Essays on Financial Derivatives

Essay One: Predicting future volatility in currency options market

Essay Two: Calibration of Heston's model in currency options market

Essay Three: Pricing European options and calibration in equity-linked derivatives market

by

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A dissertation submitted to the Graduate Faculty in Economics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

2007

UMI Number: 3283203

ProQuest Information and Learning Company
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, MI 48106-1346

This manuscript has been read and accepted for the
Graduate Faculty in Economics in satisfaction of the
dissertation requirement for the degree of Doctor of Philosophy.

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Abstract

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Atilim Murat

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This paper examines the relation between New Zealand dollar (NZD) - US dollar (USD) exchange rate volatility implied in option prices and realized volatility. To the best of our knowledge, this is the first paper which uses Garman-Klass estimator of realized volatility to predict future volatility in currency options. Using Generalized Method of Moments estimation (GMM) consistent with observations evidence suggest that implied volatilities give better forecasts of future volatility than the GARCH model and historical models. However, using Garman-Klass estimator increases the efficiency of the realized volatility and provides some additional information to implied volatility.

My second essay investigates the calibration technique applied to Heston's stochastic volatility model in currency options markets. I use closed form solution to price European vanilla call and put options and calibrate the Heston's model to market data. My study differs from the previous studies since I didn't use a penalty function on parameters. when I calibrated the Heston's model to market volatility smile. Another difference from the previous studies is that instead of using spot market prices, I'm using the derivative prices and I calibrate the model to derivative prices. I find that calibrating Heston's model to market data, under no penalization, gives good fitting for the maturities between one month to two years.

My third essay investigates the European vanilla option pricing and calibration procedure in equity options market with using Heston's stochastic volatility model. I consider Euro Stoxx 50 market volatilities for vanilla call and put options. To the best of our knowledge, wide range of call and put deltas for Risk reversals and Butterflies were firstly used in the literature in my paper. Even though there were some small fitting problems in very short maturity, the calibration method produced very good results for short, medium and long term option maturities. Important distinction of my study is, I didn't use a penalty function on calibration parameters.

ACKNOWLEDGEMENTS

I would like to express my gratefulness to my advisor Prof. Salih Neftci, for his ideas and guidance throughout my dissertation . I would like to express my thanks to Prof. Michael Grossman, for his kind support during my Ph.D. I would also like to express my thanks to Prof. Nusret Cakici, for his opinions on my dissertation.

I would like to thank my parents for their encouragement and support which has enabled me to do what I do.

I would especially like to thank my dear wife, Berken, for being an undying emotional inspiration on this life.

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Essay One:

Predicting future volatility in currency options market

1. Introduction

If we categorize the market risk of a book of options, it is divided into spot risk (changes in the price of the underlying), volatility risk, and interest rate risk. Among these risks, traders, investors are mostly concerned with volatility risk.

In the Black- Scholes model, the parameter volatility σ is the actual unknown. The Black- Scholes price of a vanilla option is strictly increasing in its input parameter volatility and one can set the price he thinks is right by changing the volatility. But how can we find this parameter? We can calculate this parameter by looking at the history and calculate what is called historical volatility.

However; this quantity tells us something only about the past but nothing about the current situation where the asset will be. We can find this volatility from a traded vanilla option. The volatility implied by a traded vanilla option is called the implied volatility. This implied volatility also changes with different degrees of moneyness.[1]

Furthermore, implied volatility backed out from option price is likely to be a good predictor of subsequent observed volatility if the option market is efficient and the option pricing model is correct. If the market is efficient (no manipulation and no insider information) and complete (if hedging opportunities exist), the market's estimate, the implied volatility is the best possible forecast given the currently available information.

[1] Moneyness could be explained by spot price/strike price

All information necessary to explain future realized volatility generated by all other explanatory variables in the market should be included in the implied volatility.

Most of the studies have focused on stock index options, whereas this paper concentrates on the predictive power of implied volatility in the currency options market. An advantage of using currency options is that simultaneous recording of exchange rates and option prices makes easier the synchronization trading problem usually seen in the studies of stock options.

Implied volatility could be explained as the prediction of the market throughout the option's remaining life. How good is the implied volatility found from the current option prices to predict future volatility? Are they better than the time series forecasts like GARCH or historical volatility? Whether implied volatility predicts future volatility and whether it does so efficiently, could be empirically tested. The time series literature has produced very different results. Several authors have suggested that the implied volatility in option prices is the market's forecast of the future volatility.

Jorion (1995) reports that implied volatility is an efficient, though biased, predictor of future return volatility for foreign currency futures but outperform standard time-series models in terms of informational content. He performs tests for the Swiss franc and for the Japanese yen. He used daily observations and long data. He found overlapping problems and used White procedure to correct this problem.

Fleming (1998) studies options on the S&P 100 equity index traded at the Chicago Board Options Exchange. He found that implied volatilities are upward-biased predictors.

In contrast, Day and Lewis (1992), who observe S&P 100 index options with expiries from 1985-1989, and Lamoreux and Lastrapes (1993), who study options on ten stocks with expiries from 1982 to 1984, conclude that implied volatility is biased and inefficient: past volatility contains predictive information about future volatility superior than that included in implied volatility.

These studies are characterized by a maturity fitting problem, in that Lamoreux and Lastrapes (1993) examine one day ahead and Day and Lewis (1992) observe one week ahead predictive power of implied volatilities computed from options that have a much longer remaining life.

