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**SOME PROPERTIES OF 3-POLYTOPAL GRAPHS**

*City University of New York*

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**SOME PROPERTIES OF 3-POLYTOPAL GRAPHS**

**by**

**PETER JOFFE**

**A dissertation submitted to the Graduate Faculty  
in Mathematics in partial fulfillment of the  
requirements for the degree of Doctor of Philosophy,  
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This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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## Abstract

### SOME PROPERTIES OF 3-POLYTOPAL GRAPHS

by

Peter Joffe

Advisor: Professor Joseph Malkevitch

By a well-known theorem of Steinitz, a graph (without multiple edges or loops) is the edge graph of a convex 3-dimensional polytope if and only if it is planar and 3-connected. Plane graphs of this type (i.e., 3-polytopal), which we shall also refer to as p-graphs, are the objects of our investigation.

Theorem (Steinitz). Each p-graph may be generated from  $K_4$  (the tetrahedron) by a succession of edge-additions. Equivalently: each p-graph, other than  $K_4$ , contains an edge which is deletable (i.e., its removal, followed by the suppression of any resulting 2-valent vertices, yields a smaller p-graph).

We strengthen this theorem (and thus also the findings of Kotzig and Jucovic concerning edges of "low weight" in p-graphs) by demonstrating that:

**THEOREM.** Each p-graph  $\neq K_4$  contains a deletable edge shared by an  $i$ -gon and a  $j$ -gon such that  $i \leq 5$  and  $i+j \leq 13$ .

Some of the theory developed to obtain this is also employed to determine sets of generating operations for the bicubic p-graphs, for the cyclically-4-connected bicubic p-graphs, and-in a sense-for the Hamiltonian bicubic p-graphs. These are of interest in view of Barnette's well-known conjecture that all bicubic p-graphs are Hamiltonian.

A second line of inquiry is that of determining the existence-or otherwise- of a HIST (a spanning tree without 2-valent vertices) in graphs of

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various sorts-usually 3-polytopal. In this connection the following are proved: if  $G$  is connected with  $\geq 4$  vertices then  $G^2$  has a HIST; if  $G$  has  $n \geq 6$  vertices, each of valence  $\geq \frac{1}{2}n$ , then  $G$  has a HIST; the HIST problem for non-cubic 3-polytopal graphs is NP-hard. Further, we derive a necessary condition for a cubic plane graph to have a HIST - one that is also a sufficient condition for Halin graphs. In addition we construct two special (infinite) classes of non-cubic  $p$ -graphs whose members have no HIST's; one comprised of 4-valent cyclically-4-connected  $p$ -graphs, and the other comprised of  $p$ -graphs whose duals also possess no HIST. Also presented are two Eberhard type theorems concerning the face vectors of Halin graphs.

## ACKNOWLEDGEMENTS

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CHAPTER 1  
INTRODUCTION

1.1 PRELIMINARIES

The objects investigated in this work are, almost exclusively, the planar 3-connected graphs. These graphs are especially interesting because they are isomorphic to the graphs arising from convex 3-dimensional polytopes, and thus are also referred to as '3-polytopal graphs'. In developing a body of results concerning the structure of such graphs, we pursue two major lines of inquiry.

(I) Locating those 3-polytopal graphs that do, or do not, have a HIST (a spanning tree with no 2-valent vertices). Much of the stimulus for this work came from the ideas, and enthusiasm, of Joseph Malkevitch. (See [M1] of bibliography). Such questions (and related material on Halin graphs) appear in Chapter 2, where a 'Dirac-type' existence theorem is also proved.

(II) Defining sets of generating operations (via edge deletion) for certain subclasses of the 3-polytopal graphs; the bicubic 3-polytopal graphs, and the cyclically 4-connected bicubic 3-polytopal graphs, are two such classes. This material appears in Chapters 3 and 4, where various other theorems about deletable edges are to be found - including a 'Kotzig-type' result concerning deletable [contractible] edges of low weight. This result makes it possible to show that any plane 3-connected graph can be obtained from the tetrahedron by splitting faces with relatively few sides.

In order to make the work essentially self-contained, we utilize this chapter to define most of the terms (and notation) to be used. With very few exceptions, our usage is 'standard' to graph theory and in agreement with that of F. Harary's introductory text "Graph Theory"

(see [H1] of bibliography); however, we do use "path" and "circuit" to mean, respectively, "open path" and "simple circuit". The last sections of Chapter 1 deal with 3-polytopal graphs and also include some 'non-standard' terminology, and a few theorems. Someone familiar with the field will probably find it adequate to read just the last two sections of Chapter 1.

For convenience the sections of each chapter are numbered, but independantly of the numbering of the theorems. All numbering is 'within chapter', and all diagrams are located after Chapter 4.

## 1.2 GRAPHS

A graph  $G = (V, E)$  is comprised of a finite nonempty set  $V = V(G)$  of vertices (or, points) and a prescribed set  $E = E(G)$  of unordered pairs of distinct vertices of  $V$ . Each pair  $e = (x, y)$  of  $E$  is an edge (or line) which joins its endpoints  $x$  and  $y$ . (Edge  $e$  may also be denoted by  $xy$ ). Further: vertices joined by an edge are adjacent, and each endpoint of an edge is incident to that edge; 2 edges sharing a common vertex are incident.

By definition a graph  $G$  has no loop (a vertex joined to itself) and cannot have several (multiple) edges joining 2 vertices. In brief, both loops and multiple edges occur in a pseudograph whereas a multi-graph has no loops. Henceforth (unless indicated otherwise)  $G, G^*, G'', G', G_0, G_1, G_2 \dots$  and so forth will always denote graphs; and, for purely practical reasons, the first three (that is  $G, G^*$  and  $G''$ ) will be used most frequently.

$G$  and  $G''$  are isomorphic graphs if there exists a 1-to-1 correspondence between their vertex sets which preserves adjacency.

Obviously, isomorphism is an equivalence relation for graphs . Thus we will often refer to some diagram as "the graph"  $G$ . Two graphs are disjoint if they have no common vertex or edge.  $G$  is the union of  $G'$  and  $G''$ , written  $G = G' \cup G''$ , if:  $G'$  and  $G''$  are any two graphs - disjoint or not - such that  $V = V' \cup V''$  and  $E = E' \cup E''$ . The following will also prove useful. The number of edges (of  $G$ ) at, or incident to, a vertex  $x$  is the valence of  $x$  (in  $G$ ) or the degree of  $x$  (in  $G$ ) and is denoted by  $\deg x$ . Vertex  $x$  is isolated if  $\deg x = 0$ . If  $\deg x = 1$  then that edge incident to it is sometimes called a pendant edge. If  $\deg x = k$  then  $x$  is  $k$ -valent. If all vertices of  $G$  are  $k$ -valent, then  $G$  is a (regular) graph of degree  $k$  or a  $k$ -valent graph. A 3-valent graph is sometimes called cubic.

### 1.3 SUBGRAPHS

A graph  $G'' = (V'', E'')$  is a subgraph of the graph  $G = (V, E)$  if all its vertices and edges are in  $G$ . Further:

- (a) If  $V'' = V$ , then  $G''$  spans  $G$ .
- (b) If  $x$  and  $y$  are adjacent in  $G''$  whenever they are adjacent in  $G$ , then  $G''$  is an induced subgraph of  $G$ .
- (c) If  $G''$  is comprised only of edges (and their endpoints) of  $G$ , then  $G''$  is an edge subgraph.
- (d) If  $G''$  consists of all edges and vertices of  $G$ , except for one vertex  $x$  and all edges of  $G$  incident to  $x$ , then  $G''$  is said to be obtained from  $G$  by removing the vertex  $x$ , and is denoted by  $G - x$ . Equivalently  $G = G'' + x$ . (See figure 1.4).
- (e) If  $G''$  contains all vertices and edges of  $G$ , but for edge  $e$ , then  $G''$  was obtained from  $G$  by removing edge  $e$ , and we write  $G'' = G - e$  and  $G = G'' + e$ . (See figure 1.4).

(f) By the complement of  $G''$  in  $G$  is meant that subgraph  $G/G''$  comprised of all edges - and their endpoints - and vertices of  $G$  that are not in  $G''$ .

If  $S$  is some subset of  $V(G) \cup E(G)$  we may use  $G - S$  to denote that subgraph obtained from  $G$  by removing all vertices and edges in  $S$ .

#### 1.4 WALKS AND CONNECTEDNESS

A walk of  $G$  is an alternating sequence of vertices and edges, with at least one edge, such that each element is incident to the next. Such a sequence

$$W = v_1, (v_1, v_2), v_2, \dots, (v_{n-1}, v_n), v_n$$

is usually denoted by the briefer  $v_1, v_2, \dots, v_n$  (or even  $v_1 v_2 \dots v_n$  may be used) and is said to 'join'  $v_1$  and  $v_n$ . The walk  $W$  is a trail if all its edges are distinct, and an open path if all its vertices are distinct. If all points of the walk  $W$ , except for  $v_1 = v_n$ , are distinct then it is a simple circuit (or cycle). Note, unless indicated otherwise, the abbreviations path and circuit will henceforth be used in place of "path" and "simple circuit".

$G$  is connected if every pair of vertices is joined by a path, and each maximal connected subgraph of  $G$  is a component of  $G$ . Two subgraphs of  $G$  will be said to be separated if they lie within different components.  $G$  is  $n$ -connected [ $n$ -edge-connected] if one must remove at least  $n$  vertices [edges] to obtain a disconnected graph, or a vertex of degree 0.

THEOREM 1.1 (Proved in Harary [H1, p. 48].)

A graph  $G$  is  $n$ -connected [ $n$ -edge-connected] if and only if every pair of points is joined by  $n$  vertex-disjoint [edge-disjoint] paths.

REMARK: The first of these 2 results is generally attributed to both H. Whitney and K. Menger (circa 1930); strangely enough, the other (concerning edge-connectedness) was not formulated until about 1956 - almost 30 years later. Note that, by definition, if a graph is  $n$ -connected [ $n$ -edge-connected] it must also be  $i$ -connected ( $i$ -edge-connected) for all  $i$  less than  $n$ .

Lastly: a graph  $G$  is cyclically- $n$ -connected if one must 'cut' (i.e., remove) at least  $n$  of its edges in order to obtain a graph with 2 components that each contain a circuit.

There are several special types of graphs which will be dealt with quite frequently, and can now be defined.

A Hamiltonian graph is one containing a circuit which passes through every vertex - a Hamiltonian circuit or HC. Analogously a path of a graph is a Hamiltonian path if it contains all vertices of that graph. Note that, as shown in figure 1.5, we generally indicate Hamiltonian circuits and paths by using 'squiggled edges'.

A graph  $G = (V, E)$  is a bipartite graph (or bigraph) if  $V$  can be partitioned into 2 subsets  $V'$  and  $V''$  such that every edge of  $G$  joins a vertex of  $V'$  to a vertex of  $V''$ . If  $G$  is bipartite and cubic (that is, 3-valent) then it is sometimes called a bicubic graph.

THEOREM 1.2 [H1, p. 18]

A graph is bipartite if and only if all its circuits are even.

A graph is acyclic if it has no circuits, and a tree is a connected acyclic graph (see figure 1.6). Any graph, connected or not, without circuits is a forest.

THEOREM 1.3 [H1, p. 32]

The following statements are equivalent for a graph  $G = (V, E)$ :  
G is a tree; every 2 vertices of G are joined by a unique path;  
G is connected and  $|E| = |V| - 1$ ; G is acyclic and  $|E| = |V| - 1$ .

### 1.5 EMBEDDINGS AND PLANAR GRAPHS

Let  $S$  be a topological surface lying in  $R^n$  (n-dimensional Euclidean space) and let  $G = (V, E)$  be some given graph. To say that G is drawn on S is to identify  $G$  with some specific configuration  $D = (N, X, f)$  which 'realizes'  $G$  in the following way:  $N$  is a set of  $|V|$  points on  $S$  (the nodes of  $D$ ) and  $f$  is some bijection of  $N$  onto  $V$ ,  $X$  is a collection of  $|E|$  arcs (the 'edges' of  $D$ ) that also lie on  $S$  and have their endpoints in  $N$ , and nodes  $p$  and  $q$  of  $N$  are joined by an arc of  $X$  if and only if  $f(p)$  and  $f(q)$  are adjacent in  $G$ . If so we may say that " $D$  is the graph  $G$ ". If  $G = D$  is such that edges intersect only at nodes, then it is said to be a geometric graph, and to be embedded in S.

Clearly a graph  $G$ , whether abstract or not, can always be embedded in  $R^3$  and can also be drawn on the plane  $R^2$  so that 2 edges 'cross' at most once; thus  $G$  can be embedded in the surface of a sphere which has at most  $|E(G)|^2$  handles. If  $G$  can be embedded in the plane then it is a planar graph. (Kuratowski characterized such graphs in 1930. See [H1, p. 108].) Finally, by "the plane graph  $G$ " is meant some particular embedding of  $G$  in the plane. For example,

the isomorphic graphs in figure 1.7 are both drawn on the plane (the plane of the page), but the one on the left is an embedding - and thus a plane graph - whereas the other is neither.

### 1.6 FACES OF PLANE GRAPHS

If a graph  $G = (V,E)$  is embedded in a surface  $S$  which is either a plane or a sphere in  $R^3$ , then consider the topologically open region  $S - G$ . It is the disjoint union of one or more connected open regions of  $S$ . Each of these, plus its (topological) closure - which must 'coincide' with some subset of  $G$ 's edges and vertices, is called a face of  $G$  and the set of all faces of  $G$  may be denoted by  $F(G)$ . If  $F$  is a face of  $G$  then that subgraph of  $G$  that 'coincides with' (and defines) the closure of  $F$  is the boundary of  $F$ , and may be denoted by  $dF$ : each edge and vertex of  $dF$  is said to be incident to, or with, face  $F$  and to be an edge or vertex of  $F$ . Lastly, 2 faces of  $G$  are adjacent faces if their boundaries share an edge; an edge on (the boundary of) only 1 face is sometimes called a 1-sided edge (see  $xy$  in figure 1.8).

REMARK 1. If  $G$  is embedded in the plane then it follows from the definition of "face" that  $G$  has precisely 1 face with an infinite area (usually called the exterior face and sometimes denoted by  $X$ ) and any other face  $F$ , regardless of the nature of its boundary  $dF$ , must be 'enclosed within' a unique circuit comprised of 3 or more edges of  $dF$ . For example, consider the 10-vertex graph of figure 1.8. The boundary of face  $F$  is comprised of 7 distinct edges (squiggled), yet  $F$  is 'bounded by' the 3-circuit  $abca$  which is not a boundary.

REMARK 2. For embeddings of 2-connected graphs (either in the plane or in the sphere) the situation is much neater. Every edge must be on a circuit and it can readily be shown (using remark 1) that every face must have a circuit for its boundary, and so there are no 1-sided edges.

A face  $F$  of a plane 2-connected graph is said to be a  $k$ -gon if (the boundary of) that face contains  $k$  edges, and we may write  $|F| = k$ . By a triangle is meant a 3-gon (in plane pseudographs 1-gons and 2-gons are called, respectively, monogons and digons) and by a small face will be meant one with 5 or fewer edges. For 2-connected graphs, a circuit bounding a face may itself be referred to as a "face" (see [H1, p.103]).

Two plane graphs may have the same 'underlying' graph (that is, they are isomorphic), yet their face sets may differ. (See figure 1.9, for example). A plane graph  $G$  is said to be combinatorially equivalent to plane graph  $G''$  if  $f$  is some isomorphism of  $G$  onto  $G''$  and  $r$  some bijection of  $F(G)$  onto  $F(G'')$  such that:  $e = (xy)$  is an edge of face  $A$  of  $G$  if and only if edge  $(f(x), f(y))$  is an edge of face  $r(A)$  of  $G''$ . Speaking informally: "f preserves faces and their adjacencies". (See figure 1.10, for example).

THEOREM 1.4 [H1, p. 105]

Any 2-connected plane graph  $G$  can be 're-embedded' in the plane so that any specified face  $F$ , that is not the exterior face, becomes the exterior face of a 'new' plane graph  $G''$  that is combinatorially equivalent to  $G$ .

PROOF: Assume  $F$  is not the exterior face, or we are done.

Pick some point  $N$  within the interior of  $F$  and consider a sphere  $S$

beneath the plane  $P$  (containing  $G$ ) which is tangent to it at  $N$ . Embed  $G$  onto  $S$  ( $S$  should be sufficiently large) via a perpendicular projection downwards. Next consider another plane,  $P''$ , which is parallel to the first and also tangent to the sphere. Let the spherical embedding of  $G$  be stereographically projected onto plane  $P''$ , with  $N$  as the 'pole'. This is  $G''$  which, obviously, is combinatorially equivalent to  $G$ .

REMARK. A plane graph  $G''$  will be called an elementary re-embedding of some plane graph  $G$  if it can be derived from the latter via the process described in theorem 1.4.

Suppose  $G$  is a connected plane graph. Let  $p, v, e, p_i$  denote the number of faces, vertices, edges and  $i$ -gons of  $G$  respectively. The following are essentially due to Euler:

THEOREM 1.5 [H1, p. 103]

$$e + 2 = p + v .$$

THEOREM 1.6 (Proved in Grunbaum [G3, p. 208].)

If  $G$  is 3-valent then  $\sum_1 (6 - i)p_i = 12$ . Equivalently:

$$3p_3 + 2p_4 + p_5 = 12 + \sum_{i \geq 6} (6 - i)p_i .$$

PROOF:  $3v = 2e = \sum_1 ip_i$  and  $p = \sum_1 p_i$  .

Also, by theorem 1.5,  $6e + 12 = 6p + 6v$ . So, substituting in this for  $e, p$  and  $v$  we obtain our result.

### 1.7 3-CONNECTED PLANE GRAPHS

A 3-polytope is the convex hull of a finite non-coplanar set of points in Euclidean 3-space.  $G$  is said to be a 3-polytopal graph if it is isomorphic to the graph defined by the edges and vertices of some 3-polytope.

THEOREM 1.7 (Proved in [G1, p. 235] and in [G3, p. 204].)

Every 3-polytopal graph is planar and 3-connected. (There are 10 non-isomorphic plane graphs that are 3-connected and have 6 or fewer faces; they are presented in figure 1.11.) It is the converse of theorem 1.7 which is the difficult half of Steinitz's well known characterization of 3-connected planar graphs:

THEOREM 1.8 (E. Steinitz; proved in [G1, p. 235] and in [BG1,p.27].)

A graph is 3-polytopal if and only if it is planar and 3-connected.

One can also characterize those plane graphs that are 3-connected in terms of incidence amongst faces. This will prove quite useful, even though the proof is by elementary means.

DEFINITION. A plane graph  $G$  will be said to have property-p if, and only if, both of the following hold:

(a)  $G$  is 2-connected and each vertex has valence 3 or more.

(b) No pair of faces can 'touch twice' (i.e., can have a multiply-connected union); that is, if two faces share vertices  $u$  and  $v$  then  $u$  and  $v$  are adjacent and edge  $uv$  is shared by both faces.

REMARK. It follows readily from this definition that a graph with property-p cannot have a pair of faces which: share 3 or more vertices; share 2 or more edges.

The graphs of figure 1.9 obviously do not have property-p, whereas all 10 graphs of figure 1.11 do. Now consider the 3 theorems which follow; the first is obvious and well known whereas the next two are, surprisingly, not too well known.

**THEOREM 1.9.** If  $G$  is a plane graph with at least 3 vertices, then it is 2-connected if and only if each face has a boundary that is a circuit.

**PROOF:** Immediate by the definition of 2-connectedness and remark 2 of section 1.5.

**THEOREM 1.10** (D. Barnette [B3, p. 37].)

If plane  $G$  has property-p (see definition preceding theorem 1.9) then  $G$  is 3-connected.

**REMARK.** Barnette's proof is by elementary means and does not invoke Steinitz's theorem. It is presented at the conclusion of this chapter.

**THEOREM 1.11** If plane  $G$  is 3-connected, then it must have property-p.

**REMARK.** In general one appeals to Steinitz's theorem to justify this (hardly startling) assertion, but the following elementary proof does not do so; and thus, with theorem 1.10, yields an independent characterization of those 2-connected plane graphs that are 3-connected.

**PROOF:** Assume that  $G$  is a plane 3-connected graph, but that it does not have property-p. Then (since condition (a) is obviously satisfied) it must be the case that: there exist 2 vertices, call them  $x$  and  $y$ , that are shared by 2 faces, say  $L$  and  $M$ ; but  $L$  and  $M$  do not share an edge joining  $x$  and  $y$  (which may, or may not, be adjacent). We will face the contradiction that  $G$  is not 3-connected.

Since  $G$  is 3-connected it is also 2-connected. So, by theorem 1.9, both  $L$  and  $M$  are bounded by circuits. Thus we can draw some simple closed curve,  $C$ , in the plane so that it passes through the interiors of both  $L$  and  $M$  without touching any edges or vertices - except for  $x$  and  $y$ . The edges of  $L$  inside [outside] curve  $C$  comprise a path joining  $x$  to  $y$ . With respect to the 2 paths lying within [exterior to]  $C$ , one on  $L$  and one on  $M$ : they may share an edge or one of them may have only 1 edge-joining  $x$  and  $y$ . Regardless, these 2 paths define a closed walk which can be 'traced' as follows: move from  $x$  to  $y$  along one and then back to  $x$  on the other. This walk must contain at least one vertex distinct from  $x$  and  $y$ ; or else  $x$  and  $y$  would be joined either by multiple edges (impossible) or by a single edge, the edge  $xy$  itself. This would mean that each of the aforementioned paths (one on  $L$  and one on  $M$ ) is actually the edge  $xy$ ; but then  $xy$  is shared by both  $L$  and  $M$  (we assumed it was not). Thus we have shown that there are 2 vertices distinct from  $x$  and  $y$  - one inside  $C$  and the other outside  $C$ . But then these 2 vertices could be separated by removing  $x$  and  $y$  from  $G$ , and so  $G$  is not 3-connected!

### 1.8 OPERATIONS ON PLANE GRAPHS

Given plane  $G$  we can construct a graph, or pseudograph,  $G^*$ , called the dual of  $G$  by doing as follows: place a vertex of  $G^*$  in each face of  $G$  and join 2 such vertices with one (or more) arcs if and only if the corresponding faces of  $G$  meet on an edge - precisely one arc across each edge shared by the 2 faces (see figure 1.13). Clearly  $G^*$  is always connected (and it is a graph if  $G$  has property-p).

We say that a plane graph  $G''$  is obtained from (a plane graph)  $G$  by face splitting provided we can get  $G''$  by adding an edge across some face of  $G$ . There are essentially 3 ways in which this can be done (see figure 1.14), depending on how many new vertices are produced. If  $e''$  denotes the edge added then we may write  $G'' = G \& e''$ . Henceforth face splitting and adding an edge (across a face) will refer to the same operation.

By deleting edge  $e$  of a plane graph  $G$  we mean, essentially, the 'inverse' operation to edge addition. That is, the plane graph (or multigraph)  $G \sim e$  is obtained from  $G$  by removing edge  $e$  (see section 1.3) and then 'suppressing' any 2-valent vertices. (See figure 1.4.) Note: if  $G' = G \sim e$  then  $G = G' \& e$ . We observe that the addition of an edge preserves property-p (and 3-connectedness) of a graph, whereas edge deletion may not (see figure 1.16).

If  $G$  is some 3-connected plane graph then, because it has property-p, its dual  $G^*$  is also a plane graph. Now if we split some face  $F$  of  $G$ , by placing a 'new' edge  $e$  across it, into two adjacent faces  $F_1$  and  $F_2$ , then the dual of  $G \& e$  can be obtained from  $G^*$  by: 'splitting' that vertex  $f^*$  of  $G^*$  which corresponds to face  $F$ , into two new vertices  $f_1^*$  and  $f_2^*$ , such that the new edge  $e^*$  joining them corresponds to edge  $e$ . To be precise (and also more general), by splitting vertex  $x$  of a plane 3-connected graph  $G$  will be meant the following: adding a vertex  $x'$  to  $G$ ; joining  $x$  and  $x'$  in the new graph, call it  $G'$ ; joining each vertex that was adjacent to  $x$  (in  $G$ ) to one, or to both, of  $x$  and  $x'$  of  $G'$  so that  $G'$  is also a plane graph with property-p; leaving all other adjacencies unchanged.

By contracting (or shrinking) an edge of a plane 3-connected graph is meant that reduction that is the inverse of vertex splitting: two adjacent vertices are identified, or 'coalesced', and any resulting set of multiple edges is replaced by a single edge. (Note that, as opposed to vertex splitting, this operation may not preserve 3-connectedness - see figure 1.15.)

Obviously both face and vertex splitting preserve property-p (and thus 3-connectedness) whereas the reductions that are their inverses (edge deletion and edge contraction, respectively) may not. In view of this we will say that: an edge  $e$  of a plane 3-connected graph  $G$  is deletable [contractible] if its deletion [contraction] results in a graph that is also 3-connected. In the literature one sometimes finds "remove" and "removable" used in place of "delete" and "deletable". The following theorem, which will be proved in Chapter 3, is one of the most well known of numerous 'edge-adding-generating-operations' theorems in the literature and was proved by Steinitz in 1922.

THEOREM 1.12 [BG1, p. 29]

If  $G$  is a plane 3-connected graph with more than 6 edges (that is, not the tetrahedron  $K_4$ ) then  $G$  has a deletable edge and, equivalently: there exists a sequence of plane 3-connected graphs

$$K_4 = G_0, G_1, \dots, G_{n-1}, G_n = G$$

such that each is obtained from its predecessor by the addition of an edge.

REMARK 1. This result does not provide (except in the 3-valent case) for a quick induction proof of the difficult half of Steinitz's theorem; although there have been several such improper "proofs" (see [G2, p.1137])

and [G3, p. 206] for a discussion of this).

REMARK 2. However Theorem 1.12 is sufficient to establish, via simple induction arguments (that utilize face and vertex splitting) the following - all ramifications of Steinitz's theorem:

THEOREM 1.13 (see [H1, p. 105 and p. 106] and [G3, p. 206].)

If  $G$  is a plane 3-connected graph then:

- (1) The dual of  $G$  is also 3-connected and (as can be shown by elementary means) its dual is just  $G$  itself.
- (2) The face of  $G$  are uniquely determined by its underlying graph; that is, they are 'preserved' regardless of how  $G$  may be re-embedded or re-drawn in the plane. (Originally proved by H. Whitney.)
- (3)  $G$  can be re-embedded in the plane so that the boundary of each  $n$ -gon is a convex polygon - but perhaps not a convex  $n$ -gon (to demonstrate the latter does require Steinitz's theorem).

REMARK: Because of (1) above, each theorem about plane 3-connected graphs obviously has a 'dual' formulation.

THEOREM 1.14 (H. Whitney; see [H1, p. 105] and [G3, p. 206].)

A plane graph, with 4 or more vertices, that is maximally planar (i.e., all faces are triangles) is 3-connected; and each such graph has as its dual a plane 3-valent 3-connected graph.

PROOF: Assume  $G^*$  is some maximally planar graph (with at least 4 vertices) that is not 3-connected. Thus  $G^*$  does not have property-p; and so some pair of vertices are common to 2 triangles, but no edge joining the former is common to both faces! But that is clearly

impossible - contradiction. One can similarly invoke "property-p" to obtain an elementary proof of the theorem's second assertion.

THEOREM 1.10 (PROOF)

Assume  $G$  is a plane graph with property-p and that  $V$  is a minimal set of vertices that disconnects  $G$ . Consider some vertex, call it  $v$ , of  $V$  and all the faces incident to  $v$ . Their boundaries are simple circuits containing  $v$  and thus, by property-p: the collection of all edge on these faces, minus those edges incident to  $v$ , comprises a simple circuit - call it  $T$ . The argument (and we quote directly from D. Barnette's proof) is concluded as follows:

"By the minimality of  $V$ , each component of the separated graph [i.e.,  $G - V$ ] will be joined to  $v$  by an edge, thus  $T$  has vertices in every component. Since  $T$  is a circuit, it required at least two vertices to separate it, thus  $V$  has at least three vertices and  $G$  is 3-connected."

CHAPTER 2

SPANNING TREES IN 3-POLYTOPAL GRAPHS

2.1 INTRODUCTION

If  $G$  denotes a plane 2-connected graph, or a convex 3-polytope, then one generally uses  $p_k(G)$ , or merely  $p_k$  (if no ambiguity results), to denote the number of  $k$ -sided faces ( $k$ -gons) of  $G$ . Analogously if  $T$  is a tree then by  $t_i(T)$ , or just  $t_i$ , will be meant the number of  $i$ -valent vertices of  $T$ .

THEOREM 2.1 (D. Barnette; see [B1, p. 731].)

Each 3-polytopal graph  $G$  has a spanning tree  $T$  with maximum valence 3.

REMARK. Obviously if  $T$  is a Hamiltonian path spanning  $G$ , then:

$$t_1 = 2, t_2 = |V(G)| - 2, t_3 = 0 \quad (2.1)$$

As J. Malkevitch observed (see [M1]), the following Euler type relation - from which  $t_2$  is absent - must be satisfied by every tree  $T$ :

$$t_1 = 2 + \sum_{i \geq 2} (i - 2)t_i \quad (2.2)$$

Note: if  $t_1$  and  $t_3$  are non-zero, then equation (2.2) becomes:

$$t_1 = 2 + t_3 \quad (2.2a)$$

In analogy with Eberhard's classic result (see theorem 2.3 below) we have the following theorem.

THEOREM 2.2 (Malkevitch; see [M1]).

If  $t_1^i, t_3^i, t_4^i, \dots, t_n^i$  is a sequence of non-negative integers satisfying equation (2.2) then there exists a non-negative integer  $t_2^i$ , an  $n$ -valent convex 3-polytope  $P$ , and a spanning tree  $T$  of  $P$ , such

that  $t_1(T) = t_1'$  for  $1 \leq i \leq n$ . ( $n = 3, 4, 5$ .)

If  $G$  is a 3-valent plane graph then it follows easily from Euler's formula ( $E + 2 = V + F$ ) that:

$$3p_3 + 2p_4 + 1p_5 = 12 + \sum_{k \geq 6} (k - 6)p_k \quad (2.3)$$

An early (and difficult) result which has given rise to much fruitful work, is the following.

THEOREM 2.3 (V. Eberhard; see [G1, p. 253].)

If  $p_3', p_4', p_5', p_7', p_8', \dots, p_n'$  is a non-negative integer solution of equation (2.3), then there exists a non-negative integer  $p_6'$  and a convex 3-valent 3-polytope  $P$  such that  $p_k(P) = p_k'$  for  $3 \leq k \leq n$ , and  $p_k(P) = 0$  for  $k > n$ .

REMARK. Note that  $p_6$  is absent from equation (2.3). For a non-trivial lower bound for  $p_6$  see theorem 2.4 below. Note also that the 4-valent analogue to Eberhard's theorem was first proved by B. Grünbaum (see [G1, p. 254].)

THEOREM 2.4 (D. Barnette [B2, p. 99].)

If  $P$  is a 3-valent 3-polytope, with  $p_k$   $k$ -gonal faces ( $k = 3, 4, 5, 6, \dots$ ) then:

$$2p_6 \geq 4 + p_3 - p_5 - 2 \sum_{k \geq 7} p_k, \text{ if } \sum_{k \geq 7} p_k \geq 3.$$

A theorem in the spirit of theorem 2.4 is the following result.

THEOREM 2.5 (Malkevitch [M1]).

If  $G$  arises from  $G^*$ , a plane 3-valent, 2-connected graph by 'truncation' of all of  $G^*$ 's vertices, then: each spanning tree of  $G$  has at least  $|V(G)|/3 - 2$  vertices of valence 2. (See figure 2.1 for example.)

REMARK. It will be proved later that  $G$  must actually possess some spanning tree with precisely  $|V(G)|/3 - 2$ , 2-valent vertices. (See figure 2.1(b).)

The discussion above (and we now quote Malkevitch [M].):

"... suggests studying the role that 2-valent vertices play in the theory of spanning trees, since the work on Eberhard's theorem shows the special role  $p_6$  plays in the realizability of 3-valent polytopes."

Furthermore, it seems to be the case that when a graph obeys a condition sufficient to guarantee a Hamiltonian circuit the same condition (or even a weaker related one) is sufficient to guarantee a spanning tree (for the graph) with no 2-valent vertex. The remainder of this chapter develops some of these ideas.

## 2.2 DEFINITIONS

By a p-graph will be meant a plane, 3-connected (and thus 3-polytopal) graph. See section 1.7 for the definition of "property-p" which provides a useful characterization of those plane graphs that are 3-connected.

By a proper tree will be meant a tree with no 2-valent vertex. More generally: a graph  $G$  will be said to be a proper graph if it is spanned by a proper tree  $T$  (in which case  $T$  may also be referred to as a HIST, or homeomorphically irreducible spanning tree).

By truncating a vertex  $v$  of valence  $k$  of a plane graph [pseudograph]  $G$  is meant that operation - see figure 2.2(a)- which 'replaces' the vertex  $v$  with a  $k$ -gon. By truncating a plane graph [pseudograph]  $G$  is meant the truncation of all the vertices of  $G$ ; the resulting plane graph may sometimes be denoted by truncation( $G$ ).

See figure 2.2(b).

Suppose  $G$  is a plane 2-connected graph (or pseudograph) and  $k$  is some positive integer ( $\geq 2$ ), then:  $G$  is a multi-k-gon graph if each face has some multiple of  $k$  sides; analogously,  $G$  is multi-k-valent if each vertex has a valence that is a multiple of  $k$ . Euler's formula can be used to show that  $k \leq 5$  if  $G$  is a graph.

A plane 4-valent graph  $G$  is said to be the medial graph of  $G''$  if  $G$  may be obtained from the plane graph  $G''$  by the following procedure (see figure 2.24 for example): let  $V(G) = E(G'')$  and join two vertices of  $G$  with an edge if and only if their 'corresponding' edges (in  $G''$ ) share a common endpoint, and belong to some common face.

