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CATEGORICAL SEMANTICS OF MODAL DOCTRINES

by

MIRCO ANTONIO MANNUCCI

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

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ABSTRACT

CATEGORICAL SEMANTICS OF MODAL DOCTRINES

by

Mirco A. Mannucci

Adviser: Alex Heller, Distinguished Professor of Mathematics

We consider an elementary topos together with a lex endofunctor acting on it (*a modal topos* in our notation) and show that such a pair provides a universe appropriate for a semantics of first order modal theories. A comparison between this categorical structure and the classical Kripke frames is established. We also prove a representation theorem of modal doctrines and discuss the notion of classifying modal topos for a given modal geometric theory. Lastly, we introduce some generalisations of this approach to multimodal operators and investigate connections with other categorical approaches to modality.

Acknowledgements

What can be learned I learn, what can be found I seek, what can
be prayed for I ask from the gods.

Sophocles, fragment

The long Odyssey through an endless career as a student of Mathematics is coming to a close. Harking back to the past, I cannot help regretting the many opportunities I had for learning, which for the most part I clumsily missed. I have had good instructors everywhere during my pilgrimage (Milano, Pisa, Siena, Praha, Amsterdam, New York), many more than I could have reasonably deserved. To all of them, my hearty thanks. I wish to name at least the ones that stand in my mind as milestones: Prof. G. Mazzacua, who back in Liceo taught me to look at mathematics from the viewpoint of philosophy and conversely. Prof. E. De Giorgi, whose wednesday seminar on Foundation of Mathematics at the Scuola Normale Superiore was for me a unique chance to see a real mathematician cheerfully wrestling with his art. Finally my adviser, Prof. A. Heller, through countless shining conversations and flavoursome cigars, moulded my confused stutter and chaotic exuberance into a new taste for clarity and order. This dissertation has been made possi-

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This dissertation is dedicated to the memory of my beloved parents, Luigi e Antonietta.

Contents

Chapter 1

Introduction

Sunt bona, sunt quaedam mediocria, sunt mala plura quae legis
hic; non aliter fit, Avite, liber.
Martial, *Epigrammaton Libri.*

This dissertation belongs to categorical logic, namely to the field of mathematical research that aims at providing foundations to Mathematics inside the framework of category theory. This field has seen the light in 1963 with Lawvere's seminal dissertation on universal algebra and has been developed by Lawvere himself and several other authors throughout the last two decades. The introduction of elementary topos theory in the late sixties has dominated categorical logic ever since. The discovery of the internal logic of a topos has brought forth surprising connections to the limelight and has forced a reinterpretation of old topics, model theory to name one (perhaps the most complete account of model theory from the categorical standpoint is the recent monograph by Makkai and Parè [17]). The attention of researchers has focused largely on the paradigma of topos theory, which has been developed to a considerable extent in many different directions. On the other hand, the risk is present that such robust paradigm might monopolize the entire spectrum of categorical analysis of logic. For instance, the topos provides a natural universe of discourse in which models of intuitionistic theories can be found. It would be desirable to reproduce the same pattern for a broader class of logics, i.e. to be able to isolate categorical structures for which an internal non-classical logic still makes sense (as a matter of fact, logicians have been considering for quite a while a large variety of such non classical logics, like modal logics, paraconsistent logics, n-valued logics, minimal logic etc..). If this program can be carried out, it is to be expected that the topos would still remain prominent, much in the same way as the ring of integers is the central and motivating example of ring in algebraic number theory, without being by any means the only relevant one. This dissertation is intended to be a step in this direction, dealing with one kind of non-classical logics, the so-called modal ones. A simple categorical entity is defined, the *modal topos*, which is essentially a topos together with an exact endofunctor acting on it. The modal topos possesses an internal modal logic of a rather arbitrary type. We will show that these modal topoi are natural and are implicitely suggested by standard semantics à la Krikpe. Loosely speaking, modal topoi are the dynamic counterpart of Kripke structures. Dually, the Kripkean viewpoint can be recovered once we assume the statical point-like perspective (where of course points are taken in the proper topos-theoretical meaning). Here is the plan of the work:

Chapter 2 introduces our modal version of geometric theories, a class of theories in the modal first order language expecially suited for this context.

Chapter 3 describes modal doctrines, which can be regarded as the invariant notion of first order modality.

Chapter 4 sets up the bridge between modal geometric theories and Category Theory, using a construction of categories out of syntax quite familiar to experts in the field.

Chapter 5 is dedicated to modal topoi and some relevant facts and constructions pertaining to them. The category of fixed points of a modal topos allows the production of a modal doctrine of the desired type.

Chapter 6 compares modal topoi and Kripke structures via the notion of relation between points. This notion is in keeping with classical algebraic semantics of modal algebras.

Chapter 7 proves a representation theorem for modal doctrines and analyses in which sense modal topoi are classifying objects for modal theories.

Chapter 8 extends the previous perspective to the multimodal case: the notion of modal topos is subsumed into the ampler notion of topos acted upon by a monoid.

Chapter 9 establishes some comparison with alternative "toposophical" treatment of modal semantics. Also, some indication of future work is given.

We conclude the introduction with a word of warning: this work is not self-contained. It assumes from the reader some background in basic category theory plus a certain familiarity with methods and tecniques of topos theory. On the other hand, the required preliminaries are for the most part standard and can be found in several books, like the classic reference CWM ([5]) on elementary category theory (our notation will conform to it as far as possible) and the recent joint work of Mac Lane and Moerdijk for topos theory ([24]). The bibliographical references spread throughout the text should suffice for orientation in specific matters.

Chapter 2

Modal geometric theories

Geometric theories are by now a well-established topic in categorical logic; they happen to be the right fragment of first order logic for which a fully satisfactory model theory can be obtained by topos-theoretic methods (see [18]). We recall that a theory in the infinitary multisorted first-order language is *geometric* iff its non-logical axioms consist of sequents of the form

$$\Phi \Longrightarrow \Psi$$

where Φ and Ψ are *geometric formulae*, i.e. formulae built up from $\bigvee$, $\wedge$, $\exists$ (notice that we allow infinite disjunctions but only finite conjunctions). This fragment of first-order logic is powerful enough to deal with some relevant mathematical structures, such as finitary algebras or local rings; moreover, several authors have pointed out the computational side of geometric theories, which makes them appealing to the theoretical computer scientist as well as to the computationally oriented logician (see [39]). We are now going to caracterize a class of theories in the *modal* first- order language which will play a role entirely analogous to geometric theories in a non-modal enviroment. Our language $L_\infty(\Box)$ is just the infinitary language (allowing only finitary quantifiers) with equality, enriched with a unary modal operator $\Box$; a theory T in this language is said to be *modal geometric* iff the axioms of T consist of sequents

$\Phi \Longrightarrow \Psi$, where Φ and Ψ are built up from $\bigvee$, $\wedge$,$\exists$ *and* $\Box$; moreover T contains non-logical axioms of the form

1)$\Box\phi \wedge \Box\psi \Longrightarrow \Box(\phi \wedge \psi)$

2)$\top \Longrightarrow \Box\top$

3)$\exists(\vec{x})\Box\phi(\vec{x}) \Longrightarrow \Box\exists(\vec{x})\phi(\vec{x})$

4)$\Box(x = y) \Longrightarrow (x = y)$

5)$(x = y) \Longrightarrow \Box(x = y)$

and is closed under the usual rules (substitution and modus ponens) plus the necessitation rule: if $\Longrightarrow \psi$ then $\Longrightarrow \Box\psi$ and the box-monotonicity rule: $\phi \Longrightarrow \psi$ entails $\Box\phi \Longrightarrow \Box\psi$. The reader will recognize the non-classical analogous of *normal* modal logics in axioms 1) and 2) (the monotonicity rule is required because we are working in a non-classical environment and internal implication is not available to us); 3) establish a certain compatibility between $\exists$ and $\Box$. This axiom will be satified in any modal topos and is thus required for completeness result. Finally, 4) and 5) express the *absolute* character of equality. The link with what is ahead of us is the following observation: if T is modal geometric and one-sorted, we can associate to it a *polyadic algebra* in the spirit of Halmos (see [1]), reflecting its algebraic features. More specifically , let T be modal geometric; to each n we associate $D_T(n) = \{[\phi]\}$ where $[\phi]$ indicates the equivalence class of ϕ under T-provability and the free variables of ϕ are contained in $\{x_1, \ldots.., x_n\}$; if

$$n \xrightarrow{f} m$$

is a map in N (meaning by this just the skeletal category of finite sets), we have

$$D(n) \xrightarrow{D(f)} D(m)$$

by substituting along f. Also, for each n, $D(n)$ is a *distributive lattice* and the transition maps are distributive lattice morphisms (actually, in case T is infinitary , $D(n)$ is a *frame* and the transition maps become frame-maps: see [3] for definitions). Furthermore, each $D(n)$ is endowed with an additional operation

$$D(n) \xrightarrow{\Box(n)} D(n)$$

$$\phi \longmapsto \Box\phi$$

commuting with the transition maps: D_T is a *functor*:

$$N \xrightarrow{D_T} MDL$$

where MDL denotes the (algebraic) category of *modal distributive lattices*: $(MDL)_0 \equiv \{(L, \sigma)\}$, where L a distributive lattice and

$$L \xrightarrow{\sigma} L$$

such that $\sigma(a \wedge b) = \sigma a \wedge \sigma b$, while maps are distributive lattice maps preserving σ. As shown in ([13]), the presence of equality in our theory is equivalent to assuming that every transition map comes equipped with a left adjoint: if

$$n \xrightarrow{f} m$$

we have:

$$D(m) \xrightarrow{\exists f} D(n)$$

and $\exists f \dashv D(f)$. Finally, by axiom 3) $\forall a \in D(n) \exists f \Box_n(a) \leq \Box_m \exists f(a)$, where

$$\begin{tikzcd}
D(n) \arrow[r, "\exists f"] \arrow[d, "\Box_n"'] & D(m) \arrow[d, "\Box_m"'] \\
D(n) \arrow[r, "\exists f"'] & D(m)
\end{tikzcd}$$

The functor verifies also certain diagrammatical equations which will be spelled out in the next chapter. Summing up, we have constructed a categorical object reflecting the invariant features of T: the polyadic algebra of T. Of course in this process, known to logicians as the *Lindenbaum construction*, we have completely lost the proof-theoretical side of T. To be sure, our original T can be looked upon as a particular presentation of its polyadic algebra, much in the same way as an assignment of generators and relations presents a given group; the pay-off of this move is that we are now in the position of thinking about semantics for T in terms of *representations* of the underlying polyadic algebra D_T: standard model theory becomes a chapter of the general theory of representation of mathematical structures, a topic that possesses an intrinsic categorical flavour. Though we could develop our work in an infinitary environment, we intend to restrict our attention to geometric theories in the finitary version of the language $L_\infty(\Box)$: we shall denote such theories as *modal coherent theories.*

Chapter 3

Modal Doctrines

We are now going to introduce *modal doctrines*, in the spirit of [13]; these doctrines are essentially generalized polyadic algebras coming from *multisorted* geometric theories. We thus replace N^{op} with an arbitrary small category with finite products, say T. The objects of T will represent *types* and morphisms in T will be *terms*.

Remark 3.1 *We just notice in passing that more general categories are indeed possible, i.e. lex categories or even cartesian closed categories; however, in the context of the present work, we are concerned only with first-order logic, so a finite-product term-theory is largely satisfactory for our purpose.*

We begin by recalling the definition of doctrines:

Definition 3.1 *A distributive lattice doctrine based on a term-theory T is a contravariant functor D from the finite-product small category T to the algebraic category DL (distributive lattices):*

$$T^{op} \longrightarrow DL$$

satisfying the following conditions:

1) for every map

$$t \xrightarrow{f} t_1$$

in T the map of ordered sets

$$D(t_1) \xrightarrow{D(f)} D(t)$$

has a (necessarily unique) left adjoint:

$$D(t) \xrightarrow{\exists f} D(t_1)$$

2) the " Frobenius " condition holds :

$\forall a \in D(t) \forall b \in D(t_1) \exists f(a \wedge D(f)(b)) = \exists f(a) \wedge b$

3) the " Beck-Chevalley" condition also holds: from

$$\begin{tikzcd}
t \times t \arrow[r, "\pi_2"] \arrow[d, "t \times f"'] & t \arrow[d, "f"] \\
t \times t_1 \arrow[r, "\pi_2"'] & t_1
\end{tikzcd}$$

in T, we get a commutative diagram:

$$\begin{tikzcd}
D(t \times t) \arrow[d, "\exists(t \times f)"'] & D(t) \arrow[l, "D(\pi_2)"'] \arrow[d, "\exists f"'] \\
D(t \times t_1) & D(t_1) \arrow[l, "D(\pi_2)"]
\end{tikzcd}$$

Given two T-doctrines D_1 and D_2, a morphism

$$\mu : D_1 \longrightarrow D_2$$

is a *natural transformation* from D_1 to D_2 preserving $\exists$: if $f : t \longrightarrow t_1$ the diagram

$$\begin{tikzcd}
D_1(t) \arrow[r, "\mu_t"] \arrow[d, "\exists f"'] & D_2(t) \arrow[d, "\exists f"'] \\
D_1(t_1) \arrow[r, "\mu_{t_1}"] & D_2(t_1)
\end{tikzcd}$$

commutes. These data determine a (large) category, $T - DOC$, a non-full subcategory of $DL^{T^{op}}$, which in turn is precisely the category of distributive lattices in the topos of presheaves on T (for a more precise discussion, see [14]). It is important to realize that any change of the term-category induces a functor between the corresponding categories of doctrines: if $F : T \longrightarrow T'$ is a morphism of term-categories (i.e.if F preserves all finite products), and $D : T' \longrightarrow DL^{op}$ is any T'-doctrine, we obtain a T-doctrine by composition with F: an easy checking shows that this produces a functor between the two large categories, the functor *"change of base along F"* .

