The composition of the finite Hilbert transform with the differentiation operator

by

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Abstract

THE FINITE HILBERT TRANSFORM AND THE DIFFERENTIATION OPERATOR

by

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Solutions of the Neumann problem for the Laplace and Helmholtz operators in the exterior of a compact plane curve without self-intersections depends on a formally symmetric operator defined on a dense subspace of the L^2 functions on a closed interval. Within the subspace the operator is differentiation composed with the finite Hilbert transform. We find the self-adjoint extension of this operator and investigate its properties thereby developing the theory of the Neumann problem to its natural limit.

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Contents

1 Introduction

1.1 The operator $\mathcal{T}$

The first sections here are concerned with the operator $\mathcal{T} = \mathcal{F}\mathcal{D}$ defined on a suitable domain by the composition of the operator $\mathcal{D} = -d/dx$, where d/dx is the differentiation operator followed by the ***finite Hilbert transformation*** $\mathcal{F}$. This transformation is defined by the following *singular integral* :

$$\mathcal{F}(f)(x) = \frac{1}{\pi} \int_{-1}^{*1} \frac{f(y)}{y - x} dy,$$

where the star "$*$" in the integral sign means that the integral is taken in the sense of the Cauchy principal value, i.e.,

$$\int_{-1}^{*1} \frac{f(y)}{y - x} = \lim_{\epsilon \to 0} \left(\int_{-1}^{x-\epsilon} + \int_{x+\epsilon}^{1} \right) \frac{f(y)}{y - x} dy .$$

One motivation for studying the operator $\mathcal{T}$ is its connection with a *Neumann problem for the Laplace equation* in the exterior of the line segment $[-1, 1]$ in the complex plane. The form of the problem considered here can be stated as follows :

Let g be a function in $L^2([-1, 1])$. Find a solution $u(z)$, $z = x+iy$, of the Laplace equation in the exterior of [−1, 1],

$$\left.\begin{array}{rcl} \Delta u & \equiv & 0\,, \quad \text{on } \mathbb{C}\setminus[-1,1] \\ \lim\limits_{y\to 0} \partial_y u(z) = & & g(x), \quad \text{a.e.} \quad x\in[-1,1]\,. \end{array}\right\} (NP)$$

Here we shall be concerned with solutions constructed by means of double layer potentials. Such solutions automatically satisfy certain boundary conditions at infinity that are often included with the statement of the Neumann problem. As will be seen later these solutions also have limiting values as the segment $[-1,+1]$ is approached from above or from below.

The next section indicates how solving this kind of problem by constructing a suitable double layer potential is connected with the operator $\mathcal{T}$.

1.2 Link of $\mathcal{T}$ with the Neumann problem

In order to show precisely the connection between the operator $\mathcal{T}$ and the *Neumann problem* we need to introduce some definitions and make some remarks.

We will consider functions f defined on the interval $[-1,1]$, such that f is absolutely continuous, f' in L^{1+} and $f(-1) = f(1) = 0$, where $f' \in L^{1+}$ means that f' is in some space $L^{1+\epsilon}$, for $\epsilon > 0$. Our basic interval will be $[-1,1]$, oriented from left to right, if not otherwise indicated. Let $z = (x,y)$ and $z_1 = (t,s)$ be two points of $\mathbb{C}$ and denote the square of their distance by $r^2 = |z-z_1|^2 = (x-t)^2 + (y-s)^2$. Let z be outside $[-1,1]$. A *double layer* *potential with density* f is defined as the integral

$$Kf(x,y) = -\frac{1}{\pi}\int_{-1}^{1} f(t)\,\frac{\partial}{\partial s}(\ln r)\bigg|_{s=0} dt.$$

We can write

$$\frac{\partial}{\partial s}(\ln|z - z_1|)\bigg|_{s=0} = \frac{\partial}{\partial t}(\arctan\frac{x-t}{y}),$$

where, when $y = 0$ but $x \neq t$, the right side is interpreted as zero. After integrating by parts and taking z_1 in the segment $[-1,1]$, by the assumption $f(1) = f(-1) = 0$, we get

$$Kf(x,y) = \frac{1}{\pi}\int_{-1}^{1} f'(t)\,\arctan\frac{x-t}{y}dt\,.$$

We calculate the normal derivative of Kf, with respect to the normal vector $\nu_o = (0,1)$ to $[-1,1]$, and we get

$$\frac{\partial Kf}{\partial y}(x,y) = \frac{1}{\pi}\int_{-1}^{1} f'(t)\frac{x-t}{(x-t)^2+y^2}dt.$$

We know that the limit when y approaches zero exists at every Lebesgue point of f' (see [Stein], Lemma 1.2, Chapter VI,) and the integral is equal to

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f'(t)}{x-t}dt.$$

Thus, we have obtained the equality

$$\frac{\partial Kf}{\partial y}(z)\bigg|_{y=0} = \mathcal{T}f(x) \quad \text{a.e. in } [-1,1] \tag{1}$$

We have shown with eq.(1) that (NP) can be expressed in terms of the operator $\mathcal{T}$ as follows :

> Let g be a function in $L^2([-1, 1])$. Find a function $f(x)$ that satisfies the following condition :
>
> $$\mathcal{T}f(x) \;=\; g(x), \qquad \text{for almost all } x \text{ in } [-1,1]. \tag{2}$$

Therefore, solutions of (NP) will be given by Kf, where f is a solution of eq.(2). The set of admissible f's, that is, the domain of $\mathcal{T}$, will be vague for a while. Much of the point of what we will do is to determine what the domain should be.

We conclude that, in order to solve the *Neumann problem* expressed this way, it is important to understand and characterize the operator $\mathcal{T}$ as completely as possible, find its most general domain of definition, and place all questions about it into the context of appropriate function spaces. One of our goals then, is to investigate the existence of an inverse of $\mathcal{T}$ to solve the problem (NP), as eq.(2) suggests.

1.3 Problems about the domain of definition of $\mathcal{T}$

The classical formulas in [Tri1], show how to solve the problem of finding an inverse for $\mathcal{T}$ for a very limited class of functions. [Boc] extends somewhat those results, but still they do not allow one to work in functional spaces where the spectral theory can be applied. In that case, we could deduce many of the properties of $\mathcal{T}$ we need to find its right domain of definition, and therefore, an inverse. Our strategy is thus to bring the spectral theory of unbounded operators to bear on the problem.

On the whole line, the problem of finding a suitable domain to make $\mathcal{T}$ self-adjoint and invertible is fairly simple : When $\mathcal{H}$, the *Hilbert transform on the line,*

$$\mathcal{H}(f)(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(y)}{y-x}dy\,,$$

is considered as an operator on L^p into L^p, $1 < p < \infty$, one has

$$\mathcal{H}^2 = -I_d\,,$$

(see [Titch], Chapter V, Section 5.10) where I_d denotes the identity operator. The last equality yields an obvious inversion formula of $\mathcal{H}\mathcal{D}$, operator which is defined on a dense set of L^2. We have that $\mathcal{H}$ satisfies $\mathcal{H}^{-1} = -\mathcal{H}$, and we can obtain

$$(\mathcal{H}\mathcal{D})^{-1} = \mathcal{D}^{-1}\mathcal{H}^{-1} = \mathcal{D}^{-1}\mathcal{H}\,,$$

where, recalling our notation, $-\mathcal{D} = d/dx$. The symbol $\mathcal{D}^{-1}$ denotes the anti-derivative that vanishes at $\pm\infty$. (It can be shown that such an anti-derivative exists on the range of $\mathcal{HD}$.)

On the segment $[-1, 1]$ however, there are difficulties that are not present on the whole line. An important one is that if we consider as domain of $\mathcal{T}$ the whole space L^2, the kernel of $\mathcal{T}$ is not trivial : The operator $\mathcal{T}$ maps the constants to zero, so

$$\mathcal{T}(c) = \mathcal{FD}(c) = 0. \tag{3}$$

The operator $\mathcal{T}$ also vanishes on a wider class of functions. For instance, for $f(x) = \arccos(x)$ we have that

$$(\mathcal{D}f)(x) = -\frac{d \arccos}{dx}(x) = \frac{1}{\sqrt{1-x^2}} \, ,$$

and then,

$$(\mathcal{T}f)(y) = \mathcal{F}\left(\frac{1}{\sqrt{1-x^2}}\right)(y) = \int_{1}^{*1} \frac{1}{\sqrt{1-x^2}} \frac{dx}{x-y} = \int_0^{*\pi} \frac{d\theta}{\cos\theta - y} = 0 \, , \tag{4}$$

where the integral vanishes by the known formula

$$\int_0^{*\pi} \frac{\cos(n\xi) d\xi}{\cos\xi - \cos\theta} = \pi \frac{\sin(n\theta)}{\sin\theta}, \tag{5}$$

(apply the formula with $n = 0$. See for instance [Tri2], Chapter IV, Section 4.3). Therefore, an inverse of $\mathcal{T}$ would not be uniquely defined if f were included in the domain.

Let us give another example showing that the properties valid for $\mathcal{H}$ are not, in general, inherited by $\mathcal{F}$. The operators $\mathcal{H}$ and $\mathcal{D}$ commute :

$$\mathcal{H} \circ \mathcal{D} = \mathcal{D} \circ \mathcal{H} . \tag{6}$$

Indeed, applying the Fourier transform and assuming f sufficiently smooth, from the right-hand side of eq.(6) we have

$$\widehat{(\mathcal{H}f)'}(x) = 2\pi i\, x\, \widehat{\mathcal{H}}(x) = 2\pi\, |x|\, \widehat{f}(x)\,,$$

and from the left we have

$$\widehat{\mathcal{H}(f')}\,(x) = -i \,\mathrm{sgn}\, x\, \widehat{f'}(x) = 2\pi\, |x|\, \widehat{f}(x)\,.$$

However, the operators $\mathcal{F}$ and $\mathcal{D}$ do not commute on $L^2([-1,1])$. We have seen in eq.(3) that on constant functions the composition $\mathcal{FD}$ vanishes. On the other hand, if $c(x)$ denotes a constant function, we obtain

$$(\mathcal{DF})c(x) = \frac{c}{\pi}\,\mathcal{D}\,(\ln\frac{1-x}{1+x}) = \frac{2c}{\pi}\,\frac{1}{1-x^2}\,, \tag{7}$$

which is not even a function in L^2.