Canina and Figlewski (1993) claimed that implied volatility does not incorporate information contained in recent observed volatility and have no correlation with future return volatility on S&P 100 index option. Their conclusion is not in parallel to the base of option pricing theory because option price is based on the underlying asset price. It is unreasonable that volatility implied from option market has no relationship with the volatility of the underlying asset unless the index option market is inefficient. This conclusion of Canina and Figlewski is unlikely given the depth, liquidity and trading activity in the OEX[2] options market. Another possibility, which leads Canina and Figlewski to find that conclusion, is that the Black and Scholes (1973) option pricing model, which is used to compute implied volatility, cannot be used to price index options because of excessive transaction costs associated with hedging of options in the cash index market.

This conclusion is also unlikely, though. Black- Scholes formula does not necessarily require continuous trading in cash markets. Constantinides (1994) shows that transaction costs cannot explain the flaw of the Black-Scholes pricing model.

Black-Scholes can be thought of as volatility forecast, it can be also interpreted as a measure of an option's price- one that controls for option-specific characteristics such as the moneyness of an option, time to expiry, etc.

Hull and White (1987) show that when volatility is constant, the Black-Scholes implied volatility of an ATM (at-the-money) option approximately equals to the expected future realized volatility during the option life. It means that if option market is information efficient in reflecting underlying asset market, then the implied volatility should form an unbiased forecast of future realized volatility over the option life.

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[2] OEX options: the Standard & Poor's 100, a stock index on the Chicago Board Options Exchange.

Christensen and Prabhala (1998) find that in the period after the 1987 stock market crash, the implied volatility from one-month at the money OEX call options in fact is an unbiased and efficient forecast of ex-post realized index volatility. Their sample covers the period beginning in November 1983 through May 1995. Their research is different from previous studies on two things.

First they use volatility data sampled over a long period of time than in previous studies. This increases statistical power. Second, Christensen and Prabhala, sample the implied and realized volatility series at a lower frequency (monthly). According to their results, using overlapping data affects the prediction power of historical volatility. They find that implied volatility, sampled at monthly level, outperforms past volatility in forecasting future volatility and even contains the information content of past volatility in some specification.

Amin and Ng (1997) focus on the euro dollar options. According to their results, implied volatilities contain more information about future volatility than statistical time series models. Another result of their research is that explanatory power of implied volatilities is improved by the use of historical information.

Vasilellis and Meade (1996) find that volatility predictions implied by the current options market are better forecasts than those based on stock market. They found that GARCH forecasts have better results, significant marginal information.

Campa and Chang (1998) compared correlations implied from option prices with subsequent realized correlations. They work with over-the-counter options on spot currencies, and obtain results in line with the related research on implied volatilities, i.e., historically based forecasts contribute no incremental information to implied correlations.

In conclusion, recent literature offers clear evidence that option prices contains information about future asset returns volatility that cannot be extracted from past returns.

In this paper, I examine whether this conclusion also apply to calls on the New Zealand dollar- US dollar spot exchange rate traded in Chicago Mercantile Exchange (CME). I extract implied volatility from short term at the money NZD-USD options by employing the Garman and Kohlhagen (1983) currency option valuation model, the extension of the Black- Scholes (1973) model.

I try to examine the ability of implied volatilities computed with this model to provide information about future volatility and to test whether time series predictions contain additional information to implied volatilities. Stochastic volatility models present numerical difficulties and some additional parameters should be estimated with these models. Moreover, the gain from using stochastic volatility models may be limited for short term at-the-money options.[3]

The new contribution to the finance literature from this paper is, to the best of our knowledge, Garman-Klass estimator of realized volatility was used first time in the literature in this paper to predict future volatility in currency options market. Furthermore, the Garman- Klass estimator's predictive power was compared with the predictive power of implied volatility, time series volatilities and other measures of realized volatilities. Another new contribution is, I use closest to money options and at-the-money options together.

The out-of-sample tests focus on examining the quality of the implied volatility as a beforehand volatility forecast over a fixed horizon. Statistical problems could be seen in testing of unbiasedness. Serial correlation occurs and orthogonality restrictions could be seen. One of the best efficient ways to solve these problem in the literature is using the the Generalized Method of Moment (GMM) estimator. I will apply GMM estimator of Hansen (1982). These tests suggest that the implied volatility from NZD/USD options is an upward biased forecast of future volatility of the NZD/USD exchange rate, and the GARCH volatility and historical volatility (Standard deviation realized volatility, Parkinson estimator, Garman & Klass estimator) observations do not present significant incremental information. But Garman-Klass estimator of realized volatility increases the efficiency of the prediction power of historical volatilities.

[3] According to Jorion (1995),stochastic volatility models don't perform well in short-term

2. Data and Sampling Procedure

Data in this paper are, daily New Zealand dollar-US dollar closest to the money and at-the-money calls closing prices from 12 December 2001 to 12 December 2006, provided by Chicago Mercantile Exchange (CME).[4] NZD/USD options are highly liquid and the NZD/USD exchange rate is highly volatile.

NZD/USD calls are of the European style, and mature on the first business day of the corresponding month of expiration. Our data covers 60 expiration cycles. The calls in our very first cycle maturing on the first business day of January 2002 and the last one of the calls maturing on the first business day of January 2006.