### 2.3 THE SQUARE OF A CONNECTED GRAPH (WITH $\geq 4$ VERTICES) HAS A HIST.

By the square  $G^2$  of a graph  $G$  is meant that graph having:  $V(G^2) = V(G)$  and vertices  $x$  and  $y$  adjacent in  $G^2$  if and only if they are either adjacent or have a common adjacency (i.e., distance from  $x$  to  $y$  is 1 or 2) in  $G$ . As shown in figure 2.3, if  $G$  is connected with just 3-vertices then  $G^2$  is a 3-circuit (and so cannot have a HIST).

The theorem we shall prove is analogous to the well known result of H. Fleischner [F1, p. 29] that the square of every 2-connected graph is Hamiltonian. We commence with 2 short lemmas.

#### LEMMA 2.6

If  $H$  is a subgraph of  $G$ , then  $H^2$  is a subgraph of  $G^2$ .

PROOF: Immediate, by definition.

#### LEMMA 2.7

If  $P$  is a path (open) with  $n \geq 4$  vertices, then  $P^2$  has a HIST.

PROOF: The diagrams of figure 2.4 show an obvious way in which the 'new' edges of  $P^2$  (those not in  $P$  - in the diagrams they are curved arcs) may be used to draw a HIST.

THEOREM 2.8

If  $T$  is a tree with  $n \geq 4$  vertices, then  $T^2$  has a HIST.

PROOF: We will induct on  $m = |V(T)| \geq 4$ . The theorem is obviously true for  $m = 4$  (see figure 2.5). Now assume the result is true for all  $m$  such that  $4 \leq m \leq n-1$ . Consider a tree  $T$  with  $n \geq 5$  vertices. If  $T$  is simply a path then, by lemma 2.7, we are done; so assume not. So  $T$  contains a 'branch point', that is some vertex - call it  $x$  - of degree  $k \geq 3$ . Thus  $k$  distinct subtrees 'originate' at  $x$  (see figure 2.6); denote them by  $T_1, T_2, \dots, T_k$ . Any two have precisely one element of  $T$  in common - the vertex  $x$ , and their union is just  $T$ . By lemma 2.6 each  $(T_i)^2$  is a subgraph of  $T^2$ , so their union must also be a subgraph of  $T^2$ . So we want to 'cover' each  $(T_i)^2$  with a tree and then join them all together at  $x$ , so as to have a HIST for  $T^2$ . Well, if  $T_i$  has 4 or more vertices then - by the induction hypothesis - we can span  $(T_i)^2$  with a proper tree. If  $T_i$  is just an edge (as is  $T_2$  of figure 2.6) then that edge alone will be used. If one of the  $k$  trees has just 3-vertices - including  $x$  - then just use the construction of figure 2.7. Then, putting all the  $k$   $(T_i)^2$ 's back together, we obviously get a HIST.

THEOREM 2.9

If  $G$  is a connected graph with at least 4 vertices, then  $G^2$  has a HIST.

PROOF: Every connected graph  $G$  has a spanning tree,  $T$ , whose

square is a subgraph of  $G^2$  (by lemma 2.6); thus, by theorem 2.8, we are done.

#### 2.4 A DIRAC TYPE THEOREM

##### THEOREM 2.10

If a graph  $G$  has  $n \geq 6$  vertices, each of valence  $\geq \frac{1}{2}n$ , then  $G$  has a HIST.

REMARK. The proof that follows may be extended (see the remark following that proof for details) to prove the stronger Ore type result:

THEOREM. If  $G$  has  $n \geq 8$  vertices and for each pair of nonadjacent vertices  $x$  and  $y$ ,  $\deg x + \deg y \geq n$ , then  $G$  has a HIST.

PROOF (of theorem 2.10): We require a few preliminary definitions.

Let  $V$  denote  $V(G)$  and define  $L = \{x \in V: \deg x < \frac{1}{2}n\}$ ,  
 $M = \{x \in V: \deg x = \frac{1}{2}n\}$  and  $H = \{x \in V: \deg x > \frac{1}{2}n\}$ . Also, define a tree  $T$  in  $G$  to be a "good tree" or "good" if:  $T$  is proper (i.e., has no 2-valent vertex) and  $V(T)$  contains more than  $\frac{1}{2}n$  vertices - including all vertices of  $L$ , but some positive even number of vertices (of  $G$ ) are not in  $T$ .

Strategy of the proof: to construct a good tree,  $T$ , in  $G$  (see lemma 2.13 for the case in which  $L$  is empty) because then, by lemmas 2.11 and 2.12, it follows that either  $T$  can be extended to a HIST or - if not - another HIST must exist.

##### LEMMA 2.11

If  $T$  is a proper tree in  $G = (V, E)$  with vertex set  $W = V(T)$  (and  $|V| = n$ ,  $|W| = t$ ) such that  $n - t \geq 3$  and  $t > \frac{1}{2}n$  then: some vertex of  $T$  is adjacent (in  $G$ ) to 2 vertices of  $V - W$ , and so  $T$  may be 'extended' to a larger proper tree with 2 more vertices.

(Note:  $n - t$  need not be even.)

PROOF: We can invoke the "pigeon-hole Principle" if it can be shown that: the number of edges joining vertices of  $V - W$  to vertices of  $W$  is greater than the cardinality,  $t$ , of  $W$ . (For then some vertex in  $W$  will be joined by 2 edges to distinct vertices of  $V - W$ .) Now:  $V - W$  contains  $n - t$  vertices all of valence  $\geq \frac{1}{2}n$ , and obviously each has at least  $\frac{1}{2}n - (n - t - 1)$  adjacencies in  $W$ . So at least  $(n - t)(t + 1 - \frac{1}{2}n)$  edges 'run from'  $V - W$  to  $W$ . Thus we wish to demonstrate that:  $0 > (t) - (n - t)(t + 1 - \frac{1}{2}n)$ . If we substitute  $n - r$  for  $t$  (where, consequently,  $3 \leq r < \frac{1}{2}n$ ) we get:  $0 > (n - r) - (n - r)(\frac{1}{2}n - r + 1)$ . Which in turn simplifies to:  $0 > (2r - n)(r - 2)$ . And this last inequality must hold, since  $3 \leq r < \frac{1}{2}n$ .

LEMMA 2.12

If  $T$  is a proper tree in  $G$  and contains all but 2 vertices of the latter - say  $p$  and  $q$ , then  $G$  has a HIST.

PROOF: If  $p$  and  $q$  share a common adjacency in  $W = V(T)$  then, clearly, we are done. So assume not. But then it follows that  $p$  and  $q$  are adjacent. Reason: if not adjacent, each would have at least  $\frac{1}{2}n$  adjacencies in  $W$  (a set of  $n - 2$  vertices) but none in common - impossible! Thus the following must be the case:  $p$  and  $q$  are adjacent and each has at least  $\frac{1}{2}n - 1$  adjacencies in  $W$ , but none are shared; and  $|W| = n - 2$ . But such a situation can occur only if:  $n \geq 6$  and  $n$  is even; each vertex of  $T$  is adjacent to either  $p$  or  $q$ , and each of  $p$  and  $q$  is adjacent to precisely  $\frac{1}{2}$  of the 4 or more vertices of  $T$ . If such is the case (see figure 2.9) then, the set of all edges (of  $G$ ) containing  $p$  and/or  $q$  as an endpoint (and the edge  $pq$  is obviously one such edge) comprises a HIST in  $G$ .

LEMMA 2.13

If  $L$  is empty, then  $G$  contains a good tree  $T$ .

PROOF: We will consider 4 cases, depending on whether  $n \equiv 0, 1, 2$  or 3 modulo 4. (Note: each vertex of  $G$  has valence at least  $\frac{1}{2}n$ .)

CASE (i):  $n = 4k + 2$  ( $k \geq 1$ ). Each vertex is incident to at least  $2k + 1$  edges. To construct  $T$  we will just use some (any) vertex and precisely  $2k + 1$  ( $\geq 3$ ) of the edges incident to it. Thus  $T$  will have  $2k + 2 > \frac{1}{2}n$  vertices, and  $2k$  vertices of  $G$  will not be in  $T$ . So  $T$  is a 'good' tree.

CASE (ii):  $n = 4k + 3$  ( $k \geq 1$ ). In this case  $\frac{1}{2}n = 2k + 1\frac{1}{2}$ . So we will pick some vertex and use precisely  $2k + 2$  of the edges incident to it to form the tree  $T$ . Clearly  $T$  is 'good'.

CASE (iii):  $n = 4k + 1$  ( $k \geq 2$ ). Again we will just use edges incident to some vertex - precisely  $2k$  of them, to be exact.

CASE (iv):  $n = 4k$  ( $k \geq 2$ ). In this case we need a good tree  $T$  with an even number of vertices, and at least  $2k + 2$  of them. Now each vertex of  $G$  has at least  $2k$  edges incident to it, so define the proper tree  $T^*$  to be comprised of precisely  $2k$  ( $\geq 4$ ) edges that share a common vertex - call it  $x$ . But then:  $T^*$  has more than  $\frac{1}{2}n$  vertices and does not contain  $2k - 1 \geq 3$  vertices of  $G$ . So  $T^*$  satisfies the hypothesis of lemma 2.11. Thus, some vertex of  $T^*$  - call it  $y$  - is adjacent to 2 vertices not in  $T^*$ . Add these 2 edges (from  $y$  to the 2 vertices) to  $y$ , and so enlarge  $T^*$  by 2 vertices. Then 'discard' one of the  $2k$  edges incident to  $x$  (not edge  $xy$ ) to form the tree  $T$  we seek. (See figure 2.10.) Thus lemma 2.13 has been proved.

To conclude the theorem's proof, consider the following. By

lemma 2.13 we can find a 'good' tree,  $T$ , in  $G$  which does not contain some even number,  $r \geq 2$ , of vertices of  $G$ . If  $r = 2$  then lemma 2.12 completes the proof. If  $r = 4, 6, 8, \dots$ , then we can extend the proper tree  $T$  - two vertices at a time - until  $r$  is reduced to 2. The proof of theorem 2.10 is thus complete.

REMARK. In order to prove the stronger 'Ore-type' theorem one proves (and it is rather lengthy) the counterpart of lemma 2.13: that is, one proves that  $G$  contains a good tree when  $L \neq \emptyset$ . This may be done by considering 4 distinct cases:  $L \neq \emptyset$  and  $M \neq \emptyset$ ;  $M = \emptyset$  and  $|L| \geq 3$ ;  $M = \emptyset$  and  $|L| = 2$ ;  $M = \emptyset$  and  $|L| = 1$ .

## 2.5 3-POLYTOPAL GRAPHS: COMPARING THE HAMILTONIAN CIRCUIT PROBLEM AND THE HIST PROBLEM

### THEOREM 2.14

Given a cubic 3-polytopal graph  $G^*$  on  $n$  vertices, one can construct a (non-cubic) 3-polytopal graph  $G$ , with  $12n - 8$  vertices, such that:

(i) If  $G^*$  contains a Hamiltonian circuit  $H^*$ , then  $G$  contains a HIST,  $T$ , which is 'induced' by  $H^*$ .

(ii) Conversely, if  $G$  has a HIST,  $T$ , then  $T$  'induces' a Hamiltonian circuit in the cubic graph  $G^*$ .

PROOF: The diagrams of figure 2.11 indicate how  $G$  is to be obtained from  $G^*$ . Replace all but one (any one) vertex of  $G^*$  with the 12-vertex configuration shown in figure 2.11(a); replace the 'last' vertex, call it  $p^*$ , with a tetrahedron - as shown in diagram (b) - and let  $p$  denote the 3-valent vertex inside. Observe that  $G$  has property-p and is thus 3-connected. (In figure 2.12 we see an example

in which  $G^*$  is the tetrahedron.) We shall demonstrate that a HIST in  $G$  must span that graph in essentially the manner indicated in figure 2.13 (where  $G$  is a 40-vertex graph obtained from the tetrahedron).

The 3 diagrams of figure 2.13 indicate in a "typical case" precisely how an (arbitrary) HC in  $G^*$  induces an HC in  $G$ , an intermediate graph obtained by truncating all vertices of  $G^*$ , which in turn is easily extended to a HIST in  $G$ . Thus (i) of the theorem is proved.

The proof of (ii) is as follows.

Let us consider  $T$  to be a fixed HIST in  $G$ . Direct all of  $T$ 's edges in a well-defined manner, by starting at vertex  $p$  (viewed as the 'origin') and directing all edges (of  $T$ ) incident to  $p$  away from that vertex - and continue in the natural manner. Let each 12-vertex subgraph of  $G$  which replaced a vertex of  $G^*$  be called a "big unit"; and let the tetrahedron which replaced  $p^*$  be called the "small unit".

#### LEMMA 2.15

The directed edges of  $T$  induce a directed spanning tree (a subtree of  $T$ ) in each big unit. Moreover: each big unit must be 'visited' by  $T$  in the manner illustrated in figure 2.14, either as shown in (a) or as shown in (b). (Note that exactly 2 of the edges  $e_1, e_2, e_3$  are visited in each case.)

PROOF: The diagrams of figure 2.15 lead to an easy proof. In each case assume  $T$  contains precisely those edges of attachment (of the unit) that are squiggled - and uses them with the indicated directions; recall that  $T$  'starts' at  $p$  (which is not a vertex of the unit) and contains

no 2-valent vertices. But then it is easily seen that, in each case  $T$  could not possibly reach all 12 vertices of the unit. Thus it follows that  $T$  must use either 2 or 3 of the edges of attachment, in the manner shown in the 2 diagrams of figure 2.16; and this forces  $T$  to span the unit as illustrated in figure 2.14. Thus the lemma is proved.

LEMMA 2.16

The directed tree  $T$  must have valence 1 at vertex  $p$  and must use the edges and vertices of the small unit as illustrated in figure 2.17 (as shown in (a), as shown in (b), or as shown in (c)).

PROOF: Consider the 2 diagrams of figure 2.18. Clearly  $p$  cannot have valence 3 in  $T$ , or else  $T$  would 'not advance' - see diagram (a); so assume edge  $pq$  is the only edge of  $T$  incident to  $p$ . This implies that  $q$  has valence 3 or 4 (in  $T$ ), as shown in figures 2.17 and 2.18(b); but actually the last cannot occur or else  $T$  'could not advance'. Thus the lemma is proved.

PROOF (of theorem 2.14): Let us continue with the proof of assertion (ii) of the theorem. Consider  $G$  spanned by the HIST  $T$ , which satisfies lemmas 2.15 and 2.16, and collapse each of the  $n-1$  big units to a point; call the resulting graph  $G'$ .  $T$  has become a connected spanning subgraph of  $G'$ ; call it  $T'$ , because it is in fact a tree (since shrinking the big units could obviously not create circuits 'from' the remaining edges of the tree). Now consider the tree  $T'$  spanning  $G'$ .  $T'$  has just 2 or 3 vertices of valence 1 (all vertices of the small unit that was not collapsed). By equation (2.2) we know  $T'$  satisfies  $t'_1 = 2 + t'_3 + 2t'_4$ . Since  $t'_1 = 2$  or 3 it follows that  $t'_3 + 2t'_4 = 0$  or 1. This in turn means that  $t'_4 = 0$

and  $t_3' = 0$  or  $1$ . So, getting back to  $G$ , we immediately know that only configurations 2.14(a) and 2.17(a) can occur in  $G$ . Having sharpened lemmas 2.15 and 2.16 in this way, we conclude the theorem's proof by observing that: by collapsing each of the  $n$  units of  $G$  to a point (and thus 'recovering'  $G^*$ ), we cause the tree  $T$  to become a connected spanning subgraph of  $G^*$  of regular valence 2 - that is, an HC in  $G^*$ .

REMARK 1. By a simple parity argument it is easily demonstrated that one cannot strengthen the theorem - while still using the same general strategy - by constructing a  $G$  (from  $G^*$ ) that is also cubic.

Argument: Assume one can, and that some 3-valent configuration  $K$  has replaced a vertex of  $G^*$  in the new cubic graph  $G$ ;  $K$  must contain an odd number of vertices because 3 of them are endpoints of the 3 edges of attachment; but if  $G$  is spanned by a HIST,  $T$ , which uses only 2 of  $K$ 's edges of attachment, then - by equation (2.2) -  $K$  must have an even number of vertices!

REMARK 2. In 1976 Garey, Johnson and Tarjan [GJT1] proved that: the problem of determining if a cubic 3-polytopal graph contains an HC is an NP-complete problem. Thus, by theorem 2.14, it follows that: the problem of determining if a non-cubic 3-polytopal graph has a HIST is an NP-hard problem.

## 2.6 NECESSARY CONDITIONS FOR A CUBIC PLANE GRAPH TO HAVE A HIST

Recall that if  $G = (V,E,F)$  is a plane cubic graph with  $|V| = n$  vertices, then:  $n$  is even;  $|E| = 3n/2$ ;  $|F| = n/2 + 2$ .

If  $G = (V,E,F)$  is a plane cubic graph and  $M$  some subset of its face set  $F$ , then  $M$  will be said to be a mask of  $G$  if each of the following hold:

- (i) The faces of  $M$  are disjoint.
- (ii) Each face of  $G$  is either in  $M$  or contains an edge of some face of  $M$ .
- (iii) The edges, and vertices, of (the faces of  $M$ ) each total  $\frac{1}{2}|V| + 1$  in number.
- (iv) A vertex on one face of  $M$  cannot be joined by an edge of  $G$  to a vertex on another face of  $M$ .

Further, a mask  $M$  will be said to be a strong mask if and only if, the edges of  $G$  not in  $M$  comprise a HIST. By way of example, consider the strong masks (with faces shaded) in each of the graphs of figure 2.19. Note that not every mask is strong, as shown by the graph of figure 2.20 - a 38-vertex, bipartite and 3-polytopal graph.

The next result is hardly surprising.

THEOREM 2.17

If  $G = (V,E)$  is a plane cubic graph spanned by a HIST,  $T$ , then the edges not in  $T$  must comprise a strong mask.

PROOF: Removing the  $|V| - 1$  edges of  $T$  from  $G$  leaves a set of  $\frac{1}{2}|V| + 1$  edges, each with endpoints of valence 0 or 2. So these edges must define a subgraph that is the disjoint union of circuits of  $G$ ; let this set of circuits be denoted by  $M$ .

Now consider all those vertices of  $G$  that are not in some circuit of  $M$ . Every pair must be connected by a path lying wholly in the spanning tree  $T$ ; but, since we are in the plane,  $T$  obviously cannot 'cross' a circuit of  $M$  to join any such pair. So  $M$  must be comprised of faces of  $G$ , and is thus a mask. (Since each face not in  $M$  must be adjacent to a face of  $M$ , or else the tree  $T$  would not be acyclic!) Moreover,  $M$  must be a strong mask (because

the tree  $T$  is connected).

REMARK 1. As noted earlier, not every mask is strong (i.e., gives rise to a HIST). However, as will be shown in theorem 2.25, every mask of a cubic Halin graph is strong.

REMARK 2. It follows from theorem 2.17 that if a plane cubic graph has no mask, it can not have a HIST. (The cube is such a graph).

## 2.7 CUBIC 3-POLYTOPAL GRAPHS ARISING FROM TRUNCATIONS

In this section we obtain results concerning (the absence of a HIST in) those plane cubic graphs which arise by truncating all vertices of a  $k$ -valent ( $k = 3, 4$ , or  $5$ ) plane graph. (See Malkevitch [M1] for results, and conjectures, concerning this topic.) Note: no plane graph can have regular valence  $k \geq 6$  (see [H1, p. 104]). For the case in which  $k = 3$  we already have theorem 2.5, so we begin with the following.

### THEOREM 2.18

If  $G$  is a connected, bipartite, cubic, plane graph with  $n \equiv 0 \pmod{4}$  vertices, then  $G$  has no HIST.

PROOF:  $G$  cannot have a mask since all circuits are even in length, but  $n/2 + 1$  is odd.

REMARK. If  $n$  is not a multiple of 4 then  $G$  may have a HIST (see the bipartite 3-polytopal graph of figure 2.21).

### THEOREM 2.19

If  $G$  is a plane cubic graph obtained by truncating all vertices of some 4-valent plane graph, then  $G$  has no HIST.

PROOF: The proof is immediate by theorem 2.18, since  $G$  must be bipartite with  $|V(G)| \equiv 0 \pmod{4}$  vertices.

THEOREM 2.20

If  $G = (V, E)$  is a plane cubic graph obtained by truncating all vertices of some plane 5-valent graph  $G^*$ , then:  $|V(G^*)| \equiv 4 \pmod{6}$  is a necessary condition for the existence of a HIST in  $G$ .

PROOF: Let  $n = |V(G^*)|$ . Note that  $|V(G)| = 5n$  and that each vertex of  $G^*$  is 'replaced' by a pentagon in  $G$  (all pentagons of  $G$  arise thus, and are disjoint). Furthermore, each face  $F^*$  of  $G^*$  is replaced by one,  $F$ , with twice as many edges in  $G$  - and every other edge of  $F$  is an edge of a 'new' pentagon. Now assume  $M$  is a strong mask of  $G$  that is the complement (in  $G$ ) of some HIST.

There are precisely  $n$  pentagons in  $G$ ; assume  $k$  of them belong to  $M$  (and thus that  $n - k$  of them are not in  $M$ ). Thus, of  $M$ 's  $5n/2 + 1$  edges, precisely  $5k$  lie on the  $k$  pentagons of  $M$ . If  $P$  is one of the pentagons not in  $M$ , then  $P$  must be adjacent to a face of  $M$  (by definition of  $M$ ) that is not a pentagon; in fact  $P$  must touch just one face of  $M$  (because if  $P$  did touch say faces  $A$  and  $B$  of  $M$ , then one of  $P$ 's edges would join a vertex of  $A$  to a vertex of  $B$  - contradiction! See figure 2.22).

Now, there are  $n - k$  pentagons like  $P$  (in the sense that they are not in  $M$ ). Thus, by the preceding, they 'collectively' contribute  $n - k$  edges to the mask  $M$ ; and these edges must comprise precisely  $\frac{1}{2}$  of the edges on those mask faces which are not pentagons - that is, precisely  $\frac{1}{2}$  of  $(5n/2 + 1 - 5k)$ . Equivalently:

$$2(n - k) = 5n/2 + 1 - 5k,$$

which simplifies to  $n = 6k - 2$ .

CONJECTURE (see [M1]). Truncating all vertices of a plane, 5-valent, 2-connected graph produces a graph with no HIST.

THEOREM 2.21

Given:  $G^*$  a plane 2-connected cubic graph with  $n$  vertices;  
and that plane cubic graph  $G$ , with  $3n$  vertices, which arises from  
 $G^*$  by truncation of all its vertices.

Then:  $G$  contains a spanning tree  $T$  which has precisely  
 $|V(G)|/3 - 2$  vertices of valence 2.

REMARK. In view of theorem 2.5 this result is best possible in that no  
spanning tree of  $G$  could have fewer 2-valent vertices.

PROOF: By Petersen's theorem (see [H1, p. 89]) we know that  $G^*$   
contains a set  $S$  of  $n/2$  disjoint edges which, obviously, span the  
vertices of  $G^*$ . Let us start with these  $n/2$  edges of  $G^*$  as the  
'beginning' of the tree  $T$ . (See figure 2.23,) Next: when  $G^*$  is  
truncated to produce  $G$ , we can 'extend' each endpoint of each edge of  
 $S$ , so as to produce a spanning graph of  $G$  that is a forest - with no  
vertices of valence 2. This forest, call it  $F$ , has  $n$  3-valent  
vertices. If we now construct  $T$  by adding edges of  $G$  to  $F$  (and,  
as is well known and readily verified, this can be done) we will have  
a spanning tree  $T$  with  $t_3(T) \geq n$ . Since  $t_1 = 2 + t_3$ , we know that:

$$t_2 \leq 3n - (2n + 2) = n - 2 = |V(G)|/3 - 2.$$

Since theorem 2.5 tells us that:  $t_2 \geq |V(G)|/3 - 2$ , we have our result,  
since equality must hold.

2.8 4-VALENT, c4-CONNECTED 3-POLYTOPAL GRAPHS WHICH CONTAIN NO HIST

THEOREM 2.22

Given a plane cubic 3-polytopal graph  $G$ , with  $2n \geq 4$  vertices,  
one can construct a 4-valent 3-polytopal graph  $G^*$  (with  $27n$  vertices)  
such that  $G^*$  has no HIST.

REMARKS. It will be obvious that  $G^*$  (as it is to be defined) is  $c_4$ -connected but not 4-connected. Also, by refining the proof below, one may obtain stronger results in terms of the minimum number of 2-valent vertices contained in any spanning tree of  $G^*$ .

PROOF: The diagrams of figure 2.25 indicate how  $G^*$  is to be obtained from  $G$ : truncate all vertices of  $G$  to obtain  $G'$  with  $6n$  vertices; then truncate the vertices of  $G'$  to obtain  $G''$  with  $18n$  vertices; last, define  $G^*$  to be the medial graph of  $G''$ . (Also see figure 2.29, where  $G$  is a 2-vertex multigraph.)

Now consider  $G'$ : each of its  $6n$  vertices (as shown in figures 2.26 and 2.29) 'gives rise to' a triangle in  $G''$ ; and each such triangle is, in turn, replaced by a 6-vertex subgraph in  $G^*$ . Let each such subgraph of  $G^*$  be called a "unit" (see figure 2.27). Obviously there are  $6n$  units, each sharing its 3 'outer vertices' with other units.

Now assume that  $T$  is a HIST in  $G^*$ . (We wish to force a contradiction.)  $T$  must use  $27n-1$  edges of  $G^*$ ; which means an average of  $4\frac{1}{2} - 1/(6n)$  edges of  $T$  per unit. Obviously a 6-vertex unit  $U$  can contain at most 5 edges of  $T$ , in which case  $T$  would induce a spanning tree in  $U$  - call it  $T(U)$ . Obviously  $T(U)$  is connected and cannot have a vertex of valence 2 (in  $U$ ) unless it is an 'outer vertex' of  $U$ . But then it is trivial to verify that  $T(U)$  must appear as shown in figure 2.28! Assume that there are in all  $k$  units that are 'full' (i.e., contain 5 edges of  $T$ ). Then the  $6n-k$  remaining units each contain at most 4 edges of  $T$ . Consequently we have the inequality:

$$5(k) + 4(6n - k) \geq |E(T)| = 27n - 1$$

which simplifies to:  $k \geq 3n - 1$ . We obtain a contradiction as follows:

No 2 'full' units can share a vertex, or else (obviously)  $T$  would have a 2-valent vertex. Now it is clear that 2 units of  $G^*$  share an 'outer vertex' if and only if, the 2 vertices of  $G'$  from which they were 'obtained' are adjacent. But since all  $6n$  vertices of  $G'$  are on triangles, there are at most  $2n$  of them that are mutually non-adjacent; consequently  $G^*$  can have at most  $2n$  disjoint units. Thus it must be that  $k \leq 2n$  which, combined with  $k \geq 3n - 1$ , implies that:  $n = 1$  or, equivalently, that  $G$  has 2 vertices!

REMARK. Malkevitch [M1.] has conjectured that: every plane, 4-connected graph has a HIST. This appears to be a very hard question.

2.9 A CLASS OF 3-POLYTOPAL GRAPHS, NONE OF WHOSE MEMBERS - OR THEIR DUALS - HAVE A HIST

The idea underlying the next theorem is, roughly, to: 'stitch together'  $n$  ( $\geq 2$ ) configurations of the sort shown in figure 2.30 with  $n$  configurations of the sort shown in figure 2.31, to produce a 3-polytopal graph. (See figure 2.32 where we use  $n = 1$  by starting with a multigraph.) Note that the configurations in figure 2.30 and figure 2.31 are duals.

THEOREM 2.23

Given a plane cubic 3-polytopal graph  $G$  with  $2n$  vertices, one can construct a (non-regular) 3-polytopal graph  $G^*$ , with  $22n$  vertices, such that:

(a) Neither  $G^*$  nor its dual (which has  $20n + 2$  vertices) has a HIST.

(b) A tree spanning  $G^*$ , or its dual, must contain at least  $n$  2-valent vertices.

PROOF: Consider cubic  $G$  with  $2n$  vertices. Truncate all of its vertices and obtain  $2n$  triangles. (See figure 2.32 where  $n = 1$ .) Replace  $n$  of them (each) with the 14-vertex configuration of figure 2.30; then replace each of the other  $n$  triangles with the 9-vertex 'triangulation' of figure 2.31. (Note: the  $3n$  'old' edges of  $G$  are used to piece together these  $2n$  configurations.) This is  $G^*$ , which has property-p and is thus 3-connected.

With respect to the size of  $G^*$ : it obviously has  $13n + 9n = 22n$  vertices, and  $(n+2) + 6n + 13n = 20n + 2$  faces; and so its dual has  $20n + 2$  vertices.

Obviously any tree spanning  $G^*$  must have a 2-valent vertex within each 13-vertex configuration, if it is to 'reach' that point  $p$  in the middle; so the theorem is proved for  $G^*$ .

Now consider the dual of  $G^*$ : in that graph we must also find  $n$  configurations of the type shown in figure 2.30; and thus the theorem is proved for the dual of  $G^*$ . (in order to demonstrate the preceding claim it is sufficient to look at figure 2.33.) The proof of the theorem is complete.

CONJECTURE (Malkevitch [M1, p. 7]). If  $G$  is  $k$ -valent ( $k = 3,4,5$ ) and 3-polytopal, then  $G$  or its dual has a HIST.

## 2.10 HALIN GRAPHS AND TWO EBERHARD TYPE THEOREMS

A plane graph  $G$  is said to be a Halin graph if it may be obtained from some plane tree  $T$  (with at least 4 vertices, and none of valence 2) in the following manner (see, for example, figure 2.34): a simple closed curve is drawn (in the plane of  $T$ ) so as to pass through all the 1-valent vertices of  $T$  and to contain  $T$  within its closed interior. Such a graph  $G$  will sometimes be said to skirt the tree  $T$ ,

and we may refer to the latter as the underlying tree of  $G$ .

REMARK. Naturally every Halin graph has a HIST; further, as may readily be verified, all Halin graphs are 3-polytopal.

Each interior face of a Halin graph contains 1 edge of the exterior face; the edges and vertices of the exterior face are sometimes referred to as "rim edges" and "rim vertices"; and the exterior face itself may be referred to as the "rim" of  $G$ .

LEMMA 2.24 (To be used for theorems 2.27 and 2.28.)

If  $G$  is a Halin graph (other than a wheel) whose underlying tree is a (1,r)-tree (i.e., has valences 1 and  $r$  only) then: the face set of  $G$  contains 2 (disjoint) sets of  $r - 2$  'consecutive' triangles.

PROOF: Let  $T$  denote the underlying (1,r)-tree of  $G$ , which has at least 2  $r$ -valent vertices. If we remove all pendant edges from  $T$  we obtain a smaller tree which must have at least 2 vertices of valence 1. But then each of 2 such vertices must be incident (in  $T$ ) to precisely  $r - 1$  pendant edges (of  $T$ ), whose other endpoints appear 'consecutively' about the rim of  $G$  - and thus give rise to  $r - 2$  consecutive triangles. Thus the lemma is proved.

THEOREM 2.25

If  $G$  is a cubic Halin graph containing a mask  $M$ , then  $M$  is a strong mask. (See section 2.6 for definitions.) That is: the edges not in  $M$ , which is a subset of  $F(G)$ , comprise a HIST.

PROOF: Let  $M$  be a mask of  $G$  and let  $X$  denote the exterior face. Assume the theorem false; that is, the  $|V(G)| - 1$  edges not in  $M$  do not comprise a tree. Thus some of these edges must comprise a circuit - call it  $C$ . We force a contradiction as follows.

Within the closed region bounded by  $C$  there must lie some interior face of  $G$  that is not a face of  $M$  - call it  $A$ ; and  $A$  must be touched by some face of  $M$  - call it  $A^*$ . Now,  $A^*$  cannot touch  $A$  'across' the circuit  $C$ , and so  $A^*$  must also lie within the region bounded by  $C$  - but cannot contain an edge of  $C$ . But then how can  $A^*$ , which must be an interior face of  $G$ , contain an edge of the exterior face  $X$ ? It cannot, since it is 'bounded away' from  $X$  by the circuit  $C$  - contradiction! The theorem is proved.

We conclude this section with 2 Eberhard type theorems concerning the interior faces of certain 'regular' Halin graphs: those that are cubic, and those whose underlying tree has valencies 1 and 4 only.

Define a (1,r)-tree to be a tree with vertices of valence  $r$  and 1 only (and at least one of each).

THEOREM 2.26

Let  $G$  be a Halin graph with its underlying tree a  $(1,r)$ -tree (where  $r \geq 3$ ) and let  $p_k$  denote the number of interior faces of  $G$  that are  $k$ -gons, then:

$$\sum_{k \geq 3} (k - 3 - 2/(r - 2))p_k + (2 + 4/(r - 2)) = 0 \quad (1)$$

and

$$p_3 \geq 2r - 4 \quad (2)$$

Moreover, if  $G$  is a cubic Halin graph (that is,  $r = 3$ ) then:

$$2p_3 + p_4 = 6 + \sum_{k \geq 5} (k - 5)p_k, \text{ and } p_3 \geq 2 \quad (3)$$

Analogously, if  $r = 4$  then:

$$p_3 = 4 + \sum_{k \geq 4} (k - 4)p_k, \text{ and } p_3 \geq 4 \quad (4)$$

REMARK 1. By inspection of equation (1), it is clear that no coefficient (of some  $p_k$ ) is identically zero unless  $r = 3$  or  $4$ .

REMARK 2. Equality holds in equation (2) if and only if the following is the case: removing all the pendant edges from  $G$ 's underlying tree produces a simple path.

PROOF: The proof of equation (2) is immediate by lemma 2.24. To prove (1) let  $x$  denote the number of edges (and thus also of vertices) on the exterior face; also let  $n$  denote the number of vertices of valence  $r$  that do not lie on the exterior face.

From equation (2.2) we have:  $x = 2 + (r - 2)n$ , which is equivalent to:

$$n = (1/(r - 2))(x - 2) \quad (5)$$

Now:

$$x + \sum_{k \geq 3} k(p_k) = 2|E(G)| = rn + 3x \quad (6)$$

Thus:

$$\sum_{k \geq 3} k(p_k) = rn + 2x = r(1/(r-2))(x-2) + 2x \quad (7)$$

Or:

$$\sum_{k \geq 3} k(p_k) = (3 + 2/(r - 2))x - 2r/(r - 2) \quad (8)$$

But the infinite face contains 1 edge of each of the other faces.

