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INDUCED SHAPE FIBRATIONS
AND
FIBER SHAPE EQUIVALENCE

by

MAHENDRA JANI

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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ABSTRACT

INDUCED SHAPE FIBRATIONS
AND
FIBER SHAPE EQUIVALENCE

by

Mahendra Jani

Adviser: Professor Eldon Dyer

In this paper we prove that a map induced from a shape fibration is a shape fibration. We define an equivalence relation between shape fibrations. Also, generalizing the homotopy relation, we define a strong equivalence relation in the set of maps between compacta. Then we prove that two strongly equivalent maps induce fiber shape equivalent shape fibrations. As a corollary we show that the fibers over two points connected by a strong shape path are of the same shape. Finally, we prove that a fiber shape equivalence induces a relative shape map which induces an appropriate isomorphism on relative shape groups.

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INTRODUCTION

In 1968, K. Borsuk introduced the notion of a shape of a compact metric space [1], generalizing the notion of homotopy type. The idea was to consider only global properties of compact metric spaces and ignore the local ones. This idea evolved into an effective theory for compact metric spaces, called shape theory, as homotopy theory for locally nice spaces like ANR's or cw-complexes. Both coincide for ANR's.

S. Mardesic and J. Segal approached shape theory using ANR-systems [9] and generalized the notion of shape to compact Hausdorff spaces. Mardesic [8] and Morita [13] have generalized it to topological spaces. However, they all agree over compact metric spaces. In this paper we will restrict ourselves to compact metric spaces and will use ANR-sequence approach [9] to the shape theory.

Shape theory is rapidly progressing and has attracted mathematicians from different branches of topology. Recently, S. Mardesic and T.B. Rushing [10] have defined an important notion of 'shape fibration' $p: E \to B$ for compact metric spaces. One expands p into a map $\underline{p}: \underline{E} = (E_n, r_{nm}) \to \underline{B} = (B_n, q_{nm})$ of inverse sequences of compact ANR's. The map p is a shape fibration if $\underline{p}$ has the following approximate homotopy lifting property:

Each n and each $\epsilon > 0$ admit an index $m \geq n$ and $\delta > 0$ such that for any topological space X, whenever the maps $h: X \to E_m$ and $H: X \times I \to B_m$ satisfy $d(p_m h, H_0) \leq \delta$ then there is a homotopy $G: X \times I \to E_n$ satisfying $d(G_0, r_{nm} h) < \epsilon$ and $d(p_n G, q_{nm} H) < \epsilon$.

Analogously to fibrations, one may ask the following questions for shape fibrations: Is a map induced from a shape fibration, a shape fibration? Is there a notion of fiber shape map? In what sense are two shape fibrations fiber shape equivalent? Is it true that two homotopic maps induce equivalent shape fibrations?

In this paper we have studied all these questions and have found positive answers.

Section 1 contains basic definitions and some basic results that we need.

In Section 2 we prove that a map induced from a shape fibration is a shape fibration.

Section 3 contains the definition of a fiber shape equivalence.

Let $p: E \to B$ and $p': E' \to B$ be shape fibrations expanded to maps $\underline{p}: \underline{E} = (E_n, r_{nm}) \to \underline{B} = (B_n, q_{nm})$ and $\underline{p}': \underline{E}' = (E'_n, r'_{nm}) \to \underline{B}$ of inverse sequences of compact ANR's respectively.

Roughly, a fiber shape map $[\underline{f}]: p \to p'$ is an equivalence class of a map $\underline{f}: \underline{E} \to \underline{E}'$ of ANR-sequences satisfying the following condition:

For every n and for every $\epsilon > 0$ there is $n^* \geq n$ such that for all $\ell \geq m \geq n^*$ there is a homotopy $H: r'_{m\ell} f_\ell \simeq f_m r_{\alpha(m)\alpha(\ell)}$ such that for every $t \in I$ the maps $q_{nm} p'_m H_t$ and $q_{n\alpha(\ell)} p_{\alpha(\ell)}$ are ϵ-close.

Two such maps $\underline{f}, \underline{g}: \underline{E} \to \underline{E}'$ are said to be equivalent if for every n and for every $\epsilon > 0$ there is $\hat{n} \geq n$ such that for all $m \geq \hat{n}$, the following is true:

There exist $\ell \geq \alpha(m)$ and a homotopy $L: f_m r_{\alpha(m)\ell} \simeq g_m r_{\alpha(m)\ell}$ such that for every $t \in I$, the maps $q_{nm} p'_m L_t$ and $q_{n\ell} p_\ell$ are ϵ-close.

Finally, two shape fibrations p and p' are fiber shape equivalent if there are fiber shape maps $[\underline{f}]: p \to p'$ and $[\underline{g}]: p' \to p$ such that $\underline{g}\,\underline{f} \sim \underline{1}_{\underline{p}}$ and $\underline{f}\,\underline{g} \sim \underline{1}_{\underline{p}'}$.

For example, the shape fibration $p: W \to W/A \approx S^1$, where W is the Warsaw circle and A is its limit arc, is fiber shape equivalent to the obvious shape fibration $1_{S^1} : S^1 \to S^1$.

In Section 4 we prove the main result that strongly equivalent maps induce fiber shape equivalent shape fibrations.

Two maps $f,g: C \to B$ between compact metric spaces are strongly equivalent if there are maps $\underline{f},\underline{g}: \underline{C} \to \underline{B}$ of inverse sequences of compact ANR's such that

i) For every n there exists $n(*)$ such that for all $m \geq n(*)$ there is a homotopy $H^m: q_{nm}f_m \simeq q_{nm}g_m$ and

ii) for $m' \geq n'(*)$ where $n' \geq n$ there is a homotopy

$$q_{nn'}H^{m'} \simeq H^m(q_{mm'} \times 1_I) \text{ (rel. } 0,1).$$

Two homotopic maps are strongly equivalent. Hence, in particular, the theorem says that two induced shape fibrations by homotopic maps are fiber shape equivalent.

One immediate corollary of the theorem is that if B is of a trivial shape then a shape fibration $p: E \to B$ is fiber shape equivalent to a trivial shape fibration $\pi_B: F_x \times B \to B$ where $F_x = p^{-1}(x)$ for any $x \in B$.

The strong equivalence notion leads naturally to a notion of strong shape path connectedness. We have discussed this fully in Section 5.

Two points x,y of a compact metric space B are connected by a strong shape path if for any ANR-sequence $\underline{B} = (B_n, q_{nm})$ with $\varprojlim \underline{B} = B$ there is a family $\underline{\omega} = \{\omega: I \to B_n \mid \omega_0 = x, \omega_1 = y\}$ of paths such that for all $m \geq n$, $q_{nm}\,\omega_m \simeq \omega_n$ (rel. x,y).

A space is strongly shape path connected if any two of its points can be connected by a strong shape path. Clearly path connected spaces are strongly shape path connected. Moreover, plane compact connected metric spaces and pointed 1-movable compact connected metric spaces are strongly shape path connected. But the fact that dyadic solenoid is not strongly shape path connected shows that unlike to Borsuk's approximate 0-connectedness [3], not all compact connected spaces are strongly shape path connected.

S. Mardesic and T.B. Rushing have asked the following question in [10]: For a shape fibration, is it true that two fibers over points lying in the same component, are of the same shape?

We have partially answered this question by the following corollary of the main theorem:

For a shape fibration, if two points are connected by a strong shape path then two fibers over those points are of the same shape.

Finally, in Section 6 we have proved that a fiber shape equivalence $[\underline{f}]: p \to p'$ induces a pointed relative shape map $\underline{f}: (E,F,e) \to (E',F',e')$ which induces an 'appropriate' isomorphism

$$\underline{f}_* : \check{\pi}_q(E,F,e) \to \check{\pi}_q(E',F',e')$$

of shape groups.

Section 1

PRELIMINARIES

All spaces considered will be metric spaces. Denote by $d(x,y)$ the distance between two points x and y. For a number $\delta > 0$, two maps (continuous functions) $f,g\colon X \to Y$ are δ-close if for each $x \in X$, $d(f(x),g(x)) < \delta$. For such f and g we will write $d(f,g) < \delta$. The maps f and g are δ-homotopic if there is a homotopy $H\colon X \times I \to Y$ such that $H_0 = f$, $H_1 = g$ and for each $x \in X$,

$$d(H(x,t), H(x,t')) < \delta \quad \text{for all } t, t' \in I .$$

For the interior of a space X we write int. X. Let A,Y be subspaces of the space X. Then Y is said to be a neighborhood of A in X if $A \subset \text{int}.Y$. By an ANR we mean absolute neighborhood retract for metric spaces. It is well-known that if Y is a compact ANR then for every $\epsilon > 0$ there exists a $\delta > 0$ such that any two δ-close maps from a space X to Y are ϵ-homotopic [6]. We use this result freely without mentioning it further.

Following Moszynska [14], Coram and Duvall [5] have defined a convenient ANR: A space E is a convenient ANR if each compact metric space X in E has the following property: for each neighborhood U of X in E there is a compact ANR $M \subset U$ with $X \subset \text{int}.M$.

Every polyhedron is convenient and by the triangulation theorem [4] every Q-manifold is convenient where Q is the Hilbert cube. Also if E is a locally compact ANR then $E \times Q$ is convenient [5].

An ANR-sequence $\underline{E} = (E_n, r_{nm})$ is an inverse sequence of compact ANR's. A level map $\underline{P} = (p_n)\colon \underline{E} \to \underline{B} = (B_n, q_{nm})$ of ANR-sequences is a map of inverse sequences, i.e. $\underline{p}$ is a family of maps $p_n\colon E_n \to B_n$ such that for

$m \geq n$, $p_n r_{nm} = q_{nm} p_m$.

$$\begin{tikzcd} E_n \arrow[d, "p_n"] & E_m \arrow[l, "r_{nm}"'] \arrow[d, "p_m"] \\ B_n & B_m \arrow[l, "q_{nm}"'] \end{tikzcd}$$

Let $\underset{\leftarrow}{\lim}\ \underline{E} = (E, r_n)$ and $\underset{\leftarrow}{\lim}\ \underline{B} = (B, q_n)$. Then the unique map $p\colon E \to B$ is said to be the limit map of the level map $\underline{p}$ if for each n, $q_n p = p_n r_n$.

$$\begin{tikzcd} E_n \arrow[d, "p_n"] & E \arrow[l, "r_n"'] \arrow[d, "p"] \\ B_n & B \arrow[l, "q_n"] \end{tikzcd}$$

Generalizing Coram and Duvall's approximate fibrations [5], S. Mardesic and T.B. Rushing [10] have defined shape fibrations.

A map $p\colon E \to B$ between compact metric spaces is called a shape fibration if it is a limit map of a level map $\underline{p}\colon \underline{E} \to \underline{B}$ of ANR-sequences which has the following approximate homotopy lifting property (AHLP):

A level map $\underline{p}\colon \underline{E} \to \underline{B}$ of ANR- sequences has the AHLP if each n and each $\epsilon > 0$ admit an $m \geq n$ and a $\delta > 0$ such that for given maps $h\colon X \to E_m$ and $H\colon X \times I \to B_m$ with

(1) $$d(p_m h, H_0) \leq \delta$$

there is a homotopy $G\colon X \times I \to E_n$ such that

(2) $$d(G_0, r_{nm} h) < \epsilon$$

and

(3) $$d(p_n G, q_{nm} H) < \epsilon .$$

Every such m is called a lifting index for (n,ϵ) and δ is called a lifting mesh for (n,ϵ). We refer to (m,δ) as a lifting pair for (n,ϵ).

It has been shown in [10] that if a map $p\colon E \to B$ between compact metric spaces is a limit map of level maps $\underset{\sim}{p}\colon \underset{\sim}{E} \to \underset{\sim}{B}$ and $\underset{\sim}{p}'\colon \underset{\sim}{E}' \to \underset{\sim}{B}'$ of ANR-sequences and if $\underset{\sim}{p}$ has AHLP then so does $\underset{\sim}{p}'$.

Also, in [10], the authors have defined a homotopy lifting property (HLP) for a level map as follows:

A level map $\underset{\sim}{p}\colon \underset{\sim}{E} \to \underset{\sim}{B}$ of ANR-sequences has the HLP if each n admits an $m \geq n$ such that for given maps $h\colon X \to E_m$ and $H\colon X \times I \to B_m$ with

$$H_0 = p_m h \tag{4}$$

there is a homotopy $G\colon X \times I \to E_n$ such that

$$G_0 = r_{nm} h \quad \text{and} \tag{5}$$

$$p_n G = q_{nm} H . \tag{6}$$

For a level map $\underset{\sim}{p}\colon \underset{\sim}{E} \to \underset{\sim}{B}$ of ANR-sequences, we will state two results from [10] which relate the stronger lifting properties with the weaker properties.

(I) If $\underset{\sim}{p}$ has the AHLP then it has the stronger lifting property obtained where (2) is replaced by (5).

(II) $\underset{\sim}{p}$ has the AHLP if it has the weaker lifting property obtained where (1) is replaced by (4).

The following proposition follows immediately from (II).

<u>Proposition 1.1</u>: Let $\underset{\sim}{p}\colon \underset{\sim}{E} \to \underset{\sim}{B}$ be a level map of ANR-sequences. If $\underset{\sim}{p}$ has the HLP then the limit map $p\colon E \to B$ is a shape fibration.

□

Section 2

INDUCED SHAPE FIBRATIONS

In this section we will prove that analogously to pullbacks of fibrations, a map induced from a shape fibration by a continuous map is a shape fibration.

For this we need the following:

Proposition 2.1: For maps $p: E \to B$ and $f: C \to B$ between compact metric spaces, there are level maps $\underline{p}: \underline{E} \to \underline{B}$ and $\underline{f}: \underline{C} \to \underline{B}$ of ANR-sequences with limit maps p and f respectively.

Proof: Embed E,B,C in the Hilbert cube Q. Since Q is an AR and E,C are compact spaces, maps p and f can be extended to $\tilde{p}: Q \to Q$ and $\tilde{f}: Q \to Q$. Choose for $\underline{B}$ a decreasing sequence of compact ANR-neighborhoods B_n of B with $\bigcap_n B_n = B$. By induction we can choose decreasing sequences of compact ANR-neighborhoods E_n of E and C_n of C with $\bigcap_n E_n = E$, $\bigcap_m C_m = C$, $\tilde{p}(E_n) \subseteq B_n$ and $\tilde{f}(C_n) \subseteq B_n$.

Let $r_{nm}: E_m \to E_n$ and $q'_{nm}: C_m \to C_n$ denote inclusions and for each n let $p_n = \tilde{p}|E_n$ and $f_n = \tilde{f}|C_n$. Hence $\underline{p} = (p_n): \underline{E} \to \underline{B}$ and $\underline{f} = (f_n): \underline{C} \to \underline{B}$ are level maps of ANR-sequences with limit maps p and f respectively.

For maps $p: E \to B$ and $f: C \to B$ between compact spaces, a triple $(Z; p', f')$ is a pull-back of $(B;p,f)$ in the category of compact spaces if $Z = \{(e,c) \in E \times C | p(e) = f(c)\}$ and $p': Z \to C$ and $f': Z \to E$ are the projections, i.e. $p'(e,c) = c$ and $f'(e,c) = e$ for all $(e,c) \in Z$.

Note that Z is also compact.

We say that p' is a map induced from p by f.

The main theorem of this section is

<u>Theorem 2.1</u>: Let $p: E \to B$ be a shape fibration and $f: C \to B$ be a map between compact metric spaces. Then the map $p': E' \to C$ induced from p by f is also a shape fibration.

<u>Proof</u>: Let $(Z;p',f')$ be the pull-back of $(B;p,f)$. We want to show that p' is a shape fibration.

Let $\underline{p}: \underline{E} \to \underline{B}$ and $\underline{f}: \underline{C} \to \underline{B}$ be level maps of ANR-sequences with limit maps p and f respectively. Without loss of generality we can assume that for each n, $E_n \times C_n$ is a convenient ANR. If this is not the case then as in [10] consider an ANR-sequence $\underline{Q} = (Q_n, \alpha_{nm})$ where for each n, $Q_n = Q$, the Hilbert cube and $\varprojlim \underline{Q}$ is a point. Then $\varprojlim \underline{E} \times \underline{Q} = E$. Also, if for each n, $\pi_n: E_n \times Q_n \to E_n$ is a projection map then $\underline{\pi} = (\pi_n): \underline{E} \times \underline{Q} \to \underline{E}$ is a level map of ANR-sequences and hence $\underline{\pi} \cdot \underline{p} = (\pi_n \cdot p_n): \underline{E} \times \underline{Q} \to \underline{B}$ is a level map of ANR-sequences with limit map p. Since $\underline{p}$ has the AHLP, $\underline{\pi}\, \underline{p}$ has the AHLP [10]. Now, by the triangulation theorem [4] $E_n \times Q_n$ is a Q-manifold, hence $E_n \times C_n \times Q_n$ is a Q-manifold and therefore is a convenient ANR with $\varprojlim \underline{E} \times \underline{C} \times \underline{Q} = E \times C$.

Consider an ANR-sequence of compact convenient ANR's $\underline{E} \times \underline{C} = (E_n \times C_n, r'_{nm})$ where $r'_{nm} = r_{nm} \times q'_{nm}: E_m \times C_m \to E_n \times C_n$. Then $\varprojlim \underline{E} \times \underline{C} = \varprojlim \underline{E} \times \varprojlim \underline{C} = E \times C$.

Let $\underline{p}' = (p'_n): \underline{E} \times \underline{C} \to \underline{C}$ and $\underline{f}' = (f'_n): \underline{E} \times \underline{C} \to \underline{E}$ be level maps where $p'_n: E_n \times C_n \to C_n$ and $f'_n: E_n \times C_n \to E_n$ are projections.

For each n, let

$$Z_n = \{(e_n, c_n) \in E_n \times C_n \,|\, p'_n(e_n) = f'_n(c_n)\}$$

be the indicated subspace of $E_n \times C_n$. Then $(Z_n; p'_n|Z_n, f'_n|Z_n)$ is a pull-back of $(B_n; p_n, f_n)$. Note that each Z_n is a closed and hence compact subspace of $E_n \times C_n$. Thus, $\underline{Z} = (Z_n, r'_{nm}|Z_m)$ is an inverse sequence of compact spaces. Also note that $\varprojlim \underline{Z} = Z$ and the limit maps of $\underline{p}'|\underline{Z}$ and $\underline{f}'|\underline{Z}$ are p' and f' respectively.

Now, by induction, for each $n = 1,2,3,\ldots$ we will define a closed ANR-neighborhood E'_n of Z_n in $E_n \times C_n$; numbers $\epsilon_n > 0$ and $\delta_n > 0$ and an integer m such that (m,δ_n) is a lifting pair for (n,ϵ_n) with respect to $\underline{p}$ and $\epsilon_n \to 0$. Also, we require

(1) $r'_{nm}(E'_m \subset E'_n)$ where $r'_{nm} = r_{nm} \times q'_{nm}|E'_m$.

(2) $\varprojlim \underline{E}' = Z$ where $\underline{E}' = (E'_n, r'_{nm})$.

(3) $d(f_m p'_m|E'_m \, , \, p_m f'_m|E'_m) < \delta_n$ and

(4) for $(e_n,c_n) \in E_n \times C_n$,

$d(p_n(e_n), f_n(c_n)) < \epsilon_n \Rightarrow (e_n,c_n) \in E'_n$.

Let $d_1 \colon E_1 \times C_1 \xrightarrow{p_1 \times f_1} B_1 \times B_1 \xrightarrow{d} \mathbb{R}$ be the composition, where d is the distance function and $\mathbb{R}$ is the set of reals.

By definition $d_1^{-1}(0) = Z_1$. Since $E_1 \times C_1$ is compact, $d_1(E_1 \times C_1)$ is a bounded subset of $\mathbb{R}$. Let η_1 be the least upper bound. We can assume that $Z_1 \subset \text{int. } E_1 \times C_1$. (If $Z_1 = E_1 \times C_1$ then, Z_1 itself is a compact ANR. So $E'_1 = Z_1$. If $Z_n \not\subset \text{int. } E_n \times C_n$ for each n, consider as above $E_n \times Q_n \times C_n$. Then we can consider $Z_n \subset \text{int. } E_n \times Q_n \times C_n$ and proceed as suggested in the beginning of the proof.)

Then $d_1(E_1 \times C_1) \subseteq [0,\eta_1]$ where $\eta_1 > 0$. Also, note that for every $\epsilon \in [0,\eta_1)$, $d_1^{-1}[0,\epsilon)$ is an open set of $E_1 \times C_1$ containing Z_1 and

$$\bigcap_{\epsilon\in[0,\eta)} d_1^{-1}[0,\epsilon) = d_1^{-1}\Big(\bigcap_{\epsilon} [0,\epsilon)\Big) = d_1^{-1}(0) = Z_1 .$$

Hence, for every open set u of $E_1 \times C_1$ containing Z_1 there exists an $\epsilon \in (0,\eta_1)$ such that for all ϵ', $0 < \epsilon' < \epsilon$,

$$Z_1 \subset d_1^{-1}[0,\epsilon') \subset u .$$

Select $\epsilon_1' > 0$ such that $Z_1 \subset d_1^{-1}[0,\epsilon_1') \subset \text{int.}\ E_1 \times C_1$. Since $E_1 \times C_1$ is a convenient ANR, for the neighborhood $d_1^{-1}[0,\epsilon_1')$ of Z_1, there is a compact ANR E_1' such that

$$Z_1 \subset \text{int.}E_1' \subset E_1' \subset d_1^{-1}[0,\epsilon_1') . \tag{5}$$

Let us denote $p_1'|E_1'$ by p_1' and $f_1'|E_1'$ by f_1'. Then (5) implies that

$$d(p_1 f_1', f_1 p_1') < \epsilon_1' .$$

(i.e. for every $(e_1,c_1) \in E_1'$, $d(p_1(e_1),f_1(c_1)) = d_1(e_1,c_1) < \epsilon_1'$).

Also, select $\epsilon_1 > 0$; $0 < \epsilon_1 < \epsilon_1'$ such that

$$Z_1 \subset d_1^{-1}[0,\epsilon_1) \subset \text{int.}\ E_1' \subset E_1' . \tag{6}$$

(6) implies that for $(e_1,c_1) \in E_1 \times C_1$

$$d(p_1(e_1), f_1(c_1)) = d_1(e_1,c_1) < \epsilon_1 \Rightarrow (e_1,c_1) \in E_1' .$$

Hence for $n = 1$ conditions (3) and (4) are satisfied. Now we will select $\delta_1 = \epsilon_2'$, E_2', ϵ_2 satisfying the conditions (1)-(4). The rest of the induction follows from these two steps.

Let us denote the metric on B_2 by d^{B_2} and the metric on B_1 by d^{B_1}. Since $q_{12}: B_2 \to B_1$ is uniformly continuous, for ϵ_1 there exists a number $\epsilon_2' > 0$ such that $\epsilon_2' < \text{Min}(\epsilon_1,\eta_1)$ and

$$d^{B_2}(x,y) < \epsilon_2' \Rightarrow d^{B_1}(q_{12}(x), q_{12}(y)) < \epsilon_1 \quad \text{for} \quad (x,y) \in B_2 \times B_2 . \tag{7}$$

Let d_2 be the composition

$$E_2 \times C_2 \xrightarrow{p_2 \times f_2} B_2 \times B_2 \xrightarrow{d^{B_2}} \mathbb{R} .$$

The open set $d_2^{-1}[0,\epsilon_2')$ of $E_2 \times C_2$ contains Z_2. Since $E_2 \times C_2$ is a convenient ANR, for the neighborhood $d_2^{-1}[0,\epsilon_2')$ of Z_2, there exists a compact ANR E_2' such that

(8) $$Z_2 \subset \text{int. } E_2' \subset E_2' \subset d_2^{-1}[0,\epsilon_2') \ .$$

Choose $\epsilon_2 > 0$; $0 < \epsilon_2 < \epsilon_2'$ such that

(9) $$Z_2 \subset d_2^{-1}[0,\epsilon_2) \subset \text{int. } E_2' \subset E_2' \ .$$

If $(2,\delta_1')$ is a lifting pair for $(1,\epsilon_1)$ then choose $\epsilon_2' > 0$ such that $\epsilon_2' < \text{Min}(\eta_1,\delta_1',\epsilon_1)$. By definition $(2,\delta_1 = \epsilon_2')$ will be lifting pair for $(1,\epsilon_1)$.

Now, conditions (3) and (4) are satisfied by (8) and (7) respectively.

Also, by (8) and (7), $r_{12}'(E_2') \subset E_1'$. Since $E_2' \subset d_2^{-1}[0,\epsilon_2')$,

$$r_{12}'(E_2') \subset r_{12}'(d_2^{-1}[0,\epsilon_2')) \ .$$

But (7) implies that $r_{12}'(d_2^{-1}[0,\epsilon_2')) \subset d_1^{-1}[0,\epsilon_1)$.

Let $(e_2,c_2) \in d_2^{-1}[0,\epsilon_2')$ then

$$d_2(e_2,c_2) = d^{B_2}(p_2(e_2),\ f_2(c_2)) < \epsilon_2'$$

$$\Rightarrow d^{B_1}(q_{12}p_2(e_2),\ q_{12}f_2(c_2)) < \epsilon_2'$$

i.e. $$d_1(r_{12}'(e_2),\ r_{12}'(c_2)) < \epsilon_2'$$

$$\Rightarrow r_{12}'(e_2,c_2) \in d_1^{-1}[0,\epsilon_2') \ .$$

$$\begin{tikzcd}
E_1\times C_1 \arrow[r, "p_1 \times f_1"] & B_1\times B_1 \arrow[dr, "d^{B_1}"] & \\
 & & \mathbb{R} \\
E_2\times C_2 \arrow[uu, "r_{12}'"] \arrow[r, "p_2 \times f_2"'] & B_2\times B_2 \arrow[uu, "q_{12}\times q_{12}"'] \arrow[ur, "d^{B_2}"'] &
\end{tikzcd}$$

By assumption $d_1^{-1}[0,\epsilon_1) \subset E_1'$.

Putting everything together, we have

$$r_{12}'(E_2') \subset r_{12}'(d_2^{-1}[0,\epsilon_2')) \subset d_1^{-1}[0,\epsilon_1) \subset E_1' .$$

Thus the condition (1) is satisfied.

Similarly for each n, we can select ϵ_n', E_n' and ϵ_n such that

$$r_{n\ell}(E_\ell') \subset E_n' \quad \text{for} \quad \ell \le n ,$$

$$d(f_n p_n', p_n f_n') < \epsilon_n' \quad \text{where} \quad \epsilon_n' = \delta_\ell ,$$

if (n,δ_ℓ) is a lifting pair for (ℓ,ϵ_ℓ) and if $(e_n,c_n) \in E_n \times C_n$,

$$d(p_n(e_n), f_n(c_n)) < \epsilon_n \Rightarrow (e_n,c_n) \in E_n' \quad \text{and}$$

$$0 < \epsilon_n < \epsilon_n' < \epsilon_{n-1} < \epsilon_{n-1}' < \cdots .$$

Since for each n, $Z_n \subset E_n' \subset d_n^{-1}[0,\epsilon_n')$ and $\varprojlim(Z_n) = Z$, $\varprojlim(E_n') = Z$.

To show that p' is a shape fibration, by Proposition 1.1, it is enough to show that $\underline{p}'$ has the HLP.

By I of Section 1, we can assume that $\underline{p}$ has the stronger lifting property where (2) is replaced by (5).

Let $(m,\delta_n = \epsilon_m')$ be the lifting pair for (n,ϵ_n). Let $h: X \to E_m'$ and $H: X \times I \to B_m$ be maps such that

$$H_0 = p_m' h . \tag{10}$$

$$\begin{tikzcd}
X\times 0 \arrow[r, "h"] \arrow[dd] & E'_m \arrow[rr, "f'_m"] \arrow[dr, "r'_{nm}"] \arrow[dd, "p'_m"] & & E_m \arrow[dr, "r_{nm}"] \arrow[dd] & \\
& & E'_n \arrow[rr, "f'_n"] \arrow[dd, "p'_n"] & & E_n \arrow[dd, "p_n"] \\
X\times I \arrow[r, "H"'] \arrow[urr, dashed, bend left, "G"] \arrow[urrrr, dashed, bend right, "G'"'] & C_m \arrow[rr, "f_m"] \arrow[dr, "q'_{nm}"'] & & B_m \arrow[dr, "q_{nm}"] & \\
& & C_n \arrow[rr, "f_n"'] & & B_n
\end{tikzcd}$$

By (3), $d(p_m f'_m h,\ f_m p'_m h) < \delta_n = \epsilon'_m$. By (10),

(11) $$d(p_m f'_m h,\ f_m H_0) < \delta_n \ .$$

Since (m,δ_n) is a lifting pair for (n,ϵ_n), there is a map $G'\colon X\times I \to E_n$ such that

(12) $$G'_0 = r_{nm}\, f'_m h$$

and

(13) $$d(p_n G',\ q_{nm} f_m H) < \epsilon_n \ .$$

Define $G\colon X\times I \to E'_n$ by $G(x,t) = (G'(x,t),\ q'_{nm}H(x,t))$, for $(x,t)\in X\times I$. Note that by (13), (4) and $q_{nm}f_m = f_n q'_{nm}$ for every $(x,t)\in X\times I$, $G(x,t)\in E'_n$. Hence $G(X\times I)\subset E'_n$. Also by (12) and (10)

$$G_0 = (G'_0\ ,\ q'_{nm}H_0) = (r_{nm}f'_m h,\ q'_{nm}p'_m h) = r'_{nm}h$$

and

$$p'_n G = p'_n(G',\ q'_{nm}H) = q'_{nm}H \ .$$

Thus $\underline{p}'$ has the HLP.

p' is called a shape fibration induced from p by f.

□

Section 3

FIBER SHAPE EQUIVALENCE

In this section we will define the basic concept of this paper - that of a fiber shape equivalence.

First we recall some of the definitions from [9] and establish notation.

Let $\underline{E} = (E_n, r_{nm})$ and $\underline{E}' = (E'_n, r'_{nm})$ be ANR-sequences. A map (not necessarily a level map) $\underline{f} = (\alpha, f_n): \underline{E} \to \underline{E}'$ of ANR-sequences consists of an increasing function $\alpha: N \to N$ (N is the set of natural numbers, $\alpha(n) \geq n$ for every $n \in N$ and α is divergent) and a collection of maps (continuous functions) $f_n: E_{\alpha(n)} \to E'_n$ such that for all $m \geq n$,

$$r'_{nm} f_n \simeq f_n \cdot r_{\alpha(n)\alpha(m)} ,$$

i.e., the diagram

$$\begin{tikzcd}
E'_m \arrow[d, "r'_{nm}"] & E_{\alpha(m)} \arrow[l, "f_m"'] \arrow[d, "r_{\alpha(n)\alpha(m)}"] \\
E'_n & E_{\alpha(n)} \arrow[l, "f_n"']
\end{tikzcd}$$

commutes up to homotopy.

If $K: E_{\alpha(m)} \times I \to E'_n$ is a homotopy such that $K_0 = r'_{nm} f_m$ and $K_1 = f_n r_{\alpha(n)\alpha(m)}$ then we will write

$$K: r'_{nm} f_m \simeq f_n \cdot r_{\alpha(n)\alpha(m)} .$$

If $\underline{f} = (\alpha, f_n): \underline{E} \to \underline{E}'$ and $\underline{g} = (\gamma, g_n): \underline{E}' \to \underline{E}'' = (E''_n, r''_{nm})$ are two maps of ANR-sequences then clearly

$$\underline{g}\,\underline{f} = (\alpha\gamma, g_n f_{\gamma(n)}) : \underline{E} \to \underline{E}''$$ is a map of ANR-sequences.

Also, $\underline{1}_{\underline{E}} = (1_N, 1_{E_n}): \underline{E} \to \underline{E}$ is a map of ANR-sequences.

Note that if $\alpha = 1_N$ and if for all $m \geq n$, $r'_{nm} f_m = f_n \cdot r_{nm}$ then $\underline{f}$ will be a level map. We denote the level map by $(\underline{E}, \underline{f}, \underline{B})$.

Let E be a compact ANR. An ANR-sequence $\underline{E} = (E_n, r_{nm})$ is called trivial if for every n, $E_n = E$ and for all $m \geq n$, $r_{nm} = 1_{E_n}$. We will denote this trivial ANR-sequence by $\underline{E} = (E)$. A level map between two trivial ANR-sequences is just a map $f: E \to B$. We will call this map a trivial level map and denote it by $\underline{f} = (f): \underline{E} \to \underline{B}$.

Two maps $\underline{f} = (\alpha, f_n)$, $\underline{g} = (\gamma, g_n): \underline{E} \to \underline{E}'$ of ANR-sequences are said to be equivalent (in symbol $\underline{f} \sim \underline{g}$) if for every n there exists m, $m \geq \alpha(n), \gamma(n)$ such that

$$f_n \cdot r_{\alpha(n)m} \simeq g_n \cdot r_{\gamma(n)m},$$

i.e. the diagram

$$\begin{tikzcd}
 & E_{\alpha(n)} \arrow[dl, "f_n"'] & \\
E'_n & & E_m \arrow[ul, "r_{\alpha(n)m}"'] \arrow[dl, "r_{\gamma(n)m}"] \\
 & E_{\gamma(n)} \arrow[ul, "g_n"] &
\end{tikzcd}$$

commutes up to homotopy.