We can see from these examples that the operator $\mathcal{FD}$ presents difficulties on $[-1,1]$ not found for $\mathcal{HD}$ on the whole line.

1.4 Formally symmetric properties of $\mathcal{T}$

The operator $\mathcal{T}$ behaves in a particularly simple way on functions related to Chebyshev polynomials. These polynomials of the first kind T_n ($n = 0, 1, \ldots$), and of the second kind U_n ($n = 0, 1, \ldots$), are defined respectively by the equations

$$\begin{aligned} T_n(x) &= \cos(n \arccos x) , \\ U_n(x) &= \frac{\sin((n+1)\arccos(x))}{\sqrt{1-x^2}} . \end{aligned}$$

The formula given by eq.(5), together with the one below

$$\int_0^{*\pi} \frac{\sin(n+1)\xi \sin\xi d\xi}{\cos\xi - \cos\theta} = -\pi\cos(n+1)\theta ,$$

leads to the following expressions for $\mathcal{F}$ applies to Chebyshev polynomials, (see [Tri2], Chapter IV, Section 4.3). Let us introduce the notation

$$\rho(x) = \sqrt{1-x^2} ,$$

that we shall always use in this work. We have, for $n = 1, 2, \ldots$,

$$\mathcal{F}(U_{n-1}\,\rho)(x) = -T_n(x) , \tag{8}$$

$$\mathcal{F}(T_n\,\rho^{-1})(x) = U_{n-1}(x) , \tag{9}$$

and from them we can obtain the action of $\mathcal{T}$ on the $U_{n-1}\,\rho$

$$\mathcal{T}(U_{n-1}\,\rho)(x) = \mathcal{F}\mathcal{D}(U_{n-1}\,\rho)(x) = n\,U_{n-1}(x)\ .$$

Let E_o the space of finite linear combinations of Chebyshev polynomials of second kind, D_o the space of finite linear combination of the functions $U_{n-1}\,\rho$, and $\mathcal{T}_o$ the restriction of $\mathcal{T}$ to D_o. Then, $\mathcal{T}_o(D_o) = E_o$ and therefore, it is possible to define $\mathcal{T}_o^{-1}$ on E_o, by

$$(\mathcal{T}_o^{-1}U_{n-1})(x) = \frac{1}{n}\,(U_{n-1}\rho)(x)\ . \tag{10}$$

The operator $\mathcal{T}_o$ is formally self-adjoint as an operator in D_o. In order to show this, we have to show that for any finite linear combination f and g, of functions $U_{n-1}\,\rho$, we have

$$< \mathcal{T}_o f, g > \ = \ < f, \mathcal{T}_o g >\ . \tag{11}$$

Indeed, for arbitrary $n, m = 1, 2, \ldots$, we have

$$\begin{aligned} &< \mathcal{T}_o\ (U_{n-1}\rho), U_{m-1}\rho > \\ &= \ \frac{n}{\pi} < U_{n-1}, U_{m-1}\rho > = \frac{n}{\pi}\int_0^{\pi} \sin(m\theta)\sin(n\theta)\,d\theta \ = \ \frac{n}{2}\,\delta_{m,n} \\ &= \ < U_{n-1}\rho, \mathcal{T}_o(U_{m-1}\rho) >\ . \end{aligned} \tag{12}$$

Clearly, from the above calculation, equation (11) is also satisfied by any function in D_o.

We note finally that on D_o the operators $\mathcal{F}$ and $\mathcal{D}$ commute. On the one hand we have

$$\mathcal{F}\mathcal{D}(U_{n-1}\,\rho)(x) = n\,\mathcal{F}(T_n\,\rho^{-1}) = n\,U_{n-1}(x)\;,$$

and on the other

$$\mathcal{D}\mathcal{F}(U_{n-1}\,\rho)(x) = -\mathcal{D}(T_n) = n\,U_{n-1}(x)\,.$$

Apparently, it could appear that the natural spaces to be taken as domain and range are D_o and E_o. But, despite having many of the properties we desire for a self-adjoint invertible operator, $\mathcal{T}_o$ is not closed on D_o, viewed as a subset of L^2.

Therefore, it seems natural to try to consider $\mathcal{T}_o$ as an unbounded, symmetric operator on L^2, find its closed extensions in the biggest possible domain, study the existence of a self-adjoint extension and, if possible, find an inverse. In this way the tools provided by the spectral theory of unbounded operators become available.

1.5 A Commutation Formula

We introduce here a formula that will be used frequently throughout this work. This formula is a special case of a general theorem, whose proof can be found in [Prsf], Chapter II, Section 4 (Theorem 4.4). It deals with the order of integration when the integrals involved are singular and of Cauchy principal value type. We shall state it here in our context, after the introduction of the following notation.

Let $\rho(x) = \sqrt{1-x^2}$ and suppose α is a real number. The Hilbert space $L^2(\rho^\alpha)$ is the space of measurable functions f such that

$$\int_{-1}^{1} |f(t)\rho^\alpha(t)|^2 < +\infty ,$$

with norm

$$\|f\|_{L^2(\rho^\alpha)} = \left(\int_{-1}^{1} |f(x)\rho^\alpha(x)|^2 dx\right)^{1/2} .$$

Proposition 1.1 *Let α be a real number, f in $L^2(\rho^\alpha)$ and g in $L^2(\rho^{-\alpha})$. If $-1/2 < \alpha < 1/2$, then*

$$\int_{-1}^{1} f(x)dx \int_{-1}^{*1} \frac{g(y)}{y-x}\,dy = \int_{-1}^{1} g(y)dy \int_{-1}^{*1} \frac{f(x)}{y-x}\,dx .$$

2 Properties of the operator $\mathcal{T}$

2.1 Domain of definition of $\mathcal{T}$

We are concerned with finding a self-adjoint extension of the operator $\mathcal{T}_o$, defined by

$$\mathcal{T}_o(U_{n-1}\,\rho) = n\,U_{n-1}\ .$$

Let us recall the definitions given previously of $E_o \subseteq L^2$, the space of finite linear combinations of Chebyshev polynomials U_{n-1}, so that $\mathcal{T}_o$ is defined on E_o. The space D_o is that of finite linear combinations of functions $U_{n-1}\,\rho$. We define the operator S_o from E_o into L^2 by the formula

$$(S_oU_{n-1})(x) = \frac{1}{n}\,U_{n-1}(x)\,\rho(x)\ . \tag{13}$$

Theorem 2.1 *The operator S_o viewed as an operator from a subset of L^2 into L^2 is bounded. Its bounded closure S, whose domain is the space L^2, is compact, self-adjoint, and positive definite.*

Proof:

It is a known fact that the functions $\sqrt{2/\pi}\,U_{n-1}(x)\,\rho^{1/2}(x)$, $n = 1, 2, \ldots$, where $\rho^{1/2}(x) = (1-x^2)^{1/4}$, form an orthonormal set in L^2, which can be shown to be complete. We will write $\tilde{U}_{n-1}(x) = \sqrt{2/\pi}\,U_{n-1}(x)\rho^{1/2}(x)$ in order to simplify the notation. Let h be a function whose Fourier coefficients with respect to the orthonormal system defined above are b_1, b_2, $\ldots$, and define

Bh as the function whose n^{th} Fourier coefficient is b_n/n. Then, if

$$h(x) = \sum_{n=1}^{\infty} b_n \tilde{U}_{n-1}(x) ,$$

the operator B acts on such an h as

$$(Bh)(x) = \sum_{n=1}^{\infty} \frac{b_n}{n} \tilde{U}_{n-1}(x) . \tag{14}$$

The operator $M_{\rho^{1/2}}$ is multiplication by $\rho^{1/2}(x) = (1 - x^2)^{1/4}$, that is,

$$(M_{\rho^{1/2}} f)(x) = \rho^{1/2}(x) f(x) .$$

We claim that we can write the operator S_o as the composition

$$S_o = M_{\rho^{1/2}} \circ B \circ M_{\rho^{1/2}} . \tag{15}$$

Indeed, we can verify that the linear operators $M_{\rho^{1/2}} \circ B \circ M_{\rho^{1/2}}$ and S_o are equal on the Chebyshev polynomials U_{n-1}. We obtain from the right-hand side of eq.(15), the equality

$$(M_{\rho^{1/2}}\, B\, M_{\rho^{1/2}})(U_{n-1})(x) = \rho^{1/2}(x)\, B(\rho^{1/2} U_{n-1})(x) = \frac{1}{n} U_{n-1}(x) \rho(x) .$$

From this and the definition of S_o in eq.(13), we get

$$(M_{\rho^{1/2}}\, B\, M_{\rho^{1/2}})(U_{n-1})(x) = S_o(U_{n-1})(x).$$

Therefore, $S_o(f) = M_{\rho^{1/2}} B M_{\rho^{1/2}}(f)$, for f any finite linear combination of Chebyshev polynomials U_{n-1}, that is, for $f \in E_o$, as we have claimed.

It is clear that both operators $M_{\rho^{1/2}}$ and B are bounded. Therefore, their composition S_o is a bounded operator on E_o, also. The closure of E_o is L^2, since the finite linear combinations of Chebyshev polynomials of the second kind are dense in L^2. As a consequence, we can extend S_o to the whole space L^2, preserving its boundedness. We will denote the bounded closure of S_o by S. Thus, we can write, for all $f \in L^2$

$$S(f) = M_{\rho^{1/2}} \circ B \circ M_{\rho^{1/2}}(f) \ . \tag{16}$$

The operator B is compact. This can be verified by observing that B is the limit on the uniform topology of a sequence of operators of finite rank. Since $M_{\rho^{1/2}}$ is bounded, it follows that S is compact.

Let us show that S is self-adjoint. We observe first that for arbitrary $n, m = 1, 2, \ldots$, we have the equality, as in eq.(12)

$$< S(U_{n-1}), U_{m-1} > \ = \ \frac{1}{n} < U_{n-1}\,\rho, U_{m-1} > \ = \ \frac{1}{2n}\,\delta_{m,n} \tag{17}$$

$$= \ < U_{n-1}, S(U_{m-1}) > . \tag{18}$$

Therefore, for all f and g in E_o, we can verify also

$$< S(f), g > \ = \ < f, S(g) >, \tag{19}$$

by the continuity of the scalar product. We conclude that S is symmetric on L^2. Since the domain of S is the whole space L^2, this implies that S is self-adjoint.