Remark 3.2 *The previous considerations implicitely suggest that there exists an underlying category of base-free doctrines, whose fibers are the various $T - DOC$ as T changes. This is indeed the case: define DOC as the huge category whose objects are T-doctrines for* arbitrary T *and maps*

$$\psi : D \longrightarrow D'$$

where $D : T \longrightarrow DL^{op}$ *and* $\acute{D} : T \longrightarrow DL^{op}$ *are pairs* $\psi \equiv (F, \mu)$ *such that*

$$F : T \longrightarrow T_1$$

is a product-preserving functor and μ *is a 2-cell (natural transformation):*

$$\mu : D \longrightarrow D_1 \circ F$$

. The category DOC *is tecnically a Grothendieck fibration over the category* FP *of finite product small categories.*

A doctrine depends also on the target category: in the foregoing we have implicitly assumed that it is just the dual of the algebraic category of distributive lattices in the ambient category *Set*. We could replace *Set* with an arbitrary category with finite products (for instance every elementary topos) C and consider doctrines $D : T \longrightarrow DL^{op}(C)$, where $DL^{op}(C)$ denotes the category of distributive lattices objects of C (or, in different words, the models in C of the equational theory of distributive lattices); again, if

$$G : C \longrightarrow C'$$

is an arbitrary finite product-preserving functor (like the left or right image part of a geometric morphism between topoi) we can transfer algebraic objects along G:

$$DL(C) \longrightarrow DL(C')$$

Furthermore, if

$$\begin{tikzcd}
D_1 \arrow[r, shift left, "\phi"] \arrow[r, phantom, "\perp"] & D_2 \arrow[l, shift left, "\psi"]
\end{tikzcd}$$

is a couple of adjoint maps between two objects of $DL(C)$ (or, equivalently, a pair of adjoint functors between the corresponding *internal categories*) then

$$\begin{tikzcd}
G(D_1) \arrow[r, shift left, "G(\phi)"] \arrow[r, phantom, "\perp"] & G(D_2) \arrow[l, shift left, "G(\psi)"]
\end{tikzcd}$$

is an adjoint pair in $\acute{C}$, simply because in this context adjointness is *equational.* Summing up, if

$$D : T \longrightarrow DL(C)^{op}$$

is a C-based T-doctrine and

$$G : C \longrightarrow C'$$

a functor preserving finite products, $F \circ D$ is a C'-based doctrine. An even more drastic standpoint, which we just mention here, would be when T *itself* is an internal finite-product category-object in a lex category C and D an internal functor satisfying suitable equations: this train of thought would lead us to the notion of an *internal doctrine* in the ambient category C.

We now turn our attention to *modal doctrines*; here we have at least two options (indeed several more !) depending on how many restrictions we impose on such objects. Let us start first with a sufficiently general one:

Definition 3.2 *A functor* $D : T^{op} \longrightarrow MDL$ *is a modal doctrine iff the composition* $U \circ D$ *is a* T*-doctrine, where* U *is the standard forgetful functor from* MDL *to* DL.

. This definition requires that the modal operator commutes with substitution along any map in T, but no compatibility conditions are required between $\exists$ and $\Box$.

Definition 3.3 *A morphism* $\mu : D \longrightarrow D^1$ *of modal doctrines is a natural transformation* μ *such that* $U \circ \mu$ *is a morphism of* T*-doctrines*

. The category so defined will be denoted as $T - MODOC$. We now come to a more restrictive definition, which will be adopted in our exposition as the standard one:

Definition 3.4 *A* regular *modal doctrine is a modal doctrine* $D : T^{op} \longrightarrow MDL$ *such that, for an arbitrary map* $t \longrightarrow t_1$ *in* T *we have* $\forall a \in D(t) \exists f \sigma_t(a) \leq \sigma_{t_1} \exists f(a)$, *where*

$$\begin{tikzcd}
D(t) \arrow[r, "\exists f"] \arrow[d, "\sigma_t"] & D(t_1) \arrow[d, "\sigma_{t_1}"] \\
D(t) \arrow[r, "\exists f"] & D(t_1)
\end{tikzcd}$$

Furthermore, $\forall t \in T_0 \sigma_{t \times t}(eq_t) = eq_t$

Remark 3.3 *Notice that modal doctrines coming from modal geometric theories are always regular.*

We thus have the full subcategory $T - REGMODOC$ of $T - MODOC$ whose objects are regular modal doctrines. Going back to doctrines, we say that $D : T^{op} \longrightarrow DL$ is *geometric* if it factors as

$$\begin{tikzcd}
T^{op} \arrow[r, "D"] \arrow[dr, "D"'] & DL \\
& FRAMES \arrow[u, "U"']
\end{tikzcd}$$

where $FRAMES$ is the category of frames and frame-maps (we recall that a frame is a complete distributive lattice and maps between frames are distributive lattice maps preserving arbitrary sups. The category $FRAMES$ is the dual of the category of $LOCALES$, or generalized spaces: see [4] for a full exposition on this subject).

Definition 3.5 *A* modal geometric doctrine *is a regular modal doctrine D such that UD is geometric.*

Taking up the thread of the previous chapter we can say that to every modal geometric theory is associated a geometric modal doctine, indeed a regular one: via this association we can freely identify the theory with the corresponding doctrine , which can be thought of as the invariant counterpart of the theory: of course, one should first show that this association is functorial once one takes the pain of defining a suitable category of theories (in the same language $L_\infty(\Box)$ and their morphisms: without entering details which would lead us far from our present endeavour, we limit ourself to saying that theories would become the objects of such a category and *interpretations* between theories its maps; it goes without saying that this can be accomplished in different ways according to our bias in selecting the appropriate notion of interpretation; one possible indication is contained in the book of Lambek-Scott [8] quoted in the bibliography.

Chapter 4

Modal Sites

To avoid ambiguities, we begin by saying that a **site** is a pair (C, J) where C is a *lex* category and J a Grothendieck topology on C (see [3] or [6] on this standard material). Observe that we have dropped the assumption concerning the smallness of C because we want to think of a topos as a particular site (possibly large) endowed with the *standard* topology of epimorphic families. We could even drop the restriction concerning the exactness of C, but we will refrain from doing so just to make our life considerably easier. The next actors on stage are *modal sites*:

Definition 4.1 *A modal site is a triple* (C, J, σ) *where* (C, J) *is a site and* $\sigma : C \longrightarrow C$ *is an endofunctor of* C *preserving finite limits (*lex endofunctor *for short).*

. Notice that σ is *not* in general an endomorphism of the site (it needn't preserve covers).

Definition 4.2 *A morphism* ψ *from the modal site* (C, J, σ) *to the modal site* (C', J', σ') *is a pair* $\psi \equiv (\psi_0, \psi_1)$ *where* ψ_0 *is a morphism of the underlying sites and*

$$\begin{tikzcd}
C \arrow[r, "\psi_0"] \arrow[d, "\sigma"'] & C_1 \arrow[d, "\sigma'"] \\
C \arrow[r, "\psi_0"'] \arrow[ur, Rightarrow, "\psi_1"] & C_1
\end{tikzcd}$$

ψ_1 *is a natural isomorphism between* $\psi_0 \circ \sigma$ *and* $\sigma' \circ \psi_0$.

Composition goes the obvious way: if

$$\begin{tikzcd}
C \arrow[r, "\psi_0"] \arrow[d, "\sigma"] & C' \arrow[r, "\psi_0"] \arrow[d, "\sigma'"] & C'' \arrow[d, "\sigma''"] \\
C \arrow[r, "\psi_0"'] \arrow[ur, Rightarrow, "\psi_1"] & C' \arrow[r, "\phi_0"'] \arrow[ur, Rightarrow, "\phi_1"] & C''
\end{tikzcd}$$

we have

$$\begin{tikzcd}
C \arrow[r, "\phi_0 \circ \psi_0"] \arrow[d, "\sigma"] & C" \arrow[d, "\sigma"'] \\
C \arrow[r, "\phi_0 \circ \psi_0"'] \arrow[ur, Rightarrow, "(\phi \circ \psi)_1"] & C''
\end{tikzcd}$$

where $(\phi \circ \psi)_1 \equiv \phi_1 \psi_0 \circ \phi_0 \psi_1$. We denote the category defined by the previous data as $MODSITE$ (modal sites). The motivating reason for this name is the following elementary observation: if (L, σ) is a modal distributive lattice in the sense of Chapter 2, we can regard it as a canonical example of modal site; indeed L itself is an ordered set and so a category in the trivial way, whereas the conditions imposed on σ make it a lex endofunctor on this category (the only limits being meets of elements and the top of the lattice). A moment thought will convince the reader that via the previous identification the algebraic category MDL "lives" as a full subcategory of $MODSITE$. By the same token, many algebraic categories relevant to modal logicians are also examples of subcategories of $MODSITE$: perhaps the most typical of these is $MBOOL$, the full subcategory of MDL whose objects have a ***boolean algebra*** as the underlying lattice (a thorough presentation of modal algebras and their duality theory can be found in [37]); this category is the algebraico-logical version of the ground normal modal logic K and it contains a number of interesting varietal subcategories which in turn represent various possible extensions of the logic K (like S_4-algebras, diagonalisable algebras etc...). There is more to it: the key point here is that to *every* regular modal doctrine we can associate functorially a modal site. We are now going to sketch this construction, which will become essential in Chapter 7. The main idea belongs to the folklore of categorical logic and has been known to several autors for quite some time (see for instance [15] for an essentially analogous construction. Ideas of a similar type appear also in [9] in the representation theorem for p-categories). It can be summarized by saying that out of any doctrine a *syntactic category* is constructed, which in turn is a site (in fact even a logical category in the sense of Makkai-Reyes: see below for the definition). Here is the recipe : start from a doctrine

$$D : T^{op} \longrightarrow DL$$

and define the associated syntactic category $C(D)$ thus:

0) objects are pairs (t,a) where $t \in T_0$ and $a \in D(t)$ (think of adding formally all subobjects prescribed by the doctrine)

1) a map $\phi : (t,a) \longrightarrow (s,b)$ is an element $\phi \in D(t \times s)$, to be thought of as a formal relation between t and s, such that is " functional " and the "domain"

of ϕ is a, the "codomain" is b. Spelling out some details, this means that

1_0) $a = \exists\pi_1(\phi \wedge D(\pi_2)(b))$ where π_1 and π_2 are the two canonical projections from $t \times s$ to t and s (in common language this expresses the fact that for every x in a there exists a y in b such that $\phi(x,y)$)

1_1) if $f : s' \longrightarrow s$ is any map in T

$$D(<\pi_1,\pi_2>)(\phi) \wedge D(<\pi_1,\pi_3> \circ <t \times f>)(\phi) \leq D(<\pi_2,\pi_3> \circ <s,f>)(eq_s)$$

where eq_s is the equality on s (equality is defined at every t by the formula $\exists\Delta(\top)$ where Δ is the diagonal map) and

$$\begin{tikzcd}
D(t \times s) \arrow[r, "D(<\pi_1,\pi_2>)"] \arrow[dr, "D(<t,f>)"'] & D(t \times s \times s') & D(s \times s') \arrow[l, "D(<\pi_2,\pi_3>)"'] \\
& D(t \times s') \arrow[u, "D(<\pi_1,\pi_3>)"'] & D(s \times s) \arrow[u, "D(<s,f>)"']
\end{tikzcd}$$

(1_1 expresses diagrammatically the functionality of ϕ; in logicians' notation $\phi(x,y) \wedge \phi(x,y') \leq (y = y')$) A note for the reader: the previous definitions may look at first quite esoteric or perhaps rather arbitrary; some sort of abstract categorical nonsense. In order to dissipate the apparent darkness we suggest to stick to the set-theoretical analogy: maps are graphs of functions and existential quantification is the ordinary projection onto one coordinate. This game is well known to logicians when they speak for instance of provably functional relations or the structure of r.e. sets in a given arithmetical theory; in such cases, even though in disguised form, syntactic categories are conjured up . If

$$\phi : (t, a) \longrightarrow (t', a')$$

and

$$\psi : (t', a') \longrightarrow (t'', a'')$$

are two maps, composition $\psi \circ \phi$ is given by

$$\exists(\pi_1 \times \pi_3)((\pi_1 \times \pi_1) \wedge D(\pi_2 \times \pi_3))$$

where

$$\begin{tikzcd}
D(t \times t') \arrow[r, "D(<\pi_1{,}\pi_2>)"] & D(t \times t' \times t'') \arrow[r, "\exists(<\pi_1{,}t_3>)"] & D(t \times t'') \\
& D(t' \times t'') \arrow[u, "D(<\pi_2{,}\pi_3>)"'] &
\end{tikzcd}$$

The identity map $id_{(t,a)} : (t, a) \longrightarrow (t, a)$ is just the equality on t "restricted" to a: $\exists\Delta(a)$. The category so defined has finite products :

$$(t, a) \times (s, b) \equiv (t \times s, D(\pi_1)(a) \wedge D(\pi_2)(b))$$

(the projections maps are the graphs of π_1 and π_2, again suitably restricted) and equalizers: if ϕ and ψ are a parallel pair between (t, a) and (s, b) their equalizer is given by

$$(t, \exists\pi_1(\phi) \wedge \exists\pi_2(\psi) \wedge a)$$

. In fact $C(D)$ is *logical*, meaning that:

1) $C(D)$ has all finite limits (this is immediate from the existence of finite product and equalizers)

2) the functor $Sub_C(D) : C(D) \longrightarrow Set^{op}$ (such a functor is well-defined because $C(D)$ is a lex category) factors through DL

3) $C(D)$ has images which are stable under pull-backs:

$$\begin{tikzcd}
(t,a) \arrow[r, "\phi"] \arrow[dr] & (t',a') \\
& (t', \exists\pi_2\phi) \arrow[u, "\exists\Delta(\exists\pi_2\phi)"']
\end{tikzcd}$$

Finally, T embeds faithfully in $C(D)$ via a finite-product preserving functor which we shall call G (G stands for global subobject functor):

$$t \mapsto (t, \top)$$

$$f \mapsto Graph(f)$$

where $Graph(f)$ is defined by the following procedure: if

$$f : t \longrightarrow t'$$

$$Graph(f) \equiv \exists(< t, f >)(eq_t)$$

Thus far this construction exploits only the fact that D is a T-doctrine: modality comes into play by observing that there exist a lex endofunctor

$$\sigma_D : C(D) \longrightarrow C(D)$$

given by

$$\sigma_D(t, a) \equiv (t, \sigma(t)(a))$$

and for every map ϕ

$$\sigma_D(\phi) = \sigma(\phi)$$

. One can check that σ_D is well-defined. This amounts to showing that if ϕ is a "functional relation" with domain a and codomain b, $\sigma_D(\phi)$ is also functional with domain σa and codomain σb; that σ_D preserves composition: all these come easily using the regularity of D. Again, by regularity, one can show that σ_D preserve finite products, equalizers and hence all finite limits; finally, the endofunctor σ_D commutes with the embedding of T:

$$\begin{tikzcd}
T \arrow[r, "G"] \arrow[d, "G"] & C(D) \\
C(D) \arrow[ur, "\sigma_D"'] &
\end{tikzcd}$$

Indeed, global subobject are sent over to themselves and

$$\sigma_D(Graph(f)) = Graph(f)$$

(it suffices to unravel the definition of $Graph(f)$ and remember the absoluteness of equality). It is also an easy matter to verify the functoriality of our construction, so we can summarize it by saying that we have produced a functor:

$$SC : T - REGMODOC \longrightarrow T - MODLOG$$

(SC stands for **syntactic category functor**) where $T - MODLOG$ is the category of logical categories equipped with a lex endofunctor "under" the fixed term-category T: the objects of this category are finite product preserving functors: $F : T \longrightarrow C$ such that

$$\begin{tikzcd}
T \arrow[r, "F"] \arrow[d, "F"'] & C \\
C \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

commutes and maps are modal logical functors $\psi : C \longrightarrow C'$ (i.e. preserving all the logical structure and commuting with the lex endofunctors) such that:

$$\begin{tikzcd}
 & T \arrow[dl, "F"'] \arrow[dr, "F'"] & \\
C \arrow[rr, "\psi"'] & & C'
\end{tikzcd}$$

is a commutative diagram.