Clearly if $m' \geq m$ then

$$f_n \cdot r_{\alpha(n)m'} \simeq g_n \cdot r_{\gamma(n)m'}.$$

Two ANR-sequences $\underline{E}$ and $\underline{E}'$ are said to be equivalent (in symbol $\underline{E} \sim \underline{E}'$) if there are maps $\underline{f}: \underline{E} \to \underline{E}'$ and $\underline{g}: \underline{E}' \to \underline{E}$ of ANR-sequences such that

$$\underline{g}\,\underline{f} \sim \underline{1}_{\underline{E}} \quad \text{and} \quad \underline{f}\,\underline{g} \sim \underline{1}_{\underline{E}'}.$$

<u>Definition 3.1</u>: A morphism

$$F = (\underline{f},\underline{h}) : (\underline{E},\underline{p},\underline{B}) \to (\underline{E}',\underline{p}',\underline{B}')$$

between two level maps of ANR-sequences is defined to be a pair of maps

$$\underline{f} = (\alpha, f_n) : \underline{E} = (E_n, r_{nm}) \to \underline{E}' = (E'_n, r'_{nm})$$

and

$$\underline{h} = (\beta, h_n) : \underline{B} = (B_n, q_{nm}) \to \underline{B}' = (B'_n, q'_{nm})$$

of ANR-sequences such that for every n and for every $\epsilon > 0$ there is an index $n^* = F(n,\epsilon)$ satisfying the following conditions:

(A) for all $m \geq n^*$

$$d(q'_{nm}p'_m f_m r_{\alpha(m)\ell},\ q'_{nm}h_m q_{\beta(m)\ell}p_\ell) < \epsilon$$

where $\ell \geq \text{Max}(\alpha(m), \beta(m))$ and

(B) if $m' \geq m$ there are homotopies

$$K\colon f_m r_{\alpha(m)\,\alpha(m')} \simeq r'_{mm'} f_{m'}$$

$$H\colon h_m q_{\beta(m)\beta(m')} \simeq q'_{mm'} h_{m'}$$

such that for every $t \in I$

(i) $d(q'_{nm}h_m q_{\beta(m)\beta(m')},\ q'_{nm}H_t) < \epsilon$ and

(ii) $d(q'_{nm}p'_m K_t r_{\alpha(m')\ell'},\ q'_{nm}H_t q_{\beta(m')\ell'}p_{\ell'}) < \epsilon$

where $\ell' \geq \text{Max}(\alpha(m'), \beta(m'))$.

$$\begin{tikzcd}
 & E_\ell \arrow[rr, "r_{\alpha(m)\ell}"] \arrow[dd, dashed, "p_\ell"] & & E_{\alpha(m)} \arrow[rr, "f_m"] & & E'_m \arrow[dd, "p'_m"] & \\
E_{\ell'} \arrow[ur, "r_{\ell\ell'}"] \arrow[rr] \arrow[dd, "p'_\ell"'] & & E_{\alpha(m')} \arrow[ur] \arrow[rr, "f_{m'}"'] \arrow[urrr, dashed, "K_t"] & & E'_{m'} \arrow[ur, "r'_{mm'}"'] \arrow[dd, "p'_{m'}"] & & \\
 & B_\ell \arrow[rr, dashed, "q_{\beta(m)\ell}"] & & B_{\beta(m)} \arrow[rr, dashed, "h_m"] & & B'_m \arrow[r, "q'_{nm}"'] & B'_n \\
B_{\ell'} \arrow[ur, dashed] \arrow[rr, "q_{\beta(m')\ell'}"'] & & B_{\beta(m')} \arrow[rr, "h_{m'}"'] \arrow[urrr, dashed, "H_t"] & & B'_{m'} \arrow[ur] & &
\end{tikzcd}$$

i) Clearly, $1_{\underline{p}} = (\underline{1}_{\underline{E}}, \underline{1}_{\underline{B}}): \underline{p} \to \underline{p}$ is a morphism.

ii) If in the definition 3.1, $\underline{B} = \underline{B}'$, $\underline{h} = \underline{1}_{\underline{B}}$ then the conditions reduce to the following:

(A) for all $m \geq n^*$

$d(q_{nm}p'_m f_m, q_{n\alpha(m)}p_{\alpha(m)}) < \epsilon$ and

(B) if $m' \geq m$ then there is a homotopy

$K: f_m r_{\alpha(m)\alpha(m')} \simeq r'_{mm'} f_{m'}$

such that for every $t \in I$

$d(q_{nm}p'_m K_t, q_{n\alpha(m')}p_{\alpha(m')}) < \epsilon$.

iii) Let E, E', B be compact ANR's and $\underline{E} = (E)$, $\underline{B} = (B)$ and $\underline{E}' = (E')$ be the trivial ANR-sequences in the definition 3.1.

Then $\underline{p}: \underline{E} \to \underline{B}$ and $\underline{p}': \underline{E}' \to \underline{B}$ are just continuous functions $p: E \to B$ and $p': E' \to B$ respectively, the same for each n.

A morphism $F: \underline{p} \to \underline{p}'$ is a family $\{f_\epsilon: E \to E'\}_\epsilon$ of continuous functions, one for each $\epsilon > 0$ such that $d(p'f_\epsilon, p) < \epsilon$ for every $\epsilon > 0$.

Denote this morphism by $F_\epsilon: \underline{p} \to \underline{p}'$ and call it a morphism over B.

Recall that a map $f: p \to p'$ over B is a continuous function $f: E \to E'$ such that $p'f = p$.

Then clearly, the trivial morphism $\underline{f} = (f): \underline{E} \to \underline{E}'$ is a morphism over B.

<u>Definition 3.2</u>: Two morphisms

$F = (\underline{f}, \underline{h})$, $G = (\underline{g}, \underline{k}): (\underline{E}, \underline{p}, \underline{B}) \to (\underline{E}', \underline{p}', \underline{B}')$ where

$\underline{f} = (\alpha, f_n)$, $\underline{h} = (\beta, h_n)$, $\underline{g} = (\gamma, g_n)$, $\underline{k} = (\delta, k_n)$,

of level maps of ANR-sequences are said to be equivalent (in symbols $F \sim G$) if for every n and for every $\epsilon > 0$ there is an index

$n' = (F,G)(n,\epsilon)$ with the following property: for every $m \geq n'$ there is an index ℓ; $\ell \geq \operatorname{Max}(\alpha(m),\beta(m),\gamma(m),\delta(m))$ and there are homotopies

$$L : f_m \cdot r_{\alpha(m)\ell} \simeq g_m r_{\gamma(m)\ell}$$

$$M : h_m q_{\beta(m)\ell} \simeq k_m q_{\delta(m)\ell}$$

such that for every $t \in I$

$$d(q'_{nm} h_m q_{\beta(m)\ell} \, , \, q'_{nm} M_t) < \epsilon$$

and

$$d(q'_{nm} p'_m L_t \, , \, q'_{nm} M_t p_\ell) < \epsilon \, .$$

$$\begin{tikzcd}
 & & E_{\alpha(m)} \arrow[dr, "f_m"] & & \\
E_{\ell'} \arrow[r, dashed, "r_{\ell\ell'}"] \arrow[ddd, "p_{\ell'}"] & E_{\ell} \arrow[ur, "r_{\alpha(m)\ell}"] \arrow[rr, dashed, "L_t"] \arrow[dr, "r_{\gamma(m)\ell}"'] \arrow[ddd, "p_{\ell}"] & & E'_m \arrow[ddd, "p'_m"] & \\
 & & E_{\gamma(m)} \arrow[ur, "g_m"'] & & \\
 & & B_{\beta(m)} \arrow[dr, "h_m"] & & \\
B_{\ell'} \arrow[r, dashed, "q_{\ell\ell'}"] & B_{\ell} \arrow[ur, "q_{\beta(m)\ell}"] \arrow[rr, dashed, "M_t"'] \arrow[dr, "q_{\delta(m)\ell}"'] & & B'_m \arrow[r, "q'_{nm}"] & B'_n \\
 & & B_{\delta(m)} \arrow[ur, "k_m"'] & &
\end{tikzcd}$$

Remarks:

iv) Denote the set of all such ℓ by $(F,G)(m;n,\epsilon)$. Clearly if $\ell' \geq \ell$ then $\ell' \in (F,G)(m;n,\epsilon)$ since $p_\ell r_{\ell\ell'} = q_{\ell\ell'} p_{\ell'}$.

v) If $F = (\underline{f},\underline{h}) \sim G = (\underline{g},\underline{k})$ then $\underline{f} \sim \underline{g}$ and $\underline{h} \sim \underline{k}$.

Since for every n there is an ℓ , $\ell \geq \operatorname{Max}(\alpha(m),\gamma(m))$, such that

$f_m r_{\alpha(m)\ell} \simeq g_m r_{\gamma(m)\ell}$ where $m \geq n$. Since $\underline{f}$ is a map of ANR-sequences $r'_{nm} f_m \simeq f_n r_{\alpha(n)\alpha(m)}$ and $r'_{nm} g_m \simeq g_n r_{\gamma(n)\gamma(m)}$. Hence

$$\begin{aligned} f_n r_{\alpha(n)\ell} &= f_n r_{\alpha(n)\alpha(m)} r_{\alpha(m)\ell} \\ &\simeq r'_{nm} f_m r_{\alpha(m)\ell} \\ &\simeq r'_{nm} g_m r_{\gamma(m)\ell} \\ &\simeq g_n r_{\gamma(n)\gamma(m)} r_{\gamma(m)\ell} \\ &= g_n r_{\gamma(n)\ell} \,. \end{aligned}$$

Similarly $h_n q_{\beta(n)\ell} \simeq k_n q_{\delta(n)\ell}$.

Proposition 3.1: The relation $\sim$ on the morphisms of level maps of ANR-sequences is an equivalence relation.

Proof: By definition 3.2, it is clear that $\sim$ is reflexive and symmetric. Only transitivity requires a proof.

$$\text{Let}\quad \begin{aligned} F &= (\underline{f},\underline{h}) \\ F' &= (\underline{f}',\underline{h}') \\ F'' &= (\underline{f}'',\underline{h}'') \end{aligned} \quad : \ (\underline{E},\underline{p},\underline{B}) \to (\underline{E}',\underline{p}',\underline{B}')$$

be morphisms of level maps such that $F \sim F'$ and $F' \sim F''$.

For given n and $\epsilon > 0$ let $n' = (F,F')(n,\epsilon/2)$ and $n'' = (F',F'')(n,\epsilon/2)$. Let $\tilde{n} = \mathrm{Max}(n',n'')$. Since $F \sim F'$ and $F' \sim F''$ there are indices $\ell' \in (F,F')(m;n,\epsilon/2)$, $\ell'' \in (F',F'')(m;n,\epsilon/2)$ and there are homotopies

$$L' : f_m r_{\alpha(m)\ell'} \simeq f'_m r_{\alpha'(m)\ell'}$$

$$M' : h_m q_{\beta(m)\ell'} \simeq h'_m q_{\beta'(m)\ell'}$$

and

$$L'' : f'_m r_{\alpha'(m)\ell''} \simeq f''_m r_{\alpha''(m)\ell''}$$

$$M'' : h'_m q_{\beta'(m)\ell''} \simeq h''_m q_{\beta''(m)\ell''}$$

such that for every $t \in I$

(1) $d(q'_{nm} h_m q_{\beta(m)\ell'} , q'_{nm} M'_t) < \epsilon/2$

(2) $d(q'_{nm} p'_m L'_t , q'_{nm} M'_t p_{\ell'}) < \epsilon/2$

(3) $d(q'_{nm} h'_m q_{\beta'(m)\ell''} , q'_{nm} M''_t) < \epsilon/2$ and

(4) $d(q'_{nm} p'_m L''_t , q'_{nm} M''_t p_{\ell''}) < \epsilon/2$.

Let $\ell \geq \mathrm{Max}(\ell',\ell'')$. Note that $r_{\alpha'(m)\ell'} r_{\ell'\ell} = r_{\alpha'(m)\ell} = r_{\alpha'(m)\ell''} r_{\ell''\ell}$. Similarly, $q_{\beta'(m)\ell'} q_{\ell'\ell} = q_{\beta'(m)\ell} = q_{\beta'(m)\ell''} q_{\ell''\ell}$.

Define homotopies $L: f_m r_{\alpha(m)\ell} \simeq f''_m r_{\alpha''(m)\ell}$

$$M : h_m q_{\beta(m)\ell} \simeq h''_m q_{\beta''(m)\ell}$$

by

$$L(x,t) = \begin{cases} L'(r_{\ell'\ell}(x), 2t) & 0 \leq t \leq \tfrac{1}{2} \\ L''(r_{\ell''\ell}(x), 2t-1) & \tfrac{1}{2} \leq t \leq 1 . \end{cases}$$

$$M(y,t) = \begin{cases} M'(q_{\ell'\ell}(y), 2t) & 0 \leq t \leq \tfrac{1}{2} \\ M''(q_{\ell''\ell}(y), 2t-1) & \tfrac{1}{2} \leq t \leq 1 . \end{cases}$$

We want to show that for every $t \in I$

(5) $d(q'_{nm} h_m q_{\beta(m)\ell} , q'_{nm} M_t) < \epsilon$ and

(6) $d(q'_{nm} p'_m L_t , q'_{nm} M_t p_{\ell}) < \epsilon$.

<u>Let $0 \leq t \leq \frac{1}{2}$</u>

By (1)

(7) $d(q'_{nm} h_m q_{\beta(m)\ell'} q_{\ell'\ell} , q'_{nm} M'_{2t} q_{\ell'\ell}) < \epsilon/2$ which is (5).

By (2)

(8) $d(q'_{nm} p'_m L'_{2t} r_{\ell'\ell} , q'_{nm} M'_{2t} p_{\ell'} r_{\ell'\ell}) < \epsilon/2$.

Since $p_{\ell'} r_{\ell'\ell} = q_{\ell'\ell} p_{\ell}$

(9) $d(q'_{nm}p'_m L'_{2t} r_{\ell'\ell} \, , \, q'_{nm} M'_{2t} q_{\ell'\ell} p_{\ell}) < \epsilon/2$ which is (6).

<u>Let</u> <u>$\frac{1}{2} \leq t \leq 1$</u>.

For $t = \frac{1}{2}$, (7) implies

(10) $d(q'_{nm} h_m q_{\beta(m)\ell'} q_{\ell'\ell} \, , \, q'_{nm} h'_m q_{\beta'(m)\ell'} q_{\ell'\ell}) < \epsilon/2$. Also (3) implies

(11) $d(q'_{nm} h'_m q_{\beta'(m)\ell''} q_{\ell''\ell} \, , \, q'_{nm} M''_{2t-1} q_{\ell''\ell}) < \epsilon/2$.

By (10) and (11)

(12) $d(q'_{nm} h_m q_{\beta(m)\ell} \, , \, q'_{nm} M''_{2t-1} q_{\ell''\ell}) < \epsilon$ which is (5).

By (4)

(13) $d(q'_{nm} p'_m L''_{2t-1} r_{\ell''\ell} \, , \, q'_{nm} M''_{2t-1} p_{\ell''} r_{\ell''\ell}) < \epsilon/2$.

Since $p_{\ell''} r_{\ell''\ell} = q_{\ell''\ell} p_{\ell}$, (13) implies

(14) $d(q'_{nm} p'_m L''_{2t-1} r_{\ell''\ell} \, , \, q'_{nm} M''_{2t-1} q_{\ell''\ell} p_{\ell}) < \epsilon/2$ which is (6).

<u>Remark</u>: Let $F = (\underline{f}, \underline{h}) : (\underline{E}, \underline{p}, \underline{B}) \rightarrow (\underline{E}', \underline{p}', \underline{B}')$ be a morphism of level maps where $\underline{f} = (\alpha, f_n)$ and $\underline{h} = (\beta, h_n)$.

For convenience we can define an index function $\hat{\alpha} = \mathrm{Max}(\alpha, \beta) : N \rightarrow N$ and maps

$$\underline{\hat{f}} = (\hat{\alpha}, \hat{f}_n) \quad \text{and} \quad \underline{\hat{h}} = (\hat{\alpha}, \hat{h}_n)$$

(with the same index function) by $\hat{f}_n = f_n r_{\alpha(n)\hat{\alpha}(n)}$ and $\hat{h}_n = h_n q_{\beta(n)\hat{\alpha}(n)}$ for every n . Then clearly

$$\hat{F} = (\underline{\hat{f}}, \underline{\hat{h}}) : (\underline{E}, \underline{p}, \underline{B}) \rightarrow (\underline{E}', \underline{p}', \underline{B}')$$

is a morphism of level maps and is equivalent to F.

Considering this remark, from now on we will always consider morphism with the same index function between two level maps.

Now we will prove that composition of two morphisms of level maps is also a morphism of level maps.

Proposition 3.2: Let

$$F = (\underline{f},\underline{h}) : (\underline{E},\underline{p},\underline{B}) \to (\underline{E}',\underline{p}', \underline{B}') \text{ and}$$

$$G = (\underline{g},\underline{k}) : (\underline{E}',\underline{p}',\underline{B}') \to (\underline{E}'',\underline{p}'',\underline{B}'')$$

be morphisms of level maps of ANR-sequences then

$$GF = (\underline{gf},\underline{kh}) : (\underline{E},\underline{p},\underline{B}) \to (\underline{E}'',\underline{p}'',\underline{B}'')$$

is a morphism of level maps.

First we will prove a useful lemma.

Lemma 3.1: Let all the spaces in the following diagram be compact ANR's and the arrows be continuous functions. Let

(1) $d(k_0q'_0, q''_0k_1) < \epsilon/4$ and also suppose that

$$\begin{tikzcd}
 & E_1 \arrow[rr, "f_1"] \arrow[ddd, "p_1"] & & E'_1 \arrow[rr, "g_1"] \arrow[ddd, "p'_1"] & & E''_1 \arrow[ddd, "p''_1"] & \\
E_2 \arrow[ur, "r"] \arrow[rr, "f_2"] \arrow[urrr, dashed, "K'_t"] \arrow[ddd, "p_2"] & & E'_2 \arrow[ur, "r'"] \arrow[rr, "g_2"] \arrow[urrr, dashed, "K''_t"] \arrow[ddd, "p'_2"] & & E''_2 \arrow[ur, "r''"] \arrow[ddd, "p''_2"] & & \\
 & & & & B'_0 \arrow[rr, dashed, "k_0"] & & B''_0 \\
 & B_1 \arrow[rr, "h_1"] & & B'_1 \arrow[rr, "k_1"] \arrow[ur, dashed] & & B''_1 \arrow[ur, dashed, "q''_0"] & \\
B_2 \arrow[ur, "q"] \arrow[rr, "h_2"] \arrow[urrr, dashed, "H'_t"] & & B'_2 \arrow[ur, "q'"] \arrow[rr, "k_2"] \arrow[urrr, dashed, "H''_t"] & & B''_2 \arrow[ur, "q''"] & &
\end{tikzcd}$$

there are homotopies

$$K' : f_1r \simeq r'f_2$$

$$H' : h_1q \simeq q'h_2$$

$$K'' : g_1r' \simeq r''g_2$$

$$H'' : k_1q' \simeq q''k_2$$

such that for every $t \in I$

(1') $d(q_0'h_1q, q_0'H_t') < \delta$ where $\delta \in \Lambda(\epsilon/4, k_0)$.

(2) $d(q_0'p_1'K_t', q_0'H_t'p_2) < \delta$

(3) $d(q_0''k_1q', q_0''H_t'') < \epsilon/4$ and

(4) $d(q_0''p_1''K_t'', q_0''H_t''p_2') < \epsilon/4$,

and we are also given

(4A) $d(q_0''K_1p_1', q_0''p_1''g_1) < \epsilon/4$

(4B) $d(q_0'q'p_2'f_2, q_0'q'h_2p_2) < \delta$.

then there are homotopies

$$K\colon g_1f_1r \simeq r''g_2f_2$$

$$H\colon k_1h_1q \simeq q''k_2h_2$$

such that for every $t \in I$

(5) $d(q_0''k_1h_1q, q_0''H_t) < \epsilon$ and

(6) $d(q_0''p_1''K_t, q_0''H_tp_2) < \epsilon$

<u>Proof</u>: Define K and H by

$$K(x,t) = \begin{cases} g_1K'(x,2t) , & 0 \le t \le \frac{1}{2} \\ K''(f_2(x),2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

$$H(y,t) = \begin{cases} k_1H'(y,2t) , & 0 \le t \le \frac{1}{2} \\ H''(h_2(y),2t-1) , & \frac{1}{2} \le t \le 1 \end{cases}$$

for $x \in E_2$, $y \in B_2$.

<u>Let $0 \le t \le \frac{1}{2}$</u>

By (1) and choice of δ

(7) $$d(k_0q_0'h_1q,\ k_0q_0'H_{2t}') < \epsilon/4\ .$$

By (1)

(8) $$d(k_0q_0'h_1q,\ q_0''k_1h_1q) < \epsilon/4 \quad \text{and}$$

(9) $$d(k_0q_0'H_{2t}'\ ,\ q_0''k_1H_{2t}') < \epsilon/4\ .$$

By (8), (7), (9)

$$d(q_0''k_1h_1q,\ q_0''k_1H_{2t}') < 3\epsilon/4 \quad \text{which is (5).}$$

By (2) and the choice of δ

(10) $$d(k_0q_0'p_1'K_{2t}'\ ,\ k_0q_0'H_{2t}'p_2) < \epsilon/4\ .$$

By (1)

(11) $$d(k_0q_0'p_1'K_{2t}'\ ,\ q_0''k_1p_1'K_{2t}') < \epsilon/4$$

(12) $$d(k_0q_0'H_{2t}'p_2\ ,\ q_0''k_1H_{2t}'p_2) < \epsilon/4\ .$$

By (11), (10) and (12)

(13) $$d(q_0''k_1p_1'K_{2t}'\ ,\ q_0''k_1H_{2t}'p_2) < 3\epsilon/4\ .$$

By (4A)

(14) $$d(q_0''k_1p_1'K_{2t}'\ ,\ q_0''p_1'g_1K_{2t}') < \epsilon/4\ .$$

By (13) and (14)

$$d(q_0''p_1'g_1K_{2t}'\ ,\ q_0''k_1H_{2t}'p_2) < \epsilon \quad \text{which is (6).}$$

Let $\frac{1}{2} \le t \le 1$

By (1)

(15) $$d(k_0q_0'q'h_2\ ,\ q_0''k_1q'h_2) < \epsilon/4\ .$$ For $t = 1$

(1') implies

$$d(q_0'h_1q\ ,\ q_0'q'h_2) < \delta\ .$$

By the choice of δ ,

(16) $$d(k_0q_0'h_1q\ ,\ k_0q_0'q'h_2) < \epsilon/4\ .$$

By (1)

(17) $d(k_0q_0'h_1q\ ,\ q_0''k_1h_1q) < \epsilon/4$ and

(18) $d(k_0q_0'q'h_2\ ,\ q_0''k_1q'h_2) < \epsilon/4$.

Also by (3)

(19) $d(q_0''k_1q'h_2\ ,\ q_0''H_{2t-1}''h_2) < \epsilon/4$.

By (17), (16), (18) and (19)

$d(q_0''k_1h_1q\ ,\ q_0''H_{2t-1}''h_2) < \epsilon$ which is (5).

Now, by (4)

(20) $d(q_0''p_1''K_{2t-1}''f_2\ ,\ q_0''H_{2t-1}''p_2'f_2) < \epsilon/4$.

We will show that

(21) $d(q_0''H_{2t-1}''p_2'f_2\ ,\ q_0''H_{2t-1}''h_2p_2) < \epsilon + \epsilon/4$.

Then by (20) and (21)

$d(q_0''p_1''K_{2t-1}''f_2\ ,\ q_0''H_{2t-1}''h_2p_2) < \epsilon + \epsilon/2$ which is (6).

We have to show that (21) holds.

By (4B) and the choice of δ

(22) $d(k_0q_0'q'p_2'f_2\ ,\ k_0q_0'q'h_2p_2) < \epsilon/4$.

By (1)

(23) $d(k_0q_0'q'p_2'f_2\ ,\ q_0''k_1q'p_2'f_2) < \epsilon/4$ and

(24) $d(k_0q_0'q'h_2p_2\ ,\ q_0''k_1q'h_2p_2) < \epsilon/4$.

By (3)

(25) $d(q_0''k_1q'h_2p_2\ ,\ q_0''\ H_{2t-1}''h_2p_2) < \epsilon/4$ and

(26) $d(q_0''k_1q'p_2'\ f_2\ ,\ q_0''H_{2t-1}''p_2'f_2) < \epsilon/4$.

By (26), (23), (22), (24) and (25)

$d(q_0''H_{2t-1}''p_2'f_2\ ,\ q_0''H_{2t-1}''h_2p_2) < \epsilon + \epsilon/4$ which is (21).

We will set up some notations.

<u>Notations</u>: Let $f\colon X \to Y$ be a map between compact ANR's. For given

$\epsilon > 0$, $\Lambda(f,\epsilon)$ denotes the set of all δ's such that

$$d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon$$

and $\Gamma(Y,\epsilon)$ denotes the set of all η's such that two η-close maps from a metric space to Y are ϵ-homotopic.

Note that if $0 < \delta' < \delta$ and $0 < \eta' < \eta$ then $\delta' \in \Lambda(f,\epsilon)$ and $\eta' \in \Gamma(Y,\epsilon)$ if $\delta \in \Lambda(f,\epsilon)$ and $\eta \in \Gamma(Y,\epsilon)$. By convention if $\delta \in \Lambda(f,\epsilon)$ and $\eta \in \Gamma(Y,\epsilon)$ then $\delta, \eta < \epsilon$.

<u>Proof of the Proposition 3.2</u>:

For given n and $\epsilon > 0$, let $m \geq G(n,\epsilon/6)$ and for $\delta \in \Lambda(\epsilon/6 , q''_{nm}h_m)$, $m' \geq F(\alpha(m),\delta)$.

Choose $n' \geq m$ such that $\alpha(n') \geq m'$. (e.g. $n' = m'$). We want to show that

$$d(q''_{nn'}p''_{n'}g_{n'}f_{\alpha(n')} , q''_{nn'}h_{n'}k_{\alpha(n')}p_{\beta\alpha(n')}) < \epsilon$$

$$\begin{tikzcd}
 & E_{\beta(m')} \arrow[rr, "f_{m'}"] \arrow[dd, dashed] & & E'_{m'} \arrow[r, "r'"] \arrow[dd, dashed] & E'_{\alpha(m)} \arrow[rr, "g_m"] \arrow[dd, dashed] & & E''_m \arrow[dd, "p''_m"] & \\
E_{\beta\alpha(n')} \arrow[ur, "r"] \arrow[rr, "f_{\alpha(n')}"] \arrow[dd, "p_{\beta\alpha(n')}"] & & E'_{\alpha(n')} \arrow[ur, dashed] \arrow[urr, "r'"] \arrow[rrr, "g_{n'}"] \arrow[dd, "p'_{\alpha(n')}"] & & & E''_{n'} \arrow[ur, "r''_{mn'}"] \arrow[dd, "p''_{n'}"] & & \\
 & B_{\beta(m')} \arrow[rr, dashed, "h_{m'}"] & & B'_{m'} \arrow[r, dashed] & B'_{\alpha(m)} \arrow[rr, dashed, "k_m"] & & B''_m \arrow[r, "q''_{nm}"] & B''_n \\
B_{\beta\alpha(n')} \arrow[ur, dashed, "q"] \arrow[rr, "h_{\alpha(n')}"] & & B'_{\alpha(n')} \arrow[ur, dash] \arrow[urr, "q'_{\alpha(m)\alpha(n')}"] \arrow[rrr, "k_{n'}"] & & & B''_{n'} \arrow[ur, "q''_{mn'}"] & &
\end{tikzcd}$$

Since $n' \geq G(n, \epsilon/6)$, by definition of G

(1) $$d(q''_{nn'}p''_{n'}g_{n'} \, , \, q''_{nn'}k_{n'}p'_{\alpha(n')}) < \epsilon/6 \quad \text{and}$$

(2) $$d(q''_{nm}k_m q'_{\alpha(m)\alpha(n')} \, , \, q''_{nn'}k_{n'}) < \epsilon/6 \, .$$

Since $\alpha(n') \geq F(\alpha(m), \delta)$, by definition of F

(3) $$d(q'_{\alpha(m)\alpha(n')}p'_{\alpha(n')}f_{\alpha(n')} \, , \, q'_{\alpha(m)\alpha(n')}h_{\alpha(n')}p_{\beta\alpha(n')}) < \delta$$

where $\delta \in \Lambda(q''_{nm}k_m, \epsilon/6)$.

By (1)

(4) $$d(q''_{nn'}p''_{n'}g_{n'}f_{\alpha(n')} \, , \, q''_{nn'}k_{n'}p'_{\alpha(n')}f_{\alpha(n')}) < \epsilon/6 \, .$$

By (3) and choice of δ

(5) $$d(q''_{nm}k_m q'_{\alpha(m)\alpha(n')}p'_{\alpha(n')}f_{\alpha(n')} \, , \, q''_{nm}k_m q'_{\alpha(m)\alpha(n')}h_{\alpha(n')}p_{\beta\alpha(n')}) < \epsilon/6.$$

By (2) we have

(6) $$d(q''_{nm}k_m q'_{\alpha(m)\alpha(n')}p'_{\alpha(n')}f_{\alpha(n')}, q''_{nn'}k_{n'}p'_{\alpha(n')}f_{\alpha(n')}) < \epsilon/6$$

and

(7) $$d(q''_{nm}k_m q'_{\alpha(m)\alpha(n')}h_{\alpha(n')}p_{\beta\alpha(n')}, q''_{nn'}k_{n'}h_{\alpha(n')}p_{\beta\alpha(n')}) < \epsilon/6 \, .$$

By (4), (6), (5) and (7) we have

$$d(q''_{nn'}p''_{n'}g_{n'}f_{\alpha(n')}, q''_{nn'}k_{n'}h_{\alpha(n)}p_{\beta\alpha(n')}) < 4\epsilon/6 < \epsilon \, .$$

Let $\ell \geq n'$. Recall (2)

$$d(q''_{nm}k_m q'_{\alpha(m)\alpha(n')} \, , \, q''_{nn'}k_{n'}) < \epsilon/6$$

since $n' \geq m \geq G(n, \epsilon/6)$, by definition of G

(8) $$d(q_{nn'}k_{n'}p'_{\alpha(n')}, q_{nn'}p''_{n'}g_{n'}) < \epsilon/6$$

$$\begin{tikzcd}
 & & E_{\beta\alpha(n')} \arrow[rrrr, "f_{\alpha(n')}"] \arrow[ddd, "p_{\beta\alpha("] & & & & E'_{\alpha(n')} \arrow[rrrr, "g_{n'}"] \arrow[ddd, "p'_{\alpha(n')}"] & & & & E''_{n'} \arrow[ddd, "p''_{n'}"] & & & & \\
E_{\beta\alpha(\ell)} \arrow[urr, "r"] \arrow[rrrr, "f_{\alpha(\ell)}"] \arrow[ddd, "p_{\beta\alpha(\ell)}"] & & & & E'_{\alpha(\ell)} \arrow[urr, "r'"] \arrow[rrrr, "g_\ell"] \arrow[ddd] & & & & E''_\ell \arrow[urr, "r''_{n'\ell}"] \arrow[ddd] & & & & & & B''_n \\
 & & & & & B_{m'} \arrow[rr, dashed] & & B_{\alpha(m)} \arrow[rrrrr, dashed, "k_m"] & & & & & B''_m \arrow[urr, dashed, "q''_{nm}"] & & \\
 & & B_{\beta\alpha(n')} \arrow[rrrr, "h_{\alpha(n')}"] & & & & B'_{\alpha(n')} \arrow[rrrr, "k_{n'}"] \arrow[ul, dashed] \arrow[ur, dashed] & & & & B''_{n'} \arrow[urr, dashed, "q''_{mn'}"] & & & & \\
B_{\beta\alpha(\ell)} \arrow[urr, "q"] \arrow[rrrr, "h_{\alpha(\ell)}"] & & & & B'_{\alpha(\ell)} \arrow[urr, "q'_{\alpha(n')\alpha(\ell)}"] \arrow[rrrr, "k_\ell"'] & & & & B''_\ell \arrow[urr, "q''_{n'\ell}"'] & & & & & &
\end{tikzcd}$$

Since $\alpha(\ell) \geq \alpha(n') \geq F(\alpha(m),\delta)$, by definition of F

(9) $$d(q'_{\alpha(m)\alpha(n')}q'_{\alpha(n')\alpha(\ell)}p'_{\alpha(\ell)}f_{\alpha(\ell)},q'_{\alpha(m)\alpha(n')}q'_{\alpha(n')\alpha(\ell)}h_{\alpha(\ell)}p_{\beta\alpha(\ell)} < \delta$$

where $\delta \in \Lambda(q''_{nm}k_m,\epsilon/6)$.

Also, since $\alpha(\ell) \geq \alpha(n') \geq F(\alpha(m),\delta)$ and $\ell \geq n' \geq G(n,\epsilon/6)$, by definition of F and G there are homotopies

$$K' : f_{\alpha(n')}\cdot r_{\beta\alpha(n')\beta\alpha(\ell)} \simeq r'_{\alpha(n')\alpha(\ell)}f_{\alpha(\ell)}$$

$$H' : h_{\alpha(n')}q_{\beta\alpha(n')\beta\alpha(\ell)} \simeq q'_{\alpha(n')\alpha(\ell)}h_{\alpha(\ell)}$$

and

$$K'' : g_{n'}\cdot r'_{\alpha(n')\alpha(\ell)} \simeq r''_{n'\ell}\cdot g_\ell$$

$$H'' : k_{n'}\cdot q'_{\alpha(n')\alpha(\ell)} \simeq q''_{n'\ell}\cdot k_\ell$$

such that for every $t \in I$

(10) $$d(q'_{\alpha(m)\alpha(n')}h_{\alpha(n')}q_{\beta\alpha(n')\beta\alpha(\ell)},q''_{\alpha(m)\alpha(n')}H'_t) < \delta .$$

(11) $$d(q'_{\alpha(m)\alpha(n')}p'_{\alpha(n')}K'_t,q'_{\alpha(m)\alpha(n')}H'_tp_{\beta\alpha(\ell)}) < \delta$$

where $\delta \in \Lambda(q''_{nm}k_m,\epsilon/6)$ and

(12) $$d(q''_{nn'}k_{n'}q'_{\alpha(n')\alpha(\ell)}\ ,\ q''_{nn'}H''_t) < \epsilon/6$$

(13) $$d(q''_{nn'}p''_{n'}K''_t\ ,\ q''_{nn'}H''_t p'_{\alpha(\ell)}) < \epsilon/6\ .$$

By Lemma 3.1, there are homotopies

$$K:\ g'_{n'} f_{\alpha(n')} r_{\beta\alpha(n')\beta\alpha(\ell)} \simeq r''_{n'\ell} g_\ell f_{\alpha(\ell)}$$

$$H:\ k_{n'} h_{\alpha(n')} q_{\beta\alpha(n')\beta\alpha(\ell)} \simeq q''_{n'\ell} k_\ell h_{\alpha(\ell)}$$

such that for every $t \in I$

$$d(q''_{nn'}k_{n'}h_{\alpha(n')}q_{\beta\alpha(n')\beta\alpha(\ell)}\ ,\ q''_{nn'}H_t) < \epsilon$$

and

$$d(q''_{nn'}p''_{n'}K_t\ ,\ q''_{nn'}H_t p_{\beta\alpha(\ell)}) < \epsilon\ .$$

Hence $GF = (\underline{g}\ \underline{f},\ \underline{k}\ \underline{h}) : (\underline{E}, \underline{p}, \underline{B}) \to (\underline{E}'', \underline{p}'', \underline{B}'')$ is a morphism of level maps of ANR-sequences.