It remains to be shown that S is positive definite, that is, that for all non- vanishing $f \in L^2$ we have

$$< S(f), f > \; > \; 0. \tag{20}$$

Note that B is positive definite. Indeed, if $h = \sum_{n=1}^{\infty} b_n \tilde{U}_{n-1}$, we have

$$< B(h), h > \; = \; \sum_{n=1}^{\infty} \frac{|b_n|^2}{n^2} \, ,$$

which is always positive and it is zero only if all b_n's are zero. Therefore, S is positive also, since we have

$$\begin{aligned} < S(f), f > \; &= \; < M_{\rho^{1/2}} \circ B \circ M_{\rho^{1/2}}(f), f > \\ &= \; < B(\rho^{1/2} f), \rho^{1/2} f > \; \geq \; 0 \, , \end{aligned} \tag{21}$$

and if $< S(f), f > = 0$, we have from the right-hand side of eq.(21) that $\rho^{1/2} f = 0$, and this implies that $f = 0$. This completes the proof of the theorem.

Theorem 2.1 allows us to discard the vague and provisional definition that was given in Sections 1.1 and 1.2, and define the operator $\mathcal{T}$ as the inverse of S. We will denote by $D_{\mathcal{T}}$ the domain of $\mathcal{T}$. This definition of $\mathcal{T}$ makes sense because S is injective and the range of S, which is $D_{\mathcal{T}}$, is dense in L^2.

Corollary 2.2 *On $D_{\mathcal{T}}$, the operator $\mathcal{T}$ is an unbounded self-adjoint extension of $\mathcal{T}_o$. Furthermore, $\mathcal{T}$ is the closure of the operator $\mathcal{T}_o$.*

Proof: The fact that $\mathcal{T}$ is self-adjoint is immediate from the previous theorem and the known fact that the inverse of a self-adjoint operator, if it exists, is also self-adjoint. As a consequence, we have that $\mathcal{T}$ is also closed. Obviously, $\mathcal{T}$ is an extension of $\mathcal{T}_o$, since $D_o \subset D_{\mathcal{T}}$.

It remains to be shown that $\mathcal{T}$ is actually the closure of $\mathcal{T}_o$. To see this it is only necessary to observe that $S = \mathcal{T}^{-1}$ is the closure of $S_o = \mathcal{T}_o^{-1}$ and the closure of the graph of S_o is essentially the same as the closure of the graph of $\mathcal{T}_o$.

There are several problems that will concern us for the rest of the chapter. For example: Can one get a more explicit description of the domain of $\mathcal{T}$ than that it is the range of S? Can one restore the interpretation of $\mathcal{T}$ as the composition of differentiation and the finite Hilbert transform? Do Hilbert transform and differentiation commute on the domain of $\mathcal{T}$? What can be said about the spectrum of $\mathcal{T}$?

The next section will provide us with an important tool used later, in connection with the finite Hilbert transform.

2.2 Continuity of $\mathcal{F}$ on weighted spaces

The theorem that we will present in this section expresses the boundedness of the singular integral operator $\mathcal{F}$ which operates on certain weighted spaces that are of interest. We include the statement for our particular framework. The theorem is due to Khvedeldze and its proof can be found in [Prsf], Chapter II, Section 3. We use the notation of Section 1.5.

Theorem 2.3 *Let α be a real number, $|\alpha| < 1$, and $\rho(x) = \sqrt{1-x^2}$. Then, the finite Hilbert transform $\mathcal{F}$ defines a bounded singular integral operator on the space $L^2(\rho^\alpha)$.*

2.3 Characterization of $\mathcal{T}^{-1}$

In this section we show how the operator $S = \mathcal{T}^{-1}$, defined in Section 2.1, is represented as a composition of bounded linear maps of Hilbert spaces in two different ways.

Lemma 2.4 *The operator $\mathcal{T}^{-1}$ is given by the composition*

$$\mathcal{T}^{-1} \;=\; I \circ M_{\rho^{-1}} \circ \mathcal{F} \circ M_\rho \;, \tag{22}$$

where $(Ig)(x) = \int_{-1}^{x} g(t)dt$ and $M_\tau f(x) = \tau(x)f(x)$.

Proof: First, we need to verify that the composition $I \circ M_{\rho^{-1}} \circ \mathcal{F} \circ M_\rho$ is well defined. In fact, each of the maps in the composition on the right-hand side of eq.(22) is a bounded linear map between Hilbert spaces.

Let $0 < \epsilon < 1$. Consider the following diagram describing the action of $I \circ M_{\rho^{-1}} \circ \mathcal{F} \circ M_\rho$, and let us analyze each of its segments,

$$L^2 \xrightarrow{M_\rho} L^2(\rho^{\epsilon-1}) \xrightarrow{\mathcal{F}} L^2(\rho^{\epsilon-1}) \xrightarrow{M_{\rho^{-1}}} L^2(\rho^\epsilon) \xrightarrow{I} L^2 .$$

The boundedness of all of the maps in the composition is clear, except the one involving $\mathcal{F}$, which follows from Theorem 2.3. Then, $IM_{\rho^{-1}}\mathcal{F}M_\rho$ is a well defined bounded operator.

Let us show that both the left and right-hand sides of eq.(22) operate in the same way on U_{n-1}. Recall that $(\mathcal{F}\rho U_{n-1})(x) = -T_n(x)$ and observe that $\rho^{-1} T_n = -1/n\ (U_{n-1}\,\rho)'$. We have then

$$\begin{aligned} I\Big(\rho^{-1}(x)\mathcal{F}(\rho U_{n-1})(x)\Big)(y) &= -\int_{-1}^{y} \rho^{-1}(x)T_n(x)dx = \frac{1}{n}\, U_{n-1}(y)\,\rho(y) \\ &= (\mathcal{T}^{-1}U_{n-1})(y) \;. \end{aligned} \tag{23}$$

Equation (23) shows that both operators act identically on finite linear combinations of Chebyshev polynomials of the second kind, which are dense in L^2. Therefore, identity (22) follows and the lemma is proved.

Lemma 2.5 *The operator $\mathcal{T}^{-1}$ can also be characterized as the following composition of operators*

$$\mathcal{T}^{-1} = M_\rho \circ \mathcal{F} \circ M_{\rho^{-1}} \circ I \;.$$

Proof: Let us, for a moment, call R the composition $M_\rho \circ \mathcal{F} \circ M_{\rho^{-1}} \circ I$, and consider the following diagram describing the action of R

$$L^2 \xrightarrow{I} L^2(\rho^{\epsilon-1}) \xrightarrow{M_{\rho^{-1}}} L^2(\rho^\epsilon) \xrightarrow{\mathcal{F}} L^2(\rho^\epsilon) \xrightarrow{M_\rho} L^2 .$$

We observe, as in Lemma 2.4, that the boundedness of all of the maps in the composition is clear, except that of $\mathcal{F}$, which follows from Theorem 2.3. Then, R is a well defined bounded operator. Let us show now how R acts on the Chebyshev polynomials U_{n-1} :

$$\begin{aligned} R(U_{n-1})(x) &= \rho(x)\mathcal{F}(\rho^{-1}I(U_{n-1}))(x) = \frac{1}{n}\rho(x)\mathcal{F}(\rho^{-1}T_n)(x) \\ &= \frac{1}{n}\rho(x)U_{n-1}(x) . \end{aligned} \tag{24}$$

We have already shown in eq.(10) that $\mathcal{T}^{-1}(U_{n-1}) = 1/n\ U_{n-1}\,\rho$, which is the right hand side of eq.(24) By linearity, it is clear that $R(f) = \mathcal{T}^{-1}(f)$, when f is any finite linear combination of Chebyshev polynomials of the second kind, which are dense in L^2. Therefore, $R \equiv \mathcal{T}^{-1}$ on L^2, which proves the lemma.

We now prove another consequence of Theorem 2.3, which we will need later [cf. [Tri2], Chapter V, Section 4.3].

Lemma 2.6 *On L^2, the operator $-M_{\rho^{-1}} \circ \mathcal{F} \circ M_\rho$ is a right inverse of $\mathcal{F}$,* *that is, for all $f \in L^2$*

$$\mathcal{F}(M_{\rho^{-1}}\mathcal{F}M_\rho)f = -f \ . \tag{25}$$

Proof: As in the proof of Lemma 2.4, $M_{\rho^{-1}}\mathcal{F}M_\rho$ is a bounded map from L^2 to $L^2(\rho^\epsilon)$, hence, by Theorem 2.3 $\mathcal{F}(M_{\rho^{-1}}\mathcal{F}M_\rho)$ is a bounded map from L^2 to $L^2(\rho^\epsilon)$.

Let $f = U_{n-1}$. Applying the identity (8), and then that of (9), we obtain the equation

$$\mathcal{F}(\rho^{-1}\mathcal{F}(\rho U_{n-1})) = -U_{n-1} \ .$$

Therefore, eq.(25) holds for any finite linear combination of Chebyshev polynomials of the second kind, which are dense in L^2. Thus, eq.(25) is satisfied by all $f \in L^2$.

2.4 Characterization of $D_\mathcal{T}$

In this section we will characterize the domain of definition of $\mathcal{T}$ completely.

Lemma 2.7 *Let h be a function in $L^2(\rho^\epsilon)$, for all $0 < \epsilon < 1$, such that $\int_{-1}^{1} h(x)dx = 0$, and $\mathcal{F}(h) \in L^2$. Then,*

$$\mathcal{F}I(h) = I\mathcal{F}(h) + C \ , \tag{26}$$

where C is a constant.

Proof: It is easy to verify that under our assumptions both sides of eq.(26) make sense.

