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Geometry in the non-abelian resolution of the unstable Adams spectral sequence

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by

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Abstract

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Advisor: Martin Bendersky

The use of the Adams spectral sequence in the study of stable homotopy groups, is usually done in the category of spectra using iterated push outs, in contrast the unstable Adams spectral sequence, which can be used to obtain information about unstable homotopy groups was originally developed in the category of semi-simplicial sets, or using the infinite telescope associated to a spectrum, and iterated pullbacks, this method has been developed by M. Bendersky.

The theme or inspiration of this treatise is to obtain a map from pullbacks to push outs, which will allow one to view a resolution of the unstable Adams spectral sequence as the homotopy group of push outs instead of pullbacks and thus treat the unstable situation in much the same manner as the stable case. The hope is this will provide new insights into this construction.

Preface

A Principle goal of Relative Homological Algebra is to obtain Ext terms in both abelian and non-abelian categories. One typically works in a category $\mathfrak{M}$ with a functor $F:\mathfrak{M} \longrightarrow \mathfrak{Y}$, for each $M \in \mathfrak{M}$, $F(X)$ is an extended object or relative injective and there is a structure map $m:M \longrightarrow F(M)$, as well as a multiplication $\varepsilon_M:F^2M \longrightarrow FM$, which satisfy certain commutative, diagrams. Classic references are,(*A) and (*EM1). The particular case of interest is for $\mathfrak{M}$ = $\mathfrak{R}$-modules or more accurately (1)-connected free $\mathfrak{R}$-modules and $\mathfrak{Y}$ = the category of unstable Γ-coalgebras, where Γ is the analog of the dual to the Steenrod Algebra, and $\mathfrak{R}$ is the coefficient ring.

The category of Γ - coalgebras can be defined through the use of the language of triples, and cosimplicial resolutions, a thorough treatment of which can be found in (*A) and (*EM2). A brief sketch for which (*B) is a good reference follows.

Let $R(-)$ an endofunctor of Topological spaces, be the mapping telescope , $R(X) = \underrightarrow{\text{hocolim}}_{p \to \infty} \Omega^p(R_p \wedge X)$ where $\underline{R} = \{R_n, \varepsilon_{Rn}\}$ is a Ω ring spectrum, and $\eta_{(-)}:I(-) \longrightarrow R(-)$ is the Hurewicz map. Observing that the product map of a ring spectra $\underline{R}$ induces a multiplication $\mu:R^2(X) \longrightarrow R(X)$ s.t. μ and η satisfy certain commutative diagram, i.e. unitary and associative and thus define a triple $\{R(-),\eta,\mu\}$.

Following (*A) one can form a cosimplicial space, that is a diagram of spaces $\mathbf{R}(X)$;

$$\mathbf{R}(X)\ :\ R(X) \underset{d^1}{\overset{d^0}{\rightrightarrows}} R^2(X) \rightrightarrows R^3(X)\ \ldots$$

with $d^i = R^i\eta R^{n-i}:R^nX \longrightarrow R^{n+1}X$, then $ch(\pi_*\mathbf{R}(X))$ is a cochain complex

$$0 \longrightarrow \pi_*R(X) \xrightarrow{\delta_0} \pi_*R^2X \xrightarrow{\delta_1} \ldots$$ where δ is given as the sum of the alternating values of the d^i .

The category of unstable Γ - coalgebras, labelled $\mathfrak{M}(G)$ is formed, where $\pi_*(\eta_X)$ induces the structure map and $\pi_*(\mu_X)$ the multiplication map, here the functor $G:\mathfrak{M} \longrightarrow \mathfrak{M}(G)$ which is defined in (B*) as well as in chapter #2, is the R-homology of the R-analog of an Eilenberg-MacLane space. There is then an adjunction; $Hom_{\mathfrak{M}}(\pi_*R(S^0),\pi_*R(X)) \cong Hom_{\mathfrak{M}(G)}(\pi_*R(S^0),\pi_*R^2(X))$. Using this adjunction and the cosimplicial resolution one defines $Ext_{\mathfrak{M}(G)}(\mathfrak{K},\pi_*R(X)) \cong H^*(ch\pi_*\mathbf{R}(X))$.

The theory of double complexes is then used to establish that the E_2 term of the unstable Adams Spectral Sequence is isomorphic to $H^*(\mathrm{ch}\pi_* R(X))$.

However the theory of double complexes does not provide a realization of this isomorphism. This is in contrast to the methods used in (*EM1), carried out in the stable case in (*Sw, pp.#467) and in (*A1), where the structure map defines an isomorphism between the E_2 terms and the cohomology of an appropriate resolution of comodules over a stable coalgebra.

This method is not immediately applicable for the unstable Adams spectral sequence because the E_1 terms are defined as the homotopy groups of iterated pull backs $E_1^{s,t} = \pi_{t-s}\Gamma^s_{\eta X}$ where $\Gamma^s_{\eta X}$ is defined as the pullback over an internal Hurewicz map, $\Gamma^{s-1}(\eta_X, \eta_{RX}):\Gamma^{s-1}_{\eta X} \longrightarrow \Gamma^{s-1}_{R\eta X}$, which is not formally the same as the external Hurewicz map, that is the structure map of $\mathfrak{M}(G)$.

The definition of both the internal and external Hurewicz map as well as that of the internal iterated pullback are given in appendix A.

also defined there are the internal iterated pushout $\mathfrak{P}_f^{s+1}$, the pushout under the internal Hurewicz map $\mathfrak{P}^s(\eta_X,\eta_{RX}):\mathfrak{P}_f^s \longrightarrow \mathfrak{P}_{Rf}^s$ and the external iterated pushout $\mathfrak{L}_f^{s+1}$, which is the pushout under the Hurewicz map $\eta_{\mathfrak{L}_f^s}:\mathfrak{L}_f^s \longrightarrow R\mathfrak{L}_f^s$, as noted earlier $\pi_*\eta_{(-)}$ is the structure map in the category $\mathfrak{M}(G)$, For this reason the homotopy groups of the external iterated push outs $\pi_*\mathfrak{L}_f^s$ provide a resolution in the category $\mathfrak{M}(G)$ for $\pi_*R(X)$. The aim of this treatise is to define morphisms, $\varphi_f^s : \Gamma_f^s \longrightarrow \Omega^s\mathfrak{P}_f^s$,for $s \geq 1$, (proposition #3.8.b) and $\chi_f^{s,n}:\mathfrak{P}_{R^nf}^s \longrightarrow R^n\mathfrak{L}_f^s$ with $n \geq 0$, $s \geq 1$, (Theorem #3.20) , such that $\varphi_{Rf}^s:\Gamma_{Rf}^s \longrightarrow \Omega^sR\mathfrak{P}_f^s$ (Corollary #3.11) is a weak equivalence under certain restrictions, likewise $\chi_f^{s,n}$ is shown to be a weak equivalence under certain restrictive conditions (proposition #3.24). Inherit in the manner in which φ_f^s is constructed that is apriori one has;

$$\Omega^s\mathfrak{P}(\eta_X,\eta_Y)\circ\varphi_f^s = \varphi_{Rf}^s\circ\Gamma(\eta_X,\eta_Y)$$

by equation #3.21 one has

$\chi_{f}^{\underline{s},1} \circ \mathfrak{L}^{s}(\eta_X, \eta_Y) = \eta_{\mathfrak{L}_{f}^{\underline{s}}} \circ \chi_{f}^{\underline{s},0}$ and hence the pair of morphisms $\{\Omega^{s}\chi_{f}^{\underline{s},0} \circ \varphi_{f}^{\underline{s}}, \Omega^{s}\chi_{f}^{\underline{s},1} \circ \varphi_{Rf}^{\underline{s}}\}$ commute with the internal Hurewicz map, that is the boundary maps of the unstable E_1 term and the external Hurewicz maps, which are the boundary maps of the resolution in $\mathfrak{M}(G)$ of $\pi_* R(X)$ by the external iterated push outs. Since the map $\Omega^{s}\chi_{f}^{\underline{s},1} \circ \varphi_{Rf}^{\underline{s}}$ is an equivalence this map is the desired realization of the isomorphism $E_2 \cong \mathrm{Ext}_{\mathfrak{M}(G)}(\mathfrak{K}, \pi_* R(X))$.

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Notation

<u>Category</u> For a category $\mathfrak{X}$ denote by Obj($\mathfrak{X}$) the objects of $\mathfrak{X}$, and $\text{Hom}_{\mathfrak{X}}(X,Y)$ the set of morphisms with domain X and range or image Y, $f \in \text{Hom}_{\mathfrak{X}}(X,Y)$ will be written as $f:X \longrightarrow Y$, or $X \xrightarrow{f} Y$. To say that $f \in \text{Morph}(\mathfrak{X})$ will mean there exists X,Y in $\mathfrak{X}$, s.t. $f \in \text{Hom}_{\mathfrak{X}}(X,Y)$.

<u>Functors</u> will be denoted by uppercase letters, while <u>natural transformations</u> abbreviated n.t. will be denoted by lower case Greek letters.

<u>Let</u> $\mathfrak{T}$op be the category whose objects are compactly generated Hausdorff spaces and whose morphisms are continuous functions.

<u>Let</u> $\mathfrak{T}$op* be the based sub category of $\mathfrak{T}$op.

<u>Let</u> $\mathfrak{CW}$* be the sub category of $\mathfrak{T}$op* consisting of pointed C.W. complexes and cellular functions.

<u>Let</u> $\mathfrak{C}_T = \mathfrak{C}_{\mathfrak{T}op*}$, be the category of morphisms in $\mathfrak{T}$op*, that is the category with objects maps in $\mathfrak{T}$op* and morphisms pairs of maps $(h,k):f \longrightarrow f'$ in $\mathfrak{T}$op* s.t. the following diagram commutes;

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ f\downarrow & & \downarrow f' \\ Y & \xrightarrow{k} & Y' \end{array}$$

<u>Define</u> $\mathfrak{C}^T$ to be the functor category with objects functors $X(-):\mathfrak{C}_T \longrightarrow \mathfrak{T}op*$. A morphism in $\mathfrak{C}^T$, between objects X,Y is a n.t. $\alpha_{(-)}:X(-) \longrightarrow Y(-)$, s.t. $\alpha_{(f)}:X(f) \longrightarrow Y(f)$ is a continuous function. Like many functor categories , $\mathfrak{C}^T$ is not a small category.

<u>Let</u> $D(-)$, $I(-) \in Obj(\mathfrak{C}^T)$ be given on $f:X \longrightarrow Y$, an object in $\mathfrak{C}_T$ as $D(f) = X$, the domain of f, and $I(f) = Y$, the image or range of f. For $(h,k):f \longrightarrow f'$ a morphism between objects in $\mathfrak{C}_T$, $D(h,k) = h$, while $I(h,k) = k$. Let $Id(-):D(-) \longrightarrow I(-)$ be the morphism in $\mathfrak{C}^T$, given by $Id(f) = f:D(f) \longrightarrow I(f)$.

Following Switzer (*Sw) or (*A1,pp.#11) for $\mathbb{Z}$ the integers define,

A <u>Spectrum</u> <u>E</u> is a sequence of spaces and maps in $\mathfrak{C}\mathcal{W}*$ $\{E_n, \varepsilon^n_E : S^1 \wedge E_n \longrightarrow E_{n+1}\}_{n \in \mathbb{Z}}$. Let <u>S</u> be the sphere spectrum.

A <u>C.W.</u> <u>Spectrum</u> is a spectrum where the structure maps ε^n_E are closed homeomorphisms of $S^1 \wedge E_n$ into a closed subcomplex of E_{n+1} .

A morphism between spectra $\underline{E}$, $\underline{F}$ is a sequence of maps, $\{f_n\}_{n\in\mathbb{Z}}$ $f_n : E_n \longrightarrow F_n$ s.t. the following diagram commutes.

$$\begin{array}{ccc} E_{n+1} & \xrightarrow{\quad f_{n+1} \quad} & F_{n+1} \\ \uparrow \varepsilon_E^n & & \uparrow \varepsilon_F^n \\ S^1 \wedge E_n & \xrightarrow{\quad S^1 \wedge f_n \quad} & S^1 \wedge F_n \end{array} \qquad \{f_*\}$$

For all $n \geq n_o$ where n_o is an arbitrary value of $\mathbb{Z}$. Such a morphism will be called a strict map of spectra. Let $\mathfrak{S}$pec be the category with objects C.W. spectrum, which are required to be both connective, (-1)-connected and Ω - spectrum. That is to say one can take $R_n = (*)$ for $n < 0$, and $\pi_r R_n = (0)$ for $r < n$, the morphisms of $\mathfrak{S}$pec are strict maps of spectra.

Note Given any collection of spaces and maps $\{E_n, \varepsilon_E^n\}_{n\in\mathbb{Z}}$ in $\mathfrak{CB}$* one constructs the Telescope spectrum Tel(E) $\in$ $\mathfrak{S}$pec s.t. E_n is a natural deformation retract of $TelE_n$ (Sw. Ch.#8).

As with $\mathfrak{T}$op* the category of $\mathfrak{S}$pec has a notion of homotopy, For $\underline{E}$ an object in $\mathfrak{S}$pec, define $(\underline{E}x'I)_n =$ $E_n x'I$, an object in $\mathfrak{S}$pec with structure maps, $\varepsilon_E^n x'I : S^1 \wedge E_n x'I \longrightarrow E_{n+1} x'I$.

Let $i_0:\underline{E}\longrightarrow \underline{E}x'I$ and $i_1:\underline{E}\longrightarrow \underline{E}x'I$ be the inclusions in $\mathfrak{S}$pec induced by the inclusion of 0,1 into I, then for $f,g:\underline{E}\longrightarrow \underline{F}$ morphisms in $\mathfrak{S}$pec, $f \simeq g$, means there exists $h:\underline{E}x'I\longrightarrow \underline{F}$ in $\mathfrak{S}$pec, s.t. $h\circ i_0 = f$ and $h\circ i_1 = g$.

Proposition Let $\underline{E}$, $\underline{F}$ be objects in $\mathfrak{S}$pec, and let $\{f_n\}_{n\in\mathbb{N}^+}$,with $f_n:E_n\longrightarrow F_n$ be a sequence of morphisms in $\mathfrak{CB}^*$, which make diagram $\{f_*\}$ commute up to homotopy , where $\mathbb{N}^+$ are the whole numbers. Then one can find a morphism f′ in $\mathfrak{S}$pec s.t.;

1) $f'_n \simeq f_n$ for all $n \in \mathbb{N}^+$

2) If g is another such morphism then $f \simeq g$.

The inductive proof of #1) uses the fact that objects in $\mathfrak{S}$pec are (-1)-connected, which begins the induction and that the structure maps are homeomorphisms of closed sub complexes hence are closed cofibrations. While the proof of #2) uses the connectivity property of objects in $\mathfrak{S}$pec to show that an appropriate Lim^1 term is trivial. The details for both part #1) and #2) are carried out in (Sw. Ch #10) For $\underline{E} \in$ Obj(Spec) and X $\in \mathfrak{T}$op* .

Following (*A2) Let $\hat{E}(X) = \underset{p \to \infty}{\text{colimit}}\, \Omega^p(E_p \wedge X)$ be the infinite loop space associated to the spectrum $\underline{E}$ when there is no possibility of confusion with spectra I will use the notation $E(X)$ instead of $\hat{E}(X)$. By this discussion if $H:f \simeq g$ from $\underline{E}\wedge X$ to $\underline{F}\wedge Y$, as morphisms in $\mathfrak{S}$pec then $Ad^p \circ (Ad^{-p} \circ \Omega^p(E_p \wedge X)x'I)$ induces,
$\hat{E}(X)x'I \longrightarrow \hat{E}(Xx'I) \xrightarrow{\hat{H}} \hat{F}(X)$ a homotopy in Top* .

Let $(X,x_0) \in \mathfrak{T}op*$, let $i_{x_0}:x_0 \hookrightarrow X$ be the inclusion of the base point, while $\bar{*}:X \longrightarrow x_0$ will be the constant map $\bar{*}(x) = x_0$, on all $x \in X$. Let I denote the unit interval and S^p the standard p-sphere.
$X \approx Y$ means X is <u>homeomorphic</u> to Y
$X \simeq Y$ means X is <u>homotopy equivalent</u> to Y.
For $Y \subseteq X$, $i_Y = i_Y^X : Y \hookrightarrow X$ is the canonical inclusion X/Y is the quotient space, and $q_Y = q_Y^X : X \longrightarrow X/Y$, is the canonical quotient.
<u>Cartesian Product</u> For $(X,x_0),(Y,y_0) \in \mathfrak{T}op*$, let XxY represent the Cartesian product with base point (x_0,y_0)
<u>Half Smash</u> For $(X,x_0),(Y,y_0) \in \mathfrak{T}op*$,
let $Xx'Y = \{XxY/(x_0,y) \sim *\}$, where $*$ is the base point of XxY. One can define X'xY in a similar manner.

Let $q_{X_o xY}: X\times Y \longrightarrow X\times' Y$ be the canonical quotient map an element $q_{X_o xY}(x,y)$ will be written as $[x,'y]$.

When $Y = (I,0)$, $r_{X\times' I}: X\times' I \longrightarrow X$ will be the retraction given by $r[x,'t] = (x,1)$, for $t \in I$, $X\times 1 \approx X$.

<u>Coproduct</u> For $X,Y \in \mathfrak{T}op*$, $X\vee Y = X\times y_o \cup x_o \times Y$ is the wedge of X and Y .