<u>Proposition 3.3</u>: Let $\left.\begin{array}{l} F = (\underline{f}, \underline{h}) \\ F' = (\underline{f}', \underline{h}') \end{array}\right\} : (\underline{E}, \underline{p}, \underline{B}) \to (\underline{E}', \underline{p}', \underline{B}')$

and

$$G = (\underline{g}, \underline{k}) : (\underline{E}', \underline{p}', \underline{B}') \to (\underline{E}'', \underline{p}'', \underline{B}'')$$

where $\underline{f} = (\alpha, f_n)$, $\underline{f}' = (\alpha, f'_n)$, $\underline{h} = (\alpha, h_n)$, $\underline{h}' = (\alpha, h'_n)$, $\underline{g} = (\beta, g_n)$, $\underline{k} = (\beta, k_n)$, be morphism of level maps of ANR-sequences. Then

$$F \sim F' \Rightarrow GF \sim GF'\ .$$

<u>Proof</u>: For given n and $\epsilon > 0$, let $m \geq G(n, \epsilon/4)$. Choose $m' \geq m$ such that

$$\beta(m') \geq (F, F')(\beta(m), \delta)\ ,$$

where $\delta \in \Lambda(q''_{nm}k_m, \epsilon/4)$. Since $F \sim F'$, there is $\ell \geq \alpha\beta(m')$ and there are homotopies

$$L' :\ f_{\beta(m')} r_{\alpha\beta(m')\ell} \simeq f'_{\beta(m')} r_{\alpha\beta(m')\ell}$$

$$M' :\ h_{\beta(m')} q_{\alpha\beta(m')\ell} \simeq h'_{\beta(m')} q_{\alpha\beta(m)\ell}$$

such that for every $t \in I$

(1) $$d(q'_{\beta(m)\beta(m')}h_{\beta(m')}q_{\alpha\beta(m')\ell}\ ,\ q'_{\beta(m)\beta(m')}M'_t) < \delta$$

and

(2) $$d(q'_{\beta(m)\beta(m')}p'_{\beta(m')}L'_t\ ,\ q'_{\beta(m)\beta(m')}M'_t p_\ell) < \delta\ .$$

Since $m' \geq m \geq G(n,\epsilon/4)$,

(3) $$d(q''_{nm'}p''_{m'}g_{m'}\ ,\ q''_{nm'}k_{m'}p'_{\beta(m')}) < \epsilon/4$$

and

(4) $$d(q''_{nm}k_m q'_{\beta(m)\beta(m')}\ ,\ q''_{nm'}k_{m'}) < \epsilon/4\ .$$

$$\begin{tikzcd}
 & & & E'_{\beta(m)} \arrow[rr, "g_m"] \arrow[ddd] & & E''_m \arrow[ddd] & \\
E_\ell \arrow[r, "r_{\alpha\beta(m')\ell}"] \arrow[ddd, "p_\ell"] & E_{\alpha\beta(m')} \arrow[r, shift left, "f_{\beta(m')}"] \arrow[r, shift right, "f'_{\beta(m')}"'] \arrow[ddd, "p_{\alpha\beta(m')}"] & E'_{\beta(m')} \arrow[ur] \arrow[rr, "g_{m'}"] \arrow[ddd, "p'_{\beta(m')}"] & & E''_{m'} \arrow[ur] \arrow[ddd, "k''_{m'}"] & & \\
 & & & & & & B''_n \\
 & & & B'_{\beta(m)} \arrow[rr, "k_m"] & & B''_m \arrow[ur, "q''_{nm}"'] & \\
B_\ell \arrow[r, "q_{\alpha\beta(m)\ell}"'] & B_{\alpha\beta(m')} \arrow[r, shift left, "h_{\beta(m')}"] \arrow[r, shift right, "h'_{\beta(m')}"'] & B'_{\beta(m')} \arrow[ur] \arrow[rr, "k_{m'}"] & & B''_{m'} \arrow[ur, "q''_{mm'}"'] & &
\end{tikzcd}$$

Define homotopies

$$L : g_{m'}f_{\beta(m')}r_{\alpha\beta(m')\ell} \simeq g_{m'}f'_{\beta(m')}r_{\alpha\beta(m')\ell}$$

$$M : k_{m'}h_{\beta(m')}q_{\alpha\beta(m')\ell} \simeq k_{m'}h'_{\beta(m')}q_{\alpha\beta(m')\ell}$$

by $L = g_{m'}L'$ and $M = k_{m'}M'$.

We want to show that for every $t \in I$

(5) $$d(q''_{nm'}k_{m'}h_{\beta(m')}q_{\alpha\beta(m')\ell}\ ,\ q''_{nm'}M_t) < \epsilon$$

and

(6) $$d(q''_{nm'}p''_{m'}L_t\ ,\ q''_{nm'}M_tp_\ell) < \epsilon\ .$$

By (1) and the choice of δ , for every $t \in I$

(7) $$d(q''_{nm}k_mq'_{\beta(m)\beta(m')}h_{\beta(m')}q_{\alpha\beta(m')\ell}\ ,\ q''_{nm}k_mq'_{\beta(m)\beta(m')}M'_t) < \epsilon/4\ .$$

By (4)

(8) $$d(q''_{nm}k_mq'_{\beta(m)\beta(m')}h_{\beta(m')}q_{\alpha\beta(m')\ell}, q''_{nm'}k_{m'}h_{\beta(m')}q_{\alpha\beta(m')\ell}) < \epsilon/4$$

and

(9) $$d(q''_{nm}k_mq'_{\beta(m)\beta(m')}M'_t\ ,\ q''_{nm'}k_{m'}M'_t) < \epsilon/4$$

for every $t \in I$.

By (8), (7) and (9)

$$d(q''_{nm'}k_{m'}h_{\beta(m')}q_{\alpha\beta(m')\ell}\ ,\ q''_{nm'}k_{m'}M'_t) < 3\epsilon/4 < \epsilon$$

which is (5) since $k_{m'}M'_t = M_t$ for every $t \in I$.

Now for every $t \in I$, by (3)

(10) $$d(q''_{nm'}p''_{m'}g_{m'}L'_t\ ,\ q''_{nm'}k_{m'}p'_{\beta(m')}L'_t) < \epsilon/4\ .$$

By (4)

(11) $$d(q''_{nm}k_mq'_{\beta(m)\beta(m')}p'_{\beta(m')}L'_t\ ,\ q''_{nm'}k_{m'}p'_{\beta(m')}L'_t) < \epsilon/4$$

and

(12) $$d(q''_{nm}k_mq'_{\beta(m)\beta(m')}M'_tp\ell\ ,\ q''_{nm'}k_{m'}M'_tp\ell) < \epsilon/4\ .$$

By (2) and the choice of δ

(13) $$d(q''_{nm}k_mq'_{\beta(m)\beta(m')}p'_{\beta(m')}L'_t\ ,\ q''_{nm}k_mq'_{\beta(m)\beta(m')}M'_tp_\ell) < \epsilon/4\ .$$

Since $L = g_{m'}L'$ and $M = k_{m'}M'$, by (10), (11), (13) and (12) for every $t \in I$

$$d(q''_{nm'}p''_{m'}L_t\ ,\ q''_{nm'}M_tp_\ell) < \epsilon$$

which is (6).

If $m'' \geq m'$ then $\beta(m'') \geq \beta(m') \geq (F,F')(\beta(m),\delta)$ and we can repeat the same argument for $\beta(m'')$. Hence $m' = (GF,GF')(n,\epsilon)$ and so $GF \sim GF'$.

<u>Proposition 3.4</u>: Let

$$F = (\underline{f},\underline{h}) : (\underline{E},\underline{p},\underline{B}) \to (\underline{E}',\underline{p}',\underline{B}')$$

and

$$\left.\begin{array}{r} G = (\underline{g},\underline{k}) \\ G' = (\underline{g}',\underline{k}') \end{array}\right\} : (\underline{E}',\underline{p}',\underline{B}') \to (\underline{E}'',\underline{p}'',\underline{B}'')$$

be morphisms of level maps of ANR-sequences. Then

$$G \sim G' \Rightarrow GF \sim G'F .$$

<u>Proof</u>: For given n and $\epsilon > 0$, let $m \geq G(n,\epsilon/4)$, $G'(n,\epsilon/4)$ and $(G,G')(n,\epsilon/4)$. Let $m' \geq m$ be such that $\alpha(m') \geq F(\alpha(m),\delta)$ and $(G,G')(m;n,\epsilon/4)$ (which is $\geq \alpha(m)$) where $\delta \in \Lambda(q''_{nm}k_m,\epsilon/4) \cap \Lambda(q''_{nm}k'_m,\epsilon/4)$ (for example, $m' \geq F(\alpha(m),\delta)$ and $(G,G')(m;n,\epsilon/4)$) .

Let $\ell' \geq F(\alpha(m'),\lambda)$ where

$$\lambda \in \Lambda(q''_{nm'}k_{m'},\epsilon/4) \cap \Lambda(q''_{nm'}k'_{m'},\epsilon/4) .$$

We want to show that $\beta(\ell') \in (GF,G'F)(m';n,\epsilon)$. Since $m' \geq m \geq G(n,\epsilon/4)$, $G'(n,\epsilon/4)$ and $(G,G')(n,\epsilon/4)$ by definitions of G,G' and $G \sim G'$

(1) $$d(q''_{nm'}p''_{m'}g_{m'} , q''_{nm'}k_{m'}p'_{\alpha(m')}) < \epsilon/4 .$$

(2) $$d(q''_{nm'}p''_{m'}g'_{m'} , q''_{nm'}k'_{m'}p'_{\alpha(m')}) < \epsilon/4 .$$

(3) $$d(q''_{nm}k_m q'_{\alpha(m)\alpha(m')} , q''_{nm'}k_{m'}) < \epsilon/4 .$$

(4) $$d(q''_{nm}k'_m q'_{\alpha(m)\alpha(m')} , q''_{nm'}k'_{m'}) < \epsilon/4 .$$

$$\begin{tikzcd}
 & E_{\beta\alpha(m')} \arrow[rr, "f_{\alpha(m')}"] \arrow[ddd] & & E'_{\alpha(m')} \arrow[rr, shift left, "g_{m'}"] \arrow[rr, shift right, "g'_{m'}"'] \arrow[ddd, "p'_{\alpha(m')}"] & & E''_{m'} \arrow[ddd, "p''_{m'}"] & & B''_{n} \\
E_{\beta(\ell')} \arrow[ur, "r"] \arrow[rr, "f_{\ell'}"] \arrow[ddd, "p_{\beta(\ell')}"] & & E'_{\ell'} \arrow[ur] \arrow[ddd, "p'_{\ell'}"] & & & & & \\
 & & & & B'_{\alpha(m)} \arrow[rr, dashed, shift left, "k_{m}"] \arrow[rr, dashed, shift right, "k'_{m}"'] & & B''_{m} \arrow[uur] & \\
 & B_{\beta\alpha(m')} \arrow[rr] & & B'_{\alpha(m')} \arrow[ur, dashed] \arrow[rr, shift left, "k_{m'}"] \arrow[rr, shift right, "k'_{m'}"'] & & B''_{m'} \arrow[ur, "q''_{mm'}"'] & & \\
B_{\beta(\ell')} \arrow[ur, "q"] \arrow[rr, "h_{\ell'}"'] & & B'_{\ell'} \arrow[ur, "q'_{\alpha(m')\ell}"'] & & & & &
\end{tikzcd}$$

There are homotopies

$$L' : g_{m'} r'_{\alpha(m')\ell'} \simeq g'_{m'} r'_{\alpha(m')\ell'}$$

$$M' : k_{m'} q'_{\alpha(m')\ell'} \simeq k'_{m'} q'_{\alpha(m')\ell'}$$

such that for every $s \in I$

$$d(q''_{nm'} k_{m'} q'_{\alpha(m')\ell'} \, , \, q''_{nm'} M'_{s}) < \epsilon/4 \tag{5}$$

and

$$d(q''_{nm'} p''_{m'} L'_{s} \, , \, q''_{nm'} M'_{s} p'_{\ell'}) < \epsilon/4 \, . \tag{6}$$

Since $\ell' \geq F(\alpha(m'),\lambda)$ and $\alpha(m') \geq F(\alpha(m),\delta)$ by definition of F,

$$d(q'_{\alpha(m')\ell'} p'_{\ell'} f_{\ell'} \, , \, q'_{\alpha(m')\ell'} p_{\beta(\ell')}) < \lambda \tag{7}$$

and

$$d(q'_{\alpha(m)\alpha(m')} p'_{\alpha(m')} K_{s} \, , \, q'_{\alpha(m)\alpha(m')} H_{s} p_{\beta(\ell')}) < \delta \tag{8}$$

$$d(q'_{\alpha(m)\alpha(m')} h_{\alpha(m')} q_{\beta\alpha(m')\beta(\ell')} \, , \, q'_{\alpha(m)\alpha(m')} H_{s}) < \delta \tag{8A}$$

since $\alpha(m') \geq (G,G')(m;n,\epsilon/4)$

(8B) $$d(q''_{nm}k_m q'_{\alpha(m)\alpha(m')}\ ,\ q''_{nm}k'_m q'_{\alpha(m)\alpha(m')}) < \epsilon/4$$

for every $s \in I$, where

$$K : f_{\alpha(m')}r_{\beta\alpha(m')\beta(\ell')} \simeq r'_{\alpha(m')\ell'}f_{\ell'}$$

$$H : h_{\alpha(m')}q_{\beta\alpha(m')\beta(\ell')} \simeq q'_{\beta(m')\ell'}h_{\ell'} \quad .$$

Define homotopies

$$L : g_{m'}f_{\alpha(m')}r_{\beta\alpha(m')\beta(\ell')} \simeq g'_{m'}f_{\alpha(m')}r_{\beta\alpha(m')\beta(\ell')}$$

$$M : k_{m'}h_{\alpha(m')}q_{\beta\alpha(m')\beta(\ell')} \simeq k'_{m'}h_{\alpha(m')}q_{\beta\alpha(m')\beta(\ell')}$$

by

$$L(x,t) = \begin{cases} g_{m'}K(x,3t) & 0 \le t \le \frac{1}{3} \\ L'(f_{\ell'}(x),3t-1) & \frac{1}{3} \le t \le \frac{2}{3} \\ g'_{m'}K(x,3t-2) & \frac{2}{3} \le t \le 1 \end{cases}$$

$$M(y,t) = \begin{cases} k_{m'}H(y,3t) & 0 \le t \le \frac{1}{3} \\ M'(h_{\ell'}(y),3t-1) & \frac{1}{3} \le t \le \frac{2}{3} \\ k'_{m'}H(y,3t-2) & \frac{2}{3} \le t \le 1 \ . \end{cases}$$

where $x \in E_{\beta(\ell')}$, $y \in B_{\beta(\ell')}$.

We want to show that for every $t \in I$

(9) $$d(q''_{nm'}k_{m'}h_{\alpha(m')}q_{\beta\alpha(m')\beta(\ell')}\ ,\ q''_{nm'}M_t) < \epsilon$$

and

(10) $$d(q''_{nm'}p''_{m'}L_t\ ,\ q''_{nm'}M_t p_{\beta(\ell')}) < \epsilon \ .$$

<u>Let $0 \le t \le \frac{1}{3}$</u>

By (8A) and a choice of δ , for every $t \in [0,\frac{1}{3}]$

(11) $$d(q''_{nm}k_m q'_{\alpha(m)\alpha(m')}h_{\alpha(m')}q_{\beta\alpha(m')\beta(\ell')}\ ,\ q''_{nm}k_m q'_{\alpha(m)\alpha(m')}H_{3t}) < \epsilon/4.$$

By (3)

(12) $$d(q''_{nm}k_m q'_{\alpha(m)\alpha(m')}h_{\alpha(m')}q_{\beta\alpha(m')\beta(\ell')}, q''_{nm'}k_{m'}h_{\alpha(m')}q_{\beta\alpha(m')\beta(\ell')}) < \epsilon/4$$

and

(13) $$d(q''_{nm}k_m q'_{\alpha(m)\alpha(m')}H_{3t}\ ,\ q''_{nm'}k_{m'}H_{3t}) < \epsilon/4\ ,\ \forall\ t \in [0,\tfrac{1}{3}]\ .$$

By (12), (11) and (13)

$$d(q''_{nm'}k_{m'}h_{\alpha(m')}q_{\beta\alpha(m')\beta(\ell')}\ ,\ q''_{nm'}k_{m'}H_{3t}) < 3\epsilon/4$$

which is (9) since $k_{m'}H_{3t} = M_t$ for every $t \in [0,\frac{1}{3}]$.

Now for every t, by (1)

(14) $$d(q''_{nm'}p''_{m'}g_{m'}K_{3t}\ ,\ q''_{nm'}k_{m'}p'_{\alpha(m')}K_{3t}) < \epsilon/4\ .$$

By (8) and the choice of δ ,

(15) $$d(q''_{nm}k_m q'_{\alpha(m)\alpha(m')}p'_{\alpha(m')}K_{3t}, q''_{nm}k_m q'_{\alpha(m)\alpha(m')}H_{3t}p_{\beta(\ell')}) < \epsilon/4\ .$$

By (3),

(16) $$d(q''_{nm}k_m q'_{\alpha(m)\alpha(m')}p'_{\alpha(m')}K_{3t}\ ,\ q''_{nm'}k_{m'}p'_{\alpha(m')}K_{3t}) < \epsilon/4$$

and

(17) $$d(q''_{nm}k_m q'_{\alpha(m)\alpha(m')}H_{3t}p_{\beta(\ell')}\ ,\ q''_{nm'}k_{m'}H_{3t}p_{\beta(\ell')}) < \epsilon/4\ .$$

By (14), (16), (15) and (17)

$$d(q''_{nm'}p''_{m'}g_{m'}K_{3t}\ ,\ q''_{nm'}k_{m'}H_{3t}p_{\beta(\ell')}) < \epsilon$$

which is (10) for every $t \in [0,\frac{1}{3}]$.

<u>Let $\frac{1}{3} \leq t \leq \frac{2}{3}$</u>

For every t we have , by (5)

(18) $$d(q''_{nm'}k_{m'}q'_{\alpha(m')\ell'}h_{\ell'}\ ,\ q''_{nm'}M'_{3t-1}h_{\ell'}) < \epsilon/4\ .$$

By (3)

(19) $$d(q''_{nm}k_m q'_{\alpha(m)\alpha(m')}q'_{\alpha(m')\ell'}h_{\ell'}\ ,\ q''_{nm'}k_{m'}q'_{\alpha(m')\ell'}h_{\ell'}) < \epsilon/4$$

and

(20) $$d(q''_{nm}k_m q'_{\alpha(m)\alpha(m')}h_{\alpha(m')}q_{\beta\alpha(m')\beta(\ell')}\ ,\ q''_{nm'}k_{m'}h_{\alpha(m')}q_{\beta\alpha(m')\beta(\ell')}) < \epsilon/4.$$

For $s = 1$, by (8A) and by the choice of δ

(21) $$d(q''_{nm}k_m q'_{\alpha(m)\alpha(m')}h_{\alpha(m')}q_{\beta\alpha(m')\beta(\ell')}, q''_{nm}k_m q'_{\alpha(m)\alpha(m')}q''_{\alpha(m')\ell'}h_{\ell'}) < \epsilon/4.$$

By (20), (21), (19) and (18),

$$d(q''_{nm'}k_{m'}h_{\alpha(m')}q_{\beta\alpha(m')\beta(\ell')}\,,\ q''_{nm'}M'_{3t-1}h_{\ell'}) < \epsilon$$

for every $t \in [\frac{1}{3},\frac{2}{3}]$ which is (9).

Now by (6)

$$d(q''_{nm'}p''_{m'}L'_{3t-1}f_{\ell'}\,,\ q''_{nm'}M'_{3t-1}p'_{\ell'}f_{\ell'}) < \epsilon/4\,. \tag{22}$$

By (7) and the choice of λ

$$d(q''_{nm'}k_{m'}q'_{\alpha(m')\ell'}p'_{\ell'}f_{\ell'}\,,\ q''_{nm'}k_{m'}q'_{\alpha(m')\ell'}h_{\ell'}p_{\beta(\ell')}) < \epsilon/4\,. \tag{23}$$

By (5)

$$d(q''_{nm'}k_{m'}q'_{\alpha(m')\ell'}p'_{\ell'}f_{\ell'}\,,\ q''_{nm'}M'_{3t-1}p'_{\ell'}f_{\ell'}) < \epsilon/4 \tag{24}$$

and

$$d(q''_{nm'}k_{m'}q'_{\alpha(m')\ell'}h_{\ell'}p_{\beta(\ell')}\,,\ q''_{nm'}M'_{3t-1}h_{\ell'}p_{\beta(\ell')}) < \epsilon/4\,. \tag{25}$$

By (22), (24), (23) and (25)

$$d(q''_{nm'}p''_{m'}L'_{3t-1}f_{\ell'}\,,\ q''_{nm'}M'_{3t-1}h_{\ell'}p_{\beta(\ell')}) < \epsilon$$

for every $t \in [\frac{1}{3},\frac{2}{3}]$ which is (10).

Let $\frac{2}{3} \le t \le 1$

For every t, by (3)

$$d(q''_{nm}k_m q'_{\alpha(m)\alpha(m')}h_{\alpha(m')}q_{\beta\alpha(m')\beta(\ell')}, q''_{nm'}k_{m'}h_{\alpha(m')}q_{\beta\alpha(m')\beta(\ell')}) < \epsilon/4. \tag{26}$$

By (4)

$$d(q''_{nm}\ k'_m q'_{\alpha(m)\alpha(m')}H_{3t-2}\,,\ q''_{nm'}k'_{m'}H_{3t-2}) < \epsilon/4\,. \tag{27}$$

By (8A) and the choice of δ,

$$d(q''_{nm}k_m q'_{\alpha(m)\alpha(m')}h_{\alpha(m')}q_{\beta\alpha(m')\beta(\ell')}, q''_{nm}k_m q'_{\alpha(m)\alpha(m')}H_{3t-2}) < \epsilon/4 \tag{28}$$

and

$$d(q''_{nm}k_m q'_{\alpha(m)\alpha(m')}H_{3t-2}\,,\ q''_{nm}k'_m q'_{\alpha(m)\alpha(m')}H_{3t-2}) < \epsilon/4\,. \tag{29}$$

Now, by (26), (28), (29), (27)

$$d(q''_{nm'}k_{m'}h_{\alpha(m')}q_{\beta\alpha(m')\beta(\ell')}\,,\ q''_{nm'}k'_{m'}H_{3t-2}) < \epsilon$$

which is (9) since $k'_{m'}H_{3t-2} = M_t$.

Now, by (2) for every t,

(30) $$d(q''_{nm'}p''_{m'}g'_{m'}K_{3t-2}\ ,\ q''_{nm'}k'_{m'}p'_{\alpha(m')}K_{3t-2}) < \epsilon/4$$

by (8) and the choice of δ

(31) $$d(q''_{nm}k'_{m}q'_{\alpha(m)\alpha(m')}p'_{\alpha(m')}K_{3t-2}\ ,\ q''_{nm}k'_{m}q'_{\alpha(m)\alpha(m')}H_{3t-2}p_{\beta(\ell')}) < \epsilon/4.$$

By (4)

(32) $$d(q''_{nm}k'_{m}q'_{\alpha(m)\alpha(m')}p'_{\alpha(m')}K_{3t-2}\ ,\ q''_{nm'}k'_{m'}p'_{\alpha(m')}K_{3t-2}) < \epsilon/4$$

and

(33) $$d(q''_{nm}k'_{m}q'_{\alpha(m)\alpha(m')}H_{3t-2}p_{\beta(\ell')}\ ,\ q''_{nm}k'_{m'}H_{3t-2}p_{\beta(\ell')}) < \epsilon/4 .$$

By (30), (32), (31) and (33)

$$d(q''_{nm'}p''_{m'}g'_{m'}K_{3t-2}\ ,\ q''_{nm}k'_{m'}H_{3t-2}p_{\beta(\ell')}) < \epsilon$$

which is (10).

<u>Definition 3.3</u>: Two level maps $(\underline{E},\underline{p},\underline{B})$ and $(\underline{E}',\underline{p}',\underline{B}')$ are equivalent (in symbol $(\underline{E},\underline{p},\underline{B}) \cong (\underline{E}',\underline{p}',\underline{B}')$) if there are morphisms

$F : (\underline{E},\underline{p},\underline{B}) \to (\underline{E}',\underline{p}',\underline{B}')$ and

$G : (\underline{E}',\underline{p}',\underline{B}') \to (\underline{E},\underline{p},\underline{B})$ of level maps such that

$GF \sim 1_{(\underline{E},\underline{p},\underline{B})}$ and

$FG \sim 1_{(\underline{E}',\underline{p}',\underline{B}')}$.

<u>Proposition 3.6</u>: $\cong$ is an equivalence relation on the set of level maps of ANR-sequences.

<u>Proof</u>: Only transitivity requires proof.

Let $(\underline{E},\underline{p},\underline{B})$, $(\underline{E}',\underline{p}',\underline{B}')$ and $(\underline{E}'',\underline{p}'',\underline{B}'')$ be level maps of ANR-sequences such that

$$(\underline{E},\underline{p},\underline{B}) \cong (\underline{E}',\underline{p}',\underline{B}') \text{ and } (\underline{E}',\underline{p}',\underline{B}') \cong (\underline{E}'',\underline{p}'',\underline{B}'').$$

There are morphisms of level maps

$$F : (\underline{E},\underline{p},\underline{B}) \to (\underline{E}',\underline{p}',\underline{B}')\ ;\ F' : (\underline{E}',\underline{p}',\underline{B}') \to (\underline{E}'',\underline{p}'',\underline{B}'')$$

$$G : (\underline{E}',\underline{p}',\underline{B}') \to (\underline{E},\underline{p},\underline{B})\ ;\ G' : (\underline{E}'',\underline{p}'',\underline{B}'') \to (\underline{E}',\underline{p}',\underline{B}')$$

such that $GF \sim 1_{\underline{p}}$, $FG \sim 1_{\underline{p}'}$ $G'F' \sim 1_{\underline{p}'}$, $F'G' \sim 1_{\underline{p}''}$. By Proposition 3.4, $GG'F' \sim G \cdot 1_{\underline{p}'} = G$ and by Proposition 3.5, $GG'F'F \sim GF \sim 1_{\underline{p}}$. Similarly, $F'FGG' \sim F'G' \sim 1_{\underline{p}''}$. Hence $F'F : \underline{p} \to \underline{p}''$ and $GG' : \underline{p}'' \to \underline{p}$ are the required morphisms so that

$$(\underline{E},\underline{p},\underline{B}) \cong (\underline{E}'',\underline{p}'',\underline{B}'') .$$

Now we can define the concept of fiber shape equivalence.

Definition 3.4: Let E, E', B be compact metric spaces. Two shape fibrations $p : E \to B$ and $p' : E' \to B$ are said to be *fiber shape equivalent* if their associated level maps $(\underline{E},\underline{p},\underline{B})$ and $(\underline{E}',\underline{p}',\underline{B}')$ respectively are equivalent. (A continuous function $f : E \to B$ is said to be associated with a level map $\underline{p} : \underline{E} \to \underline{B}$ if $\varprojlim \underline{p} = p$). The following theorem justifies the Definition 3.4.

Theorem 3.1: Let $(\underline{E},\underline{p},\underline{B})$ and $(\underline{E}',\underline{p}',\underline{B}')$ be level maps of ANR-sequences with

$$\varprojlim (\underline{E},\underline{p},\underline{B}) = \varprojlim (\underline{E}',\underline{p}',\underline{B}')$$

then

$$(\underline{E},\underline{p},\underline{B}) \cong (\underline{E}',\underline{p}',\underline{B}') .$$

First we will state one useful lemma from [10].

Lemma M: Let $\underline{X} = (X_n, r_{nm})$ be an ANR-sequence with $\varprojlim \underline{X} = (X, r_n)$ and let Y be a compact ANR. Then the following assertions hold:

i) For every $\epsilon > 0$ and for every map $f: X \to Y$ there is an n^* such that for each $n \geq n^*$ there is a map $f_n : X_n \to Y$ with

$$d(f_n r_n, f) < \epsilon .$$

ii) If $\epsilon > 0$ and $f_n, g_n : X_n \to Y$ are maps such that

$$d(f_n r_n, g_n r_n) < \epsilon$$

then there exists $\hat{n} \geq n$ such that

$$d(f_n r_{nm}, g_n r_{nm}) < \epsilon$$

for every $m \geq \hat{n}$.

Proof of the Theorem 3.1: By using the Lemma M we will construct morphisms

$$F = (\underline{f},\underline{h}), F' = (\underline{f}',\underline{h}') : (\underline{E},\underline{p},\underline{B}) \to (\underline{E}',\underline{p}',\underline{B}')$$

and

$$G = (\underline{g},\underline{k}) : (\underline{E}',\underline{p}',\underline{B}') \to (\underline{E},\underline{p},\underline{B})$$

such that

$$FG \sim 1_{\underline{p}'}$$

$$GF' \sim 1_{\underline{p}} \text{ and } F \sim F' .$$

Construction of F: Let $\epsilon_1 = 1$. By induction we will select numbers $(\epsilon_m, \delta_m; \tilde{\epsilon}_m, \tilde{\delta}_m)$ with the following properties:

a) For every n and $\epsilon > 0$ there is an index n^* such that for all $m \geq n^*$,

$$\epsilon_m \in \Lambda(q_{nm}; \epsilon) .$$

b) There are maps $\underline{f} = (\alpha, f_n) : \underline{E} \to \underline{E}'$ and $\underline{h} = (\alpha, h_n) : \underline{B} \to \underline{B}'$ of ANR-sequences such that

i) $d(r'_m, f_m r_{\alpha(m)}) < \tilde{\delta}_m /2$

ii) $d(q'_m, h_m q_{\alpha(m)}) < \delta_m/2$

iii) $d(p'_m f_m, h_m p_{\alpha(m)}) < \delta_m$

and

iv) for every $m' \geq m$ there are $\tilde{\epsilon}_m$- and $\epsilon_{m/3} \sim$ homotopies

$$K : f_m \cdot r_{\alpha(m)\alpha(m')} \simeq r_{mm'} f_{m'}$$

$$H : h_m \cdot q_{\alpha(m)\alpha(m')} \simeq q_{mm'} h_{m'}$$

respectively such that for every $t \in I$

$$d(p'_m K_t , H_t p_{\alpha(m')}) < \epsilon_m .$$

Note that, since $\epsilon_m \in \Lambda(q_{nm},\epsilon)$, by (iii) and (iv) for every $m \geq n^*$

$$d(q_{nm}p'_m f_m , q_{nm}h_m p_{\alpha(m)}) < \epsilon$$

and for every $t \in I$

$$d(q_{nm}p'_m K_t , q_{nm}H_t p_{\alpha(m')}) < \epsilon$$

and so, $F = (\underline{f},\underline{h}) : \underline{p} \to \underline{p}'$ is a morphism of level maps.

Let $m = 1$.

Select $\delta_1 \in \Gamma(B'_1 , \epsilon_{1/3})$

$\tilde{\epsilon}_1 \in \Lambda(p'_1 , \delta_{1/3})$

$\tilde{\delta}_1 \in \Gamma(E'_1 , \tilde{\epsilon}_{1/2})$.

Now by Lemma M (i), for $\tilde{\delta}_{1/2}$ and $\delta_{1/2}$ there is an index n_1 and there are continuous functions $\tilde{f}_1 : E_{n_1} \to E'_1$ and $\tilde{h}_1 : B_{n_1} \to B'_1$ such that

(1) $d(r'_1,\tilde{f}_1 r_{n_1}) < \tilde{\delta}_{1/2}$

and

(2) $d(q'_1,\tilde{h}_1 q_{n_1}) < \delta_{1/2}$.

By the choice of $\tilde{\delta}_1$, (1) implies

(3) $d(p'_1 r'_1 , p'_1 \tilde{f}_1 r_{n_1}) < \delta_{1/3}$.

Since $p'_1 r'_1 = q'_1 p$, we have

(3') $d(q'_1 p , p'_1 \tilde{f}_1 r_{n_1}) < \delta_{1/3}$.

By (2)

(4) $d(q'_1 p , \tilde{h}_1 q_{n_1} p) < \delta_{1/2}$.

By (3'), (4) and $q_{n_1} p = p_{n_1} \cdot r_{n_1}$

(5) $d(q'_1 \tilde{f}_1 r_{n_1} , \tilde{h}_1 p_{n_1} r_{n_1}) < \delta_{1/2} + \delta_{1/3} < \delta_1$.

By Lemma M (ii) there is an index, say $\alpha(1)$ such that

(5') $d(p_1'\tilde{f}_1 r_{n_1\alpha(1)}, \tilde{h}_1 p_{n_1} \cdot r_{n_1\alpha(1)}) < \delta_1$.

Since $p_{n_1} \cdot r_{n_1\alpha(1)} = q_{n_1\alpha(1)} p_{\alpha(1)}$,

(5") $d(p_1'\tilde{f}_1 r_{n_1\alpha(1)}, \tilde{h}_1 q_{n_1\alpha(1)} p_{\alpha(1)}) < \delta_1$.

Write $\tilde{f}_1 \cdot r_{n_1\alpha(1)} = f_1$ and $\tilde{h}_1 \cdot q_{n_1\alpha(1)} = h_1$. Also since $r_{n_1} = r_{n_1\alpha(1)} \cdot r_{\alpha(1)}$, $q_{n_1} = q_{n_1\alpha(1)} \cdot q_{\alpha(1)}$, by (1), (2), (5")

(6) $d(r_1', f_1 r_{\alpha(1)}) < \tilde{\delta}_{1/2}$

(7) $d(q_1', h_1 q_{\alpha(1)}) < \delta_{1/2}$

(8) $d(p_1' f_1, h_1 p_{\alpha(1)}) < \delta_1$.

<u>Let $m = 2$</u>

Select $\epsilon_2 \in \Lambda(q_{12}', \delta_1/2^2)$

$\delta_2 \in \Gamma(B_2', \epsilon_{2/3})$

$\tilde{\epsilon}_2 \in \Lambda(p_2', \delta_{2/3}) \cap \Lambda(r_{12}', \tilde{\delta}_1/2^2)$

$\tilde{\delta}_2 \in \Gamma(E_2', \tilde{\epsilon}_{2/2})$.

As in the case $m = 1$, for $\tilde{\delta}_{2/2}$ and $\delta_{2/2}$ there is an index n_2 and there are maps $\tilde{f}_2 : E_{n_2} \to E_2'$ and $\tilde{h}_2 : B_{n_2} \to B_2'$ such that

(9) $d(r_2', \tilde{f}_2 r_{n_2}) < \tilde{\delta}_{2/2}$

(10) $d(q_2', \tilde{h}_2 q_{n_2}) < \delta_{2/2}$

and

(11) $d(p_2'\tilde{f}_2, \tilde{h}_2 p_{n_2}) < \delta_2$.

By the choices of $\tilde{\delta}_2$ and δ_2 , (9) and (10) imply

(12) $d(r'_{12}\, r'_2,\ r'_{12}\tilde{f}_2 r_{n_2}) < \tilde{\delta}_1/2^2$

(13) $d(q'_{12}\, q'_2,\ q'_{12}\, \tilde{h}_2 q_{n_2}) < \delta_1/2^2$.

Since $r'_{12}\cdot r'_2 = r'_1$ and $q'_{12}\cdot q'_2 = q'_1$

(12') $d(r'_1, r'_{12}\, \tilde{f}_2 r_{n_2}) < \tilde{\delta}_1/2^2$

(13') $d(q'_1,\ q'_{12}\tilde{h}_2 q_{n_2}) < \delta_1/2^2$.

By (6), (12')

(14) $d(f_1 r_{\alpha(1)}\ ,\ r'_{12}\, \tilde{f}_2 r_{n_2}) < \tilde{\delta}_{1/2} + \tilde{\delta}_1/2^2$.

By (7) and (13')

(15) $d(h_1 q_{\alpha(1)},\ q'_{12}\, \tilde{h}_2 q_{n_2}) < \delta_{1/2} + \delta_1/2^2$.

Since $r_{\alpha(1)} = r_{\alpha(1)n_2} \cdot r_{n_2}$ and $q_{\alpha(1)} = q_{\alpha(1)n_2} q_{n_2}$.

(14') $d(f_1 r_{\alpha(1)n_2} \cdot r_{n_2}, r'_{12}\, \tilde{f}_2 r_{n_2}) < \tilde{\delta}_1$

(15') $d(h_1 q_{\alpha(1)n_2}\, q_{n_2},\ q'_{12}\, \tilde{h}_2 q_{n_2}) < \delta_1$.

By Lemma M (ii) there is an index n_2^* such that for all $\alpha(2) \geq n_2^*$

(14") $d(f_1 \cdot r_{\alpha(1)n_2}\, r_{n_2\alpha(2)},\ r'_{12}\, \tilde{f}_2\, r_{n_2\alpha(2)}) < \tilde{\delta}_1$

(15") $d(h_1 q_{\alpha(1)n_2}\, q_{n_2\alpha(2)},\ q'_{12}\, \tilde{h}_2 q_{n_2\alpha(2)}) < \delta_1$.