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I also used daily exchange rate and interest rate futures of NZD/ USD which were also provided by from Chicago Mercantile Exchange's data mine. Interest rate futures are due on the first business days of the equivalent month. Open, average, high and low prices of the NZD/USD exchange rate were provided by Bloomberg.

Another important point is, I wanted to diminish the effect of option expirations. I chose options which are the nearest, but with at least 15 business days, to maturity.

3. Computing implied volatility

[4] http://www.cme.com

Implied volatility was calculated from the call option price. I calculated the volatility with using closing prices.Our option pricing model is the Garman- Kohlhagen model; it is an extension of the Black-Scholes model. To limit the effect of option expirations, in our sampling procedure we wanted to pick options which are the nearest to maturity.

The closest to the money call was chosen in each cycle, considering the NZD-USD futures prices in the market on that day. The closest to the money option was picked since the nearest to the money option for each maturity day, has the most price sensitivity with respect to volatility.

Another reason for picking the nearest-to-the money option could be related to imperfection of Black- Scholes. In other words, clear inconsistency of recovering a volatility prediction of the Black-Scholes .[5]

According to Feinstein (1989) under the assumption that volatility is uncorrelated to returns, the linearity turns Black-Scholes implied volatility into a virtually unbiased estimator o future volatility for those options.[6]

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[5] Black-Scholes formula is very successful option pricing formula but it has a widely known flaw, Black-Scholes option pricing model assumes that volatility is known and constant.

Synchronization of prices is a big problem in the options market. To reduce the effect of these errors, I compute implied volatilities using the price of the NZD/USD future contract expiring in the same day of the option contract, instead of using directly the spot market price. In other words, I used future price instead of spot price in the Garman-Kohlhagen model.[7]

Therefore, from daily vanilla call price C_t, implied volatility $\sigma_{i.T}$ is calculated by :

$$C_t = \frac{1}{(1+r_t)^{T_t}} \left[F_t N(d_t) - E_t N\left(d_t - \sigma_{i,t}\sqrt{T_t}\right)\right] \quad (1)$$

Where

$$d_t = \frac{\ln\left(F_t / E_t \right)}{\sigma_{i,t}\sqrt{T_t}} + \frac{1}{2}\sigma_{i,t}\sqrt{T_t} \quad (2)$$

The number of days to maturity is shown as T_t , r_t is the daily interest rate, F_t is the future price of the NZD/USD expiring in T_t days, and $N(.)$ is the standard normal

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[6] Feinstein (1989) expressed that, for the short-term at-the- money options, the Black-Scholes formula is almost linear in its volatility argument.

[7] To connect the futures to spot prices, cost of carry arbitrage equation could be used . Fleming (1998) used this formula.

distribution function. The interest rate is the short term interest rate future contract that expires in T_t days.

The foreign currency option pricing model of Garman and Kohlhagen (1983) Barone-Adesi and Whaley (1987) adjustment of early exercise premium is adopted to extract implied volatilities from the NZD/USD options.

-The market is frictionless: trading takes place continuously in time, there are no transaction costs, taxes, or short sale restrictions.

-The instantaneous risk-free interest rate r_f and domestic/foreign interest differential $b = r_d - r_f$ are known and non-stochastic.

-The spot exchange rate follows a Geometric Brownian Motion process.

$$dS(t) = \mu S(t) dt + \sigma S(t) dz, \tag{3}$$

The price of an European call option on foreign currency is

$$C_t = S_t e^{-r_f T} N(d_1) - K e^{-rT} N(d_2), \tag{4}$$

$$d_1 = \frac{\log(S_t / K) + (r_d - r_f + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau} \tag{5}$$

Where S_t is the spot exchange rate at time t in every unit of foreign currency; μ is the instantaneous rate of appreciation of the foreign currency; r_d , r_f is the domestic and foreign risk-free interest rates , which is known and nonstochastic ; $\tau = T - t$, the time to maturity of the option ; K is the striking price of an option on one unit of foreign currency; $N(.)$ is the cumulative standard normal distribution function ; C_t is the domestic currency price at time t of a call written on one unit of foreign currency.

4. Time series predictors of future volatility

I test the predictive power of implied volatilities and compare this with GARCH and historical volatilities. Returns are computed using the average daily prices of the NZD/USD spot exchange rate, and I first consider observing GARCH time series model as in my tests.

The GARCH (1, 1) model of Bollerslev (1986) is specified as

$$R_t = \lambda_o + \varepsilon_t, \qquad \varepsilon_t \mid \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots \sim \mathrm{N}\,(0, \mathrm{h}_t^2) \tag{6}$$

$$h_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1}^2 \quad (7)$$

$$h_{t+k,t}^2 = \alpha_0 + \alpha_1 E\left[\varepsilon_{t+k-1}^2 \middle| \omega_t\right] + \beta_1 h_{t+k-1,t}^2$$

$$= \alpha_0 + (\alpha_1 + \beta_1) h_{t+k-1,t}^2 \quad , \quad k = 1,\ldots,N \quad (8)$$

$R_t = \sqrt{252}\ln\left(S_t / S_{t-1} \right)$ is the change in the logarithm of the exchange rate ratios from time t-1 to t, ε_t is assumed to be conditionally normal mean zero and conditional variance h_t^2, ω_t is the information set. The exchange rate is measured as U.S. dollars per New Zealand dollar. In the variance equation, the parameter estimates should be positive; $\alpha_0 \succ 0, \alpha_1 \succ 0, \beta_1 \succ 0, \alpha_1 + \beta_1 \prec 1$. The GARCH models are estimated by Maximum Likelihood: the t-statistics are computed using the robust inference procedure proposed by Bollerslev and Wooldridge (1992).