<u>Smash</u> <u>Product</u> For $X,Y \in \mathfrak{T}op*$ let $X\wedge Y = X\times Y/(X\vee Y \sim *)$.

Let $q_{X\times y_o}: X\times' Y \longrightarrow X\wedge Y$ and $q_{X\vee Y}: X\times Y \longrightarrow X\wedge Y$, be the canonical quotient maps, let an element of $X\wedge Y$, $q_{X\vee Y}(x,y)$ will be written as $[x,y]$.

Let $\curlywedge$ represent one of the three products $\times, \times'$ or $\wedge$ then fixing X respectively Y one has endofunctors $X\curlywedge(-)$, $(-)\curlywedge Y$.

Let $t: X\wedge Y \approx Y\wedge X$ be the twisting homeomorphism.

<u>Suspension</u> $\Sigma(X)$, the right suspension of X, $\Sigma_r(X) = X\wedge S^1$ while the left suspension of X, $\Sigma_l(X) = S^1\wedge X$.

For $X \in \mathfrak{T}op$, let X^I be the unbased paths of X, for $(-) \in I$, let $ev_{(-)}: X^I \longrightarrow X$, be the evaluation map, let $\sigma,\tau: X^I \longrightarrow X$ be the initial and terminal points, $\sigma = ev_{(0)}$ and $\tau = ev_{(1)}$.

Path Space For $X = (X,x_0) \in \Sigma op*$, let $P(X) = (X,x_0)^{(I,0)} = \{\alpha \in X^I : \sigma(\alpha) = x_0\}$, $P(-)$ is an endofunctor of $\Sigma op*$

Path Space of a pair For $A \subseteq X$, $B \subseteq X$, Let $(X;A,B)^{(I,1,0)} = \{\alpha \in X^I : \tau\alpha \in A, \sigma\alpha \in B\}$, be the space of paths into X, which map 1 into A.

Loop Space For $X \in \Sigma op*$ let $\Omega(X) = (X,x_0,x_0)^{(I;1,0)} = (X,x_0)^{(S^1,*)}$, $\Omega(-)$ is an endofunctor of $\Sigma op*$.

Adjunction There is an adjunction between Σ and Ω that is a bijection $\Psi : Hom_{\Sigma op*}(\Sigma X,Y) \approx Hom_{\Sigma op*}(X,\Omega Y)$

Where Σ is either the right or the left suspension, Ψ is given on $f \in Hom_{\Sigma op*}(\Sigma X,Y)$ by;

$\Psi(f)_{(x)} = f[-,x] \in \Omega(Y)$.

Where $ev_{(t)}(\Psi(f)_{(x)}) = f[t,x]$, for $x \in X$, $t \in I$.

The inverse Ψ^{-1} is defined on $g \in Hom_{\Sigma op*}(X,\Omega Y)$ by

$(\Psi^{-1}(g)) \circ [t,x] = g(x)_{(t)} = ev_{(t)}(g(x))$.

Extending this notation if $h : S^q \wedge X \longrightarrow Y$,

let $Ad^q(h) : X \longrightarrow \Omega^q(Y)$ be the q-th iterate of the adjunct Ψ.

Similarly for $g:X \longrightarrow \Omega^p(Y)$, let

$Ad^{-p}(g):S^p \Lambda X \longrightarrow Y$, be the p-th iterate of the inverse adjunction Ψ^{-1}. Let $\rho_X^1 : X \longrightarrow \Omega\Sigma X$ be given by

$\rho_X^1 = Ad_{\Sigma X} = Ad\circ(Id_{\Sigma X})$, thus $\rho_X^1(x) = [-,x] \in \Omega(\Sigma X)$

where $ev_{(t)}([-,x]) = [t,x]$.

Let $\rho_X^p(x) = \Omega^{p-1}(Ad\circ Id_{\Sigma^{p-1}}) : \Omega^{p-1}(\Sigma^{p-1}X) \longrightarrow \Omega^p(\Sigma^p X)$,

let $\rho_X = \underset{p \longrightarrow \infty}{\underline{colimit}}\ \Omega^p(\Sigma^p \rho_X^p):X \longrightarrow Q(X)$ where $\Omega^0(X) = \underline{X}$

$Q(X) = (\underline{S}\Lambda X)\hat{}$, if $\underline{i}:\underline{S} \longrightarrow \underline{R}$ is the unit of a spectrum

let $\eta_X = (\underline{i}\Lambda X)\hat{}\circ\rho_X : X \longrightarrow Q(X) \longrightarrow \hat{R}(X)$ be the Hurewicz map. Taking $\Sigma = \Sigma_1$, one has ,

$(Ad^{-1}\circ\Omega\Sigma X)\curlywedge I = (Ad^{-1}\circ id_{\Omega\Sigma X})\curlywedge I : (\Sigma(\Omega\Sigma X)\curlywedge I) \longrightarrow \Sigma X\curlywedge I$

Let $\theta_X = \theta_X^1 = Ad\circ((Ad^{-1}\Omega\Sigma X)\curlywedge I) : (\Omega\Sigma X\curlywedge I) \longrightarrow \Omega(\Sigma X\curlywedge I)$

let $\theta_X^p = Ad^p\circ((Ad^{-p}\circ\Omega^p\Sigma^p X)\curlywedge I) : (\Omega^p\Sigma^p X)\curlywedge I \longrightarrow \Omega^p(\Sigma^p X\curlywedge I)$

thus on $\gamma(-) \in \Omega^p\Sigma^p X$, $t \in I$, $\theta_X^p[\gamma,t] = [\gamma,t]_{(-)}$.

By passing to colimits θ^p induces a delooping map, $\sigma_{S,I} : Q(X)\Lambda I \longrightarrow Q(X\Lambda I)$ see Appendix B.

<u>Pull Back over f</u> Let $\Gamma_{Id(-)} = \Gamma_{(-)}$ be given on $Id(f) = f$, by $\Gamma_f = \{(x,\alpha) \in X x P(Y) : \tau\alpha = f(x)\}$.

another common notation for Γ_f is $X_f x_\tau P(Y)$.

Let $p_{D(-)} = p^{\Gamma}_{D(-)}:\Gamma_{Id(-)} \longrightarrow D(-)$, and

$p_{I(-)} = p^{\Gamma}_{I(-)}:\Gamma_{Id(-)} \longrightarrow P(I(-))$ be the canonical

projections, the pullback is characterized by the

universal property that given $h:Z \longrightarrow X$, $k:Z \longrightarrow P(Y)$

with $\tau\circ k = f\circ h$, there exist a map unique up to

homeomorphism $\theta:Z \longrightarrow \Gamma_f$ with $p^{\Gamma}_X\circ\theta = h$, $p^{\Gamma}_Y\circ\theta = k$.

Let $1_{D(-)} = 1^{\Gamma}_{D(-)}:\Omega(-) \hookrightarrow \Gamma_{Id(-)}$ be the canonical

inclusion induced by the inclusion of the loop

space into the path space. If $(h,k):f \longrightarrow f'$ is a

morphism in $\mathfrak{C}_T$, The universal property of Γ_f induces a

unique map $\Gamma(h,k) = h_f x_\tau P(k):\Gamma_f \longrightarrow \Gamma_{f'}$ this follows

since $f'\circ h\circ p^{\Gamma}_X = \tau\circ P(k)\circ p^{\Gamma}_Y$, for this reason one can

view $\Gamma_{Id(-)}$ as an object in $\mathfrak{C}^T$, in fact as a pullback

in the category $\mathfrak{C}^T$, over the map $Id:D(-) \longrightarrow I(-)$, the

details are carried out in Appendix.A.

<u>Pushout</u> For $f:X \longrightarrow Y$, $g:X \longrightarrow Y'$ the pushout of f & g

$\mathfrak{P}_{(f,g)}$ is the space $(Y \vee Y')/(f(x) \sim g(x))$ a pushout is

characterized by the universal property dual to

pullbacks.

That given a pair of maps $k:Y \longrightarrow Z$, $h:Y' \longrightarrow Z$ with $k \circ f = h \circ g$ there exists a unique map $\theta:\mathfrak{P}(f,g) \longrightarrow Z$ s.t. $\theta \circ j_{Y'} = h$ and $\theta \circ j_Y = k$, here $j_{Y'}:Y' \hookrightarrow \mathfrak{P}(f,g)$ and $j_Y:Y \hookrightarrow \mathfrak{P}(f,g)$ are the canonical inclusions.

Let $\curlywedge$ be Λ or x' the smash respectively the half smash product. Let $\mathfrak{P}^{\curlywedge}_{Id(-)} = \mathfrak{P}^{\curlywedge}_{(-)}$ be given on

$Idf = f:X = D(f) \longrightarrow I(f) = Y$, by $\mathfrak{P}^{\curlywedge}_f = \mathfrak{P}^{\curlywedge}_f = \mathfrak{P}(i^1_X, f)$

where $i^k_{D(f)}:X \longrightarrow X \curlywedge I$, is induced by the inclusion of k into I with base point 0, for $\curlywedge = \Lambda$, k is restricted to 1 another notation for $\mathfrak{P}^{\curlywedge}_f$ is $(X \curlywedge I) \cup_f Y$,

let $j_{D(-)I} = j_{D(-)\curlywedge I} :X \curlywedge I \hookrightarrow \mathfrak{P}^{\curlywedge}_f$ and $j_{I(f)}:Y \hookrightarrow \mathfrak{P}^{\curlywedge}_f$ be the canonical inclusions.

For $\curlywedge = \Lambda$, $\mathfrak{P}^{\curlywedge}_{Id(-)}$ is the <u>mapping cone of f</u>

while for $\curlywedge = x'$ $\mathfrak{P}^{\curlywedge}_{Id(-)}$ is the <u>mapping cylinder of f</u> .

If $(h,k):f \longrightarrow f'$ is a morphism in $\mathfrak{C}_T$, by the universal property of the pushout $\mathfrak{P}^{\curlywedge}_f$, and because $(j_{X \curlywedge I}) \circ (h \curlywedge I) = (j_Y \circ k)$ there exists a unique map,

$\mathfrak{P}^{\curlywedge}(h,k) = (h \curlywedge I) \cup_f k:\mathfrak{P}^{\curlywedge}_f \longrightarrow \mathfrak{P}^{\curlywedge}_{f'}$.

For this reason $\mathfrak{P}^{\curlywedge}_{Id(-)}$ is an object in the category $\mathfrak{C}^T$

$\mathfrak{P}^{\wedge}_{Id(-)}$ is also a pushout in $\mathfrak{C}^T$, under the map $Id:D(-) \longrightarrow I(-)$ the details are carried out in Appendix.A, simultaneously with those of the pullback.

<u>Cone</u> <u>of</u> <u>f</u>, For $f:X \longrightarrow Y$ in $\mathfrak{T}op*$, let $C(f) = \mathfrak{P}_{(Id,f)} = (X \vee Y)/(x \sim f(x)) = C(f) = X \cup_f Y$.

Let $j_X = j_X^{Cf}:X \hookrightarrow C(f)$ and $j_Y:Y \hookrightarrow C(f)$ be the canonical inclusions. As with $\mathfrak{P}^{\wedge}_{(-)}$ $C(-)$ may be viewed as an object in $\mathfrak{C}^T$.

Chapter 1

In the category of spectra pull backs are equivalent to push outs

In chapter 1 we begin the process of realizing the isomorphism mentioned in the preface between the E_1 term of the unstable A.S.S. and a non-abelian resolution of unstable co- algebras in $\mathfrak{M}(G)$.

The first step is a realization of the fact stated (*A1. pp. #41) that in the category of Spectra, in particular for R-homology that pull backs and push outs are equivalent.

This realization is a morphism in $\mathfrak{C}^T$, theorem #1.1 establishes the existence of this morphism $\varphi_{Id(-)}:\Gamma_{Id(-)} \longrightarrow \Omega\mathfrak{P}^{\Lambda}_{Id(-)}$, using $\varphi_{Id(-)}$ one defines $\varphi_{R,Id(-)}:\Gamma_{R(-)} \longrightarrow \Omega R\mathfrak{P}^{\Lambda}_{Id(-)}$ in $\mathfrak{C}^T$, which by theorem #1.3 is a weak equivalence. The morphism $\varphi_{R,Id(-)}$ is the realization of Adam's assertion that R-homology pull-backs and push outs are equivalent.

The proof of theorem #1.1 requires the use of a third morphism also in $\mathfrak{C}^T$,

$$\psi_{Id(-)} = \psi^{\Lambda}_{Id(-)}:\mathfrak{P}^{\Lambda}_{R(-)} \longrightarrow R\mathfrak{P}^{\Lambda}_{Id(-)}$$

which is introduced in appendix.B.

This map is proven to be a weak equivalence in chapter #2 given the following restrictions.

$f:X \longrightarrow Y$ in Top* is said to be retractable w.r.t. $\underline{R}$ or simply retractable when $\underline{R} \in$ Obj($\mathfrak{S}$pec) is understood, if $\pi_* R(f)$ has a retract.

<u>Theorem</u> #2.1 For $f:X \longrightarrow Y$, in Top* , f retractable w.r.t. $\underline{R}$ X,Y torsion free, simply connected, and of finite type ψ_f^Λ is a weak equivalence.

<u>Theorem</u> #1.1 For $f:X \longrightarrow Y$ in $\mathfrak{T}$op* . There is a map $\varphi_f:\Gamma_f \longrightarrow \Omega(\mathfrak{P}_f^\Lambda)$ which induces $\varphi_{Id(-)}:\Gamma_{Id(-)} \longrightarrow \mathfrak{P}_{Id(-)}^\Lambda$ which is a morphism in $\mathfrak{C}^T$.

The morphism φ_f is actually the composition of three morphisms For $\mathfrak{P}_{(-)}^{X'} = \mathfrak{P}(i_{D(-)}^1, Id(-))$, with

$j_{DI} = j_{D(-)x'I}:D(-)x'I \longrightarrow \mathfrak{P}_{(-)}^{X'}$ one has

$j_{DI} \circ i_{D(-)}^k = j_{Dx'I} \circ i_{D(-)}^k : D(-) \longrightarrow \mathfrak{P}_{(-)}^{X'}$ and one can form $\Gamma_{j_{DI} \circ i_{D(-)}^k}$ the first map an inclusion is given by,

$\Gamma(id_{D(-)}, j_{I(-)}):\Gamma_{Id(-)} \longrightarrow \Gamma_{j_{DI} \circ i_{D(-)}^1}$ and has inverse $\Gamma(id_{D(-)}, r_{D(-)})$ where $r_{D(-)} \circ j_{I(-)} = Id_{I(-)}$ and $j_{I(-)} \circ r_{D(-)} \simeq Id_{I(-)}$.

The next map $\theta_{Id(-)}:(\Gamma j_{DI}\circ i^{1}_{D(-)})' \longrightarrow \Gamma j_{DI}\circ i^{0}_{D(-)}$ where $(\Gamma j_{DI}\circ i^{1}_{D(-)})'$ is the restriction to the image of $\Gamma(id_{D(-)}, j_{I(-)})$ and is defined by the loop space multiplication structure, The third and final map is a quotient map;

$\Gamma(*D(-), \mathfrak{P}(q_{D(-)x'0}, Id_{I(-)})):\Gamma j_{DI}\circ i^{0}_{D(-)} \longrightarrow \Omega\mathfrak{P}^{\Lambda}_{Id(-)}$.

Because the loop space functor is a degenerate pull back, it follows by the work in appendix A, that the following maps,

$\Gamma(Id_{D(-)}, j_{I(-)})$ and $\Gamma(*D(-), \mathfrak{P}(q_{D(-)x'0}, Id_{I(-)}))$ are both morphisms in $\mathfrak{C}^{T}$. In order to complete the proof of theorem #1.1, it suffices to define $\theta_{Id(-)}$ restricted to the image of $\Gamma(id_{D(-)}, j_{I(-)})$, and prove that $\theta_{Id(-)}$ is natural w.r.t. morphisms in $\mathfrak{C}_{T}$.

Because I is locally compact the evaluation map $ev_{(-)}:Y^{I}x'I \longrightarrow Y$ is continuous, $(-) \in I$, furthermore since $j_{XI}\circ i^{0}_{X}(X)$, $j_{XI}\circ i^{1}_{X}(X)$ are subsets of $\mathfrak{P}^{X'}_{f}$ with $j_{XI}\circ i^{0}_{X}(X) \approx Xx'0 \approx X$, although this is not true for $j_{XI}\circ i^{1}_{X}(X)$, one then has continuous maps,

$ev_{(-)}:(\mathfrak{M}_{f}, j_{XI}\circ i^{1}_{X}X)^{(I,1)}x'I \longrightarrow \mathfrak{M}_{f}$

and $ev_{(-)}:(\mathfrak{M}_{f}, j_{XI}\circ i^{1}_{X}X, j_{XI}\circ i^{0}_{X}X)^{(I,1,0)}x'I \longrightarrow \mathfrak{M}_{f}$.