Write $\tilde{f}_2 \cdot r_{n_2\alpha(2)} = f_2$ and $\tilde{h}_2 \cdot q_{n_2\alpha(2)} = h_2$. Since $r_{\alpha(1)n_2} \cdot r_{n_2\alpha(2)} = r_{\alpha(1)\alpha(2)}$, $q_{\alpha(1)n_2} \cdot q_{n_2\alpha(2)} = q_{\alpha(1)\alpha(2)}$, $r_{n_2} = r_{n_2\alpha(2)} \cdot r_{\alpha(2)}$, $q_{n_2} = q_{n_2\alpha(2)} \cdot q_{\alpha(2)}$ and $p_{n_2} r_{n_2\alpha(2)} = q_{n_2\alpha(2)} p_{\alpha(2)}$

by (9), (10), (11), (14"), (15")

(16) $d(r'_2 , f_2 r_{\alpha(2)}) < \tilde{\delta}_{2/2}$

(17) $d(q'_2 , h_2 q_{\alpha(2)}) < \delta_{2/2}$

(18) $d(p'_2 f_2 , h_2 p_{\alpha(2)}) < \delta_2$

(19) $d(f_1 r_{\alpha(1)\alpha(2)} , r'_{12} f_2) < \tilde{\delta}_1$

(20) $d(h_1 q_{\alpha(1)\alpha(2)} , q'_{12} h_2) < \delta_1$.

[<u>Note</u>: (11) $\Rightarrow$

$$d(p'_2 \tilde{f}_2 r_{n_2\alpha(2)} , \tilde{h}_2 p_{n_2} r_{n_2\alpha(2)}) < \delta_2$$

where $\tilde{h}_2 p_{n_2} r_{n_2\alpha(2)} = q_{n_2\alpha(2)} p_{\alpha(2)}$] . By the choices of $\tilde{\delta}_1$ and δ_1, there are $\tilde{\epsilon}_1$- and $\epsilon_{1/3}$-homotopies;

$$K : f_1 r_{\alpha(1)\alpha(2)} \simeq r'_{12} f_2$$

$$H : h_1 q_{\alpha(1)\alpha(2)} \simeq q'_{12} h_2$$

such that for any pairs (t',t) and (s',s), $(t,t',s,s' \in I)$

$$d(K_{t'},K_t) < \tilde{\epsilon}_1$$

and

$$d(H_{s'},H_s) < \epsilon_{1/3} .$$

In particular, for $t' = 0$, $s' = 0$, for every $t \in I$

(21) $d(f_1 r_{\alpha(1)\alpha(2)} , K_t) < \tilde{\epsilon}_1$

(22) $d(h_1 q_{\alpha(1)\alpha(2)} , H_t) < \epsilon_{1/3}$.

By the choice of $\tilde{\epsilon}_1$, for every $t \in I$

(21') $d(p'_1 f_1 r_{\alpha(1)\alpha(2)}, p'_1 K_t) < \delta_{1/3} < \epsilon_{1/3}$.

By (22), for every $t \in I$

(22') $d(h_1 q_{\alpha(1)\alpha(2)} p_{\alpha(2)} , H_t p_{\alpha(2)}) < \epsilon_{1/3}$.

Also, by (8)

$$(23)\quad d(p_1'f_1r_{\alpha(1)\alpha(2)}\ ,\ h_1p_{\alpha(1)}r_{\alpha(1)\alpha(2)}) < \delta_1 < \epsilon_{1/3}$$

since $p_{\alpha(1)}r_{\alpha(1)\alpha(2)} = q_{\alpha(1)\alpha(2)}p_{\alpha(2)}$, by (21'), (22') and (23), for every $t \in I$

$$(24)\quad d(p_1'K_t\ ,\ H_tp_{\alpha(2)}) < \epsilon_1\ .$$

Now, for any $m \geq 3$ select

$$\epsilon_m \in \bigcap_{n=1}^{m-1} \Lambda(q_{nm}'\ ,\ \delta_{n/2}\ m - n + 1)$$

$$\delta_m \in \Gamma(B_m'\ ,\ \epsilon_{m/3})$$

$$\tilde{\epsilon}_m \in \Lambda(p_m'\ ,\ \delta_{m/3}) \cap \bigcap_{n=1}^{m-1} \Lambda(r_{nm}';\ \tilde{\delta}_{n/2}\ m - n + 1)$$

$$\tilde{\delta}_m \in \Gamma(E_m'\ ,\ \tilde{\epsilon}_{m/2})\ .$$

By the following remark (vii) it is clear that $F = (\underline{f},\underline{h}) : \underline{p} \to \underline{p}'$ where $\underline{f} = (\alpha, f_n)$, $\underline{h} = (\alpha, h_n)$ is a morphism of level maps.

<u>Note</u> that for any n and any $\epsilon > 0$, there is n^* such that for all $m \geq n^*$, $\delta_{n/2}\ m - n + 1 < \epsilon$. Then $\epsilon_m \in \Lambda(q_{nm}', \epsilon)$.

<u>Remark (vii)</u>: Let all the spaces in the diagram be compact ANR's and arrows be maps such that $p_1'r_{12}' = q_{12}'p_2'$ and $p_2r_{23} = q_{23}p_3$.

$$\begin{tikzcd}
 & & E_1 \arrow[rrr, "f_1"] \arrow[ddd, dashed] & & & E_1' \arrow[ddd, "p_1'"] \\
 & E_2 \arrow[ur] \arrow[rrr, "f_2"] \arrow[ddd, dashed, "p_2"] & & & E_2' \arrow[ur] \arrow[ddd, "p_2'"] & \\
E_3 \arrow[ur] \arrow[rrr, "f_3"] \arrow[ddd, "p_3"] & & & E_3' \arrow[ur] \arrow[ddd, "p_3'"] & & \\
 & & B_1 \arrow[rrr, dashed, "h_1"] & & & B_1' \\
 & B_2 \arrow[ur, dashed] \arrow[rrr, dashed, "h_2"] & & & B_2' \arrow[ur, dashed] & \\
B_3 \arrow[ur, dashed] \arrow[rrr, "h_3"] & & & B_3' \arrow[ur, dashed] & &
\end{tikzcd}$$

Also, let there be homotopies

$$K' : f_1 r_{12} \simeq r'_{12} f_2$$

$$H' : h_1 q_{12} \simeq q'_{12} h_2$$

$$K'' : f_2 r_{23} \simeq r'_{23} f_3$$

$$H'' : h_2 q_{23} \simeq q'_{23} h_3$$

such that for every $t \in I$

(1) $d(p'_1 K'_t \ , \ H'_t p_2) < \epsilon_1$

(2) $d(p'_2 K''_t \ , \ H''_t p_3) < \epsilon_2$

where $\epsilon \in \Lambda(q'_{12} \ , \ \epsilon_1)$ then we can define homotopies

$$K : f_1 r_{13} \simeq r'_{13} f_3$$

$$H : h_1 q_{13} \simeq q'_{13} h_3$$

by

$$K(x,t) = \begin{cases} K'(r_{23}(x), 2t) & 0 \le t \le \frac{1}{2} \\ r'_{12} K''(x, 2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

and

$$H(x,t) = \begin{cases} H'(q_{23}(x), 2t) & 0 \le t \le \frac{1}{2} \\ q'_{12} H''(x, 2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

such that for every $t \in I$

$$d(p'_1 K_t \ , \ H_t p_3) < \epsilon_1 \ .$$

<u>Check</u>: <u>Let $0 \le t \le \frac{1}{2}$</u>

By (1), $d(p'_1 K'_{2t} r_{23} \ , \ H'_{2t} p_2 r_{23}) < \epsilon_1$. since $p_2 \cdot r_{23} = q_{23} p_3$

$$d(p'_1 k'_{2t} r_{23} \ , \ H'_{2t} q_{23} p_3) < \epsilon_1$$

i.e. $d(p'_1 K_t \ , \ H_t p_3) < \epsilon_1$ for every $t \in [0, \frac{1}{2}]$.

Let $\frac{1}{2} \le t \le 1$

By (2)

$$d(q'_{12}p'_2K''_t \, , \, q'_{12}H''_tp_3) \; < \; \epsilon_1 \; .$$

Since $q'_{12}p'_2 = p'_1r'_{12}$

$$d(p'_1r'_{12}K''_t \, , \, q'_{12}H''_tp_3) \; < \; \epsilon_1 \; ;$$

i.e. $d(p'_1K_t \, , \, H_tp_3) \; < \; \epsilon_1$ for every $t \in [\frac{1}{2},1]$.

Construction of G: For every n, similar to the numbers $(\epsilon_n, \delta_n, \tilde{\epsilon}_n, \tilde{\delta}_n)$, we can select numbers $(\lambda_n, \mu_n, \tilde{\lambda}_n, \tilde{\mu}_n)$ with respect to the level map $\underline{p} : \underline{E} \to \underline{B}$ with the following additional properties: For every $n = \alpha(m)$,

(C)
$$\lambda_{\alpha(m)} \in \Lambda\,(h_m, \delta_{m/2})$$

$$\mu_{\alpha(m)} \in \Gamma\,(B_{\alpha(m)}, \lambda_{\alpha(m)/3})$$

$$\tilde{\lambda}_{\alpha(m)} \in \Lambda\,(f_m, \tilde{\delta}_{m/2})$$

and

$$\tilde{\mu}_{\alpha(m)} \in \Gamma\,(E_{\alpha(m)}, \tilde{\lambda}_{\alpha(m)/2}) \; .$$

Also, the numbers are so selected that if we construct a morphism, say $G = (\underline{g},\underline{k}) : \underline{p}' \to \underline{p}$ where $\underline{g} = (\beta, g_n) : \underline{E}' \to \underline{E}$ and $\underline{k} = (\beta, k_m) : \underline{B}' \to \underline{B}$ similar to F , then $FG = (\underline{fg}, \underline{hk}) : \underline{p}' \to \underline{p}'$ is a morphism of level maps, i.e. for given n and $\epsilon > 0$ there is n^* such that for all $m \ge n^*$,

$$d(p'_mf_mg_{\alpha(m)} \, , \, h_mk_{\alpha(m)}p'_{\beta\alpha(m)}) \; < \; \epsilon_m$$

and for every $t \in I$,

$$d(h_mk_{\alpha(m)}q'_{\beta\alpha(m)\beta\alpha(m')} \, , \, H_t) \; < \; \epsilon_m$$

$$d(p'_mK_t \, , \, H_tp'_{\beta\alpha(m')}) \; < \; \epsilon_m$$

where
$$H = H' \cup H''$$
$$K = K' \cup K''$$
$$K' : f_m r_{\alpha(m)\alpha(m')} \simeq r'_{mm'} f_{m'}$$
$$K'' : g_{\alpha(m)} r'_{\beta\alpha(m)\beta\alpha(m')} \simeq r_{\alpha(m)\alpha(m')} g_{\alpha(m')}$$
$$H' = h_m q_{\alpha(m)\alpha(m')} \simeq q'_{mm'} h_{m'}$$
$$H'' = k_{\alpha(m)} q'_{\beta\alpha(m)\beta\alpha(m')} \simeq q_{\alpha(m)\alpha(m')} k_{m'}$$

$$\begin{tikzcd}
 & E'_{\beta\alpha(m)} \arrow[rr, "g_{\alpha(m)}"] \arrow[dd, dashed, "p'_{\beta\alpha(m)}"] & & E_{\alpha(m)} \arrow[rr, "f_m"] \arrow[dd, dashed] & & E'_m \arrow[dd, "p'_m"] & \\
E'_{\beta\alpha(m')} \arrow[ur] \arrow[rr] \arrow[dd, "p'_{\beta\alpha(m')}"] & & E_{\alpha(m')} \arrow[ur] \arrow[rr] \arrow[dd] & & E'_{m'} \arrow[ur] \arrow[dd] & & B'_n \\
 & B'_{\beta\alpha(m)} \arrow[rr, dashed] & & B_{\alpha(m)} \arrow[rr, dashed] & & B'_m \arrow[ur, dashed] & \\
B_{\beta\alpha(m')} \arrow[ur] \arrow[rr, "k_{\alpha(m')}"] & & B_{\alpha(m')} \arrow[ur, dashed] \arrow[rr, "h_{m'}"] & & B'_{m'} \arrow[ur, dashed, "q'_{mm'}"'] & &
\end{tikzcd}$$

Now we will show that $FG \sim 1_{\underline{P}'}$.

For given n and $\epsilon > 0$ let n' be an index such that for all $m \geq n'$,
$$\epsilon_m \in \Lambda\,(q'_{nm}, \epsilon) .$$

By construction of F and G, we have

(1) $$d(r'_m\,,\ f_m r_{\alpha(m)}) < \tilde{\delta}_{m/2} \quad \text{where}$$
$$\tilde{\delta}_m \in \Gamma\,(E'_m, \tilde{\epsilon}_{m/2})$$
$$\tilde{\epsilon}_m \in \Lambda\,(p'_m, \delta_{m/3} \cap \bigcap_{n=1}^{m-1} \Lambda\,(r'_{nm}; \tilde{\delta}_{n/2^{m-n+1}}) .$$

(2) $d(r_{\alpha(m)}, g_{\alpha(m)} r'_{\beta\alpha(m)}) < \tilde{\mu}_{\alpha(m)/2}$

where $\tilde{\mu}_{\alpha(m)} \in \Gamma\ (E_{\alpha(m)} , \tilde{\lambda}_{\alpha(m)/2})$

$\tilde{\lambda}_{\alpha(m)} \in \Lambda\ (f_m , \tilde{\delta}_{m/2} \cap \Lambda\ (p_{\alpha(m)} , \lambda_{m/3}) \cap \cap \ldots$

(3) $d(q'_m , h_m q_{\alpha(m)}) < \delta_{m/2}$

where $\delta_m \in \Gamma\ (B'_m , \epsilon_{m/3})$.

(4) $d(q_{\alpha(m)}, k_{\alpha(m)} q'_{\beta\alpha(m)}) < \mu_{\alpha(m)/2}$

where $\mu_{\alpha(m)} \in \Gamma\ (B_{\alpha(m)}, \lambda_{\alpha(m)/3})$

$\lambda_{\alpha(m)} \in \Lambda\ (h_m , \delta_{m/2}) \cap \ \cap \Lambda\ (q_{n\alpha(m)}, \ldots)$.

By the choice of $\tilde{\mu}_{\alpha(m)}$, (2) implies

(5) $d(f_m r_{\alpha(m)}, f_m g_{\alpha(m)} r'_{\beta\alpha(m)}) < \tilde{\delta}_{m/2}$

$$\begin{tikzcd}
 & & E \arrow[dl, "r'_{\beta\alpha(m)}"'] \arrow[d, "r_{\alpha(m)}"] \arrow[dr, "r'_m"] & & \\
E'_{\ell} \arrow[r] \arrow[d, "p'_{\ell}"] \arrow[rrr, dashed, bend right, "r'_{m\ell}"'] & E'_{\beta\alpha(m)} \arrow[r, "g_{\alpha(m)}"] \arrow[d] & E_{\alpha(m)} \arrow[r, "f_m"] \arrow[d] & E'_m \arrow[d, "p'_m"] & B'_n \\
B'_{\ell} \arrow[r] \arrow[rrr, dashed, bend left, "q'_{m\ell}"] & B'_{\beta\alpha(m)} \arrow[r, "k_{\alpha(m)}"] & B_{\alpha(m)} \arrow[r, "h_m"] \arrow[d, "q_{\alpha(m)}"] & B'_m \arrow[ur] & \\
 & & B \arrow[ul, bend left, "q'_{\beta\alpha(m)}"] \arrow[ur, bend right, "q'_m"'] & &
\end{tikzcd}$$

By (1) and (5)

(6) $$d(r'_m , f_m g_{\alpha(m)} r'_{\beta\alpha(m)}) < \tilde{\delta}_m$$

since $r'_m = r'_{m,\beta\alpha(m)} \cdot r'_{\beta\alpha(m)}$.

(6') $$d(r'_{m,\beta\alpha(m)} \cdot r'_{\beta\alpha(m)} , f_m g_{\alpha(m)} r'_{\beta\alpha(m)}) < \tilde{\delta}_m .$$

Similarly by the choice of $\mu_{\alpha(m)}$, (4) implies

(7) $$d(h_m q_{\alpha(m)}, h_m k_{\alpha(m)} q'_{\beta\alpha(m)}) < \delta_{m/2} .$$

By (3) and (7) and $q'_m = q'_{m,\beta\alpha(m)} \cdot q'_{\beta\alpha(m)}$

(8) $$d(q'_{m,\beta\alpha(m)} \cdot q'_{\beta\alpha(m)} , h_m k_{\alpha(m)} q'_{\beta\alpha(m)}) < \delta_m .$$

By Lemma M (ii), there is an index m^* such that for all $\ell \geq m^*$

(9) $$d(r'_{m\ell} , f_m g_{\alpha(m)} r'_{\beta\alpha(m)\ell}) < \tilde{\delta}_m \text{ and}$$

(10) $$d(q'_{m\ell} , h_m k_{\alpha(m)} q'_{\beta\alpha(m)\ell}) < \delta_m .$$

Since $\delta_m \in \Gamma(B'_m, \epsilon_{m/3})$ and $\tilde{\delta}_m \in \Gamma(E'_m, \tilde{\epsilon}_{m/2})$,there are $\tilde{\epsilon}_{m/2}$- and $\epsilon_{m/2}$-homotopies

$$L : r'_{m\ell} \simeq f_m g_{\alpha(m)} r'_{\beta\alpha(m)\ell}$$

$$M : q'_{m\ell} \simeq h_m k_{\alpha(m)} q'_{\beta\alpha(m)\ell} .$$

Hence for every $t \in I$,

(11) $$d(r'_{m\ell} , L_t) < \tilde{\epsilon}_{m/2} \text{ and}$$

(12) $$d(q'_{m\ell} , M_t) < \epsilon_{m/2} .$$

Since $\tilde{\epsilon}_m \in \Lambda(p'_m ; \delta_{m/3})$, (11) implies

(13) $$d(p'_m r'_{m\ell} , p'_m L_t) < \delta_{m/3} < \epsilon_{m/3}$$

for every $t \in I$.

By (12)

(14) $$d(q'_{m\ell} p_\ell , M_t p_\ell) < \epsilon_{m/2} \text{ for every } t \in I .$$

By (13) and (14) and $p'_m r'_{m\ell} = q'_{m\ell} p_\ell$,

(15) $\quad d(p'_m L_t , M_t p_\ell) < \epsilon_m$ for every $t \in I$.

Since $\epsilon_m \in \Lambda\ (q'_{nm}, \epsilon)$, for every $t \in I$ by (12) $d(q'_{n\ell}, q'_{nm} M_t) < \epsilon$ and $d(q'_{nm} p'_m L_t , q'_{nm} M_t p_\ell) < \epsilon$. Also, for every $m \geq n'$ there is such ℓ and such homotopies L, M, $FG \sim 1_{\underline{p}'}$. Similarly, we can construct a morphism $F' : \underline{p} \to \underline{p}'$ such that $GF' \sim 1_{\underline{p}}$ and $F \sim F'$.

For every n, similar to the numbers $(\epsilon_n, \delta_n; \tilde{\epsilon}_n, \tilde{\delta}_n)$, we can select numbers $(\epsilon'_n, \delta'_n; \tilde{\epsilon}'_n, \tilde{\delta}_n)$ with $\epsilon'_n < \epsilon_n$; $\delta'_n < \delta_n$

$\tilde{\epsilon}'_n < \tilde{\epsilon}_n$, $\tilde{\delta}'_n < \tilde{\delta}_n$ and similar to the morphisms F and G we can construct a morphism

$$F' = (\underline{f}', \underline{h}') : \underline{p} \to \underline{p}$$

where

$$\underline{f}' = (\alpha', f'_n) : \underline{E} \to \underline{E}'$$

$$\underline{h}' : (\alpha', h'_n) : \underline{B} \to \underline{B}'$$

such that the conditions (A), (B) and (C) are satisfied with appropriate changes. In other words, for conditions (a) and (b) replace ϵ_n, δ_n, $\tilde{\epsilon}_n$, $\tilde{\delta}_n$ by ϵ'_n, δ'_n, $\tilde{\epsilon}'_n$, $\tilde{\delta}'_n$ respectively. $\alpha(m)$, f_m, h_m by $\alpha'_{(m)}$, f'_m, h'_m respectively. $\forall$ m and for condition (c) replace $\lambda_{\alpha(m)}$, $\mu_{\alpha(m)}$, $\tilde{\lambda}_{\alpha(m)}$, $\tilde{\mu}_{\alpha(m)}$ by $\epsilon'_{\beta(m)}$, $\delta_{\beta(m)}$, $\tilde{\epsilon}'_{\beta(m)}$, $\tilde{\delta}'_{\beta(m)}$ and replace δ_m, $\tilde{\delta}_m$ by μ_m and $\tilde{\mu}_m$ respectively for every m .

Since F' satisfied the condition (c) by construction

$$GF' \sim 1_{\underline{p}} \ .$$

Now we have to prove that $F \sim F'$.

By conditions (a), for every n and $\epsilon > 0$ there exists indices n_1^* and n_2^* such that for all $m_1 \geq n_1^*$ and $m_2 \geq n_2^*$;

$$\epsilon_{m_1} \in \Lambda\ (q_{nm};\epsilon) \quad \text{and}$$

$$\epsilon'_{m_2} \in \Lambda\ (q_{nm_2};\epsilon) \ .$$

Let $n^* = (F,F')(n) = \text{Max}(n_1^*, n_2^*)$. Then for all $m \geq n^*$

(16) $\epsilon'_m \, , \, \epsilon_m \in \Lambda \, (q_{nm}, \epsilon)$.

Now conditions (b)(i) are

$$d(r'_m, \, f_m r_{\alpha(m)}) \; < \; \tilde{\delta}_{m/2}$$

$$d(r'_m, \, f'_m r_{\alpha'(m)}) \; < \; \tilde{\delta}'_{m/2} \; .$$

Hence $d(f_m r_{\alpha(m)}, \, f'_m r_{\alpha'(m)}) < \tilde{\delta}_m$. (Note $\tilde{\delta}'_m < \tilde{\delta}_m$) . By Lemma M (ii) there is an index ℓ , $\ell \geq \alpha(m), \, \alpha'(m)$ such that

(17) $d(f_m r_{\alpha(m)\ell} \, , \, f'_m r_{\alpha'(m)\ell}) \; < \; \tilde{\delta}_m$.

Similarly, by condition (b)(ii) we can find ℓ so large that

(18) $d(h_m q_{\alpha(m)\ell} \, , \, h'_m q_{\alpha'(m)\ell}) \; < \; \delta_m$.

By choice,

$$\tilde{\delta}_m \in \Gamma \; (E'_m, \, \tilde{\epsilon}_{m/2})$$

$$\delta_m \in \Gamma \; (B'_m, \, \epsilon_{m/2}) \quad .$$

There are $\tilde{\epsilon}_{m/2}$ - and $\epsilon_{m/2}$-homotopies

$$L' : f_m r_{\alpha(m)\ell} \; \simeq \; f'_m r_{\alpha'(m)\ell}$$

$$M' : h_m q_{\alpha(m)\ell} \; \simeq \; h'_m q_{\alpha'(m)\ell}$$

respectively. i.e., for every $t \in I$

(19) $d(f_m r_{\alpha(m)\ell} \, , \, L'_t) \; < \; \tilde{\epsilon}_{m/2}$ and

(20) $d(h_m q_{\alpha(m)\ell}, \, M'_t) \; < \; \epsilon_{m/2}$.

Since $\tilde{\epsilon}_m \in \Lambda \; (p'_m; \, \delta_{m/3})$, (19) implies

(21) $d(p'_m f_m r_{\alpha(m)\ell} \, , \, p'_m L'_t) \; < \; \delta_{m/3} \; < \; \epsilon_{m/3}$.

Also by (20),

(22) $d(h_m q_{\alpha(m)\ell} p_\ell \, , \, M'_t p_\ell) \; < \; \epsilon_{m/2}$.

Since $p'_m f_m r_{\alpha(m)\ell} = h_m q_{\alpha(m)\ell} p_\ell$, by (21) and (22)

(23) $d(p'_m L'_t, M'_t p_\ell) < \epsilon_m$.

Finally, by the choice of $\epsilon_m \in \Lambda (q_{nm}, \epsilon)$, (20)and (23) implies

$$d(q_{nm} h_m q_{\alpha(m)\ell} , q_{nm} M'_t) < \epsilon \quad \text{and}$$

$$d(q_{nm} p'_m L'_t , q_{nm} M'_t p_\ell) < \epsilon \quad \text{for every} \quad t \in I .$$

Hence $F \sim F'$. Note that $(FGF') \cdot G \sim FG \sim 1_{\underline{p}'}$ and $G \cdot (FGF') \sim GF' \sim 1_{\underline{p}}$. Hence $(\underline{E}, \underline{p}, \underline{B}) \cong (\underline{E}', \underline{p}', \underline{B}')$.

Example:

1) Consider in the plane $\mathbb{R}^2$, the closure c of the diagram of the function $y = \sin \frac{\pi}{x}$ for $0 < x \leq 1$ and let ℓ denote a simple arc with endpoints $(0,1)$, $(1,0)$ and with interior lying in $\mathbb{R}^2 - c$. Denote the resulting space by W.

Let S^1 be the unit circle in $\mathbb{R}^2$. The quotient map $p : W \to W/A \approx S^1$ is a shape fibration. $p' = 1_{S^1} : W/A \to W/A$ is also a shape fibration. $(A = \{(0,y) \mid -1 \leq y \leq 1\} \subset \mathbb{R}^2)$.

Claim: p and p' are fiber shape equivalent.

Consider annula $E_1 \supset E_2 \supset \ldots$ being neighborhoods of W in $\mathbb{R}^2$ shrinking to W. For every n, there is a retract $p_n : E_n \to W/A$. Then $\underline{p} = (p_n) : \underline{E} \to W/A$ is a level map.

Consider $\underline{p}' = (p' = 1_{S^1}) : W/A \to W/A$ to be a trivial level map.

Let $F = (\underline{f}) : \underline{p}' \to \underline{p}$ be a morphism of level maps where for each n , $f_n : S^1 \to E_n$ is a simple closed curve, $f_n \simeq f_{n+1}$ in E_n and for every $\epsilon > 0$ there exists $n^* \geq n$ such that for $\ell \geq m \geq n^*$.

$d(p_n H_t , 1_{S^1}) < \epsilon$ for every $t \in I$ where $H : f_\ell \simeq f_m$ in E_n.

Let $G = (\underline{g}) : \underline{p} \to \underline{p}'$ be a morphism of level maps where for each n, $g_n : E_n \to S^1$ is a retract of $f_n(S^1)$. Clearly $FG \sim 1_{\underline{p}}$ and $GF \sim 1_{\underline{p}'}$.

Section 4

STRONGLY EQUIVALENT MAPS INDUCE EQUIVALENT SHAPE FIBRATIONS

In this section we will define a notion called 'strongly equivalence' and will prove the main theorem of the paper that two strongly equivalent continuous functions induce fiber shape equivalent shape fibrations.

Definition 4.1: Two level maps

$$\underline{f}, \underline{g} : \underline{C} = (C_n, q'_{nm}) \to \underline{B} = (B_n, q_{nm})$$

of ANR-sequences are said to be strongly equivalent (in symbol $\underline{f} \simeq \underline{g}$) if for every n there is an index $n^* \geq n$ such that for all $m \geq n^*$ there is a homotopy

$$H : C_m \times I \to B_n$$

with $H_0 = f_n \cdot q'_{nm} = q_{nm} \cdot f_m$, $H_1 = g_n \cdot q'_{nm} = q_{nm} \cdot g_m$ and if $n' \geq n$,

$$H' : C_{m'} \times I \to B_{n'}$$

with $H'_0 = f_{n'} \cdot q'_{n'm'}$; $H'_1 = g_{n'} \cdot q'_{n'm'}$

then there is a homotopy

$$G : C_{m'} \times I \times I \to B_n$$

such that

$$G_0 = q_{nn'} \cdot H'$$

$$G_1 = H \cdot (q'_{mm'} \times 1_I)$$

and for every $x \in C_{m'}$; $t \in I$,

$$G(x,0,t) = q_{nm'} f_{m'}(x)$$

$$G(x,1,t) = q_{nm'} g_{m'}(x) \quad .$$

Proposition 4.1: The relation $\simeq$ of the Definition 4.1 is an equivalence relation.

Proof: Only transitivity requires proof. Let $\underline{f}, \underline{g}, \underline{h} : \underline{C} \to \underline{B}$ be level maps of ANR-sequences such that $\underline{f} \simeq \underline{g}$ and $\underline{g} \simeq \underline{h}$. For any n, let $n^*(\underline{f},\underline{h}) = \mathrm{Max}(n^*(\underline{f},\underline{g}), n^*(\underline{g},\underline{h}))$. Then for every $m \geq n^*(\underline{f},\underline{h})$ there is a homotopy

$$H : C_m \times I \to B_n$$

defined by

$$H(x,s) = \begin{cases} H^m(x,2s) & 0 \leq s \leq \tfrac{1}{2} \\ K^m(x,2s-1) & \tfrac{1}{2} \leq s \leq 1 \end{cases}$$

$x \in C_m$, $s \in I$ where $H^m, K^m : C_m \times I \to B_n$ with

$$H_0^m = f_n \cdot q'_{nm}$$

$$H_1^m = g_n \cdot q'_{nm} = K_0^m \quad \text{and}$$

$$K_1^m = h_n \cdot q'_{nm} \quad .$$

Also, for $n' \geq n$ let $H^{m'} : C_{m'} \times I \to B_{n'}$ be a homotopy defined by

$$H'(x,s) = \begin{cases} H^{m'}(x,2s) & 0 \leq s \leq \tfrac{1}{2} \\ K^{m'}(x,2s-1) & \tfrac{1}{2} \leq s \leq 1 \end{cases}$$

$x \in C_{m'}$, $s \in I$ where $H^{m'}, K^{m'} : C_{m'} \times I \to B_{n'}$ are homotopies with

$$H^{m'}_0 = f_{n'} \cdot q'_{n'm'}$$

$$H^{m'}_1 = g_{n'} \cdot q'_{n'm'} = K_0^{m'}$$

$$K^{m'}_1 = h_{n'} q'_{n'm'} \quad .$$

Since $\underline{f} \simeq \underline{g}$ and $\underline{g} \simeq \underline{h}$ there are homotopies

$$G^H, G^K : C_{m'} \times I \times I \to B_n$$

such that

$$G^H(x,s,0) = q_{nn'} H^{m'}(x,s)$$

$$G^H(x,s,1) = H^m \cdot (q'_{mm'} \times 1_I)$$

$$G^H(x,0,t) = f_n q'_{nm'}(x)$$

$$G^H(x,1,t) = g_n q'_{nm'}(x)$$

for $x \in C_{m'}$, $s,t \in I$ and

$$G^K(x,s,0) = q_{nn'}K^{m'}(x,s)$$
$$G^K(x,s,1) = K^m \cdot (q'_{mm'} \times 1_I)(x,s)$$
$$G^K(x,0,t) = g_n \cdot q'_{nm'}(x)$$
$$G^K(x,1,t) = h_n \cdot q'_{nm'}(x) \ .$$

Define a homotopy

$$G : C_{m'} \times I \times I \to B_n$$

by

$$G(x,s,t) = \begin{cases} G^H(x,2s,t) & 0 \le s \le \frac{1}{2} \\ G^K(x,2s-1,t) & \frac{1}{2} \le s \le 1 \end{cases}$$

$x \in C_{m'}$, $s,t \in I$. Note that

$$G(x,s,0) = \begin{cases} G^H(x,2s,0) = q_{nn'}H^{m'}(x,2s) & 0 \le s \le \frac{1}{2} \\ G^K(x,2s-1,0) = q_{nn'}K^{m'}(x,2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

Hence $G(x,s,0) = q_{nn'}H'(x,s)$. For $t = 1$,

$$G(x,s,1) = \begin{cases} G^H(x,2s,1) = H^m \cdot (q'_{mm'} \times 1_I) & 0 \le s \le \frac{1}{2} \\ G^K(x,2s-1,1) = K^m \cdot (q'_{mm'} \times 1_I) & \frac{1}{2} \le s \le 1 \end{cases}$$

Hence $G(x,s,1) = H \cdot (q'_{mm'} \times 1_I)$. Also, for every $t \in I$, $x \in C_{m'}$,

$$G(x,0,t) = G^H(x,0,t) = f_n q'_{nm'}(x)$$
$$G(x,1,t) = G^K(x,1,t) = h_n q'_{nm'}(x) \ .$$

Hence $\underline{f}$ and $\underline{h}$ are strongly equivalent $(\underline{f} \simeq \underline{h})$.

Now we will define strongly equivalence relation.

Definition 4.2: Two continuous functions $f,g : C \to B$ between compact metric spaces are said to be strongly equivalent (in symbol $f \sim g$) if there are level maps

$\underline{f},\underline{g} : \underline{C} \to \underline{B}$ of ANR-sequences such that $\underline{f} \simeq \underline{g}$ and $\varprojlim \underline{f} = f$, $\varprojlim \underline{g} = g$.

(Note that $\varprojlim \underline{C} = C$ and $\varprojlim \underline{B} = B$).

<u>Remark</u>: Proposition 4.1 implies that $\sim$ is an equivalence relation. (For proof, see appendix).

<u>Proposition 4.2</u>: Two homotopic maps $f,g: C \to B$ between compact metric spaces are strongly equivalent.

<u>Proof</u>: Let $H: C \times I \to B$ be the homotopy with $H_0 = f$ and $H_1 = g$. Embed $C \times I$ in $Q \times I \approx Q$ and B in Q. Select a decreasing sequence $\underline{B}$ of compact ANR-neighborhoods B_n of B in Q with $\cap B_n = B$. Let $\tilde{H}: Q \to Q$ be an extension of H. For each n, we can select a compact ANR-neighborhood $C_n \times I$ of $C \times I$ in $\tilde{H}^{-1}(B_n)$ such that $C_{n+1} \times I \subset C_n \times I$ and $\cap (C_n \times I) = C \times I$. Write $H_n = \tilde{H}|C_n \times I$. Note that for every n, $H_n|C \times I = H$. Write $f_n = H_n(0)$ and $g_n = H_n(1)$. Thus $\underline{f} = (f_n) \simeq \underline{g} = (g_n)$ showing that f and g are strongly equivalent.

<u>Remark (1)</u>: Two strongly equivalent maps may not be homotopic.

Let X be the $\sin \frac{1}{x}$ curve in $\mathbb{R}^2$ with domain $(0,1]$ and let $A = cl(X) \supset X$. Let c be a point. Define f and g such that $f(c) \in A - X$ and $g(c) \in X$.

Choose a decreasing sequence $\underline{A} = (A_n)$ of compact ANR-neighborhoods of A in $\mathbb{R}^2$ such that for each n, A_n is contractible and $\cap A_n = A$. Hence for every n there is a path from $f(c)$ to $g(c)$ in A_n which shows that f and g are strongly equivalent.

But there is no path from $f(c)$ to $g(c)$ in A. So f is not homotopic to g.

(2) The concept of strongly equivalence leads to a definition of strongly shape path connectedness which we will discuss in Section 5.

Now, we will prove the main theorem.

Theorem 4.1: Let E,B,C be compact metric spaces; $p : E \to B$ be a shape fibration and $f,g : C \to B$ be strongly equivalent continuous functions then the shape fibrations p' and p'' induced from p by f and g respectively are fiber shape equivalent.

Proof: Let $\underline{p} : \underline{E} \to \underline{B}$ be any level map of ANR-sequences and $\underline{f},\underline{g}: \underline{C} \to \underline{B}$ be level maps of ANR-sequences such that $\underline{f} \simeq \underline{g}$.