I found the results for GARCH (1, 1) estimation for the period of December 2001 to December 2006. Results are on Table I.

I have to emphasize that from my GARCH (1, 1) results, $(\alpha_1 + \beta_1)$ equals 0.98, we could say that the process is stationary. In parallel with my previous expectations our results are showing that GARCH (1, 1) model is highly significant.

GARCH (1,1) model's forecast horizon should be identical to that of the option maturity, the T-day GARCH variance is found by substitution of the (T+1) ,one period, ahead GARCH (1,1) variance forecast $h^2_{t+1,t}$ to retrieve a k-period ahead prediction of the conditional variance $h^2_{t+k,t}$, then the predictions are averaged over the forecast period.

If we want to add something on out of sample prediction, parameters could be estimated from the previous year are used in the GARCH specification to predict future volatility at each time t.

It can be shown as:

$$h^2_{t+k,t} = \alpha_0 \left[\frac{1-(\alpha_1+\beta_1)^{k-1}}{1-(\alpha_1+\beta_1)} \right] + (\alpha_1+\beta_1)^{k-1} h^2_{t+1,t} \tag{9}$$

If we want to show the T period ahead average GARCH volatility, it could be shown as:

$$\sigma^2_{G,t} = \frac{1}{T}\left(\sum_{k=1}^{T} h^2_{t+k,t} \right) \tag{10}$$

5. Methods of evaluating realized volatility:

Volatility cannot be directly observed. Realized (historic) volatility shows past variations in a share price or an option price. There are several methods to estimate realized volatility. The basic approach is to use the concept of standard deviation of return on exchange rate,

$$\sigma_{st} = \sqrt{\frac{1}{n-1}\sum_{t=1}^{n}(r_t - \bar{r})^2} \tag{11}$$

where $r_t = \ln(\frac{S_t}{S_{t-1}})$ is the yield of the exchange rate calculated as the natural log of the ratio of close prices for the current and previous days , S_t is the close price at a day t

$r_t = \ln(\frac{S_t}{S_{t-1}})$,

t =1: n ; $\bar{r} = \frac{1}{n}\sum_{t=1}^{n} r_t$ is the average price of the NZD-USD per an n-day period.

Thus, the realized volatility calculated as the standard deviation of return on exchange rate, characterizes the spread of possible returns of exchange rate about the mean value of return.

Taylor and Xu (1997) and Andersen and Bollerslev (1998) claims that measurement errors in the estimation of realized volatility may misrepresent conclusions about the informational content of volatility forecasts.[8]

To get better results , I use the Parkinson and Garman and Klass estimators, which increases the efficiency of realized volatility measures by using daily low and daily high prices of financial asset during a day. To the best of our knowledge, Garman-Klass estimator was firstly used here in the literature to increase efficiency of prediction of future volatility for currency options.

The value of the indicator based on maximum/minimum prices allows tracking volatility extrema and predicting its further variation. The qualities of our measures are improved by using the Parkinson indicator and Garman and Klass indicator.

Parkinson estimator is:

$$\sigma_{pt} = k\sqrt{\frac{1}{n}\sum_{t=1}^{n} p_t^2} \tag{12}$$

[8] One important difference about their research was, they used the high frequency intra-day data.

Where $p_t = \ln \frac{H_t}{L_t}$, L_t and H_t are the low and high prices of financial asset, respectively, at day t , t = 1 : n, and $k = \frac{1}{2\sqrt{\ln 2}} \approx 0.601$

Garman and Klass (1980) have proposed an approach to estimate volatility using open prices. A formula for calculating volatility (open-high-low-close estimator-OHLC) is

$$\sigma_{gk} = \sqrt{1/n \sum_{t=1}^{n} \left[\frac{1}{2} p_t^2 - (2\ln 2 - 1) q_t^2 \right]} \qquad (13)$$

where $q_t = \ln \frac{S_t}{O_t}$, O_t is the open price at day t , t = 1: n.

6. EMPIRICAL RESULTS

6.1. Implied volatility versus realized volatility:

I want to find out the predictive ability of implied volatilities by regressing realized volatility on implied volatility.

$$\text{Realized}_t = \alpha_1 + \beta_1 \sigma_{i,t} + \varepsilon_{1,t} \tag{14}$$

In both of our series, we do have a high serial correlation.[9] Serial correlation will not affect the unbiasedness or consistency of OLS estimators, but it affects its efficiency. It arises by using overlapping observations rather than restricting the analysis to independent observations. The reason of overlapping is, we only have sixty option cycles for our prediction. By contrast we have 1320 days of data. Describing statistics of implied volatility, GARCH volatility and realized volatilities can be seen on Table II.

I perform Augmented Dickey-Fuller (1979) and Phillips-Perron (1988) tests to observe the existence of unit root. If volatility series owns a unit root, regression is spurious. Using the ADF and Phillips-Perron tests, I reject the unit root hypothesis for all series. The results of the unit root tests can be seen on Table III.