Proof of theorem #1.1; Consider $\Gamma_{(\tau,\tau)}$ the subset of $(\mathfrak{M}_f, j_{XI}\circ i_X^1X,*)^{(I,1,0)} x (\mathfrak{M}_f, j_{XI}\circ i_X^1X, j_{XI}\circ i_X^0X)^{(I,1,0)}$ consisting of pairs of maps (α,β) s.t. $\tau\beta = \tau\alpha$. Then one can form the continuous map, $\Theta: Ix'\Gamma_{(\tau,\tau)} \longrightarrow \mathfrak{M}_f$ given by the loop space multiplication,

$$\Theta(s,\alpha,\beta) = \alpha{*}\beta^{-1} = \begin{cases} \alpha(2s) & 0\le s \le 1/2 \\ \beta(2-2s) & 1/2 \le s \le 1 \end{cases}$$

Because $\alpha(1) = \beta(1)$ this map is continuous. If f is an embedding then keeping β fixed as $\beta(-) = j_{XI}\circ i_X^1[\tau\alpha,-]$, or simply $[\tau\alpha,-]$ one has a continuous map $\Theta(-,-)_\beta: Ix'P(\mathfrak{M}_f, j_{XI}\circ i_X^1(X),*) \longrightarrow \mathfrak{M}_f$ with adjoint $\theta_f: P(\mathfrak{M}_f, j_{XI}\circ i_X^1X,*) \longrightarrow P(\mathfrak{M}_f, Xx0,*)$.

Because $[x,-] = [x,-]^{-1}$, since they are both the constant map the definition of θ_f becomes,

$$\theta_f(\alpha) = \alpha{*}[\tau\alpha,-] = \left\{ \begin{array}{ll} \alpha(2s) & 0\le s \le 1/2 \\ {[\tau\alpha, 2-2s]}, & 1/2 \le s \le 1 \end{array} \right\}$$

If the map f, is not an embedding there may be no way to make an identification of $\tau\alpha$ with a well defined value of $x \in X$, That is to say since $j_{XI} \circ i^1_X(X) \approx X$ is false, there can be distinct values $x_1, x_2 \in X$, with $fx_1 = fx_2 = \tau\alpha$, however this restriction is temporary and does not carry through to the composition of $\theta_f \circ \Gamma(id, j_I)$ since in Γ_f the pairs (x_1,α) and (x_2,α) are distinct, and hence the composition $\theta_f \circ \Gamma(id, j_I)$ is well defined even when f is not an embedding.

To see that $\theta_{Id(-)} \circ \Gamma(id_{D(-)}, j_{I(-)}) \in Morph(\mathfrak{C}^T)$ let $(x,\alpha) \in \Gamma_f$ and $(h,k): f \longrightarrow f'$ be a morphism in $\mathfrak{C}_T$.
Then one has, $\Gamma(h, \mathfrak{P}(h,k)) \circ \theta_f(x,\alpha) =$
$\Gamma(h, \mathfrak{P}(h,k)) \circ (\alpha_*[\tau\alpha, -]) = k \circ \alpha_*[\tau hx, -]$ while,
$\theta_{f'} \circ \Gamma(h, \mathfrak{P}(h,k)) = \theta_{f'}(hx, k \circ \alpha) = k \circ \alpha_*[\tau hx, -]$.
Theorem #1.1 has now been completed ■

Let $\psi^\Lambda_{Id(-)} : \mathfrak{P}^\Lambda_{R(-)} \longrightarrow R\mathfrak{P}^\Lambda_{Id(-)}$ be the delooping map introduced in appendix.B, $\psi^\Lambda_{Id(-)} \in Morph(\mathfrak{C}^T)$.
Define $\varphi_{R,f} = \Omega(\psi^\Lambda_f) \circ \varphi_{Rf} : \Gamma_{R(-)} \longrightarrow \Omega R \mathfrak{P}^\Lambda_{Id(-)}$, as the composition of morphisms in $\mathfrak{C}^T$ $\varphi_{R,Id(-)}$ is a morphism in $\mathfrak{C}^T$. The n.t. $\varphi_{R,Id(-)}$ is our candidate for a morphism to establish the equivalence between pullbacks and push outs in R - homology.

<u>Lemma</u> #1.2 The squares in diagram {phi-f} commute at least up to homotopy, where $\Sigma = \Sigma_1$, $f:X \longrightarrow Y$;

$$\begin{array}{ccccccc}
\Omega X & \xrightarrow{\Omega f} & \Omega Y & \xrightarrow{l_f} & \Gamma_f & \xrightarrow{p_X} & X \\
\Big\downarrow = & \text{I} & \Big\downarrow \Omega j_X & \text{II} & \Big\downarrow \varphi_f & \text{III} & \Big\downarrow \Sigma_X \\
\Omega X & \xrightarrow[\ ]{\Omega j_{XI} \circ i_X^1} & \Omega \mathfrak{p}_f^{X'} & \xrightarrow[\mathfrak{p}(q_{XX'0},\, id_Y)]{} & \Omega \mathfrak{p}_f^{\Lambda} & \xrightarrow[q_Y]{} & \Omega\Sigma X
\end{array} \qquad \text{\{phi-f\}}$$

That square I commutes is straightforward, while square II commutes up to homotopy to see this, let $\beta \in \Omega Y$ since $[y,-]^{-1} = [y,-]$, $\varphi_f \circ l_f \circ (\beta) = \beta * [y_0,-]$ where y_0 is the base point of Y identified with $*\mathfrak{p}_f^{\Lambda}$ the base point of $\mathfrak{p}_f^{\Lambda}$. In the loop space of Y, ΩY, $\beta \simeq \beta * [y_0,-]$ through a straightforward collapsing homotopy. This same homotopy induces $q_{XX'0} \circ j_Y(\beta) \simeq \beta * [y_0,-]$ and hence square II commutes up to homotopy. Finally for $(x,\alpha) \in \Gamma_f$ $\Sigma(x) = [x,-]$ while $\varphi_f(x,\alpha) = \alpha * [x,-]$, and $q_Y \circ (\alpha * [x,-]) = (q_Y \circ \alpha) * [x,-] = \bar{y}_0 * [x,-] \simeq [x,-]$, where $\bar{y}$ is the constant path of ΩY with value $y \in Y$ for all $t \in I$, hence $\Omega(q_Y) \circ \varphi_f(x,\alpha) \simeq [x,-]$ therefore square III commutes up to homotopy. ∎

<u>Theorem</u> #1.3 $\varphi_{R,f}$ is a weak equivalence for all $f:X \longrightarrow Y$. Proof; Consider the following diagram #1.4

$$
\begin{array}{ccccccc}
\Omega RX & \xrightarrow{\Omega Rf} & \Omega RY & \xrightarrow{1_{Rf}} & \Gamma_{Rf} & \xrightarrow{p_{RX}} & RX \\
\Big\downarrow = & \text{I} & \Big\downarrow \Omega j_{RY} & \text{II} & \Big\downarrow \varphi_{Rf} & \text{III} & \Big\downarrow \Sigma_{RX} \\
\Omega RX & \xrightarrow{\Omega j_{RXI}\circ i^{1}_{RX}} & \Omega \mathfrak{p}^{X'}_{Rf} & \xrightarrow{\Omega \mathfrak{p}(q_{RXx'0},Id_{RY})} & \Omega \mathfrak{p}^{\Lambda}_{Rf} & \xrightarrow{q_{RY}} & \Omega \Sigma RX \\
\Big\downarrow = & \text{IV} & \Big\downarrow \Omega\psi^{X'}_{f} & \text{V} & \Big\downarrow \Omega\psi^{\Lambda}_{f} & \text{VI} & \Big\downarrow \sigma_{S^1;RX} \\
\Omega RX & \xrightarrow{\Omega R(j_{XI}\circ i^{1}_{X})} & \Omega R\mathfrak{p}^{X'}_{f} & \xrightarrow{\Omega R\mathfrak{p}(q_{Xx'0},Id_{Y})} & \Omega R\mathfrak{p}^{\Lambda}_{f} & \xrightarrow{Rq_{Y}} & \Omega R\Sigma X
\end{array}
$$

Square I , II and III all commute at least up to homotopy by lemma 1.2. That IV commutes is due to property B.6 diagram σ-j of appendix.B, while square V commutes by property B.5 diagram σ-q.1. The definition of $\sigma_{S^1;RX}$ is given in appendix B, while square VI commutes by property B.3 , from appendix.B. Thus each square commutes at least up to homotopy, In diagram #1.4 both the top and the bottom rows induce long exact sequences in homotopy where the bottom row is equivalent to that of the long exact sequence in R - homology of the pair $(\mathfrak{M}_f, Xx'0)$. Consider the vertical maps, we have that $\psi^{X'}_{f}$ is a w.h.e. which is because 1) $\psi^{X'}_{f}\circ j_{RY} = Rj_Y$, 2) both j_{RY} , Rj_Y are w. e. and 3) Y is a deformation retract of $\mathfrak{p}^{X'}_{f}$. While $(\sigma_{S^1;Y})\circ\Sigma_{RY}$ is the homology suspension and induces an isomorphism in homotopy, thus by the 5 - lemma Theorem #1.3 is proven true. ∎

Chapter #2

Two R-homology push outs which are equivalent.

The goal of chapter #2 is to establish conditions for which ψ_f will be a weak equivalence, this is accomplished in the following theorem #2.1.

Theorem #2.1 For X,Y torsion free, simply connected and of finite type, (tf.sc.ft) and f retractable w.r.t. $\underline{R}$ abbreviated f is tf.sc.ft and retractable when $\underline{R}$ is understood, then $\psi_f : \mathfrak{P}_{Rf} \longrightarrow R\mathfrak{P}_f$, is a weak equivalence. In chapter #2 $\psi_f = \psi_f^\wedge$ and $\mathfrak{P}_f = \mathfrak{P}_f^\wedge$.

Proof of Theorem #2.1; For X,Y, $f:X \longrightarrow Y$, tf.sc.ft and retractable w.r.t $\underline{R}$ as above, the general theory of spectra yields a long exact homology sequence for the pair $(\mathfrak{P}_f^\wedge ,X)$ or more accurately for the map f

let $\pi_n(RX) = [S^n,RX]$, the homotopy class of maps in $\mathfrak{T}op*$, $[S^n,RX] \cong [\underline{S}(S^n),\underline{R}\wedge X]$, the the homotopy class of maps of spectra. By the the retract hypothesis on Rf this l.e.s. becomes a s.e.s; [#2.2,f]

$$0 \longrightarrow \pi_*RX \rightarrowtail \pi_*RY \twoheadrightarrow \pi_*R\mathfrak{P}_f \longrightarrow 0$$

Let $\mathfrak{K} = \pi_*R(S^0)$ the coefficient ring for the category $\mathfrak{M} = {}_{\mathfrak{K}}\mathfrak{M}$ of 1-connected free $\mathfrak{K}$ - modules of finite type then $\pi_*RX \cong \mathrm{Hom}_{\mathfrak{M}}(\mathfrak{K},\pi_*RX)$.

Let $\mathfrak{X}'$ be the subcategory of $\mathfrak{X}op*$ obtained by restricting the objects to those spaces which are tf.sc.ft. Define a functor $M(-):\mathfrak{X}' \longrightarrow \mathfrak{M}$ by $M(X) = \pi_* R(X)$, by the work in (*B) each object in $\mathfrak{M}$ can be realized as $M(X)$ for a space $X \in \mathfrak{X}'$ furthermore one can identify the image of $M(-)$ with $\mathfrak{M}'$ and for $Z \in \mathfrak{X}'$ there is an adjunction,

$\mathrm{Hom}_{\mathfrak{M}}(MZ,MX) \cong \mathrm{Hom}_{\mathfrak{M}(G)}(MZ,GMX)$, where $G(-):\mathfrak{M} \longrightarrow \mathfrak{M}(G)$ is a functor induced by $G(MX) = \pi_* R^2 X$ and $\mathfrak{M}(G)$ is the category of unstable co - algebras.

In order to simplify notation let $H_G(MZ,MX) = \mathrm{Hom}_{\mathfrak{M}(G)}(MZ,MX)$. For $\mathfrak{K}, M(X) \in \mathfrak{M}(G)$, this adjunction together with [2.2,f] results in the s.e.s.; [#2.3]

$$0 \longrightarrow H_G(\mathfrak{K},\pi_* R^2 X) \rightarrowtail H_G(\mathfrak{K},\pi_* R^2 Y) \twoheadrightarrow H_G(\mathfrak{K},\pi_* R^2 \mathfrak{P}_f) \longrightarrow 0$$

In order to extend this s.e.s to general $Z \in \mathfrak{X}'$ observe that MX, MZ are objects in $\mathfrak{M}G$ as well as $\mathfrak{M}$ that $\mathrm{Hom}_{\mathfrak{M}G}(MZ,MX) \subset \mathrm{Hom}_{\mathfrak{M}}(MZ,MX)$ as abelian groups, and that the isomorphisms;

$\Phi:\mathrm{Hom}_{\mathfrak{M}}(MZ,\prod_\alpha \mathfrak{K}) \cong \prod_\alpha \mathrm{Hom}_{\mathfrak{M}}(MZ,\mathfrak{K})$ and

$\Psi:\mathrm{Hom}_{\mathfrak{M}}(\coprod_\alpha \mathfrak{K},MX) \cong \prod_\alpha \mathrm{Hom}_{\mathfrak{M}}(\mathfrak{K},MX)$ with Φ induced by the projections $p_\alpha:\prod_\alpha \mathfrak{K} \longrightarrow \mathfrak{K}$, while Ψ is induced by the inclusions $\lambda_\alpha:\mathfrak{K} \longrightarrow \coprod_\alpha \mathfrak{K}$

restrict to isomorphisms

$\Phi : \mathrm{Hom}_{\mathfrak{M}G}(MZ, \prod_\alpha \mathfrak{K}) \cong \prod_\alpha \mathrm{Hom}_{\mathfrak{M}G}(MZ, \mathfrak{K})$ and

$\Psi : \mathrm{Hom}_{\mathfrak{M}}(\coprod_\alpha \mathfrak{K}, MX) \cong \prod_\alpha \mathrm{Hom}_{\mathfrak{M}}(\mathfrak{K}, MX)$.

Furthermore because the objects in $\mathfrak{M}$ and hence those in $\mathfrak{M}G$ are of finite type direct sums are equivalent to direct products. We are now able to extend the s.e.s. [2.3] to the following s.e.s. [#2.3,b]

$$0 \longrightarrow H_G(MZ, \pi_* R^2 X) \rightarrowtail H_G(MZ, \pi_* R^2 Y) \twoheadrightarrow H_G(MZ, \pi_* R^2 \mathfrak{P}_f) \longrightarrow 0$$

Consider the s.e.s. [#2.2,Rf], that is the s.e.s.,

$$0 \longrightarrow \pi_* R^2 X \rightarrowtail \pi_* R^2 Y \twoheadrightarrow \pi_* R \mathfrak{P}_{Rf} \longrightarrow 0$$

Using the language of (*A), if X = RX′ for some X′ ∈ $\mathfrak{T}$op*, then one has a model in the category of $\mathfrak{M}$(G) and hence for such X, the Ext term is trivial. For this reason if one applies the functor $H_G(MZ,-)$ to this s.e.s. [#2.3,Rf] with $\mathrm{Ext}^0(MZ,-) = \mathrm{Hom}_{\mathfrak{M}(G)}(MZ,(-))$ one obtains a s.e.s. [#2.4]

$$0 \longrightarrow H_G(MZ, \pi_* R^2 X) \rightarrowtail H_G(MZ, \pi_* R^2 Y) \twoheadrightarrow H_G(MZ, \pi_* R \mathfrak{P}_{Rf}) \longrightarrow 0$$

Combining the s.e.s. [#2.3,b] with that of [#2.4], one has the following commutative diagram which implies that $H_G(MZ, \psi_f)$ is an isomorphism.

[#2.5]

$$\begin{array}{ccccccccc}
0\to & H_G(MZ,\pi_*R^2X) & \longrightarrow & H_G(MZ,\pi_*R^2Y) & \longrightarrow & H_G(MZ,\pi_*R\mathfrak{P}_{Rf}) & \to & 0 \\
 & \Big\downarrow = & & \Big\downarrow = & & \Big\downarrow H_G(MZ,\pi_*R\psi_f) & & \\
0\to & H_G(MZ,\pi_*R^2X) & \longrightarrow & H_G(MZ,\pi_*R^2Y) & \longrightarrow & H_G(MZ,\pi_*R^2\mathfrak{P}_f) & \to & 0
\end{array}$$

If X is torsion free and of finite type then so also is R(X). Since π_*Rf has a retract π_*RY is the direct sum of π_*RX and $\pi_*R\mathfrak{P}_f$, therefore when both X,Y are torsion free, 1 - connected and of finite type, $\mathfrak{P}_f$ is also torsion free, 1 - connected and of finite type, and $\mathfrak{P}_{Rf}$, $R\mathfrak{P}_f$ as well.

Let $N = \pi_*R^2\mathfrak{P}_f$ in diagram #2.6, then $H_G(\pi_*R^2\mathfrak{P}_f,\pi_*R\mathfrak{P}_{Rf}) \cong H_G(\pi_*R^2\mathfrak{P}_f,\pi_*R^2\mathfrak{P}_f)$ therefore there exists $g:\pi_*R^2\mathfrak{P}_f \longrightarrow \pi_*R\mathfrak{P}_{Rf}$ in $\mathfrak{M}(G)$ s.t. $(R\psi_f)_*\circ(g) = Id_{\pi_*R\mathfrak{P}_f}$ and hence $H_G(N,R\psi_f)\circ H_G(N,g) = Id$, since $H_G(N,R\psi_f)$ is an isomorphism so also is $H_G(N,g)$ and as before there exists $h:\pi_*R\mathfrak{P}_{Rf} \longrightarrow \pi_*R^2\mathfrak{P}_f$ s.t. $g\circ h = Id$, therefore g is itself an isomorphism and hence $\pi_*(R\psi_f)$ and as a result $\pi_*(\psi_f)$ is also. Theorem #2.1, has now been proven true.