For each n, let $(Z'_n; f'_n,p'_n)$ be the pull-back of $(B_n; f_n,p_n)$ and $(Z''_n; g''_n,p''_n)$ be the pull-back of $(B_n;g_n,p_n)$. Then $\underline{Z}' = (Z'_n, r'_{n,n+1})$ and $\underline{Z}'' = (Z''_n, r''_{n,n+1})$ are inverse sequences of compact metric spaces with limits $(Z'; f',p')$ and (Z'', g'',p'') where $(Z'; f',p')$ is a pull-back of $(B; f,p)$ and (Z'', g'', p'') is a pull-back of $(B; g,p)$.

As in the proof of the Theorem 2.1, by induction, for each $n = 1,2,3,\dots$ select compact ANR-neighborhoods E'_n and E''_n of Z'_n and Z''_n respectively in $E_n \times C_n$; positive numbers ϵ_n and η_n $(\eta_n < \epsilon_n)$ such that

(1) $r'_{n,n+1}(E'_{n+1}) \subset E'_n \; ; \; r''_{n,n+1}(E''_{n+1}) \subset E''_n$.

(2) $\varprojlim \underline{E}' = Z' \; ; \; \varprojlim \underline{E}'' = Z''$.

(3) $d(p_n f'_n|E'_n \, , \, f_n p'_n|E'_n) < \epsilon_n$,

$d(p_n g''_n|E''_n \, , \, g_n p''_n|E''_n) < \epsilon_n$, where $f'_n : E_n \times C_n \to E_n$; $p'_n : E_n \times C_n \to C_n$

(4) for $e', e'' \in E_n$ and $x,y \in C_n$,

$d(p_n(e'), f_n(x)) < \eta_n \Rightarrow (e',x) \in E'_n$ and

$d(p_n(e''), g_n(y)) < \eta_n \Rightarrow (e'',y) \in E''_n$.

(5) As $n \to \infty$, $\epsilon_n \to 0$ (so does η_n) , and $\epsilon_{n+1} \in \Lambda\,(q_{n,n+1}, \eta_n)$.

We will define morphisms

$$\underline{h}' : \underline{E}' \to \underline{E}'' \text{ and } \underline{h}'' : \underline{E}'' \to \underline{E}'$$

such that

$$(\underline{h}'', \underline{1}_{\underline{c}}) \cdot (\underline{h}', \underline{1}_{\underline{c}}) \sim \underline{1}_{\underline{p}'} \text{ and}$$

$$(\underline{h}', \underline{1}_{\underline{c}}) \cdot (\underline{h}'', \underline{1}_{\underline{c}}) \sim \underline{1}_{\underline{p}} \ .$$

<u>Morphism $\underline{h}'' : \underline{E}'' \to \underline{E}'$</u>

Since $\underline{p}$ has AHLP and $\epsilon_n \to 0$ as $n \to \infty$, every (n,ϵ) has a lifting pair (m, ϵ_m) . Let (i, ϵ_i) be a lifting pair for (n, η_n) ; (j, ϵ_j) be a lifting pair for (i, η_i). Since $\underline{f} \simeq \underline{g}$ for j, there is an index m and a homotopy

$$H' : C_m \times I \to B_j$$

such that

(6) $H'_0 = f_j \cdot q'_{jm} = q_{jm} f_m$ and

(7) $H'_1 = g_j \cdot q'_{jm} = q_{nm} g_m$

$$\begin{tikzcd}
C_m \arrow[r, "q'_{jm}"] \arrow[dr, "f_m"] \arrow[dr, bend right, "g_m"'] & C_j \arrow[r, shift left, "f_j"] \arrow[r, shift right, "g_j"'] & B_j \\
& B_m \arrow[ur, "q_{jm}"'] &
\end{tikzcd}$$

Define a homotopy $H^m : E''_m \times I \to B_j$ by $H^m = H' \cdot (p''_m \times 1_I)$. Let $h^m = r_{jm} \cdot g''_m : E''_m \to E_m \to E_j$.

Consider the following diagram

$$\begin{tikzcd}
E''_m \times 1 \arrow[r, "h^m"] \arrow[d] & E_j \arrow[r, "r_{ij}"] \arrow[d, "p_j"] & E_i \arrow[d, "p_i"] \\
E''_m \times I \arrow[r, "H^m"] \arrow[urr, dashed, "G^m"] & B_j \arrow[r, "q_{ij}"'] & B_i
\end{tikzcd}$$

We will denote $f'_n|E'_n$ by f'_n etc. Then for $n = m$, by (3) and the choice of $\epsilon_m \in \Lambda\ (q_{jm}, \epsilon_j)$,

(8) $$d(q_{jm}p_mg''_m , q_{jm}g_mp''_m) < \epsilon_j \ .$$

Since $\underline{p}$ is a level map, $q_{jm}p_m = p_jr_{jm}$. So (8) becomes

(8') $$d(p_jr_{jm}g''_m , q_{jm}\ {}_mp''_m) < \epsilon_j$$

Since $h^m = r_{jm}g''_m$ and $H^m_1 = q_{jm}g_mp''_m$

(8") $$d(p_jh^m, H^m_1) < \epsilon_j \ .$$

Since (j, ϵ_j) is a lifting pair for (i, η_i) there is a map $G^m : E''_m \times I \to E_i$ such that

(9) $$G^m_1 = r_{ij} \cdot h^m \quad \text{and}$$

(10) $$d(p_iG^m , q_{ij}H^m) < \eta_i \ .$$

In particular, by (10) we have

(11) $$d(p_iG^m_0 , q_{ij}H^m_0) < \eta_i \ .$$

Now $q_{ij}H^m_0 = q_{ij}H'_0p''_m = q_{ij}q_{jm}f_mp''_m = q_{im}f_mp''_m = f_iq'_{im}p''_m$. So by (11)

(11') $$d(p_iG^m_0 , f_i \cdot q'_{im}p''_m) < \eta_i \ .$$

i.e. for every $e''_m \in E''_m$,

$$d(p_i(G^m_0(e''_m)), f_i(q'_{im}p''_m(e''_m)) < \eta_i \ .$$

By the choice of η_i and (4)

$$(G^m_0(e''_m), q'_{im}p''_m(e''_m)) \in E'_i \ .$$

Hence we can define a continuous function

$$\widetilde{h}''_n = (G^m_0, q'_{im}p''_m) : E''_m \to E'_i \ .$$

Write $h''_n = r'_{ni} \cdot \widetilde{h}''_n : E''_m \to E'_n$. Now for $n' \geq n$, let $(i', \epsilon_{i'})$ be a lifting pair for $(n', \eta_{n'})$ and $(j', \epsilon_{j'})$ be a lifting pair for $(i', \eta_{i'})$ such that $i' \geq i$ and $j' \geq j$.

Let $H'' : C_{m'} \times I \to B_j$ be a homotopy such that

$$H''_0 = f_{j'} \cdot q'_{j'm'} = q_{j'm'}f_{m'} \quad \text{and} \tag{12}$$

$$H''_1 = g_{j'} \cdot q'_{j'm'} = q_{j'm'}g_{m'} \ . \tag{13}$$

Define $H^{m'} : E''_{m'} \times I \to B_{j'}$ by $H^{m'} = H'' \cdot (p''_{m'} \times 1_I)$. Let $h^{m'} = r_{j'm'} \cdot g''_{m'} : E''_{m'} \to E_{m'} \to E_{j'}$.

By AHLP of $\underline{p}$, let $G^{m'} : E''_{m'} \times I \to E_{i'}$ be a homotopy such that

$$G^{m'}_1 = r_{i'j'}h^{m'} \quad \text{and} \tag{14}$$

$$d(p_{i'}G^{m'} , q_{i'j'}H^{m'}) < \eta_{i'} \ . \tag{15}$$

As before there is a continuous function

$$\widetilde{h}''_{n'} = (G^{m'}_0, q'_{i'm'}p''_{m'}) : E''_{m'} \to E'_{i'} \ .$$

Write $h''_{n'} = r'_{n'i'} \cdot \widetilde{h}''_{n'} : E''_{m'} \to E_{n'}$. We want to show that there is a homotopy $H : r'_{nn'}h''_{n'} \simeq h''_n r''_{mm'}$ such that for every $t \in I$ and for every $\epsilon > 0$, $d(p'_n H_t, q'_{nm'}p''_{m'}) < \epsilon$. We will show that $p'_n H_t = q'_{nm'}p''_{m'}$.

$$\begin{tikzcd}
E''_{m'} \arrow[rr, "h''_{n'}"] \arrow[dr, "r''_{mm'}"] \arrow[dd, "p''_{m'}"'] & & E'_{n'} \arrow[dr, "r'_{nn'}"] & \\
& E''_{m} \arrow[rr, "h''_{n}"] & & E'_{n} \arrow[d, "p'_{n}"] \\
C_{m'} \arrow[rrr, "q'_{nm'}"'] & & & C_{n}
\end{tikzcd}$$

Since $\underline{f} \simeq \underline{g}$ there is a homotopy

$$\hat{H} : q_{jj'}H'' \simeq H' \cdot (q'_{mm'} \times 1_I)$$

such that

(16) $\hat{H}(x,0,t) = f_{j'}q'_{jm'}(x) = q_{jm'}f_{m'}(x)$

(17) $\hat{H}(x,1,t) = g_{j'}q'_{jm'}(x) = q_{jm'}g_{m'}(x)$ for every $x \in C_{m'}$ and $t \in I$.

$$\begin{tikzcd}
C_{m'} \times I \arrow[rr, "H''"] \arrow[dr, "q'_{mm'} \times 1_I"] & & B_{j'} \arrow[dr, "q_{jj'}"] & \\
& C_{m} \times I \arrow[rr, "H'"] & & B_{j}
\end{tikzcd}$$

Let $H : E''_{m'} \times I \times I \to B_i$ be $H = q_{ij} \cdot \hat{H} \cdot (p''_{m'} \times 1_I \times 1_I)$. Note that $J = I \times 1 \cup 1 \times I \cup I \times 0$ is homeomorphic to $I \times 1$. Define a continuous function $g : E''_{m'} \times J \to E_i$ by

$$g(x,s,t) = \begin{cases} G^{m} \cdot (r''_{mm'} \times 1_I)(x,s) & t = 1 \\ r_{im'} \cdot g''_{m'}(x) & s = 1 \\ r_{ii'}G^{m'}(x,s) & t = 0 \end{cases}$$

$$d(p_i g,\ H|E''_{m'} \times J) = \begin{cases} d(p_i G^m\cdot(r''_{mm'} \times 1_I),\ H|E''_{m'} \times I \times 1) & t = 1 \\ d(p_i r_{im'} g''_{m'},\ H|E''_{m'} \times 1 \times I) & s = 1 \\ d(p_i r_{ii'} G^{m'},\ H|E''_{m'} \times I \times 0) & t = 0 \end{cases}$$

Now by (10)

$$d(p_i G^m\cdot(r''_{mm'} \times 1_I),\ q_{ij} H^m \cdot (r''_{mm'} \times 1_I)) < \eta_i$$

since

$$\begin{aligned} H|E''_{m'} \times I \times 1 &= q_{ij} \cdot \hat{H} \cdot (p''_{m'} \times 1_I \times 1_I)|E''_{m'} \times I \times 1 \\ &= q_{ij} \cdot H' \cdot (q'_{mm'} \times 1_I)(p''_{m'} \times 1_I) \\ &= q_{ij} \cdot H' \cdot (p''_{m'} \times 1_I)(r''_{mm'} \times 1_I) \\ &= q_{ij}\ H^m \cdot (r''_{mm'} \times 1_I) \end{aligned}$$

$$\begin{tikzcd} E''_{m'} \arrow[r, "r''_{mm'}"] \arrow[d, "p''_{m'}"] & E''_m \arrow[d, "p''_m"] \\ C_{m'} \arrow[r, "q'_{mm'}"] & C_m \end{tikzcd}$$

Hence $d(p_i g,\ H|E''_{m'} \times J) < \eta_i$ for $t = 1$. Now,

$$\begin{aligned} H|E''_{m'} \times 1 \times I &= q_{ij} \cdot \hat{H}\ (p''_{m'} \times 1_I \times 1_I)|E''_{m'} \times 1 \times I \\ &= q_{ij}\ \hat{H}|p''_{m'}(E''_{m'}) \times 1 \times I \\ &= q_{ij} q_{jm'} g_{m'} p''_{m'} \\ &= q_{im'} g_{m'} p''_{m'} \qquad \text{and} \\ p_i r_{im'} &= q_{im'} p_{m'} \ . \end{aligned}$$

Hence for $s = 1$,

$$d(p_i g,\ H|E''_{m'} \times J) = d(q_{im'} p_{m'} g''_{m'},\ q_{im'} g_{m'} p''_{m'}) \ .$$

By construction, for $n = m'$ in (3) and the choice of $\epsilon_{m'}$ in (5) we have

$$d(p_{m'}g''_{m'}, g_{m'}p''_{m'}) < \epsilon_{m'} \Rightarrow d(q_{im'}p_{m'}g''_{m'}, q_{im'}g_{m'}p''_{m'}) < \eta_i$$

for $t = 0$,

$$\begin{aligned} H|E''_{m'} \times I \times 0 &= q_{ij}\cdot\hat{H}\cdot(p''_{m'} \times 1_I \times 1_I)|E''_{m'} \times I \times 0 \\ &= q_{ij}\cdot q_{jj'}\, H''\cdot(p''_{m'} \times 1_I) \\ &= q_{ij'}\, H''\cdot(p''_{m'} \times 1_I) = q_{ij'}H^{m'} \end{aligned}$$

By the choice of $\eta_{i'}$,

$$d(p_{i'}G^{m'}, q_{i'j'}H^{m'}) < \eta_{i'} \Rightarrow d(q_{ii'}p_{i'}G^{m'}, q_{ii'}q_{i'j'}H^{m'}) < \eta_i ,$$

i.e., $d(p_i\cdot r_{ii'}G^{m'}, q_{ij'}H^{m'}) < \eta_i$. So, for $t = 0$,

$$d(p_i g, H|E''_{m'} \times J) < \eta_i .$$

Since (i,ϵ_i) is a lifting pair for (n,η_n), there is a homotopy $G\colon E''_{m'} \times I \times I \to E_n$ such that

(18) $\qquad G|E''_{m'} \times J = r_{ni}g$ and

$$\begin{tikzcd} E''_{m'} \times J \arrow[r, "g"] \arrow[d] & E_i \arrow[r, "r_{ni}"] \arrow[d, "p_i"] & E_n \arrow[d, "p_n"] \\ E''_{m'} \times I \times I \arrow[r, "H"'] \arrow[urr, dashed, "G"] & B_i \arrow[r, "q_{ni}"'] & B_n \end{tikzcd}$$

(19) $\qquad d(p_nG(x,s,t), q_{ni}H(x,s,t)) < \eta_n$ for every $(x,s) \in E''_{m'} \times I$ and $t \in I$. Write $\hat{G} = G|E''_{m'} \times 0 \times I$. We want to show that

(20) $\qquad (\hat{G}, q'_{nm'} \cdot (p''_{m'} \times 0 \times 1_I))(E''_{m'} \times 0 \times I) \subset E'_n$

(21) $\qquad (\hat{G}, q'_{nm'} \cdot (p''_{m'} \times 0 \times 1_I))\colon r'_{nn'}h''_{n'} \simeq h''_n \cdot r''_{mm'}$ and

(22) $\quad p'_n(\hat{G}_t, q'_{nm'} \cdot (p''_{m'} \times 0 \times 1_I)) = q'_{nm'} \cdot p''_{m'}$ for every $t \in I$

(by definition). For (20) we have to show that for every $x \in E''_{m'}$ and $t \in I$; $d(p_n\hat{G}(x,0,t), f_n \cdot q'_{nm'} \cdot p''_{m'}(x,0,t)) < \eta_n$ which by the choice of η_n implies that

$$(\hat{G}(x,0,t), q'_{nm'}p''_{m'}(x,0,t)) \in E'_n .$$

By (19) $\quad d(p_nG(x,0,t), q_{ni}H(x,0,t)) < \eta_n .$

By definition of H, $d(p_n\hat{G}(x,0,t), q_{ni} \cdot q_{ij}\hat{H}(p''_{m'}(x),0,t)) < \eta_n .$

By (16), $d(p_n\hat{G}(x,0,t), q_{nj} \cdot q_{jm'}f_{m'}p''_{m'}(x,0,t)) < \eta_n ,$

i.e. $\quad d(p_n\hat{G}(x,0,t), q_{nm'}f_{m'}p''_{m'}(x,0,t)) < \eta_n$ since $q_{nm'}f_{m'} = f_n \cdot q'_{nm'}$

$$d(p_n\hat{G}(x,0,t), f_nq'_{nm'}p''_{m'}(x,0,t)) < \eta_n$$

so by (4)

$$(\hat{G}(x,0,t), q'_{nm'}p''_{m'}(x,0,t)) \in E'_n .$$

Now for $t = 0$

$$\begin{aligned}
(\hat{G}(x,0,0), q'_{nm'}p''_{m'}(x,0,0) &= (G(x,0,0), q'_{nm'}p''_{m'}(x)) \\
&= r_{ni}g(x,0,0), q'_{nm'}p''_{m'}(x)) \\
&= (r_{ni} \cdot r_{ii'}G^{m'}(x,0), q'_{nm'}p''_{m'}(x)) \\
&= (r_{ni'}G^{m'}(x,0), q'_{nm'}p''_{m'}(x)) \\
&= (r_{nn'} \cdot r_{n'i'}G^{m'}(x,0), q'_{nn'} \cdot q'_{n'i'} \cdot q'_{i'm'}p''_{m'}(x)) \\
&= r'_{nn'}(r_{n'i'}G^{m'}(x,0), q'_{n'i'} \cdot q'_{i'm'}p''_{m'}(x)) \\
&= r'_{nn'} \cdot r'_{n'i'}(G^{m'}(x,0), q'_{i'm'}p''_{m'}(x)) \\
&= r'_{nn'} \cdot h''_{n'} .
\end{aligned}$$

Similarly,

$$\begin{aligned}
(\hat{G}(x,0,1), q'_{nm'}p''_{m'}(x,0,1)) &= (G(x,0,1), q'_{nm'}p''_{m'}(x)) \\
&= (r_{ni}g(x,0,1), q'_{nm'}p''_{m'}(x)) \\
&= (r_{ni}G^{m} \cdot (r''_{mm'} \times 1_I)(x,0), q'_{nm'}p''_{m'}(x))
\end{aligned}$$

$$\begin{aligned}
&= r'_{ni}(G^m(r''_{mm'}(x),0),q'_{im'}p''_{m'}(x)) \\
&= r'_{ni}(G^m(r''_{mm'}(x),0),q'_{im}\cdot q'_{mm'}p''_{m'}(x)) \\
&= r'_{ni}(G^m(r''_{mm'}(x),0),q'_{im}\cdot p''_m\cdot r''_{mm'}(x)) \\
&= r'_{ni}(G^m_0,q'_{im}p''_m)\ r''_{mm'}(x) \\
&= h''_n \cdot r''_{mm'}(x)\ .
\end{aligned}$$

Hence $r'_{nn'} \cdot h''_{n'} \simeq h''_n \cdot r''_{mm'}$, which proves (21). Hence $(\underline{h}'',\underline{1}_{\underline{c}})$: $(\underline{E}'',\underline{p}'',\underline{C}) \to (\underline{E}',\underline{p}',\underline{C})$ is a morphism of level maps.

<u>Morphism $\underline{h}' : \underline{E}' \to \underline{E}''$</u>

Let (i_m,ϵ_{i_m}) be a lifting pair for (m,η_m) and (j_m,ϵ_{j_m}) be a lifting pair for (i_m,ϵ_{i_m}).

Let $K' : C_{m_0} \times I \to B_{j_m}$ be a homotopy with

$$K'_0 = f_{j_m} \cdot q'_{j_m m_0} = q_{j_m m_0} = q_{j_m m_0} f_{m_0}$$

$$K'_1 = g_{j_m} q'_{j_m m_0} = q_{j_m m_0} g_{m_0} \quad .$$

$$\begin{tikzcd}
C_{m_0} \arrow[r, "q'_{j_m m_0}"] \arrow[dr, shift left, "f_{m_0}"] \arrow[dr, shift right, "g_{m_0}"'] & C_{j_m} \arrow[r, shift left, "f_{j_m}"] \arrow[r, shift right, "g_{j_m}"'] & B_{j_m} \\
& B_{m_0} \arrow[ur, "q_{j_m m_0}"'] &
\end{tikzcd}$$

Define a homotopy $K : E'_{m_0} \times I \to B_{j_m}$ by $K = K' \cdot (p'_{m_0} \times 1_I)$.

Let $k = r_{j_m m_0} \cdot f'_{m_0} : E'_{m_0} \to E_{m_0} \to E_{j_m}$

$$\begin{tikzcd}
E'_{m_0} \times 0 \arrow[r, "k"] \arrow[d] & E_{j_m} \arrow[r, "r_{i_m j_m}"] \arrow[d, "p_{j_m}"] & E_{i_m} \arrow[d, "p_{i_m}"] \\
E'_{m_0} \times I \arrow[r, "K"'] \arrow[urr, dashed, "J"] & B_{j_m} \arrow[r, "q_{i_m j_m}"'] & B_{i_m}
\end{tikzcd}$$

By construction, for $n = m_0$ in (3)

$$d(p_{m_0} f'_{m_0} , f_{m_0} p'_{m_0}) < \epsilon_{m_0} .$$

Since $\epsilon_{m_0} \in \Lambda (q_{j_m m_0}, \eta_{j_m})$,

$$d(q_{j_m m_0} p_{m_0} f'_{m_0} , q_{j_m m_0} f_{m_0} p'_{m_0}) < \eta_{j_m} .$$

Since $q_{j_m m_0} p_{m_0} = p_{j_m} \cdot r_{j_m m_0}$,

$$d(p_{j_m} \cdot r_{j_m m_0} f'_{m_0} , q_{j_m m_0} f_{m_0} p'_{m_0}) < \eta_{j_m} < \epsilon_{j_m} ,$$

i.e. $d(p_{j_m} k, K_0) < \epsilon_{j_m}$. Since (j_m, ϵ_{j_m}) is a lifting pair for (i_m, ϵ_{i_m}), there is a map $J : E'_{m_0} \times I \to E_{i_m}$ such that

(23) $J_0 = r_{i_m j_m} \cdot k$ and

(24) $d(p_{i_m} J, q_{i_m j_m} K) < \eta_{i_m}$.

In particular, $d(p_{i_m} J_1, q_{i_m j_m} K_1) < \eta_{i_m}$. Now

$$\begin{aligned} q_{i_m j_m} K_1 &= q_{i_m j_m} K'_1 \cdot p'_{m_0} \\ &= q_{i_m j_m} q_{j_m m_0} g_{m_0} p'_{m_0} \\ &= q_{i_m m_0} g_{m_0} p'_{m_0} \\ &= g_{i_m} \cdot q'_{i_m m_0} p'_{m_0} . \end{aligned}$$

By the choice of η_{i_m}, there is a continuous function

$$\widetilde{h}'_m = (J_1, q'_{i_m m_0} p'_{m_0}) : E'_{m_0} \to E''_{i_m} .$$

Write $h'_m = r''_{mi_m} \cdot \widetilde{h}'_m : E'_{m_0} \to E''_{i_m} \to E''_m$. Now we want to show that

$$\underline{h}'' \cdot \underline{h}' \sim \underline{1}_{\underline{p}'} .$$

Consider the following diagram

$$\begin{tikzcd}
E'_{m_0} \arrow[r, "h'_m"] \arrow[d, "p'_{m_0}"] \arrow[rr, dashed, bend right, "r'_{nm_0}"'] & E''_m \arrow[r, "h''_n"] \arrow[d, "p''_m"] & E'_n \arrow[d, "p'_n"] \\
C_{m_0} \arrow[r, "q'_{mm_0}"'] & C_m \arrow[r, "q'_{nm}"'] & C_n
\end{tikzcd}$$

Observe that $p'_n \cdot r'_{nm_0} = q'_{nm_0} \cdot p'_{m_0}$. Also,

$$\begin{aligned}
p'_n \cdot h''_n \cdot h'_m &= p'_n \cdot r'_{n_i} \cdot \widetilde{h}''_n \cdot r''_{mi_m} \cdot \widetilde{h}'_m \\
&= p'_n \cdot r'_{ni} \cdot (G^m_0, q'_{i_m} p''_m) \cdot r''_{mi_m} \cdot (J_1, q'_{i_m m_0} p'_{m_0}) \\
&= q'_{ni} p'_i \cdot (G^m_0, q'_{i_m} p''_m) \cdot r''_{mi} (J_1, q'_{i_m m_0} p'_{m_0}) \\
&= q'_{ni} \cdot q'_{i_m} p''_m \cdot r''_{mi_m} \cdot (J_1, q'_{i_m m_0} p'_{m_0}) \\
&= q'_{nm} \cdot q'_{mi_m} \cdot p''_{i_m} \cdot (J_1, q'_{i_m m_0} p'_{m_0}) \\
&= q'_{ni_m} \cdot q'_{i_m m_0} p'_{m_0} \\
&= q'_{nm_0} \, p'_{m_0} .
\end{aligned}$$

Hence, $p'_n \cdot h''_n \cdot h'_m = p'_n \cdot r'_{nm_0}$.

We only need to show that there is a homotopy

$$L : h''_n \cdot h'_m \simeq r'_{nm_0}$$

such that for every $t \in I$ and for every $\epsilon > 0$

$$d(p'_n L_t,\ q'_{nm_0} p'_{m_0}) < \epsilon .$$

We will show that $p'_n L_t = q'_{nm_0} p'_{m_0}$. Since f and g are strongly equivalent, there is a homotopy

$$\hat{M} : C_{m_0} \times I \times I \to B_j$$

such that $\hat{M}_0 = q_{jj_m} K'$ and $\hat{M}_1 = H' \cdot (q'_{mm_0} \times 1_I)$,

$$\hat{M}(x,0,t) = q_{jj_m} f_{j_m} \cdot q'_{j_m m_0} \quad \text{and} \quad \hat{M}(x,1,t) = q_{jj_m} g_{j_m} \cdot q'_{j_m m_0} ,$$

$\forall\, x \in C_{m_0}$, $t \in I$.

$$\begin{tikzcd}
C_{m_0} \times I \arrow[r, "K'"] \arrow[d, "q'_{mm_0} \times 1_I"] & B_{j_m} \arrow[r, dashed] \arrow[d, "q_{jj_m}"] & B_{i_m} \arrow[d, "q_{ii_m}"] \\
C_m \times I \arrow[r, "H'"] & B_j \arrow[r, dashed, "q_{ij}"'] & B_i
\end{tikzcd}$$

Let $M = q_{ij} \cdot \hat{M} \cdot (p'_{m_0} \times 1_I \times 1_I)$:

$$\begin{tikzcd}
E'_{m_0} \times I \times I \arrow[r, "p'_{m_0} \times 1_I \times 1_I"] & C_{m_0} \times I \times I \arrow[r, "\hat{M}"] & B_j \arrow[r, "q_{ij}"] & B_i
\end{tikzcd}$$

Let $J = I \times 1 \cup 1 \times I \cup I \times 0 \approx I \times 0$. Define a continuous function $\ell : E'_{m_0} \times J \to E_i$ by

$$\ell(x,s,t) = \begin{cases} r_{ii_m}\ J(x,s) & t = 0 \\ r_{ii_m}\ J_1(x) & s = 1 \\ G^m \cdot (h'_m \times 1_I)(x,s) & t = 1 \end{cases}$$

$$\begin{tikzcd}
E'_{m_0} \times J \arrow[r, "\ell"] \arrow[d] & E_i \arrow[r, "r_{ni}"] \arrow[d, "p_i"] & E_n \arrow[d, "p_n"] \\
E'_{m_0} \times I \times I \arrow[r, "M"'] \arrow[urr, dashed, "\hat{L}"] & B_i \arrow[r, "q_{ni}"'] & B_n
\end{tikzcd}$$

for $s = 1$,

$$\begin{aligned}
G^m \cdot (h'_m \times 1_I)(x,1) &= G_1^m \cdot h'_m \\
&= r_{ij} h^m \cdot r''_{mi_m} \cdot (J_1, q'_{i_m m_0} p'_{m_0}) \\
&= r_{ij} \cdot r_{jm} \cdot g''_m \cdot r''_{mi_m} (J_1, q'_{i_m m_0} p'_{m_0}) \\
&= r_{im}\, g''_m \cdot r''_{mi_m} (J_1, q'_{i_m m_0} p'_{m_0}) \\
&= r_{im}\, g''_{i_m} \, (J_1, q'_{i_m m_0} p'_{m_0}) \\
&= r_{ii_m} \, J_1 \,.
\end{aligned}$$

Hence ℓ is well-defined. Also,

$$d(p_i\ell,\ M|E'_{m_0} \times J) = \begin{cases} d(p_i\ell,\ M|E'_{m_0} \times I \times 0) & t = 0 \\ d(p_i\ell,\ M|E'_{m_0} \times 0 \times I) & s = 0 \\ d(p_i\ell,\ M|E'_{m_0} \times I \times 1) & t = 1 \end{cases}$$

<u>For $t = 0$</u>

$$\begin{aligned}
d(p_i\ell,\ M|E'_{m_0} \times I \times 0) &= d(p_i r_{ii_m} J, q_{ij} \hat{M}_0 \cdot (p'_{m_0} \times 1_I)) \\
&= d(q_{ii_m} p_{i_m} J,\ q_{ij} q_{jj_m} K' \cdot (p'_{m_0} \times 1_I)) \\
&= d(q_{ii_m} p_{i_m} J,\ q_{ij_m} K) \,.
\end{aligned}$$

By (24) and $\eta_{i_m} < \epsilon_{i_m} \in \Lambda\,(q_{ii_m}, \eta_i)$,

$$d(q_{ii_m} p_{i_m} J,\ q_{ii_m} \cdot q_{i_m j_m} K) < \eta_i \ .$$

For $s = 1$

$$\begin{aligned} d(p_i \ell, M | E'_{m_0} \times 1 \times I) &= d(p_i r_{ii_m} J_1, q_{ij} \hat{M} \cdot (p'_{m_0} \times 1_I \times 1_I) | E'_{m_0} \times 1 \times I) \\ &= d(q_{ii_m} p_{i_m} J_1,\ q_{ij} q_{jj_m} q'_{j_m m_0} p'_{m_0}) \\ &= d(q_{ii_m} p_{i_m} J_1,\ q_{ij_m} \cdot q_{j_m m_0} \cdot g_{m_0} p'_{m_0} \\ &= d(q_{ii_m} p_{i_m} J_1,\ q_{ii_m} \cdot q_{i_m m_0} g_{m_0} p'_{m_0}) \\ &= d(q_{ii_m} p_{i_m} J_1,\ q_{ii_m} \cdot q_{i_m j_m} K_1) \ . \end{aligned}$$

By (24) and $\epsilon_{i_m} \in \Lambda\,(q_{ii_m}, \eta_i)$,

$$d(q_{ii_m} p_{i_m} J_1,\ q_{ii_m} \cdot q_{i_m j_m} K_1) < \eta_i \ .$$

For $t = 1$

$$\begin{aligned} d(p_i \ell, M | E'_{m_0} \times I \times 1) &= d(p_i G^m \cdot (h'_m \times 1_I), q_{ij} \hat{M}_1 \cdot (p'_{m_0} \times 1_I)) \\ &= d(p_i G^m (h'_m \times 1_I), q_{ij}\ H'(q'_{mm_0} \times 1_I)(p'_{m_0} \times 1_I)) \ . \end{aligned}$$

Since $q'_{mm_0} \cdot p'_{m_0} = p''_m \cdot h'_m$,

$$d(p_i \ell, M | E'_{m_0} \times I \times 1) = d(p_i G^m (h'_m \times 1_I), q_{ij} H' \cdot (p''_m \times 1_I)(h'_m \times 1_I))$$

$$\begin{tikzcd} E'_{m_0} \arrow[r, "h'_m"] \arrow[dr, "p'_{m_0}"] & E''_m \arrow[dr, "p''_m"] & \\ & C_{m_0} \arrow[r, "q'_{mm_0}"] & C_m \end{tikzcd}$$

By (10)

$$d(p_i G^m \cdot (h'_m \times 1_I),\ q_{ij} H^m(h'_m \times 1_I)) < \eta_i ,$$

i.e., $$d(p_i G^m \cdot (h'_m \times 1_I),\ q_{ij}\ H' \cdot (p''_m \times 1_I)(h'_m \times 1_I)) < \eta_i .$$

Hence $d(p_i \ell,\ M|E'_{m_0} \times J) < \eta_i$. Since (i, ϵ_i) is a lifting pair for (n, η_n) there is a map $\hat{L} : E'_{m_0} \times I \times I \to E_n$ such that

(25) $\hat{L}|E'_{m_0} \times J = r_{n_i} \ell$

(26) $d(p_n \hat{L},\ q_{ni} M) < \eta_n$.

Define a homotopy $L : E'_{m_0} \times I \to E'_n$ by $L(x,t) = (\hat{L}(x,0,t),\ q'_{nm_0} p'_{m_0}(x))$ for $x \in E'_{m_0}$, $t \in I$. We have to show that $L(E'_{m_0} \times I) \subset E'_n$, i.e., that for every $x \in E'_{m_0}$ and $t \in I$

$$d(p_n \hat{L}(x,0,t),\ f_n \cdot q'_{nm_0} p'_{m_0}(x)) < \eta_n .$$

By (26), $d(p_n \hat{L}(x,0,t),\ q_{ni} M(x,0,t)) < \eta_n$ where

$$\begin{aligned} q_{ni}\ M(x,0,t) &= q_{ni} q_{ij}\ \hat{M} \cdot (p'_{m_0}(x),0,t) \\ &= q_{nj} \cdot q_{jj_m} \cdot f_{j_m} \cdot q'_{j_m m_0} p'_{m_0}(x) \\ &= q_{nj_m} \cdot f_{j_m} \cdot q'_{j_m m_0} p'_{m_0}(x) \\ &= f_n \cdot q'_{nj_m} \cdot q'_{j_m m_0} p'_{m_0}(x) \\ &= f_n\ q'_{nm_0}\ p'_{m_0}(x) . \end{aligned}$$

Now,

$$\begin{aligned} L(x,0) &= (\hat{L}(x,0,0),\ q'_{nm_0} p'_{m_0}(x)) \\ &= (r_{ni} \ell(x,0,0),\ q'_{nm_0} p'_{m_0}(x)) \quad \text{by (25)} \\ &= (r_{ni} \cdot r_{ii_m}\ J_0(x),\ q'_{nm_0} p'_{m_0}(x)) \end{aligned}$$

$$
\begin{aligned}
&= (r_{ni_m} \cdot r_{i_m m_0} \cdot f'_{m_0}(x),\ q'_{nm_0} p'_{m_0}(x)) \\
&= (r_{nm_0} f'_{m_0}(x),\ q'_{nm_0} p'_{m_0}(x)) \\
&= r'_{nm_0}(x)
\end{aligned}
$$

and

$$
\begin{aligned}
L(x,1) &= (\hat{L}(x,0,1),\ q'_{nm_0} p'_{m_0}(x)) \\
&= (r_{ni} \ell(x,0,1),\ q'_{nm_0} p'_{m_0}(x)) \\
&= (r_{ni} G_0^m \cdot (h'_m(x)),\ q'_{nm} \cdot p''_m \cdot (h'_m \times 1_I)(x,1)) \\
&= (r_{ni} G_0^m (h'_m(x)),\ q'_{nm} \cdot p''_m (h'_m(x))) \\
&= h''_n \cdot h'_m(x) .
\end{aligned}
$$

Hence $L : r'_{nm_0} \simeq h''_n \cdot h'_m$ is the required homotopy.