I want to find whether the volatility prediction gives information about future volatility, if it contains information, then the slope should be statistically different from zero. If the prediction is unbiased, then the intercept should be zero and the slope should be one. Since the possibility of measurement errors in independent variables could be seen in these types of series, Scott (1992) and Fleming (1998) used GMM estimation in order to deal with some errors in the variables. Following Scott and Fleming, I performed GMM estimation, using lagged independent variables as instruments.

[9] Serial correlation occurs in time-series studies when the errors associated with a given time period carry over into future time periods.

Residual autocorrelation could also be seen because of using overlapping observations rather than restricting the analysis to independent observations. If we examined the independent observations, we wouldn't have this problem. This could result in inefficient slope estimates and spurious explanatory power. I correct this employing the Generalized Method of Moments estimator (GMM) of Hansen (1982), along with Newey and West (1987) approach to estimate heteroskedasticity and autocorrelation consistent variance-covariance matrix.

The variance-covariance matrix S_t is not need fully positive semi-definite, kernel estimators are then applied,

$$\hat{S}_t = \hat{\Omega}_0 + \sum_{j=1}^{m} w(j,m)(\hat{\Omega}_j + \hat{\Omega}'_j) \qquad (15)$$

Where w (.) is a weighting function (kernel) and m is bandwidth parameter. The Bartlett kernel in Newey and West (1987) is w (j, m) = 1-(j/m+1), the weighted sum in calculating $\hat{S}_t$ with the weight attached to auto covariance matrix $\hat{\Omega}_j$ by a sequence of weights that decrease as j increase. A plug-in bandwidth estimator for Bartlett kernel is used. . (Nielsen (2005), Guo (2001), Wooldridge (2001) used the same approach)

The regressions of realized volatility outcomes, as we measured with the Parkinson, Garman - Klass, and standard deviation measured realized volatility, on implied volatility $(\sigma_{i,t})$ are illustrated in Table IV.

Slope coefficients less than one suggest that volatility is too high on implied volatility, that means on average a change in implied volatility does not fully translate into changes in realized volatility, but it should be reduced proportionally.

According to my prior expectations , the R^2 of regression suggest that the Parkinson estimator and the Garman-Klass estimators are better indicators in measuring realized volatility than the standard deviation of returns.

T-statistics on the coefficients of implied volatilities are very high, 7.30 for SD_t, 9.27 for PK_t and 12.1 for GK_t, I strongly reject the null hypothesis and I can say that implied volatilities carry no information about future volatility. Implied volatility tend to exaggerate the importance of future volatility, in other words, it is an upward biased estimator of future volatility.

6.2. Comparing implied volatility and time series volatility forecasts

It can be seen that valuable information about future market volatility could be gained from implied volatility because I found that implied volatility is an upward biased estimator. We can now compare the informational content of implied volatility and time series models.

I perform regressions of realized volatility (SD_t, PK_t, GK_t) on time series forecasts ($GARCH_t$) and compare adjusted R^2's with the regressions using implied volatility.

I also perform regressions of realized volatility on implied volatility and on time-series forecasts at the same time.

$$\text{Realized}_t = \alpha_0 + \beta \; timeseriesforecast_t + \varepsilon_t \qquad (16)$$

$$\text{Realized}_t = \alpha_0 + \beta_1 implied_t + \beta_2 timeseriesforecast_t + \varepsilon_t \qquad (17)$$

The R^2 of the regressions from equations in (16) and (17) with one independent variable suggest that implied volatility includes more information about future volatility than historical predictions, considering the Parkinson estimator, Garman-Klass estimator and the standard deviation estimator of the realized volatility.

My results of regressing realized volatility on more than one independent variable show that the implied volatility gives valuable information about future volatility which is not observed by statistical models based on historical returns. It can be seen from our results and regressions that its coefficient is always significantly different from zero.

The results of standard deviation (SD_t) measure of the realized volatility could be seen in Table IV. According to our results, it could be said that the implied volatility is the only significant variable. When the Garman-Klass and the Parkinson estimators of realized volatility are used, Tables VI and VII show that the coefficients of historically-based forecasts are significantly different from zero. This could be observed as the volatilities from historical forecasts give some added information to implied volatility.

6.3. Orthogonal restriction tests

To measure the predictive power of implied volatility, GARCH volatility, and historical volatility, the encompassing principle of Hendry and Richard (1982), Fair and Schiller (1990) could be used. This principle shows the ability of a model to explain the behavior expressed in the relevant characteristics of rival models. If a volatility forecast doesn't include good information regarding the dependent variable, we would expect the estimate of the coefficient for the particular forecast to be statistically insignificant.

A general model for the encompassing tests is formulated as:

$$\sigma_{F,t+1}^2 = \alpha_0 + b_1\sigma_{I,t}^2 + b_2\sigma_{G,t}^2 + b_3\sigma_{H,t}^2 + \xi_{t+1} \quad (18)$$

The orthogonal restriction implies that the historical volatility and average GARCH volatility ought to contain no information contained in the implied volatility for predicting future volatility.

Tables VI, VII, VIII show the encompassing tests for implied volatility, average GARCH volatility and realized (historical) volatilities. If an independent variable carries no useful information regarding the evolution of the dependent variable, we would say the coefficient of that independent variable to be insignificantly indifferent from zero.