In the previous chapter we have remarked that, for our purposes, we can identify a theory T with its associated doctrine D_T. Now it is time to replace D_T itself with the syntactic category $C(D_T)$ under T; the reason being that to such a category is attached a T-doctrine isomorphic to the original one, namely

$$SUB_{C(D_T)} \circ G$$

. To be more precise, we need a preliminary definition:

Definition 4.3 *A logical category C is replete iff every functional relation in C is the graph of a function (the converse is always true). The full subcategory of LOG whose objects are replete will be denoted as $REPLOG$.*

The standard category of sets is obviously replete as a logical category. This feature of Set is precisely what enables us to identify a function with its associated graph. The first observation to be made is that the functor SC actually lands in the category of *replete* modal logical category under T. We will adopt as the name of such a category $T - MODREPLOG$. The second observation is that there is a functor going the other way around, namely the one that associates to (C, F, σ) the modal doctrine $SUB_C \circ F$. The following theorem holds:

Theorem 4.1 *The functor SC is left adjoint to the functor*

$$SUB_C(-) \circ (-) : T - MODREPLOG \longrightarrow T - MODOC$$

Moreover the unit of such an adjunction is a natural isomorphism (this last statement is equivalent to saying that T-MODOC sits in T-MODREPLOG as a coreflective subcategory).

Here is a sketch of the proof: the unit of the adjunction at D is given by

$$\eta_D(t) : D(t) \longrightarrow SUB_{SC(D)} \circ G$$

$$\eta_D(t)(a) = [(t,a), \exists\Delta(a)]$$

The map $\eta_D(t)$ is obviously a morphism of distributive lattices; moreover it is also epi: for every monomorphism with codomain $G(t)$ there exist an equivalent monomorphism in the range of η_D, namely its image. The map is also injective. Indeed, suppose $(t,a) \leq (t,b)$ in $SC(D)$. This means that there is a map $\phi : (t,a) \rightarrow (t,b)$, which necessarily is a monomorphism in $SC(D)$, such that $\exists\Delta(a) = \phi \circ \exists\Delta(b)$. Now we are in a replete category, so we can read the preceding as saying that $\exists\Delta(a) = \phi \circ \exists\Delta(b)$ as *graphs*. Furthermore, it is easy to show that ϕ is contained in the identity id_t as subobjects of $(t, \top) \times (t, \top)$. From this last fact we get $\exists\Delta(a) \leq \exists\Delta(b)$. But by adjointness, this is equivalent to $a \leq D(\Delta)\exists(b)$ which in turn implies that $a \leq b$. The counit of the adjunction is given by:

$$SC(SUB_C \circ F) \longrightarrow C$$

$$(t, [A, f]) \mapsto A$$

(notice that a choice has to be made here)

$$\phi = Graph(g) \mapsto g$$

(C is replete by hypothesis).

Theorem 3.1 enables us to proceed further by thinking of $C(D_T)$ as the *cat-*

egorical counterpart of the theory T, much as the doctrine $D(T)$ was its algebraic analog. In the light of the previous theorem is not surprising that several authors skip entirely over the subject of doctrines in their exposition of categorical logic (for instance, in Makkai-Reyes [18] the syntactic category is constructed directly from the theory. A quick mention of doctrines is in the concise exposition by Kock and Reyes in the Handbook of Mathematical Logic). It should be pointed out though that doctrines, far from being a pleonastic concept, are as yet a rather unexplored universe and that this representation theorem does not trivialize them as the proper set-up for a unified categorical analysis of generalized logic (the reader should perhaps think of the notion of internal doctrine as sketched in the previous chapter). Coming back to our discussion, we find ourselves with a new object beforehand: a site with a lex endofunctor acting on it. Sites are very familiar to topos-theorists as presentations of topoi: it seems rather natural to think that modal sites should also be presentations of some invariant entity; indeed, this is the case, as we shall see in the next chapter.

Chapter 5

From Modal Sites to Modal Topoi

A starting point of Topos Theory is the observation that to every (small) site corresponds a (large) category, namely the category of sheaves on that site and their natural transformations: a Grothendieck topos (see [2]). Of course, the same topos can be presented by different sites; this fact suggests that a topos can and must be studied in its own terms, as a category of a particular kind. In the late sixties Lawvere and Tierney succeeded in axiomatising topos theory with the notion of *elementary topos* (it is worth reading Lawvere's introduction to [13] for an ample perspective on the whole topic). In our "modal" context, it is necessary to introduce *modal topoi*:

Definition 5.1 *An elementary modal topos is a pair* (E, σ) *where* E *is an elementary topos and* $\sigma : E \longrightarrow E$ *is a lex endofunctor on* E. *A map* $\psi : (E, \sigma) \longrightarrow (E', \sigma')$ *is a pair* $\psi \equiv (\psi_0, \psi_1)$ *where*

$$\begin{tikzcd} E \arrow[r, "\perp", "(\psi_0)_*"'] & E' \arrow[l, bend right, "(\psi_0)^*"'] \end{tikzcd}$$

is a geometric morphism of topoi and

$$\begin{tikzcd} E \arrow[r, "\sigma"] & E \\ E' \arrow[u, "\psi_0^*"] \arrow[r, "\sigma'"] \arrow[ur, Rightarrow, "\psi_1"] & E' \arrow[u, "\psi_0^*"'] \end{tikzcd}$$

so that ψ_1 *is a natural isomorphism between* $\psi_0^* \circ sigma'$ *and* $\sigma \circ \psi_0^*$.

The reader has certainly noticed that this definition coincides with the one of the previous chapter once one identifies a topos with the corresponding site (the left image part of a geometric morphism is indeed a continuous functor and so a legitimate morphism in the category of sites).

The category of modal topoi will be denoted as *MODTOP*. Let (C, J, σ) be an arbitrary modal site. We can associate to it $(Sh(C, J), \sigma^{\#})$ where $\sigma^{\#} : Sh(C, J) \longrightarrow Sh(C, J)$ is given by

$$\begin{tikzcd}
\hat{C} \arrow[r, "Lan_\sigma"] & \hat{C} \arrow[d, "a"] \\
Sh(C,J) \arrow[u, "i"] \arrow[r, "\sigma^{\#}"'] & Sh(C,J)
\end{tikzcd}$$

Theorem 5.1 *The previous association defines a pseudofunctor*

$$Sh : ModSite \to MODTOP^{op}$$

where ModSite denotes the full subcategory of MODSITE whose objects have an undelying small site.

Proof: Observe that $\sigma^{\#}$ is lex, because a and i are (i is the embedding of the category of sheaves and a the lex reflection) and Lan_σ inherits exactness from σ (for a proof of this general fact, see [5]).

We must show the functoriality of the construction: if

$$\psi : (C, J, \sigma) \longrightarrow (C', J', \sigma')$$

we have:

$$\begin{tikzcd}
\hat{C} \arrow[rrr, "Lan\sigma"] \arrow[dr, "\psi^*"] & & & \hat{C} \arrow[dl, "\psi^*"] \arrow[ddd, "\acute{a}"] \\
& \hat{C} \arrow[r, "Lan\sigma"] & \hat{C} \arrow[d, "a"] & \\
& Sh(C,J) \arrow[u, "i"'] \arrow[r, "\sigma^{\#}"] & Sh(C,J) & \\
Sh(\acute{C},\acute{J}) \arrow[uuu, "\acute{\imath}"] \arrow[ur, "\psi^*"'] \arrow[rrr, "\sigma^{\#}"'] & & & Sh(\acute{C},\acute{J}) \arrow[ul, "\psi^*"']
\end{tikzcd}$$

the largest square commutes by hypothesis, the two trapezoids on the sides commute because ψ is a morphism of sites; the top just expresses the naturality of the left Kan extension: putting all the pieces together we get the commutativity of the bottom trapezoid, as desired.

Remark 5.1 *the previous theorem can be thought as a kind of* cocompletion *for modal sites: indeed the dual of the category of Grothendieck modal topoi "sits" in $MODSITE$ as the subcategory of objects whose undelying site is cocomplete. The pseudofunctor Sh describes the universal way of mapping a small modal site into one object of that type.*

It is worth noticing that in the particular case in which (C, J) happens to be a site with a *subcanonical* topology

$$\begin{tikzcd}
C \arrow[r, "\sigma"] \arrow[d, "Y"'] & C \arrow[d, "Y"'] \\
Sh(C,J) \arrow[r, "Lan\sigma"'] & Sh(C,J)
\end{tikzcd}$$

commutes up to isomorphism (in other terms, Lan_{σ} is here $\sigma^{\#}$ and extends in the only possible way the action of σ to the whole sheaf-category when we think of C as sitting in Sh(C,J) via the Yoneda embedding).

The previous theorem provides us with a multitude of modal topoi: indeed, to every modal distributive lattice is associated canonically a modal topos (we have already said that each modal distributive lattice is also a modal site). Obviously, ordinary elementary topoi themselves can be regarded as trivial examples of modal topoi (σ is just the identity).

A lex endofunctor on a topos is a kind of *action* and so it makes good sense to ask for the category of objects and maps left fixed by such an action, pretty much in the same way as we talk of invariant subspaces under a given endomorphism of, say, a vector space; this lead us to introduce a "fixed point category", with one proviso: things are left fixed *up to isomorphims* (we don' t want to be too restrictive at the risk of finding an empty structure!):

Definition 5.2 *The fixed point category, denoted as* $FIX(E,\sigma)$*, has objects* (A,f) *where* $A \in E$ *and* $f : A \longrightarrow \sigma A$ *and* f *is an* isomorphism *amd maps* $\psi : (A,f) \longrightarrow (A',f')$ *are morphisms in* E $\psi : A \longrightarrow A'$ *such that the following diagram commutes:*

$$\begin{tikzcd}
A \arrow[r, "\psi"] \arrow[d, "f"] & A' \arrow[d, "f'"] \\
\sigma A \arrow[r, "\sigma\psi"'] & \sigma A'
\end{tikzcd}$$

•

Remark 5.2 *the fixed point category is never empty: any terminal object is mapped to another isomorphic terminal object; see also the lemma below (the reader may now appreciate the advantage of being relaxed about* $FIX(E,\sigma)$*).*

There is an obvious forgetful functor

$$U : FIX(E,\sigma) \longrightarrow E$$

$$(A,f) \mapsto A$$

Lemma 5.1 $FIX(E,\sigma)$ *is a lex category and* U *preserves and creates finite limits.*

Proof: let (A, f) and (A', f') be two elements in $FIX(E, \sigma)$; their product is $A \times A' \longrightarrow \sigma A \times \sigma A' \equiv \sigma(A \times A')$ and if

$$\begin{tikzcd}
(A, f) \arrow[r, shift left, "\psi"] \arrow[r, shift right, "\phi"'] & (A', f')
\end{tikzcd}$$

is a parallel pair in $FIX(E, \sigma)$, their equalizer is $(eq(\psi, \phi), t) \hookrightarrow (A, f)$ where

$$\begin{tikzcd}
eq(\psi, \phi) \arrow[r, "i"] \arrow[d, "t"'] & A \arrow[r, shift left, "\psi"] \arrow[r, shift right, "\phi"'] \arrow[d, "f"] & A' \arrow[d, "f'"'] \\
\sigma eq(\psi, \phi) \arrow[r, "\sigma i"'] & \sigma A \arrow[r, shift left, "\sigma\psi"] \arrow[r, shift right, "\sigma\phi"'] & \sigma A'
\end{tikzcd}$$

t exists because the bottom is also an equalizer (σ is exact!) QED.

