academic Press, New York 1966

[2] Cohn, H. "Dyadatropic Polynomials", Math of Comp, Vol. 30, Number 136, October 1976 (854-862)

[3] Cohn, H. "Dyadatropic Polynomials II", Math of Comp, Vol. 33, Number 145, January 1979 (359-367)

[4] Cohn, H. "A Note on Dyadatropic Cubics", J of Pure and Applied Algebra, Vol. 13, 1978 (37-40)

[5] Cohn, H. A Classical Invitation to Algebraic Numbers and Class Fields, Springer-Verlag, New York, 1978

[6] Jacobson, N. Lectures on Abstract Algebra, Springer-Verlag, New York, 1975

[7] Mordell, L. Diophantine Equations, Academic Press, New York 1969

[8] Zimmer, H.G. Computational Problems, Methods, and Results in Algebraic Number Theory, Springer Lecture Notes, 262 (1972)