Observe that for every $x \in E'_{m_0}$ and for every $t \in I$,

$$
\begin{aligned}
p'_n L(x,t) &= p'_n(L(x,0,t),\ q'_{nm_0} p'_{m_0}(x)) \\
&= q'_{nm_0} p'_{m_0}(x) .
\end{aligned}
$$

By similar arguments one can show that $\underline{h}' \cdot \underline{h}'' \sim \underline{1}_{\underline{P}}$.

Important corollaries of this theorem are in the next section.

Let B be a compact metric space of a trivial shape. Then one can embed B in the Hilbert cube in such a way that there is a decreasing sequence of neighborhoods B_n of B such that each B_n is homeomorphic to Q and

$$\bigcap_n B_n = B .$$

Let $\underline{1}_{\underline{B}} = (1_{B_n}) : \underline{B} \to \underline{B}$ be an identity level map and for any point $b_0 \in B$,

$$\underline{b}_0 = (\bar{b}_0) : \underline{B} \to \underline{B}$$

be the trivial level map. (For each n, $\bar{b}_0 : B_n \to B_n$ is a trivial continuous function).

For every n and for every $b_n \in B_n$ there is a path, $\omega_{n,b} : I \to B_n$ such that $\omega_{n,b}(0) = b_0$ and $\omega_{n,b}(1) = b_n$. $\omega_{n,b}$ induces a homotopy

$$H^n : B_n \times I \to B_n$$

defined by $H^n(b_n,s) = \omega_{n,b}(s)$ for $b_n \in B_n$, $s \in I$.

Again since B_n is homeomorphic to Q for each n,

$$q_{n,n+1}(\omega_{n+1,b}) - \omega_{n,b} \text{ is a contractible loop in } B_n$$

(i.e. $q_{n,n+1}(\omega_{n+1,b}) - \omega_{n,b} \in [0] \in \pi_1(B_n,b_0))$. Hence for each b_{n+1} there is a homotopy $H^n_b : I \times I \to B_n$ such that

$$H^n_b(s,0) = \omega_{n,b}(s)$$
$$H^n_b(s,1) = q_{n,n+1}(\omega_{n+1,b}(s))$$
$$H^n_b(0,t) = b_0$$
$$H^n_b(1,t) = q_{n,n+1}(b_{n+1})$$

for $b_{n+1} \in B_{n+1}$, $s,t \in I$. Define a homotopy $H^n : B_{n+1} \times I \times I \to B_n$ by $H^n(b_{n+1},s,t) = H^n_b(s,t)$.

This proves that 1_B and $\bar{b}_0 : B \to B$ are strongly equivalent.

This fact together with the Theorem 4.1 implies the following:

Corollary 4.1.1: Let $p: E \to B$ be a shape fibration and B be of trivial shape, then p is fiber shape equivalent to a trivial shape fibration

$$\pi_B : F_{b_0} \times B \to B$$

where $F_{b_0} = p^{-1}(b_0)$.

Section 5

STRONG SHAPE PATH CONNECTEDNESS

In this section we will define strong shape path connectedness and will show classes of such spaces. At the end we will partially answer the question raised by Mardesic [10].

Let B be a compact metric space and $\underline{B} = (B_n, q_{nm})$ be an ANR-sequence with $\underleftarrow{\lim}\, \underline{B} = (B, q_n)$. Let $b, c \in B$ be any two points and $\underline{b} = (b_n)$, $\underline{c} = (c_n)$ where for every n, $b_n = q_n(b) \in B_n$, $c_n = q_n(c) \in B_n$.

We will define a path from $\underline{b}$ to $\underline{c}$ in $\underline{B}$.

Definition 5.1: A path from $\underline{b} = (b_n)$ to $\underline{c} = (c_n)$ in an ANR-sequence $\underline{B} = (B_n, q_{nm})$ is a family of paths $\underline{\omega} = \{\omega_n : I \to B_n \mid n \in N,\ \omega_n(0) = b_n;\ \omega_n(1) = c_n\}$ such that for each $m \geq n$ there is a homotopy $H: I \times I \to B_n$ such that

$$H(s,0) = \omega_n(s)$$

$$H(s,1) = q_{nm}\omega_m(s)$$

$$H(0,t) = b_n = q_{nm}(b_m)$$

$$H(1,t) = c_n = q_{nm}(c_m) \quad \text{for all} \quad s,t \in I\ .$$

Proposition 5.1: Let $\underline{B} = (B_n, q_{nm})$ and $\underline{B}' = (B'_n, q'_{nm})$ be ANR-sequences such that $\underleftarrow{\lim}\, \underline{B} = \underleftarrow{\lim}\, \underline{B}' = B$. For points $b, c \in B$, let

$$\underline{b} = (b_n = q_n(b));\ \underline{c} = (c_n = q_n(c));$$

$$\underline{b}' = (b'_n = q'_n(b));\ \underline{c}' = (c'_n = q'_n(c)).$$

If there is a path $\underline{\omega}$ from $\underline{b}$ to $\underline{c}$ in $\underline{B}$ then there is a path $\underline{\omega}'$ from $\underline{b}'$ to $\underline{c}'$ in $\underline{B}'$.

Proof: As in the proof of the Theorem 3.1, by induction we can choose a sequence (ϵ_n) of positive numbers ϵ_n , and construct a map $\underline{f} = (f_n, \alpha): \underline{B} \to \underline{B}'$ of ANR-sequences such that for all m,n $(m \geq n)$

(1) $\quad d(q'_{nm}f_m, f_n q_{\alpha(n)\alpha(m)}) < \epsilon_n$ where ϵ_n is so small that there is an ϵ_n-homotopy

$$G : q'_{nm} f_m \simeq f_n q_{\alpha(n)\alpha(m)}$$

and

(2) for every $b \in B$,

$$b'_n = q'_n(b) \text{ and } q'_{nm} f_m(b_{\alpha(m)}) = q'_{nm} f_m q_{\alpha(m)}(b)$$

lie in a contractible ϵ_n-ball.

By assumption for each n there is a path $\omega_{\alpha(n)}: I \to B_{\alpha(n)}$ such that $\omega_{\alpha(n)}(0) = b_{\alpha(n)}$ and $\omega_{\alpha(n)}(1) = c_{\omega(n)}$. Hence $f_n \cdot \omega_{\alpha(n)}: I \to B_{\alpha(n)} \to B'_n$ is a path in B'_n from $f_n(b_{\alpha(n)})$ to $f_n(c_{\alpha(n)})$.

By (2) for $n = m$, there are paths

$$\lambda^n_b, \lambda^n_c : I \to B'_n$$

such that $\lambda^n_b(0) = b'_n$; $\lambda^n_b(1) = f_n(b_{\alpha(n)})$; $\lambda^n_c(0) = f_n(c_{\alpha(n)})$; $\lambda^n_c(1) = c'_n$ and each path lies in an ϵ_n-ball. So, $\omega'_n = \lambda^n_c \cdot f_n \omega_{\alpha(n)} \cdot \lambda^n_b : I \to B'_n$ is a path from b'_n to c'_n in B'_n defined by

$$\omega'_n(t) = \begin{cases} \lambda^n_c(3t) & 0 \le t \le \frac{1}{3} \\ f_n \omega_{\alpha(n)}(3t-1) & \frac{1}{3} \le t \le \frac{2}{3} \\ \lambda^n_b(3t-2) & \frac{2}{3} \le t \le 1 \end{cases}$$

Again by assumption for $m \ge n$ there is a homotopy

$$H : I \times I \to B_{\alpha(n)}$$

such that

$$H(s,0) = \omega_{\alpha(n)}(s)$$

$$H(s,1) = q_{\alpha(n)\alpha(m)} \omega_{\alpha(m)}(s)$$

$$H(0,t) = b_{\alpha(n)} = q_{\alpha(n)\alpha(m)}(b_{\alpha(m)})$$

$$H(1,t) = c_{\alpha(n)} = q_{\alpha(n)\alpha(m)}(c_{\alpha(m)})$$

for all $s,t \in I$. Hence there is a homotopy

$$\hat{H} : I \times I \to B'_n$$

such that

$$\hat{H}(s,t) = \begin{cases} \lambda_b^n(3s) & 0 \le s < \frac{1}{3} \\ f_n H(3s-1,t) & \frac{1}{3} \le s \le \frac{2}{3} \\ \lambda_c^n(3s-2) & \frac{2}{3} \le s \le 1 . \end{cases}$$

By (1) and (2) there is a homotopy

$$G' : I \times I \to B_n'$$

such that

$$G'(s,t) = \begin{cases} K_b(3s,t) & 0 \le s \le \frac{1}{3} \\ G(3s-1,t) & \frac{1}{3} \le s \le \frac{2}{3} \\ K_c(3s-2,t) & \frac{2}{3} \le s \le 1 \end{cases}$$

where by (2), $K_b : q_{nm}'\lambda_b^m \simeq \lambda_b^n$ and $K_c : q_{nm}'\lambda_c^m \simeq \lambda_c^n$ such that $K_b(0,t) = b_n'$, $K_c(1,t) = c_n'$, $K_b(1,t) = G(0,t)$ and $K_c(0,t) = G(1,t)$ for every $t \in I$.

Now, we define a homotopy

$$H' : I \times I \to B_n'$$

by

$$H'(s,t) = \begin{cases} G'(s,2t) & 0 \le t \le \frac{1}{2} \\ \hat{H}(s,2t-1) & \frac{1}{2} \le t \le 1 . \end{cases}$$

Note that

$$H'(s,0) = G'(s,0) = q_{nm}'\omega_m'$$

$$H'(s,1) = \hat{H}(s,1) = \lambda_c^n \cdot f_n\omega_{\alpha(n)} \cdot \lambda_c^n = \omega_n'$$

$$H'(0,t) = K_b(0,t) = b_n'$$

$$H'(1,t) = K_c(1,t) = c_n' \quad \text{for all } s,t \in I.$$

Hence H' is the required homotopy between $q_{nm}'\omega_m'$ and ω_n' which proves that $\underline{\omega}' = (\omega_n' = \lambda_c^n \cdot f_n\omega_{\alpha(n)} \cdot \lambda_b^n)$ is a path from $\underline{b}'$ to $\underline{c}'$.

Now we can define the following.

<u>Definition 5.2</u>: A compact metric space B is said to be strongly shape path connected if there is an ANR-sequence $\underline{B} = (B_n, q_{nm})$ with

$\underleftarrow{\lim}\ \underline{B} = (B,q_n)$ and for every pair of points $b,c \in B$, there is a path $\underline{\omega} = (\omega_n)$ from $\underline{b} = (b_n = q_n(b))$ to $\underline{c} = (c_n = q_n(c))$.

Remarks: (1) Clearly if X is a path-connected compact metric space then X is strongly shape path connected.

(2) Let $c = \{*\}$ be a space with one point and B be a strongly shape path connected space.

Then any two maps $f,g\colon C \to B$ will be strongly equivalent.

Let $f(*) = x$ and $g(*) = y$. If $p\colon E \to B$ is a shape fibration then the induced shape fibrations from p by f and g respectively are $p_f\colon F_x \to C$ and $p_g\colon F_y \to C$.

Analogous to the path component we can define strong shape path component of a compact metric space. Two points are in the same strong shape path component if there is a strong shape path connecting them.

Combining this with the Theorem 4.1 we have

Corollary 4.1.2: Let $p\colon E \to B$ be a shape fibration and $x,y \in B$ be any points in the same strong shape path component then the fibers $p^{-1}(x)$ and $p^{-1}(y)$ are of the same shape.

Definition 5.3: A compact metric space Y is said to be strongly shape dominated by a compact metric space X if there are ANR-sequences $\underline{X} = (X_n, q_{nm})$, $\underline{Y} = (Y_n, r_{nm})$ and if there exist a level map

$$\underline{f} = (f_n, 1_N) : \underline{Y} \to \underline{X}$$

and a map $\quad \underline{g} = (g_n, \alpha) : \underline{X} \to \underline{Y}$

of ANR-sequences satisfying the following condition:

For every n there is an index n^* such that for all $n' \geq n^*$ there is a homotopy

$$H^n : Y_{n'} \times I \to Y_n$$

such that

$$H^n_0 = g_n f_{\alpha(n)} r_{\alpha(n)n'}$$

$$H^n_1 = r_{nn'}$$

and if for $m \geq n$,

$$H^m : Y_{m'} \times I \to Y_m \qquad (m' \geq n')$$

with

$$H^m_0 = g_m f_{\alpha(m)} r_{\alpha(m)m'}$$

$$H^m_1 = r_{mm'}$$

then there is a homotopy

$$H : Y_{m'} \times I \times I \to Y_n$$

such that

$$H(x,s,0) = r_{nm} H^m(x,s)$$

$$H(x,s,1) = H^n \cdot (r_{n'm'} \times 1_I)(x,s)$$

$$H(x,1,t) = r_{nm'}(x)$$

$$H(x,0,t) = G \cdot (f_{\alpha(m)} \cdot r_{\alpha(m)m'} \times 1_I)(x,t)$$

for $x \in r_{m'}(Y)$, $s,t \in I$, where the homotopy

$$G : X_{\alpha(m)} \times I \to Y_n$$

is given by the definition of $\underline{g}$ with $G_0 = r_{nm} g_m$, $G_1 = g_n \cdot q_{\alpha(n)\alpha(m)}$.

$$\begin{tikzcd}
Y \arrow[r, "r_{m'}"] & Y_{m'} \arrow[r, "r_{\alpha(m)m'}"] \arrow[d, "r_{n'm'}"] \arrow[rrrr, bend left, dashed, "r_{mm'}"] & Y_{\alpha(m)} \arrow[r, "f_{\alpha(m)}"] \arrow[d, "r_{\alpha(n)\alpha(m)}"] & X_{\alpha(m)} \arrow[r, "g_m"] \arrow[d, "q_{\alpha(n)\alpha(m)}"] & Y_m \arrow[d, "r_{nm}"] \\
& Y_{n'} \arrow[r, "r_{\alpha(n)n'}"] \arrow[rrrr, bend right, dashed, "r_{nn'}"'] & Y_{\alpha(n)} \arrow[r, "f_{\alpha(n)}"] & X_{\alpha(n)} \arrow[r, "g_n"] & Y_n
\end{tikzcd}$$

Theorem 5.1: Let a compact metric space Y be strongly shape dominated by a compact metric space X. If X is strongly shape path connected, so is Y.

Proof: Let $\underline{X} = (X_n, q_{nm})$ and $\underline{Y} = (Y_n, r_{nm})$ be ANR-sequences with $\underleftarrow{\lim}\, \underline{X} = (X, q_n)$ and $\underleftarrow{\lim}\, \underline{Y} = (Y, r_n)$. Also, let $\underline{f} = (f_n, 1_N): \underline{Y} \to \underline{X}$ be a level map and $\underline{g} = (g_n, \alpha) : \underline{X} \to \underline{Y}$ be a map of ANR-sequences with the property defined in the definition 5.3.

Let $\underleftarrow{\lim}\, \underline{f} = f : X \to Y$ be a continuous function and for any pair of points $y, y' \in Y$; $f(y) = x$, $f(y') = x' \in X$. For every n, denote $q_n(x) = x_n$, $q_n(x') = x'_n \in X_n$, $r_n(y) = y_n$; $r_n(y') = y'_n \in Y_n$. Then $\underline{x} = (x_n)$, $\underline{x}' = (x'_n)$, $\underline{y} = (y_n)$ and $\underline{y}' = (y'_n)$ are sequences of points. Note that for all $m \geq n$, $q_{nm}(x_m) = x_n$ and $r_{nm}(y_m) = y_n$.

By assumption there is a path $\underline{\omega} = \{\omega_n : I \to X_n\}$, $\omega_n(0) = x_n$; $\omega_n(1) = x'_n$ $\}$ from $\underline{x}$ to $\underline{x}'$ in $\underline{X}$. Hence $\omega_{\alpha(n)}: I \to X_{\alpha(n)}$ is a path with

$$\omega_{\alpha(n)}(0) = x_{\alpha(n)}$$
$$\omega_{\alpha(n)}(1) = x'_{\alpha(n)}$$

and

$$g_n \cdot \omega_{\alpha(n)} : I \to Y_n$$

is a path with

$$g_n\, \omega_{\alpha(n)}(0) = g_n(x_{\alpha(n)})$$
$$g_n\, \omega_{\alpha(n)}(1) = g_n(x'_{\alpha(n)}) \ .$$

Also, by assumption, for every n there is an index n^* such that for every $n' \geq n^*$ there is a homotopy

$$H^n : Y_{n'} \times I \to Y_n$$

such that

$$H_0^n = g_n f_{\alpha(n)} r_{\alpha(n)n'}$$
$$H_1^n = r_{nn'} \ .$$

Hence for $y_{n'}$ and $y'_{n'} \in Y_{n'}$ there are paths

$$H^n(y_{n'}) = H^n_y : I \to Y_n$$
$$H^n(y'_{n'}) = H^n_{y'} : I \to Y_n$$

such that

$$H^n_y(0) = g_n f_{\alpha(n)} r_{\alpha(n)n'}(y_{n'}) = g_n(x_{\alpha(n)})$$
$$H^n_y(1) = r_{nn'}(y_{n'}) = y_n$$

and

$$H^n_{y'}(0) = g_n f_{\alpha(n)} r_{\alpha(n)n'}(y'_{n'}) = g_n(x'_{\alpha(n)})$$
$$H^n_{y'}(1) = r_{nn'}(y'_{n'}) = y'_n \ .$$

Let $\hat{H}^n_y(s) = H^n(1-s)$. Then there is a path

$$\omega'_n : I \to Y_n$$

defined by

$$\omega'_n(s) = \begin{cases} \hat{H}^n_y(3s) & 0 \le s \le \frac{1}{3} \\ g_n \omega_{\alpha(n)}(3s-1) & \frac{1}{3} \le s \le \frac{2}{3} \\ H^n_{y'}(3s-2) & \frac{2}{3} \le s \le 1 \ . \end{cases}$$

Note that $\omega'_n(0) = \hat{H}^n_y(0) = H^n_y(1) = y_n$, $\omega'_n(1) = H^n_{y'}(1) = y'_n$.

For $m \ge n$ let $m' \ge m^*$ and

$$H^m : Y_{m'} \times I \to Y_m$$

be the homotopy such that

$$H^m_0 = g_m f_{\alpha(m)} r_{\alpha(m)m'}$$

and

$$H^m_1 = r_{mm'} \ .$$

For $y_{m'}, y'_{m'} \in Y_{m'}$ there are paths $H^m(y_{m'}) = H^m_y$; $H^m(y'_{m'}) = H^m_{y'}$: $I \to Y_m$ such that

$$H^m_y(0) = g_m(x_{\alpha(m)})$$
$$H^m_y(1) = y_m$$
$$H^m_{y'}(0) = g_m(x'_{\alpha(m)})$$

and

$$H^m_{y'}(1) = y'_m \ .$$

Let $\hat{H}^m_y(s) = H^m_y(1-s)$. Define a path

$$\omega'_m(s) = \begin{cases} \hat{H}^m_y(3s) & 0 \le s \le \frac{1}{3} \\ g_m \omega_{\alpha(m)}(3s-1) & \frac{1}{3} \le s \le \frac{2}{3} \\ H^m_{y'}(3s-2) & \frac{2}{3} \le s \le 1 \end{cases}$$

Since $\underline{X}$ is strongly shape path connected there is a homotopy

$$K : I \times I \to X_{\alpha(n)}$$

such that

$$K(s,1) = \omega_{\alpha(n)}(s)$$

$$K(s,0) = q_{\alpha(n)\alpha(m)}\ \omega_{\alpha(m)}(s)$$

$$K(0,t) = x_{\alpha(n)}$$

$$K(1,t) = x'_{\alpha(n)} \quad .$$

Define a homotopy $K' : I \times I \to Y_n$ by

$$K'(s,t) = \begin{cases} \hat{H}^n_y(3s) & 0 \le s \le \frac{1}{3} \\ g_n K(3s-1,t) & \frac{1}{3} \le s \le \frac{2}{3} \\ H^n_{y'}(3s-2) & \frac{2}{3} \le s \le 1 \end{cases}$$

Also, by assumption there is a homotopy

$$H : Y_{m'} \times I \times I \to Y_n$$

such that $\quad H(z,s,0) = r_{nm} H^m(z,s)$

$$H(z,s,1) = H^n \cdot (r_{n'm'} \times 1_I)(z,s)$$

$$H(z,1,t) = r_{nm'}(z)$$

and $\quad H(z,0,t) = G \cdot (f_{\alpha(m)} \cdot r_{\alpha(m)m'} \times 1_I)(z,t)$

for $z \in Y_{m'}$, $s,t \in I$, where by definition of $\underline{g}$ and by assumption there is a homotopy

$$G : X_{\alpha(m)} \times I \to Y_n$$

such that $\quad G_0 = r_{nm} g_m$

and $\quad G_1 = g_n q_{\alpha(n)\alpha(m)} \quad .$

Define a homotopy $G' : I \times I \to Y_n$ by

$$G'(s,t) = \begin{cases} H(y_{m'}, 1-3s, t) & 0 \le s \le \frac{1}{3} \\ G\cdot(\omega_{\alpha(m)} \times 1_I)(3s-1) & \frac{1}{3} \le s \le \frac{2}{3} \\ H(y'_{m'}, 3s-2, t) & \frac{2}{3} \le s \le 1 \end{cases}$$

Note that $y_{m'} = r_{m'}(y)$ and $y'_{m'} = r_{m'}(y'))$. Hence the required homotopy $H' : I \times I \to Y_n$ is defined by

$$H'(s,t) = \begin{cases} G'(s,2t) & 0 \le t \le \frac{1}{2} \\ K'(s,2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

Note that

$$H'(s,0) = G'(s,0) = \begin{cases} H(y_{m'}, 1-3s, 0) & 0 \le s \le \frac{1}{3} \\ G(\omega_{\alpha(m)}(3s-1), 0) & \frac{1}{3} \le s \le \frac{2}{3} \\ H(y'_{m'}, 3s-2, 0) & \frac{2}{3} \le s \le 1 \end{cases}$$

Now $H(y_{m'}, 1-3s, 0) = r_{nm} H^m(y_{m'}, 1-3s) = r_{nm} H^m_y(1-3s) = r_{nm} \hat{H}^m_y(3s)$

$G(\omega_{\alpha(m)}(3s-1), 0) = r_{nm} g_m \omega_{\alpha(m)}(3s-1)$

$H(y'_{m'}, 3s-2, 0) = r_{nm} H^m_{y'}(3s-2)$.

Hence $H'(s,0) = r_{nm}\, \omega'_m(s)$.

$$H'(s,1) = K'(s,1) = \begin{cases} \hat{H}^n_y(3s) & 0 \le s \le \frac{1}{3} \\ g_n K(3s-1,1) = g_n \omega_{\alpha(n)}(3s-1) & \frac{1}{3} \le s \le \frac{2}{3} \\ H^n_{y'}(3s-2) & \frac{2}{3} \le s \le 1 \end{cases}$$

So, $H'(s,1) = \omega'_n(s)$. Also,

$$H'(0,t) = \begin{cases} G'(0,2t) & 0 \le t \le \frac{1}{2} \\ K'(0,2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

$G'(0,2t) = H(y_{m'}, 1, 2t) = r_{nm'}(y_{m'}) = y_n$

$K'(0,2t-1) = \hat{H}^n_y(0) = H^n_y(1) = y_n$

and

$$H'(1,t) = \begin{cases} G'(1,2t) & 0 \le t \le \frac{1}{2} \\ K'(1,2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

$$G'(1,2t) = H(y'_{m'},1,t) = r_{nm'}(y'_{m'}) = y'_n$$

$$K'(1,2t-1) = H^n_{y'}(1) = y'_n \,.$$

Hence $\underline{\omega}' = (\omega'_n)$ is a path from $\underline{y}$ to $\underline{y}'$ in $\underline{Y}$.

Corollary: If a compact metric space Y is homotopy dominated by a compact metric space X and if X is strongly shape path connected then so is Y.

Proof: We just have to show that Y is strongly shape dominated by X.

Embed Y and X into the Hilbert cube Q. By assumption there are continuous functions $f: Y \to X$ and $g: X \to Y$ and a homotopy $H: Y \times I \to Y$ such that $H_0 = gf$; $H_1 = 1_Y$.

Extend the continuous function $g: X \to Y$ to $\tilde{g}: Q \to Q$ say.

First, select a decreasing sequence of compact ANR-neighborhoods Y_n of Y in Q such that $\bigcap_n Y_n = Y$. Then select a decreasing sequence of compact ANR-neighborhoods X_n of X in Q such that $\bigcap_n X_n = X$ and $\tilde{g}|X_n \subseteq Y_n$ for each n.

Let $q_{nm}: X_m \to X_n$ and $r_{nm}: Y_m \to Y_n$ be the inclusions for all $m \geq n$. Write $\tilde{g}|X_n = g_n$ for each n. Then $\underline{g}: \underline{X} = (X_n,q_{nm}) \to \underline{Y} = (Y_n,r_{nm})$ is a level map of ANR-sequences such that $g_n|X = g$ for each n. Similarly extend the continuous function $f: Y \to X$ to $\tilde{f}: Q \to Q$ say.

For each n there is an index $\hat{n}$ such that for all $n' \geq \hat{n}$, $Y_{n'} \subseteq \tilde{f}^{-1}(X_n)$. Hence for each n we can select an index $\alpha(n)$ such that $\tilde{f}(Y_{\alpha(n)}) \subseteq X_n$ and for $m \geq n$, $Y_{\alpha(m)} \subseteq Y_{\alpha(n)}$. Write $\tilde{f}|Y_{\alpha(n)} = f_n$ for each n. Then $\underline{f} = (\alpha;\, f_n): \underline{Y} \to \underline{X}$ is a map of ANR-sequences such that $q_{nm}f_m = f_n r_{\alpha(n)\alpha(m)}$ for $m \geq n$, and $f_n|Y = f$ for each n.

Since for each n, Y_n is an ANR, Y is a closed subset of $Y_{\alpha(n)}$, $g_n f_n|Y = gf$ and $r_{n\alpha(n)}|Y = 1_Y$ the homotopy $H: Y \times I \to Y$ can be extended to a homotopy

$$H^{\alpha(n)} : Y_{\alpha(n)} \times I \to Y_n$$

such that

$$H_0^{\alpha(n)} = g_n f_n \quad \text{and} \quad H_1^{\alpha(n)} = r_{n\alpha(n)} \quad .$$

Also, note that if $H^{\alpha(m)} : Y_{\alpha(m)} \times I \to Y_m$ is an extension of H with $H_0^{\alpha(m)} = g_m f_m$ and $H_1^{\alpha(m)} = r_{m\alpha(m)}$ then

$$r_{nm} H^{\alpha(m)} | Y \times I = H = H^{\alpha(n)} \cdot (r_{\alpha(n)\alpha(m)} \times 1_I) \; .$$

Therefore by the definition 5.3, X strongly shape dominates Y.

Let (X,x) be a pointed compact metric space.

A pointed ANR-sequence $(\underline{X},\underline{x})$ is a sequence $(X_n, q_{nm}; x_n)$ of pointed compact-ANR's (X_n, x_n) such that for all $m \geq n$, $q_{nm}(x_m) = x_n$.

$(\underline{X},\underline{x})$ is said to be associated with (X,x) if $\varprojlim (\underline{X},\underline{x}) = (X,x)$. For a pointed metric space (X,x), in [15] M. Moszynska has defined limit homotopy groups

$$\check{\pi}_n(X,x) = \varprojlim_m (\pi_n(X_m,x_m), q_{m\ell *}) \quad , \quad n \in N \; ,$$

and has proved that $\check{\pi}_n(X,x)$ is independent of the choice of pointed ANR-sequence $(\underline{X},\underline{x})$ associated with it.

In [15] it has been shown that a pointed map $\underline{f}: (\underline{X},\underline{x}) = (X_n,x_n) \to (\underline{Y},\underline{y}) = (Y_n,y_n)$ (i.e. for each n, $f_n: (X_{\alpha(n)}, x_{\alpha(n)}) \to (Y_n,y_n)$ is a pointed continuous function and for $m \geq n$, there is a pointed homotopy

$$H: (X_{\alpha(m)}, x_{\alpha(m)}) \times I \to (Y_n, y_n)$$

such that $H_0 = r_{nm} f_m$; $H_1 = f_n \cdot q_{\alpha(n)\alpha(m)}$) induces a homomorphism

$$\underline{f}_* : \pi_n(\underline{X},\underline{x}) \to \pi_n(\underline{Y},\underline{y})$$

and if $\underline{f}$ is a 'homotopy equivalence' in the category of ANR-sequences then $\underline{f}_*$ is an isomorphism in the category of inverse systems of groups.

Definition 5.4: A pointed ANR-sequence is said to have property (*) if for every n there is an index $n(*)$ such that for all $m \geq n(*)$ the following condition is satisfied:

For every $\omega_n \in \pi_1(X_n,x_n)$ where $\omega_n = q_{nm*}(\omega_m)$ for some $\omega_m \in \pi_1(X_m,x_m)$ there is $\omega_\ell \in \pi_1(X_\ell,x_\ell)$ for all $\ell \geq m$ such that

$$q_{n\ell*}(\omega_\ell) = \omega_n .$$

Proposition 5.2: Let $(\underline{X},\underline{x})$ and $(\underline{X}',\underline{x}')$ be pointed ANR-sequences associated with a pointed compact metric space (X,x).

If $(\underline{X},\underline{x})$ has the property $*$ then so does $(\underline{X}',\underline{x}')$.

Proof: By assumption there are pointed maps

$$\underline{f} : (\underline{X},\underline{x}) \to (\underline{X}',\underline{x}') \quad \text{and} \quad \underline{g} : (\underline{X}',\underline{x}') \to (\underline{X},\underline{x})$$

such that $\underline{gf} \simeq \underline{1}_{(\underline{X},\underline{x})}$ and $\underline{fg} \simeq \underline{1}_{(\underline{X}',\underline{x}')}$, $\underline{f}$ and $\underline{g}$ induce homomorphisms

$$\underline{f}_* : \pi_1(\underline{X},\underline{x}) \to \pi_1(\underline{X}',\underline{x}') \quad \text{and}$$

$$\underline{g}_* : \pi_1(\underline{X}',\underline{x}') \to \pi_1(\underline{X},\underline{x})$$

such that $\underline{g}_*\underline{f}_* \sim \underline{1}_{(\underline{X},\underline{x})*}$

$$\underline{f}_*\underline{g}_* \sim \underline{1}_{(\underline{X}',\underline{x}')*} .$$

Note that for every $m \geq n$ and $\ell \geq n$

$$q'_{nm}q'_{nm*}\, f_{m*} = f_{n*}\, q_{\alpha(n)\alpha(m)*}$$

and

$$q_{n\ell*}\, g_{\ell*} = g_{n*}\, q'_{\beta(n)\beta(\ell)*} ,$$

i.e. the following diagram commutes.

$$\begin{tikzcd}
\pi_1(X_{\alpha(m)}, x_{\alpha(m)}) \arrow[r, "f_{m*}"] \arrow[d, "q_{\alpha(n)\alpha(m)*}"] & \pi_1(X'_m, x'_m) \arrow[d, "q'_{nm*}"] \\
\pi_1(X_{\alpha(n)}, x_{\alpha(n)}) \arrow[r, "f_{n*}"] & \pi_1(X'_n, x'_n)
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_1(X'_{\beta(\ell)}, x'_{\beta(\ell)}) \arrow[r, "g_{\ell*}"] \arrow[d, "q'_{\beta(n)\beta(\ell)*}"] & \pi_1(X_\ell, x_\ell) \arrow[d, "q_{n\ell*})"] \\
\pi_1(X'_{\beta(n)}, x'_{\beta(n)}) \arrow[r, "g_{n*}"] & \pi_1(X_n, x_n)
\end{tikzcd}$$

For any n, let $m \geq \alpha(n)(*)$. Since $\underline{f}_*\underline{g}_* \sim \underline{1}_{(\underline{X}', \underline{x}')*}$ there is an index $m' \geq \beta(m)$ such that $f_{n*}g_{\alpha(n)*}q_{\beta\alpha(n)m'} = q_{nm'*}$ We want to show that $m' = n(*)$. Let $\ell \geq m'$. Clearly $\alpha(\ell) \geq m$. Consider the following diagram.

$$\begin{tikzcd}
 & & & \pi_1(X_{\alpha(\ell)}, x_{\alpha(\ell)}) \arrow[r] \arrow[dd, "q_{\alpha(n)\alpha(\ell)*}"] & \pi_1(X'_\ell, x'_\ell \arrow[dd, "q'_{n\ell*}"] \\
\pi_1(X'_{m'}, x'_{m'}) \arrow[r] \arrow[dr, "q'_{\beta\alpha(n)m'*}"'] & \pi_1(X'_{\beta(m)}, x'_{\beta(m)}) \arrow[r, "g_{m*}"] \arrow[d] & \pi_1(X_m, x_m) \arrow[dr, "q_{\alpha(n)m*}"'] & & \\
 & \pi_1(X'_{\beta\alpha(n)}, x'_{\beta\alpha(n)}) \arrow[rr, "g_{\alpha(n)*}"] \arrow[rrr, bend right, dashed, "q'_{n\beta\alpha(n)*}"'] & & \pi_1(X_{\alpha(n)}; x_{\alpha(n)}) \arrow[r, "f_{n*}"] & \pi_1(X'_n, x'_n)
\end{tikzcd}$$

Let $\lambda_n \in \pi_1(X'_n, x'_n)$ such that $\lambda_n = q'_{nm'*}(\lambda_{m'})$ for some $\lambda_{m'} \in \pi_1(X'_{m'}\, ,\, x'_{m'})$. Let $\omega_m = g_{m*} \cdot q'_{\beta(m)m'*}(\lambda_{m'}) \in \pi_1(X_m, x_m)$ and

$$\omega_{\alpha(n)} = g_{\alpha(n)*} \cdot q'_{\beta\alpha(n)m'*}(\lambda_{m'}) = q_{\alpha(n)m*}\, h_{m*}\, q'_{\beta(m)m'*}(\lambda_{m'})$$
$$= q_{\alpha(n)m*}(\omega_m) \in \pi_1(X_{\alpha(n)}, x_{\alpha(n)}) \text{ .}$$

Since $m \geq \alpha(n)(*)$ and $\alpha(\ell) \geq m$. There is $\omega_{\alpha(\ell)} \in \pi_1(X_{\alpha(\ell)}, x_{\alpha(\ell)})$ such that $q_{\alpha(n)\alpha(\ell)*}(\omega_{\alpha(\ell)} = q_{\alpha(n)m*}(\omega_m) = \omega_{\alpha(n)}$. Denote $\lambda_\ell = f_{\ell*}(\omega_{\alpha(\ell)})$, then

$$\begin{aligned} q'_{n\ell*}(\lambda_\ell) &= q'_{n\ell*}\, f_{\ell*}(\omega_{\alpha(\ell)}) \\ &= f_{n*}\, q_{\alpha(n)\alpha(\ell)*}(\omega_{\alpha(\ell)}) \\ &= f_{n*}(\omega_{\alpha(n)}) \\ &= f_{n*}\, g_{\alpha(n)*}\, q'_{\beta\alpha(n)m'*}(\lambda_{m'}) \\ &= q'_{nm'*}(\lambda_{m'}) \text{ .} \end{aligned}$$

Hence $m' = n(*)$, which proves that $(\underline{X}', \underline{x}')$ has the property (*). Now we can define

<u>Definition 5.5</u>: A pointed compact metric space (X,x) has property (*) if there is a pointed ANR-sequence $(\underline{X}, \underline{x})$ associated with (X,x) and has a (*) property.