Notice that, if $\psi : (E, \sigma) \longrightarrow (E'\sigma')$ is a morphism of modal topoi, we get a functor in the opposite direction between the corresponding fixed-point categories; indeed, let $(A, f) \in FIX(E', \sigma')_0$;then

$$\psi_0^*(A)(f) \xrightarrow{\psi_0^*} \psi_0^*(\sigma')A)(A) \xrightarrow{\psi_1} \sigma(\psi_0^* A)$$

so we define $FIX(\psi)(A,f) \equiv (\psi_0(A), \psi_1(A) \circ \psi_0^*(f))$ and if $\lambda : (A,f) \longrightarrow (B,g)$ is a map in $FIX(E',\sigma')$

$$\begin{tikzcd}
\psi_0^*(A) \arrow[rrr, "\psi(\lambda)"] \arrow[dr, "\psi(f)"] \arrow[dd] & & & \psi_0^*(B) \arrow[dl, "\psi(g)"'] \arrow[dd] \\
& \psi_0^*(\sigma' A) \arrow[r, "\psi(\sigma\lambda)"] \arrow[dl, "\psi_{1(A)}"] & \psi_0^*(\sigma' B) \arrow[dr, "\psi_{1(B)}"] & \\
\sigma\psi_0^*(A) \arrow[rrr, "\sigma\psi(\lambda)"] & & & \sigma\psi_0^*(B)
\end{tikzcd}$$

Summing up, we have a functor:

$$FIX : MODTOP^{op} \longrightarrow CAT$$

from the large category of modal topoi to the ultralarge category of (not necessarily small) categories: in fact it can be shown that such a functor factors through LEX, the category of lex categories and lex functors between them. The reader may wonder why we introduced the fixed point category in our present context; the reason is that, loosely speaking, the restriction of the subobjects-functor of E to $FIX(E,\sigma)$ generates a modal doctrine; more precisely, consider $SUB_E \circ U : FIX(E,\sigma) \longrightarrow DL^{op}$.

Theorem 5.2 *The functor $SUB_E \circ U$ is in a canonical way a modal doctrine with term category $FIX(E,\sigma)$.*

Proof: this functor is a doctrine because U preserves products; we now define a modality on this doctrine in the following fashion: let (A, f) be in $FIX(E, \sigma)_0$ and $m : A_0 \hookrightarrow U(A, f) = A$ then

$$\sigma A_0 \xrightarrow{\sigma(m)} \sigma A \xrightarrow{f^{-1}} A$$

is a monomorphism (σ preserves monos and f^{-1} is the inverse of an iso) so the map $(A_0, m) \longmapsto (\sigma A_0, f^{-1} \circ \sigma m)$ is clearly a map in $SUB_{E(A)}$:

$$SUB_{E(A)} \xrightarrow{\sigma^{\#}_{A}} SUB_{E(A)}$$

and preserves infs. Indeed, this operation is natural in A and so it defines an *endofunctor* $\sigma^{\#}$ of $SUB_E \circ U$ regarded as a functor taking values in the category of meet-semilattices; let us check this: if $\lambda : (A, f) \longrightarrow (B, g)$ and $h : B_0 \hookrightarrow B$ then $\lambda^{-1}\sigma^{\#}_{B(B_0)}$ is computed as

$$\begin{tikzcd}
\lambda^{-1}\sigma^{\#}(B_0) \arrow[rr] \arrow[d] & & A \arrow[d, "\lambda"] \\
\sigma B_0 \arrow[r, "\sigma m"'] & \sigma B \arrow[r, "g^{-1}"'] & B
\end{tikzcd}$$

(observe that the diagram above is a pull-back). On the other hand, $\sigma^{\#}_{A}(\lambda^{-1}B_0)$ is computed in the following way: first pulling back

$$\begin{tikzcd}
\lambda^{-1}B_0 \arrow[r] \arrow[d] & B_0 \arrow[d] \\
A \arrow[r, "\lambda"'] & B
\end{tikzcd}$$

Since σ is exact

$$\begin{tikzcd}
\sigma\lambda^{-1}B_0 \arrow[r] \arrow[d] & \sigma B_0 \arrow[d] \\
\sigma A \arrow[r, "\sigma\lambda"'] & \sigma B
\end{tikzcd}$$

is also a pull-back: hence

$$\begin{tikzcd}
\sigma\lambda^{-1}B_0 \arrow[r] \arrow[d] & \sigma B_0 \arrow[d] \\
\sigma A \arrow[r, "\sigma\lambda"] \arrow[d, "f^{-1}"'] & \sigma b \arrow[d, "g^{-1}"'] \\
A \arrow[r, "\lambda"'] & B
\end{tikzcd}$$

the composition of the two pull- backs is also a pull-back

$$\begin{tikzcd}
\sigma_A\lambda^{-1}B_0 \arrow[r] \arrow[d] & \sigma B_0 \arrow[d] \\
A \arrow[r] & B
\end{tikzcd}$$

The uniqueness of pull-backs (up to isos) completes the argument. QED

It is important to realize that the modal doctrine produced in the foregoing is far from being "free": a certain compatibility of $\exists$ and $\Box$ is forced, as it were, by the topos itself. Indeed, let $m : A_0 \hookrightarrow UA$ and $U\lambda : UA \longrightarrow UB$

Then we have a image-factorisation in E

$$\begin{tikzcd}
A_0 \arrow[r] \arrow[d] & UA \arrow[d, "U\lambda"'] \\
\exists U\lambda A_0 \arrow[r] & UB
\end{tikzcd}$$

Applying σ to the previous diagram, we obtain

$$\begin{tikzcd}
 & \sigma A_0 \arrow[r, "\sigma m"] \arrow[d] \arrow[ddl] & \sigma A \arrow[r, "f^{-1}"] \arrow[d, "\sigma U f"] & A \arrow[d, "U\lambda"] \\
 & \sigma \exists U f(A_0) \arrow[r] & \sigma B \arrow[r] & B \\
\exists(U \circ f^{-1} \circ \sigma m)A_0 \arrow[ur, dashed] \arrow[urrr] & & &
\end{tikzcd}$$

(the dashed arrow comes from the universal property of the image-factorization). Summing up, we get $\exists\sigma^{\#} \leq \sigma^{\#}\exists$; the converse inequality is not true in general, unless we postulate that σ preserves images of a certain class of maps, containing the ones coming from $FIX(E\sigma)$. The reader could be (rightly) unhappy with the fact that this type of doctrine admits a (possibly) large term-category; worse, this category changes from topos to topos. It seems far more profitable to keep it fixed by selecting once for all a small term-category T:

Definition 5.3 *A T-modal topos is a triple (F, E, σ) where $F : T \longrightarrow E$ is a finite product preserving functor and*

$$\begin{tikzcd}
 & T \arrow[dl, "F"'] \arrow[dr, "F"] & \\
E \arrow[rr, "\sigma"'] & & E
\end{tikzcd}$$

commutes. A map $\psi : (F, E, \sigma) \longrightarrow (F', E', \sigma')$ *is a modal topos map* $\psi : (E, \sigma) \longrightarrow (E', \sigma')$ *such that*

$$\begin{tikzcd}
 & T \arrow[dl, "F'"'] \arrow[dr, "F"] & \\
E' \arrow[rr, "\psi_0^*"'] & & E
\end{tikzcd}$$

commutes.

The previous data determine a large category, denoted by $T - MODTOP$. We are now in a better position: all doctrines will have the same term-category, namely T. We can reformulate the construction described by theorem 5.2 as:

Theorem 5.3 *The association* $(F, E, \sigma) \mapsto SUB_E \circ F$ *defines a functor*

$$DOC : (T - MODTOP)^{op} \longrightarrow T - MODOC$$

Proof: Indeed, if (F, E, σ) is any T-modal topos there is a factorization

$$\begin{tikzcd}
T \arrow[r, "F"] \arrow[dr, "F^{\#}"'] & E \\
 & FIX(E, \sigma) \arrow[u, "U"']
\end{tikzcd}$$

where $F^{\#}(t) \equiv (F(t), id_{F(t)}$ for $t \in T_0$. But this means that $SUB_E \circ F$ is obtained from the doctrine $SUB_E \circ U$ by changing of term-theory along $F^{\#}$ and so is itself a doctrine. The naturality of the association is straightforward checking. QED

The preceding theorem can be read off as saying that modal topoi provide a *sound* semantics for modal theories. In Chapter 7 we will show that in some sense modal theories are complete with respect to this semantics. To be more precise, we would like to produce a left adjoint to the functor DOC; this is indeed possible, if we replace $T - MODTOP$ and $T - DOC$ with suitable subcategories.

Chapter 6

On points and relations

We are now going to show that modal topoi are in some sense like generalized *Kripke frames* (see [27]), establishing in this way a line of continuity with the traditional semantics of modal logics. Moreover, we will see how a suitable subcategory of modal topoi can be regarded as constituted by "spaces with continuous relations", much in the same spirit as topoi themselves can be treated as generalized topological spaces (the reader is referred to the introduction of [2] for Grothendieck's view of the topos as the natural object of study for topology). We intend to stress though, that the results we will present in the next chapter are entirely independent from this connection. Let us first remind the reader of the fundamental notion of *point* for an arbitrary topos:

Definition 6.1 *A point p of a topos E is just a geometric morphism:*

$$\begin{tikzcd} Set \arrow[r, "\perp", "p_*"'] & E \arrow[l, bend right, "p^*"'] \end{tikzcd}$$

By *pt*(E) we shall denote the class of points of E; to rid ourselves of set-theoretic intricacies we will work only within the full subcategory of TOP whose objects are topoi having a *set* of points (we are not being too restrictive if we think that such a category contains for instance all *localic* topoi); we shall refer to it as $S - TOP$ (analogously, $MOD[S - TOP]$ will be the corresponding modal version). This notion makes good sense if we confine our attention to Grothendieck topoi, where Set is indeed the terminal object; also, when a topos happens to be spatial (in other terms is just the topos of sheaves for a topological space), there is a bijection between points of the topos and points of the space. Now we turn our attention to modal topoi. The key observation is that the lex endofunctor forces a *relation* on the set of points of the underlying topos:

Definition 6.2 *Let* (E, σ) *be a modal topos and*

$$\begin{tikzcd} Set \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & E \end{tikzcd}$$

two arbitrary points of E. *We say that* p_1 *is* σ*-related to* p_2 ($p_1 R_\sigma P_2$) *iff the following diagram commutes as a meet-semilattice diagram :*

$$\begin{tikzcd}
SUB_E(1) \arrow[rr, "\sigma"] \arrow[dr, "p_2^*"'] & & SUB_E(1) \arrow[dl, "p_1^*"] \\
& 2 = SUB_{Set}(1) &
\end{tikzcd}$$

Remark 6.1 *Observe that* σ, *being lex, preserves monomorphisms and terminal object and so it defines a meet-semilattice map (meets are computed as pull-backs) from* $SUB_E(1)$ *to itself.* p_1 *and* p_2 *are left image parts of geometric morphisms and so lex functors ipso facto: this tells us that the previous diagram is well-defined; its commutativity is of course not guaranteed, which makes our definition a meaningful one.*

Via the definition above, we associate to a modal topos (E, σ) a pair $(pt(E), R_\sigma)$. The next Lemma shows the functoriality of the association:

Lemma 6.1 *The covariant functor*

$$pt : S - TOP \longrightarrow Set$$

extends to a functor

$$MOD[S - TOP] \longrightarrow RelSet$$

where RelSet denotes the category of pairs (X, R), $R \subseteq X^2$ *and maps* $\psi : (X, R) \longrightarrow (Y, S)$ *such that* $\psi : X \longrightarrow Y$ *is a set-map preserving the relation:* $\forall x, y \in X (xRy \Rightarrow \psi(x) S \psi(y))$.

Proof: we must show that if $\psi : (E, \sigma) \longrightarrow (E', \sigma')$ is a map in $MODS{-}TOP$ then the map $pt(\psi) : pt(E) \longrightarrow pt(E')$ preserves the relation; but if $p_1, p_2 \in pt(E)$ and $p_1 R_\sigma p_2$ we have that

$$\begin{tikzcd}
SUB_E(1) \arrow[rr, "\sigma"] \arrow[dr, "p_2^*"'] & & SUB_E(1) \arrow[dl, "p_1^*"] \\
& 2 &
\end{tikzcd}$$

commutes. Let us now take a look at

$$\begin{tikzcd}
SUB_{E'}(1) \arrow[rr, "\sigma'"] \arrow[dr, "(\psi o p_1)^*"] \arrow[dd, "\psi_0^*"'] & & SUB_{E'}(1) \arrow[dl, "(\psi o p_2)^*"'] \arrow[dd, "\psi_0^*"] \\
& 2 & \\
SUB_E(1) \arrow[rr, "\sigma"'] \arrow[ur, "p_1^*"] & & SUB_E(1) \arrow[ul, "p_2^*"']
\end{tikzcd}$$

The external rectangle commutes because ψ is a morphism in $MODTOP$. The two triangles on the sides commute by definition: we obtain the commutativity of the upper triangle. But this just expresses the fact that $\psi(p_1) R_\sigma \psi(p_2)$. We have just seen that to any modal topos corresponds a Kripke frame; but frames are adequate only in dealing with propositional modal logic, whereas a modal topos is a carrier of a full-fledged first order modal doctrine; it is to be expected that to such an object there corresponds a family of frames. This is indeed the case, as we shall see in a moment. First, let us notice that we can "slice" a modal topos, obtaining a variety of new modal topoi (it is a classical result of topos theory that slicing a topos by any of its elements we always get a topos: our next theorem can thus be seen as a lifting of the Slicing Lemma to the modal context). Let (E, σ) be a modal topos and (A, f) an arbitrary element of the fixed point category; the topos E/A possesses a natural endofunctor, namely

$$\sigma_{(A,f)} : E/A \longrightarrow E/A$$

$$\sigma_{(A,f)}[p : B \rightarrow A] = [f^{-1} \circ \sigma p : \sigma B \rightarrow A]$$

and if

$$\begin{tikzcd}
B \arrow[rr, "\psi"] \arrow[dr, "p"'] & & \acute{B} \arrow[dl, "\acute{p}"] \\
& A &
\end{tikzcd}$$

$\sigma_{(A,f)}(\psi) = \sigma\psi$. Moreover $\sigma_{(A,f)}$ is a lex functor, just because σ is.