■

Chapter 3

In R-homology the Internal Iterated Push Out is Equivalent to the External iterated Push out

The results of chapter 1 that $\varphi_{R,f}:\Gamma_{R(f)} \longrightarrow \Omega R\mathfrak{P}_f$ is a w.e. and chapter 2 that $\psi_f = \psi_f^{\Lambda}:\mathfrak{P}_{Rf} \longrightarrow R\mathfrak{P}_f$ is a w.e. whenever X,Y are restricted to spaces which are tf.sc.ft. , and f is retractable , has as a corollary that φ_{Rf} is a w.e. for such conditions.

In appendix.A , the internal theorem iterated functor construct, (Theorem itfun.$\mathfrak{C}$) defines for each s ≥ 1, iterated pull backs and push outs $\Gamma^s_{Id(-)}$, $\mathfrak{P}^s_{Id(-)}$ over the internal hurewicz map $\Gamma^s(\eta_D,\eta_I)$ respectively under the internal hurewicz map $\mathfrak{P}^s(\eta_D,\eta_I)$.

The first objective in chapter 2 is to define;

$\varphi^s_{Id(-)}:\Gamma^s_{Id(-)} \longrightarrow \Omega^s\mathfrak{P}^s_{Id(-)}$, $\varphi^s_{R,Id(-)}:\Gamma^s_{R(-)} \longrightarrow \Omega^s R\mathfrak{P}^s_{Id(-)}$ and $\psi^s_{Id(-)}:\mathfrak{P}^s_{R(-)} \longrightarrow R\mathfrak{P}^s_{Id(-)}$ all morphisms in $\mathfrak{C}^T$ with $\Omega^s(\psi^s_{Id(-)})\circ(\varphi^s_{R,Id(-)}) = \varphi^s_{Id(-)}$ and to extend the results of chapter 1 and 2, by proving that $\varphi^s_{R,(-)}$ is a w.e. while $\psi^s_{R,f}$ and hence $\varphi^s_{R(f)}$ are w.e. subject to the above conditions.

The second objective of this chapter (Theorem #3.20,#3.24) is the result that the internal iterated push outs $\mathfrak{P}^s_{R^n(-)}$ are naturally equivalent to the external iterated push out $R^n\mathfrak{g}^s_{Id(-)}$, through a morphism in $\mathfrak{C}^T$,

$\chi^{s,n}_{(-)}:\mathfrak{P}^s_{R^n(-)} \longrightarrow R^n\mathfrak{g}^s_{Id(-)}$ s.t.

$(\chi^{s,1})\circ\mathfrak{P}(\eta_D,\eta_I) = (\eta_{\mathfrak{g}}s)\circ(\chi^{s,0})$.

Combined with the results of chapter 1,

$\Omega^s(\eta_{\mathfrak{g}^s_f})\circ\Omega^s(\chi^{s,0}_{f})\circ(\varphi^s_f) = \Omega^s(\chi^{s,1}_{f})\circ(\varphi^s_{Rf})\circ\Gamma(\eta_X,\eta_Y)$ and the following diagram #3.1 commutes. [#3.1]

$$\begin{array}{ccc}
\Gamma^s_f & \xrightarrow{\Gamma(\eta_D,\eta_I)} & \Gamma^s_{Rf} \\
\downarrow{\scriptstyle \Omega^s\chi^{s,0}_f\circ\varphi^s_f} & & \downarrow{\scriptstyle \Omega^s\chi^{s,1}_f\circ\varphi^s_{Rf}} \\
\Omega^s\mathfrak{g}^s_f & \xrightarrow{\Omega^s\eta_{\mathfrak{g}^s_f}} & \Omega^s R\mathfrak{g}^s_f
\end{array}$$

In diagram #3.1 the right vertical map $\Omega^s\Omega^{s,1}_f\circ\varphi^s_{Rf}$ is a weak equivalence for $f = \eta_X$ and X, t.f., simply connected, and of finite type by theorem #2.1, while the lower horizontal map in diagram #3.1, $\eta_{\mathfrak{g}^s_f}$ is the external Hurewicz map that is the desired structure map in the category $\mathfrak{M}(G)$ of unstable co-algebras.

The top map of diagram #3.1 is the internal Hurewicz used in constructing the tower of iterated pull backs over the map f, with $E_1^{s,t} = \pi_{t-s}\Gamma^{s}_{Rf}$.

Furthermore $\pi_* R\mathfrak{L}^s_f \cong Hom_{\mathfrak{K}}(\mathfrak{K}, \pi_* R\mathfrak{L}^s_f) \cong Hom_{\mathfrak{M}(G)}(\mathfrak{K}, \pi_* R^2\mathfrak{L}^s_f)$ therefore diagram #3.1 represents an equivalence on the E_1 terms of the unstable A.S.S. with a resolution in $\mathfrak{M}(G)$ as a result one has $E_2 \cong Ext_{\mathfrak{M}(G)}(\mathfrak{K}, \pi_* RX)$ through the natural equivalence $\Omega^s(\chi^{s,*}_{\eta_X}) \circ (\varphi^s_{R\eta_X})$ whenever X is torsion free ,simply connected and of finite type, this completes the major result of this thesis.

In appendix.A, it is established that for any morphism $\alpha(-):X(-) \longrightarrow Y(-)$ in $\mathfrak{C}^T$, the functor $\mathfrak{P}_{\alpha(-)}$ is an object in $\mathfrak{C}^T$ and has the pushout property. Therefore the pair of morphisms

$\{Rj_{XI}(-) \circ \sigma_{X(-)}, Rj_{Y(-)}\}$ in $\mathfrak{C}^T$ induce a morphism

$\psi_{\alpha(-)}:\mathfrak{P}_{R\alpha(-)} \longrightarrow R\mathfrak{P}_{\alpha(-)}$, in $\mathfrak{C}^T$.

Where $j_{Y(-)}:Y(-) \longrightarrow \mathfrak{P}_{\alpha(-)}$ is the canonical inclusion.

From Property B.2 diagram σ-η in appendix.B, and lemma Tcom in appendix A, one observes that the following diagram #3.2 in $\mathcal{C}^T$ commutes.

Diagram #3.2

$$\begin{array}{ccc} X(-)\Lambda I & \xrightarrow{\eta_{X(-)}\Lambda I} & R\circ X(-)\Lambda I \\ {\scriptstyle \eta_{X(-)\Lambda I}}\downarrow & & \downarrow{\scriptstyle \sigma_{RX;I}} \\ R\circ(X(-)\Lambda I) & \xrightarrow[\mathrm{Id}]{} & R\circ(X(-)\Lambda I) \end{array}$$

By diagram #3.2 one has that,

$R(j_{XI(-)})\circ\sigma_{XI(-)}\circ(\eta_{X(-)}\Lambda I) = R(j_{XI(-)})\circ\eta_{X(-)\Lambda I}$ and

hence that the pair of maps,

$(Rj_{XI(-)}\circ\sigma_{XI(-)}\circ(\eta_{X(-)}\Lambda I), Rj_{Y(-)}\circ\eta_{Y(-)}) =$

$(Rj_{XI(-)}\circ\eta_{X(-)\Lambda I}, Rj_{Y(-)}\circ\eta_{Y(-)})$. Which implies that

$\psi_{\alpha(-)}\circ\mathfrak{P}(\eta_{D(-)},\eta_{I(-)}) = \eta_{\mathfrak{P}_{\alpha(-)}}$ and therefore that

diagram #3.3 below commutes.

Diagram #3.3

$$\begin{array}{ccc} \mathfrak{P}_{\alpha(-)} & \xrightarrow[\mathfrak{P}(\eta_X,\eta_Y)_{(-)}]{} & \mathfrak{P}_{R\circ\alpha(-)} \\ =\downarrow & & \downarrow{\scriptstyle \psi_{\alpha(-)}} \\ \mathfrak{P}_{\alpha(-)} & \xrightarrow{\eta_{\mathfrak{P}_{\alpha(-)}}} & R\circ\mathfrak{P}_{\alpha(-)} \end{array}$$

Of particular interest is the case $\alpha(-) =$ $Id(-):D(-)\longrightarrow I(-)$, $\psi_{Id(-)}$ will also be labelled ψ_X , where $X = D(f)$.

<u>Lemma</u> #3.4 There is a morphism in $\mathfrak{C}^T$, $\psi^s_\alpha : \mathfrak{P}_{R^s\alpha} \longrightarrow R^s\mathfrak{P}_\alpha$

s.t. $R^s\eta\mathfrak{P}_{\alpha(-)} \circ \psi^s_{\alpha(-)} = \psi^{s+1}_{\alpha(-)} \circ \mathfrak{P}(R^s\eta_{D\alpha}, R^s\eta_{I\alpha})(-)$

for all $s \geq 0$, which is a weak equivalence when α_f is retractable, with $D(\alpha_f)$, $I(\alpha_f)$ torsion free and simply connected spaces of finite type.

Proof by induction,

case $s = 0$, with $\psi^0_{\alpha(-)} = Id$ and $\psi^1_{\alpha(-)} = \psi_{\alpha(-)}$ here lemma #3.4 reduces to diagram #3.3 which begins the induction, suppose then that for some fixed $s \geq 1$ the map $\psi^s_{\alpha(-)} : \mathfrak{P}_{R^s\alpha(-)} \longrightarrow R^s\mathfrak{P}_{\alpha(-)}$ has been defined and is a morphism in $\mathfrak{C}^T$, define

$\psi^{s+1}_{\alpha(-)} = R^s\psi_{\alpha(-)} \circ \psi^s_{R\circ\alpha(-)}$ as the

composition of morphisms in $\mathfrak{C}^T$, $\psi^{s+1}_{\alpha(-)}$ is itself in $\mathfrak{C}^T$, using induction if one assumes that $\psi^s_{\alpha(-)}$ is a weak equivalence when $\alpha(f)$ is restricted to retractable maps with $X = D(\alpha_f)$ and $Y = I(\alpha_f)$ tf.sc.ft , then $R\circ\alpha_f$ is also retractable and $\psi^{s+1}_{\alpha(-)}$ as the composition of weak equivalences is a weak equivalence.

Consider the following diagram #3.5

#3.5

$$\begin{array}{ccccc} \mathfrak{p}_{R^s\alpha} & \xrightarrow{\psi^s_\alpha} & R^s\mathfrak{p}_\alpha & \xrightarrow{Id} & R^s\mathfrak{p}_\alpha \\ \Big\downarrow \mathfrak{p}(R^s\eta_{D\alpha}, R^s\eta_{I\alpha}) & 3.5.I & \Big\downarrow R^s\mathfrak{p}(\eta_{D\alpha}, \eta_{I\alpha}) & 3.5.II & \Big\downarrow R^s\eta_{\mathfrak{p}_\alpha} \\ \mathfrak{p}_{R^{s+1}\alpha} & \xrightarrow{\psi^s_{R\alpha}} & R^s\mathfrak{p}_{R\alpha} & \xrightarrow{R^s\psi_\alpha} & R^s\mathfrak{p}_\alpha \end{array}$$

Square #3.5.I commutes because $\psi^s_{\alpha(-)}$ is in $\mathfrak{C}^T$ while #3.5.II is simply the functor $R^s(-)$ applied to diagram #3.3. This completes the induction and proof of lemma #3.4 ∎

<u>Lemma</u> #3.6 A commutative diagram of morphisms in $\mathfrak{C}^T$ such as diagram #3.7.a ,

induces the commutative diagram #3.7.b, also in $\mathfrak{C}^T$.

$$\begin{array}{ccc} X(-) & \xrightarrow{\alpha(-)} & Y(-) \\ \Big\downarrow \beta_X(-) & & \Big\downarrow \beta_Y(-) \\ X'(-) & \xrightarrow{\alpha'(-)} & Y'(-) \end{array} \qquad \#3.7.a$$

$$\begin{array}{ccc} \mathfrak{p}_{R^n\alpha(-)} & \xrightarrow{\mathfrak{p}_{R^n(\beta_X,\beta_Y)}(-)} & \mathfrak{p}_{R^n\alpha(-)} \\ \psi^n_{\alpha(-)} \Big\downarrow & & \Big\downarrow \psi^n_{\alpha'(-)} \\ R^n\mathfrak{p}_{\alpha(-)} & \xrightarrow{R^n\mathfrak{p}(\beta_X,\beta_Y)(-)} & R^n\mathfrak{p}_{\alpha'(-)} \end{array} \qquad \#3.7.b$$

Proof of lemma #3.6;

given the commutative diagram #3.7.a) of morphisms in $\mathfrak{C}^T$, there exists, two morphism $\mathfrak{P}R^n(\beta_{D\circ\alpha(-)},\beta_{I\circ\alpha(-)})$, $R^n\mathfrak{P}(\beta_{D\circ\alpha(-)},\beta_{I\circ\alpha(-)})$ both of which are morphisms in $\mathfrak{C}^T$ this is listed as a corollary in appendix A .
Because $\psi^n_{\alpha(-)}$ is a morphism in $\mathfrak{C}^T$, in particular taking $\alpha(-) = Id(-)$ one has that for $(h,k):f \longrightarrow f'$ in $\mathfrak{C}_T$ that $\psi^n{}_{Id(f')}\circ\mathfrak{P}R(h,k) = R\mathfrak{P}(h,k)\circ\psi^n{}_{Id(f)}$.
If diagram #3.7.b is evaluated on $f \in$ Object $\mathfrak{C}_T$ the result is a morphism $(\beta_X(f),\beta_Y(f)):\alpha_{(f)} \longrightarrow \alpha'_{(f)}$ in $\mathfrak{C}_T$. In particular then,

$$\psi_{Id\circ\alpha'(f)}\circ\mathfrak{P}R(\beta_X(f),\beta_Y(f)) = R\mathfrak{P}(\beta_X(f),\beta_Y(f))\circ\psi_{Id\circ\alpha(f)}$$

since $Id\circ\alpha(-) = \alpha(-)$, the above line and lemma Tcom (appendix A) suffice to show that diagram #3.7.b commutes.

∎

The first objective of chapter #3 will have been accomplished when it has been established that,

$$(\Omega^S\psi^S_{(-)})\circ(\varphi^S_{R(-)}) = \varphi^S_{R,(-)} \quad .$$

Proposition #3.8.a For each $s \geq 1$ there exists $\varphi^s_{R,(-)}:\Gamma^s_{R(-)} \longrightarrow \Omega^s R\mathfrak{P}^s_{Id(-)}$ a morphism in $\mathfrak{C}^T$, and a weak equivalence, on all $f \in Obj(\mathfrak{C}_T)$.

Proof; Let $\delta^t_f:\Gamma_{\Omega^t f} \approx \Omega^t \Gamma_f$ be the t-th iterate of the loop-space pull back homeomorphism. By induction, using δ^t one defines $\varphi^s_{R,(-)}:\Gamma^s_{R(-)} \longrightarrow \Omega^s R\mathfrak{P}^s_{(-)}$ case $s = 1$ is already defined, given $\varphi^s_{R,(-)}$ a morphism in $\mathfrak{C}^T$ and a weak equivalence define $\varphi^{s+1}_{R,(-)} =$

$$\Omega^s(\varphi_{R,\mathfrak{P}(\eta_D,\eta_I)})\circ\delta^s_{R\mathfrak{P}^s(\eta_D,\eta_I)}\circ\Gamma(\varphi^s_{R,(-)},\varphi^s_{R,R(-)})$$

Since $\Gamma(\varphi^s_{R,(-)},\varphi^s_{R,R(-)})$ is a weak equivalence by the 5-lemma, therefore $\varphi^{s+1}_{R,(-)}$ is a weak equivalence and a morphism in $\mathfrak{C}^T$.

Proposition #3.8.b) There exists $\varphi^s_{(-)}:\Gamma^s_{Id(-)} \longrightarrow \Omega^s\mathfrak{P}^s_{Id(-)}$ a morphism in $\mathfrak{C}^T$.

Proof, using induction, case $s = 1$ is already defined, given $\varphi^s_{(-)}$ a morphism in $\mathfrak{C}^T$ define $\varphi^{s+1}_{(-)} =$

$\Omega^s(\varphi_{\mathfrak{P}(\eta_D,\eta_I)})\circ\delta^s_{\mathfrak{P}^s(\eta_D,\eta_I)}\circ\mathfrak{P}(\varphi^s_{(-)},\varphi^s_{R(-)})$ as the composition of morphism in $\mathfrak{C}^T$, φ^{s+1} is itself $\in$ Morph$(\mathfrak{C}^T)$.

<u>Lemma</u> #3.9 For each $s\geq 1$, $\Omega^s(\psi^s_{Id(-)})\circ\varphi^s_{R(-)} = \varphi^s_{R,(-)}$,

proof by induction;

the case $s = 1$, is the definition of $\varphi_{R,(-)}$.