So:

$$x = \sum_{k \geq 3} p_k \quad (9)$$

If we substitute this into equation (8) we obtain:

$$\sum_{k \geq 3} k(p_k) = (3 + 2/(r - 2)) \sum_{k \geq 3} p_k - 2r/(r - 2) \quad (10)$$

And equation (1) of the theorem follows from (10). The proof is complete.

**THEOREM 2.27**

If  $(p_3^*, p_5^*, p_6^*, \dots, p_n^*)$  is a non-negative integer solution of

equation (4), of theorem 2.26, then: there exists a Halin graph  $G$  with an underlying  $(1,4)$ -tree, such that  $G$  has precisely  $p_k^*$  interior  $k$ -gons ( $k = 3, 5, 6, \dots, n$ ).

PROOF: It is immediate by lemma 2.24 that a Halin graph with an underlying  $(1,4)$ -tree must contain a pair of adjacent 3-gons. As shown in figure 2.37, the 4-wheel (which is the smallest such graph) obviously has this property. In figure 2.38 we illustrate how one can 'enlarge a 3-gon to a 7-gon'. If we can construct a Halin graph with an underlying  $(1,4)$ -tree and  $p_k^*$   $k$ -gons (for each  $k$  such that  $5 \leq k \leq n$ ), then that graph proves the theorem; since, by theorem 2.26, the number of 3-gons (that is,  $p_3^*$ ) will automatically be 'taken care of'.

The presence of 2 adjacent 3-gons will always permit us to enlarge one of them to a  $k$ -gon (for any particular  $5 \leq k \leq n$ ) without changing the number of faces of other sizes (other than 3-gons, 4-gons, and  $k$ -gons, that is). Note that the operation in figure 2.38, which acts on adjacent 3-gons, creates 2 more adjacent 3-gons (labelled A and B) so that the process can be iterated. The proof is complete.

THEOREM 2.28

Given:  $(f_3, f_4, f_6, f_7, \dots, f_n)$  a non-negative integer solution of equation (3) of theorem 2.26. That is:

$$2f_3 + f_4 = 6 + \sum_{k \geq 5}^n (k - 5)f_k \quad \text{and} \quad f_3 \geq 2 \quad (***)$$

Then: there exists a cubic Halin graph,  $G$ , with precisely  $f_k$  interior  $k$ -gons ( $k = 3, 4, 6, 7, \dots, n$ ).

PROOF: Throughout this proof the word "face", unless indicated otherwise, will always mean "interior face". Now, consider the integer solution of the "Given". Define:  $s = \sum_{6 \leq k, k \text{ even}} (f_k)$ .

LEMMA 2.29

$$s \equiv f_4 \pmod{2} .$$

PROOF: The summation  $\sum_{k \geq 5}^n (k - 5)f_k$  of equation (\*\*\*) may be

re-written as:

$$\sum_{6 \leq k \text{ odd}}^n (k-5)f_k + \sum_{6 \leq k \text{ even}}^n (k-6)f_k + \sum_{6 \leq k \text{ even}}^n (1)f_k$$

where, obviously, the first 2 summations contribute even numbers to the sum. Thus by (\*\*\*) we have:

$$2f_3 + f_4 \equiv 6 + \sum_{6 \leq k \text{ even}} f_k \pmod{2}$$

And, since  $2f_3$  and 6 are each even, the lemma is proved.

By way of introducing all but one of the methods of construction, consider the graph of figure 2.39 (where all unlabelled faces are 3-gons or 5-gons). We will consider, in all, 9 cases. (Actually there are essentially 3 general cases, but some overlap will occur because it will permit ease of presentation.) With respect to  $s$  and  $f_4$  they are:

Case 1:  $f_4 = 0$  ,  $s = 0$  (see figure 2.40).

Case 2:  $f_4 = 2$  ,  $s = 0$  or  $2$  (see figure 2.41).

Case 3:  $f_4 = 0$  ,  $s = 2$  (see figure 2.42).

Case 4:  $f_4 = 1$  ,  $s = 1$  (see figure 2.43).

Case 2':  $f_4 = 2$  ,  $s$  is even.

Case 3':  $f_4 = 0$  ,  $s \geq 2$  and even.

Case 4':  $f_4 = 1$  ,  $s$  is odd.

Case 4'':  $f_4 \geq 3$  ,  $s \geq 1$ , and both are odd.

Case 2'':  $f_4 \geq 4$  ,  $s \geq 0$ , and both are even.

By the lemma it is clear that all cases are subsumed under the 9 above.

We now proceed to provide constructions for each case. The graph of

figure 2.39 conveys the general strategy, except for our first case.

Case 1: This is the exceptional case for which we define a particular 10-vertex cubic Halin graph - see  $G_1$  of figure 2.40 - with only 3-gons and 5-gons (as interior faces). By equation (\*\*\*) we know that  $f_3 \geq 3$  when  $f_4 = s = 0$ . Thus  $G_1$  'solves' the theorem if  $f_k = 0$  for all  $k \geq 6$ . If we wish to construct a  $G$  with only odd faces and including, say a 9-gon, then we start with  $G_1$  and successively 'add large faces' as indicated by the construction of figure 2.40. This construction depends on finding a 3-gon and a 5-gon that are adjacent (which is indeed the case for  $G_1$ ), and then providing another such pair ( see faces A and B of figure 2.40); which, as shown, can always be done.

Case 2: The construction is clear if one just considers the 2 graphs of figure 2.41.

Case 3: Again, the example (see figure 2.42) makes the construction clear. The 2 even faces appear on the 'ends'.

Case 4: See figure 2.43 for the method of construction. The 1 even face is on the left.

Next we consider all cases for which  $f_4 \leq 2$ . That is:

Case 2', Case 3', Case 4': Clearly these 3 cases subsume (respectively) Cases 2,3 and 4 treated above. Actually the latter 3 cases are basic, because: to construct the desired  $G$ , just 'insert pairs of large even faces' (as required) by using the method illustrated in figure 2.41.

Case 4'': First construct the 'corresponding' graph by using the construction described for case 4'. (That is: construct a graph with  $f_4 = 1$  but all other  $f_k$ 's as desired - except for  $f_3$  of course.)

Next: 'trade' triangles for pairs of 4-gons as is necessary - using either of the methods shown in figure 2.44.

Case 2'': Again we will construct a graph that will be the precursor of  $G$  in the sense that:  $f_3$  and  $f_4$  will not be as desired ( $f_4$  will be just 2), but all other  $f_k$ 's will be satisfied. To do this we use the construction employed in Case 2', and thus get  $f_4 = 2$ . Then we obtain  $G$  by 'trading' 3-gons for pairs of 4-gons, by using the methods illustrated in figure 2.44.

This completes the proof of the theorem.

REMARK. Using Barnette's inequality for  $p_6$  (see theorem 2.4), it follows readily that: if  $G$  is a cubic Halin graph with precisely  $f_k$  interior  $k$ -gons ( $k = 3, 4, 6, 7, \dots$ ), then

$$f_5 \geq 2 + f_3 - 2 \sum_{k \geq 6} f_k \quad \text{if} \quad \sum_{k \geq 7} f_k \geq 2 .$$

CHAPTER 3

DELETABLE EDGES IN 3-POLYTOPAL GRAPHS

3.1 INTRODUCTION

In this chapter, by a p-graph we will mean a plane 3-connected graph. An edge of such a graph is a deletable edge if the graph resulting from its deletion (see chapter 1.8 - that is, section 8) is also a p-graph. By a k-edge of a plane graph will be meant an edge shared by an i-gon and a j-gon, where  $i \leq 5$  and  $i + j \leq 13$ .

In 1974 E. Jucovic [J1, p. 233], in pursuing a line of inquiry initiated by A. Kotzig (see [G3, p. 233]), proved that a cubic p-graph contains a k-edge.

The major result presented in this chapter is (see theorems 3.17 and 3.19):

**THEOREM.** A p-graph, other than the tetrahedron, contains a deletable k-edge.

**REMARK.** It is worthwhile to point out the intuitive significance of this result. Theorem 1.12 shows that every 3-polytope can be built up from the tetrahedron by a sequence of face splits (or equivalently edge additions). If some 3-polytope  $P$ , say, has a 50-gon as its largest face, it is conceivable that one would have to build up a 3-polytope  $P'$  with say a 90-gon and then split that face to get  $P$ . Theorems 3.17 and 3.19 show that this would not be necessary, because one need never split a face with more than 11 sides.

The work needed to establish theorems 3.17 and 3.19 also leads, quite quickly to several other new results about deletable edges in 3-valent p-graphs (see theorems 3.7 through 3.12), and also prepares the way for chapter 4.

NOTATION: The following will prove convenient. If  $S$  is some simple closed curve in the plane then by int  $S$  and Int  $S$  we will mean, respectively: the bounded open region of the plane which lies within  $S$ ; and the closure of that open region, that is  $(\text{int } S \cup S)$ . We define ext  $S$  and Ext  $S$  analogously with respect to the region outside  $S$ . Further, if  $G$  is some plane graph drawn in the same plane as  $S$  then: by  $S(G)$  [ $Sx(G)$ ] we will mean that (planar) sub-graph defined by all edges and vertices of  $G$  lying wholly in Int  $S$  [Ext  $S$ ].

### 3.2 DEFINITIONS

Let  $G$  be a  $p$ -graph and let  $R = F_{j_1}, F_{j_2}, \dots, F_{j_n}$  be a sequence of  $n \geq 3$  distinct faces of  $G$ .  $R$  will be said to be a ring if and only if some reordering of  $R$  yields a sequence  $F_1, F_2, \dots, F_n$  for which:

(a) For  $i = 1, 2, \dots, n-1$  faces  $F_i$  and  $F_{i+1}$  meet in an edge or just a vertex - and similarly for  $F_n$  and  $F_1$ . (Each such shared edge, or vertex, to be called a common edge, or common vertex, of  $R$ .)

(b) Faces of  $R$  can meet only as specified in (a), and those  $n$  common elements of  $R$  must be disjoint.

By a  $k$ -ring, where  $k$  is some integer  $\geq 3$ , we will mean a ring with  $k$  faces.

To illustrate some of these ideas consider the  $p$ -graph of figure 3.2.  $KEM$  is a 3-ring with 3 common vertices and  $AEFJD$  is a 5-ring with 3 common edges and 2 common vertices. Neither  $FJM$  nor  $AEBCD$  are rings (because condition (b) is not satisfied, although (a) is). Also,  $XALC$  is a 4-ring.

Each edge of (a face of) a ring  $R$  which is not a common edge will be called a rim edge of  $R$ . For example, the squiggled edges in figure 3.4 are the rim edges of the 4-ring ABCD. The rim edges of a ring  $R$  define, in a very natural way, 2 'non-crossing' edge-disjoint circuits. To be precise: draw some simple closed curve  $C^*$  through all of ring  $R$ 's faces so that  $C^*$  crosses each common edge just once - and not at an endpoint - and touches no vertex except for the common vertices of  $R$ . Then: the rim edges (of  $R$ ) lying within curve  $C^*$  define a circuit to be called a rim (of  $R$ ); analogously, the rim edges outside  $C^*$  form another circuit - also to be called a rim of  $R$ . (See figure 3.3.)

LEMMA 3.0

The 2 rims of a ring  $R$ , of a  $p$ -graph  $G$ , are edge-disjoint 'non-crossing' circuits, and are concentric in the plane if and only if no face of ring  $R$  is the exterior face of  $G$ .

PROOF: Immediate by definition.

Making use of curve  $C^*$  again, consider all edges and vertices of  $G$  that lie on  $C^*$  or within the plane interior of  $C^*$ . They define a connected (and planar) subgraph  $C^*(G)$  which, by theorem 1.7, must be 2-connected. Similarly for  $C_x^*(G)$  which is comprised of all edges and vertices lying on and outside  $C^*$ . Each of these 2 edge-disjoint subgraphs, to be called the cores of ring  $R$ , contains one rim of  $R$  (which shall sometimes be referred to as the rim of the core).

REMARK 1. If  $S$  and  $T$  are say the rims of ring  $R$  then clearly:  $S(G)$  and  $T(G)$  are the cores of  $R$  if  $S$  and  $T$  are not concentric; whereas  $S(G)$  and  $T_x(G)$  are the cores if they are concentric and  $S$

is the inner one - in which case  $S(G)$   $[T_x(G)]$  may be called the inner [outer] core, and  $S[T]$  called the inner [outer] rim. Note: a core may be just a rim (that is, just a circuit that is the boundary of a face), which is the case for both cores defined by the ring of  $k$  4-gons of a  $k$ -prism. Also: except for the graph of the tetrahedron, each  $p$ -graph must have at least 1 ring.

Given some nonempty subset  $Z$  of the set of all rings of a  $p$ -graph  $G$ , let us refer to each of  $Z$ 's rings as a  $z$ -ring and to each of the  $2|Z|$  distinct cores they define as a  $z$ -core. Then: a core will be said to be  $z$ -minimal, or minimal with respect to all  $z$ -cores, if it does not properly contain (as a subgraph) another core that is a  $z$ -core. For example consider the graph of figure 3.2, and let  $Z$  be say the set of all 4-rings of this graph. Neither core of the 4-ring ADJB is  $z$ -minimal. Reason: its inner core properly contains the inner core of ring ADMB, while its outer core properly contains the outer core of ring ABCD. On the other hand the inner core of ring EFJK is the union of the boundaries of the 3-gons  $L$  and  $M$ , and is  $z$ -minimal.

REMARK 2. If a core is just a face then, automatically, it is minimal with respect to all other cores of the graph.

REMARK 3. By definition each  $z$ -ring 'gives rise' to 2 or more edge-disjoint  $z$ -minimal cores. Reason: if a particular  $z$ -core is not  $z$ -minimal then it must contain a smaller  $z$ -core which in turn is either  $z$ -minimal or ... ; thus each of the 2-edge-disjoint  $z$ -cores defined by a  $z$ -ring must either be  $z$ -minimal or contain a  $z$ -core that is.

Lastly, consider the rings and cores of some  $p$ -graph  $G$ : a ring

$R$  will be said to be a normal ring if its rims are concentric (or, equivalently - by lemma 3.0, if none of its faces is the exterior face); a core  $H$  will be said to be a normal core if it is the inner core defined by a normal ring.

### 3.3 PRELIMINARY LEMMAS

Given: a  $p$ -graph  $G$  and some nonempty subset  $Z$  of its rings; core  $H^*$  defined by ring  $R^*$  of  $Z$ ; some other  $p$ -graph  $G''$  that is an 'elementary re-embedding' of  $G$  (see theorem 1.2 and the remark following it), and the natural bijection  $f$  from  $V(G) \cup E(G) \cup F(G)$  onto  $V(G'') \cup E(G'') \cup F(G'')$  which preserves vertices, edges and faces (and incidence and adjacency amongst them).

Then: the following facts about  $G$  and  $G''$  follow almost immediately from the definitions of section 3.1.

#### LEMMA 3.1

(i)  $R = F_1, \dots, F_k$  is a  $k$ -ring in  $G$  if and only if  $f(F_1), \dots, f(F_k)$  - denote it by  $f(R)$  - is a  $k$ -ring in  $G''$ .

(ii) Subgraph  $H$  is a core of  $G$ , with rim  $C$ , defined by ring  $R$  if and only if its image  $f(H)$  is a core of  $G''$  defined by ring  $f(R)$  with rim  $f(C)$ .

(iii) Core  $J$  properly contains core  $L$  (as a subgraph) in  $G$  if and only if core  $f(J)$  properly contains core  $f(L)$  in  $G''$ .

PROOF: Immediate by the definition of an elementary re-embedding.

#### LEMMA 3.2

Core  $H^*$  is  $z$ -minimal in  $G$  if and only if core  $f(H^*)$  is minimal with respect to all other cores of  $G''$  which are defined by rings of  $Z''$  - where by  $Z''$  we mean the set of all rings of  $G''$

whose corresponding rings in  $G$  are members of  $Z$ .

PROOF: By definition and by lemma 3.1.

LEMMA 3.3

If core  $H^*$ , defined by ring  $R^*$ , is not normal in  $G$  then  $G''$  can be 'chosen' so that core  $f(H^*)$  is normal in it.

PROOF: Pick some face  $M$  of  $G$  whose boundary lies wholly in the other core - not  $H^*$  - defined by ring  $R^*$ . (Such a face must exist.) Then 'make'  $M$  the exterior face of  $G''$ . So, in  $G''$ , all faces of ring  $f(R^*)$  are interior faces and obviously core  $f(H^*)$  cannot contain the boundary of  $G''$ 's exterior face - and thus is normal in  $G''$ .

LEMMA 3.4

If  $R^{**}$  is some other ring of  $Z$  such that (the boundary of) one of its faces  $F$  lies in the  $z$ -minimal core  $H^*$ , then another of its faces  $F''$  must lie neither in ring  $R^*$  nor in core  $H^*$  - but outside both (as shown in figure 3.5).

PROOF: By the preceding lemmas we can assume, without loss of generality, that core  $H^*$  (and its defining ring  $R^*$ ) are both normal in  $G$ . This is not essential to the proof but does make it simpler and more intuitive. So let  $S$  and  $T$  denote, respectively, the inner and outer rims of ring  $R^*$ . Now assume the lemma is false. Then all faces of  $R^{**}$  lie within the closed plane region bounded by rim  $T$  of ring  $R^*$  - call that region  $D$  - and so none of them can be the exterior face of  $G$ . Thus the rims of  $R^{**}$  must also be concentric in the plane and must also lie in  $D$ . Since  $R^{**}$  is different from  $R^*$ , the inner rim of the former must be distinct from  $S$  - the inner rim of the latter - even though it must lie within the closed plane region

bounded by  $S$ . But since this inner rim of  $R^{**}$  bounds the inner core of  $R^{**}$ , the latter is a  $z$ -core of  $G$  that is properly contained in  $H^*$  - contradiction!

REMARK. More generally we have just shown that: if  $K$  is any normal core of  $G$ , with defining ring  $R$ , and  $R''$  is another ring with all its faces lying inside the outer rim of  $R$ , then  $R''$  is normal and its inner core is properly contained in  $K$ .

### 3.4 DELETABLE EDGES: SOME THEOREMS

#### THEOREM 3.5

An edge  $e$  of a  $p$ -graph  $G$  is not a deletable edge if and only if it is common to 2 faces of some 3-ring of  $G$ .

PROOF: By theorems 1.10 and 1.11, and by the definition of "deletable".

#### LEMMA 3.6

If  $G$  is a 3-valent  $p$ -graph and core  $H^*$  is defined by ring  $R^*$  with 3,4 or 5 faces, then  $H^*$  contains (the boundary of) some 'small' face of  $G$  - that is, a face with 5 or fewer edges.

PROOF: By lemmas 3.1 and 3.3 we can assume, without loss of generality, that core  $H^*$  and ring  $R^*$  are both normal in  $G$ . Let  $S$  and  $T$  denote, respectively, the inner and outer rims of  $R^*$ . Consider the plane graph  $T(G)$  and suppress any 2-valent vertices it might have. (See figure 3.6, for example.) The resulting plane cubic graph has a  $k$ -ring (that 'was'  $R^*$ ) with  $k = 3, 4$  or 5 faces - each a 4-gon or bigger. These  $k$  faces, plus the exterior  $k$ -gonal face cannot alone satisfy the inequality  $3p_3 + 2p_4 + p_5 \geq 12$  which must hold. So it must have another small face lying within the inner rim of the  $k$ -ring it 'inherited' from  $G$ . But such a face must also

lie in  $G$ .

**THEOREM 3.7**

Given: a 3-valent p-graph  $G = (V, E, F(G))$  with  $L$  some non-empty subset of its edge set  $E$  such that some edges of  $L$  are common edges of 3-rings, and thus are not deletable;  $Z$  denoting the nonempty set of all 3-rings of  $G$  having an edge of  $L$  as a common edge.

Then:  $G$  contains 2 disjoint cores each defined by a ring of  $Z$  and neither is properly contained in any other core defined by a ring of  $Z$ ; so it follows that no edge of  $L$  lying in either of these 2  $z$ -minimal cores can be a non-deletable edge.

**PROOF:** The first assertion is immediate. The second follows by theorem 3.5 and the observation that 2 3-rings of a cubic graph cannot 'interlock' (see figure 3.7).

**REMARK.** Obviously this theorem proves (just let  $L = E$ ) that each 3-valent p-graph, other than the tetrahedron, has a deletable edge.

(See theorem 1.12). Thus we have also proved theorem 1.13 for the 3-valent case.

**THEOREM 3.8**

If  $G$  is a 3-valent p-graph (not the tetrahedron) and  $C$  is a circuit - perhaps the boundary of a face or perhaps a Hamiltonian circuit - which uses 4 or more edges of  $G$ , then  $C$  uses a pair of disjoint edges which are deletable.

**REMARK.** If  $C$  has just 3 edges then all must be deletable.

**PROOF:** If all edges of  $C$  are deletable then we are done, so assume not and apply theorem 3.7 by letting  $L$  denote the edges of  $C$ . (See statement of theorem 3.7). Thus  $Z$  is nonempty. Now,

every core defined by a ring of  $Z$  must contain an edge of  $L = C$  since the latter is a circuit. So, by the assertion of theorem 3.7 we have our result.

THEOREM 3.10

If  $C$  is a  $k$ -circuit (where  $k \geq 6$ ) in a 3-valent  $p$ -graph  $G$  (not the tetrahedron), then 2 disjoint edges not used by  $C$  are deletable and each is an edge on a small face.

PROOF: Immediate by theorem 3.8.

REMARK 1. It can be shown (by using theorems 1.4, 3.7, 3.8, 3.9, and the remark following lemma 3.6) that the restriction  $k \geq 6$  may be dropped unless  $C$  is a 3-circuit and  $G$  is the 3-prism. (See figure 3.8.)

REMARK 2. As shown by the graphs of figure 3.9 (where squiggled edges represent a Hamiltonian circuit), there may only be 2 such edges - and both may lie on the exterior face. In figure 3.9 'they' are the 2 'unused' edges of the 3-gons.

THEOREM 3.11

If  $G = (V, E)$  is a 3-valent  $p$ -graph (other than the tetrahedron) with  $|E|$  edges, then at least  $|E|/3 + 3$  of them are deletable.

PROOF: Since  $G$  is cubic we have, by the Euler formula that  $|E| = 3(\frac{1}{2}|V|)$  and  $|F(G)| = \frac{1}{2}|V| + 2$ . By theorem 3.9 each face has 2 or more deletable edges; by theorem 3.8 two of the faces have 3 or more deletable edges. So if we multiply  $|F(G)|$  by 2, add 2, and then divide that 'count' by 2 we obtain a lower bound of  $\frac{1}{2}|V| + 3$ .

REMARK 1. In figure 3.10 we look again at the family of graphs displayed in figure 3.9. This time the deletable edges are squiggled and we observe that each has the minimum number.

REMARK 2. It can be proved that a cubic p-graph has the minimum number of deletable edges if and only if it can be obtained from the tetrahedron by successive truncations of vertices of 3-gon. (See figure 3.11.)

THEOREM 3.12

Given: a Hamiltonian p-graph  $G^*$  that is maximally planar (all faces are triangles) but is not the tetrahedron, and some particular Hamiltonian circuit  $H$  in  $G^*$ .

Then: at least 4 edges used by  $H$  - each with an endpoint of values 5 or less - are contractible edges of  $G^*$ .

PROOF: By theorem 3.8 we know that  $G^*$  contains 2 nonadjacent vertices, each of valence 5 or less, and each incident to only contractible edges.

REMARK. In figure 3.12 we see how this theorem allows us to reduce a triangulation - and its Hamiltonian circuit - to the tetrahedron, by sequentially contracting one of the edges guaranteed by the theorem. The significance of this is that a hamiltonian simplicial 3-polytopal (i.e., all faces triangles) graph is obtainable from the tetrahedron by a sequence of vertex splits which 'stays within' the class.

3.5 DELETABLE k-EDGES IN 3-POLYTOPAL GRAPHS

In this section it is proved that (theorem 3.17): every 3-valent p-graph, other than the tetrahedron, contains a deletable k-edge. (See section 1 for definitions.) Unless stated otherwise, all p-graphs will assumed to be different from the tetrahedron.

In view of theorem 3.7 this result will follow readily once we have proved that (theorem 3.16): a core must contain a k-edge if it

(the core) is defined by a 3-ring with some  $k$ -edge as a common edge. This result, in turn, depends on a lemma (see lemma 3.14 and the definition preceding it) that utilizes the following theorem.

THEOREM 3.13 (E. Jucovic [J1, p. 233])

If  $G$  is a 3-valent  $p$ -graph (not the tetrahedron) and  $e(i,j)$  denotes the number of  $k$ -edges of  $G$  that are shared by an  $i$ -gon and a  $j$ -gon, then:  $120 \leq \sum_{3 \leq i \leq j} w(i,j) \cdot e(i,j)$ , where the 18 positive coefficients - the  $w(i,j)$ 's - are displayed in the table presented in figure 3.1.

REMARK. Observe that, for all values of  $i$  and  $j$ ,

$$w(i,j) \geq w(i,j+1), w(i+1,j) .$$

This monotonicity will be useful. Actually Jucovic's original inequality has  $w(3,3) = 20$ ,  $w(3,7) = 6\frac{2}{3}$  and  $w(3,9) = 2\frac{1}{2}$ . But increasing these 3 coefficients - see entries with asterisks - can only yield another true inequality, and the values shown are more convenient for our purposes.

DEFINITION: A  $p$ -graph  $G$  (not necessarily 3-valent) will be said to be light if and only if with appropriate labelling it satisfies the following description (see figure 3.13): its exterior face  $X$  is a 3-gon with three 3-valent vertices and contains an edge of each of the 3 faces of the 3-ring  $R = ABC$  - each a 4-gon or bigger;  $R$  has outer rim  $T$  and inner rim  $S$  and defines the (normal) core  $H = S(G)$ , and this core does not contain any  $k$ -edges of  $G$  - except perhaps on its rim  $S$ ; if face  $M$  is some  $m$ -gon of  $G$  whose boundary lies in  $H$  and contains an edge of rim  $S$ , then  $m \geq 9$  if  $M$  contains an edge of  $A$  and  $m \geq 4$  if  $M$  contains an edge of  $B$ .

REMARK. From the last condition it follows immediately that some

vertex must lie within the interior of core  $H$  (that is, in  $H$  but not on rim  $S$ ) if  $G$  is a light graph. And the triangular prism, for example, is not light.

LEMMA 3.14

If  $G$  is a 3-valent  $p$ -graph then  $G$  cannot be light.

PROOF: Assume that  $G$  is light and assign a weight of  $w(i,j)$  to each edge of  $G$  that is shared by an  $i$ -gon and a  $j$ -gon. To obtain a contradiction to theorem 3.13 it clearly will suffice to show that:

(i) the 3 common edges of ring  $R = ABC$  have a combined weight of at most 30.

(ii) the total weight of the rim edges (on  $S$  and  $T$ ) on the boundary of each face of  $R$  is at most 28.

Proof of (i): If some face of  $R$  is a 6-gon or bigger then, by the monotonicity of weights,  $R$ 's 3 common edges have a total weight of at most  $w(4,6) + w(6,4) + w(4,4) = 5 + 5 + 20 = 30$ . Alternately, if each of  $R$ 's faces has no more than 5 edges then all 3 are 5-gons.

(Reason: by the remark following the definition of a light graph, some vertex must lie interior to rim  $S$ ; so by 3-connectedness it follows that  $|A| + |B| + |C| \geq (4 + 4 + 4) + 3 = 15$ .) In this case each of the 3 common edges (see figure 3.14) has a weight of  $w(5,5) = 8$ .

Proof of (ii): Let  $F$  denote that face of  $R = ABC$  being considered, and let  $|F| = k$ . Recall,  $k$  is at least 4 (as shown in figure 3.15). Regardless of whether  $F = A, B,$  or  $C$ , we argue as follows.

The rim edge shared by  $F$  and  $X$  has weight  $w(3,k)$ . The other  $k-3$  rim edges about  $F$  are on rim  $S$  and comprise a simple path - call it  $P$ . We consider 2 cases.

First: if  $k = 4$  then  $P$  has just 1 edge which has one (or both)

of its endpoints on face A. Thus that edge is shared by F and some face - call it M - which contains an edge of A. Now, by definition of a light graph  $|M| \geq 9$ , and it follows that this single edge comprising the path P has a weight of at most  $w(4,9)$ . So the 2 rim edges of F (one on S, one on T) have a combined weight of at most  $w(3,4) + w(4,9) = 25 + 3 = 28$ .

Second:  $k \geq 5$ . So path P has a 'first' and a 'last' edge (as indicated by the arrows in figure 3.15). One must contain a vertex of A and is thus shared by F and a face with 9 or more sides, and the other has an endpoint on B and is thus shared by F and some face with 4 or more sides. Each of the remaining  $k-5$  edges of path P obviously cannot weigh more than  $w(3,k)$ . So in this case (when  $|F| = k \geq 5$ ), all of F's  $k-2$  rim edges (on both T and S) have a total weight of at most:

$$w(3,k) + (w(9,k) + (k-5) w(3,k) + w(4,k)) ,$$

which simplifies to  $(k-4) w(3,k) + w(4,k) + w(9,k)$ .

And, for  $k = 5, 6, 7, 8, 9, 10 \geq 11$ , this expression yields values (respectively) of: 27, 25, 26, 25, 18, 12, and 0. This proves case (ii), and thus the lemma is proved.

LEMMA 3.15 (Used for theorem 3.19, but not for 3.17).

No p-graph, whether 3-valent or not, can be light.

PROOF: Assume the lemma's assertion is false, and that G is some light graph for which  $\#(G) = \sum_{v \text{ in } V(G)} (\text{deg } v - 3)$  is a minimum.

Since G cannot be 3-valent (by lemma 3.14, it must have some vertex x of valence 4 or more (and thus  $\#(G)$  is positive) which lies in core H. The assumption that  $\#(G)$  is a minimum means that if we split vertex x so as to reduce  $\#(G)$  by 1, then we must create a

p-graph that is no longer light. But we will show that we can always split vertex  $x$  so as to reduce  $\#(G)$  and still obtain a light graph - this will be the contradiction!

There are 3 cases to consider (see diagram(s) in (a) of figures 3.17, 3.18 and 3.19) and each can be disposed of very quickly by splitting  $x$  to form, in a 'suitable' manner, the new edge  $(x,x')$ . This is displayed in parts (b) of figures 3.17, 3.18 and 3.19.

Case (i):  $x$  is a vertex of face  $A$  but not of  $B$  or  $C$ , as shown in figure 3.17. Then splitting  $x$  'inwards' (as shown) cannot reduce the size of any face; further, the new edge  $(x,x')$  cannot be a  $k$ -edge since (from the definition of a light graph) edge  $(x,x')$  'becomes' shared by 2 faces that each have  $9 + 1 = 10$  or more edges. Thus the new graph is light.

Case (ii): Here  $x$  is a vertex on one or both of the boundaries of  $B$  and  $C$ , and may also belong to  $A$  (see figure 3.18). In this case we split  $x$  'along' rim  $S$  so that  $x$  and  $x'$  are both on  $B$ , or are both on  $C$ . It is clear that within core  $H$  of the new graph: no 3-gons have an edge on  $B$  or  $C$  and no face with 8 or fewer edges has been made adjacent to  $A$ . Hence  $(x,x')$  is not a  $k$ -edge, and the new graph is light.

Case (iii): In this case  $x$  is in core  $H$  but is not a vertex of rim  $S$  (see figure 3.19). If all of the 4 or more faces about  $x$  are 6-gons or bigger, we just split  $x$  in any 'direction' and get a light graph. So assume face  $F$  is incident to  $x$  (see figure 3.19) and has 5 or fewer sides. Then the 2 faces flanking  $F$  about  $x$  must both be 9-gons or bigger, call them  $M$  and  $Q$ . Thus we can split  $x$  (as shown) so that  $M$  and  $Q$  share the new edge  $(x,x')$ , which obviously cannot be a  $k$ -edge. Thus the new graph is light.

This concludes the proof of lemma 3.15.

THEOREM 3.16

Each core of a p-graph (be the graph 3-valent or not) that is defined by a 3-ring with a common k-edge must contain a k-edge.

REMARK. As shown in figure 3.20, the 3-ring of 11-gons has no k-edge as a common edge - and its core H has no k-edges!

PROOF: Assume that the theorem is false for some core H of some p-graph G, where H is defined (see figure 3.21) by the 3-ring  $R = ABC$  for which:  $|A| \leq 5$  and A and B share a k-edge (and thus B has at most 10 sides).

The strategy of the proof is to obtain from G a light p-graph, and thus the assumption of the theorem's falsity will contradict lemma 3.15 (lemma 3.14, if G is cubic)!

Once again (by lemmas 3.1 and 3.3) we can assume, without loss of generality, that R and H are both normal in G - as depicted in figure 3.21 or 3.22(a). So let S and T denote, respectively, the inner and outer rims of ring R. From our initial assumption (that the theorem is false) we immediately have the following 2 facts.

Fact (1): a face sharing an edge of rim S with face A [B] must have at least 9 [4] edges, or else rim S would contain a k-edge of G.

Fact (2): no k-edges (of G) may lie within the interior of core H.

If G is not 3-valent it is possible that face C may share just a vertex with one or both of faces A and B (see figure 3.22(a)). If so, we can split such common vertices of R (such as v and w in figure 3.22(a)) so as to 'give' ring R 3 common edges (as shown in 3.22(b)) - and not alter Fact (1) or Fact (2). So without loss of

generality we shall assume that  $R$  has 3 common edges, and that Facts (1) and (2) both hold.

Now consider the plane subgraph  $T(G)$  comprised of all edges and vertices of  $G$  that lie on, and within, rim  $T$ ; suppress any 2-valent vertices on its exterior face (see figure 3.23). By theorem 1.10 this graph is 3-connected (since no pair of faces 'meet twice') and Facts (1) and (2) are 'inherited' from  $G$ . Thus  $T(G)$  is certainly a light graph! This contradiction to lemma 3.15 concludes the proof.