For each $y \in [-1,1]$, let $\chi^y(t)$ be the characteristic function of the interval $[-1,y]$, i.e. $\chi^y(t) = 1$, if $-1 \leq t \leq y$, and zero otherwise. Let us fix $y \in [-1,1]$, and calculate the following integral from the right hand side of eq.(26)

$$(I\mathcal{F}h)(y) = \int_{-1}^{1} \chi^y(t)(\mathcal{F}h)(t)dt \,.$$

The characteristic function of a finite interval is in $L^p(\rho^{-\epsilon})$, for all $0 < \epsilon < 1$ and $1 \leq p \leq \infty$, and $h \in L^2(\rho^\epsilon)$. This allow us to apply Proposition 1.1 and we obtain

$$\begin{aligned}(I\mathcal{F}h)(y) &= -\int_{-1}^{1} h(t)(\mathcal{F}\chi^y)(t)dt = -\int_{-1}^{1} h(t)\ln\left|\frac{y-t}{1+t}\right|dt \\ &= -\int_{-1}^{1} h(t)\ln|y-t|dt + \int_{-1}^{1} h(t)\ln|1+t|dt \\ &= -\int_{-1}^{1} h(t)\ln|y-t|dt + C\,, \qquad (27)\end{aligned}$$

where

$$C = \int_{-1}^{1} h(t)\ln|1+t|$$

is a convergent integral, since $h \in L^(\rho^\epsilon)$, which does not depend on y. We calculate next the left-hand side of eq.(26). We have

$$\begin{aligned}(\mathcal{F}Ih)(y) &= \int_{-1}^{1}\left(\int_{-1}^{1}\chi^x(t)h(t)dt\right)\frac{dx}{x-y} \\ &= \lim_{\delta\to 0}\left[\int_{-1}^{y-\delta}\left(\int_{-1}^{1}\chi^x(t)h(t)dt\right)\frac{dx}{x-y} + \int_{y+\delta}^{1}\left(\int_{-1}^{1}\chi^x(t)h(t)dt\right)\frac{dx}{x-y}\right]\end{aligned}$$

$$= \lim_{\delta \to 0} \left(I_1(\delta, y) + I_2(\delta, y) \right) .$$

Let us fix δ. It is not difficult to verify that if $t \in [-1, 1]$ and $x \in [-1, y - \delta]$, we have

$$\frac{\chi^x(t) h(t)}{x - y} \in L^1([-1, 1] \times [-1, y - \delta]) .$$

This allows us to apply Fubini's Theorem to the double integral on the expression of $I_1(\delta, y)$ and we obtain the equality

$$I_1(\delta, y) = \int_{-1}^{1} h(t) \left(\int_{-1}^{y-\delta} \frac{\chi^x(t) dx}{x - y} \right) dt .$$

Splitting the outer integral at y, we obtain

$$\begin{aligned} I_1(\delta, y) &= \int_{-1}^{y} h(t) \left(\int_{-1}^{y-\delta} \frac{\chi^x(t) dx}{x - y} \right) dt + \int_{y}^{1} h(t) \left(\int_{-1}^{y-\delta} \frac{\chi^x(t) dx}{x - y} \right) dt \\ &= \int_{-1}^{y-\delta} h(t) \left(\int_{t}^{y-\delta} \frac{dx}{x - y} \right) dt + \int_{y}^{1} h(t) \left(\int_{t}^{y-\delta} \frac{dx}{x - y} \right) dt . \end{aligned}$$

The last integral vanishes since $\chi^x(t) = 0$, if $t \geq y > x$. Therefore, we get

$$I_1(\delta, y) = \int_{-1}^{y-\delta} h(t) \ln \delta \, dt - \int_{-1}^{y-\delta} h(t) \ln |y - t| dt . \qquad (28)$$

We observe also that

$$\frac{\chi^x(t) h(t)}{x - y} \in L^1([-1, 1] \times [y + \delta, 1]).$$

In the same way as we did for $I_1(\delta, y)$, applying Fubini's Theorem, we get

$$\begin{aligned}
I_2(\delta, y) &= \int_{y+\delta}^{1} \left(\int_{-1}^{1} h(t)\chi^x(t)dt \right) \frac{dx}{x-y} = \int_{-1}^{1} h(t) \left(\int_{y+\delta}^{1} \frac{\chi^x(t)dx}{x-y} \right) dt \\
&= \int_{-1}^{y+\delta} h(t) \left(\int_{y+\delta}^{1} \frac{\chi^x(t)dx}{x-y} \right) dt + \int_{y+\delta}^{1} h(t) \left(\int_{y+\delta}^{1} \frac{\chi^x(t)dx}{x-y} \right) dt \\
&= \int_{-1}^{y+\delta} h(t) \left(\int_{y+\delta}^{1} \frac{dx}{x-y} \right) dt + \int_{y+\delta}^{1} h(t) \left(\int_{t}^{1} \frac{dx}{x-y} \right) dt \\
&= \int_{-1}^{y+\delta} h(t) \ln \left| \frac{1-y}{\delta} \right| dt + \int_{y+\delta}^{1} h(t) \ln \left| \frac{1-y}{y-t} \right| dt \\
&= \ln|1-y| \int_{-1}^{1} h(t)dt - \int_{-1}^{y+\delta} h(t) \ln\delta \, dt - \int_{y+\delta}^{1} h(t) \ln|y-t| \, dt \, .
\end{aligned}$$

Since by assumption, $\int_{-1}^{1} h(t)dt = 0$, we get

$$I_2(\delta, y) = -\int_{-1}^{y+\delta} h(t) \ln\delta \, dt - \int_{y+\delta}^{1} h(t) \ln|y-t| dt \, . \tag{29}$$

Adding $I_1(\delta, y)$ and $I_2(\delta, y)$ from eq.(28) and (29), we have

$$\begin{aligned}
&I_1(\delta, y) + I_2(\delta, y) \\
&= -\ln\delta \int_{y-\delta}^{y+\delta} h(t)dt - \left(\int_{-1}^{y-\delta} + \int_{y+\delta}^{1} \right) h(t) \ln|y-t| \, dt
\end{aligned}$$

Since $h \in L^2(\rho^\epsilon)$, applying Hölder's inequality we obtain

$$\int_{y-\delta}^{y+\delta} |h(t)|dt < C \int_{y-\delta}^{y+\delta} (1-x^2)^{-\epsilon}(x)dx = O(\delta^\alpha)$$

where C is a constant and α is a positive real number. Therefore,

$$(\mathcal{F}Ih)(y) = \lim_{\delta \to 0}\{I_1(\delta, y) + I_2(\delta, y)\} = -\int_{-1}^{1} h(t) \ln|y - t| dt \ ,$$

which together with eq.(27), proves the lemma.

Theorem 2.8 *The space $D_{\mathcal{T}}$ is the space of absolutely continuous functions such that*

$f(1) = f(-1) = 0,$

$f' \in L^2(\rho^{\epsilon})$, *for all* $\epsilon > 0$, *and*

$\mathcal{F}\,(f') \in L^2.$

Proof: Let $f \in D_{\mathcal{T}}$. By definition, there is a function $g \in L^2$ such that $(\mathcal{T}^{-1}g)(x) = f(x)$. Then, by Lemma 2.4, we can write

$$f(x) = \int_{-1}^{x} \rho(y)^{-1} \mathcal{F}(\rho g)(y) dy \tag{30}$$

which is absolutely continuous, since $(\rho^{-1}\mathcal{F}\rho)g \in L^2(\rho^{\epsilon})$, and $L^2(\rho^{\epsilon}) \subset L^1$. We can see from eq.(30) that $f(-1) = 0$. On the other hand

$$\begin{aligned} f(1) &= \int_{-1}^{1} \rho(y)^{-1} \mathcal{F}(\rho g)(y) dy \\ &= \int_{-1}^{1} \rho(y)^{-1} \left(\int_{-1}^{*1} \frac{\rho(x) g(x)}{x - y} dx \right) dy \,. \end{aligned}$$

The facts that $\rho g \in L^2(\rho^{-\epsilon})$ and $\rho^{-1} \in L^2(\rho^{\epsilon})$ allow us to apply Proposition 1.1, and we obtain

$$f(1) = -\int_{-1}^{1} \rho(x)g(x)\left(\int_{-1}^{*1} \frac{1}{\rho(y)}\frac{1}{x-y}dy\right) dx\,.$$

As in equation (4), we can show that the innermost integral vanishes. Thus, we have shown that $f(1) = f(-1) = 0$. It is easy to see from eq.(30) that $f'(x) \in L^2(\rho^{\epsilon})$, since $\mathcal{F}(\rho g) \in L^2(\rho^{\epsilon-1})$. To complete the first part of the proof, we have to show that $\mathcal{F}(f') \in L^2$. From eq.(30) we can write, for $g \in L^2$

$$f'(x) = \rho^{-1}(x)\mathcal{F}(\rho g)(x)\,.$$

Applying $\mathcal{F}$ to both sides we obtain, by Lemma 2.6

$$(\mathcal{F}f')(x) = \mathcal{F}(\rho^{-1}\mathcal{F}\rho g)(x) = -g(x)\,.$$

Let us show the other inclusion, that is, that any function f satisfying the properties stated in the theorem is actually in $D_{\mathcal{T}}$. Let f be such a function. Since we have that $g = \mathcal{F}(f') \in L^2$, we can apply to g the operator $\mathcal{T}^{-1}$, which is defined on the whole space L^2. We have proved in Lemma 2.5 that we can write $\mathcal{T}^{-1} = M_\rho \circ \mathcal{F} \circ M_{\rho^{-1}} I$, so that

$$\mathcal{T}^{-1}(g) = M_\rho \mathcal{F} M_{\rho^{-1}} I \mathcal{F}(f')\,.$$

Since $f(-1) = f(1) = 0 = \int_{-1}^{1} f'(x)dx$, taking $h = f'$ in Lemma 2.7, we get $\mathcal{F}I(f') = I\mathcal{F}(f') + C$, for some constant C. Then

$$\mathcal{T}^{-1}(g) = (M_\rho \mathcal{F} M_{\rho^{-1}})\mathcal{F}I(f') - (M_\rho \mathcal{F} M_{\rho^{-1}})C .$$

Applying eq.(4), we see that the last term on the right-hand side vanishes. By Lemma 2.6, we know that $\mathcal{F}(M_{\rho^{-1}}\mathcal{F}M_\rho) = -Id$, and multiplying by $M_\rho M_{\rho^{-1}}$ between $\mathcal{F}$ and f, we obtain

$$\mathcal{T}^{-1}(g) = M_\rho \mathcal{F}(M_{\rho^{-1}}\mathcal{F}M_\rho) M_{\rho^{-1}}(f) = -f . \tag{31}$$

We have obtained that f is in the range of $\mathcal{T}^{-1}$, which by definition is $D_\mathcal{T}$, as we wanted to show. The proof of the theorem is completed.