Lemma 6.2 (The Modal Slicing Lemma) *The association* $(A, f) \mapsto (E/A, \sigma_{(A,f)}$ *is a pseudofunctor* $SL : FIX(E, \sigma) \longrightarrow MODTOP$

Proof: Let $\psi : (A, f) \to (B, G)$ be any map in $FIX(E, \sigma)$; from standard topos theory, we have a geometric morphism

$$\begin{tikzcd} E/A \arrow[r, bend right, "\psi_*"'] \arrow[r, phantom, "\perp"] & E/B \arrow[l, "\psi^{-1}"'] \end{tikzcd}$$

Now, we claim that (ψ_*, ψ^{-1}) is the 1-dimensional component of a morphism in $MODTOP$; in other words, we claim that

$$\begin{tikzcd} E/A \arrow[r, "\sigma_{(A,f)}"] & E/A \\ E/B \arrow[u, "\psi^{-1}"] \arrow[r, "\sigma_{(B,g)}"'] & E/B \arrow[u, "\psi^{-1}"'] \end{tikzcd}$$

commutes up to a given isomorphism; indeed, let $[\lambda : C \to B]$ be an object of E/B $\sigma_{(B,g)}[\lambda : C \to B] = [g^{-1} \circ \sigma\lambda]$. Now we pull it back along ψ and

$$\begin{tikzcd} \psi^{-1}(\sigma_{(B,g)}(C) \arrow[r] \arrow[dd] & \sigma C \arrow[d, "\sigma\lambda"'] \\ & \sigma B \arrow[d, "g^{-1}"'] \\ A \arrow[r, "\psi"'] & B \end{tikzcd}$$

On the other hand, first pulling back we get

$$\begin{tikzcd} \psi^{-1}(C) \arrow[r] \arrow[d] & C \arrow[d, "\lambda"'] \\ A \arrow[r, "\psi"'] & B \end{tikzcd}$$

Applying $\sigma_{(A,f)}$ we obtain $[\sigma\psi^{-1} \rightarrow \sigma A \rightarrow A]$ But σ preserves pull-backs, so that

$$\begin{tikzcd}
\sigma\psi^{-1}(C) \arrow[r] \arrow[d] & \sigma C \arrow[d] \\
\sigma A \arrow[r, "\sigma\psi"'] & \sigma B
\end{tikzcd}$$

commutes. On the other hand,

$$\begin{tikzcd}
\sigma A \arrow[r, "\sigma\psi"] \arrow[d, "f^{-1}"] & \sigma B \arrow[d, "g^{-1}"'] \\
A \arrow[r, "\psi"'] & B
\end{tikzcd}$$

is also a pull-back: combining the two we get a pull-back diagram, namely

$$\begin{tikzcd}
\sigma\psi^{-1}(C) \arrow[r] \arrow[d] & \sigma C \arrow[dd] \\
\sigma A \arrow[d] & \\
A \arrow[r, "\psi"'] & B
\end{tikzcd}$$

and the result follows from the universal property of pull-backs. If we turn now our attention to T-modal topoi, we can use the Modal Slicing Lemma to produce indexed modal topoi with parameter category T: more specifically, let

$$\begin{tikzcd}
& T \arrow[dl, "F"'] \arrow[dr, "F"] & \\
E \arrow[rr, "\sigma"'] & & E
\end{tikzcd}$$

be any T-modal topos. We can slice (E, σ) by elements of T:

$$t \mapsto [E/F(t), \sigma_{F(t)}, id_{F(t)}]$$

This process generates a T-indexed family of modal topoi $SL \circ F : T \longrightarrow MODTOP$. Finally, we compose with the points-functor to obtain: $pt \circ SL \circ F : T \longrightarrow RelSet$. We can regard this functor as an "object with a relation" in the topos Set^T (covariant presheaves on T), or equivalently as a parametrized family of Kripke frames canonically associated with the T-modal topos. Indeed, more is true: the association is functorial from the category $T - MODTOP$ to the category of objects and relations in Set^T; when T is the trivial category we obtain Lemma 4.1 as a special case. It would perhaps of some interest trying to reconstruct a T-modal topos out of an object with a relation in Set^T. We won't pursue this issue here; instead, we go back to our former construction; we intend to show that for suitable modal topoi, the associated Kripke frame carries an additional piece of structure (already in [28] Thomason introduced the notion of generalized frames. Standard Kripke frames are not enough if we want to avoid incompleteness phenomena).

We need first some preliminary definitions:

Definition 6.3 *A topological space with a continuous relation is a triple* (X, τ, R)*, where* (X, τ) *is a topological space and* $R \subseteq X^2$ *is such that* $\forall U \in$

$\tau R^{-1}(U) = \{x \in X, xRy \Rightarrow y \in U\}$ is open.

A map of such structures is a relation-preserving continuous map. These data define a category, denoted as *RelTop*. There is an obvious forgetful functor $U : RelTop \longrightarrow RelSet$; if (X, τ, R) is an object of *RelTop*, we can associate to it a modal distributive lattice, in the obvious way: $R^{-1} : O(X) \longrightarrow O(X)$ unfortunately, this correspondence is *not* functorial: if $\psi : (X, \tau, R) \longrightarrow (X', \tau', R')$ is a map in *RelTop*, we only have $\psi^{-1}(R^{-1}(U)) \subseteq R^{-1}(\psi^{-1}(U))$ for an arbitrary open $U \in \tau'$: in fact if $x \in \psi^{-1}(R^{-1}(U))$, we know that $\psi(x) \in R^{-1}(U)$; now suppose xRx': then $\psi(x)R'\psi(x')$ and so $\psi(x') \in U$, $x' \in \psi^{-1}(U)$; but x' was arbitrary: $x \in R^{-1}(\psi^{-1}(U))$. To force functoriality, we need the other half of the inequality, namely

$$(*)\forall U \in \tau' : R^{-1}(\psi^{-1}(U)) \subseteq \psi^{-1}(R^{-1}(U))$$

To this end, we impose a further restriction to the maps (a device well-known to modal logicians: see [26] or [36]):

Definition 6.4 *A* p-morphism $\psi : (X, \tau, R) \longrightarrow (X', \tau', R')$ *is a morphism in RelTop such that* $\forall x \in X \forall y \in X'(\psi(x)Ry \Rightarrow \exists x'(\psi(x') = y \wedge xRx')$

The category of topological spaces with continuous relations and p-morphisms will be denoted as $p - RelTop$.

Theorem 6.1 *The association* $(X, \tau, R) \mapsto (O(X), R^{-1})$ *is a functor*

$$\begin{tikzcd}(p - RelTop)^{op} \arrow[r] & MFRAMES \arrow[r, "U"] & MDL\end{tikzcd}$$

where $MFRAMES$ *is the pull-back category*

$$\begin{tikzcd}MFRAMES \arrow[r] \arrow[d] & MDL \arrow[d] \\ FRAMES \arrow[r, "U"'] & DL\end{tikzcd}$$

. Proof: We leave to the reader to check condition $(*)$.

Our goal now is to select a suitable subcategory of $MODTOP$ in such a way that applying the functor "point" we may land in $p - RelTop$. Why? Simply because we know how to go from $p - RelTop$ to $MODTOP$: compose the functor "opens" with the "sheaves" functor described in Chapter 5; we clearly aim at establishing an adjointness.

Definition 6.5 *A modal topos* (E, σ) *has enough points iff:*

1) E *has enough points in the usual sense (if* U *and* V *are two distinct subobjects of* 1*, there exists a point* p *such that* $p^*(U) \neq p^*(V)$*)*

2) for every point p *there exists another point* q *such that* $pR_\sigma q$.

By EPMODTOP we indicate the full subcategory of MODTOP whose objects are cocomplete *and have enough points.*

Theorem 6.2 *The functor pt* : *MODTOP* $\longrightarrow$ *RelSet factors through the category RelTop:*

$$\begin{tikzcd}
EPMODTOP \arrow[r, "pt"] \arrow[dr, "pt'"'] & RelSet \\
& RelTop \arrow[u, "U"']
\end{tikzcd}$$

Proof: if $(E, \sigma) \in EPMODTOP_0$ then $pt(E)$ admits a canonical topology, namely the one whose opens are of the form $U^* = \{p \in pt(E)p^*(U) = 1\}$ where $U \hookrightarrow 1$ (the cocompleteness of E ensures that such a family is closed under arbitrary unions). We must show that $R^{-1}(U^*)$ is open for any U^*; we will actually show more: $R_\sigma^{-1}(U^*) = (\sigma U)^*$; this fact, together with the assumption that E has enough points, entails precisely that $(SUB_E(1), \sigma) \longrightarrow O(pt(E), R_\sigma^{-1})$ given by $U \mapsto U^*$ is an isomorphism of modal distributive lattices (indeed of modal locales). Let us now show that $(\sigma U)^* \subseteq R_\sigma^{-1}(U^*)$. Let $x \in (\sigma U)^*$: $x^*(\sigma U) = 1$; if $xR_\sigma y y^*(U) = 1$ or $y \in U^*$; conclusion: $x \in R_\sigma^{-1}(U^*)$. Conversely, suppose $x \in R_\sigma^{-1}(U^*)$; pick any y such that xRy: then $y \in U^*$ or equivalently $y^*(U) = 1$ (our assumption makes sure that such a y actually exists); $x^*(\sigma U) = 1$ or $x \in (\sigma U)^*$ QED.

As a by-product of the preceding theorem, we have that, if $\psi : (E, \sigma) \longrightarrow (E', \sigma')$ is a morphism in $EPMODTOP$, the induced map

$$[pt(\psi)]^{-1} : (O(pt(E')), R_{\sigma'}) \longrightarrow (O(pt(E)), R_\sigma)$$

is a map of modal distributive lattices; indeed, ψ induces a map $(SUB_{E'}(1), \sigma') \longrightarrow (SUB_E(1), \sigma)$ and we have already observed that $(SUB_E(1) \approx (O(pt(E), R_\sigma^{-1})$. This fact does not imply, however, that $pt(\psi)$ is a p-morphism. To land in $p - RelTop$, we have to select suitable morphisms in $EPMODTOP$.

Definition 6.6 *A map* $\psi : (E, \sigma) \longrightarrow (\acute{E}, \acute{\sigma})$ *in* $EPMODTOP$ *is a p-map iff* $\forall p : Set \rightarrow E$ *and* $\forall q : Set \rightarrow \acute{E}$ $\psi_0 \circ pR_{\acute{\sigma}}q$ *implies that there exist* $\acute{p} : Set \rightarrow E$ *such that* $pR_\sigma\acute{p}$ *and* $\psi_0 \circ \acute{p} = q$.

We can now define $p - EPMODTOP$ as the subcategory of $EPMODTOP$ with the same class of objects but having p-maps as morphisms.

Theorem 6.3 *The functor "points" restricted to* $p - EPMODTOP$ *factors through* $p - RelTop$*; furthermore,*

$$\begin{tikzcd}
p - EPMODTOP \arrow[r, "\top", "pt"'] & p - RelTop \arrow[l, bend right, "Sh"']
\end{tikzcd}$$

Proof: Start from an object in $p - EPMODTOP$, say (E, σ). Applying the functor "points" we get $(pt(E), R_\sigma)$. We know that $O(pt(E)) \approx SUB_E(1)$, because the topos has enough points; the inclusion $O(pt(E)) \mapsto E$ is a continuous map of a site into a cocomplete topos, so it induces a geometric morphism

$$\begin{tikzcd}
Sh(O(pt(E)) \arrow[r, "\mu^*"] & E \arrow[l, bend left, "\mu_*", "\perp"']
\end{tikzcd}$$

We claim that μ is the 1-dimensional component of a morphism in $MODTOP$; in other words, that

$$\begin{tikzcd}
E \arrow[r, "\sigma"] & E \\
Sh(O(pt(E)) \arrow[u, "\mu^*"] \arrow[r, "(R^{-1})^{\#}"'] & Sh(O(pt(E)) \arrow[u, "\mu^*"']
\end{tikzcd}$$

commutes up to an isomorphism. But we certainly have that

$$\begin{tikzcd}
O(pt(E)) \arrow[r, "R_\sigma^{-1}"] \arrow[d] & O(pt(E)) \arrow[d] \\
E \arrow[r, "\sigma"'] & E
\end{tikzcd}$$

and we also have that

$$\begin{tikzcd}
O(pt(E)) \arrow[r, "R_\sigma^{-1}"] \arrow[d, "Y"] & O(pt(E)) \arrow[d, "Y"'] \\
Sh(O(pt(E))) \arrow[r, "(R_\sigma^{-1})^\#"] & Sh(O(pt(E)))
\end{tikzcd}$$

commutes up to an isomorphism (see the remark in chapter 4: $O(pt(E))$ is a subcanonical site). Let us consider an arbitrary element $F \in Sh(O(pt(E)))_0$. F is a colimit of representables, $F = \lim_{\rightarrow} U_i^*$ where $U_i \in O(pt(E))$. $(R_\sigma^{-1})^\#(F) = \lim_{\rightarrow}(R_\sigma^{-1}(U_i)$; but μ^* preserves arbitrary colimits (it is a left adjoint!) and so $\mu^*(R_\sigma^{-1})^\#(F) = \mu^* \lim_{\rightarrow} R_\sigma^{-1}(U_i)^*) = \lim_{\rightarrow} \mu^* U_i^* = \lim_{\rightarrow} R_\sigma^{-1} \mu^* U_i^* = (R_\sigma^{-1})^\# \lim_{\rightarrow} \mu^* U_i^* = (R_\sigma^{-1})^\#(F)$. On the other hand, if (X, τ, R) is an element of $p - RelTop$, $pt(Sh(X)) = X$ and clearly $(O(pt(Sh(X)), R_\sigma) \approx (O(X), R^{-1})$, so the counit is indeed an isomorphism. QED

We leave this chapter with an open question: we have seen that to an arbitrary modal topos is possible to associate a parametrized family of Kripke frames. It is not clear to us whether we can use the same construction to slice a modal topos with enough points to get a parametrized family of topological spaces with a continuous relation; the question is, we believe, of some logical interest; our Slicing Lemma does not preserve the first order modal logic of the modal topos, whereas this further refinement would.