Induction

assume that for some s, $\Omega^s(\psi^s_{Id(-)})\circ\varphi^s_{R(-)} = \varphi^s_{R,(-)}$

and consider the following diagram #3.10,

Diagram #3.10

$$
\begin{array}{ccc}
\Gamma^{s+1}_{Rf} & \xrightarrow{\Gamma(\varphi^s_{R,f},\varphi^s_{R,Rf})} & \Gamma_{\Omega^s R\mathfrak{p}^s(\eta_X,\eta_Y)} \\
\downarrow{\scriptstyle \Gamma(\varphi^s_{Rf},\varphi^s_{R^2f})} & \text{I} & \downarrow{\scriptstyle Id} \\
\Gamma_{\Omega^s\mathfrak{p}^s R(\eta_X,\eta_Y)} & \xrightarrow{\Gamma_{\Omega^s}(\psi^s_f,\psi^s_{Rf})} & \Gamma_{\Omega^s R\mathfrak{p}^s(\eta_X,\eta_Y)} \\
\downarrow{\scriptstyle \delta^s_{\mathfrak{p}^s R(\eta_X,\eta_Y)}} & \text{II} & \downarrow{\scriptstyle \delta^s_{R\mathfrak{p}^s(\eta_X,\eta_Y)}} \\
\Omega^s\Gamma_{\mathfrak{p}^s_{R(\eta_X,\eta_Y)}} & \xrightarrow{\Omega^s\Gamma(\psi^s_f,\psi^s_{Rf})} & \Omega^s\Gamma_{R\mathfrak{p}^s_{(\eta_X,\eta_Y)}} \\
\downarrow{\scriptstyle \Omega^s\varphi_{\mathfrak{p}^s_{R(\eta_X,\eta_Y)}}} & \text{III} & \downarrow{\scriptstyle \Omega^s\varphi_{R\mathfrak{p}^s_{(\eta_X,\eta_Y)}}} \\
\Omega^s\mathfrak{p}_{\mathfrak{p}^s_{R(\eta_X,\eta_Y)}} & \xrightarrow{\Omega^s\mathfrak{p}(\psi^s_f,\psi^s_{Rf})} & \Omega^s R\mathfrak{p}^s_{(\eta_X,\eta_Y)} \\
\downarrow{\scriptstyle =} & \text{IV} & \downarrow{\scriptstyle \Omega^s(\psi\mathfrak{p}^s(\eta_X,\eta_Y))} \\
\Omega^{s+1}\mathfrak{p}^{s+1}_{Rf} & \xrightarrow{\Omega^{s+1}(\psi^{s+1}_f)} & \Omega^{s+1}R\mathfrak{p}^{s+1}_f
\end{array}
$$

In diagram #3.10 square I commutes by induction hypothesis on dimension s, while square II commutes in a straightforward way using the definition of the homeomorphism δ^s .

Square III commutes because $\varphi_{Id(-)} \in Morph(\mathfrak{C}^T)$ and square IV commutes apriori of ψ_f^{s+1} .

The left vertical side of diagram #3.10 represents φ_f^{s+1} and because $\Omega\psi_f \circ \varphi_{Rf} = \varphi_{R,f}$ the top composed with right side represents $\varphi^{s+1}_{R,(-)}$, therefore the commutativity of diagram #3.10 completes the proof lemma #3.9. ■

Corollary #3.11 For $X \in \mathfrak{T}op*$, tf.sc.ft. with f retractable φ^s_{Rf} is a weak equivalence.

Having completed the first objective in chapter #3 it remains to define $\chi^{s,n}_{(-)} : \mathfrak{P}^s_{R^n(-)} \longrightarrow R^n\mathfrak{L}^s_{Id(-)}$ a morphism in $\mathfrak{C}^T$, and for f retractable a w.e.

This is accomplished in two steps, first a morphism $\gamma^n_{\mathfrak{P}^s(-)} : \mathfrak{P}^s_{R^n(-)} \longrightarrow R^n\mathfrak{P}^s_{Id(-)}$ in $\mathfrak{C}^T$, is constructed and proven to be a weak equivalence for f retractable this is the essence of lemma #3.12.

The second step in defining $\chi^{s,n}_{(-)}$ involves using $\gamma^n_{\mathfrak{P}^s(-)}$ and is accomplished in theorem #3.20. In theorem #3.24 $\chi^{s,n}_{(-)}$ is shown to be a w.e. .

<u>Lemma</u> #3.12 Let $X^s(-) = \mathfrak{P}^s_{(-)}$ thus

$X^{s+1}(-) = \mathfrak{P}_{X^s(\eta_D, \eta_I)}$ there is a morphism in $\mathfrak{C}^T$

$\gamma^n_{X^s(-)} : X^s \circ R^n(-) \longrightarrow R^n \circ X^s(-)$ for all $n \geq 0$, $s \geq 1$,

where $R^0(-) = Id(-)$ with

$\gamma^0_{X^s R^n(-)} = Id_{X^s \circ R^n(-)}$ and

$\gamma^n_{X^1(-)} = \psi^n_{Id(-)}$ s.t.;

#1.a) $\gamma^n_{X^{s+1}(-)} = \psi^n_{X^s(\eta_D, \eta_I)} \circ \mathfrak{P}(\gamma^n_{X^s(-)}, \gamma^n_{X^s \circ R(-)})$

$: X^{s+1} \circ R^n(-) \longrightarrow R^n \circ X^{s+1}(-)$, for $n \geq 1$, $s \geq 1$.

#1.b) For $n \geq 1$, $s \geq 1$, $\gamma^n_{X^s(-)} = R^{n-1}\gamma_{X^s(-)} \circ \gamma^{n-1}_{X^s R(-)}$.

#2) for all $s \geq 1$ and $n = 0$, diagram #3.13 commutes.

\# 3.13

$$\begin{array}{ccc} X^s(-) & \xrightarrow{\quad X^s(\eta_D, \eta_I)(-) \quad} & X^s \circ R(-) \\ {\scriptstyle Id = \gamma^0_{X^s(-)}} \downarrow & & \downarrow {\scriptstyle \gamma^1_{X^s(-)}} \\ X^s(-) & \xrightarrow{\quad \eta_{X^s(-)} \quad} & R \circ X^s(-) \end{array}$$

#3) If $f : X \longrightarrow Y$ is retractable w.r.t. $R(-)$ then for $n \geq 0$, and $s \geq 1$, $\gamma^n_{X^s(f)}$ is a weak equivalence. X,Y tf.sc.ft.

Proof of lemma #3.12,

Condition #1a) Induction on the degree of s, since when $s = 1$, $\gamma^n_{X^1(-)} = \psi^n_{Id(-)}$ the induction begin. Assume that for some fixed s conditions #1a) holds true. From the naturality of $\gamma^n_{(-)}$ one has for all $n \geq 0$, that the following diagram commutes.

Diagram #3.14

$$\begin{array}{ccc} X^s\circ R^n(-) & \xrightarrow{\quad X^s\circ R^n(\eta_D,\eta_I)(-) \quad} & X^s\circ R^{n+1}(-) \\ \Big\downarrow \gamma^n_{X^s}(-) & & \Big\downarrow \gamma^n_{X^s\circ R}(-) \\ R^n\circ X^s(-) & \xrightarrow{\quad R^n\circ X^s(\eta_D,\eta_I)(-) \quad} & R^n\circ X^s\circ R(-) \end{array}$$

Therefore one is able to define $\gamma^n_{X^{s+1}(-)}$ as

$\psi^n_{X^s(\eta_D,\eta_I)}\circ\mathfrak{p}(\gamma^n_{X^s(-)},\gamma^n_{X^s\circ R(-)}):X^{s+1}\circ R^n(-)\longrightarrow R^n\circ X^{s+1}(-)$

By earlier work this is the composition of morphisms in $\mathfrak{C}^T$ and hence itself in $\mathfrak{C}^T$.

Therefore condition #1.a) holds true.

Condition #1.b) Because $\gamma^n_{X^1} = \psi^n_{Id(-)}$, when $s = 1$ condition #1b) is true apriori of

$\psi^n_{Id(-)} = R^{n-1}\psi_{Id(-)}\circ\psi^{n-1}_{R\circ(-)}$.

Using induction on s, assume that #1.b) holds for any $n \geq 1$ with fixed $s > 1$ and consider the following diagram. #3.15

$$
\begin{array}{ccccc}
\mathfrak{p}_{X^sR^n}(\eta_D,\eta_I) & \xrightarrow{\mathfrak{p}(\gamma^{n-1}_{X^sR},\,\gamma^{n-1}_{X^sR^2})} & \mathfrak{p}_{R^{n-1}X^sR}(\eta_D,\eta_I) & \xrightarrow{\mathfrak{p}_{R^{n-1}}(\gamma_{X^s},\,\gamma_{X^sR})} & \mathfrak{p}_{R^nX^s}(\eta_D,\eta_I) \\
 & & \Big\downarrow{\scriptstyle \psi^{n-1}_{X^sR}(\eta_D,\eta_I)} & & \Big\downarrow{\scriptstyle \psi^{n-1}_{RX^s}(\eta_D,\eta_I)} \\
 & & R^{n-1}\mathfrak{p}_{X^sR}(\eta_D,\eta_I) & \xrightarrow{R^{n-1}\mathfrak{p}(\gamma^{n-1}_{X^s},\,\gamma^{n-1}_{X^sR})} & R^{n-1}\mathfrak{p}_{RX^s}(\eta_D,\eta_I) \\
 & & & & \Big\downarrow{\scriptstyle R^{n-1}\psi_{X^s}(\eta_D,\eta_I)} \\
 & & & & R^n\mathfrak{p}_{X^s}(\eta_D,\eta_I)
\end{array}
$$

Where $X^{s+1}\circ R^n(-) = \mathfrak{p}_{X^s}\circ R^n(\eta_D,\eta_I)$, and $R^n\circ X^s(-) = R^n\circ\mathfrak{p}_{X^s}(\eta_D,\eta_I)$. Diagram #3.15 commutes by lemma #3.6, as a result of diagram #3.15 one has for arbitrary $n \geq 1$ and fixed $s \geq 1$;

$\gamma^n_{X^{s+1}}(-) = \psi^n_{X^s}(\eta_D,\eta_I) \circ\mathfrak{p}(\gamma^n_{X^s}, \gamma^n_{X^sR}) =$

$R^{n-1}\psi_{X^s}(\eta_D,\eta_I)\circ\psi^{n-1}{}_{RX^s}(\eta_D,\eta_I)\circ \mathfrak{p}(\gamma^n_{X^s},\gamma^n_{X^sR}) =$

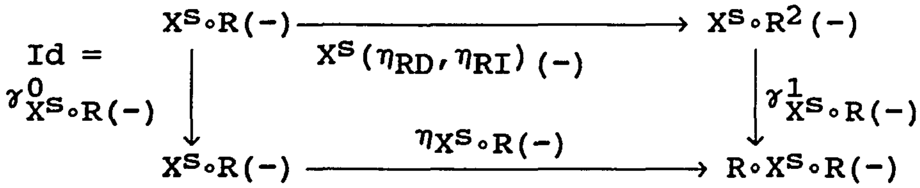
$R^{n-1}\psi_{X^s}(\eta_D,\eta_I)\circ\psi^{n-1}_{X^sR}(\eta_D,\eta_I)\circ\mathfrak{p}_{R^{n-1}}(\gamma_{X^s},\gamma_{X^sR})\circ\mathfrak{p}(\gamma^{n-1}_{X^sR},\gamma^{n-1}_{X^sR^2})$
$=$
$R^{n-1}\psi_{X^s}(\eta_D,\eta_I)\circ R^{n-1}\mathfrak{p}(\gamma_{X^s},\gamma_{X^sR})\circ\psi^{n-1}_{X^sR}(\eta_D,\eta_I)\circ\mathfrak{p}(\gamma^{n-1}_{X^sR},\gamma^{n-1}_{X^sR^2})$
$=$

$R^{n-1}\gamma_{X^{s+1}}\circ\gamma^{n-1}_{X^{s+1}R}$. Thus condition #1b has been verified.

Proof of condition #2

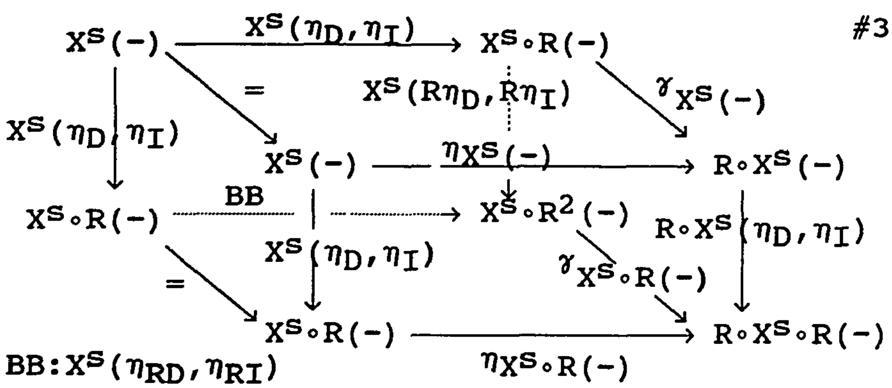
when s = 1 diagram #3.13 reduces to diagram #3.3, which commutes and hence the induction on s begins.

Assume then that for some fixed s, diagram #3.13 commutes, for all f ∈ Top* , in particular for any Rf ∈ Top*. Thus by lemma Tcom in appendix.A, diagram #3.13 induces; #3.16



To prove that diagram #3.13 commutes at dimension s+1 consider the subsequent diagram #3.17.

#3.17



The back of diagram #3.17 commutes because $R\eta\circ\eta = \eta_R\circ\eta$ the front diagram commutes because $\eta_{(-)}$ is a n.t.

while the top commutes by induction hypothesis diagram #3.13 dimension s. The bottom commutes by diagram #3.16 Last the right side commutes by the naturality of $\gamma_{X^s(-)}$ as a morphism in $\mathfrak{C}^T$.

Diagram #3.17 then induces a commutative square #3.18.

#3.18

$$
\begin{array}{ccc}
X^{s+1}(-) & \xrightarrow{X^{s+1}(\eta_D,\eta_I)(-)} & X^{s+1}\circ R(-) \\
\Big\downarrow = & & \Big\downarrow \mathfrak{P}(\gamma_{X^s(-)},\gamma_{X^s\circ R(-)}) \\
X^{s+1}(-) & \xrightarrow{\mathfrak{P}(\eta_{X^s},\eta_{X^sR})(-)} & \mathfrak{P}_{R}\circ X^s(\eta_D,\eta_I) \\
\Big\downarrow = & & \Big\downarrow \psi_{X^s}(\eta_D,\eta_I) \\
X^{s+1}(-) & \xrightarrow{\eta_{X^{s+1}(-)}} & R\mathfrak{P}_{X^s(-)} = RX^{s+1}(-)
\end{array}
$$

The right vertical map is $\gamma^1_{X^{s+1}(-)}$, hence diagram #3.13 commutes at induction dimension s + 1, and condition #2 is proven true.

Proof of condition #3;

When evaluated on a retractable map $f:X \longrightarrow Y$, X,Y tf.sc.ft. $\gamma^n_{X^s}$ is a weak equivalence for all $n \geq 0$.

Note that For such an f, R^nf is also retractable and R^nX, R^nY are both tf.sc.ft .

Proof of condition #3 Double Induction first on the degree of s, second on the degree of n. When s = 1 $\gamma^n_{X^1(-)} = \psi^n_{Id(-)}$ which is a w.e for all $n \geq 1$ by lemma #3.4 while $n = 0$, is trivial.

Primary Induction on the degree of s;
assume that for fixed s, and all $n \geq 0$, $\gamma^n_{X^s(-)}$ is a w.e. , evaluated on f tf.sc.ft and retractable.

$\gamma^n_{X^{s+1}(f)} = \psi^n_{X^s(\eta_D,\eta_I)} \circ \mathfrak{p}(\gamma^n_{X^s(-)}, \gamma^n_{X^s \circ R(-)})$ in particular

$\gamma^1_{X^{s+1}} = \mathfrak{p}(\gamma^1_{X^s}, \gamma^1_{X^s R})$ in order to establish that

$\psi^n_{X^s(\eta_D,\eta_I)}$ is a w.e. by lemma #3.14 it suffices to

prove that $X^s(\eta_D,\eta_I)$ is retractable. By condition #2, $\gamma^1_{X^s(f)} \circ X^s(\eta_D,\eta_I) = \eta_{X^s(-)}$ and since η_{X^s} is retractable, by induction hypothesus on γ^1 so also is $X^s(\eta_D,\eta_I)$.

It remains to be proven that $\gamma^1_{X^{s+1}}$ is a w.e. towards this end, consider the following diagram #3.19;

#3.19

$$\begin{array}{ccccc}
RX^sR & \xrightarrow{RX^sR(\eta_D,\eta_I)} & RX^sR^2 & \xrightarrow{R(j_{X^s}R)} & R(X^{s+1}R) \\
\downarrow{\scriptstyle R\gamma^1_{X^s}} & & \downarrow{\scriptstyle R\gamma^1_{X^s}R} & & \downarrow{\scriptstyle R\mathfrak{p}(\gamma^1_{X^s}, \gamma^1_{X^s R})} \\
R^2X^s & \xrightarrow{R^2X^s(\eta_D,\eta_I)} & R^2X^sR & \xrightarrow{R^2(j_{X^s})} & R^2(X^{s+1})
\end{array}$$

where $j_{X^s}: X^s R \hookrightarrow X^s \wedge I \cup X^s R$ is the canonical inclusion. When one applies the homotopy functor $\pi_*(-)$ to this commutative diagram, the result is two long exact sequences in homology, the 5-lemma and induction hypothesis imply that $R\mathfrak{P}(\gamma^1_{X^s}, \gamma^1_{X^s R})$ is a w.e.

thus by Whiteheads Thm. $\gamma^1_{X^{s+1}} = \mathfrak{P}(\gamma^1_{X^s}, \gamma^1_{X^s R})$ and

$R^n \gamma^1_{X^s}$ are also weak equivalences for all $n \geq 1$.