Remark: The second part of the proof also shows that on $D_\mathcal{T}$, $\mathcal{T}^{-1}$ is a left inverse of $\mathcal{F}\mathcal{D}$. Indeed, as we have defined $g = -\mathcal{F}\mathcal{D}f$, eq.(31) reads

$$\mathcal{T}^{-1}(\mathcal{F}\mathcal{D}f) = f . \tag{32}$$

We can now restore the form of $\mathcal{T}$ as the composition of the operators $\mathcal{F}$ and $\mathcal{D}$ on the domain $D_\mathcal{T}$.

Theorem 2.9 *On $D_\mathcal{T}$, the operator $\mathcal{T}$, which was defined as the inverse operator of $\mathcal{T}^{-1}$, can be written as the composition $\mathcal{F}\mathcal{D}$. That is,* (*i*) $\mathcal{FD}\circ\mathcal{T}^{-1}(g)=g,$ *for all* $g\in L^2$, *and*

(*ii*) $\mathcal{T}^{-1}\circ\mathcal{FD}(f)=f,$ *for all* $f\in D_{\mathcal{T}}$.

Proof: Let $g\in L^2$. By Lemma 2.4, we can write

$$\mathcal{FD}\mathcal{T}^{-1}(g) \quad = \quad \mathcal{FD}(IM_{\rho^{-1}}\mathcal{F}M_\rho)g\ .$$

Clearly we have

$$\mathcal{D}(IM_{\rho^{-1}}\mathcal{F}M_\rho g)=-(M_{\rho^{-1}}\mathcal{F}M_\rho)g\ .$$

Also, $(M_{\rho^{-1}}\mathcal{F}M_\rho g)\in L^2(\rho^\epsilon)$. Then, applying $\mathcal{F}$ and using Lemma 2.6 we obtain

$$\mathcal{FD}(\mathcal{T}^{-1}g)=-(\mathcal{F}M_{\rho^{-1}}\mathcal{F}M_\rho)g=g\ .$$

Condition (*ii*) is an immediate consequence of the remark after Theorem 2.8. The proof is completed

2.5 Commutation formula : $\mathcal{FD}=\mathcal{DF}$

The characterization of $D_{\mathcal{T}}$ will allow us to prove the following theorem as well.

Theorem 2.10 *On the space* $D_{\mathcal{T}}$*, the finite Hilbert transform and the dif-*

ferentiation operator commute, that is,

$$\mathcal{T} = \mathcal{F} \circ \mathcal{D} = \mathcal{D} \circ \mathcal{F}.$$

Proof: Let $f \in D_{\mathcal{T}}$. We can apply Lemma 2.7 to $h = f'$, and we get

$$\mathcal{F}I(f') = I\mathcal{F}(f') + C\ . \tag{33}$$

Since $f \in D_{\mathcal{T}}$, we have $I(f')(x) = f(x)$. Also, since $\mathcal{F}(f') \in L^2 \subset L^1$, the function $I(\mathcal{F}f')(x)$ is differentiable a.e., and

$$\mathcal{D}I\mathcal{F}(f')(x) = -\mathcal{F}(f')\ .$$

Consequently, the right-hand side of eq.(33) is also differentiable a.e. Therefore, differentiating both sides of that equation we get

$$(\mathcal{D}\mathcal{F}f)(x) = (\mathcal{F}\mathcal{D}f)(x) \text{ a.e.}$$

The proof of the theorem is complete.

2.6 Some properties of the spectrum of $\mathcal{T}$

We will prove in this section some properties of $\mathcal{T}^{-1}$ that allow us to deduce properties of the spectrum of $\mathcal{T}$. In fact, we will show a lower and an upper bound of the biggest eigenvalue of $\mathcal{T}^{-1}$, which is equivalent to showing a lower and an upper bound for the smallest eigenvalue of $\mathcal{T}$.

Theorem 2.11 *The largest eigenvalue of* $\mathcal{T}^{-1}$, λ_1 *satisfies*

$$0.816 < \lambda_1 < 0.912.$$

Proof: Recall from the proof of Theorem 2.1 that we can write

$$\mathcal{T}^{-1} = M_{\rho^{1/2}} \circ B \circ M_{\rho^{1/2}} , \tag{34}$$

and also the notation of the functions $\sqrt{2/\pi}\, U_{n-1}(x)\, \rho^{1/2}(x) = \tilde{U}_{n-1}(x)$. We have seen also in that proof that we can express $h \in L^2$ as

$$h(x) = \sum_{n=1}^{\infty} b_n \tilde{U}_{n-1}(x) = b_1 \tilde{U}_0(x) + \sum_{n=2}^{\infty} b_n \tilde{U}_{n-1}(x) ,$$

so that we can write $\|h\|^2 = \sum_{n=1}^{\infty} |b_n|^2 = 1$. We will estimate the value

$$\lambda_1 = \|\mathcal{T}^{-1}\| = \sup_{\|h\|=1} \|M_{\rho^{1/2}} B(h)\| . \tag{35}$$

When $b_1 = 0$, we have

$$\|M_{\rho^{1/2}} B(h)\| \leq \frac{1}{2} \|h\| \leq \frac{1}{2} .$$

Then, taking the supremum over the functions $\|h\| = 1$ for which $b_1 = 0$, we get

$$\sup_{b_1=0} \|M_{\rho^{1/2}} B(h)\| \leq \frac{1}{2} \; . \tag{36}$$

In the general case, that is, when $b_1 \neq 0$, we get

$$\|M_{\rho^{1/2}} B(h)\|^2 = \|b_1 \tilde{U}_0 \, \rho^{1/2} + \sum_{n=2}^{\infty} \frac{b_n}{n} \tilde{U}_{n-1} \rho^{1/2}\|^2$$

$$= \|b_1 \tilde{U}_0 \rho^{1/2}\|^2 + \|\sum_{n=2}^{\infty} \frac{b_n}{n} \tilde{U}_{n-1} \rho^{1/2}\|^2 + 2 < b_1 \tilde{U}_0 \rho^{1/2}, \sum_{n=2}^{\infty} \frac{b_n}{n} \tilde{U}_{n-1} \rho^{1/2} > \; .$$

We have that $\|\tilde{U}_0 \rho^{1/2}\|^2 = 8/3\pi$, and we get, applying eq.(36)

$$\begin{aligned} &\|M_{\rho^{1/2}} B(h)\|^2 \\ &\leq \frac{8}{3\pi}|b_1|^2 + \frac{1}{2}\sum_{n=2}^{\infty} |b_n|^2 + \frac{4}{\pi}|b_1| \sum_{n=2}^{\infty} \frac{|b_n|}{n} \left| \int_{-1}^{1} \rho^2(x) U_{n-1}(x) dx \right| \\ &= \frac{8}{3\pi}|b_1|^2 + \frac{1}{2}\sum_{n=2}^{\infty} |b_n|^2 + \frac{16}{\pi}|b_1| \sum_{n=3,\, n\, odd}^{\infty} \frac{|b_n|}{n^2(n^2-4)} \; . \end{aligned}$$

The last summation is over odd integers n, because for all even n's the integral vanishes and for all odd n's, it has the value

$$\int_{-1}^{1} \rho^2(x) U_{n-1}(x) dx = -\frac{4n}{n^2(n^2-4)} \; .$$

Changing variables in the sum over odd n's and noting that for all n

$$|b_1 \, b_n| \leq \frac{|b_1|^2 + |b_n|^2}{2} \, ,$$

we obtain

$$\begin{aligned}
\|M_{\rho^{1/2}}B(h)\|^2 &< \frac{8}{3\pi}|b_1|^2 + \frac{1}{2}\sum_{n=2}^{\infty}|b_n|^2 + \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{|b_1|^2+|b_{2n+1}|^2}{(2n+1)^2((2n+1)^2-4)} \\
&= \frac{8}{\pi}|b_1|^2(\frac{1}{3}+\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2((2n+1)^2-4)}) \\
&\quad + \frac{1}{2}\sum_{n=2}^{\infty}|b_n|^2 + \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{|b_{2n+1}|^2}{(2n+1)^2((2n+1)^2-4)} .
\end{aligned}$$

We use partial fraction expansion of $\cot(\pi z)$ to estimate the sum as shown below.

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2((2n+1)^2-4)} &= \frac{1}{4}\left(\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2-4} - \sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}\right) \\
&= \frac{1}{3} - \frac{\pi^2}{32} < 2.5\times 10^{-2} .
\end{aligned}$$

Let us write $C_1 = 2.5\times 10^{-2}$. Then,

$$\begin{aligned}
&\|M_{\rho^{1/2}}B(h)\|^2 \\
&< \frac{8}{\pi}|b_1|^2(\frac{1}{3}+C_1) + \frac{1}{2}\sum_{n=2}^{\infty}|b_n|^2 + \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{|b_{2n+1}|^2}{(2n+1)^2((2n+1)^2-4)} \\
&= |b_1|^2C_2 + \frac{1}{2}\sum_{n=2}^{\infty}|b_n|^2 + \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{|b_{2n+1}|^2}{(2n+1)^2((2n+1)^2-4)} ,
\end{aligned}$$

where

$$C_2 = \frac{8}{\pi}(\frac{1}{3}+C_1) \le 0.912 .$$

We estimate the second and third summands in the above equation as follows.

$$\begin{aligned}
&\frac{1}{2}\sum_{n=2}^{\infty}|b_n|^2 + \frac{8}{\pi}\sum_{n=3, n\, odd}^{\infty}\frac{|b_n|^2}{n^2(n^2-4)} \\
&< \frac{1}{2}|b_2|^2 + \sum_{n=3}^{\infty}|b_n|^2\left(\frac{1}{2} + \frac{8}{\pi}\frac{1}{n^2(n^2-4)}\right) \\
&\leq \sum_{n=2}^{\infty}|b_n|^2\left(\frac{1}{2} + \frac{8}{45\pi}\right) = C_3\sum_{n=2}^{\infty}|b_n|^2,
\end{aligned}$$

where $C_3 = 1/2 + 8/45\pi < 0.557$. Since $C_3 < C_2$, we get

$$\|M_{\rho^{1/2}}B(h)\|^2 < C_2|b_1|^2 + C_3\sum_{n=2}^{\infty}|b_n|^2 < C_2\|h\|^2 .$$

We are considering functions h such that $\|h\|^2 = 1$. Then, taking the supremum over these functions we get finally

$$\sup_{\|h\|=1}\|M_{\rho^{1/2}}B(h)\|^2 \leq C_2 .$$

In conclusion, the norm of $M_{\rho^{1/2}}B$ satisfies the inequality

$$\|M_{\rho 1/2}B\| < C_2 ,$$

and consequently we get

$$\|\mathcal{T}^{-1}\| = \lambda_1 < C_2 .$$

We calculate now the lower bound of λ_1. We know that the largest eigenvalue λ of a symmetric, positive definite self-adjoint operator can be expressed as

$$\lambda = \|\mathcal{T}^{-1}\| = \sup_{f \neq 0} \frac{\|\mathcal{T}^{-1}(f)\|}{\|f\|} . \tag{37}$$

We observe that for $f \equiv 1$ we obtain $\mathcal{T}^{-1}(1) = \rho(x)$. Then

$$\frac{\sqrt{6}}{3} = \frac{\|\mathcal{T}^{-1}(f)\|}{\|f\|} \leq \|\mathcal{T}^{-1}\| .$$

Therefore,

$$0.816 < \frac{\sqrt{6}}{3} \leq \lambda_1 ,$$

which completes the proof.