Chapter 7

Classifying modal topoi and the completeness theorem

Thus far we have encountered modal topoi and have seen that they provide a sound semantics for a large variety of first order modal theories; the question is still pending, as to which extent they are also a complete semantics, at least for a sufficiently large class of theories. In this chapter we take up such a question, and, after refining a bit our machinery, answer in the affirmative. Let us go back for a moment to the propositional case; it is well-known that there exist a duality, the so-called *Stone duality* between DL and $CohSpace$, a subcategory of Top. More specifically, the objects of $CohSpace$ are *coherent* *topological spaces* (we recall the definition: a sober space is coherent if the collection of compact open sets is a distributive lattice and such a collection is a basis for the topology) and maps are coherent continuous maps (the counterimage of a compact open is compact). The duality is:

$$\begin{tikzcd} DL^{op} \arrow[r, bend left, "spec"] & CohSpace \arrow[l, bend left, "KO"] \end{tikzcd}$$

where *spec* is the functor that associates to a lattice its *spectrum* (the set of its prime ideals) and KO sends a coherent space to its lattice of compact-opens (for a proof of this theorem in the language of locales, the reader can consult Johnstone's book [4]). We can recreate an adjunction in the context of topoi, instead of spaces. We need first some standard definitions from topos theory:

Definition 7.1 *An object A in a topos E is quasi-compact iff every covering of A in the canonical topology of E has a finite subcovering. An object A is* coherent *iff it is quasi-compact and for every diagram:*

$$\begin{tikzcd} B_1 \arrow[r] & A & B_2 \arrow[l] \end{tikzcd}$$

where B_1 and B_2 are quasi-compact, the pull-back is quasi-compact. The full *subcategory of coherent objects of* E *is denoted* $COH(E)$.

Definition 7.2 *A topos* E *is coherent iff*

1) the category $COH(E)$ *is a* pretopos, *i.e. is logical, has quotients of equivalence relations and finite disjoint sums*

2) the inclusion functor $COH(E) \hookrightarrow E$ *is logical.*

$COHTOP$ *is the subcategory of* TOP *whose objects are cocomplete* coherent topoi *and maps are coherent geometric morphisms, meaning that the left image part is a* coherent functor *(one which maps coherent objects to coherent objects).*

Theorem 7.1 *There is an adjunction*

$$\begin{tikzcd} DL^{op} \arrow[r, bend left, "Sh(-)"] \arrow[r, phantom, "\perp"] & COHTOP \arrow[l, "KSUB_{-}(1)"] \end{tikzcd}$$

where $KSUB_E(1)$ *denotes the distributive lattice of coherent subobjects of* 1 *(the coherence of the topos implies that it is a lattice and not merely a* $\wedge$*-semilattice), whereas* $Sh(L)$ *is the topos of sheaves on* L *regarded as a finite site (only finite covers of elements are covers).*

Stone's celebrated theorem in this new dress states that the unit of this adjunction is an isomorphism, so that DL lives as a reflective subcategory of $COHTOP$, namely the one whose objects are coherent *spatial* topoi. In turn, it can be shown that this subcategory is, up to equivalence, nothing else but $CohSpace$: the circle is closed. This detour does not seem to be very promising at first sight; indeed, insofar as we are concerned with propositional logic, nothing new appears on the horizon. However, we shall see in a moment how this topos-theoretical formulation of Stone's duality brings forth some heuristic novelties. First, let us lift the previous theorem to the modal setting: this is easily accomplished by the following diagram

$$\begin{tikzcd}
MDL^{op} \arrow[r, bend left=40] \arrow[r, phantom, bend left=20, "\top"] \arrow[d, "U^{op}"'] & COHMODTOP \arrow[l] \arrow[d, "U"'] \\
DL^{op} \arrow[r, bend right=40] \arrow[r, phantom, bend right=20, "\top"'] & COHTOP \arrow[l]
\end{tikzcd}$$

To recreate an adjoint situation, it suffices to put on the upper right corner the category $COHMODTOP$, whose objects are pairs (E, σ) where E is a coherent topos and σ is a coherent lex endofunctor of E. Maps are just modal topoi morphisms whose first component happens to be coherent. If $(E, \sigma) \in COHMODTOP_0$ the map

$$\sigma : SUB_E(1) \longrightarrow SUB_E(1)$$

restricts to

$$\sigma : KSUB_E(1) \longrightarrow KSUB_E(1)$$

so that the functor $KSUB_E(1)$ lifts properly. We know already from Chapter 5 that $(Sh(L), \tau^{\#})$ is a modal topos, indeed a coherent one (the topology on L is finite). For the proof that the two functors on top as well as the two at the bottom are adjoint to one another, we refer the reader to the last part of the present chapter, where a more general result will be given. We now turn our attention to the first order case; as usual, let us first fix a term-category T and look at the large category of T-regular modal doctrines, $T - REGMODOC$. In Chapter 4 we have seen that to any T-regular modal doctrine corresponds functorially a T-modal site, indeed a site with a finitary subcanonical topology: $D \mapsto C(D)$ equipped with an embedding $G : T \longrightarrow C(D)$ where G is a functor preserving finite products (if you wish you can think of it as a change of base theory). We can then produce a modal topos, namely

$$\begin{tikzcd} T \arrow[r, "G"] \arrow[rr, bend left, "Y \circ G"] & C(D) \arrow[r, "Y"] & Sh(C(D)) \end{tikzcd}$$

(Y is the Yoneda embedding). The foregoing construction is functorial:

$$Sh \circ SC : T - REGMODOC \longrightarrow T - MODTOP^{op}$$

Furthermore, because of the finiteness of $C(D)$, the topos so produced is indeed coherent; in this way the functor $Sh \circ SC$ factors through the opposite of the category $T\!-\!COHMODTOP$, whose objects are triples (E, F, σ), such that:

1) (E, σ) is a coherent modal topos

2) $F : T \longrightarrow E$ factors through the full subcategory of coherent objects of E:

$$\begin{tikzcd}
T \arrow[r, "F"] \arrow[d, "F"] & E \\
COH(E) \arrow[ur] &
\end{tikzcd}$$

Maps in this category are just coherent geometric morphisms under T. There is a functor going the opposite direction, namely:

$$KSUB_{-} \circ (-) : (T - COHMODTOP)^{op} \longrightarrow T - REGMODOC$$

$$(E, F, \sigma) \mapsto KSUB_E \circ F$$

Theorem 7.2 *The functor $KSUB_{(-)} \circ (-)$ is adjoint on the right to the functor $Sh \circ SC$. Moreover the unit of the adjunction is injective.*

Proof: Let D be a modal doctrine. By the adjunction of Chapter 5 we have already an isomorphism of doctrines:

$$D \xrightarrow{\eta_D} SUB_{C(D)} \circ G$$

But we also have a map of doctrines:

$$SUB_{C(D)} \circ G \xrightarrow{Y} KSUB_{Sh(C(D))} \circ G$$

given by the faithful Yoneda embedding of $C(D)$ into its category of sheaves (this embedding preserves the logical structure of $C(D)$ and furthermore maps $C(D)$ into the coherent category $COH(Sh(C(D))))$ The composition of the two preceding maps is the component at D of a natural transformation, which is injective because both of them are. This is our candidate unit. On the other hand, if we start from a topos under T, say (E, F, σ), consider the category $SC(SUB_E \circ F)$; there is a functor:

$$SC(KSUB_E \circ F) \longrightarrow E$$

given by: $(t, [A]) \mapsto A$. (of course there is a choice to be made for the representatives) Moreover, this functor is a morphism of sites under T (when we regard E as a possibly large site with its canonical topology). We then have a geometric morphism from $\epsilon_E : E \to Sh(SC(KSUB_E \circ F)$: see [24].

We have yet to check that such a geometric morphism is the first component of a modal-topos map. To this end, we just need to contemplate the following diagram

$$\begin{tikzcd}
Sh(SC(SUB_E \circ F)) \arrow[rr, "\sigma^{\#}"] \arrow[ddr, swap, "\epsilon_E"] & & Sh(SC(SUB_E \circ F)) \arrow[ddr] & \\
& SC(SUB_E \circ F) \arrow[ul, swap, "Y"] \arrow[rr, "\epsilon_E"] \arrow[d] & & SC(SUB_E \circ F) \arrow[ul, swap, "Y"] \arrow[d] \\
& E \arrow[rr, swap, "\sigma"] & & E
\end{tikzcd}$$

The top commutes (up to isomorphism) because of the remark we made in Chapter 5. The inner square commutes trivially and the two triangles on the sides commute because the left image part of ϵ_E is the natural extension of the functor from its presenting site to the topos E. All this implies the commutativity up to an isomorphism of the bottom, which is what we needed. Again, ϵ_E is the component at E of a natural transformation. We leave to the reader to check that ϵ is indeed the counit of the adjunction.

Remark 7.1 *The previous theorem is valid for every choice of term-category T; in particular, it holds true when this category is the trivial one: $*$; but in this case the category $* - REGMODOC$ is obviously equivalent to DL. On the other hand, $* - COHMODTOP$ is essentially a "pointed version" of $COHMODTOP$ (in fact equivalent to it: $*$ is mapped into the terminal* *object wich is preserved by every map); it is then easy to see that we are reduced to the adjunction at the beginning of the chapter, as promised. In the light of this observation, we find it useful to think of the theorem as a* parametrized *Stone representation.*

It is now time to turn our attention to the semantical viewpoint. We need first to define the notion of model of a modal theory in an arbitrary modal topos:

Definition 7.3 *Let $\mathcal{T}$ be a coherent modal theory and T its associated category of terms. A* model *of $\mathcal{T}$ in a T-modal topos (E, σ, F) is a modal doctrine map:*

$$\psi : D_{\mathcal{T}} \to SUB_E \circ F$$

A model is weakly faithful *iff the following holds: $\forall t \in T_0(\psi_t(a) = \top \iff a = \top)$. A model is faithful iff it is an injection in the category of T-doctrines.*

Remark 7.2 *if the doctrine D happens to be a Heyting-valued doctrine the two last conditions are equivalent to one another; in the context of distributive lattices though, congruences are not ideal-determined (in other words, the knowledge of the congruence class of $\top$ does not determine the congruence itself). The injectivity of the map becomes a stronger requirement than* *the condition above, which already encompasses the common notion of faithfulness, namely that the model makes true precisely the formulae that were true in* $\mathcal{T}$.

Definition 7.4 *The category* $\mathcal{MOD}(\mathcal{T},(E,\sigma))$, *where* (E,σ) *is a modal topos is defined by:*

1) objects are models $\psi : D_{\mathcal{T}} \rightarrow SUB_E \circ F$ *for all possible product-preserving functors* $F : T \rightarrow E$ *commuting with* σ

2) morphisms are doctrinal commutative diagrams:

$$\begin{tikzcd}
 & D_{\mathcal{T}} \arrow[dl] \arrow[dr] & \\
SUB_E \circ F \arrow[rr] & & SUB_E \circ F'
\end{tikzcd}$$

Given a map

$$\psi : (E,\sigma) \longrightarrow (E',\sigma')$$

in the category $MODTOP$, we get a "pull-back" functor

$$\mathcal{MOD}(\mathcal{T},(E,\sigma)) \longleftarrow \mathcal{MOD}(\mathcal{T},(E',\acute{\sigma}))$$

by composition. If we keep our theory $\mathcal{T}$ fixed, we can think of the category of models as a functor

$$\mathcal{MOD}(\mathcal{T},-) : MODTOP^{op} \longrightarrow CAT$$

We are now going to show that such a functor is *representable*, meaning that we can find a modal topos $\beta\mathcal{T}$ (*the classifying modal topos* of $\mathcal{T}$) so that for every *cocomplete* modal topos (E, σ)

$$\mathcal{MOD}(\mathcal{T}, (E, \sigma)) \approx MODTOP((E, \sigma), \beta(\mathcal{T}))$$

In other words, the categories above are equivalent (notice that the two categories are not required to be isomorphic, so the functor above is representable only in a weak sense). Before stating our theorem, let us see what such an equivalence entails. To the identity map on $\beta(\mathcal{T})$ corresponds a model in this topos, the *generic model* of $\mathcal{T}$, with a special property, namely that any model in an arbitrary modal topos can be obtained from this generic model by "pulling it back" along a suitable modal topos map. Conversely, the existence of the classifying modal topos together with its generic model would enable us to reconstruct the equivalence. We note in passing that the very notion of classifying topos for a geometric theory is one of the substantial contributions of topos theory to semantics. As pointed out in [24], classifying objects are rather familiar to topologists and to geometers as well: we thus meet with a topic at the crossroad of logic and geometry. It seems rather promising that such a notion extends quite naturally to the modal set-up. This result suggests perhaps that classifying categorical structures of the appropriate kind are available for a large class of logics.

Theorem 7.3 *For every modal coherent theory $\mathcal{T}$ the restriction of the functor $\mathcal{MOD}(\mathcal{T},-)$ to the full subcategory of cocomplete modal topoi is representable.*

Proof: set $\beta(\mathcal{T}) = Sh(SC(C(D_{\mathcal{T}})))$. We already know that there is a model of $\mathcal{T}$ in $\beta(\mathcal{T})$, namely :

$$D_{\mathcal{T}} \rightarrow SUB_{C(D_{\mathcal{T}})} \rightarrow SUB_{\beta(\mathcal{T})}$$

It remains to be shown that such a model is generic (if it exists, a generic model is necessarily unique up to isomorphim). So, let $\psi : D_{\mathcal{T}} \rightarrow SUB_E \circ F$ an element (E, σ, F). This implies that there exist a map f in $T - LOG$ such that

$$\begin{tikzcd}
D_{\mathcal{T}} \arrow[r] \arrow[dr] & SUB_{C(D_{\mathcal{T}})} \circ G \arrow[d] \\
 & SUB_E
\end{tikzcd}$$

commutes in $T - MODOC$. But such a F is a continuous functor and so it induces a geometric morphism $\hat{\psi}$ from E to $\beta(T)$ under T. If we pull back the generic model along the left image part of $\hat{\psi}$ we get back precisely ψ. QED

Remark 7.3 *In Chapter 3 we observed that it is possible to define a category of modal theories, say ModTh. If* e $: \mathcal{T} \rightarrow \mathcal{T}'$ *is a map of theories, by composition we get a functor*

$$\mathcal{MOD}(\mathcal{T}', (E, \sigma)) \rightarrow \mathcal{MOD}(\mathcal{T}, (E, \sigma))$$

The morphism of theories induces also a map of the corresponding classifying modal topoi (in the opposite direction) and consequently a functor

$$MODTOP((E, \sigma), \beta(\mathcal{T}')) \rightarrow MODTOP((E, \sigma, \beta(\mathcal{T}))$$

It can be verified that the equivalence of categories described above is natural with respect to such maps of theories.