<u>Definition 5.6</u>: A compact metric space X is said to have the (*) property if (X,x) has the (*) property for every $x \in X$.

<u>Theorem 5.2</u>: If a compact connected metric space X has the (*) property then it is strongly shape path connected.

<u>Proof</u>: Embed X into the Hilbert cube Q. We can select a decreasing sequence of compact connected ANR's $\underline{X} = (X_n)$ associated with X.

Since for each n, X_n is a compact connected ANR, for any two points $x_n, y_n \in X_n$ there is a chain of finite number of contractible open sets, $U_0, U_1, \ldots, U_m$ such that $x_n \in U_0$, $y_n \in U_m$ and for $0 < n < m$, $U_n \cap U_{n+1} \neq \emptyset$.

Hence x_n and y_n can be connected by a path in X_n.

So, we can assume that for each n, X_n is path connected.

Let $x,y \in X$ be any two points. For each n, write $q_n(x) = x_n$ and $q_n(y) = y_n$. Suppose for each n, $\lambda_n : I \to X_n$ is path from x_n to y_n. $(\lambda_n(0) = x_n ; \lambda_n(1) = y_n)$.

Since X has the (*) property, by definition, $(\underline{X},\underline{x})$ has the (*) property. Hence, there is a cofinal subsequence which we again denote by $(\underline{X},\underline{x}) = (X_n,x_n)$ such that for each n, $n+1 = n(*)$.

By induction we will construct a path $\underline{\omega} = (\omega_n : I \to X_n)$; $\omega_n(0) = x_n$, $\omega_n(1) = y_n$ as follows:

Since $\lambda_2 : I \to X_2$ is a path from x_2 to y_2 and $q_{12}(x_2) = x_1$; $q_{12}(y_2) = y_1$, $q_{12}\lambda_2 : I \to X_1$ is a path from x_1 to y_1. Write $\omega_1 = q_{12}\lambda_2$. Now, if there is a homotopy $H: I \times I \to X_2$ such that

$$H_0 = q_{23}\lambda_3 ; H_1 = \lambda_2$$

$$H(0,t) = x_2 ; H(1,t) = y_2 , \qquad t \in I .$$

Write $\omega_2 = q_{23}\lambda_2$. Then $q_{12}H: I \times I \to X_1$ will be the required homotopy.

[Let $\lambda,\mu : I \to X$ be two paths such that

$$\lambda_0 = x ; \lambda_1 = \mu_0 = y ; \mu_1 = z .$$

Then $\lambda + \mu : I \to X$ will be a path defined by

$$\lambda + \mu(s) = \begin{cases} \lambda(2s) & \tfrac{1}{2} \le x \le \tfrac{1}{2} \\ \mu(2s-1) & \tfrac{1}{2} \le s \le 1 \end{cases}$$

Note $(\lambda+\mu)_0 = x$; $(\lambda+\mu)_1 = z$. If $\omega : I \to X$ is a path in X such that

$$\omega_0 = z \text{ and } \omega_1 = y$$

then a path

$$-\omega : I \to X$$

is defined by

$$-\omega(s) = \omega(1-s) , \qquad s \in I$$

$$\lambda + (-\omega) = \lambda - \omega .$$

Note that $\lambda - \omega : I \to X$ is a path such that $(\lambda - \omega)_0 = x$ and $(\lambda - \omega)_1 = z$.]

If that is not the case then

$$\lambda_2 - q_{23}\lambda_3 \in \pi_1(X_2,x_2) .$$

Since $2 = 1(*)$ and $3 \geq 2$, by the (*) property there is $\mu_3 \in \pi_1(X_3,x_3)$ such that $q_{13_*}(\mu_3) = q_{12_*}(\lambda_2 - q_{23}\lambda_3)$,

i.e. $q_{13}(\mu_3) - q_{12}\lambda_2 + q_{13}\lambda_3 = 0 \in \pi_1(X_1,x_1)$.

Write $\omega_2 = q_{23}(\mu_3 + \lambda_3)$.

Now
$$\begin{aligned} q_{12}\omega_2 - \omega_1 &= q_{12}\cdot q_{23}(\mu_3 + \lambda_3) - q_{12}\lambda_2 \\ &= q_{13}\mu_3 + q_{13}\lambda_3 - q_{12}\lambda_2 \\ &= 0 \in \pi_1(X_1,x_1) . \end{aligned}$$

Hence there is a homotopy

$$K : I \times I \to X_1$$

such that

$$\begin{aligned} K(s,0) &= q_{12}\omega_2 \\ K(s,1) &= \omega_1 \\ K(0,t) &= x_1 \\ K(1,t) &= y_1 , \qquad s,t \in I , \end{aligned}$$

which we required. Now, again if $(\mu_3 + \lambda_3) - q_{34}\lambda_4 = 0 \in \pi_1(X_3,x_3)$ then write $\omega_3 = q_{34}(\lambda_4)$.

If that is not the case then $(\mu_3 + \lambda_3) - q_{34}\lambda_4 \in \pi_1(X_3,x_3)$ and since $4 \geq 3 = 2(*)$ there is $\mu_4 \in \pi_1(X_4,x_4)$ such that

$$q_{24*}(\mu_4) = q_{23*}((\mu_3 + \lambda_3) - q_{34}\lambda_4) \in \pi_2(X_2,x_2) .$$

Write $\omega_3 = q_{34}(\mu_4 + \lambda_4)$ and so on.

$\underline{\omega} = (\omega_n)$ is the required path in $\underline{X}$ from $\underline{x}$ to $\underline{y}$.

Since $x,y \in X$ are any two points, X is strongly shape path connected.

A pointed compact metric space $(X,x) \subset (Q,x)$ is 1-movable [12] if for each neighborhood U of X in Q there is a neighborhood V of X in Q $(V \subset U)$ such that for each continuous function

$$f : (S^1,*) \to (V,x)$$

and for each neighborhood W of X in Q $(W \subset V)$ there is a homotopy

$$H : (S^1,*) \times I \to (U,x)$$

such that

$$H_0 = f$$

$$H_1(S^1) \subset W$$

and

$$H|* \times I = x\ .$$

If $y \in X$ is any point in the same component of $x \in X$ then (X,x) is 1-movable $\Rightarrow$ (X,y) is 1-movable. [2].

We say that a compact metric space X is pointed 1-movable if for every $x \in X$, (X,x) is 1-movable.

Proposition 5.3: A pointed 1-movable compact connected metric space has the (*) property.

Proof: Let X be a pointed 1-movable compact connected metric space. Let $x \in X$ be any point. By assumption (X,x) is 1-movable. We want to show that there is a pointed ANR-sequence $(\underline{X},\underline{x})$ associated with (X,x) with the (*) property.

Embed X into the Hilbert cube Q. Let $\underline{X} = (X_n)$ be a decreasing sequence of compact ANR neighborhoods of X such that

$$\bigcap_n X_n = X$$

for $m \geq n$, $q_{nm}: X_m \to X_n$ is an inclusion, and for each n, $x \in X_n$.

Since (X,x) is 1-movable, by definition, for every n there is an index $n(*)$ such that for all $m \geq n(*)$ if $\omega_m \in \pi_1(X_m,x)$ there is $\omega_\ell \in \pi_1(X_\ell,x)$ for every $\ell \geq m$ such that $q_{nm*}(\omega_m) = q_{n\ell *}(\omega_\ell)$.

By this Proposition 5.3 and Theorem 5.2 we have

Corollary 5.2.1: A pointed 1-movable compact connected metric space is strongly shape path connected.

Borsuk has proved [2] that every pointed plane compact connected metric space is movable and hence 1-movable. Hence we have

Corollary 5.2.2: Every plane compact connected metric space is strongly shape path connected.

The following proposition shows that not all compact connected metric spaces are strongly shape path connected.

Proposition 5.4: A dyadic solenoid is not strongly shape path connected.

Proof: A dyadic solenoid Σ_2 is an inverse limit of $\underline{X} = (S^1, q_{n,n+1})$ where for each n, $S_n^1 = S^1$, a unit circle and $q_{n,n+1}: S^1 \to S^1$ is defined by $q_{n,n+1}(z) = z^2$ for $z \in S_{n+1}^1$.

It is well-known that Σ_2 is not path connected. Let $x,y \in \Sigma_2$ be two points in the different path components of Σ_2 .

We assume that Σ_2 is strongly shape path connected.

Let $\{\underline{\omega} = \omega_n: I \to S_n^1 \mid \omega_n(0) = x_n, \omega_n(1) = y_n\}$ be a path from $\underline{x} = (x_n)$ to $\underline{y} = (y_n)$ in $\underline{X}$ where $q_n(x) = x_n$; $q_n(y) = y_n$ for each n. We will replace each path $\omega_n: I \to S_n^1$ with a path $\lambda_n: I \to S_n^1$ where $\lambda_n(0) = x_n$, $\lambda_n(1) = y_n$ and λ_n is homotopic to ω_n keeping the end-points fixed.

Then $\underline{\lambda} = (\lambda_n : I \to S_n^1)$ will also be a path from $\underline{x}$ to $\underline{y}$ in $\underline{X}$ such that

$$q_{n,n+1}(\lambda_{n+1}) = \lambda_n$$

which shows that there is a path $\lambda = \varprojlim \lambda$ from x to y in Σ_2. This contradicts with our assumption.

For each n, let σ_n be the generator of $\pi_1(S^1_n, x) \cong \mathbf{Z}$.

If $\omega_{n+1} \notin [0] \in \pi_1(S^1_{n+1}, x)$ then $\omega_{n+1} = \omega'_{n+1} + \omega''_{n+1}$ where $\omega'_{n+1} \in [\ell_{n+1} \cdot \sigma_{n+1}]$ and $\omega''_{n+1} \in [0]$.

Since $q_{n,n+1}(\omega_{n+1})$ and ω_n are homotopic leaving end-points fixed.,

$$\omega_n \notin [0] .$$

Let $\omega_n = \omega'_n + \omega''_n$ where $\omega'_n \in [\ell_n \cdot \sigma_n]$ and $\omega''_n \in [0]$.

There are 'shortest paths' $\lambda'_n, \lambda''_n, \lambda'_{n+1}, \lambda''_{n+1}$ such that

$$\lambda'_n \in [\omega'_n] = [\ell_n \cdot \sigma_n]$$
$$\lambda''_n \in [\omega''_n] = [0]$$
$$\lambda'_{n+1} \in [\omega'_{n+1}] = [\ell_{n+1} \cdot \sigma_{n+1}]$$
$$\lambda''_{n+1} \in [\omega''_{n+1}] = [0]$$

$$\lambda''_{n+1}(0) = x_{n+1} , \quad \lambda''_{n+1}(1) = y_{n+1}$$
$$\lambda''_n(0) = x_n , \quad \lambda''_n(1) = y_n$$

and

$$\omega''_n - \lambda''_n \in [0] ; \omega''_{n+1} - \lambda''_{n+1} \in [0] .$$

Clearly $\lambda'_n + \lambda''_n$ is homotopic to $\omega_n = \omega'_n + \omega''_n$ and $\lambda'_{n+1} + \lambda''_{n+1}$ is homotopic to $\omega_{n+1} = \omega'_{n+1} + \omega''_{n+1}$, both homotopies leave end-points fixed.

By assumption $q_{n,n+1}(\omega_{n+1})$ is homotopic to ω_n leaving end-points fixed.

Hence $q_{n,n+1*}[\ell_{n+1} \cdot \sigma_{n+1}] = [\ell_n \cdot \sigma_n]$. Since $q_{n,n+1}(x_{n+1}) = x_n$; $q_{n,n+1}(y_{n+1}) = y_n$ and $\lambda'_n, \lambda''_n, \lambda'_{n+1}, \lambda''_{n+1}$ are shortest paths

$$q_{n,n+1}(\lambda'_{n+1}) = \lambda'_n$$

$$q_{n,n+1}(\lambda''_{n+1}) = \lambda''_n \quad .$$

Hence we have the paths

$$\lambda_n = \lambda'_n + \lambda''_n$$

$$\lambda_{n+1} = \lambda'_{n+1} + \lambda''_{n+1}$$

from x_n to y_n in S^1_n and x_{n+1} to y_{n+1} in S^1_{n+1} such that $q_{n,n+1}(\lambda_{n+1}) = \lambda_n$.

This is true for any n. Hence $\underline{\lambda} = (\lambda_n)$ is the required path from $\underline{x}$ to $\underline{y}$ in $\underline{X}$ which induces a path $\lambda = \underset{\leftarrow}{\lim}\, \underline{\lambda}$ from x to y in Σ_2 .

Section 6

ISOMORPHISM OF SHAPE GROUPS

Let $\mathcal{C}$ be a category of pointed pairs. Let $\mathcal{C}^*_{\sim}$ be a category of inverse sequences in $\mathcal{C}$ defined as follows:

Objects of $\mathcal{C}^*_{\sim}$ are inverse sequences $(\underline{X},\underline{A},\underline{x}) = ((X_n,A_n,x_n);q_{nm})$ in $\mathcal{C}$.

A morphism $\underline{f} : (\underline{X},\underline{A},\underline{x}) \to (\underline{Y},\underline{B},\underline{y}) = ((Y_n,B_n,y_n);r_{nm})$ consists of a pair (α,f_n) where $\alpha : N \to N$ is an increasing function and for every $n \in N$,

$$f_n : (X_{\alpha(n)},A_{\alpha(n)},x_{\alpha(n)}) \to (Y_n,B_n,y_n)$$

is a morphism in $\mathcal{C}$ such that for $m \geq n$,

$$r_{mn}f_n \sim f_m q_{\alpha(n)\alpha(m)}$$

where $\sim$ is any equivalence relation defined in the set of morphisms of $\mathcal{C}$.

Let $\underline{f} = (\alpha,f_n) : (\underline{X},\underline{A},\underline{x}) \to (\underline{Y},\underline{B},\underline{y})$ and $\underline{g} = (\beta,g_n): (\underline{Y},\underline{B},\underline{y} \to (\underline{Z},\underline{C},\underline{z})$ be morphisms in $\mathcal{C}^*_{\sim}$ then

$$\underline{g}\ \underline{f} = (\beta\alpha,g_n f_{\beta(n)}) : (\underline{X},\underline{A},\underline{x}) \to (\underline{Z},\underline{C},\underline{z})$$

is clearly a morphism in $\mathcal{C}^*_{\sim}$.

Clearly, $1_{(\underline{X},\underline{A},\underline{x})} = (1_N ; 1_{(X_n,A_n,x_n)})$.

We will be interested in the following two categories.

(I) Let $\mathcal{C} = \mathbb{R}$, be the category of pointed pairs of compact metric ANR's and the equivalence relation $\sim$ be the homotopy relation $\simeq$.

Then $\mathbb{R}^*_{\simeq}$ is category defined by Mardesic and Segal.

(II) Let $\mathcal{C} = \mathcal{G}$, be the category of groups and homomorphisms.

Let the equivalence relation be $=$. Then $\mathcal{G}^*_{=}$ or $\mathcal{G}^*$ is the category pro- $\mathcal{G}$.

For each $q \in Z^+$, there is a functor

$$\pi_q : \underset{\sim}{\mathbb{R}}^* \to \mathcal{G}^*$$

defined by $\pi_q(\underline{X},\underline{A},\underline{x}) = (\pi_q(X_n,A_n,x_n); q_{nm*})$ and

$$\pi_q(\underline{f}) = \underline{f}_* : \pi_q(\underline{X},\underline{A},\underline{x}) \to \pi_q(\underline{Y},\underline{B},\underline{y})$$

where $\underline{f} : (\underline{X},\underline{A},\underline{x}) \to (\underline{Y},\underline{B},\underline{y})$ is a morphism in $\underset{\sim}{\mathbb{R}}^*$.

<u>Equivalence Relation</u>: We will define an equivalence relation $\approx$ in $\underset{\sim}{C}^*$.

Let $\underline{f},\underline{f}' : (\underline{X},\underline{A},\underline{x}) \to (\underline{Y},\underline{B},\underline{y})$ be two morphisms in $\underset{\sim}{C}^*$. $\underline{f} = (\alpha,f_n)$; $\underline{f}' = (\alpha',f'_n)$. $\underline{f}$ is said to be equivalent to $\underline{f}'$ (in notation $\underline{f} \approx \underline{f}'$) if for every n there is an index m, $m \geq \alpha(n),\alpha'(n)$ such that

$$\begin{tikzcd}
& (X_{\alpha(n)},A_{\alpha(n)},x_{\alpha(n)}) \arrow[dr, "f_n"] & \\
(X_m,A_m,x_m) \arrow[ur, "q_{\alpha(n)m}"] \arrow[dr, "q_{\alpha'(n)m}"'] & & (Y_n,B_n,y_n) \\
& (X_{\alpha'(n)},A_{\alpha'(n)},x_{\alpha'(n)}) \arrow[ur, "f'_n"'] &
\end{tikzcd}$$

$$f_n q_{\alpha(n)m} \sim f'_n q_{\alpha'(n)m} .$$

A morphism $\underline{f} : (\underline{X},\underline{A},\underline{x}) \to (\underline{Y},\underline{B},\underline{y})$ is an equivalence in the category $\underset{\sim}{C}^*$ if there is a morphism $\underline{g} : (\underline{Y},\underline{B},\underline{y}) \to (\underline{X},\underline{A},\underline{x})$ such that

$$\underline{g}\,\underline{f} \approx 1_{(\underline{X},\underline{A},\underline{x})} \quad \text{and} \quad \underline{f}\,\underline{g} \approx 1_{(\underline{Y},\underline{B},\underline{y})} .$$

Note that if $\underline{f} \approx \underline{f}'$ in $\underset{\sim}{\mathbb{R}}^*$ then

$$\pi_q(\underline{f}) \approx \pi_q(\underline{f}') \quad \text{in} \quad \mathcal{G}^*$$

and if $\underline{f}$ is an equivalence in $\underset{\sim}{\mathbb{R}}^*$ then $\pi_q(\underline{f})$ is an isomorphism in $\mathcal{G}^*$.

There is a functor

$$\varprojlim : \mathcal{G}^* \to \mathcal{G} \ .$$

defined by $\varprojlim(\underline{G}) = \varprojlim \underline{G}$. Then $\varprojlim \pi_q((\underline{X},\underline{A},\underline{x})) = \varprojlim \pi_q(\underline{X},\underline{A},\underline{x})$ $= \varprojlim(\pi_q(X_n,A_n,x_n);\ q_{nm*})$. Again if $\underline{f}$ is an equivalence in $\underset{\sim}{\mathbb{R}}^*$ then $\varprojlim \pi_q(\underline{f})$ is an isomorphism in $\mathcal{G}$.

M. Moszynska has proved that if

$$\varprojlim(\underline{X},\underline{A},\underline{x}) = \varprojlim(\underline{Y},\underline{B},\underline{y})$$

then $\varprojlim \pi_q(\underline{X},\underline{A},\underline{x})$ and $\varprojlim \pi_q(\underline{Y},\underline{B},\underline{y})$

are isomorphic in $\mathcal{G}$.

Theorem 6.1: Let $p : E \to B$ and $p' : E' \to B$ be fiber shape equivalent shape fibrations.

Let $e' \in E'$, $p'(e') = b$ and $F' = (p')^{-1}(b)$. Then for any fiber shape equivalence $\underline{f} : p' \to p$ there is

$$e \in F = p^{-1}(b) \subset E$$

and an induced homomorphism

$$f_* : \check{\pi}_q(E',F',e') \to \check{\pi}_q(E,F,e)$$

such that $f_* = p_*^{-1} \cdot p'_*$.

Proof: Let $\underline{p}' : \underline{E}' \to \underline{B}$ and $\underline{p} : \underline{E} \to \underline{B}$

$$(\underline{E}' = (E'_n, r'_{nm}),\ \underline{B} = (B_n, q_{nm}),\ \underline{E} = (E_n, r_{nm}))$$

be level maps of ANR-sequences with limit maps p' and p respectively.

[Such $\underline{p}$ and $\underline{p}'$ exist. Embed E', E and B in the Hilbert cube Q. Extend the maps p' and p to $\tilde{p}' : Q \to Q$ and $\tilde{p} : Q \to Q$ respectively. Choose a decreasing sequence of ANR-neighborhoods B_n of B such that $\bigcap_n B_n = B$. Then choose decreasing sequences of ANR-neighborhoods E'_n and E_n of E' and E respectively such that $\bigcap_n E'_n = E'$; $\bigcap_n E_n = E$; $\tilde{p}'(E'_n) \subseteq B_n$ and $\tilde{p}(E_n) \subseteq B_n$ for each n. Let

$p'_n = \tilde{p}'|E'_n$ and $p_n = \tilde{p}|E_n$.

Thus $\underline{p}' : \underline{E}' \to \underline{B}$ and $\underline{p} : \underline{E} \to \underline{B}$ are the required level maps of ANR-sequences.]

Let $\underline{f} : \underline{E}' \to \underline{E}$ be any fiber shape equivalence. By [10] we can assume that $\underline{p}$ has HLP.

Let $e'_n = r'_n(e')$ and $b_n = q_n(b)$. By induction on n, one can define for each n a lifting index $m = m(n) > n$ and a closed neighborhood, Q_n of b_n, homeomorphic to Q and such that

(1) $$q_{nm}(Q_m) \subseteq \text{Int. } Q_n \ ; \quad m > n \ .$$

(2) $$\varprojlim \, (Q_n, q_{nm}|Q_m) = \{b\} \ .$$

Furthermore one can choose closed ANR-neighborhoods C_n of Q_n such that

(3) $$q_{nm}(C_m) \subseteq \text{Int. } Q_n \quad ; \quad m > n$$

and therefore

(4) $$\varprojlim \, (C_n, q_{nm}|C_m) = \{b\} \ .$$

Next, choose closed ANR-neighborhoods F'_n and F_n of $(p'_n)^{-1}(Q_n)$ and $p_n^{-1}(Q_n)$ respectively so small that

(5) $$F'_n \subseteq (p'_n)^{-1}(C_n) \ ; \ F_n \subseteq p_n^{-1}(C_n) \ .$$

Notice that by (3)

(6) $$\begin{gathered} q_{nm}p'_m(F'_m) \subseteq \text{Int. } Q_n \subset Q_n \\ q_{nm}p_m(F_m) \subseteq \text{Int. } Q_n \subset Q_n \end{gathered}$$

for $m > n$.

Select an $\epsilon > 0$. Since $\underline{f} : \underline{E}' \to \underline{E}$ is a fiber shape map, for each n there is an index $n^* \geq n$ such that for all $n' \geq n^*$

(7) $$d(q_{nm'}p_{n'}f_{n'} \, , \, q_{n\alpha(n')}p'_{\alpha(n')}) < \epsilon \ .$$

$$\begin{tikzcd}
E'_{\alpha(n')} \arrow[r, "f_{n'}"] \arrow[d, "p'_{\alpha(n')}"] & E_{n'} \arrow[r, "r_{nn'}"] \arrow[d, "p_{n'}"] & E_n \arrow[d, "p_n"] \\
B_{\alpha(n')} \arrow[r, "q_{n'\alpha(n')}"'] & B_{n'} \arrow[r, "q_{nn'}"'] & B_n
\end{tikzcd}$$

Also by (1) we can choose n' such that for any $x \in Q_{\alpha(n')}$ and $y \in B_n$

(8) $\qquad d(q_{n\alpha(n')}(x),y) \; < \; \varepsilon \;\Rightarrow\; y \in Q_n$.

Note that (8) is true for all $\ell \geq n'$. Let $x \in F'_{\alpha(n')}$. Then by (7)

$$d(q_{n\alpha(n')}p'_{\alpha(n')}(x),\; p_n r_{nn'}f_{n'}(x)) \; < \; \varepsilon \; .$$

By (8), $p_n r_{nn'}f_{n'}(x) \in Q_n$. Since $p_n^{-1}(Q_n) \subseteq F_n$, $r_{nn'}f_{n'}(x) \in F_n$ for any $x \in F'_{\alpha(n')}$. Hence $r_{nn'}f_{n'}(F'_{\alpha(n')}) \subseteq F_n$.

Define a function $\beta : N \to N$ by $\beta(n) = n'$.

Note that β is an increasing function. Let $f'_n = r_{nn'}f_{n'}$. Then $\underline{f}' = (\beta, f'_n) : (\underline{E}', \underline{F}') \to (\underline{E}, \underline{F})$ is a map of ANR-sequences of pairs.

Note that β and so $\underline{f}'$ depend on $\varepsilon > 0$. If we select $\varepsilon' > 0$ then we get another map $\underline{f}'' = (\beta', f''_n) : (\underline{E}', \underline{F}') \to (\underline{E}, \underline{F})$ where $f''_n : (E'_{\beta'(n)}, F'_{\beta'(n)}) \to (E_n, F_n)$ for each n and $f''_n = r_{n\beta'(n)}f_{\beta'(n)}$.

It is clear that $\underline{f}'$ and $\underline{f}''$ are equivalent.

Now, for each q, f'_n induces a homomorphism

$$f'_{n*} : \pi_q(E'_{\beta(n)}, F'_{\beta(n)}, e'_{\beta(n)}) \to \pi_q(E_n, F_n, f'_n(e'_{\beta(n)})) \; .$$

Since $\underline{f}'$ is a fiber shape map, for $\ell \geq n$, there is a homotopy

$$L^{n\ell} : E'_{\beta(\ell)} \times I \to E_n$$

such that $\quad L^{n\ell}_0 = r_{n\ell}f'_\ell$

$$L^{n\ell}_1 = f'_n r'_{\beta(n)\beta(\ell)}$$

and

$d(q_{n\beta(\ell)}p'_{\beta(\ell)},\ p_n\,L^{n\ell}_t) \;<\; \epsilon$ for every $t \in I$.

$$\begin{tikzcd}
E'_{\beta(\ell)} \arrow[r, "f_{\ell'}"] \arrow[d, "r'_{\beta(n)\beta(\ell)}"] & E_{\ell'} \arrow[r, "r_{\ell\ell'}"] \arrow[d, "r_{n'\ell'}"] & E_{\ell} \arrow[d, "r_{n\ell}"] \\
E'_{\beta(n)} \arrow[r, "f_{n'}"] & E_{n'} \arrow[r, "r_{nn'}"] & E_n
\end{tikzcd}$$

Let $x \in F'_{\beta(\ell)}$. Since $\beta(\ell) \geq \beta(n) = n'$, by (8)

$$d(q_{n\beta(\ell)}p'_{\beta(\ell)}(x),\ p_nL^{n\ell}_t(x)) \;<\; \epsilon \;\Rightarrow\; p_nL^{n\ell}_t(x) \in Q_n$$

for every $t \in I$. Hence $L^{n\ell}_t(x) \in F_n$ for every $t \in I$. Therefore $L^{n\ell}$ is a homotopy of pairs

$$(E'_{\beta(\ell)},F'_{\beta(\ell)}) \times I \to (E_n,F_n) \; .$$

For $e'_{\beta(\ell)} \in F'_{\beta(\ell)}$, $L^{n\ell}_{e'} : I \to F_n$ is a path from $r_{n\ell}f'_{\ell}(e'_{\beta(\ell)})$ to $f'_n r'_{\beta(n)\beta(\ell)}(e'_{\beta(\ell)}) = f'_n(e'_{\beta(n)})$.

The path $L^{n\ell}_e$ induces an isomorphism

$$L^{n\ell}_{e'}\,* : \pi_q(E_n,F_n,r_{n\ell}f'_{\ell}(e'_{\beta(\ell)})) \to \pi_q(E_n,F_n,f'_n(e'_{\beta(n)})) \; .$$

Now for each n, the sequence of points $\{p_nr_{n\ell}f'_{\ell}(e'_{\beta(\ell)})\}_{\ell\geq n}$ converges to b_n in B_n . Therefore the sequence of points $\{r_{n\ell}f'_{\ell}(e'_{\beta(\ell)})\}_{\ell\geq n}$ converges to a point $e_n \in F_n$. Moreover, since $\underline{p}'$ is a level map and for $m \geq n$, $q_{nm}(b_m) = b_n$, if $\{r_{m\ell}f'_{\ell}(e'_{\beta(\ell)})\}_{\ell\geq m}$ converges to a point $e_m \in F_m$ then $r_{nm}(e_m) = e_n$. So, the sequence of points (e_n) determines a point $e \in F \subset E$ such that for each n $r_n(e) = e_n$.

Notice that in the sequence $\{r_{n\ell}f'_{\ell}(e'_{\beta(\ell)})\}$ of points of F_n, any two consecutive points and so any two points are connected by a path in F_n. Also F_n is an ANR, so for every ϵ-ball there is a contractible ball around e_n in the ϵ-ball in F_n which contains all but finitely many points of the sequence. Therefore there is a path ω_n joining e_n and $f'_n(e'_{\beta(n)})$ in F_n.

Let $h_{[\omega_n]} : \pi_q(E_n,F_n,e_n) \xrightarrow{\cong} \pi_q(E_n,F_n,f'_n(e'_{\beta(n)}))$ be the induced isomorphism by ω_n.

Let $\hat{r}_{nm*} : \pi_q(E_m,F_m,e_m) \to \pi_q(E_n,F_n,e_n)$ be the homomorphism defined by

$$\hat{r}_{nm*} = h^{-1}_{[\omega_n]} \cdot L^{nm}_{e'*} \cdot r_{nm*} \cdot h_{[\omega_m]} ,$$

for all $m \geq n$.

$$\begin{tikzcd}
\pi_q(E_m,F_m,e_m) \arrow[r, "h_{[\omega_m]}"] \arrow[dd, "\hat{r}_{nm*}"] & \pi_q(E_m,F_m,f'_m(e'_{\beta(m)})) \arrow[d] \\
 & \pi_q(E_n,F_n,r_{nm}f'_m(e'_{\beta(m)})) \arrow[d] \\
\pi_q(E_n,F_n,e_n) & \pi_q(E_n,F_n,f'_n(e'_{\beta(n)})) \arrow[l, "h^{-1}_{[\omega_n]}"']
\end{tikzcd}$$

for $\ell \geq m \geq n$, $\hat{r}_{n\ell *} = \hat{r}_{nm*} \cdot \hat{r}_{m\ell *}$. Hence $(\pi_q(\underline{E},\underline{F},\underline{e});\ \hat{r}_{nm*})$ is an object of a pro-group or is an inverse sequence of groups.

Define a morphism of pro-groups

$$g_* = (g_{n*}) : (\pi_q(\underline{E}',\underline{F}',\underline{e}'),\ r'_{nm*}) \to (\pi_q(\underline{E},\underline{F},\underline{e}),\ \hat{r}_{nm*})$$

by $g_{n*} = h^{-1}_{[\omega_n]} \cdot f'_{n*}$ for every n.

We have to show that for $m \geq n$,

$$g_{n*} \cdot r'_{\beta(n)\beta(m)*} = \hat{r}_{nm*} \cdot g_{m*} .$$

First observe that for $m \geq n$ the two continuous functions

$$r_{nm} \cdot f'_m, \; f'_n \cdot r'_{\beta(n)\beta(m)} : (E'_{\beta(m)}, F'_{\beta(m)}) \to (E_n, F_n)$$

of pairs are homotopic and the homotopy is L^{nm} . Hence

(9) $$L^{nm}_{e'*} \cdot r_{nm*} \cdot f'_{m*} = f'_{n*} \cdot r'_{\beta(n)\beta(m)*}$$

Now,

$$\begin{aligned}\hat{r}_{nm*} \cdot g_{m*} &= h^{-1}_{[\omega_n]} \cdot L^{nm}_{e'*} \cdot r_{nm*} \cdot h_{[\omega_m]} \cdot h^{-1}_{[\omega_m]} \cdot f'_{m*} \\ &= h^{-1}_{[\omega_n]} \cdot L^{nm}_{e'*} \, r_{nm*} \cdot f'_{m*} \\ &= h^{-1}_{[\omega_n]} \cdot f'_{n*} \cdot r'_{\beta(n)\beta(m)*} \qquad \text{(by (9))} \\ &= g_{n*} \cdot r'_{\beta(n)\beta(m)*} \quad .\end{aligned}$$

Also note that another choice of $\epsilon > 0$, say $\epsilon' > 0$ gives another map $\underline{f}'' : (\underline{E}', \underline{F}') \to (\underline{E}, \underline{F})$ of pairs of ANR-sequences such that $\underline{f}'' \sim \underline{f}'$.

Hence the induced homomorphisms $\underline{g}'_*$ and $\underline{g}_*$ are equivalent in pro-groups.