3 Applications to Potential Theory

3.1 Double Layer Potential

In this chapter, we shall extend those results obtained in Chapter 2 for line segments, to more general curves in the complex plan. There will be a slight change of notation : The basic interval gets moved from $[-1, +1]$ to $[0, l]$, where l represents the length of the curve considered, by an affine transformation between the intervals. We shall work with functions defined on the interval $[0, l]$ rather than $[-1, +1]$, and $\mathcal{T}$, $D_{\mathcal{T}}$ and $\mathcal{F}$, will be transferred to $[0, l]$ in the obvious way by the affine map.

We introduce now the curves which we will consider from now on in this section. Let Γ be an open $\mathcal{C}^2$ curve in the plane, without self-intersections, of finite length l. We will assume that Γ is oriented and parametrized by arc length s

$$\Gamma \; : \; x = \xi(s), \quad y = \eta(s), \quad s \in [0, l].$$

If (ξ, η) is a point on Γ and $z = (x, y)$ a generic point outside Γ, we denote by r be their distance :

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2}.$$

Let ν be the unit normal vector to Γ at a point $(\xi(s), \eta(s))$, that points to the left as one looks along the curve in the direction of increasing arc length.

A *double layer potential with density* f is defined by the integral

$$Kf(z) = -\frac{1}{\pi}\int_{\Gamma} f(s)\frac{\partial}{\partial\nu}(\log r)ds \ , \tag{38}$$

where f is a smooth function of s. Proofs of the properties of a double layer potential that we summarize below can be found, for instance, in [Mik2], Chapter 3, Section 8.3.

At every point z outside Γ, the potential $Kf(z)$ is a C^{∞} function and satisfies the Laplace equation. Also, $Kf(z)$ decreases as $O(1/|z|)$, as $|z|$ goes to infinity.

The limiting value of $Kf(z)$, when z approaches a point $z_o \in \Gamma$, depends on the side of the curve, left or right, from where z approaches $z_o \in \Gamma$. This value is determined and finite, but $Kf(z)$ has a discontinuity, which can be evaluated from the formulas

$$(Kf)(z_o)^- = -\frac{1}{\pi}\int_{\Gamma} f(s)\frac{\partial}{\partial\nu}(\log r)ds + f(s_o), \tag{39}$$

$$(Kf)(z_o)^+ = -\frac{1}{\pi}\int_{\Gamma} f(s)\frac{\partial}{\partial\nu}(\log r)ds - f(s_o), \tag{40}$$

where $(Kf)(z_o)^-$ and $(Kf)(z_o)^+$ signify that the curve is approached from the left and the right respectively.

All directional derivatives of $Kf(z)$ exist when z is a point outside Γ, and can be evaluated by differentiating under the integral sign. Also, they decrease as $O(1/|z|^2)$, as $|z|$ goes to infinity.

A standard calculation allows us to express the directional derivative of $\log r$ at the point $(\xi(s), \eta(s)) \in \Gamma$ in the direction of ν, as

$$\frac{\partial (\log r)}{\partial \nu} = \frac{\eta'(s)\,[x - \xi(s)] - \xi'(s)\,[y - \eta(s)]}{r^2} .$$

Therefore, we can represent Kf in terms of the point $z = (x, y)$ by the formula :

$$Kf(z) = -\frac{1}{\pi}\int_0^l f(s)\,\frac{\eta'(s)\,[x - \xi(s)] - \xi'(s)\,[y - \eta(s)]}{(x - \xi(s))^2 + (y - \eta(s))^2}\,ds . \qquad (41)$$

Theorem 3.1 *Let Γ be an oriented C^2 open curve without self-intersections, of finite length l, parametrized by arc length s as*

$$\Gamma \; : \; x = \xi(s), \;\; y = \eta(s), \;\; s \in [0,\, l] .$$

Let $f(s)$ be in $D_{\mathcal{T}}$, $z_o \in \Gamma$ and ν_o be the unit normal vector to Γ at z_o. Then, the limit of the directional derivative of $Kf(z)$ with respect to ν_o, when z approaches z_o along the normal line to Γ through z_o, exists for almost all values of s. We denote this value by $(\mathcal{T}_\Gamma f)(z_o)$ and write $z_o = (x_o, y_o)$, so that a.e.

$$(\mathcal{T}_\Gamma f)(z_o) = -\frac{1}{\pi}\frac{\partial}{\partial \nu_o}\left(\int_\Gamma^* f(s)\,\frac{\eta'(s)\,[x_o - \xi(s)] - \xi'(s)\,[y_o - \eta(s)]}{(x_o - \xi(s))^2 + (y_o - \eta(s))^2}\,ds\right) .$$

Furthermore, $\mathcal{T}_\Gamma$ is an unbounded self-adjoint operator which can be expressed *as the sum of operators*

$$(T_\Gamma f)(z_o) = (Tf)(z_o) + (Af)(z_o),$$

where A is compact.

Proof: We can write the expression (41) as

$$Kf(z) = -\frac{1}{\pi}\int_0^l f(s)\,\frac{\partial}{\partial s}\left(\arctan\frac{y-\eta(s)}{x-\xi(s)}\right) ds\,.$$

Integrating by parts and recalling that f is continuous and vanishes at the ends of the arc Γ, we get

$$Kf(z) = \frac{1}{\pi}\int_0^l f'(s)\arctan\left(\frac{y-\eta(s)}{x-\xi(s)}\right) ds\,. \qquad (42)$$

We know that the directional derivative of $Kf(z)$ in the direction of the unit vector ν_o exists when evaluated at a point z outside Γ, and it is equal to

$$\frac{\partial Kf(z)}{\partial \nu_o} = \frac{1}{\pi}\int_0^l f'(s)\left(\frac{\xi_o'\,[\xi(s)-x]+\eta_o'\,[\eta(s)-y]}{[\xi(s)-x]^2+[\eta(s)-y]^2}\right) ds, \qquad (43)$$

where we wrote $\xi'(s_o) = \xi_o'$ and $\eta'(s_o) = \eta_o'$. When considering the unit normal vector pointing to the left of Γ, we have that $\nu_o = (-\eta_o', \xi_o')$.

Let L_{z_o} be the normal line to Γ through z_o, and take $(x,y) = z_\epsilon \in L_{z_o}$. Then, we can write for small $|\epsilon|$, $z_\epsilon = \epsilon(-\eta_o', \xi_o') + (\xi_o, \eta_o)$. We simplify the notation by writing $\Delta\xi = \xi(s) - \xi_o$ and $\Delta\eta = \eta(s) - \eta_o$. Then, the expression of the quotient inside the integral on eq.(43) becomes

$$\frac{\xi_o' [\xi(s) - x] + \eta_o' [\eta(s) - y]}{[\xi(s) - x]^2 + [\eta(s) - y]^2} = \frac{\xi_o' \Delta\xi + \eta_o' \Delta\eta}{(\Delta\xi + \epsilon\eta_o')^2 + (\Delta\eta - \epsilon\xi_o')^2} .$$

Let us introduce the notation

$$S(s, s_o, \epsilon) = \frac{\xi_o' \Delta\xi + \eta_o' \Delta\eta}{(\Delta\xi + \epsilon\eta_o')^2 + (\Delta\eta - \epsilon\xi_o')^2} - \frac{s - s_o}{(s - s_o)^2 + \epsilon^2} , \tag{44}$$

which allow us to write eq.(43) as below

$$\frac{\partial K f(z_\epsilon)}{\partial \nu_o} = \frac{1}{\pi} \int_0^l f'(s) \frac{s - s_o}{(s - s_o)^2 + \epsilon^2} ds + \int_0^l f'(s)\, S(s, s_o, \epsilon) ds . \tag{45}$$

Since $f'(s) \in L^1$, it follows from Theorem 1.4 of [Stein], Chapter VI, Section I, the existence a.e. of the limit when ϵ goes towards zero of the first integral of eq.(45), and the equality below that

$$\lim_{\epsilon \to 0} \int_0^l f'(s) \frac{s - s_o}{(s - s_o)^2 + \epsilon^2} ds = \lim_{\epsilon \to 0} \int_{|s - s_o| > \epsilon} \frac{f'(s)}{s - s_o} ds .$$

We study now the convergence to a finite limit of the second integral of eq.(45). Let us show first that the function $S(s, s_o, \epsilon)$ converges to a definite function, for all s, s_o. If $s = s_o$, we have that $S(s, s_o, \epsilon) = 0$, and therefore the limit is zero also. Let us assume, then that $s \neq s_o$. We use Taylor development of second order of the functions $\xi(s)$ and $\eta(s)$ in a neighborhood of s_o. We assume that the neighborhood is defined by $0 < |s - s_o| < \gamma$, for some positive γ. Thus,

$$\Delta\xi = \xi_o'(s-s_o)+(s-s_o)^2P_1 \tag{46}$$

$$\Delta\eta = \eta_o'(s-s_o)+(s-s_o)^2P_2\,, \tag{47}$$

where P_1 and P_2 are functions of s, bounded on $|s-s_o|<\gamma$. If both P_1 and P_2, vanished simultaneous and identically in that neighborhood, $S(s,s_o,\epsilon)$ would also and the proof of the theorem would be completed. Assume then, that they do not vanish in that sense. Replacing the eq.(46) and (47) into (44), we obtain the expression for $S(s,s_o,\epsilon)$

$$S(s,s_o,\epsilon) = \frac{(s-s_o)\,\phi_1(s)}{(s-s_o)^2\,(\phi_2(s)+\epsilon\,C_2)+\epsilon^2} - \frac{s-s_o}{(s-s_o)^2+\epsilon^2}\,, \tag{48}$$

where

$$\phi_1(s) = 1+(s-s_o)\,C_1$$

$$\phi_2(s) = 1+2(s-s_o)\,C_1+(s-s_o)^2\,C_3\,.$$

and $C_1=\xi_o'P_1+\eta_o'P_2$, $C_2=(P_2\xi_o'-P_1\eta_o')/2$ and $C_3=P_1^2+P_2^2$.