Chapter 8

The multimodal case

A great deal of what has been said thus far carries over to the case in which several unary modal operators appear simultaneously. The universes of discourse here are pairs $(E, \{\sigma_i\}_{i\in I})$ where each $\sigma_i : E \rightarrow E$ is a lex endofunctor of E and maps are geometric morphisms respecting, in the appropriate 2-categorical sense (for background on 2-categories as well as monoidal categories, the standard reference is [10]) each σ_i. More generally, we can consider topoi acted upon by an assigned monoid. Indeed, let M be any monoid

Definition 8.1 *A M-topos is a pair (E, λ), where*

$$\lambda : M \to Lex(E)$$

is a monoid morphism from M *to* $Lex(E)$*, the monoid of lex endofunctors of* E *under composition. A map of two* M*-topoi is an* equivariant *geometric morphism, again in the 2-categorical meaning:*

$$\psi : (E, \lambda) \longrightarrow (E', \lambda')$$

$$\psi \equiv (\psi_0, \{\psi_{1,i}\}_{i \in M})$$

where ψ_0 *is a geometric morphism from* E *to* $\acute{E}$ *and for all* $i \in M$

$$\begin{tikzcd}
E \arrow[r, "\lambda_i"] & E \\
E' \arrow[u, "\psi_0^*"] \arrow[r, "\lambda'_i"'] \arrow[ur, Rightarrow, "\psi_{1,i}"] & E' \arrow[u, "\psi_0^*"']
\end{tikzcd}$$

the diagram above commutes up to the given isomorphism $\psi_{1,i}$.

Remark 8.1 *It is evident that the notion of* M*-topos generalises that of modal topos: if* (E, σ) *is an arbitrary modal topos, it can be regarded as an example of* N*-topos (*N *is of course the free monoid on one generator) by defining* λ *as* $\lambda(n) = \sigma^{(n)}$ *and maps of modal topoi can be easily turned into maps of the corresponding* N*-topoi. What we have just described is an* equivalence *of categories:* $N - TOP \approx MODTOP$

In the previous definition we regarded $Lex(E)$ as being merely a monoid. It should be stressed though that this object has much more hidden structure: for instance, $Lex(E)$ is a *monoidal category* (maps are natural transformations) so we are naturally led to considering $(C, \otimes)$-actions on E where $(C, \otimes)$ is an arbitrary small monoidal category (in other terms, we may look at monoidal functors from C to $Lex(E)$). Such structures seem to be natural in dealing with the so-called *dynamic logic* (see [26]) in its first order version: the elements of the monoidal category would play the role of *commands* and the tensor would correspond to parallel execution of such commands. We do not intend to pursue this topic here; the reader is referred to our paper " Dynamic topoi " (see [40]), in preparation. Instead, we will spend some time on another area of modal logic, namely modal operators of arity $n \geq 2$, or *multimodal operators.* Such modalities, far from being just a curiosity, arise in a more or less disguised form in a variety of places, e.g. in interpretability logics, or, as new connectives, in relevant logics; even the nowadays popular linear logic can partly be seen as (but by no means reduced to) an extremal case of multimodal logic. It must be stressed though that the amount of experience in this realm of logic is at present still comparatively limited; consequently, the plausible options are (or seem to be) quite ample from the beginning. In what follows we will sketch one possible direction, without any pretence to exaustiveness. For sake of simplicity, we will limit our analysis to the case of arity 2. A simple analogy will be our guiding principle: lex functors are somewhat like linear maps so the obvious step further is to consider bilinear maps, namely *bilex functors*

Definition 8.2 *A bimodal topos is a pair* $(E, \otimes)$*, where* E *is an elementary topos and*

$$E \times E \xrightarrow{\otimes} E$$

is a functor from the topos $E \times E$ *to* E *which is* separately *lex (bilex for short), meaning that* $\forall A \in E_0$

$$E \xrightarrow{(-)\otimes A} E$$

and

$$E \xrightarrow{A\otimes(-)} E$$

are both lex endofunctors of E*, where* $A \otimes (-)$ *operates as* $B \mapsto A \otimes B$ *and if* $f : B \to B'$ *then* $id_A \otimes f : A \otimes B \to A \otimes B'$*. Analogously for* $(-) \otimes A$*. A map*

$$\psi : (E, \otimes) \longrightarrow (E', \otimes')$$

is a pair $\psi \equiv (\psi_0, \psi_1)$ *where* $\psi_0 : E \longrightarrow E'$ *is geometric and*

$$\begin{tikzcd}
E \times E \arrow[r, "\otimes"] & E \\
E' \times E' \arrow[u, "\psi_0^* \times \psi_0^*"] \arrow[r, "\otimes'"'] \arrow[ur, Rightarrow, "\psi_1"] & E' \arrow[u, "\psi_0^*"']
\end{tikzcd}$$

commutes up to the isomorphism ψ_1.

The previous definition delimits our universe of discourse. It is our task now to find out which "logic" such universes support. Because our interest is focused on the first order features of the bimodal topos, we must understand how the tensor affects its doctrine. A simple-minded observation is the following: suppose $A_0 \hookrightarrow A$ and $B_0 \hookrightarrow B$ are two arbitrary monomorphisms: then $A_0 \otimes B_0 \hookrightarrow A_0 \otimes B \hookrightarrow A \otimes B$. This means that we have a bilex map (preserving meets componentwise) $SUB_E(A) \times SUB_E(B) \longrightarrow SUB_E(A \otimes B)$. Moreover, this map is natural in A and B: if $f : A' \rightarrow A$ and $g : B' \rightarrow B$ are two arbitrary maps in the topos, we have of course a map $f \otimes g : A' \otimes B' \longrightarrow A \otimes B$ and we can pull back along this map:

$$\begin{tikzcd}
(f \otimes g)^{-1}(A_0 \otimes B_0) \arrow[r] \arrow[d] & A_0 \otimes B_0 \arrow[d] \\
A' \otimes B' \arrow[r, "f \otimes g"'] & A \otimes B
\end{tikzcd}$$

On the other hand, the tensor preserves pull-backs componentwise, so each one of the two diagrams below is a pull-back

$$\begin{tikzcd}
f^{-1} \otimes g^{-1}(A_0 \otimes B_0) \arrow[r] \arrow[d] & f^{-1} \otimes B(A_0 \otimes B_0) \arrow[r] \arrow[d] & A_0 \otimes B_0 \arrow[d] \\
A' \otimes B' \arrow[r, "A' \otimes g"'] & A' \otimes B \arrow[r, "f \otimes B"'] & A \otimes B
\end{tikzcd}$$

But composition of pull-backs is a pull-back, so $(f \otimes g)^{-1} \approx f^{-1} \otimes g^{-1}$.

What is happening? The tensor can be seen as a *type-constructor* on the term-category, namely the topos itself. The previous result tells us that if we consider two arbitrary predicates belonging to two (possibly different) types, we can tensor them to obtain a new predicate belonging to the tensor product of the preceding types: thus the tensor is somewhat like an external (!) connective. A full development of this notion would demand a complete reconsideration of our notion of doctrine. We prefer instead a more conservative approach. We would like to think of the tensor as a binary connective in the usual sense. This can be accomplished if we select suitable types in the topos, behaving properly, as it were, vis-à-vis the tensor product. As usual, it is convenient to keep fixed the term-theory:

Definition 8.3 *A T-bimodal topos is a triple: $(E, \otimes, F)$ where $(E, \otimes)$ is a bimodal topos and $F : T \rightarrow E$ is a finite-product preserving functor such that*

$$\begin{tikzcd}
T \times T \arrow[r, "\times"] \arrow[d, "F \times F"] & T \arrow[d, "F"'] \\
E \times E \arrow[r, "\otimes"'] & E
\end{tikzcd}$$

commutes. The category $T - BIMODTOP$ has T-bimodal topoi as objects and maps $\psi : (E, \otimes, F) \longrightarrow (E', \otimes', F')$ where $\psi : (E, \otimes) \longrightarrow (E', \otimes')$ is a bimodal topos map under T, namely such that the triangle

$$\begin{tikzcd}
& T \arrow[dl, "F"'] \arrow[dr, "F'"] & \\
E & & E' \arrow[ll, "\psi_0^*"]
\end{tikzcd}$$

commutes.

. If (E, F) is a T-bimodal topos, we can look at the T-doctrine

$$SUB_E \circ F : T^{op} \longrightarrow DL$$

We are now going to describe the way in which the tensor affects this doctrine: let t and $\acute{t}$ be in T_0. We have a map

$$\begin{tikzcd}
(SUB_E \circ F(t)) \times (SUB_E \circ F(t')) \arrow[r, "\otimes_{t,t'}"] & SUB_E(F(t) \otimes F(t')) \arrow[d, "id"] \\
 & SUB_E(F(t) \times F(t')) \arrow[d, "id"] \\
 & SUB_E(F(t \times t'))
\end{tikzcd}$$

(in the diagram above we are using the fact that F is a product-preserving functor and that $\forall t, s \in T_0 F(t) \times F(s) = F(t \otimes s)$) The map so constructed is natural in both variables. An additional property of this map is that $\top \otimes \top = \top$. Furthermore, the map is separately lex. As we have just seen, the doctrine $T^{op} \longrightarrow DL$ comes equipped with a substantial new piece of structure. Abstracting out, we are naturally led to the definition of *tensor doctrine*:

Definition 8.4 *A T-tensor doctrine is a pair $(D, \otimes)$ where D is a T-doctrine and $\otimes$ is a family of maps indexed by $T_0 \times T_0$*

$$\begin{tikzcd}
D_t \times D_s \arrow[r, "\otimes_{t,s}"] & D_{t \times s}
\end{tikzcd}$$

$$(a, b) \mapsto \otimes_{t,s}(a, b)$$

preserving meets componentwise and natural in the variables: $\forall f : t \longrightarrow t'$ $\forall g : s \longrightarrow s'$

$$\begin{tikzcd}
D_t \times D_s \arrow[r, "\otimes_{t,s}"] & D_{t \times s} \\
D_{t'} \times D_{s'} \arrow[u, "D(f) \times D(g)"] \arrow[r, "\otimes_{t',s'}"] & D_{t' \times s'} \arrow[u, "D(f \times g)"']
\end{tikzcd}$$

Moreover, $\forall t, s$ $\otimes_{t,s}[\top, \top] = \top$. *A map* $\psi : (D, \otimes) \longrightarrow (D', \otimes')$ *between two tensor doctrines is a doctrine morphism* $\psi : D \longrightarrow D'$ *commuting with the tensor:*

$$\begin{tikzcd}
D_t \times D_s \arrow[r, "\otimes_{t,s}"] \arrow[d, "\psi_t \times \psi_s"'] & D_{t \times s} \arrow[d, "\psi_{t \times s}"] \\
D'_t \times D'_s \arrow[r, "\otimes_{t,s}"'] & D'_{t \times s}
\end{tikzcd}$$

These data define the large category $(T, \otimes) - DOC$ of tensor doctrines (over the term theory T).

Remark 8.2 *If* D *is a tensor doctrine, through a process of "contraction of indexes" we can define a new binary connective at each type: indeed, let* $t \in T_0$ *be an arbitrary element. Then we have a map*

$$D_t \times D_t \xrightarrow{\otimes_{t,t}} D_{t \times t} \xrightarrow{D(\Delta_t)} D_t$$

The map defined above is natural in t, as easily checked. We could also express this fact by saying that such a doctrine factors through the algebraic category $(2) - MDL$, whose objects are distributive lattices endowed with a tensor product preserving meets componentwise and morphisms are distributive lattice maps preserving the new operation. Conversely, starting from a natural binary operation, say $\hat{\otimes}$, we can reconstruct a tensor structure in the sense of Definition 7.1 on the whole doctrine. In fact, for all $t, s \in T_0$, we can define $\otimes_{t,s}$ by $\otimes_{t,s} = \hat{\otimes}_{t,s} \circ < D(\pi_1), D(\pi_2) >$, where

$$\begin{tikzcd}
D_{t\times s} \times D_{t\times s} \arrow[r, "\hat{\otimes}_{t\times s}"] & D_{t\times s} \\
D_t \times D_s \arrow[u, "<D(\pi_1),D(\pi_2)>"']
\end{tikzcd}$$

Theorem 8.1 *The previous construction defines a functor: $SUB_{(-)} \circ (-) : T - BIMODTOP \longrightarrow (T, \otimes) - DOC^{op}$*

Proof: Easy checking. We now intend to adapt the results of Chap 7 to this new setting. To this end, some preliminary observations are in order. If $\otimes : C \times C \longrightarrow C$ is a site with a bilex endofunctor, we can construct a bimodal topos out of it, according to the recipe

$$\begin{tikzcd}
(C \times C) \arrow[rr, "Lan_{\otimes}"] & & \hat{C} \arrow[d, "a"] \\
Sh(C \times C, J \times J) \arrow[u, "i"] & & Sh(C, J) \\
& Sh(C, J) \times Sh(C, J) \arrow[ul, "G"] \arrow[ur, "\otimes^{\#}"'] &
\end{tikzcd}$$

where the functor G is defined by: for all $H, L \in Sh(C, J)$ $G(H, L)(C, D) = H(C) \times L(D)$. It is immediate to check that $G(H, L)$ is a contravariant presheaf on $C \times C$. Indeed, it is a sheaf with respect to the product topology $J \times J$ if H and L already are. The functor $\otimes^{\#}$ is equal to $a \circ Lan \otimes \circ i \circ G$. a, i and G are lex, whereas $Lan\otimes$ is bilex: $\otimes^{\#}$ is also bilex. Our next move is to show that to special kinds of bimodal doctrines do correspond bimodal sites, and then use the preceding fact to produce (functorially) a bimodal topos. We must first isolate a convenient subcategory of $(T, \otimes) - DOC$:

Definition 8.5 *A tensor doctrine is* regular *iff:*

1) For all $\psi \in D_{t \times t'}$ *for all* $\phi \in D_{s \times s'}$ ψ, ϕ *functional implies that* $\psi \otimes \phi$ *is also functional.*

2) If $f : t \to t'$ *and* $g : s \to s'$ *are two arbitrary maps in* T*,* $Graph(f) \otimes Graph(g) = Graph(f \times g)$.