By modifying the arguments of the Mardesic's theorem 2 in [11], we will prove that

$$\underline{p}_* = (p_{n*}) : (\pi_q(E_n, F_n, e_n), \hat{r}_{nm*}) \to (\pi_q(B_n, C_n, b_n), q_{nm*})$$

is an isomorphism of pro-groups. First we will prove

(10) $$p_{n*} \hat{r}_{nm*} = q_{nm*} \, p_{m*} \quad .$$

$$\begin{tikzcd}
\pi_q(E_m,F_m,e_m) \arrow[r, "\hat{r}_{nm*}"] \arrow[d, "p_{m*}"] & \pi_q(E_n,F_n,e_n) \arrow[d, "p_{n*}"] \\
\pi_q(B_m,C_m,b_m) \arrow[r, "q_{nm*}"] & \pi_q(B_n,C_n,b_n)
\end{tikzcd}$$

Consider the diagram 1 on page 100. Since $\underline{p}$ is a level map

$$p_{n*}\, r_{nm*} = q_{nm*} \cdot p_{m*} \quad .$$

In the first square, $h_{[\mu_m]}$ is an isomorphism induced by a path $p_m(\omega_m) = \mu_m : I \to B_m$ connecting b_m and $p_m f'_m(e'_{\beta(m')})$ where $\omega_n : I \to E_m$ is the path connecting e_m and $f'_m(e'_{\beta(m)})$. Hence

$$p_{m*} \cdot h_{[\omega_m]} = h_{[\mu_m]} \cdot p_{m*} \quad .$$

By similar arguments

$$p_{n*} \cdot h^{-1}_{[\omega_n]} = h^{-1}_{[\mu_n]} \cdot p_{n*}$$

$$p_n\, L^{nm}_{e'*} = \hat{L}^{nm}_{e'*}\, p_{n*} \quad .$$

Write $\hat{q}_{nm*} = h^{-1}_{[\mu_n]} \cdot \hat{L}^{nm}_{e'*} \cdot q_{nm*} \cdot h_{[\mu_m]*}$. Note that by the previous arguments we showed

(11) $$p_{n*}\, \hat{r}_{nm*} = \hat{q}_{nm*}\, p_{m*} \quad .$$

Also observe that the paths

$r_{nm}(\omega_m) : I \to F_n$ from e_n to $r_{nm} f'_m(e'_{\beta(m)})$

$L^{nm}_{e'} : I \to F_n$ from $r_{nm} f'_m(e'_{\beta(m)})$ to $f'_n(e'_{\beta(n)})$

and $\omega^{-1}_n : I \to F_n$ from $f'_n(e'_{\beta(n)})$ to e_n

form a loop $\sigma_n : I \to F_n$ at e_n . Hence $p_n \sigma_n : I \to C_n$ is a loop at b_n . Consider the following diagram on page 101.

$$\begin{tikzcd}
\pi_q(E_m, F_m, e_m) \arrow[r, "h_{[\omega_m]}", "\cong"'] \arrow[d, "p_{m*}"] & \pi_q(E_m, F_m, f'_m(e'_{\beta(m)})) \arrow[r, "r_{nm*}"] \arrow[d, "p_{m*}"] & \pi_q(E_n, F_n, r_{nm} f'_m(e'_{\beta(m)})) \arrow[r, "L^{nm}_{e'*}", "\cong"'] \arrow[d, "p_{n*}"] & \pi_q(E_n, F_n, f'_n(e'_{\beta(n)})) \arrow[r, "h^{-1}_{[\omega_n]}", "\cong"'] \arrow[d, "p_{n*}"] & \pi_q(E_n, F_n, e_n) \arrow[d, "p_{n*}"] \\
\pi_q(B_m, C_m, b_m) \arrow[r, "\cong", "h_{[\mu_m]}"'] & \pi_q(B_m, C_m, p_m f'_m(e'_{\beta(m)})) \arrow[r, "q_{nm*}"'] & \pi_q(B_n, C_n, p_n r_{nm} f'_m(e'_{\beta(m)})) \arrow[r, "\cong", "\hat{L}^{nm}_{e'*}"'] & \pi_q(B_n, C_n, p_n f'_n(e'_{\beta(n)})) \arrow[r, "\cong", "h^{-1}_{[\mu_n]}"'] & \pi_q(B_n, C_n, b_n)
\end{tikzcd}$$

Diagram 1.

$$\begin{tikzcd}
\pi_q(E_m,F_m,e_m) \arrow[r, "r_{nm*}"] \arrow[d, "p_{m*}"] & \pi_q(E_b,F_n,e_n) \arrow[r, "h_{[\sigma_n]}", "\cong"'] \arrow[d, "p_{n*}"] & \pi_q(E_n,F_n,e_n) \arrow[d, "p_{n*}"] \\
\pi_q(B_m,C_m,b_m) \arrow[r, "q_{nm*}"] & \pi_q(B_n,C_n,b_n) \arrow[r, "h_{[p_n\sigma_n]}", "\cong"'] & \pi_q(B_n,C_n,b_n)
\end{tikzcd}$$

where $h_{[\sigma_n]}$ and $h_{[p_n\sigma_n]}$ are the isomorphisms induced by the loops σ_n and $p_n\sigma_n$ respectively. By the construction $h_{[\sigma_n]} \cdot r_{nm*} = \hat{r}_{nm*}$ and $h_{[p_n\sigma_n]} \cdot q_{nm*} = \hat{q}_{nm*}$. But C_n is contractible, so the loop $p_n\sigma_n$ is homotopically a trivial loop which implies that

$$h_{[p_n\sigma_n]} = 1_{\pi_q(B_n,C_n,b_n)} \quad .$$

Therefore, $\hat{q}_{nm*} = h_{[p_n\sigma_n]} \cdot q_{nm*} = q_{nm*}$. By (11), $p_{n*}\hat{r}_{nm*} = q_{nm*}p_{m*}$. Next, we will show that

$$\underline{p}_* = (p_{n*}) : (\pi_q(E_n,F_n,e_n);\ \hat{r}_{nm*}) \to (\pi_q(B_n,C_n,b_n);\ q_{nm*})$$

is an isomorphism of pro-groups.

It suffices to show that for each n there is an index $\ell > n$ and a homomorphism $K : \pi_q(B_\ell,C_\ell,b_\ell) \to \pi_q(E_n,F_n,e_n)$ such that the following diagram commutes.

$$\begin{tikzcd}
\pi_q(E_\ell,F_\ell,e_\ell) \arrow[r, "\hat{r}_{n\ell*}"] \arrow[d, "p_{\ell*}"] & \pi_q(E_n,F_n,e_n) \arrow[d, "p_{n*}"] \\
\pi_q(B_\ell,C_\ell,b_\ell) \arrow[r, "q_{n\ell*}"] \arrow[ur, dashed] & \pi_q(B_n,C_n,b_n)
\end{tikzcd}$$

Let ℓ be the lifting index for m and m be the lifting index for n.

Let $[\varphi] \in \pi_q(B_\ell, C_\ell, b_\ell)$ where $\varphi : (I^q, \partial I^q, J^{q-1}) \to (B_\ell, C_\ell, b_\ell)$ is the continuous function and

$$J^{q-1} = (\partial I^{q-1} \times I) \cup (I^{q-1} \times 1) .$$

Let $h_{[\mu_\ell]}[\varphi] = [\varphi'] \in \pi_q(B_\ell, C_\ell, p_\ell f'_\ell(e'_{\beta(\ell)}))$. Let $\bar{e}' : J^{q-1} \to E_\ell$ be the constant map $f'_\ell(e'_{\beta(\ell)})$. Then $p_\ell \bar{e}' = \varphi' | J^{q-1}$. Since $(I^q, J^{q-1}) \approx (I^q, I^{q-1} \times 0)$ and ℓ is a lifting index for m there is a continuous function

$$\begin{tikzcd}
I^{q-1} \times 0 \arrow[r, "\bar{e}'"] \arrow[d, hook] & E_\ell \arrow[r, "r_{m\ell}"] \arrow[d, "p_\ell"] & E_m \arrow[d, "p_m"] \\
I^{q-1} \times I \arrow[r, "\varphi'"] \arrow[urr, dashed, "\tilde{\varphi}"] & B_\ell \arrow[r, "q_{m\ell}"'] & B_m
\end{tikzcd}$$

$\tilde{\varphi} : I^q \to E_m$ such that

(12) $$\tilde{\varphi} | J^{q-1} = r_{m\ell} \, \bar{e}'$$

and

(13) $$p_m \tilde{\varphi} = q_{m\ell} \, \varphi' .$$

Since $p_m \tilde{\varphi}(\partial I^q) \subset Q_m$, $\tilde{\varphi}(\partial I^q) \subset F_m$. Hence $\tilde{\varphi}$ is a continuous function

$$(I^q, \partial I^q, J^{q-1}) \to (E_m, F_m, r_{m\ell} f'_\ell(e'_{\beta(\ell)})) .$$

Thus $[\tilde{\varphi}] \in \pi_q(E_m, F_m, r_{m\ell} f'_\ell(e'_{\beta(\ell)}))$. Define

$$K : \pi_q(B_\ell, C_\ell, b_\ell) \to \pi_q(E_n, F_n, e_n)$$

by $$K[\varphi] = \hat{r}_{nm*} \, h^{-1}_{[\omega_m]} \cdot L^{m\ell}_{e'*} \, [\tilde{\varphi}] .$$

We have to show that (i) K is well-defined;

(ii) K is a homomorphism;

(iii) $p_{n*} K = q_{n\ell *}$ and

(iv) $K \cdot p_{\ell *} = \hat{r}_{n\ell *}$.

(i) <u>To show that</u> K <u>is well-defined</u>.

Let $\psi \in [\varphi] \in \pi_q(B_\ell, C_\ell, b_\ell)$. Then

$$\psi' = h_{[\mu_\ell]}(\psi) \in [\varphi'] \in \pi_q(B_\ell, C_\ell, p_\ell f'_\ell(e'_{\beta(\ell)}))$$

and there is a homotopy

$$H : (I^q, \partial I^q, J^{q-1}) \times I \to (B_\ell, C_\ell, b_\ell)$$

such that $H_0 = \varphi'$ and $H_1 = \psi'$.

Define a continuous function

$$k : (I^q \times 0) \cup (I^q \times 1) \cup (J^{q-1} \times I) \to E_m$$

by
$$k|I^q \times 0 = \widetilde{\varphi}$$
$$k|I^q \times 1 = \widetilde{\psi}$$
$$k|J^{q-1} \times I = r_{m\ell} f'_\ell(e'_{\beta(\ell)}) \text{ .}$$

Then
$$p_m \widetilde{\varphi} = q_{m\ell}\, \varphi'$$
$$p_m \widetilde{\psi} = q_{m\ell}\, \psi'$$

imply that $p_m k = q_{m\ell} H|(I^q \times 0) \cup (I^q \times 1) \cup (J^{q-1} \times I)$.

Since m is a lifting index for n, there is a homotopy $\widetilde{H}: I^q \times I \to E_n$ such that

$$\widetilde{H}_0 = r_{nm}\, \widetilde{\varphi} \tag{14}$$

$$\widetilde{H}_1 = r_{nm}\, \widetilde{\psi} \tag{15}$$

$$\widetilde{H}|J^{q-1} \times I = r_{nm} \cdot r_{m\ell} f'_\ell(e'_{\beta(\ell)}) \quad \text{and} \tag{16}$$

$$p_n \widetilde{H} = q_{n\ell}\, H \text{ .} \tag{17}$$

Notice that $H(\partial I^q \times I) \subseteq C_\ell$. Hence $q_{n\ell} H(\partial I^q \times I) \subset Q_n$. Consequently by (17) $p_n \widetilde{H}(\partial I^q \times I) \subset Q_n$, which implies that $\widetilde{H}(\partial I^q \times I) \subseteq p_n^{-1}(Q_n) \subseteq F_n$. In other words, $\widetilde{H}$ is a map

$$(I^q, \partial I^q, J^{q-1}) \times I \to (E_n, F_n, r_{nm} r_{m\ell} f'_\ell(e'_{\beta(\ell)})) \text{ .}$$

By (14) and (15)

$$r_{nm}\, \widetilde{\psi} \in [r_{nm}\, \widetilde{\varphi}] \in \pi_q(E_n, F_n, r_{nm} r_{m\ell} f'_\ell(e'_{\beta(\ell)})) \text{ .}$$

This implies $h^{-1}_{[\omega_n]} \cdot L^{nm}_{e'*} \cdot (r_{nm} L^{m\ell}_{e'*})[r_{nm} \widetilde{\psi}] = h^{-1}_{[\omega_n]} \cdot L^{nm}_{e'*} \cdot (r_{nm} L^{m\ell}_{e'*})[r_{nm} \widetilde{\varphi}]$,

But $h^{-1}_{[\omega_n]} \cdot L^{nm}_{e'*} \cdot (r_{nm} L^{m\ell}_{e'*})[r_{nm} \widetilde{\varphi}] = h^{-1}_{[\omega_n]} \cdot L^{nm}_{e'*}(r_{nm} L^{m\ell}_{e'*}) r_{nm*}[\widetilde{\varphi}]$

$$= h^{-1}_{[\omega_n]} \cdot L^{nm}_{e'*} \cdot r_{nm*} L^{m\ell}_{e'*}[\widetilde{\varphi}]$$

$$= \hat{r}_{nm*} \cdot h^{-1}_{[\omega_m]} \cdot L^{m\ell}_{e'*}\,[\widetilde{\varphi}]$$

$$= K[\varphi] \, .$$

Hence $K[\psi] = K[\varphi]$.

(ii) <u>To show that</u> K <u>is a homomorphism</u>

Let $[\varphi] = [\varphi_1][\varphi_2] \in \pi_q(B_\ell, C_\ell, b_\ell)$. Then φ is represented by a continuous function

$$\varphi : (I^q, \partial I^q, J^{q-1}) \rightarrow (B_\ell, C_\ell, b_\ell)$$

given by

$$\varphi(x,s,t) = \begin{cases} \varphi_1(x,2s,t) & 0 \le s \le \frac{1}{2} \\ \varphi_2(x,2s-1,t) & \frac{1}{2} \le s \le 1 \end{cases}$$

where $x \in I^{q-2}, s,t \in I$.

Let $h_{[\mu_\ell]}[\varphi] = h_{[\mu_\ell]}[\varphi_1] \cdot h_{[\mu_\ell]}[\varphi_2] = [\varphi'_1][\varphi'_2]$. φ'_1 and φ'_2 induce $\widetilde{\varphi}_1, \widetilde{\varphi}_2 : (I^q, \partial I^q, J^{q-1}) \rightarrow (E_m, F_m, r_{m\ell} f'_\ell(e'_{\beta(\ell)}))$ such that

(18) $\widetilde{\varphi}_1 | J^{q-1} = r_{m\ell} \bar{e}' = \widetilde{\varphi}_2 | J^{q-1}$ and

(19) $p_m \widetilde{\varphi}_1 = q_{m\ell} \varphi'_1 \; ; \; p_m \widetilde{\varphi}_2 = q_{m\ell} \varphi'_2$.

Define $\widetilde{\varphi}: (I^q, \partial I^q, J^{q-1}) \rightarrow (E_\ell, F_\ell, r_{m\ell} f'_\ell(e'_{\beta(\ell)}))$ by

$$\widetilde{\varphi}(x,s,t) = \begin{cases} \widetilde{\varphi}_1(x,2s,t) & 0 \le s \le \frac{1}{2} \\ \widetilde{\varphi}_2(x,2s-1,t) & \frac{1}{2} \le s \le 1 \end{cases}$$

where $x \in I^{q-2}$, $s,t \in I$. Now,

$$K[\varphi] = \hat{r}_{nm*}\, h^{-1}_{[\omega_m]} \cdot L^{m\ell}_{e'*}[\widetilde{\varphi}] = \hat{r}_{nm*}\, h^{-1}_{[\omega_m]} \cdot L^{m\ell}_{e'*}[\widetilde{\varphi}_1][\widetilde{\varphi}_2]$$

$$= \hat{r}_{nm*}\, h^{-1}_{[\omega_m]}\, L^{m\ell}_{e'*}[\widetilde{\varphi}_1] \cdot \hat{r}_{nm*}\, h^{-1}_{[\omega_m]}\, L^{m\ell}_{e'*}[\widetilde{\varphi}_2]$$

$$= K[\widetilde{\varphi}_1] \cdot K[\widetilde{\varphi}_2] \; .$$

(iii) Let $[\varphi] \in \pi_q(B_\ell, C_\ell, b_\ell)$. Then

$$\begin{aligned} p_{n*}\, K[\varphi] &= p_{n*}\, \hat{r}_{nm*}\, h^{-1}_{[\omega_m]} \cdot L^{m\ell}_{e'*}[\widetilde{\varphi}] \\ &= q_{nm*}\, p_{m*}\, h^{-1}_{[\omega_m]}\, L^{m\ell}_{e'*}[\widetilde{\varphi}] \\ &= q_{nm*}\, \hat{q}_{m\ell*}[\varphi] \\ &= q_{nm*} \cdot q_{m\ell*}[\varphi] \\ &= q_{n\ell*}[\varphi] \end{aligned}$$

(iv) Let $[\psi] \in \pi_q(E_\ell, F_\ell, e_\ell)$. Then

$p_{\ell*}[\psi] = [p_\ell \cdot \psi] = [\varphi]$ say. Let $[\widetilde{\varphi}] = r_{m\ell*} h_{[\omega_\ell]}[\psi] = r_{m\ell*}[\psi']$. Then $\widetilde{\varphi}|J^{q-1} = r_{m\ell} f'_\ell(e'_{\beta(\ell)}) = r_{m\ell}\bar{e}'$ and $p_m\widetilde{\varphi} = p_m r_{m\ell}\psi' = q_{m\ell}p_\ell\psi' = q_{m\ell}\varphi'$ where $h_{[\mu_\ell]}[\varphi] = [\varphi'] = p_{\ell*}h_{[\omega_\ell]}[\psi]$. Hence $\widetilde{\varphi}$ satisfies the conditions (12) and (13). Therefore,

$$\begin{aligned} K \cdot p_{\ell*}[\psi] &= K[\varphi] \\ &= \hat{r}_{nm*}\, h^{-1}_{[\omega_m]}\, L^{m\ell}_{e'*}[\widetilde{\varphi}] \\ &= \hat{r}_{nm*}\, h^{-1}_{[\omega_m]}\, L^{m\ell}_{e'*}\, r_{m\ell*} h_{[\omega_\ell]}[\psi] \\ &= \hat{r}_{nm*} \cdot \hat{r}_{m\ell*}[\psi] \\ &= \hat{r}_{n\ell*}[\psi] \; . \end{aligned}$$

Now we have to show that the homomorphism $\underline{g}_*$ is equal to the isomorphism

$$(\underline{p}_*)^{-1} \cdot (\underline{p}'_*) : (\pi_q(\underline{E}', \underline{F}', \underline{e}');\, r'_{nm*}) \to (\pi_q(\underline{E}, \underline{F}, \underline{e}),\, \hat{r}_{nm*}) \; .$$

It suffices to show that for each n there is an index $\ell' \geq \ell$ such that the following diagram commutes.

$$\begin{tikzcd}
& \pi_q(E'_{\beta(\ell')}, F'_{\beta(\ell')}, e'_{\beta(\ell')}) \arrow[dl, "g_{\ell'*}"'] \arrow[dr, "p'_{\beta(\ell')*}"] & \\
\pi_q(E_{\ell'}, F_{\ell'}, e_{\ell'}) \arrow[ddr, "\hat{r}_{m\ell'*}"] & & \pi_q(B_{\beta(\ell')}, C_{\beta(\ell')}, b_{\beta(\ell')}) \arrow[d, "q_{\ell\beta(\ell')*}"] \\
& & \pi_q(B_{\ell}, C_{\ell}, b_{\ell}) \arrow[dl, "K"] \\
& \pi_q(E_n, F_n, e_n) &
\end{tikzcd}$$

where the index ℓ and the homomorphism K are as defined before.

Let ϵ' be a positive number such that $\epsilon' <$ diameter C_{ℓ}. Choose $\delta \in \Gamma(B_{\ell}, \epsilon')$. Since $\underline{f}' = (\beta, f'_n) : (\underline{E}', \underline{F}') \to (\underline{E}, \underline{F})$ is a fiber shape map, for ℓ and δ, there is an index ℓ^* such that for all $\ell' \geq \ell^*$,

$$d(q_{\ell\beta(\ell')} p'_{\beta(\ell')}, q_{\ell\ell'} p'_{\ell'} f'_{\ell'}) < \delta . \tag{20}$$

$$\begin{tikzcd}
E'_{\beta(\ell')} \arrow[r, "f'_{\ell'}"] \arrow[d, "p'_{\beta(\ell')}"] & E_{\ell'} \arrow[r, dashed, "r_{\ell\ell'}"] \arrow[d, "p_{\ell'}"] & E_{\ell} \arrow[d, dashed, "p_{\ell}"] \\
B_{\beta(\ell')} \arrow[r, "q_{\ell'\beta(\ell')}"'] & B_{\ell'} \arrow[r, "q_{\ell\ell'}"'] & B_{\ell}
\end{tikzcd}$$

Notice that δ is so small that any two δ-close maps are connected by an ϵ'-homotopy.

Let $\underline{\psi} \in \pi_q(\underline{E}', \underline{F}', \underline{e}')$ and $[\psi] \in \pi_q(E'_{\beta(\ell')}, F'_{\beta(\ell')}, e'_{\beta(\ell')})$ which is represented by a continuous function

$$\psi : (I^q, \partial I^q, J^{q-1}) \to (E'_{\beta(\ell')}, F'_{\beta(\ell')}, e'_{\beta(\ell')}) .$$

Note that

(21) $$h_{[\mu_{\ell'}]} p_{\ell' *} h^{-1}_{[\omega_{\ell'}]} = p_{\ell' *}$$

where $\omega_{\ell'} : I \to E_{\ell'}$ is a path from $e_{\ell'}$ to $f'_{\ell'}(e'_{\beta(\ell')})$ and $\mu_{\ell'} = p_{\ell'} \cdot \omega_{\ell'}$.

Since $q_{\ell\ell'} p_{\ell'} = p_\ell \, r_{\ell\ell'}$

(22) $$q_{\ell\ell' *} \, p_{\ell' *} = p_{\ell *} \, r_{\ell\ell' *} .$$

Now by the choice of ℓ' and $\delta > 0$, (20) implies that there is a homotopy $G' : (E'_{\beta(\ell')}, F'_{\beta(\ell')}) \times I \to (B_\ell, C_\ell)$ such that

$$G'_0 = p_\ell \, r_{\ell\ell'} f'_{\ell'} \quad \text{and}$$
$$G'_1 = q_{\ell\beta(\ell')} p'_{\beta(\ell')} \quad .$$

Hence $G = G' \cdot (\psi \times 1_I) : (I^q, \partial I^q) \to (B_\ell, C_\ell)$ is a homotopy such that

(23) $$G_0 = G'_0 \psi = p_\ell r_{\ell\ell'} f'_{\ell'} \psi \quad \text{and}$$

(24) $$G_1 = G'_1 \psi = q_{\ell\beta(\ell')} p'_{\beta(\ell')} \psi .$$

Therefore,

(23') $$[G_0] = p_{\ell *} \, r_{\ell\ell' *} f'_{\ell' *} [\psi] \quad \text{and}$$

(24') $$[G_1] = q_{\ell\beta(\ell') *} \, p'_{\beta(\ell') *} [\psi] .$$

Now the path $\mu_{\ell'} = p_{\ell'} \omega_{\ell'} : I \to B_{\ell'}$ from $b_{\ell'}$ to $p_{\ell'} f_{\ell'} (e'_{\beta(\ell')})$ induces a homotopy $H' : (B_{\ell'}, C_{\ell'}) \times I \to (B_{\ell'}, C_{\ell'})$ such that $H'_0 = 1_{(B_{\ell'}, C_{\ell'})}$ and $H' | b_{\ell'} \times I = \mu_{\ell'}$.

Then $q_{\ell\ell'} \cdot H' : (B_{\ell'}, C_{\ell'}) \times I \to (B_\ell, C_\ell)$ is a homotopy such that $q_{\ell\ell'} H'_0 = q_{\ell\ell'}$ and $q_{\ell\ell'} H' | b_{\ell'} \times I = q_{\ell\ell'} \, \mu_{\ell'} : I \to B_\ell$ is a path from b_ℓ to $p_\ell r_{\ell\ell'} f'_{\ell'} (e'_{\beta(\ell')})$.

$$\begin{tikzcd}
\pi_q(E'_{\beta(\lambda')},F'_{\beta(\lambda')},e'_{\beta(\lambda')}) \arrow[rr, "f'_{\lambda'*}"] \arrow[dd, "p'_{\beta(\lambda')*}"] & & \pi_q(E_{\lambda'},F_{\lambda'},f'_{\lambda'}(e'_{\beta(\lambda')})) \arrow[r, "r_{\lambda\lambda'*}"] \arrow[dl] \arrow[dd, "p'_{\lambda'*}"] & \pi_q(E_\lambda,F_\lambda,r_{\lambda\lambda'}f'_{\lambda'}(e'_{\beta(\lambda')})) \arrow[r, "L^{\lambda\lambda'}_{e'*}"] \arrow[dd, "p_{\lambda*}"] & \pi_q(E_\lambda,F_\lambda,f'_\lambda(e'_{\beta(\lambda)})) \arrow[r] \arrow[dd, "p_{\lambda*}"] & {} \\
 & \pi_q(E_{\lambda'},F_{\lambda'},e_{\lambda'}) \arrow[dd, "p'_{\lambda'*}"] & & & & \\
\pi_q(B_{\beta(\lambda')},C_{\beta(\lambda')},b_{\beta(\lambda')}) \arrow[dr, "q_{\lambda'\beta(\lambda')*}"'] & & \pi_q(B_{\lambda'},C_{\lambda'},p_{\lambda'}f'_{\lambda'}(e'_{\beta(\lambda')})) \arrow[r, "q_{\lambda\lambda'*}"] \arrow[dl, "h^{-1}_{[\mu_{\lambda'}]}"] & \pi_q(B_\lambda,C_\lambda,p_\lambda r_{\lambda\lambda'}f'_{\lambda'}(e'_{\beta(\lambda')})) \arrow[r, "\hat{L}^{\lambda\lambda'}_{e'*}"] & \pi_q(B_\lambda,C_\lambda,p_\lambda f'_\lambda(e'_{\beta(\lambda)})) \arrow[r] & {} \\
 & \pi_q(B_{\lambda'},C_{\lambda'},b_{\lambda'}) \arrow[r, "q_{\lambda\lambda'*}"'] & \pi_q(B_\lambda,C_\lambda,b_\lambda) \arrow[ur, "h_{[q_{\lambda\lambda'}\mu_{\lambda'}]}"] \arrow[urr] & & &
\end{tikzcd}$$

Diagram 2.

$$\begin{tikzcd}
{} \arrow[r, "h^{-1}_{[\omega_\ell]}"] & \pi_q(E_\ell,F_\ell,e_\ell) \arrow[r, "h_{[\omega_\ell]}"] \arrow[d, "p_{\ell *}"] & \pi_q(E_\ell,F_\ell,f'_\ell(e'_{\beta(\ell)})) \arrow[r, "r_{m\ell *}"] \arrow[d, "p_{\ell *}"] & \pi_q(E_m,F_m,r_{m\ell}f'_\ell(e'_{\beta(\ell)})) \arrow[r, "L^{m\ell}_{e' *}"] \arrow[d, "p_{m *}"] & \pi_q(E_m,F_m,f'_m(e'_{\beta(m)})) \arrow[r, "h^{-1}_{[\omega_m]}"] \arrow[d, "p_{m *}"] & \pi_q(E_m,F_m,e_m) \arrow[d, "p_{m *}"] \\
{} \arrow[r, "h^{-1}_{[\mu_\ell]}"'] & \pi_q(B_\ell,C_\ell,b_\ell) \arrow[r, "h_{[\mu_\ell]}"'] & \pi_q(B_\ell,C_\ell,p_\ell f'_\ell(e'_{\beta(\ell)})) \arrow[r, "q_{m\ell *}"'] & \pi_q(B_m,C_m,p_m r_{m\ell} f'_\ell(e'_{\beta(\ell)})) \arrow[r, "\hat{L}^{m\ell}_{e' *}"'] & \pi_q(B_m,C_m,p_m f'_m(e'_{\beta(m)})) \arrow[r, "h^{-1}_{[\mu_m]}"'] & \pi_q(B_m,C_m,b_m)
\end{tikzcd}$$

Diagram 2.

Let $H = q_{\ell\ell'} \cdot H' \cdot (q_{\ell'\beta(\ell')} p'_{\beta(\ell')} \psi \times 1_I)$. Then $H : (I^q, \partial I^q) \times I \to (B_\ell, C_\ell)$ is a homotopy such that

(25) $$H_0 = G_1 \quad \text{and} \quad H|J^{q-1} \times I = q_{\ell\ell'} H'|b_{\ell'} \times I \, .$$

Note that

(26) $$h_{[q_{\ell\ell'}, \mu_{\ell'}]}[H_0] = [H_1] \in \pi_q(B_\ell, C_\ell, p_\ell r_{\ell\ell'} f'_{\ell'}(e'_{\beta(\ell)})) \, .$$

Define a homotopy $M : (I^q, \partial I^q) \times I \to (B_\ell, C_\ell)$ by

$$M(x,s) = \begin{cases} G(x,2s) & 0 \le s \le \frac{1}{2} \\ H(x,2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

for $x \in I^q$, $s \in I$.

For any $x \in J^{q-1}$, $M(x,s) : I \to C_\ell$ is a loop at $p_\ell r_{\ell\ell'} f'_{\ell'}(e'_{\beta(\ell')})$. Since C_ℓ is contractible in itself, this loop is contractible.

Hence $[M_0] = [M_1] \in \pi_q(B_\ell, C_\ell, p_\ell r_{\ell\ell'} f'_{\ell'}(e'_{\beta(\ell')}))$, i.e. $[G_0] = [H_1]$.

By (23)', (26), (25) and (24)' ,

(27) $$p_{\ell *} r_{\ell\ell' *} f'_{\ell' *}[\psi] = h_{[q_{\ell\ell'}, \mu_{\ell'}]} q_{\ell\beta(\ell') *} p'_{\beta(\ell') *}[\psi] \, .$$

Therefore

(28) $$h^{-1}_{[\mu_\ell]} \hat{L}^{\ell\ell'}_{e' *} p_{\ell *} r_{\ell\ell' *} f'_{\ell' *} [\psi]$$
$$= h^{-1}_{[\mu_\ell]} \hat{L}^{\ell\ell'}_{e' *} h_{[q_{\ell\ell'}, \mu_{\ell'}]} q_{\ell\ell' *} q_{\ell'\beta(\ell') *} p'_{\beta(\ell') *} [\psi]$$

which implies

(29) $$p_{\ell *} h^{-1}_{[\omega_\ell]} L^{\ell\ell'}_{e' *} r_{\ell\ell' *} f'_{\ell' *} [\psi]$$
$$= h^{-1}_{[\mu_\ell]} \hat{L}^{\ell\ell'}_{e' *} q_{\ell\ell' *} h^{-1}_{[\mu_{\ell'}]} q_{\ell'\beta(\ell') *} p'_{\beta(\ell') *} [\psi] \, .$$

In other words,

(30) $$p_{\ell *} \hat{r}_{\ell\ell' *} g_{\ell' *} [\psi] = \hat{q}_{\ell\ell' *} q_{\ell'\beta(\ell') *} p'_{\beta(\ell') *} [\psi] \, .$$

But $\hat{q}_{\ell\ell' *} = q_{\ell\ell' *}$. So,

$$(31) \qquad p_{\ell *}\, \hat{r}_{\ell\ell' *}\, g_{\ell' *}\, [\psi] = q_{\ell\ell' *}\, q_{\ell'\beta(\ell') *} p'_{\beta(\ell')}\, [\psi]\ .$$

Then by the same argument as in the proof of (iv) we conclude that

$$\hat{r}_{n\ell *}\, \hat{r}_{\ell\ell' *}\, g_{\ell' *}\, [\psi] = K\, q_{\ell\beta(\ell') *}\, p'_{\beta(\ell') *}\, [\psi]\ .$$

Since $\underline{\psi}$ is any element of $\pi_q(\underline{E}',\underline{F}',\underline{e}')$, this shows that $\underline{g}_*$ and $(\underline{p}_*)^{-1} \cdot (\underline{p}'_*)$ are equivalent in pro-groups. Hence

$$\lim_{\leftarrow} \underline{g}_* = \lim_{\leftarrow} (\underline{p}_*)^{-1}\, (\underline{p}'_*) : \check{\pi}_q(E,F,e') \to \check{\pi}_q(E,F,e)\ .$$

APPENDIX

Along with the Proposition 4.1, the following two lemmas are useful to show that the relation $\sim$ (of Definition 4.2) is an equivalence relation in the set of maps between compact metric spaces.

We will only consider ANR-sequences with bonding maps as inclusions. Let C and B be compact metric spaces embedded in Q and $\underline{C} = (C_n, r_{nm})$, $\underline{B} = (B_n, q_{nm})$ be ANR-sequences with $\bigcap C_n = C$ and $\bigcap B_n = B$. Let $g: C \to B$ be a map.

Lemma 1: Let $\underline{g}, \underline{g}': \underline{C} \to \underline{B}$ be level maps of ANR-sequences with $\varprojlim \underline{g} = \varprojlim \underline{g}' = g$. Then $\underline{g} \simeq \underline{g}'$ (Definition 4.1).

Proof: Consider $C \times I \subset Q \times I \approx Q$. Let $H: C \times I \to B$ be the constant homotopy, i.e. $H(c,t) = g(c)$ for all $c \in C$, $t \in I$. Since for each n, B_n is a compact ANR, with $B_{n+1} \subset B_n$, there is an index $m_n \geq n$ and a homotopy $H^n: C_{m_n} \times I \to B_n$, an extension of H such that $H^n_0 = g_n \cdot r_{nm_n}$, $H^n_1 = g'_n \cdot r_{nm_n}$ and $H^n | C_{m_{n+1}} \times I = H^{n+1}$.

By reindexing we get $\underline{g} \simeq \underline{g}'$.

Lemma 2: Let $f, g, h: C \to B$ be maps between compact metric spaces, $\underline{f}, \underline{g}: \underline{C} \to \underline{B}$ and $\underline{g}', \underline{h}: \underline{C}' = (C'_n, r'_{nm}) \to \underline{B}' = (B'_n, q'_{nm})$ be level maps of ANR-sequences such that $\varprojlim \underline{f} = f$, $\varprojlim \underline{g} = \varprojlim \underline{g}' = g$, $\varprojlim \underline{h} = h$, $\underline{f} \simeq \underline{g}$ and $\underline{g}' \simeq \underline{h}$. Then there are ANR-sequences $\underline{\tilde{C}}$, $\underline{\tilde{B}}$ and level maps $\underline{\tilde{f}}, \underline{\tilde{g}}, \underline{\tilde{g}}', \underline{\tilde{h}}: \underline{\tilde{C}} \to \underline{\tilde{B}}$ such that

$$\varprojlim \underline{\tilde{f}} = f, \ \varprojlim \underline{\tilde{g}} = \varprojlim \underline{\tilde{g}}' = g, \ \varprojlim \underline{\tilde{h}} = h, \ \underline{\tilde{f}} \simeq \underline{\tilde{g}} \text{ and } \underline{\tilde{g}}' \simeq \underline{\tilde{h}}.$$

Proof: Let $B_1 \sqcup B'_1$ be the disjoint union. For each $b \in B$, identify $b \in B_1$ to $b \in B'_1$. Let the quotient space be denoted by $B_1 \sqcup_B B'_1$. Note that for each $n = 1,2,\ldots$ $B_{n+1} \sqcup_B B'_{n+1} \subset B_n \sqcup_B B'_n$. Embed $B_1 \sqcup_B B'_1$ in Q. Choose a decreasing sequence $\underline{\tilde{B}}$ of compact ANR-neighborhoods

$\tilde{B}_n$ such that $\bigcap \tilde{B}_n = B$. For every n there is $n' \geq n$ such that $B_{n'} \underset{B}{\cup} B'_{n'} \subset \text{int.}\, \tilde{B}_n$.

Now, consider an ANR-sequence $\underline{C} \times \underline{C}' = \{C_n \times C'_n, r_{nm} \times r'_{nm}\}$. Without loss of generality we can assume that for each n, $C_n \times C'_n$ is a compact convenient ANR. Let $\Delta(C) = \{(c,c) \in C \times C\} \subset C_n \times C'_n$ be the diagonal. Since $\Delta(C)$ is a closed subspace of $C_n \times C'_n$ for each n, we can select a decreasing sequence $\tilde{\underline{C}}$ of compact ANR-neighborhoods $\tilde{C}_n$ of $\Delta(C)$ in $C_n \times C'_n$ such that for each n, $\tilde{C}_n \subset \text{int.}\, C_n \times C'_n$ and $\bigcap \tilde{C}_n = C$.

For each n, let $\pi_n: C_n \times C'_n \to C_n$ and $\pi'_n: C_n \times C'_n \to C'_n$ be the projections. We will again denote $\pi_n | \tilde{C}_n$ by π_n and $\pi'_n | \tilde{C}_n$ by π'_n . Clearly, by reindexing if necessary, $\underline{f},\underline{\pi}$, $\underline{g}\,\underline{\pi}$, $\underline{g}',\underline{\pi}'$, $\underline{h}\,\underline{\pi}'$: $\tilde{\underline{C}} \to \tilde{\underline{B}}$ are level maps of ANR-sequences with $\varprojlim \underline{f}\,\underline{\pi} = f$, $\varprojlim \underline{g}\,\underline{\pi} = \varprojlim \underline{g}'\,\underline{\pi}' = g$, $\varprojlim \underline{h}\,\underline{\pi}' = h$, $\underline{f}\,\underline{\pi} \simeq \underline{g}\,\underline{\pi}$ and $\underline{g}'\,\underline{\pi}' \simeq \underline{h}\,\underline{\pi}'$.

Note that by Lemma 1, $\underline{g}\,\underline{\pi} \simeq \underline{g}'\,\underline{\pi}'$ and by Lemma 2 and Proposition 4.1, $\underline{f}\,\underline{\pi} \simeq \underline{h}\,\underline{\pi}'$. Thus the relation $\sim$ of the Definition 4.2 is an equivalence relation. For general case Lemmas 1 and 2 are expected to be true with some generalization of the Definition 4.2. $\square$

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