THEOREM 3.17

Given: a 3-valent  $p$ -graph  $G$  (i.e.,  $G$  is a plane, 3-valent, 3-connected graph) other than the tetrahedron  $K_4$ .

Then:  $G$  contains a deletable  $k$ -edge and equivalently, there exists a sequence of  $p$ -graphs

$$K_4 = G_0, G_1, \dots, G_i, G_{i+1}, \dots, G_{n-1}, G_n = G$$

such that (for  $0 \leq i \leq n-1$ )  $G_{i+1}$  is obtained from  $G_i$  by placing an edge across some face of  $G_i$  (i.e., performing a face split) that has at most 9 sides - so that this 'new' edge is a  $k$ -edge in  $G_{i+1}$ .

REMARK 1. It will be possible, later, to drop the restriction that  $G$  be 3-valent.

PROOF:  $G$  must have  $k$ -edges (by theorem 3.13) and if none are common edges of 3-rings we are done. So let  $L$  be defined as the set of all  $k$ -edges and let  $Z$  denote the nonempty set of all 3-rings which have a  $k$ -edge as a common edge. Then by theorems 3.7 and 3.16 we are done.

REMARK 2. By truncating each vertex of the dodecahedron (the cubic p-graph with 12 5-gons) one obtains a 60-vertex graph with precisely 60 k-edges, each shared by a 10-gon and a 3-gon (and thus deletable).

REMARK 3. There exists an infinite number of cubic p-graphs having just 12 k-edges, each shared by a 3-gon and a 6-gon (and thus deletable). Such graphs may be obtained by truncating all vertices of a cubic p-graph having just 3-gonal and 6-gonal faces - and there exist infinitely many such graphs.

When a cubic p-graph has no 3-gons or 4-gons, but just 5-gons, then the following result - which is not deriveable from Jucovic's inequality (see theorem 3.13) but complements it - permits one to obtain results complementing those of theorems 3.16 and 3.17.

THEOREM 3.17(a). (Grumbaum [G4, p. 404].)

If  $G$  is a 3-valent p-graph with no 3-gons or 4-gons, then:

$$2e(5,5) + e(5,6) \geq 60 .$$

Define an (i,j)-edge of a plane graph to be an edge shared by an  $i$ -gon and a  $j$ -gon.

THEOREM 3.17(b)

Given:  $G$  a 3-valent p-graph with core  $H$  defined by a 3-ring such that each face (of  $G$ ) lying within  $H$  has 5 or more sides.

Then: at least 10 edges that are (5,5) or (5,6)-edges lie within the interior of core  $H$  (i.e., not on its rim).

PROOF: Let  $R = ABC$  denote the 3-ring which defines core  $H$  and let  $S$  and  $T$ , respectively, denote the inner and outer rims of  $R$ . (We can assume without loss of generality that  $R$  and  $H$  are both normal in  $G$ .) Consider the graph  $T(G)$  comprised of all vertices and edges of  $G$  lying within the closed plane region bounded by  $T$ ; suppress all 2-valent vertices on its exterior face and denote the

resulting cubic p-graph by  $G''$  (see figure 3.27). Some vertex must lie within the interior of core  $H$  (i.e., not on its rim  $S$ ), both in  $G$  and  $G''$ . Reason: otherwise  $H$  would contain a 3-gon and/or a 4-gon. Thus, by 3-connectedness, at least 2 of the 3 faces of ring  $R = ABC$  have 3 or more vertices on the inner rim  $S$ . So in  $G''$  at least 2 of the faces of the corresponding 3-ring are 5-gons or bigger, as indicated in figure 3.28.

Now we construct a new 3-valent p-graph,  $G^*$ , by placing edges across the infinite face of  $G''$  - as shown in figure 3.29, where the size of each face 'outside' core  $H$  (with squiggled rim) is indicated. By construction,  $G^*$  has no 3-gons or 4-gons. Now consider all edges of  $G^*$  that are (5,5) or (5,6)-edges and do not lie interior to rim  $S$ . A quick count shows there are precisely 18 (5,5)-edges and at most 4 (5,6)-edges in this set. These edges 'contribute' at most  $2(18) + 4 = 40$  towards the 'count' of 60 that is guaranteed, for  $G^*$ , by theorem 3.17(a). Thus the edges within the interior of core  $H$  of  $G^*$  (and thus of  $G$  also) must 'contribute' at least 20. So our theorem is proved.

**THEOREM 3.17(c)**

If  $G$  is a 3-valent p-graph with no 3-gons or 4-gons, then  $G$  contains a set of at least 20 deletable edges - each a (5,5), or a (5,6)-edge.

**REMARK.** Actually, 20 is not best possible.

**PROOF:** If  $G$ 's edges are all deletable then, by theorem 3.17(a), we are done. So assume not. Then  $G$  has 3-rings. So, by theorem 3.7, we know that:  $G$  contains 2 disjoint cores each defined by a 3-ring and each comprised only of deletable edges. This fact, plus the preceding result (theorem 3.17(b)), concludes the proof.

THEOREM 3.17(d)

If  $P(m)$  denotes the set of all 3-valent  $p$ -graphs which have no face with more than  $m$  sides then, for each  $m \geq 7$ ,  $P(m)$  is 'closed' in the following sense (which is not the case for  $P(6)$ ): each graph of  $P(m)$  may be obtained from the tetrahedron by a sequence of face splits - not performed on the exterior face - such that each  $p$ -graph of the corresponding sequence lies in  $P(m)$ ; and is obtained from its predecessor by either 'slicing off' a 3-gon or 4-gon, or by 'slicing off' a 5-gon from a 6-gon or 7-gon.

PROOF: Let  $G$  be some particular  $p$ -graph in  $P(m)$ , where  $m \geq 7$ . Clearly, to prove the theorem, it will suffice to demonstrate that:  $G$  contains a deletable edge not on the exterior face which either lies on a 3-gon or 4-gon, or is a (5,5) or a (5,6)-edge. To demonstrate this we consider 5 cases.

Case (i):  $G$  has no 3-gons or 4-gons. Then, by theorem 3.17(c), we are done.

Case (ii):  $G$  contains at least 2 faces with fewer than 5 edges. In this case at least one face of  $G$ , call it  $F$ , is a 3-gon or a 4-gon and is not the exterior face. By theorem 3.8 at least 2 edges of  $F$  are deletable. Since at least 1 of these edges is not on the exterior face, we are done.

Case (iii):  $G$  contains just 1 face  $F$  with fewer than 5 sides - and  $F$  is not the exterior face. Then, by theorem 3.8 again, we are done.

Case (iv):  $G$  contains just 1 face - the exterior face  $X$  - with fewer than 5 sides, and  $X$  is a 3-gon. In this case  $G$  contains some core,  $H^*$ , defined by a 3-ring such that  $H^*$  does not contain the

(boundary of) face  $X$ . If such a core  $H^*$  (which must exist in  $G$ ) is not minimal with respect to all other cores (of  $G$ ) defined by 3-rings, then consider instead some core  $H$  (also defined by a 3-ring) lying within  $H^*$  which is minimal. Now, each interior edge of  $H$  must be an interior edge of  $H^*$  - and is thus not an edge of  $X$ . By theorem 3.17(b) at least 10 such edges are (5,5) or (5,6)-edges; and, by the minimality of  $H$ , all are deletable. So we are done.

Case (v):  $G$  contains just 1 face - the exterior face  $X$  - with fewer than 5 sides, and it is a 4-gon. Either all edges of  $G$  are deletable, or not. If not, then  $G$  must contain some core  $H^*$  as described in Case (iv) - and the proof follows by precisely the same argument. So assume that all edges of  $G$  are deletable (thus  $G$  has no 3-rings). In this case the 4 faces incident to  $X$  (all 5-gons or bigger) form a 4-ring, as shown in figure 3.30. If we 'place' a 3-ring about  $G$ , as shown in figure 3.30, then the boundary of  $X$  'becomes' the inner rim of a 3-ring in the new graph - call it  $G''$ . This graph possesses property-p and is thus a p-graph. Thus, by theorem 3.17(b) - or by direct application of theorem 3.17(a), we have that: at least 10 edges (of  $G''$ ) that are (5,5) or (5,6)-edges lie interior to the inner rim (squiggled) of the 3-ring of  $G''$ . Each such edge is (obviously) a deletable (5,5) or (5,6)-edge of  $G$  also - and thus the proof is complete.

### 3.6 DELETABLE EDGES IN NON-CUBIC 3-POLYTOPAL GRAPHS

In this section we obtain a result (theorem 3.18) analogous to theorem 3.7 which is then used, with theorem 3.16, to obtain this generalization of theorem 3.17:

Every p-graph, other than the tetrahedron, has a deletable k-edge (see theorem 3.19).

Not surprisingly, things are not as simple as in the 3-valent case. To illustrate some of the problems - as well as to point the way to a strategy for theorem 3.19's proof - consider the plane graph of figure 3.24 as a subgraph of some p-graph  $G$ ; also consider the following notational convention.

Notation: (observed throughout this section): If the two endpoints of some edge  $e$  are denoted by the same (lower case) letter, one primed and the other not (for example  $e = yy'$ ), then edge  $e$  should be viewed both as an edge and as a 'pseudo-edge' - that is, with its two endpoints identified.

In figure 3.24 faces  $A$ ,  $B$  and  $C$  comprise a normal 3-ring with an inner core  $H$  that is a 3-gon. So, as previously remarked,  $H$  cannot contain any other core of  $G$ . Unfortunately edges  $yp$  and  $zp$  are common edges, respectively, of rings  $HAB^*$  and  $HA^*B$  - and thus are not deletable. Note that faces  $A$  and  $B$  share the k-edge  $pn$  and that  $yz$  (which need not be a k-edge) is the only deletable edge of this 'minimal' core  $H$  - and  $H$  is defined by a 3-ring with a common k-edge! (Compare this to the 3-valent case). Note also that edges  $yn$  and  $zn$  are deletable but need not be k-edges of  $G$ . In fact: if  $A^*$ ,  $B^*$  and  $C$  all have  $\geq 11$  sides, then  $G$  has no deletable k-edges lying on or within the outer rim of ring  $ABC$ !

#### THEOREM 3.18

Given: a p-graph  $G = (V,E)$  with normal core  $H$  (as in figure 3.25) defined by the 3-ring  $R = ABC$  (with inner rim  $S$  and outer rim  $T$ ) which has the common edge  $pn$  shared by  $A$  and  $B$  - and in which  $C$  shares  $y'y$  with  $A$  and  $zz'$  with  $B$  (and  $y,z$  and  $p$  are on the

inner rim  $S$ ); further, edge  $pn$  is a member of some prescribed subset  $L$  of  $E(G)$  such that core  $H$  does not contain any other core defined by a 3-ring with a common  $L$ -edge;  $Z$  denotes the set of all rings (of  $G$ ) which contain an  $L$ -edge as a common edge. Then:

(I) No  $L$ -edge (if any) of  $C \cap S$  can be nondeletable.

(II) If  $e = uv$  is a nondeletable  $L$ -edge that lies in  $H$  but not on rim  $S$ , then (as shown in figure 3.26)  $C$  is a 3-gon with vertices:  $y = y'$ ,  $z = z'$ , and an endpoint of  $e$  of valence 3.

(III) If either  $A$  or  $B$  contains a nondeletable  $L$ -edge on rim  $S$ , then core  $H$  is just a 3-gon.

(IV) If both  $A$  and  $B$  contain nondeletable  $L$ -edges on rim  $S$  then, in addition to  $H$ ,  $A$  and  $B$  are also 3-gons and are 'flanked' (outside the outer rim  $T$ ) by 2 faces (see  $A^*$  and  $B^*$  of figure 3.24) which each contain vertex  $n$  - and a vertex or edge of  $C$ .

REMARK. Using assertion (I) only, one can prove theorem 1.12 quite easily.

We present the proofs of theorems 3.18(I) and 3.19 now, and defer the remainder of theorem 3.18's proof till the end of this chapter - after it has been invoked to prove theorem 3.19.

PROOF (OF (I)): Consider figure 3.25 where - by convention -  $y$  and  $y'$ , and/or  $z$  and  $z'$ , may be identical. Assume that some  $L$ -edge  $e$  is not deletable, even though it is shared by face  $C$  and the inner rim  $S$ . Then (by theorem 3.5)  $e$  must be a common edge of some 3-ring - call it  $R^{**}$  - in which  $e$  is shared by  $C$  and some other face of  $R^{**}$  - call it  $F$ . Because of the minimality of  $H$  (see lemma 3.4) the third face of  $R^{**}$  - call it  $F''$  - must lie outside the outer rim of ring  $R = ABC$ . But then  $F''$  and  $F$  (which lies inside the inner rim of  $R$ ) can only meet if they and  $C$  are all incident to the same vertex;

thus  $R$  is not a 3-ring and thus we have a contradiction!

PROOF(of theorem 1.12): Let  $L = E(G)$  and assume that some of  $G$ 's edges are not deletable - or else we are done. Thus (by theorem 3.5) the set of all 3-rings (of  $G$ ) which contain an edge of  $L = E(G)$  as a common edge is not empty. Denote this set of 3-rings by  $Z$ . Let  $R = ABC$  be a ring of  $Z$  having a common edge shared by  $A$  and  $B$  such that: one of the cores defined by  $R$  - call it  $H$  - is minimal with respect to all other cores defined by rings of  $Z$  (i.e., does not properly contain any one of them). Claim: all edges of face  $C$  that lie on the rim of  $H$  are deletable. Proof of claim: one can assume without loss of generality that  $H$  and  $R$  are normal in  $G$  and so, by (I) of theorem 3.18, the claim is proved. Thus the theorem is proved.

**THEOREM 3.19**

Given: A  $p$ -graph  $G \neq K_4$ ; and  $G$  is not necessarily cubic. Then:

- (i)  $G$  contains a deletable edge that is a  $k$ -edge.
- (ii) Equivalently, there exists a sequence of  $p$ -graphs

$$K_4 = G_0, G_1, \dots, G_i, G_{i+1}, \dots, G_n = G$$

such that each is obtained from the preceeding one by splitting a face with at most  $l$  sides, and thereby creating two 'new' faces sharing a  $k$ -edge.

REMARK. For some reason Jucovic [J1] does not explicitly state that all  $p$ -graphs (cubic and non-cubic) contain a  $k$ -edge! We need this result and will obtain it very easily by employing the 'diagonalization' argument used by Grunbaum [G3, p. 213]. The theorem's proof follows the proof of the lemma.

**LEMMA 3.19(a).**

Every  $p$ -graph contains a  $k$ -edge.

PROOF: Define a  $k^*$ -edge of a graph to be an edge with endpoints of valence  $i$  and  $j$ , where  $i \geq 5$  and  $i + j \leq 13$ . Now assume the lemma is false and (equivalently) that there exists a p-graph,  $G^*$ , with no  $k^*$ -edges. By Jucovic's result (theorem 3.13)  $G^*$  must have some non-3-gonal faces. Without loss of generality we may assume that  $G^*$  is maximal in the following sense: the addition of some (any) edge  $e$  between 2 non-adjacent vertices across some non-3-gonal face would result in a p-graph which does have  $k^*$ -edges, by which it follows - obviously - that  $e$  would be a  $k^*$ -edge in the new graph. Now consider some non-3-gonal face  $F$ . By the preceding assumption, at least one of  $F$ 's (4 or more) vertices has valence  $\leq 4$ ; but then each of the two vertices of  $F$  which are adjacent to such a vertex must have a valence  $\geq 10$  - in which case a new edge placed between them would not be a  $k^*$ -edge. Contradiction!

PROOF(of theorem 3.19): Assume the theorem false and that  $G$  is some p-graph (other than the tetrahedron) which contains no deletable  $k$ -edges - and that  $G$  has the minimum number of vertices (for a p-graph rendering the theorem false). By lemma 3.19(a)  $G$  contains one or more  $k$ -edges.

Strategy (obtain a contradiction): It will be shown that  $G$  must contain the configuration of 6 faces displayed in figure 3.24, such that faces  $A^*$ ,  $B^*$  and  $C$  (of figure 3.24) will each have  $\geq 11$  sides; but then, by removing vertex  $p$  we obtain (as will be shown) a smaller p-graph  $G'' = G - p \neq K_4$  (see figure 3.31) without having 'introduced any new  $k$ -edges' - and no nondeletable  $k$ -edge (of  $G$ ) suddenly 'becomes deletable' in  $G'$  by this reduction; thus  $G''$  also has no deletable  $k$ -edges and this contradicts the assumption that  $G$  was 'smallest'!

By lemma 3.19(a)  $G$  contains  $k$ -edges. By assumption each is

nondeletable and thus (by theorem 3.5) each is a common edge of some 3-ring. Let  $Z$  denote the set of all such 3-rings. By theorem 3.16 each core defined by a ring of  $Z$  contains a  $k$ -edge.

Consider one such core - call it  $H$  - defined by some 3-ring  $R = ABC$  (of  $Z$ ) such that:  $A$  and  $B$  share a  $k$ -edge and  $H$  is  $z$ -minimal (i.e., does not properly contain another core defined by a ring of  $Z$ ). We can assume, without loss of generality, that both  $H$  and  $R$  are normal in  $G$ . (This simplifying assumption will not alter the fact that all  $k$ -edges of  $G$  - and thus of core  $H$  also - are non-deletable). Let  $np$  denote the  $k$ -edge shared by  $A$  and  $B$  (see figure 3.25); let  $yy'$  and  $zz'$  denote the 'pseudo-edges' shared, respectively, by  $A$  and  $C$  and by  $B$  and  $C$ ; let the inner and outer rims of ring  $R$  be denoted, respectively by  $S$  and  $T$ ; and let the labelling be such that vertices  $p, y$  and  $z$  lie on the inner rim  $S$ . (Recall that by the labelling convention established at the beginning of this section, one or both of the edges  $yy'$  and  $zz'$  may actually be vertices).

Now, what about the location of the (one or more) nondeletable  $k$ -edges in core  $H$ ? If we let  $L$  denote the set of all  $k$ -edges of  $G$ , then we can employ theorem 3.18 to answer this question. We proceed as follows.

(i) By (I) of theorem 3.18 we know that no nondeletable  $k$ -edge is shared by face  $C$  and rim  $S$ .

(ii) No such edge can lie within the interior (i.e., not on rim  $S$ ) of core  $H$ .

Proof (ii): Assume the contrary, and that the nondeletable  $k$ -edge  $e$  does lie within the interior of  $H$ . Then, by (II),  $e$  must have a 3-valent endpoint on the boundary of  $C$  - and  $C$  is a 3-gon. It follows

that both of the faces sharing  $e$  are 10-gons or smaller, and thus that each shares a  $k$ -edge with the 3-gon  $C$ . But, by (i), these 2  $k$ -edges must be deletable - contradiction.

Thus all of  $H$ 's nondeletable  $k$ -edges must be shared by rim  $S$  and one, or both, of faces  $A$  and  $B$ .

(iii) By the preceding it follows - using (III) of theorem 3.18 - that  $H$  is just a 3-gon. But then both  $A$  and  $B$  must share a  $k$ -edge with  $H$  (since  $H$  is a 3-gon and both  $A$  and  $B$  have 10 or fewer edges).

(iv) Thus by (IV) it follows that:  $H, A$  and  $B$  are all 3-gons; and that  $A$  and  $B$  are 'flanked' (outside the outer rim  $T$ ) by 2 faces which each contain vertex  $n$  - and a vertex or edge of  $C$ . As shown in figure 3.24, let  $A^*$  and  $B^*$  denote these 2 faces - which are shown (conveniently, and without loss of generality) as interior faces of  $G$ .

Now consider the situation portrayed in figure 3.24. Edge  $yz$ , as we already know, is a deletable edge. Thus  $|C| \geq 11$  (or else  $yz$  would be a deletable  $k$ -edge). Edge  $yn$  (shared by  $A$  and  $A^*$ ) is also deletable. Proof: if not then (by theorem 3.5)  $A, A^*$  and some third face would form a 3-ring; but by observation this is clearly not possible. Similarly edge  $zn$  is deletable. Thus it follows that both  $A^*$  and  $B^*$  also are 11-gons or bigger. (See figure 3.31.)

To conclude the proof, consider  $G'' = G - p \neq K_4$ , which is obtained from  $G$  by removing vertex  $p$  (see figure 3.31) and the 3 edges incident to it. This is a  $p$ -graph (by observation it 'inherits' property- $p$  from  $G$ ) with a new 3-gon - call it  $H''$  - formed by 'coalescing'  $A, B$  and the 3-gon  $H$ ; and  $G''$  contains one vertex less than does  $G$ . Consequently,

by our initial assumption (that  $G$  had the minimum number of vertices possible for a  $p$ -graph without deletable  $k$ -edges), it follows that  $G''$  should have a deletable  $k$ -edge. We obtain the contradiction by showing no  $k$ -edge of  $G''$  can be deletable - which contradicts the assumption that  $G$  was smallest!

Claim: No  $k$ -edge of  $G''$  is deletable.

Proof: Assume that, on the contrary,  $G''$  does contain some deletable  $k$ -edge  $e$ . Where could such an edge lie in  $G''$ , and where did it 'come from'? Obviously  $e$  cannot lie on the 'new' 3-gon  $H''$  or on any of the three surrounding faces ( $A^*$ ,  $B^*$  and  $C$ ) of  $G''$ . So it must be shared by two other faces - call them  $F_1$  and  $F_2$  - of  $G''$  which came 'unchanged' from  $G$ , and are distinct (in  $G$ ) from any of the 6 faces of  $\{A, B, C, H, A^*, B^*\}$ . Thus  $e$  was also a  $k$ -edge in  $G$  and so must have been nondeletable in that graph, yet 'became deletable' when vertex  $p$  was removed! Let  $F_1, F_2, F_3$  be some 3-ring in  $G$  which must contain the  $k$ -edge  $e$  as a common edge. Then it follows that  $F_3$  must be one of the 3-gons  $A, B$  or  $H$  - or else ring  $F_1, F_2, F_3$  would also exist in  $G''$  and cause  $e$  to be nondeletable in that graph. Thus it follows that removing vertex  $p$  'replaces' face  $F_3$  with the new 3-gon  $H''$ . Therefore, in  $G''$ , these 3 faces ( $F_1, F_2$  and  $H''$ ) would still be touching one another and yet not be a 3-ring. This could happen (by definition of a 3-ring) if, and only if those 3 faces shared a common vertex in  $G''$  (which, obviously, must be a vertex of the 3-gon  $H''$ ); that is, vertex  $y = y'$ ,  $z = z'$  or  $n$ . But then  $F_1, F_2$  and  $F_3$  must also share this vertex (since  $F_3 = A, B$  or  $H$ ) and thus cannot comprise a 3-ring in  $G$ . This proves the claim and thus completes the proof of theorem 3.19.

THEOREM 3.18 (See beginning of this section for the statement of this theorem).

PROOF(continued): Assertion (I) has already been proved (following the statement of the theorem). We now prove the rest.

(II)'s PROOF: Let  $e = uv$  be such an edge. Thus it is a common edge of some 3-ring  $R^{**} = FMQ$ , where say  $F$  and  $M$  lie within core  $H$  and share  $e$  - and  $Q$  (by lemma 3.4) lies outside the outer ring  $T$ . See figure 3.32. But then  $Q$  must touch  $F$  'across' the ring  $R$ , which means they share just a vertex -  $y = y'$  or  $z = z'$ ; assume it is  $y = y'$ . And, by precisely the same argument it follows that  $Q$  and  $M$  share the vertex  $z = z'$ . Thus the face  $Q$  contains vertices  $y$  and  $z$  of face  $C$ . Therefore, by 3-connectedness (property-p),  $Q$  and  $C$  must share an edge joining  $y$  and  $z$  (the edge  $yz$  lying on the outer rim  $T$ ). Now consider the three faces  $F, M$  and  $C$ . Each touches the other two and thus all three must share a vertex; or else  $FMC$  would be a 3-ring of  $Z$  with its inner core properly contained within  $H$  - a contradiction. That vertex can only be one of the endpoints of the L-edge  $e$ . By the preceding it follows that  $C$  must be a 3-gon containing a (3-valent) vertex that is an endpoint of  $e$ . (See figure 3.26).

(III)'s PROOF: Assume  $e$  is a nondeletable L-3dge of, say,  $S \cap A$  and that  $e$  is shared by faces  $A$  and  $M$  of some ring  $R^{**} = AMQ$  of  $Z$ . Thus (see figure 3.33)  $M$  lies within core  $H$  (i.e., inside rim  $S$ ) and, by lemma 3.4,  $Q$  lies outside the outer rim  $T$ . Then  $M$  and  $Q$  must share just a vertex - the vertex  $z = z'$  - which consequently is also shared by  $B$  and  $M$ . Thus we have the following situation:  $M$  and  $A$  share an L-edge, as do  $A$  and  $H$ , and  $B$  and  $M$  touch at

$z = z'$ . It follows that  $M, A$  and  $B$  must share a vertex - which can only be  $p$  - or else they would comprise a 3-ring of  $Z$  which contradicts the minimality of core  $H$ . Now consider  $A, M$  and  $C$ . The first two share the L-edge  $e$ , and both touch  $C$ ; consequently they must also share a common vertex - and that can only be  $y = y'$  - or else they would comprise a 3-ring of  $Z$ , which would (by lemma 3.4) contradict the minimality of core  $H$ . By the preceding we have shown that face  $M$  contains each of the vertices  $p, y = y'$  and  $z = z'$ . Thus, by 3-connectedness (property-p), core  $H$  must be just a 3-gon - as shown, for example, in figure 3.24.

(IV)'s PROOF: Assume that each of  $A$  and  $B$  contains an edge of rim  $S$  that is a nondeletable L-edge. Then by (III) of the theorem we know that  $H$  is just a 3-gon - which will also be referred to as "the face"  $H$ . From the proof of (III) we know that  $y = y'$  and  $z = z'$  and, more specifically, we know the following (see figure 3.34):

(a)  $A$  and  $H$ , and a third face - call it  $Q_1$  - which lies outside rim  $T$ , comprise a 3-ring of  $Z$  - call it  $R_1$ .

(b)  $B$  and  $H$ , and a third face - call it  $Q_2$  - also lying outside  $T$ , comprise a 3-ring of  $Z$  - call it  $R_2$ .

By property-p, neither  $Q_1$  nor  $Q_2$ , can contain both vertex  $y$  and vertex  $z$ . (Because faces  $C$  and  $H$  already share the edge  $yz$ ). Thus it follows that  $Q_1 \neq Q_2$ , and that:

(i)  $Q_1$  shares  $z$  with  $H$  and touches  $A$  - but not at  $y$ .

(See figure 3.34(a)).

(ii)  $Q_2 \neq Q_1$  shares  $y$  with  $H$  and touches  $B$  - but not at  $z$ . (See figure 3.34(b)).

Claim: By panarity,  $Q_1$  touches  $A$  only at vertex  $n$  and, similarly,

$Q_2$  touches  $B$  only at vertex  $n$ .

The proof of this claim will conclude the proof of (IV) (and thus of theorem 3.18), because it follows (by the claim) that:  $Q_1$  [ $Q_2$ ] shares vertices  $n$  and  $z$  [ $y$ ] with face  $B$  [ $A$ ] and thus (by property-p) must share the edge  $nz$  [ $ny$ ] with that face - which consequently must be a 3-gon; and  $Q_1$  and  $Q_2$  touch at  $n$  and thus cannot also touch at another vertex on  $C$ ; and thus relabelling  $Q_1$  as  $B^*$  and  $Q_2$  as  $A^*$  completes matters.

Proof of claim: Let  $n_1$  denote some vertex shared by  $Q_1$  and  $A$ , and let  $n_2$  denote some vertex shared by  $Q_2$  and  $B$ . We will show, by planarity, that  $n_1 = n = n_2$ . Well, by the preceding, the 5 vertices  $n_1, n, n_2, z, y$  must appear in that order on the outer rim  $T$  as one moves around it. (See figure 3.35). Now,  $n_1$  and  $z$  are both vertices of  $Q_1$ , and  $n_2$  and  $y$  are both vertices of  $Q_2$ . Thus it should be possible to draw 2 non-crossing arcs, both lying outside  $T$ , such that: one arc joins  $n_1$  to  $z$  and another joins  $n_2$  to  $y$ . But this can be done if and only if  $n_1 = n = n_2$ . Thus the claim is proved.

CHAPTER 4

THE STRUCTURE OF BICUBIC 3-POLYTOPAL GRAPHS

4.1 INTRODUCTION

In this chapter we study those 3-connected plane graphs that are 3-valent and bipartite - henceforth to be referred to as b-graphs. The smallest b-graph (see figure 4.1) is the 3-cube with 8 vertices and 6 faces, all 4-gons; it will sometimes be denoted by Q. (Note, by theorem 1.4 every b-graph must have at least 6 4-gons.) It is clear that each of the 3 distinct 'edge-adding' operations (which 'extend' the bicoloring of the vertices) illustrated in figures 4.2, 4.3 and 4.4 will, when applied to a b-graph, result in a cubic graph that is also bipartite (and 3-connected, since property-p is also preserved). These 3-generating operations may be referred to, and described, in the following way.

(I) Operation t1, or inserting a 4-gon, which involves placing 2 'parallel' edges between 2 edges (of a face F) so as to preserve 'evenness' (i.e., create even faces - which permits an extension of a bicoloring). As shown in figure 4.2, this can be done in several ways.

(II) Operation t2, or inserting two 4-gons (as shown in figure 4.3) by placing the 'legs of an H' on each of two nonadjacent edges (of some face) so as to preserve 'evenness'.

(III) Operation t3, or replacing a vertex with a cluster of 3 (mutually adjacent) 4-gons, which is shown in figure 4.4.

As is usually the case the results are derived in terms of 3 reducing operations that are the inverses of t1, t2 and t3. These will be referred to as follows:

(i) reduction r1, or deleting a 4-gon (see figure 4.2).

(ii) reduction r2, or deleting an H (see figure 4.3).

(iii) reduction r3, or shrinking a cluster to a vertex (see figure 4.4):

REMARK. The essential question is: when, and why, would one of these reductions - when applied to a b-graph other than the cube - fail to produce another b-graph? The answer is suggested by the following: deleting two parallel (i.e., nonadjacent) edges of some isolated 4-gon will destroy 3-connectedness if those edges are common edges of 4-ring (as shown, for example, in figure 4.14). In view of this we will study those cores of b-graph that are defined by certain rings (of size 4, and sometimes 5) which contain 4-gons; within each minimal core of this type we will be able to locate a reducible configuration of 4-gons (one to which r1, r2, or r3 may be applied to yield a smaller b-graph).

## 4.2 MAIN RESULTS

### THEOREM 4.4

If  $G$  is a b-graph (that is, a bicubic 3-connected plane graph) other than the cube  $Q$ , then:

- (a)  $G$  may be reduced to a smaller b-graph by reduction r1 or r3.
- (b) Equivalently, there exists a sequence of b-graphs

$$Q = G_0, G_1, \dots, G_i, G_{i+1}, \dots, G_N = G$$

such that each is obtained from the preceding one by the operation t1 (inserting a 4-gon) or by operation t3 (replacing a vertex with 3 mutually adjacent 4-gons).

PROOF: See section 4.3 (i.e., section 3 of this chapter).

REMARK. With a little additional work, it can be shown (see theorem 4.5) that edges need not be placed in the infinite (exterior) face when

t1 or t3 is applied.

Consider the following conjectures - the first is usually attributed to D. Barnette (see [G2, p. 1145]):

CONJECTURE 4.1. Every b-graph has an HC (Hamiltonian circuit).

CONJECTURE 4.2. If  $G$  is a b-graph with an HC, and  $e$  and  $e'$  are two edges of some face, then at least one of them must be used by some HC in  $G$ .

If conjecture 4.2 is true then theorem 4.4 implies that conjecture 4.1 must also be true (this follows easily - using induction - from part (b) of theorem 4.4); thus 4.2 is a very strong conjecture.

With respect to Barnette's conjecture we cite the following:

THEOREM (P. Goodey [GO], 1975). A b-graph with only 4-gons and 6-gons (and thus precisely 6 4-gons) is Hamiltonian.

THEOREM (D. Petersen [P1], 1981). A b-graph with 8 4-gons is Hamiltonian.

The 'Hamiltonian' theorem proved in this chapter is somewhat different in spirit from the results of Goodey and Peterson:

THEOREM 4.7

If  $G$  is some b-graph (not the cube) and  $H$  some HC in it, then: one can reduce  $G$  to a smaller b-graph containing 'that HC induced by  $H$ ', by using one of the 3 reductions  $r_1, r_2, r_3$ . (See figures 4.5-4.8 which illustrate the 'proposed' operations.)

PROOF: See section 4.5.

Before presenting our last result we note the following fact with respect to reductions  $r_1, r_2$  and  $r_3$ : any 'connected configuration' of 2 or more 4-gons (in a b-graph) may be 'satisfactorily' reduced - see lemma 4.1 and part (c) of lemma 4.3 - but that such is not necessarily the case for an isolated 4-gon  $F$  (see figure 3.14).

Our last result is the following:

THEOREM 4.8

Any cyclically-4-connected b-graph, other than the cube, may be reduced to a smaller such graph by applying either reduction r1 or reduction r2.

PROOF: See section 4.6.

REMARK. The graph(s) of figure 4.15 cannot be reduced by r1 to a smaller cyclically 4-connected b-graph. (Reason: as is easily observed, each pair of 4-gons lies in a 5-ring.) Further, there is an infinite family of such graphs. (This is made clear by the diagram.)

4.3 PRELIMINARIES

Before presenting the various proofs it will be convenient to give names to certain configurations of faces, and to prove some lemmas. Note, for the proof of theorem 4.4 one need only read this section to the end of the fifth lemma (lemma 4.2(a)).