The point here is that, if D is a regular bimodal doctrine, we can impose on $C(D)$ a tensor structure, in the following way: if (a,t) and (b,s) are two object of $C(D)$, define $(a,t)\otimes(b,s)$ as $(a\otimes b, t\times s)$ and if $\phi : (a,t) \rightarrow (a',t')$ $\psi : (b,s) \rightarrow (b',s')$ are maps in $C(D)$, then $\phi\otimes\psi : (a\otimes b, t\times s) \rightarrow (a'\otimes b', t'\times s')$. We can now obtain a bimodal topos:

$$(D,\otimes) \mapsto (C(D),\hat{\otimes}) \mapsto (Sh(C(D)),\hat{\otimes}^{\#})$$

Furthermore,

$$\begin{tikzcd}
T \times T \arrow[r, "\times"] \arrow[d, "E\times E"] & T \arrow[d, "E"'] \\
C(D) \times C(D) \arrow[r, "\otimes"'] & C(D)
\end{tikzcd}$$

commutes, and in the following

$$\begin{tikzcd}
T \times T \arrow[r, "\times"] \arrow[d] & T \arrow[d] \\
C(D) \times C(D) \arrow[r, "\otimes'"] \arrow[d, "Y\times Y"] & C(D) \arrow[d, "Y"'] \\
Sh(C(D)) \times Sh(C(D)) \arrow[r, "\#", "\otimes'"'] & Sh(C(D))
\end{tikzcd}$$

The lower square commutes (up to isomorphism). Combining the two, we get the commutativity of the external square. The reader can convince him(her)self that such an association is natural, producing a functor:

$$Sh \circ SC : (T, \otimes) - REGMODOC \longrightarrow (T - BIMODTOP)^{op}$$

In order to obtain an adjoint situation, we need to restrict ourselves to a convenient subcategory of $T - BIMODTOP$, in the spirit of Chapter 7:

Definition 8.6 *The category $T - COHBIMODTOP$ of coherent bimodal topoi, is defined by the following data:*

1) objects are $F : T \to E$ such that

$$\begin{tikzcd}
T \times T \arrow[rrr, "\times"] \arrow[dd, "F \times F"] \arrow[dr] & & & T \arrow[dl] \arrow[dd, "F \times F"'] \\
& Coh(E) \times Coh(E) \arrow[r] \arrow[dl] & Coh(E) \arrow[dr] & \\
E \times E \arrow[rrr, "\otimes"] & & & Coh(E)
\end{tikzcd}$$

In other words, $Coh(E)$ is closed under the tensor and the functor F factors through it; furthermore, For all $f : t \to t'$ for all $g : s \to s'$ in T $Graph(F(f))$ and $Graph(F(g))$ are both coherent objects of E and $Graph(F(f)) \otimes Graph(G(g)) = Graph(F(f \times g))$

2) maps are just morphisms of bimodal topoi that happen to be coherent.

We observe that the functor $Sh \circ SC$ factors through $T - COHBIMODTOP$. Moreover, we have a functor going the other way around, namely $KSUB_{(-)} \circ$

$(-)$. Following in the steps of theorem 7.2, it can be shown that:

Theorem 8.2 *The functor $Sh \circ SC$ is the left adjoint of $KSUB_{(-)} \circ (-)$*

$$\begin{tikzcd}(T,\otimes) - REGMODOC^{op} \arrow[r, "\top"] & T - COHBIMODTOP \arrow[l, bend right]\end{tikzcd}$$

To conclude the present chapter, we intend to emphasise that, in the spirit of Chapter 6, a bimodal topos is like "a space with a ternary relation". Indeed, if $(E, \otimes)$ is an arbitrary bimodal topos, we can define $R_{\otimes} \subseteq pt(E)^3$ as $\forall p_1, p_2, p_3 \in pt(E)\ R_{\otimes}(p_1, p_2, p_3)$ iff

$$\begin{tikzcd}SUB_E(1) \times SUB_E(1) \arrow[rr, "\otimes"] \arrow[dr, "p_1^* \times p_2^*"'] & & SUB_E(1) \arrow[dl, "p_3^*"] \\ & 2 & \end{tikzcd}$$

commutes. In other words $\forall U, V \hookrightarrow 1\ (p_1^* \times p_2^*)(U, V) = p_3^*(U \otimes V)$.

Notice that the definition makes good sense because $1 \otimes 1 = 1$ (an easy consequence of the fact that tensor preserves finite limits componentwise). Working out details, we obtain a functor:

$$BIMODTOP \longrightarrow Rel_3Set$$

where the category on the right side has sets with ternary relations as objects and relation-preserving maps as morphisms.

Chapter 9

Comparisons and further developments

In recent years, two proposals for a categorical analysis of logic have been brought forth, independently from one another: the first one is due to G. Reyes and some collaborators (see [22] and [23]. We will refer to it as R) and the other one to G. Meloni and his student S. Ghilardi (GM in the sequel. See [19], [20] and [21]). We intend to examine both briefly (without pretending to present them fully) and to establish some comparison with the approach described in the preceding chapters. R is motivated by an idea of relative topos theory: if

$$\begin{tikzcd} E \arrow[r, shift left, "\Gamma"] \arrow[r, phantom, "\top"] & E' \arrow[l, shift left, "\Delta"] \end{tikzcd}$$

is a geometric morpism between two topoi, we can think of E' as the topos of "constant sets" and E as the topos of "variable sets" over E'. Indeed, if E happens to be a Grothendieck topos, there exists always a (essentially unique) morphism into *Set*, namely the "global section functor"(together with its associated left adjoint). It is a basic tenet of topos theory that *any* topos can be used as the ground universe of sets: we are thus led quite naturally to the foregoing viewpoint. Now, if $A \in E'$ and $A_0 \hookrightarrow \Delta A$ we can define a new subobject of ΔA by the pull-back

$$\begin{tikzcd} \Delta\Gamma A_0 \arrow[r] & \Delta\Gamma(\Delta A) \\ \Box_A A_0 \arrow[u] \arrow[r] & \Delta A \arrow[u, "\Delta\eta_A"'] \end{tikzcd}$$

$\Box_A$ is a meet-semilattice map

$$\Box_A : SUB_E\Delta(A) \to SUB_E\Delta(A)$$

natural in A, so that we obtain a modal doctrine

$$SUB_E\Delta : (E')^{op} \to MDL$$

Our starting point was the geometric morphims (Γ, Δ). It is a classic result of topos theory that any such morphism can be factorized as

$$\begin{tikzcd}
E \arrow[r, shift left, "\Gamma"] \arrow[d, shift left, "\Gamma"] & COALG(\Delta\Gamma, E) \arrow[l, shift left, "U"] \arrow[dl, shift left, "e"] \\
E' \arrow[u, shift left, "\Delta"] \arrow[ur, shift left, "a"] &
\end{tikzcd}$$

Using this factorization, we can obtain the modal doctrine $SUB_E\Delta$ from the modal doctrine SUB_EU by composition with the functor a (which is lex and so preserves all finite products). We are thus reduced to the study of the following situation

$$\begin{tikzcd}
E \arrow[r, shift left, "\Gamma"] \arrow[r, phantom, "\top"] & COALG(\sigma, E) \arrow[l, shift left, "U"]
\end{tikzcd}$$

$$\sigma = U\Gamma$$

where σ is a lex cotriple on E. First we observe that the pair (E, σ) is a special kind of modal topos (σ obeys the equations of a cotriple). Secondly, we observe that both $COALG(\sigma, E)$ and $FIX(\sigma, E)$ are contained in the category $E \downarrow \sigma$ (the comma category of objects over σ) as finite-product subcategories. Now, a modal doctrine with term-category $E \downarrow \sigma$ can be defined for *every* modal topos, namely $SUB_E \circ U$ where U is the obvious forgetful functor. The modal operator is

$$\Box_{(A,f)} : SUB_E \circ U(A,f) \longrightarrow SUB_E \circ U(A,f)$$

$$(A_0 \hookrightarrow A) \mapsto \Box A_0$$

where

$$\begin{tikzcd}
\Box_{(A,f)}A_0 \arrow[r] \arrow[d] & A \arrow[d, "f"] \\
\sigma A_0 \arrow[r] & \sigma A
\end{tikzcd}$$

is a pull-back diagram. Notice that this modal doctrine encompasses the other two: in fact, both can be obtained by restricting along their respective embeddings. From the standpoint sketched above, R can be regarded as the study of a special class of modal topoi, corresponding to a *variety* (in the suitable algebraic sense) of modal doctrines: the fact that the endofunctor is a cotriple entails certain properties of the corresponding modal operator, namely:

1) $\Box\phi \leq \phi$

2) $\Box\Box\phi \leq \Box\phi$

(these axioms in the classical propositional environment characterize S_4-algebras).

Remark 9.1 *It would be desirable to have a general theory of axiomatic* *classes of modal topoi, extending methods and results of modal correspondence theory (see the elegant book of Van Benthem for details: [34]). Such a theory should establish a Galois connection between* varietal subcategories *of modal doctrines and classes of modal topoi whose endofunctor verifies certain diagrammatical equations. Presumably, these classes should be closed under familiar model-theoretic constructions (like ultraproducts).*

We now turn to GM. Here, the motivating idea seems to be that of *relational presheaf*:

Definition 9.1 *Let* C *be a small category. A relational presheaf on* C *is a lax functor*

$$F : C^{op} \longrightarrow REL$$

where REL *is the category having sets as objects and relations as maps. A morphism of relational presheaves is a collection indexed by* C_0 *of set-maps*

$$\psi_t : F(t) \rightarrow G(t)$$

such that, for all maps $f : t \rightarrow s$ *in* C, $f(R(f)) \subseteq G(f)$. *Relational presheaves and their maps form a large category, the* relational universe *relative to* C *(it will be denoted by* $\mathcal{R}(C)$*).*

Now, let $\mathcal{R}(C)$ be a relational universe. The natural forgetful functor into the (boolean) topos Set^{C_0} has left and right adjoints:

$$\begin{tikzcd}
\mathcal{R}(C) \arrow[r] & Set^{C_0} = E \arrow[l, bend right=40, "C"', "\perp"] \arrow[l, bend left=40, "D", "\perp"']
\end{tikzcd}$$

If $\{X_i\}$ is a C-indexed family of sets, the left and right adjoint associate repectively the discrete and codiscrete relational presheaves on C, in a way reminiscent of the familiar discrete and codiscrete topologies on a given set (this analogy should be enough for the reader to fill out details by himself). The functor U, having a left adjoint, is exact. Thus the pair (D, U) looks like a geometric morphism of topoi, even though $\mathcal{R}(C)$ is *not* itself a topos. Here is an informal description of the associated modal doctrine: start with a relational presheaf F and a subobject A of UF in E (in each coordinate t is just the assigment of a subset of $UF(t)$).The modal operator sends A to the sub(relational)-presheaf of UF generated by it.

Remark 9.2 *To be sure, there is yet another modality, considered in GM: we could send A to the largest subpresheaf contained in it. The presence of two modalities is not surprising if we think that U has adjoints on both sides. In the same way, we have dramatically oversimplified G, which con-*

siders essential *geometric morphisms between two topoi. Again, the doctrine generated has two modalities (at least): the familiar necessity and possibility operators. As we said in the preamble, this chapter is not intended as an exposition of either approach. We just wish to convey some common motives and stimulate further reflections*

Again, the same order of considerations as before applies. We have produced a modal topos, namely (E, UD). There exists a comparison functor (see Barr-Wells [6] for details)

$$\mathcal{C} : \mathcal{R}(C) \longrightarrow COALG(UD, E)$$

which preserves finite products. Once more, the doctrine examined by GM can be obtained from the one having $COALG(UD, E)$ as term-theory by composition with the comparison functor. There is yet another connection between GM and our approach, which is worth mentioning. Let us start with a given M-topos, $F : M \rightarrow LEX(E)$. We can associate to it an object in the category $\mathcal{R}(M)$, namely the functor $\hat{F} : M \rightarrow REL$ such that for all $t \in M$

$$t \mapsto R_{F(t)} \subseteq pt(E) \times pt(E)$$

where $R_{F(t)}$ is the relation on points of E induced by the endofunctor $F(t)$.

It can be shown that the association mentioned above is indeed a functor

$$(M - TOP)^{op} \longrightarrow \mathcal{R}(M)$$

There is no a priori reason for us to stick to monoids. The definition of a C-topos with C an arbitrary small category is readily made up:

Definition 9.2 *A C-topos is a functor $F : C \longrightarrow TOPLEX$, where $TOPLEX$ denotes the large category of topoi and lex functors between them. A map $(\psi, \mu) : G \rightarrow F$ of C-topoi is given by a collection of geometric morphisms $\{\psi_t\}_{t \in C_0}$ and a collection of natural isomorphisms $\{\mu_f\}_{f \in C_1}$ such that, if $f : s \rightarrow t$*

$$\begin{tikzcd}
F(t) \arrow[r, "\psi_t^*"] & G(t) \\
F(s) \arrow[u, "F(f)"] \arrow[r, "\psi_s^*"'] \arrow[ur, Rightarrow, "\mu_f"] & G(s) \arrow[u, "G(f)"']
\end{tikzcd}$$

the diagram commutes up to μ_f. The large category defined by these data will be denoted as $C - TOP$.

Mimiking what has been done before, we can produce a functor

$$(C - TOP)^{op} \longrightarrow \mathcal{R}(C)$$

At the time of writing, it is not clear whether or not this functor is part of an adjoint pair. C-topoi can be seen as an extreme generalization of modal topoi. Do they support some kind of "modal logic"? It seems plausible that the answer is in the affirmative, but perhaps a new extension of the notion of modality may be required.

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