Interestingly, when the Garman-Klass estimator (GK_t) is used, as shown in Table VI, coefficient of GARCH forecast is significantly different from zero. We could say that a GARCH prediction gives some small information to implied volatility. The coefficient of GARCH forecast is also significantly different from zero when Parkinson estimator of the realized volatility is used. However, it doesn't give better information than implied volatility and it is still less significant than implied volatility, which can be seen in Table VII . But when using standard deviation (SD_t) estimator of realized volatility is used, implied volatility is the only significant variable, results are on Table VI.

With using Garman- Klass and Parkinson estimators, the R^2 of the regression on implied volatility is higher than when using standard deviation estimator of realized volatility. According to our results, Garman-Klass estimator has higher R^2 than Parkinson and Standard deviation estimators of realized volatility.

I also performed Wald tests to check unbiasedness $(\alpha_0 = 0, \beta_1 = 1)$ and rejected the null with 99% confidence (at the 1% level) and this provides evidence that implied volatilities are biased predictors of future volatility. These results can be seen in Tables V, VI , VII ; p- values are reported in parenthesis for χ^2_A , χ^2_B statistics.

The Wald[10] statistic χ_A^2 shows that the estimates of intercept and slope parameters are significantly different from zero. The $\overline{R}^2$ statistics illustrates that implied volatility has significant explanatory power in foretelling future volatility.

7. Conclusion:

I used the Garman-Klass estimator of the realized volatility to forecast future volatility of currency options first time in the literature, to best of our knowledge. I examined the predictive ability of implied volatility and time series models for the future volatility. These implied volatilities are extracted from the New Zealand dollar/ US dollar currency options. My results showed that NZD/USD volatility implied in prices of calls with using Garman- Kohlhagen option pricing model, gives information about future volatility.

One problem is according to our results, implied volatility is an upward biased estimator of the future volatility of NZD/USD exchange rate. The reason of this upward bias could be restrictions of our model, which means our pricing model couldn't recover this biasedness.

Moreover it can be said that higher option prices exist because of exaggerated volatility or because of market inefficiencies like statistical arbitrage opportunities.[11]

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[10] Hansen(1982) used the modified Wald statistics (χ_B^2) with χ^2 distribution of two degrees of freedom

[11] It is a profit situation arising from pricing inefficiencies.

They all would lead to this upward biasedness. Time series forecast does not give enough information about future volatility of the NZD/USD cross rate. The comparisons of alternative forecasts of future volatility suggest that the implied volatilities show higher explanatory power than the alternative GARCH volatility models.

I have to emphasize that Garman-Klass estimator of realized volatility performed better than standard deviation estimator of realized volatility and Parkinson estimator of realized volatility. With using Garman Klass estimator, we provided some additional information to implied volatility but according to our results, this addition from Garman Klass estimator is marginal but increased the efficiency.

Tables:

Table I. GARCH Estimation

λ_t	α_0	α_1	β_1	$h(0)$
-0.2124e-2	0.2351e-7*	0.06481*	0.9243*	0.4705e-6**
(0.4417e-2)	(0.0771e-8)	(0.01743)	(0.0223)	(0.1751e-3)

* Rejection of the null with 99% confidence

** Rejection of the null with 95% confidence

Table II. Descriptive Statistics

	IMPLIED($\sigma_{i,t}$)	GARCH_t	REA.(PK_t)	REA.(SD_t)	REA.(GK_t)
Mean	11.476	0.0061	0.0067	0.0069	0.0072
Median	11.2	0.0218	0.0069	0.0072	0.0074
Maximum	15.3	0.4608	0.0071	0.0076	0.0077
Minimum	9.2	-0.5923	0.0036	0.0012	0.0042
Stand.dev.	1.22	0.1159	0.00045	0.00073	0.00047
Skewness	0.8768	0.8768	-2.7108	-3.1475	-2.4707
Kurtosis	3.2930	3.2930	11.437	15.187	10.144

Table III. Unit Root tests

	ADF Test Statistic	Phillips-Perron Test Statis.
IMPLIED($\sigma_{i,t}$)	-36.547	-38.561
GARCH$_t$	-35.814	-35.891
REALIZED(PK$_t$)	-3.287	-7.8276
REALIZED(SD$_t$)	-10.879	-9.6077
REALIZED(GK$_t$)	-4.741	-8.1217

Table IV. Regression of Realized Volatility on Implied Volatility

Dependent var.	Intercept	Slope	Adjusted R^2
SD_t	0.06 (0.0065)	0.5427 (0.049)	49.53 %
PK_t	0.053 (0.0041)	0.5813 (0.041)	54.44%
GK_t	0.055 (0.0037)	0.6419 (0.038)	58.16%

Table V. Regressions Using Standard Deviation Realized Volatility (SD_t)

Intercept	$\sigma_{i,t}$	$GARCH_t$	AdjustedR^2	χ^2_A	χ^2_B
0.06 (0.0065)	0.5427 (0.049)		49.53 %	342.56 [0.000]	48.71 [0.000]
0.0746 (0.017)		0.5677 (0.0904)	42.13 %	175.67 [0.000]	1.81 [0.201]
0.0358 (0.047)	0.9110 (0.3641)	-0.2151 (0.2054)	35.06 %	340.11 [0.000]	46.12 [0.000]

*Reject the null with 99% confidence

Newey-West corrected Standard errors in parenthesis.