Three mutually adjacent 4-gons will be called a cluster (see figure 4.4), and a 4-gon not adjacent to another 4-gon will be called an isolated 4-gon (or a 1-ladder). In a b-graph other than the cube we know, by 3-connectedness (property-p), that any other 'connected' configuration of k 4-gons must comprise: either a k-ring (see figure 4.1), in which case the graph is a k-prism (and  $k \geq 4$ ); or a 'ladder' (see figure 4.11). To be more precise let us define, for G any cubic p-graph (i.e., plane and 3-connected) without triangles, a k-ladder (of 4-gons) to be the following:

a set of  $k \geq 2$  4-gons  $F_1, F_2, \dots, F_i, F_{i+1}, \dots, F_k$

where each  $F_i$ , but for  $F_1$  and  $F_k$ , shares precisely 2 edges with

other 4-gons of  $G$  - one with  $F_{i-1}$  and another, disjoint from the first, with  $F_{i+1}$ ; and  $F_1$  and  $F_k$  each touch only one 4-gon of  $G$ ,  $F_2$  and  $F_{k-1}$  respectively.

It is convenient to also refer to an isolated 4-gon as a 1-ladder. Also, 2-ladders and 3-ladders will sometimes be referred to, respectively, as duos and trios. Note that by our definition, no  $k$ -ladder can contain a smaller ladder.

LEMMA 4.00

(a) If  $G$  is a connected bipartite graph or multigraph (planar or not) of regular valence 3, then  $G$  has no bridge (i.e., cut edge) and thus is 2-connected.

(b) If  $G$  is a plane, cubic, 2-connected graph then:  $G$  is bipartite if and only if each of its faces has an even number of edges.

PROOF: Suffice it to say that these two facts are well known and readily verified by elementary arguments.

LEMMA 4.0

Let  $G$  be a  $b$ -graph, then:

- (a) No  $k$ -ring of  $G$  or of any  $p$ -graph) can contain a  $(k-1)$ -ladder.
- (b) No 4-gon of  $G$ , isolated or not, can be a face of a 3-ring.
- (c) If  $k$  is odd then no  $k$ -ring of  $G$  can contain a  $(k-2)$ -ladder.

PROOF: Assertion (a) is immediate by 3-connectedness. To prove (b), assume that assertion false and that:  $e, f$ , and  $g$  are the 3 common edges of some 3-ring  $R$ , and that  $e$  and  $f$  are opposite edges of some 4-gon (which is a face of  $R$ ). Now if we delete  $e$  and  $f$  from  $G$ , we obtain a connected cubic graph or multigraph which is also bipartite (it obviously 'inherits' a bicoloring from  $G$ ) but contains edge  $g$  as a bridge! This contradicts lemma 4.00.

With respect to assertion (c): we have just proved it true for the case  $k = 3$ . Now assume the assertion false for some minimum (odd) value of  $k \geq 5$ ; and let  $R$  be some  $k$ -ring containing all 4-gons of a  $(k-2)$ -ladder. By 3-connectedness (see next lemma also) it is immediately obvious that: one can apply reduction  $r_1$  to some 4-gon in that ladder and so reduce  $G$  to a smaller b-graph in which  $R$  has become a  $(k-2)$ -ring containing a  $(k-4)$ -ladder. But this contradicts the assumption that  $k$  (odd) was a minimum value for which assertion (c) was false!

LEMMA 4.1

If  $G$  is a b-graph other than the cube then:

(a) One can apply reduction  $r_3$  to any cluster and obtain a smaller b-graph.

(b) One can apply reduction  $r_1$  to any 4-gon of a prism, or to any 4-gon of a  $k$ -ladder with 3 or more 4-gons, and obtain a smaller b-graph.

PROOF: The proof of (a) is immediate since  $r_3$  preserves property-p, and thus 3-connectedness. Assertion (b) is well known and easily verified - and is subsumed under the more general assertion of part (ii) of lemma 4.2.

REMARK. From figures 4.5 and 4.6 it is obvious that an HC in  $G$  will induce one in a b-graph which is obtained from  $G$  by: application of  $r_3$  (or of its inverse, operation  $t_3$ ); or by application of  $r_1$  to a ladder with 3 or more 4-gons.

LEMMA 4.2

If  $G$  is a cubic p-graph with no 3-gons, then:

(i) If  $F$  is an isolated 4-gon then: one can delete both edges

of either one of its two pairs of nonadjacent edges and obtain a smaller p-graph if and only if,  $F$  is not a face of some 3-ring or 4-ring (as is the case, for example, in figure 4.20 and 4.14).

(ii) If  $F$  and  $F^*$  are two adjacent 4-gons of some ladder  $L$  and share edge  $e$ , then one can delete both  $e$  and its opposite edge on  $F$  to obtain a smaller p-graph if and only if:  $F$  and  $F^*$  are not both faces of some 4-ring (in which case, obviously, all the 4-gons of ladder  $L$  are faces of that ring).

REMARK. By 3-connectedness  $G$  cannot have a 3-ring containing more than one 4-gon; but, as shown in figure 4.19, a 4-ring can.

PROOF: Both assertions will be proved together. Let  $e$  and  $f$  be 2 nonadjacent edges of the 4-gon  $F$  such that: if  $F$  and some other 4-gon  $F^*$  are indeed adjacent 4-gons in some ladder, then they share edge  $e$ . Delete  $e$  and  $f$  and consider  $G'' = (G \sim e) \sim f = (G \sim f) \sim 3$ . By hypothesis,  $G''$  has no 1-gons or 2-gons and is thus a (connected) graph. Thus,  $G''$  is a p-graph if and only if it is 3-connected. Assume it is not. That is obviously equivalent to the following: the graph  $(G \sim e)$  is not 3-connected; or  $(G \sim e)$  is 3-connected but the graph  $(G \sim e) \sim f$  is not. By theorem 3.5 this is equivalent to:  $e$  is a common edge of some 3-ring of  $G$ ; or it is not, but  $f$  is a common edge of some 3-ring of  $(G \sim e)$ . Thus we have our result.

Before applying lemma 4.2 to b-graphs (in the obvious way), we present the following.

DEFINITION. If  $R$  is a  $k$ -ring of a b-graph  $G$ , then  $R$  may be said to be a  $k$ -loop (or loop) of  $G$  if: amongst  $R$ 's faces are all the faces of some ladder,  $L$ , of 4-gons (possibly a 1-ladder); in which case each 4-gon of  $L$  may be said to be primary in loop  $R$ .

Thus lemma 4.2 may be reformulated, for b-graphs, as follows:

LEMMA 4.2(a)

Let  $G$  be a b-graph. Then:

(i) An isolated 4-gon  $F$  may be deleted in either 'direction' (i.e., in both of two ways) by reduction  $r_1$  - and  $G$  reduced to a smaller b-graph by each such reduction - if and only if  $F$  is not (primary) in some 4-loop.

(ii) A duo (or 2-ladder of 4-gons,  $FF^*$ , may be reduced by  $r_1$  - and a smaller b-graph obtained, if and only if they are not both (primary) in some 4-loop.

PROOF: Immediate by lemma 4.2 and the fact (see lemma 4.0) that a 3-ring cannot contain a 4-gon.

REMARK. An irreducible isolated 4-gon,  $F$ , and an irreducible duo,  $FF^*$ , are to be found (respectively) in figures 4.14 and 4.19.

DEFINITION. By a k-belt (of some b-graph  $G$ ) will be meant a set,  $B$ , of  $k+1$  faces - including both 4-gons of some duo  $FF^*$  - such that: the faces of  $B$ , minus either one of that duo, comprise a  $k$ -ring (in which case  $FF^*$  may also be referred to as the major duo of belt  $B$ ). See, for example, figure 4.9(a).

REMARK. By lemma 4.0 we know that a 3-belt cannot occur.

LEMMA 4.3 (not needed for theorem 4.4).

Given:  $G$  a b-graph, other than the cube, in which  $F$  and  $F^*$  are two adjacent 4-gons comprising a duo which share edge  $e$ . Then:

(a)  $G \sim e$  is a p-graph with no triangles in which the new face created by the deletion of  $e$  is an isolated 4-gon; in fact, if  $e'$  is some other edge shared by another duo of  $G$ , then  $(G \sim e) \sim e'$  is also a p-graph with no triangles.

(b) One can apply reduction  $r_2$  to the duo of  $F$  and  $F^*$  and obtain a smaller b-graph if and only if that duo is not the major duo of some 4-belt.

(c) At least one of the two reductions,  $r_1$  and  $r_2$ , can be applied to the duo  $FF^*$  so as to reduce  $G$  to a smaller b-graph.

PROOF: Part (a) is immediate by definition of  $G$  as a b-graph and the fact that (by 3-connectedness) no triangle-free p-graph can have a 3-ring which contains two 4-gons. The proof of assertion (b) is also very brief: just combine (a) above with assertion (i) of lemma 4.2. To prove (c), assume that the duo  $FF^*$  is indeed the major duo of some 4-belt  $B$ . (See assertion (b) of this lemma.) This is shown schematically in figure 4.9 (b), where we have assumed (without loss of generality) that the two 4-rings defined by  $B$  are normal. But then it is immediate, by figure 4.9 (b), that  $F$  and  $F^*$  cannot also be in a 4-loop. Thus, by lemma 4.2 (a)-part(ii), we can apply  $r_1$  to that duo.

#### 4.4 GENERATING THE BICUBIC 3-POLYTOPAL GRAPHS

##### THEOREM 4.4

If  $G$  is a b-graph, other than the cube, then  $G$  may be reduced to a smaller b-graph by applying reduction  $r_1$  or  $r_3$ .

PROOF: Let  $Z''(G)$  denote the set (possibly empty) of all 4-loops of  $G$ . (See definition preceding lemma 4.2(a).) If  $Z''$  is empty then, by lemma 4.1 and 4.2(a), the theorem is true in the 'strongest possible way'. (That is, every configuration of 4-gons may be reduced - by  $r_1$  or  $r_3$  - so as to produce a smaller b-graph!) So assume  $Z''$  is not empty and consider all the cores defined by the 4-loops of  $Z''$ .

By definition of a loop, none of these cores can be just the boundary of a 4-gon; but by lemma 3.6, each of them must contain (the boundary of) some 4-gon of  $G$ . Consider one such core  $H$ , defined by the loop  $R$  of  $Z''$ , such that  $H$  is minimal with respect to the others (i.e., does not properly contain - as a subgraph - any other core defined by a loop of  $Z''$ ). Let  $F^*$  be a 4-gon of  $G$  with boundary in  $H$ . The following lemma concludes the proof of theorem 4.4. (Actually only part(ii) is needed; (i) is used for theorem 4.5).

LEMMA 4.4(a)

(i) Loop  $R$  cannot contain two disjoint 4-gons and thus can have either 1 4-gon (isolated) or can contain 2 adjacent 4-gons (a duo).

(ii)  $F^*$  cannot be primary in some loop,  $R^*$ , of  $Z''$ ; thus (by lemmas 4.1 and 4.2(a)) one can apply either  $r_1$  or  $r_3$  to  $F^*$ , and so reduce  $G$  to a smaller  $b$ -graph.

PROOF: We prove assertion (ii) first. Assume it is false. We will obtain the contradiction that  $H$  (which is defined by  $R$ ) is not minimal (i.e., it properly contains another core of  $Z''$ ). Once again (by lemma 3.1, 3.2, 3.3) we can assume, without loss of generality, that both  $H$  and  $R$  are normal in  $G$ . It follows (by lemma 3.4) that:  $R$  and  $R^*$  must be 'interlocking' 4-loops (as shown schematically in figure 4.13) with two disjoint faces of  $R$  (call them  $W$  and  $E$ ) each meeting the 4-gon  $F^*$  on an edge. Thus neither  $W$  nor  $E$  can be 4-gons - or else  $R$  would not satisfy the definition of a 4-loop - and so  $R$  must contain some isolated 4-gon  $F$ . But then  $FWF^*E$  is a 4-loop; so the inner core of this loop is properly contained in core  $H$  - contradiction!

Proof of (i): Once again we can assume that  $R$  and  $H$  are normal. Now assume that assertion (i) is false. Thus  $R$  contains precisely two 4-gons, and they are disjoint. Let  $R = WQEQ^*$ , where  $Q$  and  $Q^*$  are both 4-gons, and the 4 faces of  $R$  appear in that order (clockwise) about  $R$ . So, for  $R$  to be a 4-loop, one of the 4-gons must be isolated - say it is  $Q$ . Now consider that unique face, call it  $A$ , which lies within  $H$  and shares an edge (of the inner rim of  $R$ ) with  $Q, W$ , and  $E$ . Claim:  $A$  cannot also touch  $Q^*$ . Proof: if it did then it would contain (by 3-connectedness/property-p) all inner rim edges of each of  $W, Q, E$  and  $Q^*$ ; thus  $H$  would be just the boundary of  $A$  (and thus a 4-gon) - contradiction! But then  $WAEQ^*$  also is a 4-loop of  $Z''$ , whose inner core is properly contained in  $H$  - contradiction! The proof of theorem 4.4 is complete.

REMARK. Several facts relating to the proof of theorem 4.4 are worth noting; but we defer comment because the proof of the next theorem (a refinement of theorem 4.4) will invoke several properties (of 4-loops in b-graphs) not yet mentioned.

#### THEOREM 4.5

If  $G$  is a b-graph, other than the cube, then  $G$  may be reduced to a smaller b-graph by applying reduction  $r_1$  or  $r_3$  - and no edges on the exterior face are deleted.

REMARK. It is implicit in the statement of the theorem that  $r_3$  (shrinking a cluster to a vertex) is also being viewed as an edge-deleting operation.

PROOF: Let  $X$  denote the exterior face of  $G$  and, as was done in theorem 4.4, let  $Z''(G)$  denote the set of all 4-loops of  $G$ . We will consider 2 cases:  $Z''$  is empty;  $Z''$  is not empty. But first

we note the following.

Note: if  $X$  is not a 4-gon but touches some cluster (see figure 4.10), then  $r_3$  - viewed as a succession of 3 edge deletions - may be applied without deleting an edge on the exterior face. (See figure 4.10 for a 'proof' of this.)

Case (i):  $Z''$  is empty. Thus all ladders are reducible - and isolated 4-gons are reducible in 2 directions. If  $G$  has no clusters but only ladders we can certainly find some 4-gon ( $G$  has at least 6) with 2 parallel edges - and neither on  $X$  - which may be deleted by  $r_1$  so that  $G$  is reduced to a smaller b-graph.

If  $G$  does have a cluster containing (as one of its 4-gons) the face  $X$ , then some other 'disjoint' configuration of 4-gons does not touch  $X$ ; and we can apply  $r_1$  or  $r_3$  there.

If  $G$  has a cluster which does not contain  $X$  but is touched by it (see figure 4.10), then that cluster can be reduced without deleting an edge of  $X$  - as was noted earlier.

Case (ii):  $Z''(G)$  is not empty. In this case consider some 4-loop,  $R^*$ , of  $Z''(G)$ . As was pointed out in chapter 3,  $R^*$  must 'give rise to' 2 or more disjoint cores - each minimal with respect to all cores defined by loops of  $Z''$ . Let  $H$  and  $H''$  denote 2 such cores. Obviously one of them, say  $H$ , does not contain the (boundary of the) exterior face  $X$ . Consider  $H$  and the 4-loop which defines it - call it  $R$ . (Note that  $R$  itself may contain, amongst its 4 faces, the face  $X$ ). By lemma 3.6,  $H$  contains - as a proper subgraph - some 4-gon of  $G$ . By part (i) of lemma 4.4(a) we know that any such 4-gon cannot be adjacent to a 4-gon of the loop  $R$ . Thus, if  $F^*$  is some 4-gon in  $H$ , then  $F^*$  belongs to some ladder or cluster which lied wholly in  $H$ . We consider 3 cases:

If  $F^*$  is one of a cluster then it can be reduced by  $r_3$  without deleting an edge of  $X$  - even if  $X$  touches that cluster. If  $F^*$  is isolated then, since it is reducible in 'both directions' (by lemmas 4.2(a) and 4.4(a)), we can always choose to delete two parallel edges (of  $F^*$ ) which are not on  $X$ . Finally, if  $F^*$  is in some  $k$ -ladder  $L$  (where  $k \geq 2$ ) but touches  $X$  (a member of loop  $R$ ), then - if necessary - we can choose to delete 2 parallel edges (neither on  $X$ ) of some other 4-gon of  $L$ , and so obtain a smaller  $b$ -graph. The proof of theorem 4.5 is complete.

#### 4.5 A THEOREM ABOUT HAMILTONIAN $b$ -GRAPHS

The next theorem is really just a strengthening of theorem 4.4 and, essentially, depends on demonstrating (with the aid of part(b) of lemma 4.3) that: if a  $b$ -graph  $G$  contains only isolated 4-gons and/or duos of 4-gons, then one can always locate such a configuration which can be reduced (via  $r_1$  and/or  $r_2$ ) 'in both directions' - as shown in figures 4.7 and 4.8 (where the squiggled edges are used by an HC). This, in brief, is to be the strategy for theorem 4.7.

DEFINITIONS. If  $G$  is a  $b$ -graph and  $B$  a  $k$ -belt (see definition preceeding lemma 4.3, and examples of figure 4.9) with its major duo of two 4-gons sharing edge  $e$ , then:  $G \sim e$  is a 3-connected graph (by lemma 4.3) in which  $B$  'has become' a  $k$ -ring, call it  $B'$ ; those subgraphs of  $G \sim e$  that are the rims [cores] of ring  $B'$  are also subgraphs of  $G$ , and will be called the rims [cores] of belt  $B$  (of  $G$ ).

Note: each of the rims and cores of  $B$  is actually defined by some  $k$ -ring of  $G$  - that ring comprised of all faces of  $B$  minus one of the 4-gons of its major duo. As expected, a belt will be said to be

a normal belt if its rims are concentric.

LEMMA 4.6

If  $B$  is a  $k$ -belt of some  $b$ -graph  $G$  then:  $B$  is normal if and only if none of its faces is the exterior face; and each of  $B$ 's two cores contains (the boundary of) some 4-gon of  $G$  if  $k \leq 5$ .

PROOF: Immediate by definition of "belt" and lemma 3.6.

THEOREM 4.7 (See section 4.2 for statement).

PROOF: If  $G$  is either a prism, or contains only clusters and/or ladders of 3 or more 4-gons, then the proof follows immediately (by lemma 4.1 and the remark following it) in the 'strongest possible way'. (That is: every configuration of 4-gons is reducible, and any HC in  $G$  will 'shrink' to one in the reduced  $b$ -graph - see figures 4.5 and 4.6). So assume  $G$  also contains isolated 4-gons and/or duos. Now, if each of these could be reduced (by  $r_1$  and/or  $r_2$ ) in 'both directions', then again our theorem would be proved in the 'strongest possible way' (see figures 4.7 and 4.8). So assume not. Thus (by lemmas 4.2(a) and 4.3)  $G$  contains 4-loops and/or 4-belts.

Define  $Z(G)$  to be the union (non-empty) of the following two sets:  $Z''(G)$ , the set of all 4-loops of  $G$ ; and the set of all 4-belts of  $G$  (see definition preceding lemma 4.3).

Consider all the cores defined by the loops and belts of  $Z$ , and choose one such core,  $H$ , that is minimal - i.e., does not properly contain any other core defined by a member of  $Z$ . (Of course there are at least 2 such cores). Let  $F^*$  denote one of the 4-gons of  $G$  which (by lemmas 3.6 and 4.6) must lie in  $H$ . If  $H$  is defined by a ring of  $Z$  call that ring  $R$ ; if it is defined by a belt, call that belt  $B$ . We want to show that if  $F^*$  is isolated or in a duo, then

it can be reduced in both directions.

Obviously our strategy will be to prove the analogue of part(ii) of lemma 4.4(a). This, with lemmas 4.1, 4.2(a), and 4.3(part b), will complete the proof. (Actually the analogue of part(i) is derivable immediately - using the same proof - but we will not use it).

LEMMA 4.7(a)

If the 4-gon  $F^*$  is isolated, or a member of some duo  $F^*F^{**}$ , then the following cannot occur:

$F^*$ , or both  $F^*$  and  $F^{**}$  - if the latter exists, are in some 4-belt or 4-loop of  $Z$  - call it  $L^*$ .

PROOF: Assume the lemma false. We can assume (by 3.1, 3.2, 3.3 and 4.6) without loss of generality that: core  $H$  and its defining loop  $R$  - or its defining belt  $B$  - are both normal. We consider 3 cases and obtain a contradiction in each.

Case (i):  $H$  is defined by the loop  $R$  of  $Z$ . Then, by lemma 3.4's remark, some face  $A$  of  $L^*$  must lie outside the outer rim of  $R$ . But then  $L^*$  also must contain 2 disjoint faces of  $R$  (to 'cross'  $R$ ), call them  $W$  and  $E$ , and each of them must touch both  $A$  and  $F^*$  (whether or nor  $F^{**}$  exists). Now  $R$  must contain all the 4-gons of some ladder, since  $R$  is a 4-loop; so both  $W$  and  $E$  must be non-4-gons, and thus  $R$  contains an isolated 4-gon - call it  $Q$ . Now consider the 4-ring  $WQEF^*$ . It is a 4-loop of  $Z$  and thus (by lemma 3.4's remark) has an inner core properly contained in  $H$  - contradiction!

Case (ii):  $H$  is defined by the 4-belt  $B$  of  $Z$  and some face,  $A$ , of  $L^*$  lies outside the outer rim of  $B$ .

Do the following: let  $Q$  and  $Q''$  denote the two 4-gons comprising the major duo of  $B$ , such that  $Q$  touches the inner rim of  $B$ . Now, as in

case (i),  $L^*$  must contain 2 disjoint faces of  $B$  - in order to 'cross' it. Call these faces  $W$  and  $E$ .  $W$  and  $E$  must both touch  $F^*$  (whether  $F^{**}$  exists or not) and thus neither can be a 4-gon of the duo  $QQ''$ , so they must 'flank' that duo (see figure 4.12(b)), in which case neither can be a 4-gon (and so both would also touch  $F^{**}$ , if it existed). If  $F^*$  were isolated then it is clear by the preceding that  $W, F, E$  and  $Q$  would comprise a normal 4-loop, whose inner core (by lemma 3.4's remark) is properly contained in  $H$  - contradiction! Similarly, if  $F^*$  and  $F^{**}$  are a duo then  $W, F^*, F^{**}, E$  and  $Q$  must comprise a normal 4-belt, whose inner core is properly contained in  $H$  - contradiction!

Case (iii):  $H$  is defined by the belt  $B$  of  $Z$ , but no face of  $L^*$  lies outside the outer rim of  $B$ .

Do the following: let  $Q$  and  $Q''$  denote the two 4-gons that comprise the major duo of  $B$ , labelled so that  $Q$  touches the inner rim. See figure 4.12(b), where  $W$  and  $E$  denote the two non-4-gons flanking  $Q$  and  $Q''$  in belt  $B$ .

If  $L^*$  contained  $Q$  - and perhaps  $Q''$  also, or contained both of them, or contained neither of them, then: since  $L^*$  contains  $F^*$  and no face lying outside  $B$ 's outer rim (by assumption), the inner core of  $L^*$  obviously is properly contained in  $H$  - contradiction! Thus one need only show that the following leads to a contradiction:  $L^*$  contains  $Q''$  but not  $Q$  (and, of course,  $L^*$  has no faces lying outside  $B$ 's outer rim). Assume this is the case. Then, obviously, both  $W$  and  $E$  must be in  $L^*$ . So  $L^*$  'already' contains the following faces:  $W, E, Q$  and  $F^*$ , if  $F^*$  is isolated; or  $W, E, Q'', F^*$ , and  $F^{**}$ , if  $F^*$  is not isolated. If the first is the case then  $L^*$  cannot be a 4-belt

because there is no 'room left' for the major duo) and so these 4 faces must 'already' comprise the 4-ring that is  $L^*$ ; if the latter is the case then those 5 faces obviously comprise the 4-belt  $L^*$ . But then, in either case, we can 'replace'  $Q''$  with  $Q$  and obtain another 4-loop or 4-belt of  $Z$  whose inner core is properly contained in  $H$  - contradiction!

The proof of the lemma, and thus of theorem 4.7, is complete.

#### 4.6 GENERATING THE $c_4$ -CONNECTED BICUBIC 3-POLYTOPAL GRAPHS

Recall that: by a b-graph is meant a 3-valent bipartite p-graph (where by a p-graph is meant a plane 3-connected graph); by a  $c_4$ -connected graph is meant a cyclically-4-connected graph (one in which two circuits cannot be separated by cutting three or fewer edges). By a c-graph will be meant a  $c_4$ -connected b-graph.

#### THEOREM 4.8

A c-graph  $G$ , other than the cube, may be reduced by  $r_1$  and/or  $r_2$  to a smaller c-graph.

REMARK. Note that the graph of figure 4.15 is reducible by  $r_2$ , but not by  $r_1$ .

PROOF: This proof contains a number of lemmas, and a reading of their respective proofs should perhaps - for the sake of continuity - be postponed until the conclusion of the 'main' proof.

The strategy, in brief, will be to demonstrate that if  $G$  is not reducible by  $r_1$  (to a smaller c-graph) then it must contain some 5-loop  $R$  (with 1 or 2 4-gons) which defines a core  $H$  containing just 3 faces - a duo and a 6-gon (essentially as shown in figure 4.16); and that in such a configuration the duo may be reduced by  $r_2$  so as

to produce a smaller c-graph.

CONVENTIONS: By "k-loop" and "loop" we will always mean - at least throughout this proof - a loop that is either a 4-loop or a 5-loop. The graph  $G$ , or a configuration of 4-gons, will be said to be "reducible" if one can apply  $r_1$  [ $r_2$ ] to  $G$ , or to the 4-gons, and obtain a smaller c-graph.

We begin by assuming that  $G$  is not a prism, or else  $r_1$  could be applied to any one of its 4-gons to prove the theorem.

LEMMA 4.9

$G$  has no 3-rings, and thus no clusters; and so by 3-connectedness contains only ladders of 4-gons. (Recall that an isolated 4-gon is also a 1-ladder).

PROOF: Immediate by definition of a c-graph.

LEMMA 4.10

If  $e$  and  $f$  are nonadjacent edges of some 4-gon  $Q$  of  $G$  such that neither of  $Q$ 's other two other edges are shared with other 4-gons, then:  $(G \sim e) \sim f = (G \sim f) \sim e$  is not a c-graph if and only if  $e$  and  $f$  are common edges of some loop (in which of course,  $Q$  is primary).

PROOF: Just use lemma 4.2(a) and the fact that  $G$  is  $c_4$ -connected.

LEMMA 4.11

If  $F$  and  $F^*$  comprise a duo (i.e., a 2-ladder), then one may reduce  $G$  to a smaller c-graph by applying  $r_2$  to that duo if and only if:

- (i) that duo is not the major duo of some 5-belt, and also ...
- (ii) that duo is not in some 4-loop (which actually cannot occur - see next lemma).

PROOF: Just use lemma 4.3 and the  $c_4$ -connectedness of  $G$ .

LEMMA 4.12

No  $k$ -loop of  $G$  can contain a ladder of  $k-1$  or  $k-2$  4-gons.

PROOF: Certainly, by 3-connectedness, each ring of  $G$  (which is not a prism) must have at least 2 non-4-gonal faces. So assume  $R$  is some loop with only two (adjacent) faces that are not 4-gons - call them  $A$  and  $B$ . By lemma 3.6 some 4-gon  $F$  lies within the inner core of  $R$  (we can assume, without loss of generality, that  $R$  is normal in  $G$ ); and  $F$  cannot touch any 4-gons of the ladder of  $R$ . But then we can cut 3 edges of  $G$  (the edge shared by  $A$  and  $B$ , and two other edges - on the inner rim of  $R$  - one each on  $A$  and  $B$ ) and separate the 4-gon  $F$  from the outer rim of  $R$  - contradiction!

MAIN PROOF (continued):

Now consider the set of all 4-loops and 5-loops of  $G$ . Assume that set is not empty, otherwise (by lemma 4.10) every 4-gon of  $G$  is reducible by  $r_1$  - and we have proved the theorem. Let  $R$  denote - for the remainder of this proof - some particular loop, which defines a core  $H$  that is not properly contained within any other core defined by a loop of  $G$ . Assume that each 4-gon (of  $G$ ) lying within  $H$  (by lemma 3.6 there is at least one) is not reducible by  $r_1$ , or else we are done; that is (see lemma 4.10) assume that each 4-gon within  $H$  is primary in some loop. By the lemmas of section 3.3 we may also assume without loss of generality that both  $R$  and  $H$  are normal. Next, let  $F$  denote - throughout the remainder of this proof - some particular 4-gon within  $H$  that is primary in, say, the loop  $L$ . We wish to demonstrate that  $R$  and  $H$  together comprise the configuration of figure 4.16.

LEMMA 4.13

$R$  must contain either an isolated 4-gon or a duo, and no other 4-gons.

PROOF: Assume the lemma false. If  $R$  is a 4-loop let  $W, S, E$  and  $N$  be its 4 faces, appearing in that order as one moves counterclockwise about it (see figure 4.17(a)); if  $R$  is a 5-loop (as in figure 4.17(b)) let  $N''$  denote its fifth face - lying between  $N$  and  $E$ . By lemma 4.12 we may assume that  $N$  is a primary 4-gon of  $R$ ,  $W$  is not a 4-gon,  $S$  is another 4-gon, and  $N''$  may or may not be a 4-gon. Now consider the (unique) face lying within  $H$  which contains edges - consecutive - of each of  $W, S$  and  $E$ . Claim: that face, call it  $A$ , cannot touch either  $N$  or  $N''$  (if the latter exists).

Proof of claim: Assume  $A$  did touch one of them. Then, by property-p (i.e., 3-connectedness) and c4-connectedness (i.e., no 3-rings),  $A$  must contain all edges of the inner rim of  $R$  - contradiction!

Thus it follows that  $WAEN$  (or  $WAEN''N$ ) is a loop in which  $N$  must be primary; and the inner core of this loop is properly contained within  $H$  - a contradiction!

LEMMA 4.14

The 4-gon  $F$  (which lies within core  $H$ ) cannot touch two disjoint faces of loop  $R$ .

PROOF: Assume it did and consider figure 4.17. Let  $W$  and  $E$  be such a pair of disjoint faces of  $R$ ; separated by say face  $S$  (of loop  $R$ ) on one side, and also separated by say face  $N$  - or by  $N$  and  $N''$  if  $R$  is a 5-loop. By lemma 4.13 no 4-gon in loop  $R$  can touch the 4-gon  $F$ . Thus it follows that neither  $W$  nor  $E$  can be a 4-gon; and that the primary 4-gon(s) of  $R$  must lie 'between' them as one moves around  $R$ . But then: faces  $W, F, E$  and  $S$  comprise a 4-loop if  $S$  is a 4-gon;  $WFEN$  is a 4-loop if  $N$  is a 4-gon;  $WFEN''N$  is a 5-loop, if  $N$  is a 4-gon and  $R$  is a 5-loop. In all three cases

the new loop (by lemma 3.4) contradicts the minimality of  $H$ !

MAIN PROOF (continued):

By lemma 3.4 we know that some face of  $L$ , call it  $X$ , lies outside the outer rim of  $R$ . By lemma 4.14 two faces of  $L$ ,  $F$  and say  $A$ , must lie within the inner rim of  $R$ . And, as expected,  $L$  also must contain two disjoint faces of  $R$  (in order to 'cross'  $R$ ), call them  $W$  and  $E$  and consider the diagrams of figure 4.17. Let us assume, henceforth, that:  $W$  shares an edge with  $F$ , which in turn shares an edge with  $A$  (which might be a 4-gon) which in turn shares an edge with  $E$ . Equivalently we are assuming that  $W, F, A$  and  $E$  appear in that order about loop  $L$ . Let us henceforth denote:  $S$  to be that unique face of  $R$  separating  $W$  and  $E$ ;  $N$  to be the other face of  $R$  adjacent to  $W$ ;  $N''$  to be the fifth face of  $R$  - if it is a 5-loop - between  $N$  and  $E$ . In view of these labels we can say the following about the distribution of 4-gons in loop  $R$  and  $L$ : By lemma 4.13  $W$  cannot be a 4-gon since  $F$  is. If  $A$  is a 4-gon then  $E$  cannot be one (see lemma 4.12), in which case  $F$  and  $A$  comprise a duo that is primary in  $L$ ; but if  $A$  is not a 4-gon then (obviously)  $F$  is an isolated 4-gon - and primary in  $L$ .

LEMMA 4.15

The edge shared by faces  $F$  and  $A$  (of loop  $L$ ) must contain, as an endpoint, a vertex of face  $S$  (of loop  $R$ ).

PROOF: Assume not. Then, obviously, the 4-gon  $F$  cannot touch  $S$ . Thus, in terms of the labels assigned (see paragraphs preceding lemma) we have: if  $A$  touched  $S$  then  $WFAS$  is a 4-loop; if  $A$  touched  $E$  but not  $S$  then  $WFAES$  is a 5-loop. In either case the 'new' loop has an inner core properly contained in  $H$  - contradiction!

LEMMA 4.16

Loop R must have 5 faces.

PROOF: Assume not. Then (using the labels already assigned)  $R = WSEN$ . But because S and N obviously 'play identical roles' in the 4-loop R, we can apply lemma 4.15 - replacing S by N in the statement of that lemma; that is, the edge shared by F and A also contains a vertex of N. So F touches both N and S. But the latter are disjoint faces of R and thus lemma 4.14 is contradicted.

LEMMA 4.17

The 4-gon F shares a vertex with faces W and S - that vertex on the inner rim of loop R which is an endpoint of the common edge shared by W and S; and F has just 1 vertex lying within the interior of loop R's inner rim.

PROOF: By lemma 4.15 F contains an edge of S (as well as an edge of W - by assumption). Since the graph G has no 3-rings, these two edges must be consecutive edges on R's inner rim. By lemma 4.14 F cannot touch another face of R, and so its 'fourth' vertex lies inside H (i.e., not on the rim).

MAIN PROOF (continued):

In view of the preceding we may say that: each 4-gon lying within core H (and F is one of them) 'monopolizes' an endpoint of a common edge of loop R; that is, no other face within H can touch that vertex. Now, by lemma 4.13, at most 3 such endpoints (of common edges of R) are 'available'. (Recall that it has been demonstrated that R is a 5-loop). Thus core H contains at most three 4-gons - including F - and, since each is primary in some loop (by assumption), they cannot comprise a 3-ladder (see lemma 4.12). Thus there are 5 distinct

cases to consider (and it will be shown that only the last can occur):

- (a) core H contains just one 4-gon, F itself.
- (b) H contains 2 isolated 4-gons and no other 4-gons.
- (c) H contains 3 isolated 4-gons.
- (d) H contains 1 isolated 4-gon and 1 duo.
- (e) H contains 1 duo and no other 4-gons.