Hence $\gamma^1_{X^{s+1}R(-)}$ is a w.e., evaluated on f tf.sc.ft and retractable.

Secondary induction on the degree of n

Assume that $\gamma^{n-1}_{X^{s+1}(-)}$ is a weak equivalence when evaluated on f tf.sc.ft. and retractable , then so also is $\gamma^{n-1}_{X^{s+1}R(-)}$ as well as

$\gamma^n_{X^{s+1}(-)} = R^{n-1}\gamma_{X^{s+1}(-)} \circ \gamma^{n-1}_{X^{s+1}} \circ R(-)$ this proves condition #3 and establishes lemma #3.12 ∎

The principle aim of this section is to establish a morphism in $\mathfrak{C}^T$ from the internal push out functor $X^s(-)$ $= \mathfrak{P}^s_{(-)} = \mathfrak{P}^s_{Id(-)}$ where $Id: D(-) \longrightarrow I(-)$ to the iterated external push out $Y^s(-) = \mathfrak{g}^s_{(-)}$ defined by $\mathfrak{g}^1_{(-)} = \mathfrak{P}(-)$ and $\mathfrak{g}^n_{(-)} = \mathfrak{P}_{\eta \mathfrak{g}^{n-1}(-)}$.

<u>Theorem</u> #3.20 For each $s \geq 1$, there is a sequence of morphisms $\chi^{s,n}(-):X^s \circ R^n(-) \longrightarrow R^n \circ Y^s(-)$ for $n \geq 0$, in $\mathfrak{C}^T$, which commute with the respective supporting maps i.e. one has;

<u>Equation</u> #3.21 $\chi^{s,n+1}_{(-)} \circ (X^s \circ R^n(\eta_D, \eta_I)) = (R^n \circ \eta_{Y^s}) \circ \chi^{s,n}(-)$

Proof by induction first on the degree s;

For $s = 1$, $n \geq 0$, $\chi^{1,n} = \psi^n_{(-)}$

and thus by lemma #3.3 the induction begins.

Assume then for some fixed s, that $\chi^{s,0}(-)$ exists

define $\chi^{s,n}_{(-)} = R^n \chi^{s,0}_{(-)} \circ \gamma^n_{X^s(-)}$

diagram #3.13 implies that $\gamma^1_{X^s} \circ X^s(\eta_D, \eta_I) = \eta_{X^s}$ and hence that;

$\eta_{Y^s} \circ \chi^{s,0} = R\chi^{s,0} \circ \eta_{X^s} = R\chi^{s,0} \circ \gamma^1_{X^s} \circ X^s(\eta_D, \eta_I) =$

$\chi^{s,1} \circ X^s(\eta_D, \eta_I)$.

The first equality is simply the naturality of the Hurewicz map, thus equation #3.21 holds with $n = 0$.

Proof of equation #3.21 and lemma #3.20 continued because of condition #1b) $\gamma^n_{X^s(-)} = R^{n-1}\gamma_{X^s(-)} \circ \gamma^{n-1}_{X^s R(-)}$ from lemma #3.12 and condition #2, that is diagram #3.13 one has the following commutative diagram.

#3.21

$$\begin{array}{ccc}
X^s\circ R^n(-) & \xrightarrow[X^s\circ R^n(\eta_D,\eta_I)(-)]{} & X^s\circ R^{n+1}(-) \\
\Big\downarrow \gamma^n_{X^s}(-) & & \Big\downarrow \gamma^n_{X^s}\circ R(-) \\
R^n\circ X^s(-) & \xrightarrow{R^n\circ X^s(\eta_D,\eta_I)(-)} & R^n\circ X^s\circ R(-) \\
\Big\downarrow R^n\circ Id(-) & & \Big\downarrow R^n\circ\gamma^1_{X^s}(-) \\
R^n\circ X^s(-) & \xrightarrow{R^n\circ\eta_{X^s}(-)} & R^{n+1}\circ X^s(-)
\end{array}$$

the commutativity of diagram #3.21 implies that

$\gamma^{n+1}_{X^s}\circ X^s R^n(\eta_D,\eta_I) = R^n\eta_{X^s}\circ\gamma^n_{X^s}$,

taking $\chi^{s,n}(-) = R^n\chi^{s,0}\circ\gamma^n_{X^s}$ one then has the following

diagram #3.23 commutes.

#3.23

$$\begin{array}{ccc}
X^s\circ R^n(-) & \xrightarrow[X^s\circ R^n(\eta_D,\eta_I)(-)]{} & X^s\circ R^{n+1}(-) \\
\Big\downarrow \gamma^n_{X^s}(-) & & \Big\downarrow \gamma^{n+1}_{X^s}\circ R(-) \\
R^n\circ X^s(-) & \xrightarrow{R^n\circ\eta_{X^s}(-)} & R^{n+1}X^s(-) \\
\Big\downarrow R^n\circ\chi^{s,0}_{(-)} & & \Big\downarrow R^{n+1}\circ\chi^{s,0}_{(-)} \\
R^n\circ Y^s(-) & \xrightarrow{R^n\circ\eta_{Y^s}(-)} & R^{n+1}\circ Y^s(-)
\end{array}$$

Diagram #3.23 establishes equation #3.21 for general n.

that is $\chi^{s,n+1}\circ X^s R^n(\eta_D,\eta_I) = R^n\eta_{Y^s}\circ\chi^{s,n}\circ\chi^{s,n}$.

In order to complete the induction it remains to establish the existence of a morphism $\chi^{s+1,0}$ in $\mathfrak{C}^T$.

Towards this end, consider the subsequent diagram #3.24 which results from equation #3.21 when n = 0;

$$\begin{array}{ccc} X^{s}(-) & \xrightarrow[X^{s}(\eta_D,\eta_I)]{} & X^{s}R(-) \\ \Big\downarrow \chi^{s,0} & & \Big\downarrow \chi^{s,1} \\ Y^{s}(-) & \xrightarrow{\eta_{Y^s}} & RY^{s}(-) \end{array} \qquad \#3.24$$

Define $\chi^{s+1,0} = \mathfrak{P}(\chi^{s,0},\chi^{s,1}):X^{s+1}(-) \longrightarrow Y^{s+1}(-)$ and Theorem #3.20 has been established ∎

<u>Theorem</u> #3.24 The map $\chi^{s,n}_{(-)}$ is a w.e. $s\geq 1$, $n \geq 0$.

Proof; First for $\chi^{s,0}_{(-)}$ by induction on the degree of s, the case s = 1 is $\chi^{1,0} = Id:\mathfrak{P}_{(-)} \longrightarrow \mathfrak{L}_{(-)} = \mathfrak{P}_{(-)}$ and $\chi^{1,n} = \psi^{n}_{f}$ which are both weak equivalences.

Induction on arbitrary s, assume that for some fixed s and all n = 0, $\chi^{s,0}$ is a weak equivalence, then $\chi^{s,n}_{(-)} = R^{n}\chi^{s,0}_{(-)}\circ\gamma^{n}_{X^{s}(-)}$ is a w. e. because both $\chi^{s,0}$ and $\gamma^{n}_{X^{s}}$ are. Since $\chi^{s,n}$, $n \geq 0$, is a weak equivalences by Whitehead theorem so also is $R(\chi^{s,n})$, and therefore $R\mathfrak{P}(\chi^{s,n},\chi^{s,n+1})$ is a weak equivalence by the 5-lemma and thus $\chi^{s+1,0} = \mathfrak{P}(\chi^{s,0},\chi^{s,1})$ is a weak equivalence and since $\chi^{s+1,n}_{(-)} = R^{n}\chi^{s+1,0}_{(-)}\circ\gamma^{n}_{X^{s+1}(-)}$ the induction is complete. ∎

Appendix. A

Recall $\mathfrak{C}^T$ the functor category with objects functors $X(-):\mathfrak{C}_T \longrightarrow$ Top* , thus a morphism in $\mathfrak{C}^T$ is a n.t. $\alpha_{(-)}:X(-) \longrightarrow Y(-)$ s.t. $\alpha_{(f)}:X(f) \longrightarrow Y(f)$ is a continuous map.

The most significant work of appendix.A is first proposition A.12 where $\Gamma_{Id(-)}$, $\mathfrak{P}_{Id(-)}$ and $C(-)$ are established to be objects in $\mathfrak{C}^T$, this is then generalized into theorem A.13, that $\mathfrak{C}^T$ has mapping cylinders, mapping cones and cones, and second the iterated functor constructions (A.27) , (A.30).

The following work establishes some corollaries of prop.A.12 which have been used throughout this treatise.

Proposition A.1 Let $F(-):\text{Top*} \longrightarrow \text{Top*}$ be an endofunctor, and $X(-) \in \text{Obj}(\mathfrak{C}^T)$ then $F \circ X(-) \in \text{Obj}(\mathfrak{C}^T)$.
Proof; For $f , f' \in \text{Obj}(\mathfrak{C}_T)$ and $(h,k):f \longrightarrow f'$ $\in \text{Morph}(\mathfrak{C}_T)$ let $F \circ X(f) = F(X(f)) \in \text{Obj}(\mathfrak{Top*})$ and $F \circ X(h,k) = F(X(h,k)):F(X(f)) \longrightarrow F(X(f'))$ $\in \text{Morph}(\mathfrak{Top*})$.

Corollary A.2 $R \circ \mathfrak{P}_{Id(-)}$, $R \circ \Gamma_{Id(-)}$ are both $\in \text{Obj}(\mathfrak{C}^T)$.
Proof, use proposition A.12 and prop. A.1 ∎

Proposition A.3,(Tcom) Given morphisms in $\mathfrak{C}^T$

$\alpha_{(-)}:X(-)\longrightarrow X'(-)\quad \gamma_{(-)}:Y(-)\longrightarrow Y'(-)$,

$\beta_1(-):X(-)\longrightarrow Y(-)$ and $\beta_2(-):X'(-)\longrightarrow Y'(-)$

in order to prove commutativity that is

$\beta_2(-)\circ\gamma_{(-)} = \beta_1(-)\circ\alpha_{(-)}:X(-)\longrightarrow Y'(-)$

it suffices to establish this on arbitrary $f \in Obj(\mathfrak{C}_T)$

$\beta_2(f)\circ\gamma_{(f)} = \beta_1(f)\circ\alpha_{(f)}$ and thus as morphisms

in Top*, and the corresponding result is true in $\mathfrak{C}^T$.

This is true apriori in the category $\mathfrak{C}^T$.

Proposition A.4 For $X(-) \in \mathfrak{C}^T$ and $F(-)$ an endofunctor of Top* then $X\circ F(-) \in obj(\mathfrak{C}^T)$. An endofunctor $F(-)$ for Top* can be considered an endofunctor for $\mathfrak{C}_T$, given on $f:X\longrightarrow Y \in Obj(\mathfrak{C}_T)$ by $F(f):F(X)\longrightarrow F(Y)$, and on $(h,k):f\longrightarrow f'$, a morphism in $\mathfrak{C}_T$, by $(F(h),F(k)):F(f)\longrightarrow F(f')$.

Corollary #A.5 $\mathfrak{P}_Rn_{(-)}$ and $\Gamma_Rn_{(-)}$ are both $\in Obj(\mathfrak{C}^T)$.

proof,

proposition A.4 together with A.12 combine to prove A.5.

∎

<u>Proposition</u> #A.6 Let $F(-)$ be an endofunctor of Top* and $\alpha(-):X(-)\longrightarrow Y(-)$ a morphism in $\mathfrak{C}^T$.

Then $F\circ\alpha(-):F\circ X(-)\longrightarrow F\circ Y(-)$ is a morphism in $\mathfrak{C}^T$.

Proof;

Let $(h,k):f\longrightarrow f'$ be a morphism between objects in $\mathfrak{C}_T$

because $\alpha_{(-)} \in \mathrm{Morph}(\mathfrak{C}^T)$

$\alpha_{(f')}\circ X(h,k) = Y(h,k)\circ\alpha_{(f)}$, while

$(F\circ\alpha_{(f')})\circ(F\circ X(h,k)) = F\circ(\alpha_{(f')}\circ X(h,k))$

by naturality of F

$= F\circ(Y(h,k)\circ\alpha_{(f)})$, by naturality of $\alpha_{(-)}$

$= (F\circ Y(h,k))\circ F(\circ\alpha_{(f)})$, by naturality of F.

∎

<u>Corollary</u> #A.7 The map $R\circ Id(-) = Id\circ R(-):R\circ D(-)\longrightarrow R\circ I(-)$, is a morphism in $\mathfrak{C}^T$.

<u>Proposition</u> #A.8 For F an endofunctor of $\mathfrak{C}_T$, $\alpha_{(-)}:X(-)\longrightarrow Y(-)$ a morphism in $\mathfrak{C}^T$, then $\alpha_{F(-)}:X\circ F(-)\longrightarrow Y\circ F(-) \in \mathrm{Morph}(\mathfrak{C}^T)$.

Proof of proposition #A.8.

because $\alpha \in Morph(\mathfrak{C}^T)$ and $F(f) \in Top*$ for all f $\in Obj(\mathfrak{C}_T)$ one has $\alpha_{(Ff')} \circ X(Fh,Fk) = Y(Fh,Fk) \circ \alpha_{(Ff)}$, and apriori $F(h,k) = (Fh,Fk)$. ∎

<u>Corollary</u> #A.9 $\mathfrak{P}(R^n\eta_D, R^n\eta_I) : \mathfrak{P}_{R^n\eta_{D(-)}} \longrightarrow \mathfrak{P}_{R^n\eta_{I(-)}}$ and $\Gamma(R^n\eta_{D(-)}, R^n\eta_{I(-)}) : \Gamma_{R^n\eta_{D(-)}} \longrightarrow \Gamma_{R^n\eta_{I(-)}}$ are both morphisms in the category $\mathfrak{C}^T$.

Proof, use prop. A.12 and A.8. ∎

<u>Proposition</u> #A.10 Let $F(-)$, $F'(-)$ be endofunctors and $g_{(-)} : F(-) \longrightarrow F'(-)$ be a n.t. between them , s.t. $g_{(X)} \in Morph(\mathfrak{T}op*)$ for arbitrary X.

If $X(-) \in Obj(\mathfrak{C}^T)$ then $g_{X(-)} : F \circ X(-) \longrightarrow F' \circ X(-)$ $\in Morph(\mathfrak{C}^T)$ is defined on arbitrary $f \in \mathfrak{T}op*$ by $g_{X(-)} \circ (f) = g_{X(f)}$

Proof, If $(h,k) : f \longrightarrow f'$ is a morphism in $\mathfrak{C}_T$, then $X(h,k) : X(f) \longrightarrow X(f')$ is a morphism in Top* , by the naturality of g , one has a commutative diagram;

#A.10 b

$$\begin{array}{ccc} F\circ X(f) & \xrightarrow{\quad g_{X(f)}\quad} & F'\circ X(f) \\ \downarrow{\scriptstyle F\circ X(h,k)} & & \downarrow{\scriptstyle F'\circ X(h,k)} \\ F\circ X(f') & \xrightarrow{\quad g_{X(f')}\quad} & F'\circ X(f') \end{array}$$

which implies the naturality requirement required to establish $g_{X(-)}$ as a morphism in $\mathfrak{C}^T$. ∎

<u>Corollary</u> #A.11 The Hurewicz map $\eta_{(-)}:Id(-) \longrightarrow R(-)$, where $Id(-)$, $R(-)$ are viewed as endofunctors in $\mathfrak{T}op*$ induces for $X(-) \in Obj(\mathfrak{C}^T)$,

$\eta_{X(-)}:X(-) \longrightarrow R\circ X(-) \in Morph(\mathfrak{C}^T)$.

The delooping map $\sigma^{\wedge}_{R(-);Y}:R(-)\wedge Y \longrightarrow R\circ(Id(-)\wedge Y)$, introduced in appendix B, between endofunctors of Top* induces for $X(-) \in Obj(\mathfrak{C}^T)$ a morphism

$\sigma^{\wedge}_{R\circ X(-);Y}:R\circ X(-)\wedge Y \longrightarrow R\circ(X(-)\wedge Y)$ in $\mathfrak{C}^T$.

<u>Proposition</u> #A.12 $\Gamma_{Id(-)}$, $\mathfrak{P}_{Id(-)}$ and $C(-)$ represent objects in the category $\mathfrak{C}^T$.

Proof; If $(Id_{D(f)},Id_{I(f)}):f \longrightarrow f$ is the identity morphism in $\mathfrak{C}_T$, then

$\mathfrak{P}^{\wedge}(Id_{D(f)},Id_{I(f)}) = Id_{D(f)}\wedge I \cup_f Id_{I(f)} = Id_{\mathfrak{P}^{\wedge}_f}$

while $\Gamma(Id_{Df},Id_{If}) = Id_{D(f)}\ {}_{f}x_{\tau}\ P(Id_{If}) = Id_{\Gamma_f}$ and similarly $C(Id_{Df},Id_{If}) = Id_{C(f)}$.