By expressing the function $S(s,s_o,\epsilon)$ as in eq.(48), it is easy to verify that, in the case $0<|s-s_o|<\gamma$, $S(s,s_o,\epsilon)$ converges, when ϵ goes towards zero, to function

$$\lim_{\epsilon\to 0} S(s,s_o,\epsilon) = -\frac{C_1+(s-s_o)C_3}{1+2\,C_1\,(s-s_o)+C_3(s-s_o)^2}\,, \tag{49}$$

On the other hand, when $|s - s_o| > \gamma$, $S(s, s_o, \epsilon)$ remains bounded when ϵ goes to zero, and the limit can be calculated from eq.(44). That is, if $|s - s_o| > \gamma$

$$\lim_{\epsilon \to 0} S(s, s_o, \epsilon) = \frac{\xi_o' \Delta\xi + \eta_o' \Delta\eta}{\Delta\xi^2 + \Delta\eta^2} - \frac{1}{s - s_o} . \tag{50}$$

We have proved that the limit of $S(s, s_o, \epsilon)$ exists for all values of s, s_o. Moreover, we can see that the limit function is bounded. That is clear for the cases $s = s_0$ and $|s - s_o| > \gamma$. For the case, $0 < |s - s_o| < \gamma$, we observe on the right-hand side of eq.(49) that close to $s = s_o$ the limit value is C_1. Let us call, for all values of s, s_o, $S(s, s_o)$ the limit function

$$\lim_{\epsilon \to 0} S(s, s_o, \epsilon) = S(s, s_o) .$$

Then, applying Lebesgue's Convergence Theorem, we have

$$\lim_{\epsilon \to 0} \int_0^l f'(s) S(s, s_o, \epsilon) ds = \int_0^l f'(s) S(s, s_o) ds .$$

Consequently we have

$$\lim_{\epsilon \to 0} \frac{\partial K f(z_\epsilon)}{\partial \nu_o} = \frac{1}{\pi} \int_0^{*l} \frac{f'(s)}{s - s_o} ds + \frac{1}{\pi} \int_0^l f'(s) S(s, s_o) ds .$$

In our notation

$$(\mathcal{T}_\Gamma f)(z_o) = \mathcal{T} f(z_o) + A f(z_o),$$

where

$$(Af)(z_o) = \frac{1}{\pi}\int_0^l f'(s)S(s,s_o)ds \ . \qquad (51)$$

Note that, as a consequence of eq.(49), the last integral is not of Cauchy's principal value type.

We still need to prove that A defines a compact operator when consider as an operator of f, and not on f'. For this purpose, we will show that the kernel defining A is bounded. Integrating by parts eq.(51), we obtain

$$(Af)(z_o) = \frac{1}{\pi}\int_0^l f(s)\frac{\partial}{\partial s}S(s,s_o)ds \ .$$

This allows us to write the operator A as $A(f) = k * f$, where

$$k(s,s_o) = \frac{\partial}{\partial s}S(s,s_o).$$

We get after differentiation of $S(s,s_o)$, the following expression for $k(s_o,s)$:

$$-\frac{1}{(s-s_o)^2} - \frac{\xi_o'\xi'(s)+\eta_o'\eta'(s)}{\Delta\xi^2+\Delta\eta^2} + 2\frac{[\xi_o'\Delta\xi+\eta_o'\Delta\eta][\xi'(s)\Delta\xi+\eta'(s)\Delta\eta]}{(\Delta\xi^2+\Delta\eta^2)^2}. \qquad (52)$$

Observe that eq.(52) implies the symmetry of the kernel k, that is,

$$k(s_o,s) = k(s,s_o) \ . \qquad (53)$$

It is not difficult to verify that k is bounded when $|s - s_o| \geq \delta$, for a given positive δ. Let us consider then, that $|s - s_o| < \delta$. Since we are assuming that Γ has continuous curvature, ξ and η are differentiable twice and we can write the equations

$$\begin{aligned}
\xi(s) &= \xi_o + \xi_o'(s - s_o) + O(|s - s_o|^2), \\
\eta(s) &= \eta_o + \eta_o'(s - s_o) + O(|s - s_o|^2), \\
\xi'(s) &= \xi_o' + O(|s - s_o|), \\
\eta'(s) &= \eta_o' + O(|s - s_o|).
\end{aligned}$$

Without loss of generality, we simplify the notation by taking $s_o = 0$. Since Γ is parametrized by arc length, we have that $\xi_o'^2 + \eta_o'^2 = 1$. We first compute the common denominator of expression (52) by inserting in it the finite developments of ξ and η given just above. This gives us the expression

$$s^2 \left((\xi_o' s + O(s^2))^2 + (\eta_o' s + O(s^2))^2\right)^2 = s^4 (1 + O(s))^2. \qquad (54)$$

Note that when divided by s^4, the last expression does not vanish in the neighborhood of zero. We will show that the numerator of eq.(52) can be written in the form $s^4 \times O(1)$, which together with eq.(54) will prove the boundedness of $k(s, s_o)$.

Noting that $\xi_o'^4 + \eta_o'^4 + 2\xi_o'^2\eta_o'^2 = 1$, the first term of eq.(52) multiplied by the common denominator given by eq.(54) takes the form

$$((\xi_o' s + O(s^2))^2 + (\eta_o' s + O(s^2))^2)^2 = -s^4(1 + O(s^2)). \qquad (55)$$

The second term of the numerator, still multiplied by the denominator given by eq.(54), reads

$$-s^2\left(s^2(\xi_o'^4 + \eta_o'^4 + 2\xi_o'^2\eta_o'^2) + O(s)\right) = -s^2\left(s^2 + O(s)\right). \qquad (56)$$

Finally, the third term multiplied by (54) is given by

$$2s^2\left(s^2(\xi_o'^4 + \eta_o'^4 + 2\xi_o'^2\eta_o'^2) + O(s)\right) = 2s^2\left(s^2 + O(s)\right). \qquad (57)$$

Adding eqs.(55), (56) and (57) we find that the numerator of eq.(52) is

$$s^2\Big(-(s^2 + O(s^3))^2 - \left(s^2 + O(s)\right) + 2\left(s^2 + O(s)\right)\Big). \qquad (58)$$

It is easy to verify in the expression above, the cancellation of all fourth and fifth order terms in s. In conclusion, the numerator of eq.(52) has the form $s^4 \times O(1)$ as claimed. This completes the proof of the theorem.

3.2 Some properties of $\mathcal{T}_\Gamma$

We have proved in the previous section that under certain conditions on the curve Γ we are able to express the operator $\mathcal{T}_\Gamma f$ on $D_\mathcal{T}$ as the sum of the operator $\mathcal{T}$ and a compact operator A :

$$\mathcal{T}_r = \mathcal{T} + A.$$

We will show now that the operator $\mathcal{T}_r$ shares some good properties of the operator $\mathcal{T}$, e.g. it is invertible.

The codim(S), or codimension of a subspace S, is defined as the dimension of its orthogonal complement in L^2 : codim(S) = dim($S^\perp$). A linear closed operator F on L^2 is said to be *Fredholm* if satisfies the following two conditions :

(i) the kernel $ker(F)$ is of finite dimension, and

(ii) the range $R(F)$ is a closed space in L^2 and of finite codimension.

The index of a Fredholm operator F is defined as the difference of the dimension of $ker(F)$ and the codimension of $R(F)$:

$$ind(F) = \dim ker(F) - \text{codim } R(F) \ .$$

Lemma 3.2 *The operator $\mathcal{T}$ is Fredholm of index* 0.

Proof: We just need to observe that in the previous Chapter, we have proved that $\mathcal{T}$ is closed, and that $ker(\mathcal{T}) = 0$ and $R(\mathcal{T}) = L^2$.

The proof of the following proposition can be found in [Kato], Chapter IV, Section 3.

Proposition 3.3 *Let F be Fredholm and A be a compact operator. Then, the operator $F + A$ is Fredholm and $ind(F + A) = ind(F)$.*

As immediate corollary we have the following :

Corollary 3.4 *The operator $\mathcal{T}_\Gamma$ is Fredholm and satisfies $ind(\mathcal{T}_\Gamma) = 0$. Hence, $ker(\mathcal{T}_\Gamma) = 0$ if and only if $R(\mathcal{T}_\Gamma) = L^2$.*

Theorem 3.5 *The operator $\mathcal{T}_\Gamma$ is invertible.*

Proof: By Corollary 3.4, in order to prove the theorem we just need to show that $ker\mathcal{T}_\Gamma = 0$.