LEMMA 4.18

Only case (e) can occur and, further, only 1 other face - a 6-gon - can lie in core H.

PROOF: Define K to be the cubic plane graph obtained from G in the following way: remove all edges and vertices of G which lie outside the outer rim of loop R, and suppress any resulting 3-valent vertices on the 'new' exterior face. Note that K (see figure 4.18) is also 3-connected (property-p is 'inherited' from G) and contains no 3-gons. The new graph K has a 5-ring (it was R) about it and this ring will still be referred to as R; and its inner core will be referred to as H.

With respect to the diagrams of figure 4.18: these contain numbers which represent weights, and will be properly defined in a moment.

Observe that each 4-gon within H 'contributes' a vertex to 2 adjacent faces of R, and thus 'makes each of them bigger than a 4-gon' in K. In figure 4.18 the various possible configurations are shown schematically. Note: even in K, at least one of the 5 faces of ring R must be a 4-gon.

Let us assign a weight of 2 to each 'possible 4-gon' of K (there are at most 3 such faces within core H of K); also assign a weight

of 1 to each 'possible 5-gon' of ring  $R$  (there cannot be any within core  $H$ ); lastly, assign a weight of 1 to the exterior 5-gonal face.

By Euler's inequality (theorem 1.4),  $2p_4(K) + 1p_5(K) \geq 12$ . It follows that the assigned weights should total at least 12. But we can get at most 11 for cases (a) through (d), and 12 can occur in case (e) only if there are just 8 vertices on the inner rim (see figure 4.18 (e)); this forces the conclusion that a 6-gon can be the only other face within core  $H$ .

To demonstrate that cases (a) through (d) yield at most 11 is not very complex.

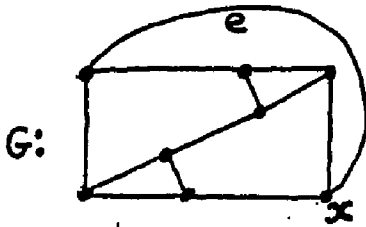
The diagrams of figure 4.18 are virtually self-explanatory and the assigned weights appear within each face. Only for case (d) need one invoke  $c_4$ -connectedness: in figure 4.18 (d), which corresponds to this case, the dotted path indicates that  $c_4$ -connectedness has 'forced' a vertex (other than the 2 endpoints of the rings common edges) to be on that face of  $R$ . Lemma 4.18 is proved.

MAIN PROOF (conclusion):

By assumption, the duo of 4-gons in  $H$  (see figure 4.16) is not reducible by  $r_1$  because it lies in the loop  $L$ . And  $L$ , as was proved, must be a 5-loop with one face - we called it  $X$  - lying outside the outer rim of  $R$ . But then, by invoking planarity, it is obvious that the presence of  $L$  prohibits the following from occurring: the duo (within  $H$ ) is the major duo of some 4-belt or 5-belt, call it  $B$ , which has to 'cross' ring  $R$ . To show this it is sufficient to show that face  $X$  (of loop  $L$ ) cannot also be a face of belt  $B$ ; and planarity forces this conclusion. By lemma 4.11 this concludes the proof of theorem 4.8.

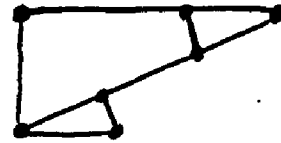
APPENDIX I. GRAPH DIAGRAMS

o  
Fig. 1.1



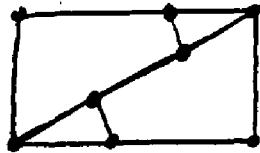
oo  
Fig. 1.2

$G-x$ :



ooo?  
Fig. 1.3

$G-e$ :



$G \sim e$ :

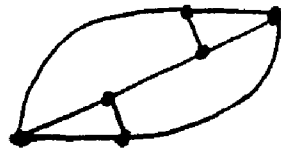


Fig. 1.4

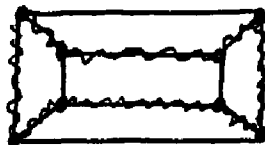


Fig. 1.5



Fig. 1.6

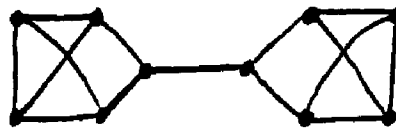
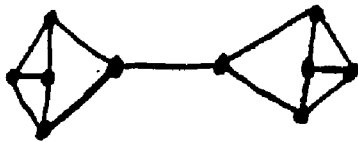


Fig. 1.7



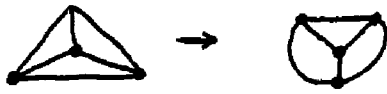


Fig. 1.12

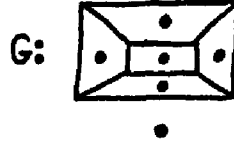


Fig. 1.13

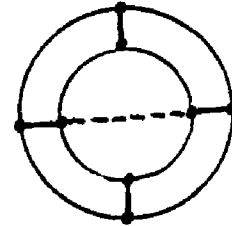
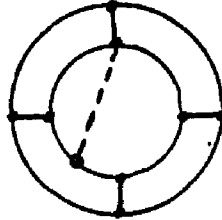
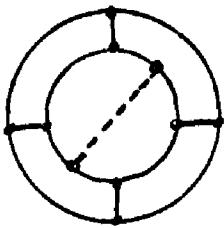


Fig. 1.14

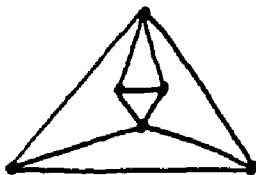
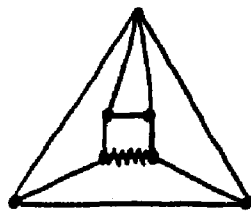


Fig. 1.15

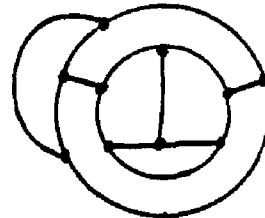
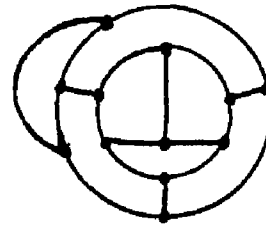


Fig. 1.16

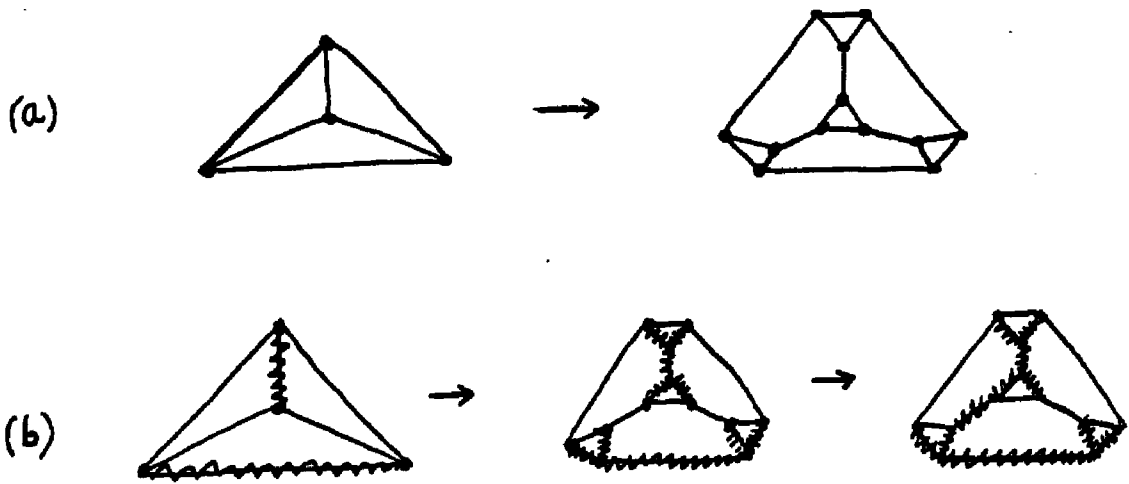


Fig. 2.1

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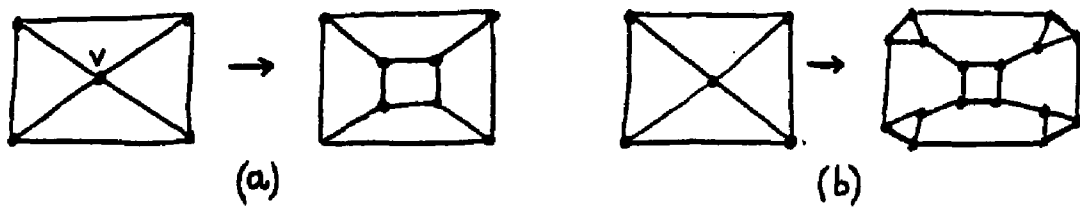


Fig. 2.2

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Fig. 2.3

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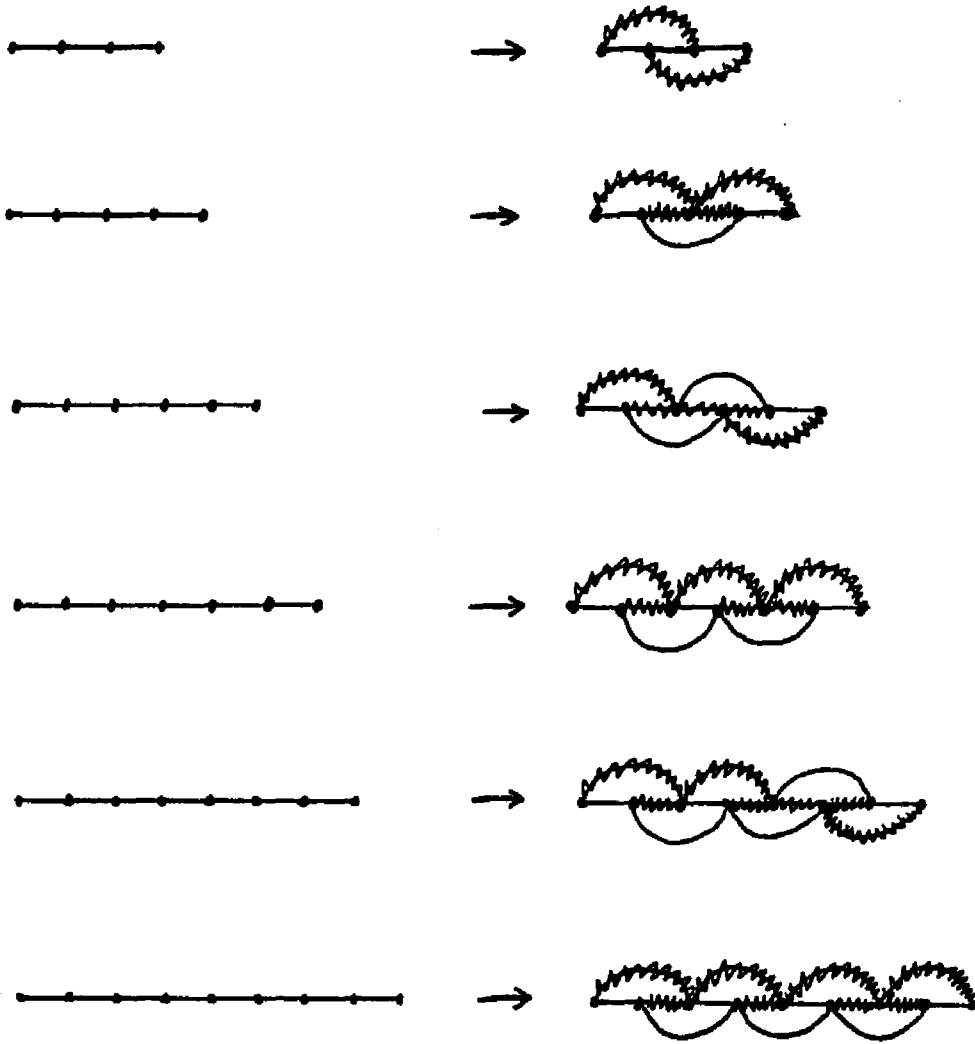


Fig. 2.4

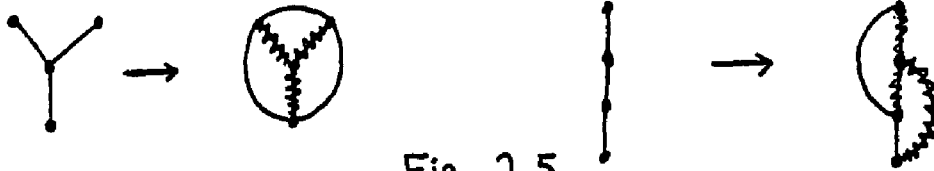
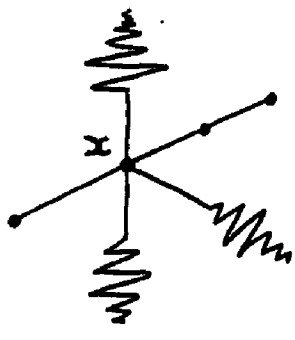
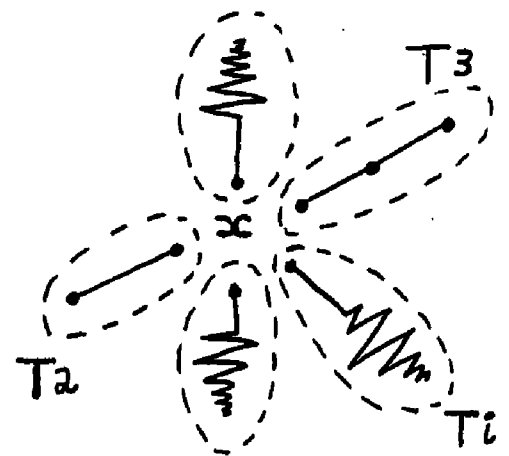


Fig. 2.5



(a)



(b)

Fig. 2.6

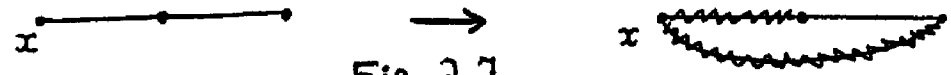


Fig. 2.7

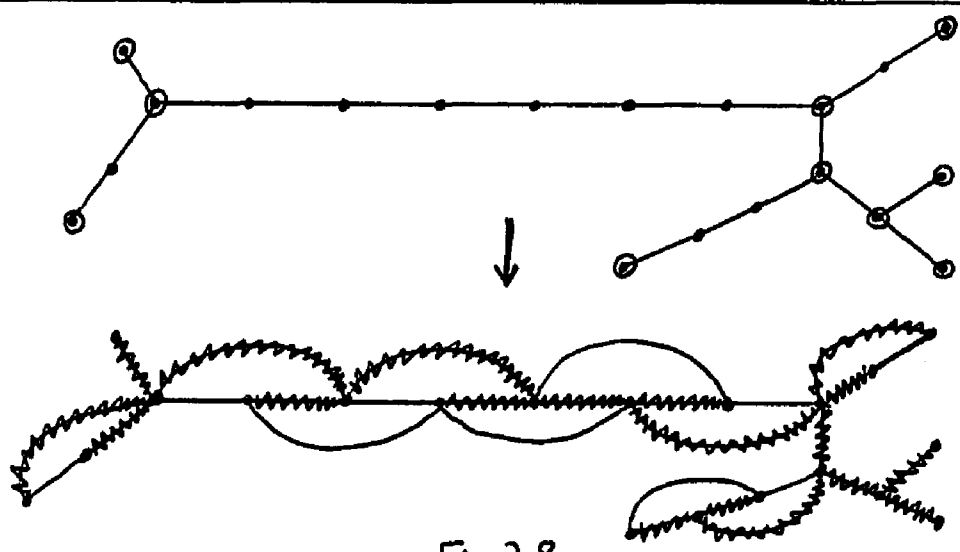


Fig. 2.8

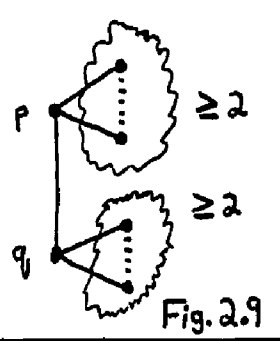


Fig. 2.9

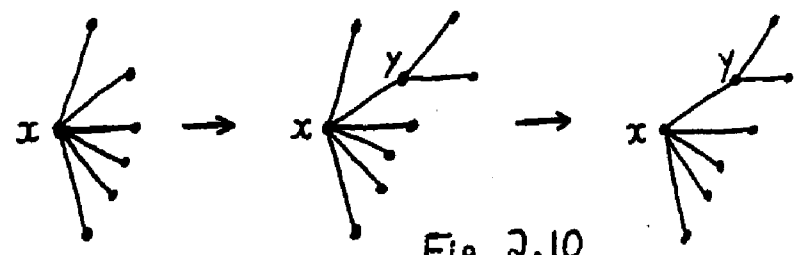
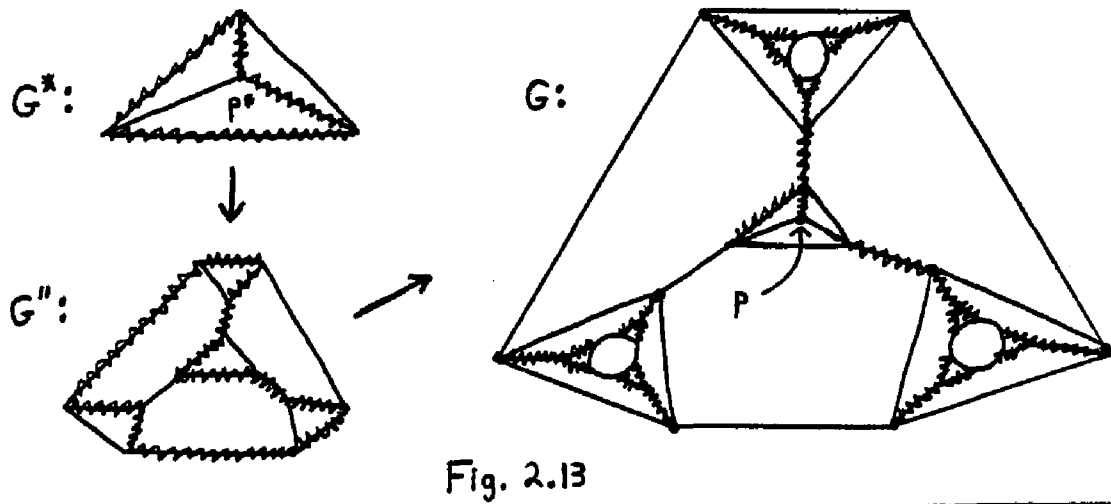
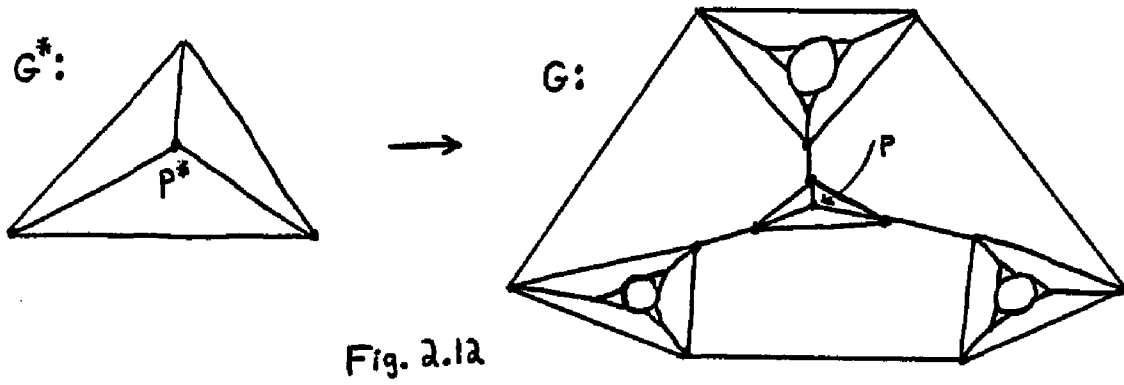
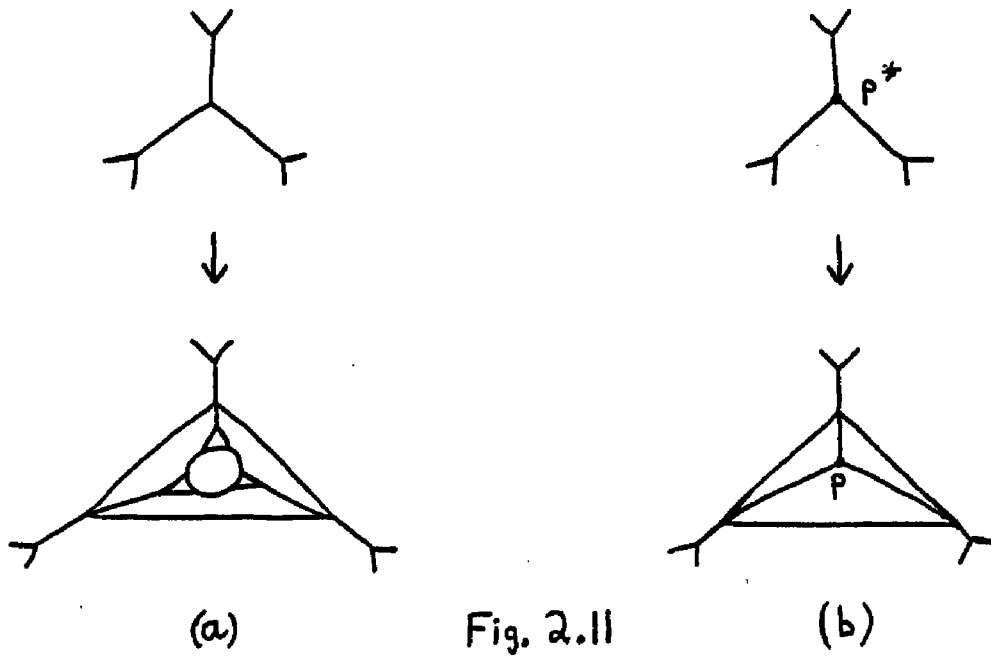
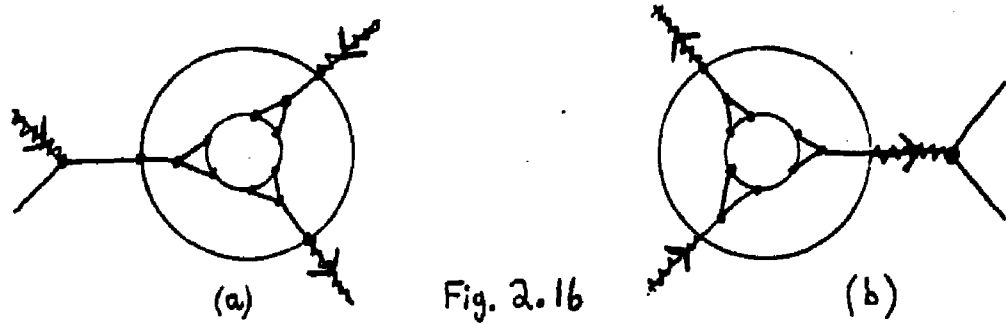
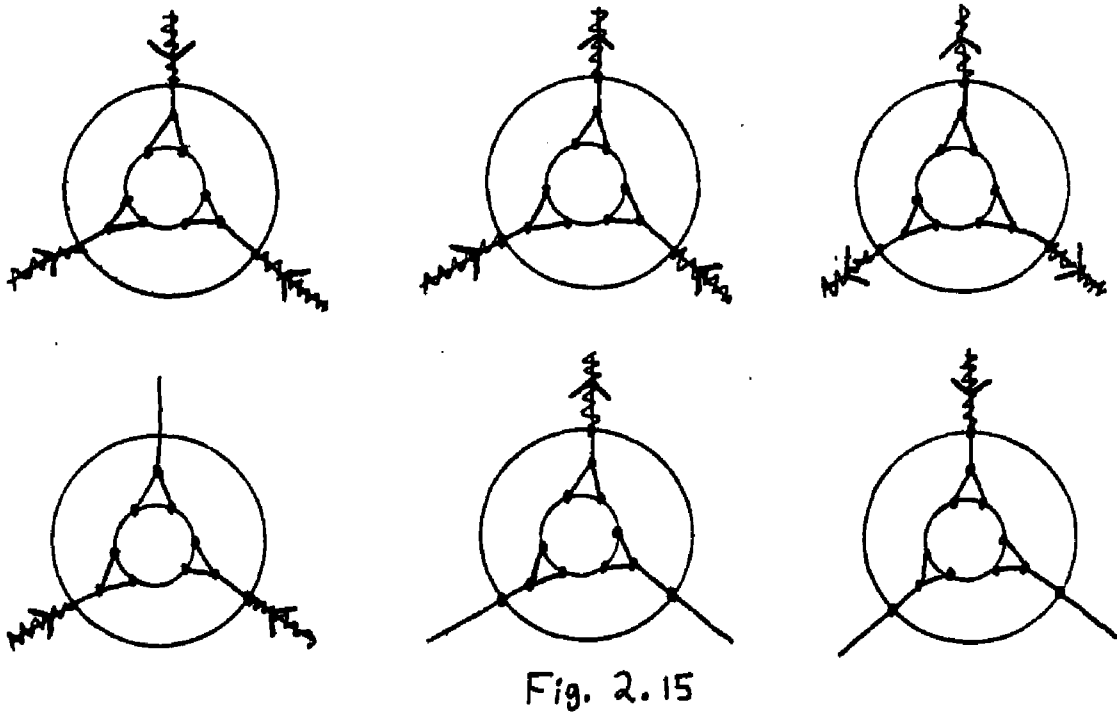
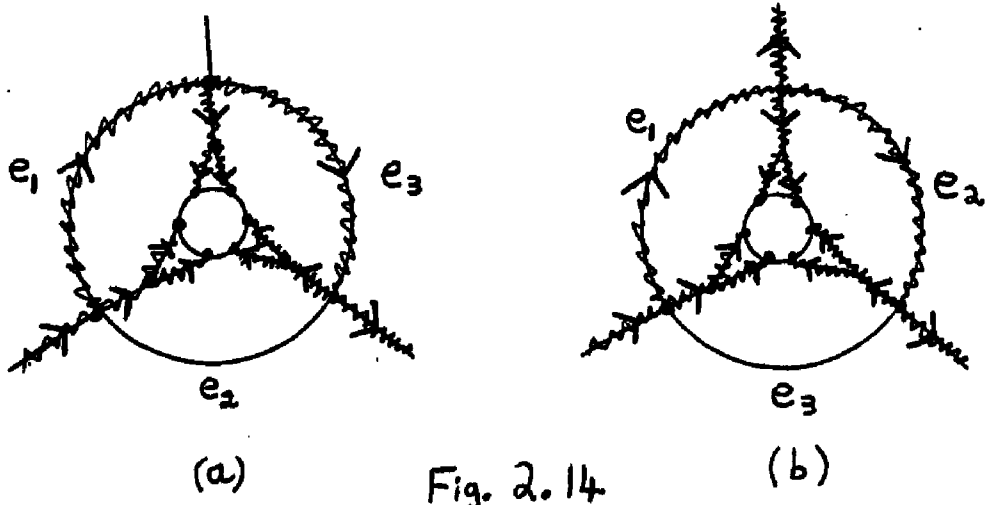


Fig. 2.10





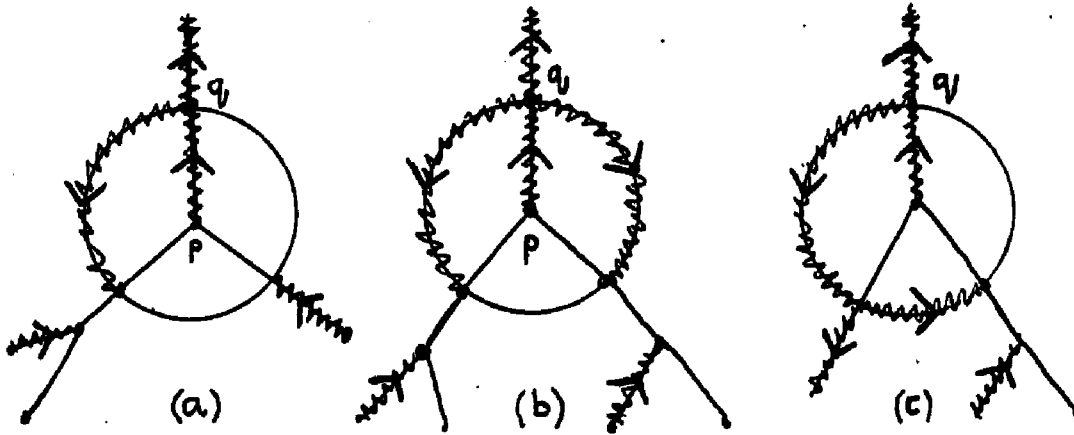


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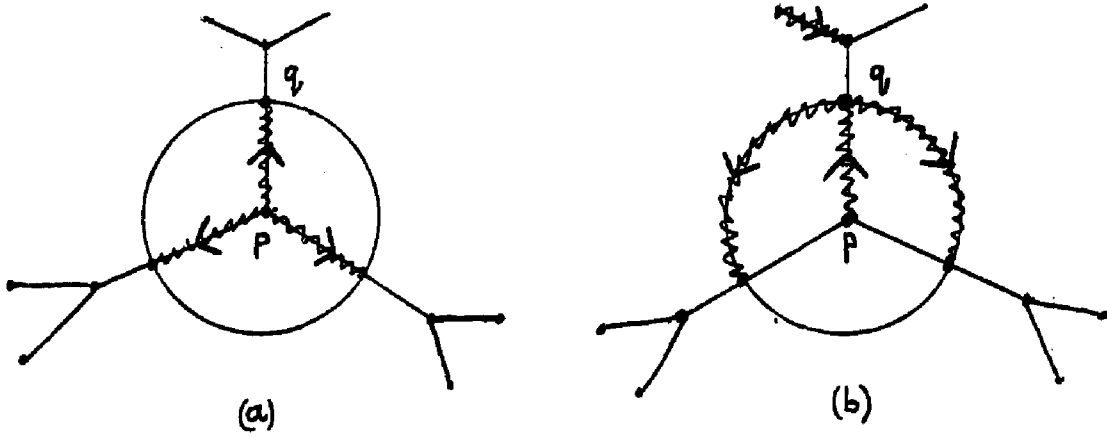


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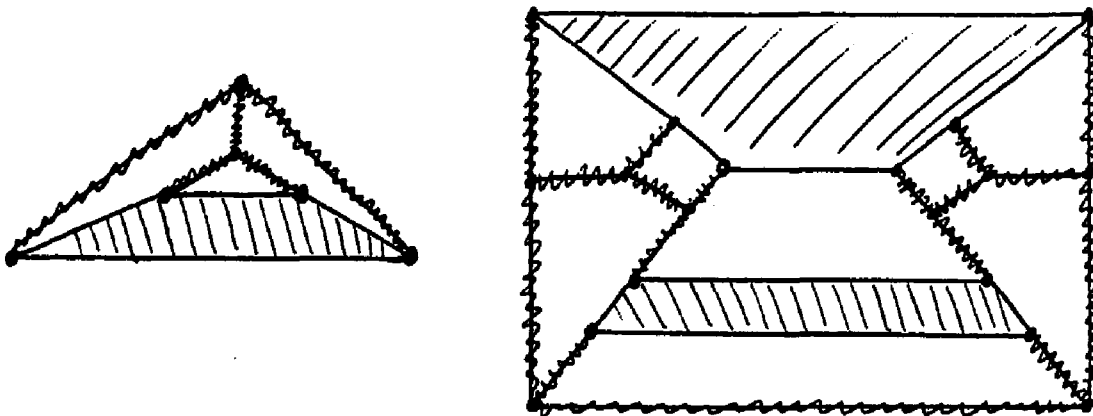


Fig. 2.19

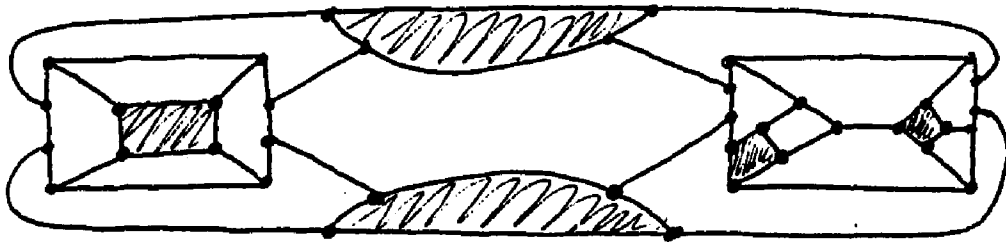


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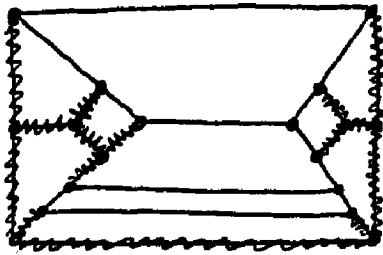


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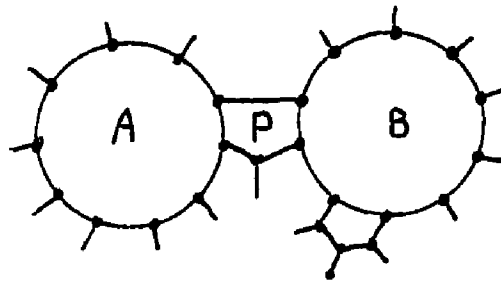


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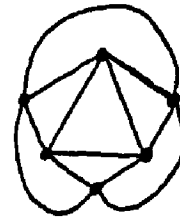
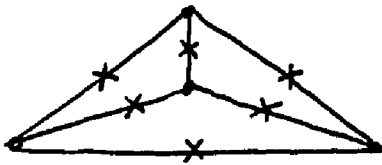