The next step is to prove naturality;

given $(h'',k'') = (h',k')\circ(h,k):f \longrightarrow f' \longrightarrow f''$

morphisms in $\mathfrak{C}_T$, $\mathfrak{P}^{\wedge}(h',k')\circ\mathfrak{P}^{\wedge}(h,k) =$

$(h'^{\wedge}I \cup_{f'} k')\circ(h^{\wedge}I \cup_f k) =$

$(h'\circ h^{\wedge}I \cup_{f''} k'\circ k) = h''^{\wedge}I \cup_{f''} k'' = \mathfrak{P}^{\wedge}(h'',k'')$

while $\Gamma(h',k')\circ\Gamma(h,k) = \Gamma(h,k) =$

$(h'\ {}_{f'}x_{\tau}\ Pk')\circ(h\ {}_{f}x_{\tau}\ Pk) =$

$h'\circ h\ {}_{f''}x_{\tau}\ P(k'\circ k) = (h''\ {}_{f''}x_{\tau}\ Pk'') = \Gamma(h'',k'')$

similarly $C(h'',k'') = C(h',k')\circ C(h,k)$. ∎

<u>Theorem</u> A.13 The category $\mathfrak{C}^T$ has pullbacks, mapping cones, mapping cylinders and mapping cones.

<u>Lemma</u> #A.14 $\Gamma_{Id(-)}$ is a pullbacks in $\mathfrak{C}^T$ while $\mathfrak{P}^{\wedge}_{Id(-)}$ and $C(-)$ are push outs in $\mathfrak{C}^T$.

Proof; Beginning with $\Gamma_{Id(-)}$

Let $\tau_{(-)} = \tau^X_{(-)}:P\circ X(-) \longrightarrow X(-)$ suppose that

$(\alpha(-):Z(-) \longrightarrow D(-),\ \beta(-):Z(-) \longrightarrow P\circ I(-)\ )$ are a pair of morphisms in $\mathfrak{C}^T$ with $\tau\circ\beta = Id(-)\circ\alpha(-)$,

<u>Lemma</u> #A.14 $\Gamma_{Id(-)}$ is a pullbacks in $\mathfrak{C}^T$ while $\mathfrak{P}^{\wedge}_{Id(-)}$ and $C(-)$ are pushouts in $\mathfrak{C}^T$.

Proof; Beginning with $\Gamma_{Id(-)}$

Let $\tau_{(-)} = \tau^X_{(-)}: P\circ X(-) \longrightarrow X(-)$ suppose that

$(\alpha(-): Z(-) \longrightarrow D(-),\ \beta(-): Z(-) \longrightarrow P\circ I(-)\ )$ are

a pair of morphisms in $\mathfrak{C}^T$ with $\tau\circ\beta = Id(-)\circ\alpha(-)$, thus for each $f \in Obj(\mathfrak{C}_T)$ one has $\tau\circ\beta_f = f\circ\alpha_f$, by the universal property of the pullback Γ_f , there exists $\theta_f: Z(f) \longrightarrow \Gamma_f$ with $p^{\Gamma}_{If}\circ\theta_f = \beta_f$ and $p^{\Gamma}_{Df}\circ\theta_f = \alpha_f$.

In order to complete the proof it remains to establish that θ is natural on pairs of maps, or morphisms $(h,k): f \longrightarrow f'$ in $\mathfrak{C}_T$. This is equivalent to the commutativity of #A.15

A.

$$\begin{array}{ccc} Z(f) & \xrightarrow{\ \theta_f\ } & \Gamma_{Id(f)} \\ \downarrow{\scriptstyle Z(h,k)} & & \downarrow{\scriptstyle \Gamma(h,k)} \\ Z(f') & \xrightarrow{\ \theta_{f'}\ } & \Gamma_{Id(f')} \end{array}$$

Because $\Gamma_{Id(f')} = \Gamma_{f'}$ is a pullback, in order to prove that diagram A.15 commutes it suffices to show that

a) $(p_{If'})\circ\Gamma(h,k)\circ\theta_f = (p_{If'})\circ(\theta_{f'})\circ Z(h,k)$ and that

b) $(p_{Df'})\circ\Gamma(h,k)\circ\theta_f = (p_{Df'})\circ(\theta_{f'})\circ Z(h,k)$

$= (\beta_{f'}) \circ Z(h,k)$ by diagram A.16

$= (p^{\Gamma}_{If'}) \circ (\theta_{f'}) \circ Z(h,k)$ because $\Gamma_{f'}$ is a pullback

Therefore condition a) has been established.

In order to prove condition b) one has;

$p^{\Gamma}_{Df'} \circ \Gamma(h,k) \circ (\theta_f) = D(h,k) \circ (p^{\Gamma}_{Df}) \circ (\theta_f)$ by diagram A.19

$= D(h,k) \circ \alpha_f\hat{}$ by property of pullback Γ_f

$= \alpha_{f'} \circ Z(h,k)$ by diagram A.17

$= (p^{\Gamma}_{Df}) \circ \Gamma(h,k) \circ \theta_f$ by property of pullback Γ_f .

To prove that $\mathfrak{P}^{\wedge}_{Id(-)}$ is a push out in $\mathfrak{C}^T$, with $j_{D\wedge I}: D(-)\wedge I \longrightarrow \mathfrak{P}^{\wedge}_{(-)}$, $j_{I(-)}: I(-) \longrightarrow \mathfrak{P}^{\wedge}_{(-)}$ and $i^1_{D(-)}: D(-) \longrightarrow D\wedge I$ the canonical inclusions, all morphisms in $\mathfrak{C}^T$. Let $\alpha(-): D(-)\wedge I \longrightarrow Z(-)$ and $\beta(-): I(-) \longrightarrow Z(-)$ be morphisms in $\mathfrak{C}^T$, with $\alpha(-) \circ j_{D\wedge I} = \beta(-) \circ j_{I(-)}$. Evaluated on each $f \in Obj(\mathfrak{C}_T)$ by the universal property of push outs one has $\theta_f: \mathfrak{P}^{\wedge}_f \longrightarrow Z(f)$ with $\theta_f \circ j_{If} = \beta_f$ and $\theta_f \circ j_{Df\wedge I} = \alpha_f$.

In order to prove that $\hat{\mathfrak{P}}_{Id(-)}$ is a push out in the category of $\mathfrak{C}^T$, one must establish that θ is a morphism in $\mathfrak{C}^T$ that is $Z(h,k)\circ\theta_f = (\theta_{f'})\circ\hat{\mathfrak{P}}_{(h,k)}$. Because $\hat{\mathfrak{P}}_{Id(-)}$ is a push out this is equivalent to the following two equalities.

condition a) $Z(h,k)\circ\theta_f\circ j_{If} = (\theta_{f'})\circ\hat{\mathfrak{P}}_{(h,k)}\circ j_{If}$

$Z(h,k)\circ\theta_f\circ j_{If} = Z(h,k)\circ\beta_f$ by property of push out

$= (\beta_{f'})\circ I(h,k)$ by naturality of β

$= (\theta_{f'})\circ j_{If'}\circ I(h,k)$ by property of push out

$= (\theta_{f'})\circ\hat{\mathfrak{P}}_{(h,k)}\circ j_{If}$ because j_I is a morphism in $\mathfrak{C}^T$.

Condition b) $Z(h,k)\circ\theta_f\circ j_{D\sim I} = (\theta_{f'})\circ\hat{\mathfrak{P}}_{(h,k)}\circ j_{D\sim I}$

$Z(h,k)\circ\theta_f\circ j_{D\sim I} = Z(h,k)\circ\alpha_{(f)}$ by property of push out

$= \alpha_{(f')}\circ D(h,k)$ by naturality of α

$= (\theta_{f'})\circ j_{D\sim I}\circ D(h,k)$ by property of push out

$= (\theta_{f'})\circ\hat{\mathfrak{P}}_{(h,k)}\circ j_{D\sim I}$ naturality of $j_{D\sim I}$

A similar proof establishes that C(-) is a push out in $\mathfrak{C}^T$.

Lemma A.14 is now proven true. ∎

The proof of theorem #A.13 continues with lemma# A.20

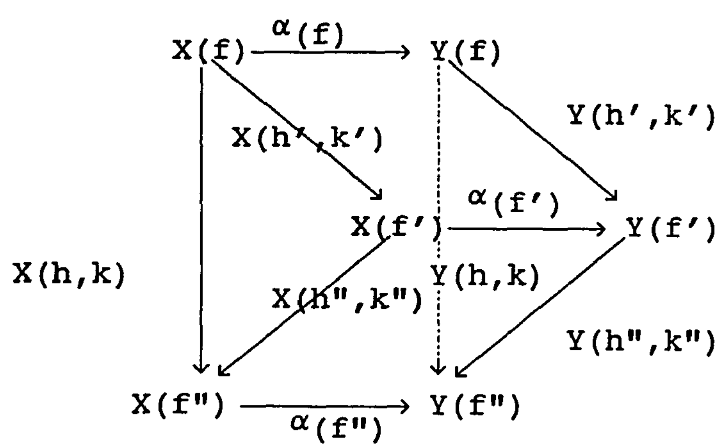
<u>Lemma</u> #A.20 Let $\alpha(-):X(-) \longrightarrow Y(-)$ be a morphism between objects in $\mathfrak{C}^T$, let $H(-) \in Obj(\mathfrak{C}^T)$.

Then $H(-)$ and $\alpha_{(-)}$ define a functor $H\circ\alpha_{(-)} \in Obj(\mathfrak{C}^T)$, given on $f \in Obj(\mathfrak{C}_T)$ by $H\circ\alpha_{(f)} = H(\alpha_{(f)})$ and on morphisms $(h,k):f \longrightarrow f'$ in $\mathfrak{C}_T$ by;

$H\circ\alpha_{(h,k)} = H(X(h,k),Y(h,k)):H\circ\alpha_{(f)} \longrightarrow H\circ\alpha_{(f')}$

Proof;

Let $f, f', f" \in Obj(\mathfrak{C}_{Top*})$ with $(h,k):f \longrightarrow f"$, $(h',k'):f \longrightarrow f'$, $(h",k"):f' \longrightarrow f"$, in $Morph(\mathfrak{C}_T)$ and $(h,k) = (h",k")\circ(h',k')$ as in the following commutative diagram. #A.21



The left and right triangular sides of diagram #A.21 commute because both $X(-)$ and $Y(-)$ are functors objects in $\mathfrak{C}^T$.

The remaining rectangular faces all commute because α is a n.t.. From diagram #A.21 one concludes that

$H\alpha_{(h',k')} \circ H\alpha_{(h'',k'')} =$

$H(X(h',k'),Y(h',k')) \circ H(X(h'',k''),Y(h'',k'')) =$

$H(X(h,k),Y(h,k)) = H\alpha_{(h,k)}$,

this completes the proof of lemma #A.20. ∎

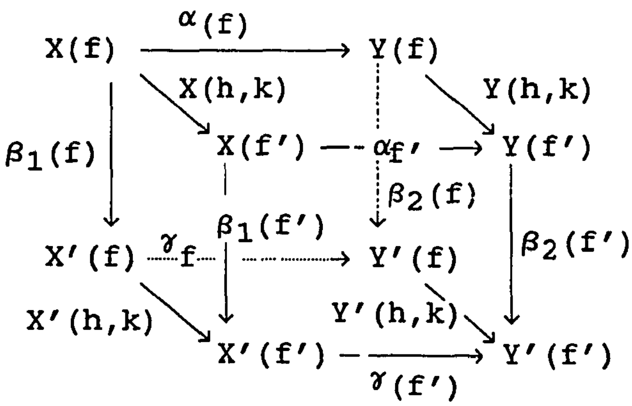
In order to complete the proof of theorem #A.13 observe that for $\alpha(-):X(-) \longrightarrow Y(-)$ in $\mathfrak{C}^T$, since α_f is a morphism in $\mathfrak{T}op*$ the objects $\mathfrak{P}^{\wedge}_{\alpha(f)} = \mathfrak{P}^{\wedge}_{Id(\alpha_f)}$, $\Gamma_{\alpha(f)} = \Gamma_{Id(\alpha_f)}$ and $C \circ \alpha(f) = C(\alpha_f)$ are respectively mapping cylinders, pullbacks and cones in $\mathfrak{C}^T$.

∎

<u>Proposition</u> A.22 Suppose that $X(-), X'(-), Y(-), Y'(-)$ and $H(-)$ are objects in $\mathfrak{C}^T$ with $\beta_1(-):X(-) \stackrel{\cdot}{\longrightarrow} X'(-)$, $\beta_2(-):Y(-) \stackrel{\cdot}{\longrightarrow} Y'(-)$, $\alpha(-):X(-) \stackrel{\cdot}{\longrightarrow} Y(-)$ and $\gamma(-):X'(-) \stackrel{\cdot}{\longrightarrow} Y'(-)$ morphisms in $\mathfrak{C}^T$, s.t. this diagram commutes #A.23

$$\begin{array}{ccc} X(-) & \xrightarrow{\beta_1(-)} & X'(-) \\ \alpha(-) \downarrow & & \downarrow \gamma(-) \\ Y(-) & \xrightarrow{\beta_2(-)} & Y'(-) \end{array}$$

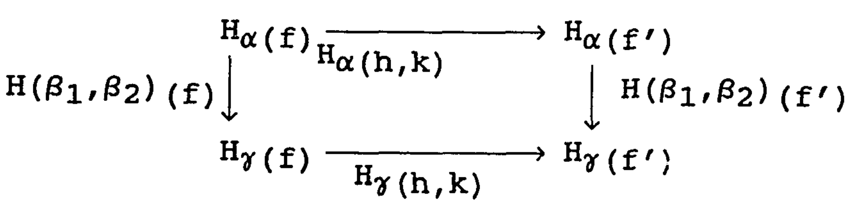
Then diagram A.23 defines a n.t. $H(\beta_1,\beta_2)_{(-)} = H(\beta_1(-),\beta_2(-)):H\alpha_{(-)} \longrightarrow H\gamma_{(-)}$, that $H(\beta_1,\beta_2)$ exists is straight forward what needs to be proven is the naturality. From #A.23 with $(h,k):f \longrightarrow f'$ one has the following diagram #A.24.



A.24

By assumption, diagram #A.23 and hence the back and front diagrams of #A.24 commute. Because α, γ, β_1 and β_2 are all n.t.'s all faces of diagram #A.24 commute, and one obtains the following diagram A.25.

#A.25



This completes the proof that $H(\beta_1,\beta_2)_{(-)}$ is a n.t.. □

The internal Hurewicz map

Corollary #A.26 Let $\eta_{D(-)}$, $\eta_{I(-)}$ be the two morphisms in $\mathbb{C}^T$ given by the Hurewicz map and Proposition A.10. For any $H(-) \in Obj(\mathbb{C}^T)$ Proposition A.22 implies for $H(-) = H \circ Id(-)$ and $H_{R(-)} = H \circ R(-)$ given by proposition A.4 the existence of the internal Hurewicz map

$H(\eta_D, \eta_I)_{(-)} : H(-) \longrightarrow H \circ R(-) \in Morph(\mathbb{C}^T)$.

The internal iterated functor construct

Theorem itfun.$\mathbb{C}$ (#A.27).

If $H(-) \in Obj(\mathbb{C}^T)$ and $\alpha(-) \in Morph(\mathbb{C}^T)$ there exists for each s; $H^s_{\alpha(-)} = H^s_{\alpha(-)} / (\eta_D, \eta_I) \in Obj(\mathbb{C}^T)$ the iterated internal functor defined inductively over The pair $(\eta_D, \eta_I) : Id(-) \longrightarrow R(-)$.

Proof, or more accurately definition.

Let $H(-)$, $X(-)$, $Y(-) \in Obj(\mathbb{C}^T)$ and let $\alpha_{(-)} : X(-) \longrightarrow Y(-) \in Morph(\mathbb{C}^T)$. Because $(\eta_D, \eta_I) : Id(-) \longrightarrow R(-)$ can be viewed, for arbitrary $f \in Obj(\mathbb{C}^T)$ as the morphism $(\eta_{Df}, \eta_{If}) : f \longrightarrow Rf$ in $\mathbb{C}^T$.

One has the following commutative diagram;

#A.28

$$\begin{array}{ccc} X(-) & \xrightarrow{X(\eta_D,\eta_I)} & X\circ R(-) \\ \downarrow{\alpha(-)} & & \downarrow{\alpha_R(-)} \\ Y(-) & \xrightarrow{Y(\eta_D,\eta_I)} & Y\circ R(-) \end{array}$$

Where each map is a morphism in $\mathfrak{C}^T$, this implies that $H_\alpha(\eta_D,\eta_I):H_\alpha(-)\longrightarrow H_{\alpha R}(-)$ exists and is a morphism in $\mathfrak{C}^T$. Using an induction argument suppose that

$H^s_{\alpha(-)} = H^s_{\alpha(-)} \;/\; (\eta_D,\eta_I)$ has been defined and is an element of Obj$(\mathfrak{C}^T)$ then proposition A.22 applied to diagram A.28 implies $H^s_{\alpha(\eta_D,\eta_I)}:H^s_{\alpha(-)}\longrightarrow H^s_{\alpha R(-)}$ exists and is a morphism in $\mathfrak{C}^T$. Define

$H^{s+1}_{\alpha(-)} = H(H^s_{\alpha(\eta_D,\eta_I)})$, in general

$H^{s+1}_{\alpha R^t(-)} = H^{s+1}_{\alpha R^t(-)}/(R^t\eta_D,R^t\eta_I)$ is defined as,

$H(H^s_{\alpha R^t(\eta_D,\eta_I)}) = H(H^s_{\alpha(R^t\eta_D,R^t\eta_I)})$.