Table VI. Regressions Using Garman-Klass Realized Volatility (GK_t)

Intercept	$\sigma_{i,t}$	$GARCH_t$	AdjustedR^2	χ^2_A	χ^2_B
0.055 (0.0037)	0.6419 (0.038)		58.16 %	339.57 [0.000]	43.57 [0.000]
0.0552 (0.0062)		0.5484 (0.032)	48.25 %	205.4 [0.000]	1.43 [0.460]
0.0317 (0.037)	0.8890	-0.2353	34.60 %	339.99 [0.000]	47.50 [0.000]

Table VII. Regressions Using Parkinson Realized Volatility (PK_t)

Intercept	$\sigma_{i,t}$	$GARCH_t$	AdjustedR^2	χ_A^2	χ_B^2
0.053 (0.0041)	0.5813 (0.041)		54.44 %	385.3 [0.000]	57.45 [0.000]
0.04921 (0.0053)		0.5347 (0.029)	51.42%	336.97 [0.000]	41.62 [0.000]
0.0291 (0.031)	0.8640 (0.3960)	-0.2582 (0.1918)	43.24 %	334.99 [0.000]	40.51 [0.000]

The tests for unbiasedness and orthogonality restrictions for implied volatility, GARCH variance (G) and historical variance (H).The implied volatilities are found from the model of Garman and Kohlhagen (1983).The GMM estimator of Hansen (1982) along with the Newey-West (1987) variance-covariance estimator is used. t-statistics are reported in each parentheses. The sample period is from December 2001 to December 2006.

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Essay Two:

Calibration of Heston’s model in currency options market

1. Introduction

The Black-Scholes formula is a very successful option pricing formula. The success of it is mainly due to the possibility of synthesizing option prices through a unique parameter which is the implied volatility. Implied volatility is so crucial for traders to be directly quoted in many financial markets. The Black-Scholes assumption of a constant volatility for pricing derivatives with the same underlying asset fails to hold true in practice.

One could introduce a time-dependent (deterministic) volatility σ_t in the Black-Scholes dynamics for the asset price

$$dS_t = \mu S_t + \sigma_t S_t dW_t, \qquad (1)$$

and, given the N increasing maturities $T_1,...,T_{N,}$ to recursively solve

$$\int_o^{T_i} \sigma_t^2 dt = v_i^2 T_i, \qquad (2)$$

Where v_i is the implied volatility for the maturity T_i.

Many researchers did several works to address the problem of a possibly approximate fitting of market option data. Smile exists because there is asymmetry in risk assessment: a big down move is more “risky” than a big up move for an asset holder and the correlation between volatility and stock is negative.

In general, this first approach does not provide sufficient flexibility to properly calibrate the whole volatility surface. An example is the general CEV process being analysed by Cox (1975) and Cox and Ross (1976).

Merton (1976) allows the volatilities to be a deterministic function of time and he also included jumps in the market. Jumps are very important since they exist in the markets. Merton’s model does not explain the smile shape for different strikes.

Dupire (1994, 1997), Rubinstein (1994) and Derman and Kani (1994, 1998) presented tree-based algorithms to extract the function $\sigma(S,t)$ from today’s quoted prices of a series of plain vanilla options of different strikes and maturities. These approaches are called local-volatility models. These models are conceptually simple and appealing but they need delicate numerical implementation and numerically it is too difficult to calibrate. Moreover it cannot explain the persistent smile shape which does not go away as time passes. Because of these difficulties, solutions presented from these models cannot be guaranteed to work in all realistic pricing environments

Hull and White (1987), Stein and Stein (1991), Heston (1993) and Tompkins (2000) allowed the volatility coefficient in the Black-Scholes diffusion equation to be random. These are the stochastic volatility two factor models with one of the factors being responsible for the dynamics of the volatility coefficient. Geometric Brownian motion and mean-reverting Ornstein- Uhlenbeck type processes have been suggested. Ornstein-Uhlenbeck processes are conditionally Gaussian. This means that not only are all the unconditional (marginal) distributions are Gaussian today but also are all the future conditional distributions. These processes have a number of important features that make them popular for financial modeling.[1]

In stochastic volatility models the value of an option is usually specified by a Partial Differential Equation (PDE). Closed-form solutions are often unavailable and Monte-Carlo simulations are too slow, PDE was used in this paper.

In this paper, for a Foreign Exchange (FX) setting, I calibrate Heston's stochastic volatility model to the volatility smile seen in FX market. Important differences of my paper from the previous studies are ; I use no penalty function for model parameters and calibration and I use Risk reversal and Butterflies for call and put options.

I will price the European vanilla call and put options using Fourier inversion based on European option pricing technique in Heston's paper and we will see how Heston's model does perform satisfactorily well after calibration.

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[1] See for example, Fouque (2000), Nielsen (1999)

I use 1Δ to 40Δ Risk-reversals and Butterflies for call and put options. In other words, I observe the respective implied volatilities to use to price the products. Another important distinction of my paper is; using wide range of deltas.[2] My underlying asset is NZD/USD exchange rate and my sample maturities are from 1 week to 2 years.