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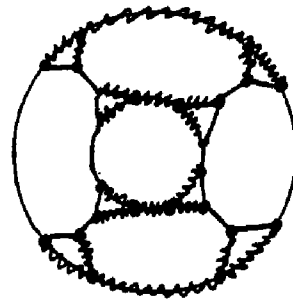
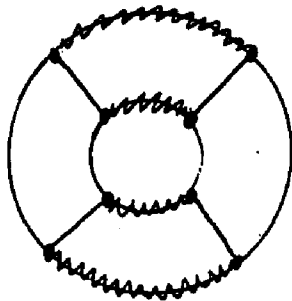
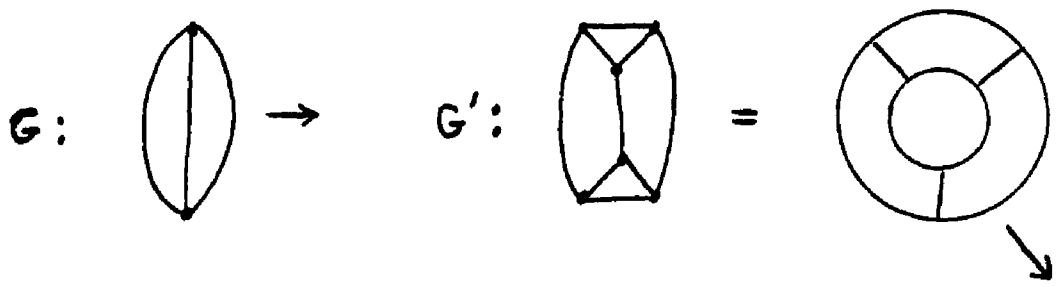
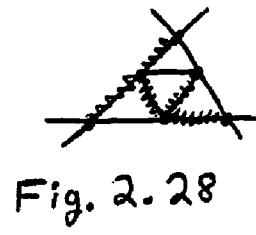
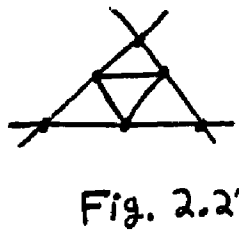
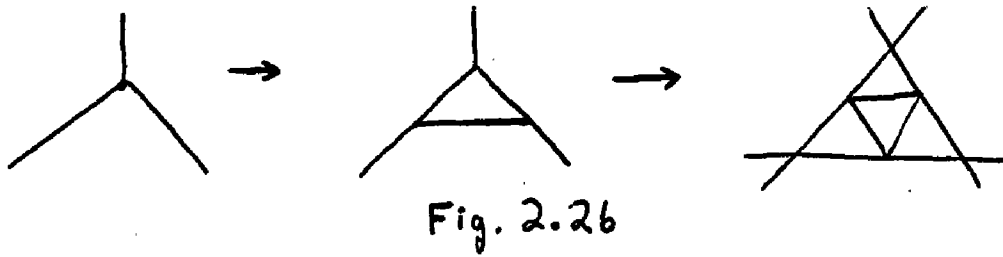
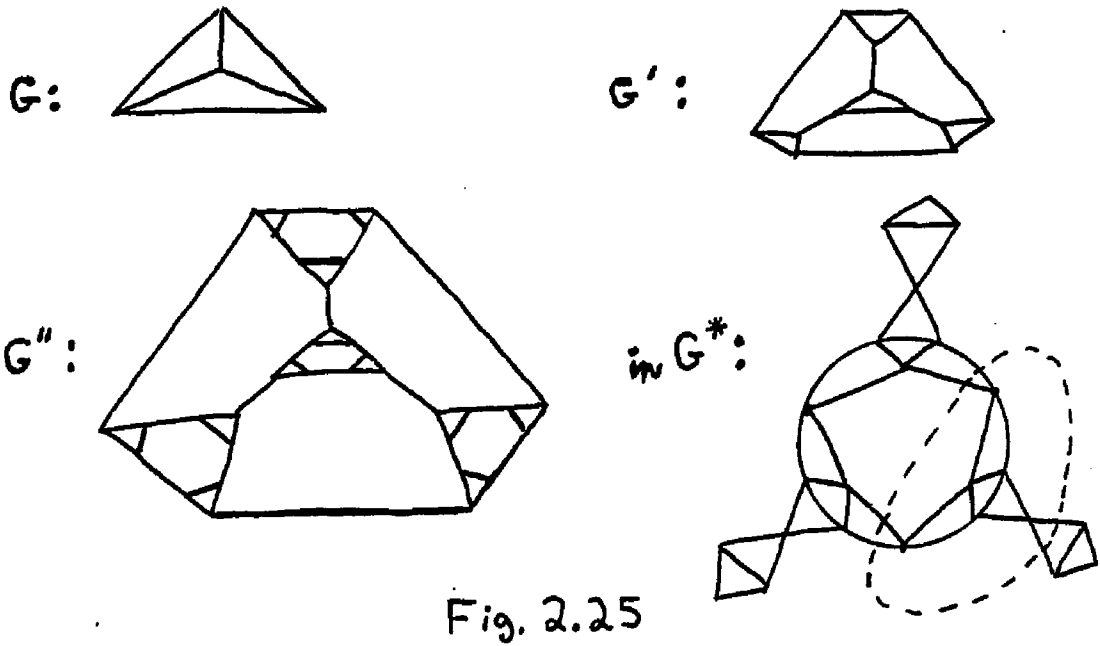


Fig. 2.23



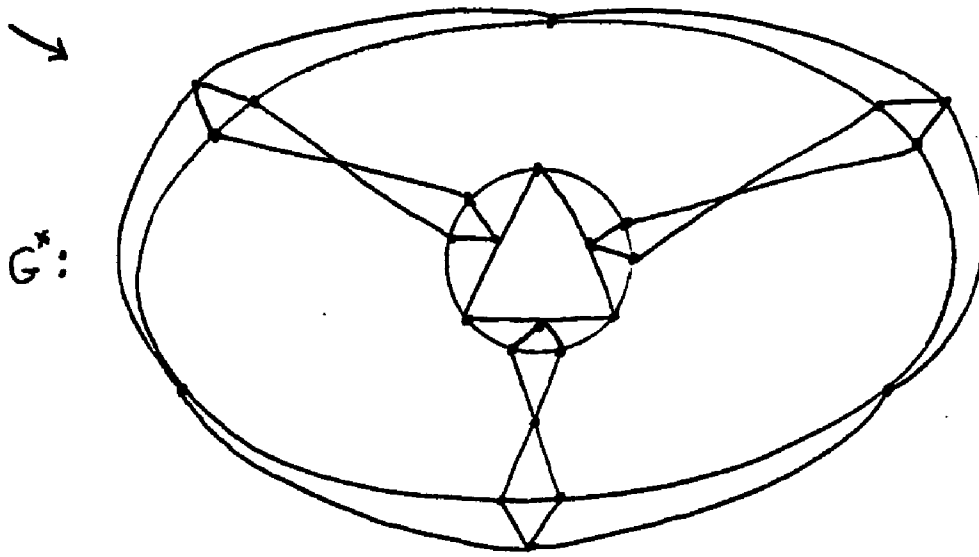


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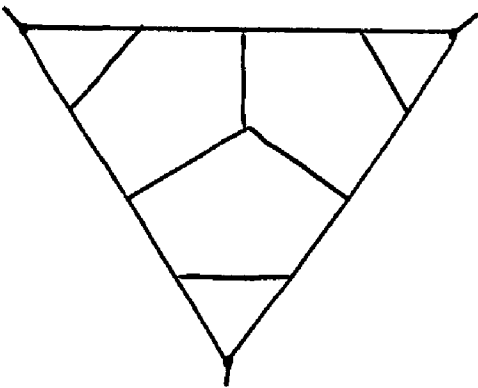


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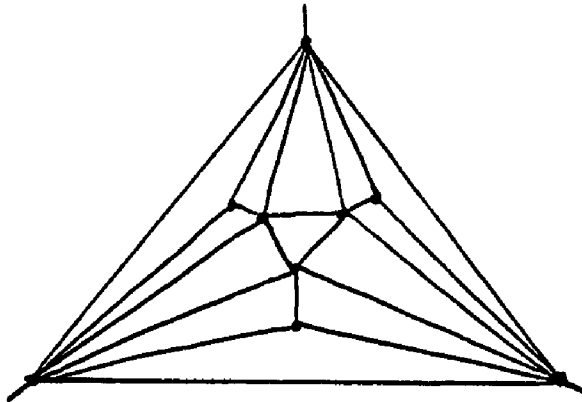


Fig. 2.31

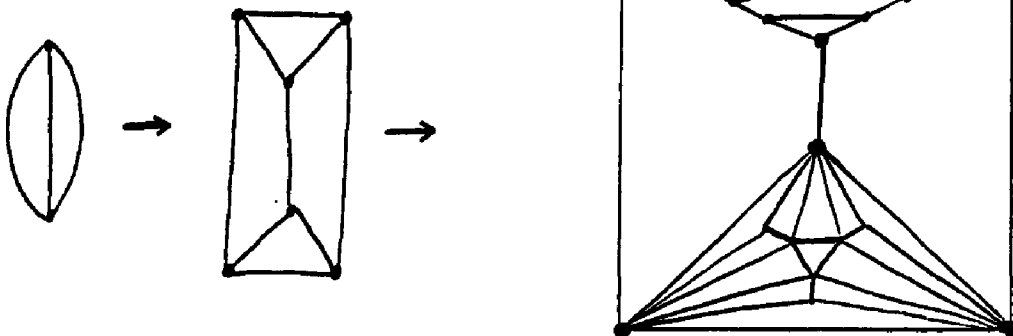


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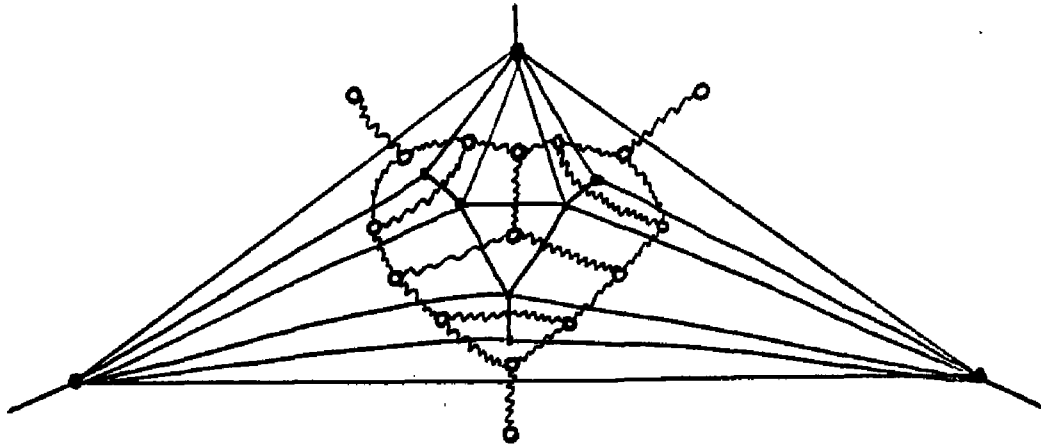


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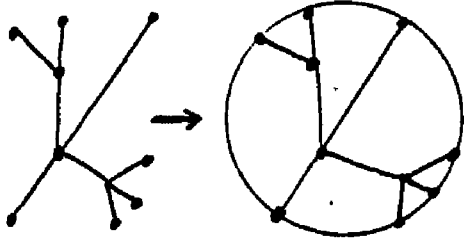


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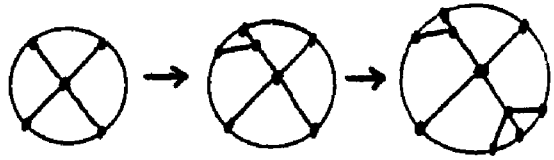


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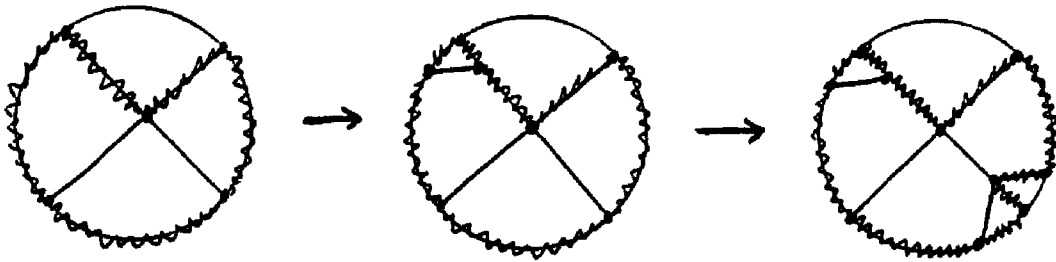


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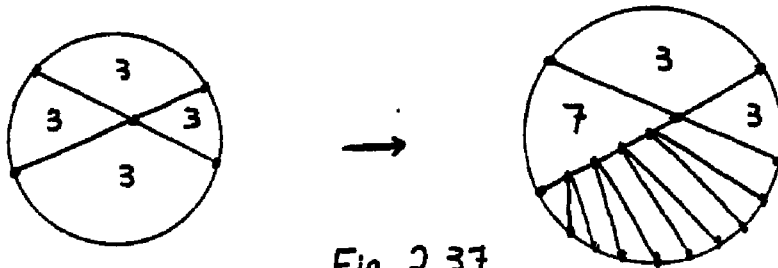


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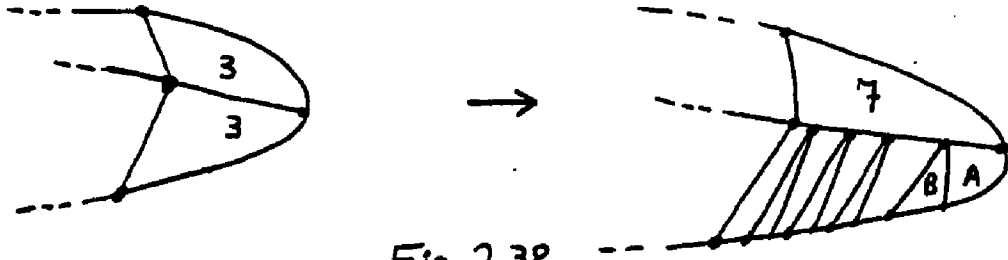


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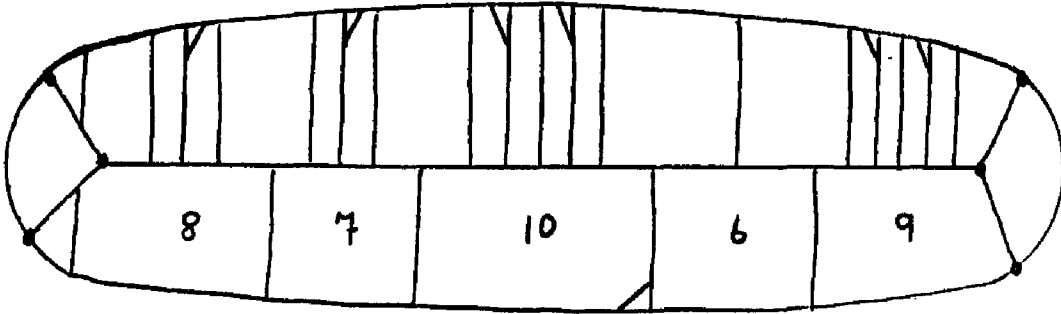


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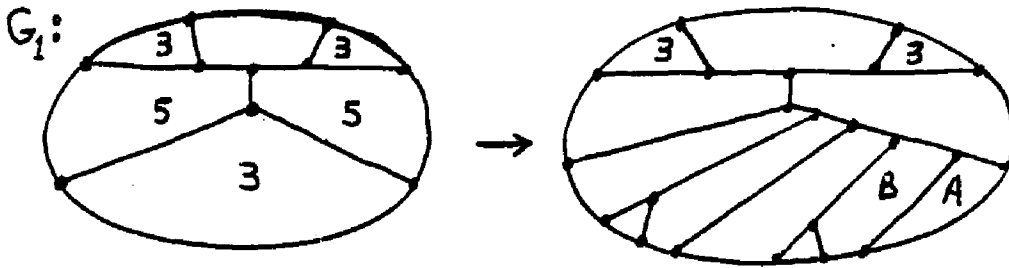


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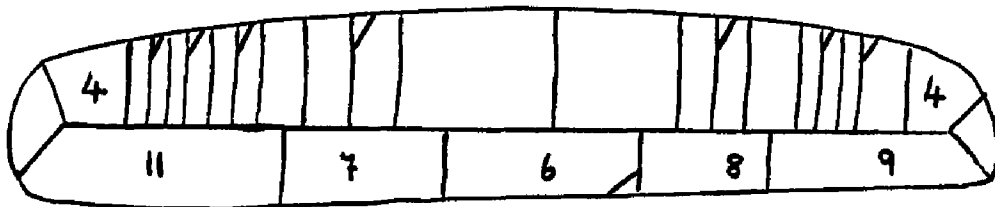
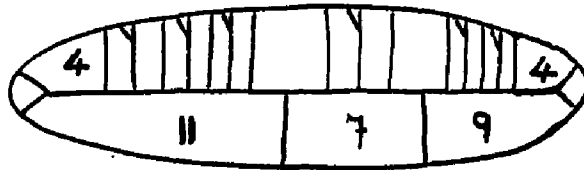


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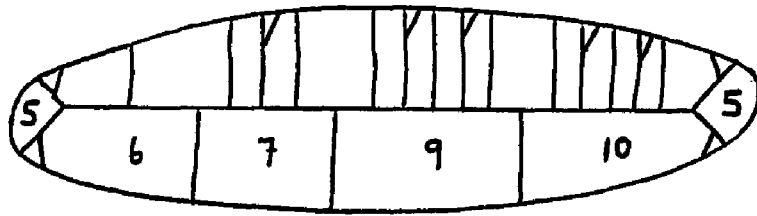


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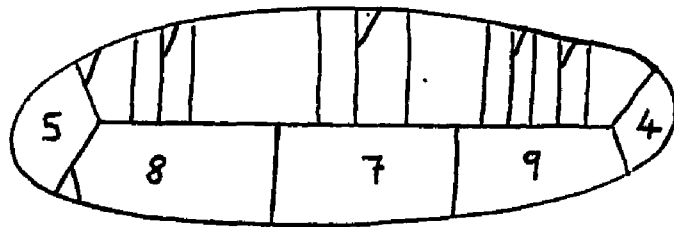


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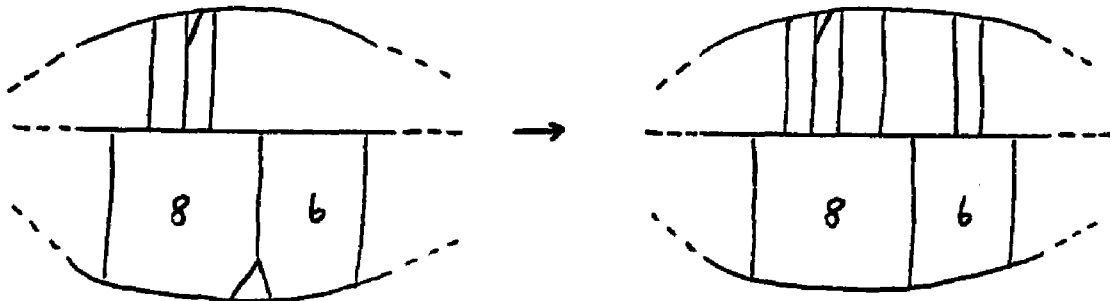
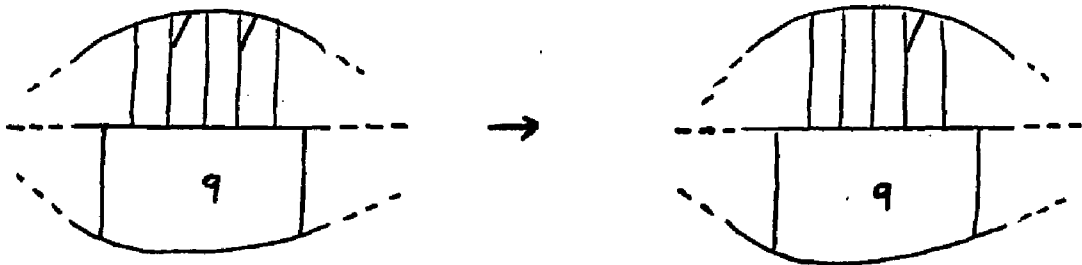


Fig. 2.44

w(i,j)-table

	j=3	=4	=5	=6	=7	=8	=9	=10	≥11
i=3	25*	25	16	10	7*	5	3*	2	0
=4	25	20	11	5	5	5	3	0	0
=5	16	11	8	2	2	2	0	0	0
=6	10	5	2	0	0	0	0	0	0
=7	7*	5	2	0	0	0	0	0	0
=8	5	5	2	0	0	0	0	0	0
=9	3*	3	0	0	0	0	0	0	0
=10	2	0	0	0	0	0	0	0	0
≥11	0	0	0	0	0	0	0	0	0

Fig. 3.1

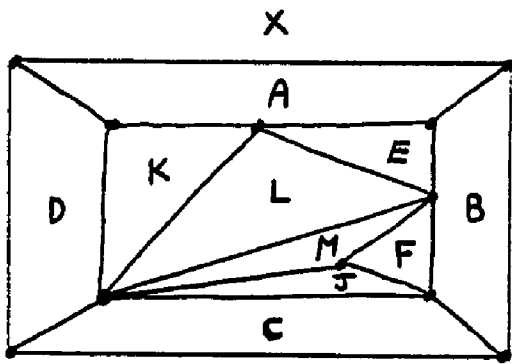


Fig. 3.2

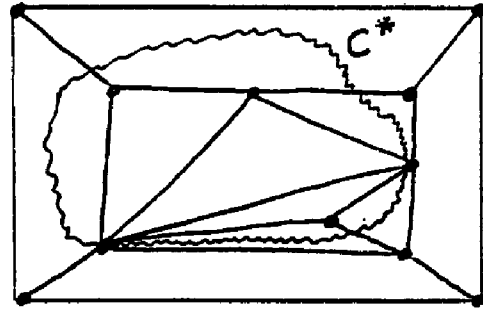


Fig. 3.3

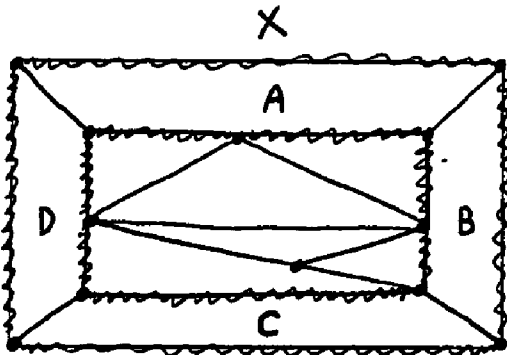


Fig. 3.4

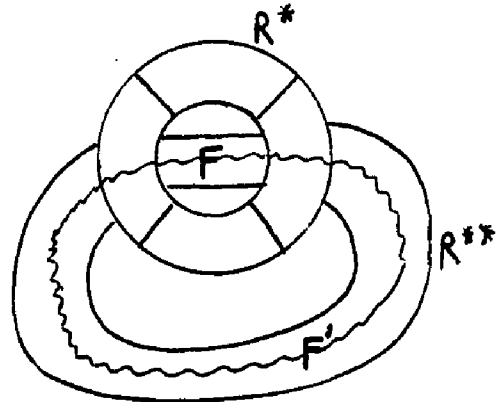


Fig. 3.5

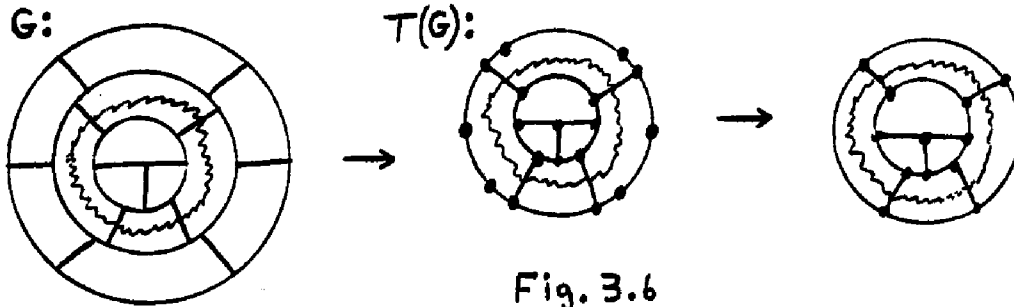


Fig. 3.6

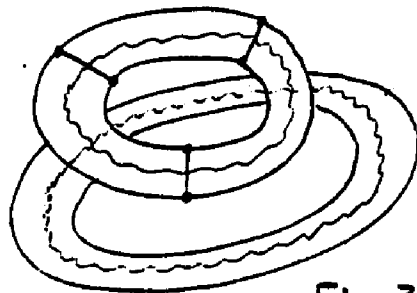


Fig. 3.7

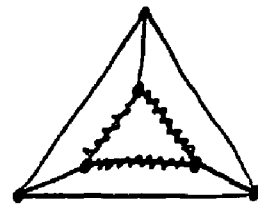


Fig. 3.8

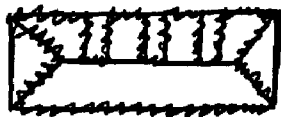
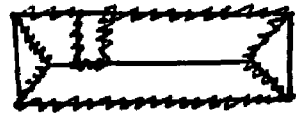
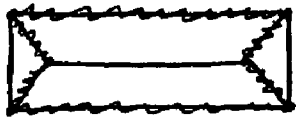


Fig. 3.9

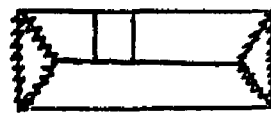
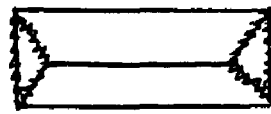


Fig. 3.10

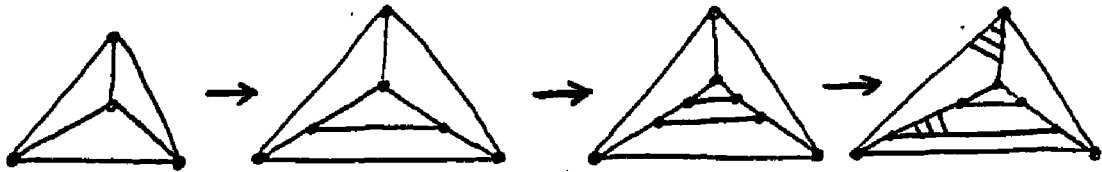


Fig. 3.11

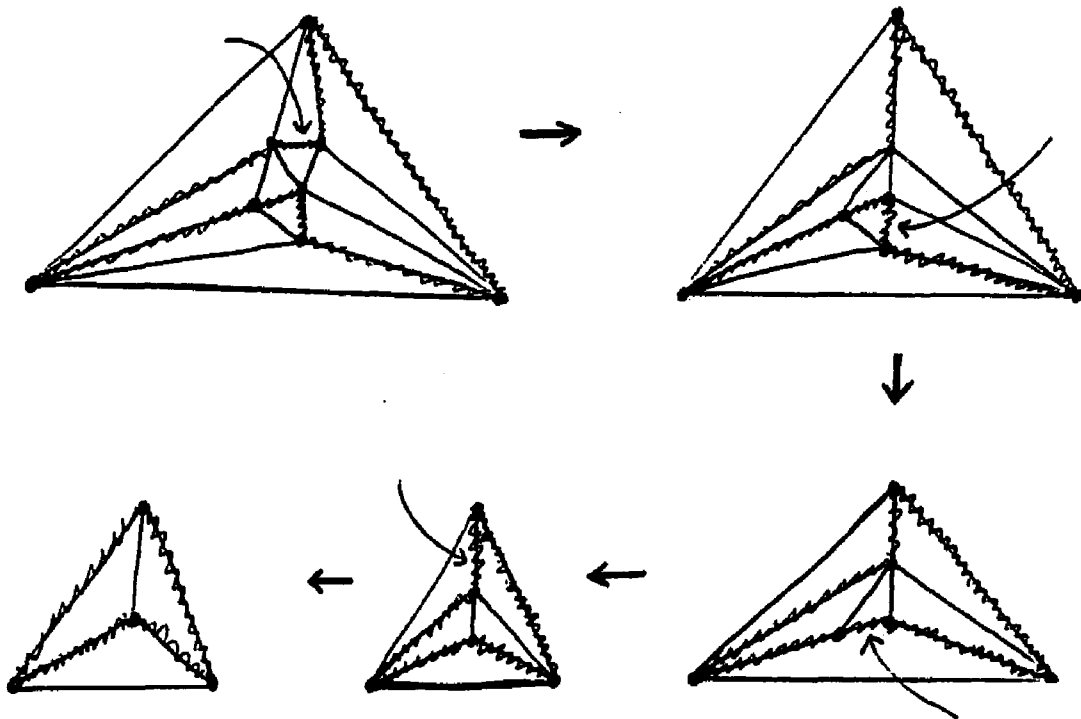


Fig. 3.12

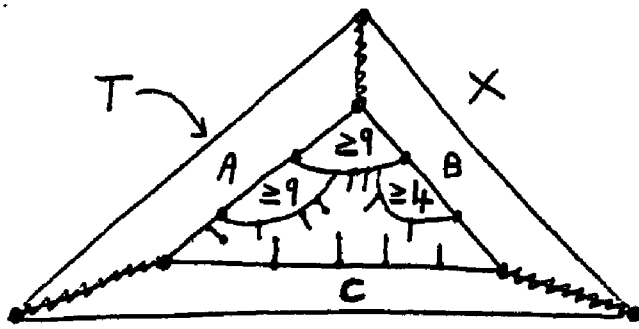


Fig. 3.13

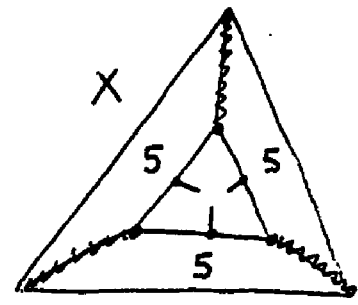


Fig. 3.14

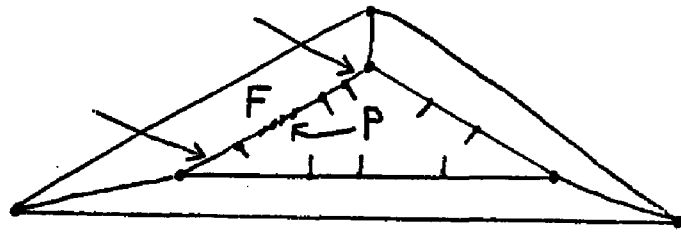
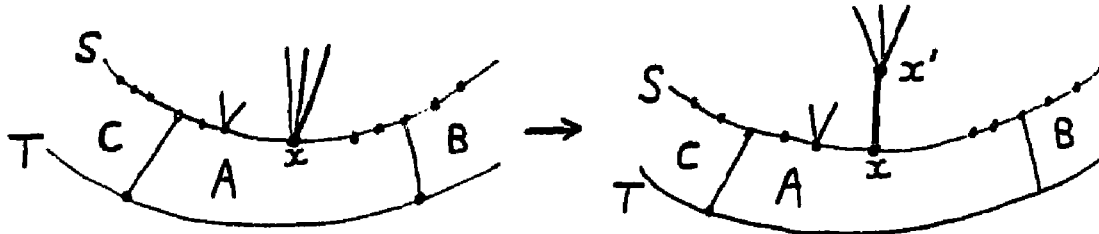


Fig. 3.15



Fig. 3.16



(a)

Fig. 3.17

(b)

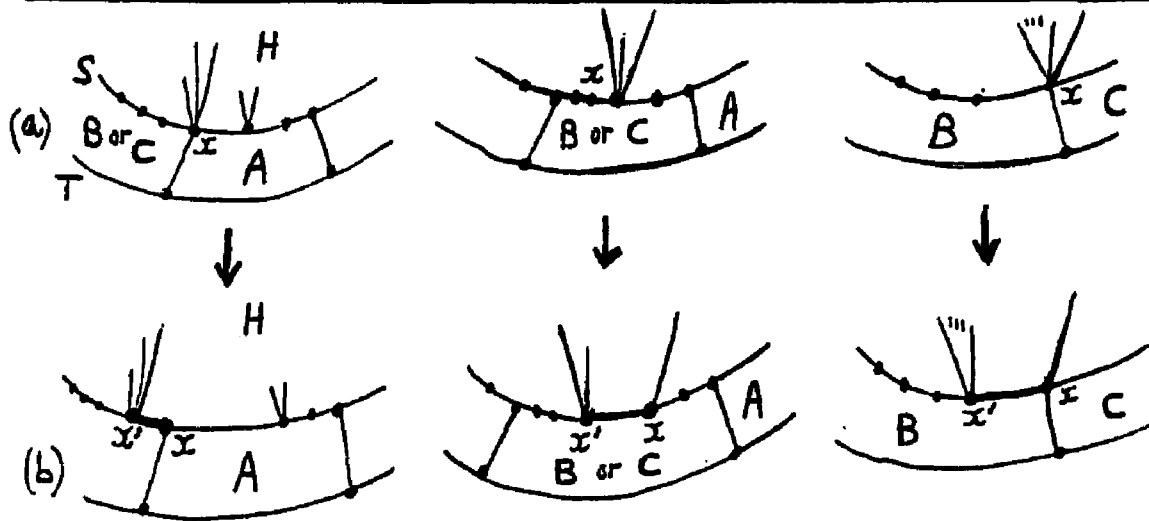


Fig. 3.18

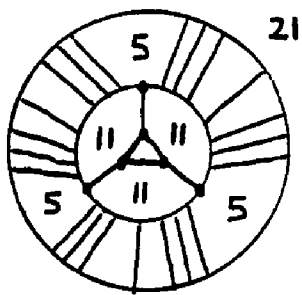
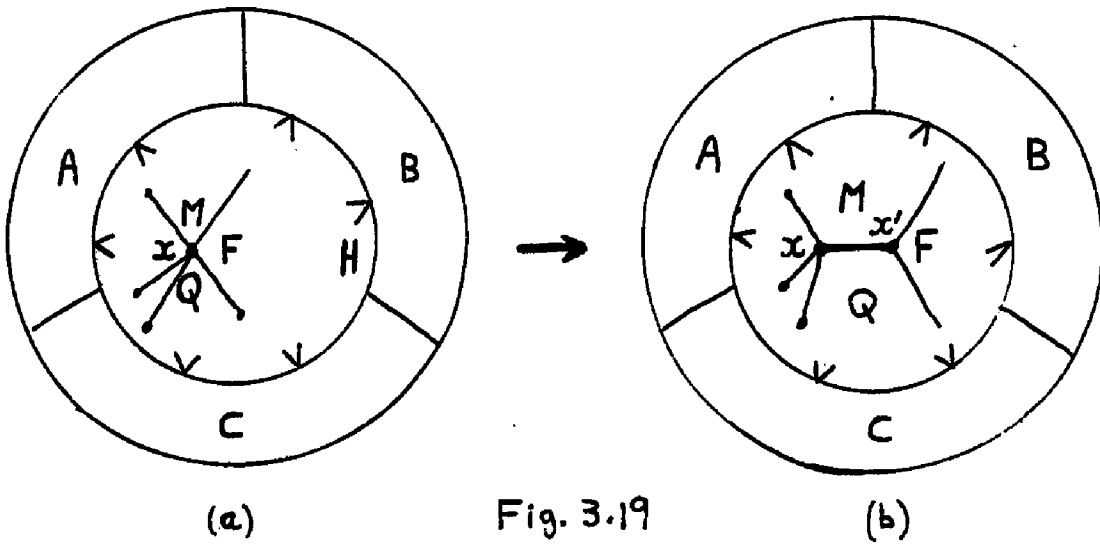