<u>Corollary</u> A.29 For $\alpha(-):X(-)\longrightarrow Y(-)$ a morphism in $\mathfrak{C}^T$ $\mathfrak{P}^s_{\alpha(-)}$ resp.. $\Gamma^s_{\alpha(-)}$ both exist as the s-th internal iterated push out under $\alpha(-)$ and respectively the s-th internal iterated pullback over $\alpha(-)$, both are objects in the category $\mathfrak{C}^T$.

The External iterated functor construct

Proposition A.30 For each $s \geq 0$, there exist an object in $\mathfrak{C}^T$ $\mathfrak{L}^s_{Id(-)}$ defined inductively as the push out under the external Hurewicz map.

Proof;

By corollary A.11 and A.7 the Hurewicz map

$\eta_{Id(-)}: Id(-) \longrightarrow R \circ Id(-)$ is a morphism in $\mathfrak{C}^T$,

let $\mathfrak{L}_{(-)} = \mathfrak{L}_{Id(-)} = \mathfrak{P}_{Id(-)}$ which is an object in $\mathfrak{C}^T$

by prop. #A.12 , proceeding by induction assume

$\mathfrak{L}^{s-1}_{Id(-)}$ has been defined and is an object in $\mathfrak{C}^T$

then $\eta_{\mathfrak{L}^{s-1}_{Id(-)}}: \mathfrak{L}^{s-1}_{Id(-)} \longrightarrow R \circ \mathfrak{L}^{s-1}_{Id(-)}$ is a morphism in $\mathfrak{C}^T$,

by Cor. #A.11,

let $\mathfrak{L}^s_{Id(-)} = \mathfrak{P}_{\eta_{\mathfrak{L}^{s-1}_{Id(-)}}}$ an object in $\mathfrak{C}^T$ by Thm. #A.13 .

∎

Appendix B

In appendix.B is defined, for $\curlywedge = \Lambda$ or x' a delooping map from $R(X)\curlywedge Y$ to $R(X\curlywedge Y)$ where $R(-) = (\underline{R}\Lambda -)^{\wedge}$, is the infinite loop space associated to a ring spectra $\underline{R}$, and some useful properties are developed, which shall be used throughout this treatise. For $X,Y \in \mathfrak{T}op*$ define, $\sigma_{RX;Y} = \sigma_{\hat{R}X;Y} =$

$$\underset{\overrightarrow{p\ \ \infty}}{\text{colimit}}\ Ad^{p}\circ(Ad^{-p}{}_{\Omega^{p}(R^{p}\curlywedge X)}\curlywedge Y):R(X)\curlywedge Y \longrightarrow R(X\curlywedge Y) \ ,$$

one also has;

$$\sigma_{X;RY} = \sigma_{\hat{X};RY} = \underset{\overrightarrow{p\ \ \infty}}{\text{colimit}}\ Ad^{p}\circ(X\curlywedge Ad^{-p}{}_{\Omega^{p}(R^{p}\curlywedge Y)}):X\curlywedge RY \longrightarrow R(X\curlywedge Y)$$

and all the succeeding work could be done using $\sigma_{X;RY}$. Using the notation that $[x,y] \in X\curlywedge Y$ represents $[x,'y]$ when $\curlywedge = x'$ or simply $[x,y]$ when $\curlywedge = \Lambda$, let $[[r;x]^{\wedge},y]$ represent an element of $R(X)\curlywedge Y$ for $r \in R_p$ in $\underline{R}$,$x \in X$, and $y \in Y$. Then $\sigma_{RX;Y}\circ([[r;x]^{\wedge},y]) = [r;x,y]^{\wedge} \in R(X\curlywedge Y)$.

<u>Property</u> #B.1) Naturality of $\sigma_{RX;Y}$, Given $(h,k):f \longrightarrow f'$ a morphism in $\mathfrak{C}_T$ the following diagram (σ-nat), commutes.

$$\begin{array}{ccc} R(X)\curlywedge Y & \xrightarrow{\ \sigma_{RX;Y}\ } & R(X\curlywedge Y) \\ {\scriptstyle R(h)\curlywedge Y}\downarrow & & \downarrow{\scriptstyle R(h\curlywedge Y)} \\ R(X')\curlywedge Y & \xrightarrow{\ \sigma_{RX';Y}\ } & R(X'\curlywedge Y) \end{array}$$

Proof;

Proof of property #B.1,

$R(h\wedge Y)\circ\sigma_{RX;Y}\circ([[r;x]^{\wedge},y]) \;=\; R(h\wedge Y)\circ[r;x,y]^{\wedge} \;=\; [r;h(x),y]^{\wedge}$, while $\sigma_{RX';Y}\circ(R(h)\wedge Y)([[r;x]^{\wedge},y]) \;=\; \sigma_{RX';Y}\circ([[r;h(x)]^{\wedge},y]) \;=\; [r;h(x),y]^{\wedge}$.

∎

<u>Property</u> #B.2) σ commutes with η, That is the following diagram $(\sigma-\eta)$ commutes;

$$
\begin{array}{ccc}
X\wedge I & \xrightarrow{\;(\eta_X)\wedge I\;} & R(X)\wedge I \\
\downarrow \eta_{X\wedge I} & & \downarrow \sigma^{\wedge}_{RX;I} \\
R(X\wedge I) & \xrightarrow{\;=\;} & R(X\wedge I)
\end{array}
$$

Proof, it suffices by naturality of the maps $(\underline{i}\wedge X)^{\wedge}$ and ρ_X which define η_X to establish that;

<u>Property</u> B.2.a) $\quad \theta^1_X\circ((\rho^1_X)\wedge I) = \rho^1_{(X\wedge I)}$

that is the following diagram $(\rho-\theta)$ commutes.

$$
\begin{array}{ccc}
X\wedge I & \xrightarrow{\;(\rho^1_X)\wedge I\;} & \Omega(\Sigma X)\wedge I \\
\downarrow \rho^1_{X\wedge I} & & \downarrow \theta^1_X \\
\Omega\Sigma(X\wedge I) & \xrightarrow{\;=\;} & \Omega(\Sigma X\wedge I)
\end{array}
$$

Proof;

For $x \in X$, $t \in I$, and $[x,t] \in X \wedge I$,

$\rho^1_X \wedge I \circ([x,t]) = [[-,x],t] \in \Omega(\Sigma X) \wedge I$, where

$ev_{(s)}[-,x] = [s,x]$, $s \in I$,

$\theta^1_X \circ([[-,x],t]) = ([[-,x],t]) \in \Omega(\Sigma X \wedge I)$ and

$ev_{(s)}[[-,x],t]) = [s,x,t]$. While

$(\rho^1_{X \wedge I})([x,t]) = [-,x,t]$ and

$ev_{(s)}([-,x,t]) = [s,x,t]$. ∎

Extending this definition of the delooping map $\sigma_{RX;I}$ to the category $\mathfrak{C}^T$ one has, by property #B.1, diagram σ-nat that σ is a morphism in $\mathfrak{C}^T$,

$\sigma_{\hat{R}D(-);I}: R \circ D(-) \wedge I \longrightarrow R \circ (D(-) \wedge I)$.

Notice that $R \circ D(-) = D \circ R(-)$, $R \circ I(-) = I \circ R(-)$ and that $R \circ Id(-) = Id \circ R(-) = R(-): R \circ D(-) \longrightarrow R \circ I(-)$,

let $\mathfrak{P}^{\wedge}_{R \circ Id(-)}$ be the push out under this morphism $R(-)$,

using $i^1_{R \circ D(-)}: R \circ D(-) \longrightarrow R \circ D(-) \wedge I$, the pair

$\{ R \circ (j_{D(-) \wedge I}) \circ (\sigma_{\hat{R}D(-);I}): R \circ D(-) \wedge I \longrightarrow R \circ \mathfrak{P}^{\wedge}_{Id(-)}$,

$R \circ j_{I(-)}: R \circ I(-) \longrightarrow R \circ \mathfrak{P}^{\wedge}_{Id(-)} \}$.

satisfy $R \circ j_{D(-) \wedge I} \circ (\sigma_{\hat{R}D(-);I}) \circ i^1_{RD(-)} = R \circ j_{I(-)} \circ R(-)$.

hence the universal property of the push out $\mathfrak{P}^{\wedge}_{R(-)}$ results in a morphism ,

$\psi^{\wedge}_{Id(-)}:\mathfrak{P}^{\wedge}_{R(-)} \longrightarrow R\circ\mathfrak{P}^{\wedge}_{Id(-)}$ in $\mathfrak{C}^{T}$,

likewise the pair of morphisms;

$R\circ j_{D(-)}:R\circ D(-) \longrightarrow R\circ C(-)$, $R\circ j_{I(-)}:R\circ I(-) \longrightarrow R\circ C(-)$

induce $\psi^{C}_{Id(-)}:C\circ R(-) \longrightarrow R\circ C(-)$ a morphism in $\mathfrak{C}^{T}$.

Therefore one has another definition for

$\psi^{\Lambda}_{Id(-)} = \psi^{C}_{Id(-)}\circ\mathfrak{P}(\sigma\hat{R}D(-);I,Id_{I(-)})$ with

$\mathfrak{P}(\sigma\hat{R}D(-);I,Id_{I(-)}):\mathfrak{P}^{\Lambda}_{Id(-)} \longrightarrow \mathfrak{P}(Ri^{1}_{A}, Rf)$ where

$\mathfrak{P}(Rj_{D(-)}\Lambda I,Rj_{I(-)}):\mathfrak{P}(Ri^{1}_{A},Rf) \longrightarrow R\mathfrak{P}^{\Lambda}_{Id(-)}$ induces $\psi^{C}_{Id(-)}$

Although the following diagram $(\sigma-_{*})$, commutes by the naturality of push outs, a proof is given.

<u>Property</u> #B.3) Diagram $(\sigma-_{*})$, commutes, $f:A \longrightarrow X$;

$$\begin{array}{ccc}
\mathfrak{P}^{\Lambda}_{Rf} = (RA)\Lambda I\cup_{Rf}RX & \xrightarrow[\mathfrak{P}(RA\Lambda I,\bar{*}_{RX})]{} & (RA)\Lambda S^{1} \\
\Big\downarrow \mathfrak{P}(\sigma^{\Lambda}_{RA;I}\ ,RX) & & \Big\downarrow \sigma^{\Lambda}_{RA;S^{1}} \\
\mathfrak{P}(Ri^{1}_{A}\ ,\ f) & \xrightarrow{\hspace{3em}} & R(A\Lambda S^{1}) \\
\Big\downarrow \mathfrak{P}(Rj_{A\Lambda I}\ ,Rj_{X}) & \mathfrak{P}(R(A\Lambda I),\bar{*}_{RX}) & \Big\downarrow = \\
R(\mathfrak{P}^{\Lambda}_{f}) & \xrightarrow[R\circ\mathfrak{P}(A\Lambda I,\bar{*}_{X})]{} & R(\Sigma A)
\end{array}$$

Proof; For $[[r;a]^\wedge,t] \cup_{Rf} [r;x]^\wedge \in \mathfrak{P}^{\Lambda}_{Rf}$,

$\sigma^{\Lambda}_{RA;S^1} \circ \mathfrak{P}(RA\Lambda I, \bar{*}_{RX}) \circ ([[r;a]^\wedge,t] \cup_{Rf} [r;x]^\wedge) =$

$\sigma^{\Lambda}_{RA;S^1} \circ ([r;a]^\wedge,t])$ (where $t \in S^1$ at this point)

$= [r;a,t]^\wedge$, $t \in S^1$. While

$\mathfrak{P} \circ R(A\Lambda I, \bar{*}_{RX}) \circ \mathfrak{P}(\sigma^{\Lambda}_{RA;I}, RX) \circ ([[r;a]^\wedge,t] \cup_{Rf} [r;x]^\wedge) =$

$\mathfrak{P} \circ R(A\Lambda I, \bar{*}_{RX}) \circ ([r;a,t] \cup_{Rf} [r;x]^\wedge) = [r;a,t]^\wedge$, $t \in S^1$.

For the bottom square;

$R\mathfrak{P}(A\Lambda I, \bar{*}_X) \circ \mathfrak{P}(Rj_{A\Lambda I}, Rj_X) \circ ([r;a,t]^\wedge \cup_{Rf} [r;x]^\wedge) =$

$R\mathfrak{P}(A\Lambda I, \bar{*}_X) \circ ([r;([a,t] \cup_f x)]^\wedge) = [r;a,t]^\wedge, t \in S^1$,

while

$\mathfrak{P}(R(A\Lambda I), \bar{*}_{RX}) \circ ([r;a,t]^\wedge \cup_{Rf} [r;x]^\wedge) = [r;a,t]$. ∎

<u>Property</u> #4 Diagram (σ-q), commutes,

$$\begin{array}{ccc} RXx'I & \xrightarrow{\sigma^{X'}_{RX;I}} & R(X \curlywedge I) \\ \downarrow{\scriptstyle q_{RXx'0}} & & \downarrow{\scriptstyle R(q_{Xx'I})} \\ RX\Lambda I & \xrightarrow{\sigma^{\Lambda}_{RX;I}} & R(X\Lambda I) \end{array}$$

Proof;

$R(q_{Xx'0}) \circ \sigma^{X'}_{RX;I} \circ ([[r;x]^\wedge,'t]) = R(q_{Xx'0}) \circ ([r;x,'t]^\wedge) =$

$[r;x,t]^\wedge$. While $\sigma^{X'}_{RX;I} \circ q_{RXx'0}([[r;x]^\wedge,'t]) =$

$\sigma^{\Lambda}_{RX;I} \circ [r;x]^\wedge,t]) = [r;x,t]^\wedge$. ∎

By the universal property of push outs, in order to establish that two morphisms $h,k:\mathfrak{P}(f,g) \longrightarrow Z$ are equivalent it suffices to show that

$h \circ j_{I(f)} \circ f = k \circ j_{I(f)} \circ f$ and $h \circ j_{I(g)} \circ g = k \circ j_{I(g)} \circ g$.

Property B.4 then implies, diagram (σ-q.1) commutes.

Where $i_X^{k,\wedge} = i_X^k : X \longrightarrow X \wedge I$, $f:X \longrightarrow Y$.

<u>Property</u> (B.5) diagram (σ-q.1) commutes.

$$
\begin{array}{ccc}
\mathfrak{P}_{Rf}^{X'} = & & R \circ \mathfrak{P}_f^{X'} = \\
\mathfrak{P}(i_{RX}^{1,X'}, Rf) & \xrightarrow[\mathfrak{P}(Rj_{XX'I} \circ \sigma_{RX;I}^{X'}, Rj_Y)]{\psi_f^{X'}} & R \circ \mathfrak{P}(i_X^{1,\wedge}, f) \\
\Big\downarrow \mathfrak{P}(q_{RXX'0}, RY) & & R \circ \mathfrak{P}(q_{XX'I}, Y) \Big\downarrow \\
\mathfrak{P}(i_{RX}^{1,\Lambda}, Rf) & \xrightarrow[\mathfrak{P}(Rj_{X\Lambda I} \circ \sigma_{RX;I}^{\Lambda} \circ Rj_Y) = \psi_f^{\Lambda}]{} & R \circ \mathfrak{P}(i_X^{1,\Lambda}, f) \\
= \mathfrak{P}_{Rf}^{\Lambda} & & = R\mathfrak{P}_f^{\Lambda}
\end{array}
$$

$R \circ \mathfrak{P}(q_{XX'0}, Y) \circ \mathfrak{P}(Rj_{XX'I} \circ \sigma_{RX;I}^{X'}, Rj_Y) \circ j_{RXX'I} =$

$(j_{X\Lambda I}) \circ R(q_{XX'0}) \circ \sigma_{RX;I}^{X'} =$

$R(j_{X\Lambda I}) \circ q_{RXX'0} \circ \sigma_{RX;I}^{X'}$ by property B.4 diagram σ-q

$= \mathfrak{P}(Rj_{X\Lambda I} \circ \sigma_{RX;I}^{\Lambda}, Rj_Y) \circ \mathfrak{P}(q_{RXX'0}, RY) \circ j_{RXX'I}$

while,

$$R\circ\mathfrak{p}(q_{XX'0},Y)\circ\mathfrak{p}(Rj_{XX'I}\circ\sigma^{X'}_{RX;I},Rj_Y)\circ j_{RY} = R(j_Y) =$$

$$\mathfrak{p}(Rj_{X\Lambda I}\circ\sigma^{\Lambda}_{RX;I},Rj_Y)\circ\mathfrak{p}(q_{RXX'0},RY)\circ j_{RY} ,$$

and diagram σ-q.1 commutes. ∎

<u>Property</u> #B.6) Diagram (σ-j), commutes; $\sigma = \sigma^{\Lambda}$

$$\begin{array}{ccc} R\circ D(-) & \xrightarrow[i^{1,\Lambda}_{RD}]{} & RD(-)\Lambda I \\ {\scriptstyle =}\downarrow & & \downarrow \sigma_{D(-)\Lambda I} \\ R\circ D(-) & \xrightarrow{Ri^{1,\Lambda}_{D}} & R\circ(D(-)\Lambda I) \end{array}$$

Proof; $\sigma_{RX;I}\circ i^{1,\Lambda}_{RX}([r;x]\hat{}) = \sigma_{RX;I}([r;x]\hat{},1]) =$

$[r;x,1]\hat{}$, while

$Ri^{1,\Lambda}_{X}\circ([r;x]\hat{}) = [r;x,1]\hat{}$.

∎

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