Suppose f is in the domain of $\mathcal{T}_\Gamma$ such that $\mathcal{T}_\Gamma(f) \equiv 0$ a.e. Let $D(r)$ be the disk of radius r, centered at some fixed point on Γ, with r sufficiently large so that the curve Γ lies inside $D(r)$, and let $\partial D(r)$ be the oriented boundary of $D(r)$. Let $X(r)$ be the set formed by the disc $D(r)$ with the curve Γ removed, that is, $X(r) = D(r) \setminus \Gamma$. The function $u(z) = Kf(z)$ is differentiable on $X(r)$. Let the symbol "$*$" denote the Hodge star operator, that is, for a differential 1-form $dv = adx + bdy$ we have that $*dv = ady - bdx$. In particular, for $u(z)$ we have that

$$*du = -\partial_y udx + \partial_x udy,$$

so that the product by u gives us the 1-form

$$U = u * du = -u\partial_y udx + u\partial_x udy.$$

Then, the exterior derivative of U is given by

$$\begin{aligned} dU &= d(-u\partial_y u) \wedge dx + d(u\partial_x u) \wedge dy \\ &= \partial_x(-u\partial_y u)dx + \partial_y(-u\partial_y u)) \wedge dx + \partial_x(u\partial_x u)dx + \partial_y(u\partial_x u)dy) \wedge dy \\ &= (\partial_x(u\partial_x u) + \partial_y(u\partial_y u))dx \wedge dy \\ &= (|\nabla u|^2 + u\Delta u)dx \wedge dy, \end{aligned}$$

where ∇u denotes the gradient of u. Notice that U is bounded on $X(r)$. Applying Stokes' Theorem to the differential form dU and the domain $X(r)$, we obtain

$$\int\int_{X(r)} dU = \int_{\partial X(r)} U .$$

Since $u(z)$ satisfies Laplace's equation at every point outside Γ, the left-hand side of eq.(59) is

$$\begin{aligned} \int\int_{X(r)} dU &= \int\int_{X(r)} |\nabla u|^2 dA + \int\int_{X(r)} u\,\Delta u\, dA \\ &= \int\int_{X(r)} |\nabla u|^2 dA , \end{aligned} \tag{59}$$

From the right-hand side of eq.(59), we obtain

$$\int_{\partial X(r)} U = \left(\int_{\partial D(r)} + \int_{+\Gamma} + \int_{-\Gamma}\right)(u * du)ds ,$$

where s is the element of arc. By assumption, $\mathcal{T}_\Gamma(f) = 0$, so the integrals along $+\Gamma$ and $-\Gamma$ vanish. Therefore,

$$\int_{\partial X(r)} U = \int_{\partial D(r)} (u * du) ds ,$$

As was mentioned at the beginning of Section 3.1, $u(|z|)$ and $du(|z|)$ decrease at infinity as $O(1/|z|)$ and $O(1/|z|^2)$ respectively. Therefore,

$$\int_{\partial D(r)} |u * du| ds < \frac{C}{r^2} .$$

As r goes to infinity, the integral above vanishes and we get

$$\lim_{r \to 0} \int_{\partial X(r)} U = 0 . \qquad (60)$$

Then, we get from eq.(59) and (60)

$$\int\int_{X(r)} |\nabla u|^2 dA = 0 .$$

This implies that $\nabla u \equiv 0$. Therefore, u is a constant function. But, when $|z|$ goes to infinity u satisfies the inequality

$$|u(|z|)| \leq \frac{C}{|z|} .$$

It follows that $u = Kf \equiv 0$. This, together with the jump relations given by eqs.(39) and (40), implies that $f \equiv 0$. The theorem is proved.

Theorem 3.6 *The operator* $\mathcal{T}_\Gamma^{-1}$ *is compact.*

Proof: We have proved in Theorem 3.1 that $\mathcal{T}_\Gamma$ can be written as $\mathcal{T}_\Gamma = \mathcal{T} + A$, where A is a compact operator. By Corollary 3.4, we know that $\mathcal{T}_\Gamma$ is a Fredholm operator, and as it was shown in the proof of Theorem 3.5, its range is the whole space L^2, and therefore closed. By the Closed Graph Theorem, we deduce that $\mathcal{T}_\Gamma^{-1}$ is bounded. Also, by Theorem 3.5, $\mathcal{T}_\Gamma$ is invertible, so that we can define B the operator given by the composition

$$B = -\mathcal{T}_\Gamma^{-1} \circ A \circ \mathcal{T}^{-1} .$$

We show now that $\mathcal{T}_\Gamma^{-1}$ can be written as

$$\mathcal{T}_\Gamma^{-1} = \mathcal{T}^{-1} + B . \tag{61}$$

Namely,

$$\begin{aligned} \mathcal{T}^{-1} + B &= \mathcal{T}^{-1} - (\mathcal{T}_\Gamma^{-1} \circ A \circ \mathcal{T}^{-1}) = (\mathcal{T}_\Gamma^{-1} \circ \mathcal{T}_\Gamma \circ \mathcal{T}^{-1}) - (\mathcal{T}_\Gamma^{-1} \circ A \circ \mathcal{T}^{-1}) \\ &= \mathcal{T}_\Gamma^{-1} \circ (\mathcal{T}_\Gamma - A) \circ \mathcal{T}^{-1} = \mathcal{T}_\Gamma^{-1} . \end{aligned}$$

Therefore, eq.(61) is satisfied. Since both B and $\mathcal{T}^{-1}$ are compact, we can conclude that $\mathcal{T}_\Gamma^{-1}$ is compact also.

3.3 Solution of the Neumann problem for the curve

We consider now the following Neumann problem for the Laplace equation in the exterior of Γ, in the complex plane :

> Given $g(s)$ a function in $L^2([0,l])$, find a solution $u(z)$, of the Laplace equation in the exterior of Γ such that
>
> (i) For almost all $z_o \in \Gamma$, the limit of the directional derivative of u, on the direction of the unitary normal to Γ through z_o, as z approaches z_o along L_{z_o}, the normal line to Γ through z_o, is $g(s)$.
>
> (ii) For all $z_o \in \Gamma$, as z approaches z_o along L_{z_o} from the left of Γ, $u(z)$ has a limiting value $u^-(z_o)$. Respectively, if z approaches z_o along L_{z_o} from the right of Γ, $u(z)$ has a limiting value $u^+(z_o)$. The limits from left and right, $u^-(z_o)$ and $u^+(z_o)$, do not need to agree, but the limits from each side depend continuously on z.

$$\left.\begin{aligned} \Delta u \equiv\ & 0\ , && \text{on } \mathbb{C}\setminus\Gamma\ , \\ \lim_{z\to z_o, z\in L_{z_o}} \partial_\nu u(z) =\ & g(z_o)\ , && \text{a.e. } z_o \in \Gamma\ . \end{aligned}\right\} (NP_\Gamma)$$

In fact, by Theorem 3.1 and the discussion just before its statement, if $f = \mathcal{T}_\Gamma^{-1}(g)$, the double layer potential Kf has the desired properties for a solution of (NP_Γ), which now can be read as below.

Given a function g in $L^2([0,l])$, find a function $f(s)$ that satisfies the condition :

$$\mathcal{T}_\Gamma f(z_o) = g(s_o), \quad \text{for almost all } s_o \in [0,l]. \qquad (62)$$

As a consequence of Theorem 3.5, we have the following result :

Theorem 3.7 *Let g be in $L^2([0,l])$. The Neumann Problem (NP_Γ) admits a solution. Such a solution is given by Kf, where $f = \mathcal{T}_\Gamma(g)$.*

Finally we note that the methods used here can be applied to the Neumann problem for the Helmholtz operator. This is possible because the fundamental solution of the Helmholtz equation has the same singularity as that for the Laplace equation.

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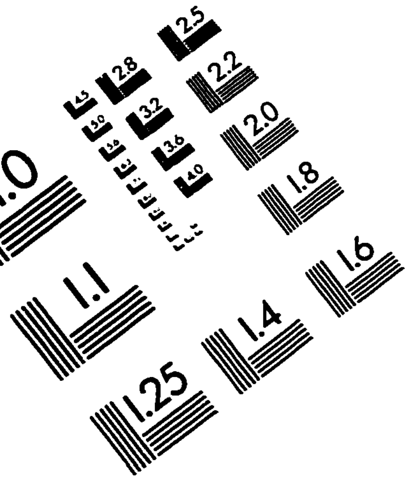

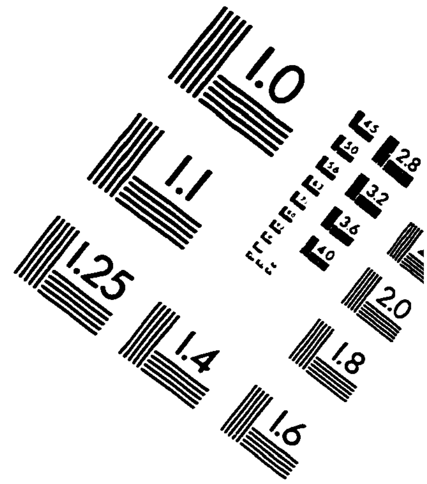

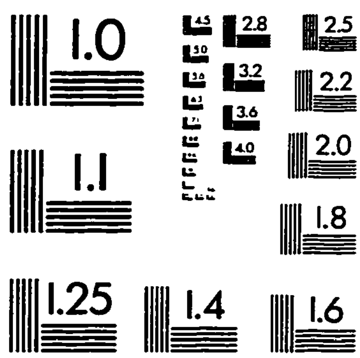

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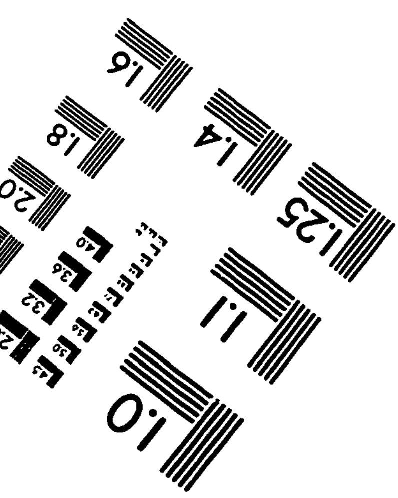

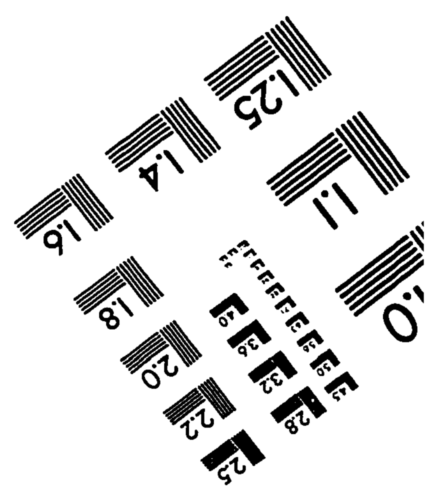

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