Fig. 3.20

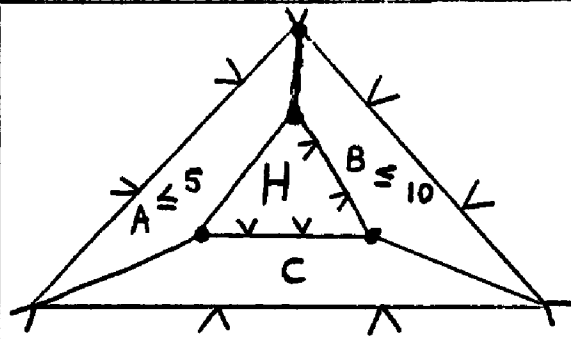


Fig. 3.21

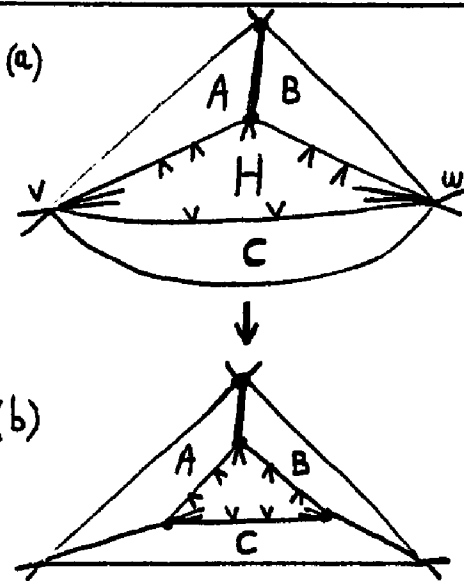


Fig. 3.22

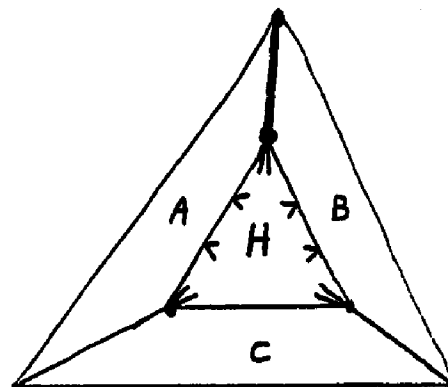


Fig. 3.23

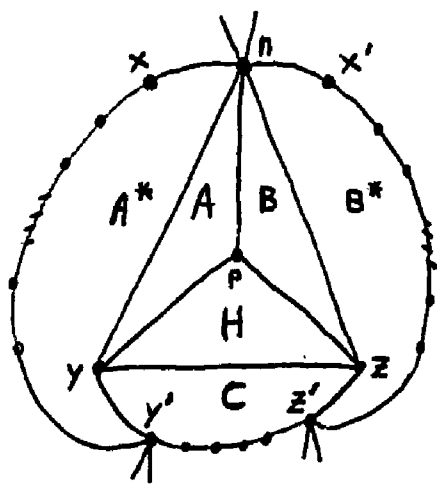


Fig. 3.24

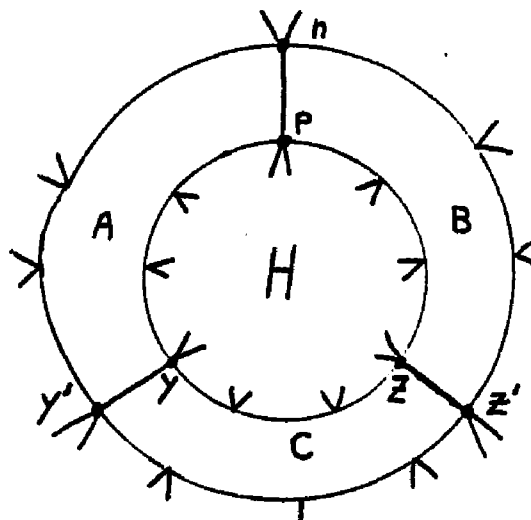


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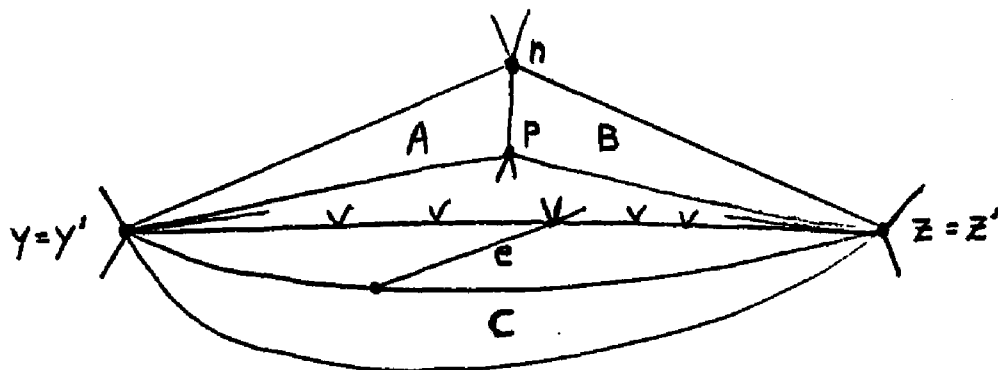


Fig. 3.26

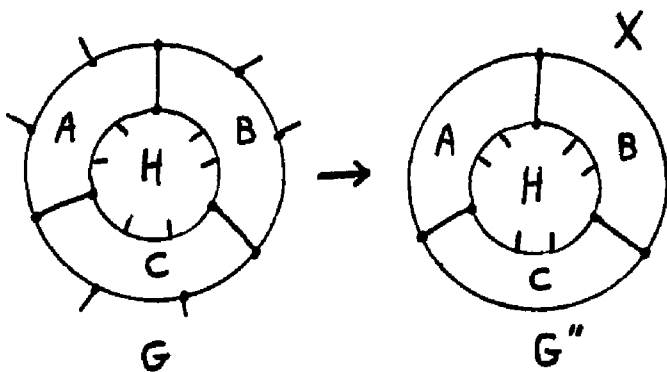


Fig. 3.27

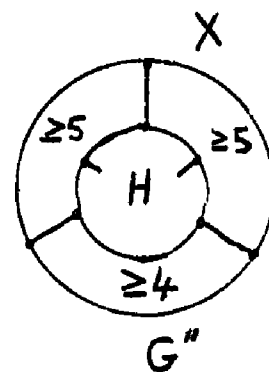


Fig. 3.28

$G^*$ :

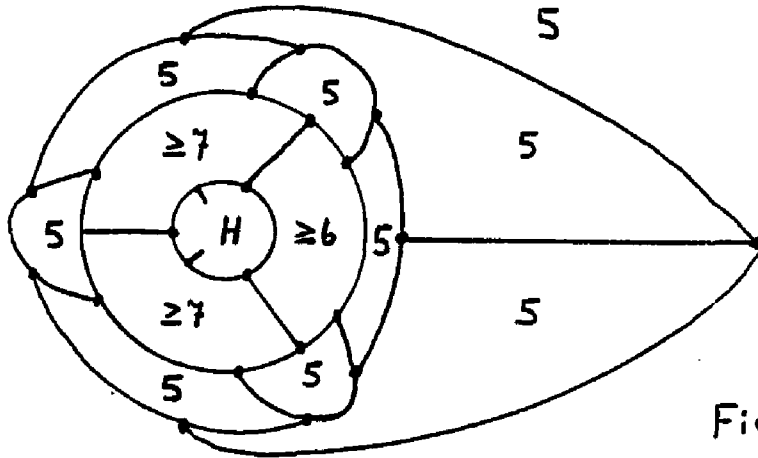
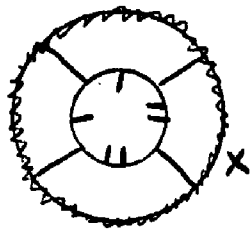


Fig. 3.29

$G$ :



$G''$ :

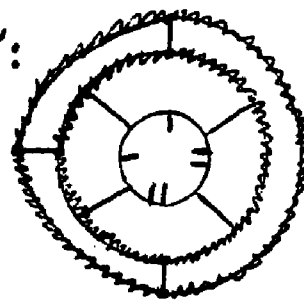


Fig. 3.30

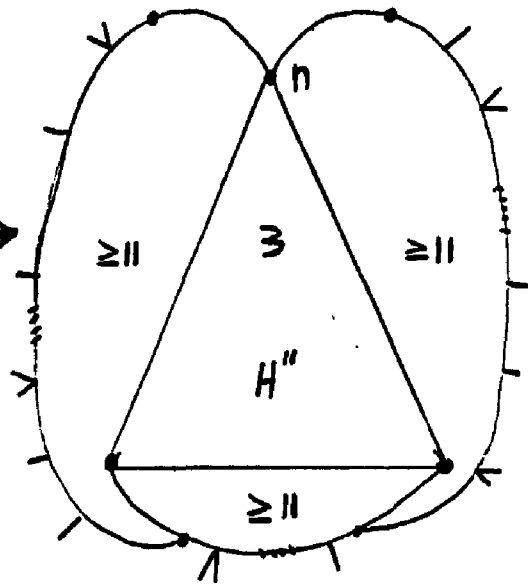
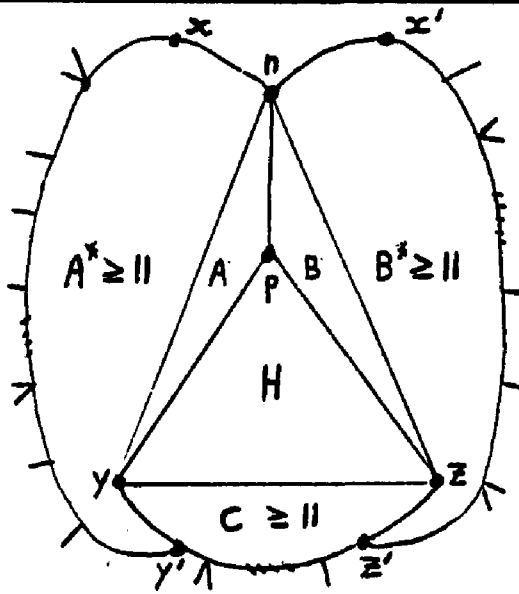


Fig. 3.31

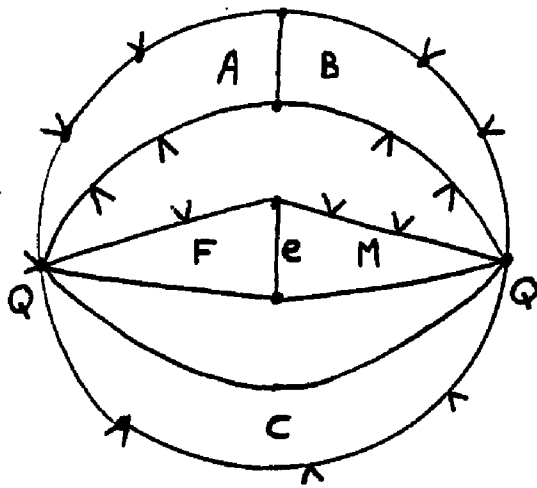


Fig. 3.32

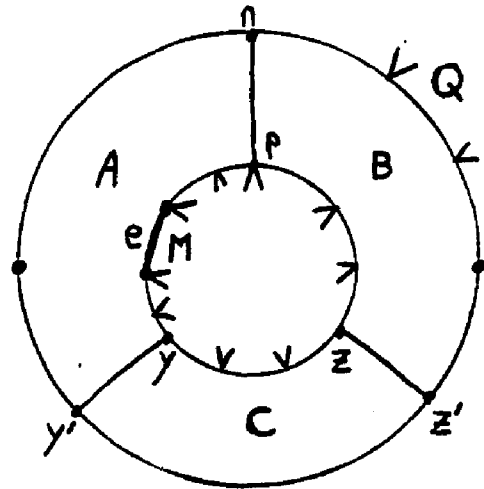
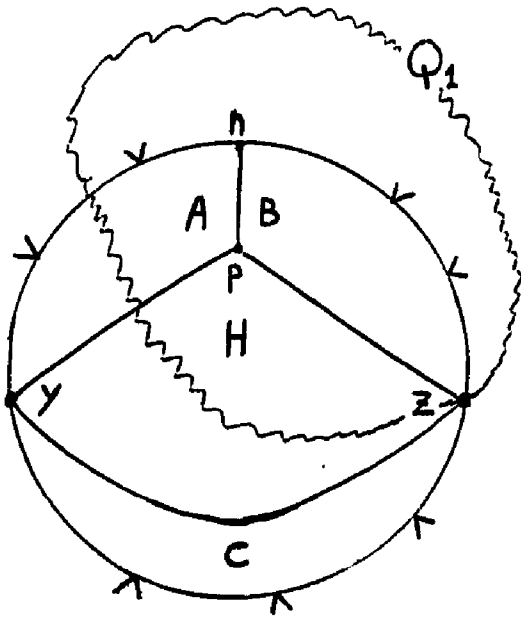
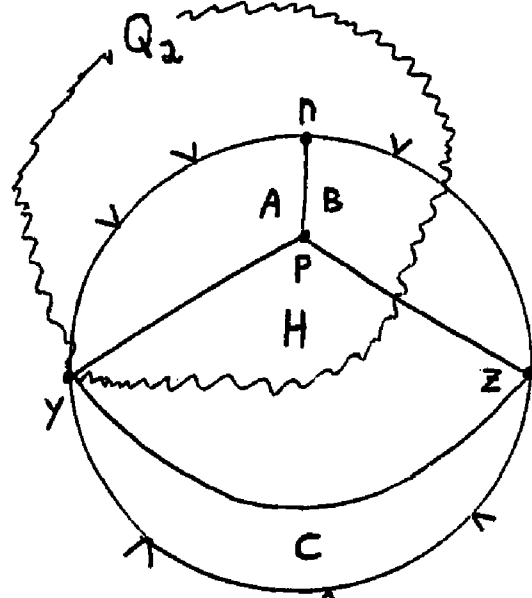


Fig. 3.33



(a)



(b)

Fig. 3.34

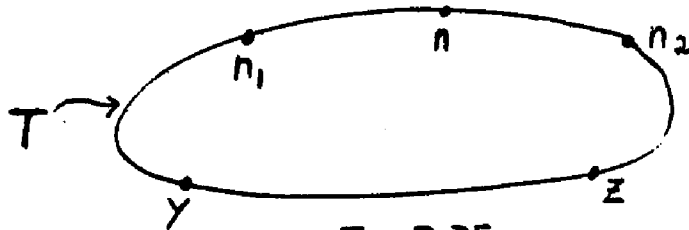


Fig. 3.35

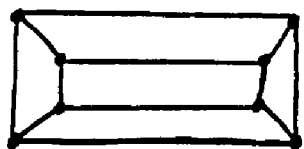


Fig. 4.1

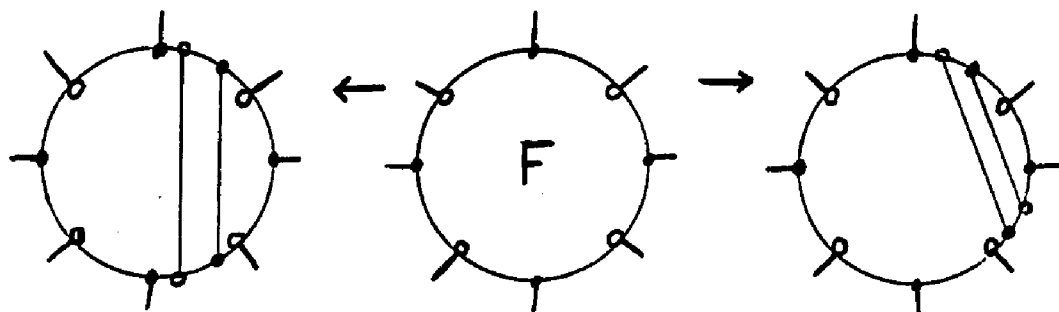
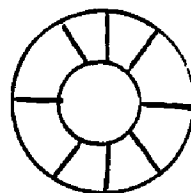


Fig. 4.2

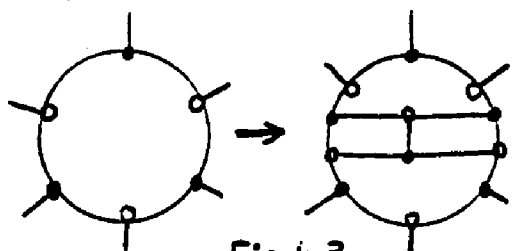


Fig. 4.3

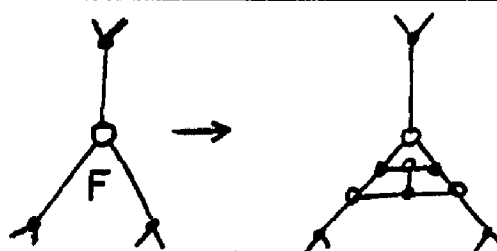


Fig. 4.4

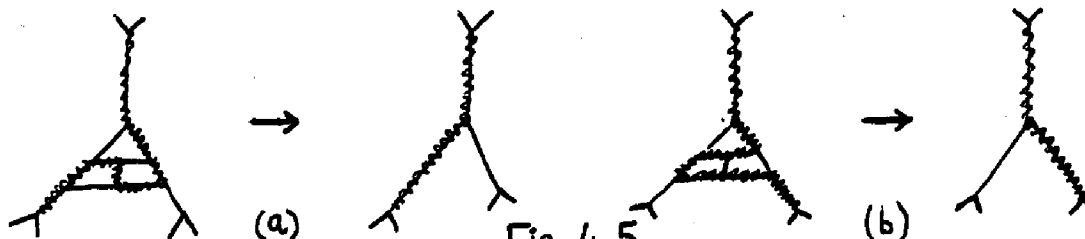


Fig. 4.5

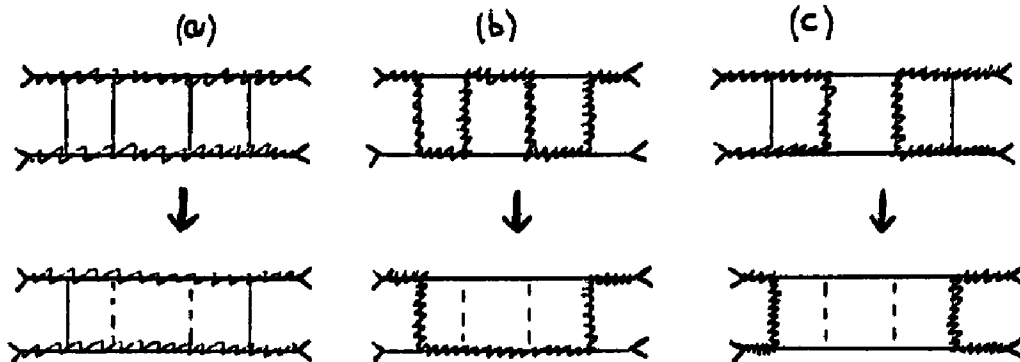


Fig. 4.6

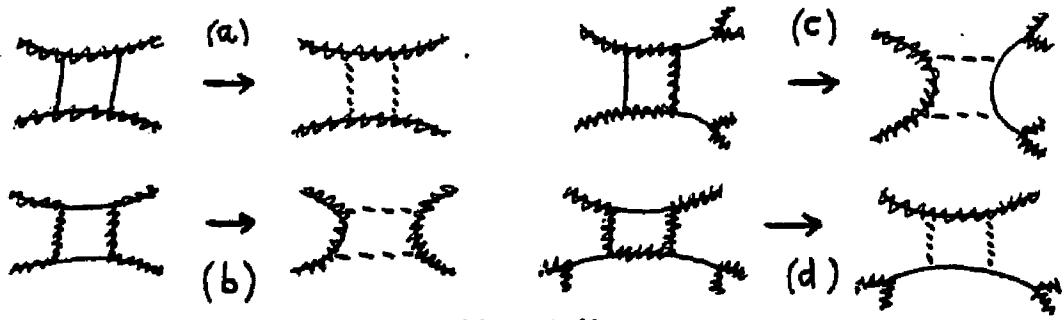


Fig. 4.7

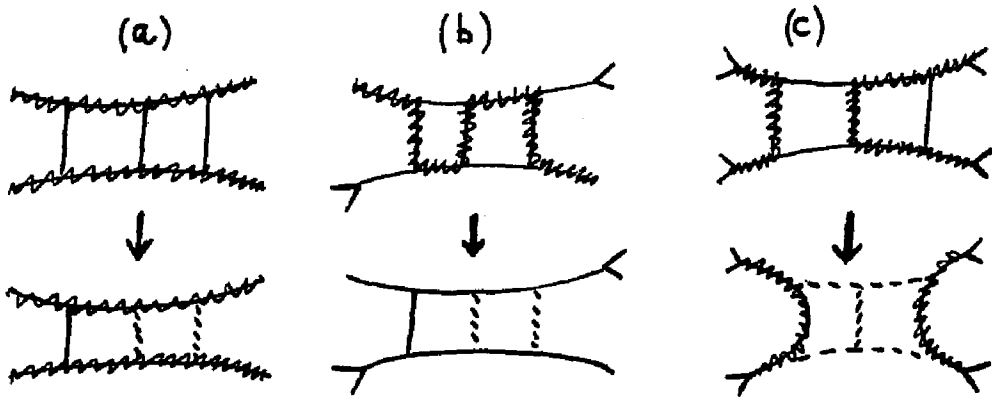


Fig. 4.8

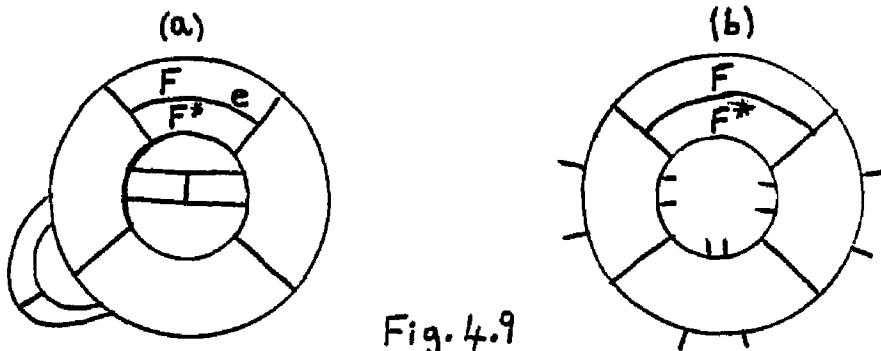


Fig. 4.9

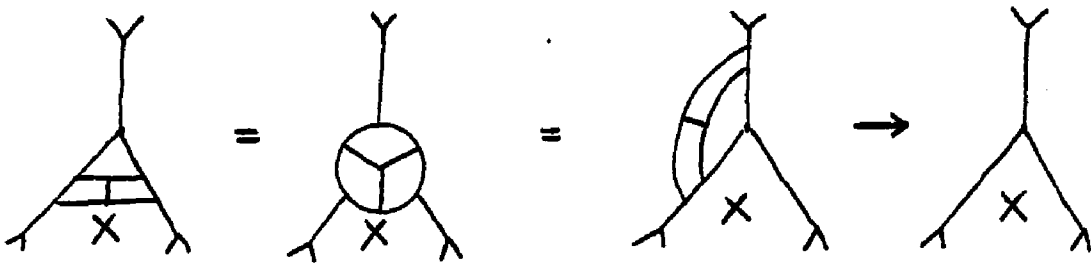


Fig. 4.10

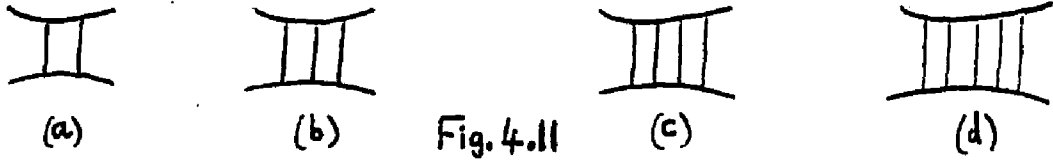
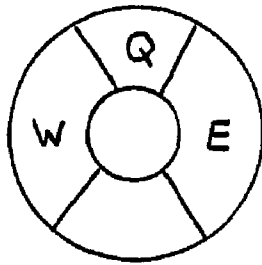
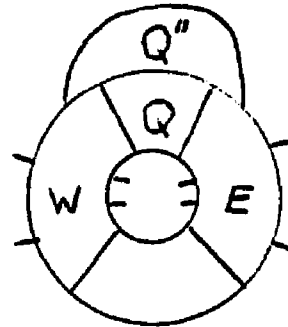


Fig. 4.11



(a)



(b)

Fig. 4.12

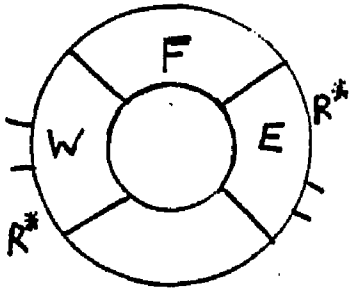


Fig. 4.13

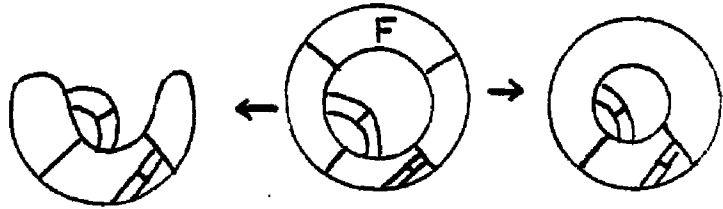


Fig. 4.14

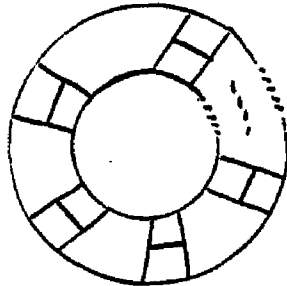


Fig. 4.15

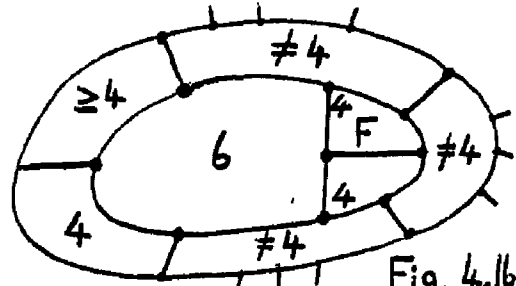
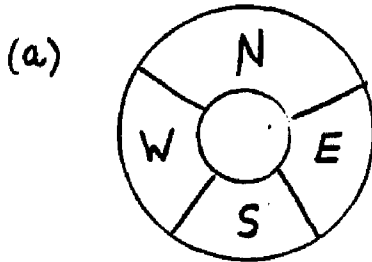
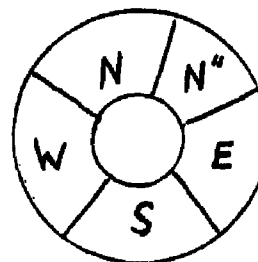


Fig. 4.16



(a)



(b)

Fig. 4.17

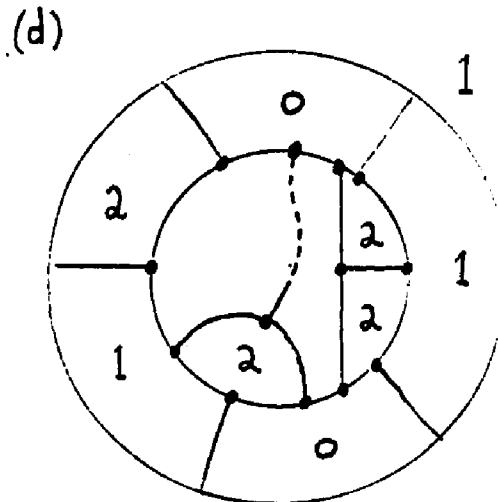
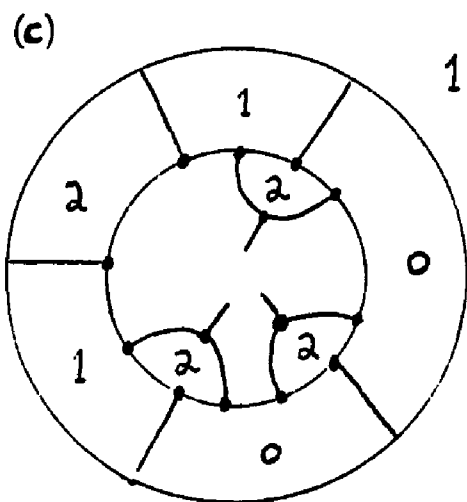
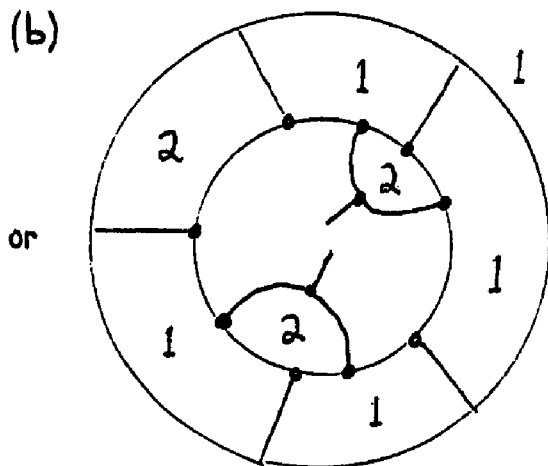
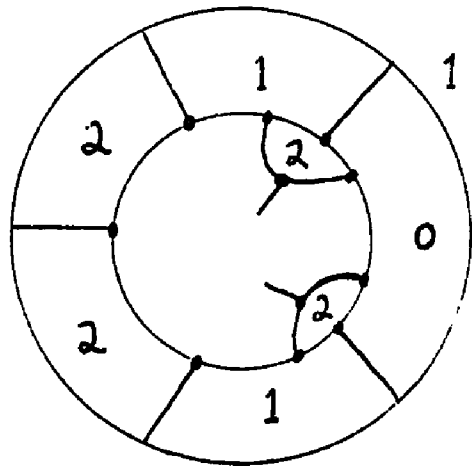
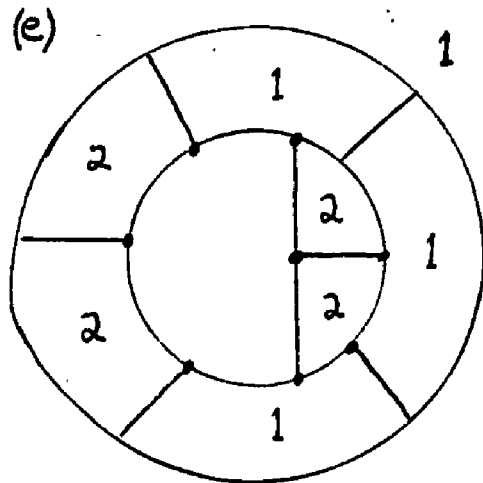
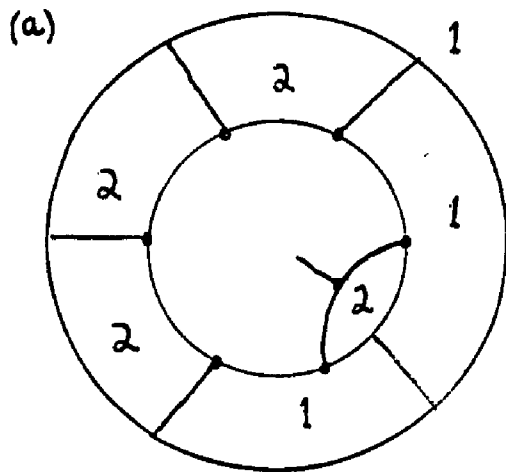


Fig. 4.18

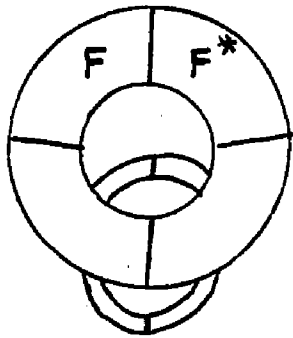


Fig. 4.19

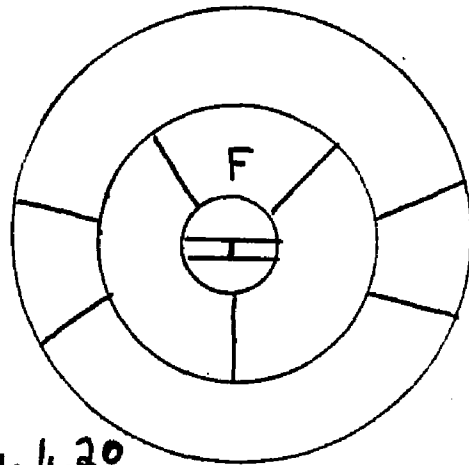


Fig. 4.20

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