2. What is Calibration?

The unknown parameters are the source of option pricing models. Let's say that we want to price vanilla or exotic options, we have to estimate the parameters of these models. How can we get these parameters? We can estimate these parameters from current option prices instead of estimating historical asset prices. Current option prices can be observed from the markets historical asset prices. In this paper, I will calibrate the Heston's parameters. These parameters are a generalization of the Black-Scholes implied volatilities. A very popular method to estimate these parameters is calibration. I will do the calibration with minimizing the distance between the model and actual option prices, in other words by minimizing the sum of squared deviations between the market-observed and model vanilla option prices on a particular day. We can also use these parameters to price more exotic contracts in a way that is consistent with the observed market prices.

[2] Typically in literature and previous studies consist of the implied volatilities for puts and calls with a delta of ϕ 25 %.

A stylized in the FX market is that options are quoted depending on their Delta, and not their strike as in other options market. This basically reflects the sticky Delta rule, according to which implied volatilities do not vary, from a day to next, if the related moneyness remains the same. We have to put different implied volatility every time into Black & Scholes formula as delta of an option changes. If our exchange rate, NZD/USD changes then delta of the option on this currency changes with respect to this change in exchange rate.

3. Characteristics of FX Volatility Smile

The volatility implied by a traded vanilla option is called the implied volatility. Implied volatility changes with different degrees of moneyness $\frac{spot}{strike}$. In my paper, for FX options it sticks to the delta and this delta is a monotone function of the moneyness. This is called the skew and term structure together with this skew is called volatility smile.

The skew part of the smile is generally a convex function. In FX markets, smile shapes tend to be more symmetric. One reason to explain smile is out-of-money vanilla options are less liquid than at-the money options if they're with a higher volatility.

From the practical point of view, vega risk is higher for out-of-the-money options than for at-the-money options since vanna explains the skew-symmetric part and volga explains the just the symmetric part of the smile. If we look for an explanation for smile from a numerical point of view, the empirical distribution of log-returns is not normal as presumed in the Black-Scholes model. The tails of the empirical distribution are usually fatter than normal tails.

4. Black-Scholes and Garman-Kohlhagen formulas

The Black& Scholes model assumes that the volatility of the underlying is constant. This assumption is now outdated, we now assume that the volatility is time dependent and stochastic. The model is now our reference. All volatilities are Black and Scholes volatilities.

The most common closed-form solution for valuing currency options is usually attributed to Garman- Kohlhagen (1983). This model is equivalent to an appropriately configured version of the generalized Black& Scholes model, where the net cost of carry parameter for all models is calculated as:

$$g = r - r_f \tag{1}$$

g = net cost of carry

r = domestic riskless interest rate

r_f = foreign riskless interest rate

Both the Garman-Kohlhagen and generalized Black Scholes models are derived under a fairly restrictive set of assumption

-The stochastic behavior of the underlying exchange rate is assumed to be well represented by a Geometric Brownian Motion process (GBM).

$$dS_t = S_t(\mu dt + \sigma dB_t) \tag{2}$$

European FX call option price (Garman and Kohlhagen, 1983):

$$C_t = S_t e^{-r_f T} N(d_1) - Ke^{-rT} N(d_2), \tag{3}$$

$$d_1 = \frac{\log(S_t / K) + (r - r_f + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau} \tag{4}$$

Where S_t is the spot exchange rate at time t in every unit of foreign currency; μ is the instantaneous rate of appreciation of the foreign currency v; r and r_f is the domestic and foreign risk-free interest rates respectively, which is known and nonstochastic ; τ = T- t, the time to maturity of the option ; K is the striking price of an option on one unit of foreign currency; N (.) is the cumulative standard normal distribution function ; C_t is the domestic currency price at time t of a call written on one unit of foreign currency.

5. Heston's model

Heston (1993) assumed that the spot price follows the diffusion:

$$dS_t = S_t(\mu dt + \sqrt{v_t}\ dW_t^{(1)}), \tag{5}$$

Heston process is geometric Brownian motion (GBM) with a non-constant instantaneous variance v_t. This instantaneous variance might be recognized as a version of the square root process presented by Cox, Ingersoll , and Ross (1985). The variance process is non-negative and mean-reverting (i.e distribution of volatility is stable as observed in the markets) stochastic process of the form:

$$dv_t = \kappa(\theta - v_t)\ d_t + \sigma\sqrt{v_t}dW_t^{(2)}\ , \tag{6}$$

and allowed the two Wiener processes to be correlated with each other:

$$dW_t^{(1)} dW_t^{(2)} = \rho d_t \qquad (7)$$

In these equations;

v_t = Variance

κ_t = Mean reversion rate

θ_t = Long term variance

σ_t = Volatility of variance

$\kappa(\theta - v_t)$ = Mean reverting term

ρ_t = Correlation between changes in variance and spot

The mean reversion rate, the long term variance and the instantaneous variance parameters control the term structure of the implied volatility surface (i.e. time to maturity direction). The correlation and the volatility of variance parameters control the smile/skew (i.e. moneyness direction).

The variance process (2) was originally used by Cox, Ingersoll and Ross (1985) for modeling the short term interest rate and it is defined by three parameters: θ, κ, and σ.

Equations (5) and (6) define a two dimensional stochastic process for the variables S_t and v_t. With setting $x_t = \log(S_t / S_o) - \mu t$, we can show it in terms of the centered (log) return x_t and v_t. The process is then characterized by the transition probability

$P_t(x,\ v|v_0)$ to have (log) return x and variance v at time t=0.

The time process of $P_t(x,v|v_0)$ is explained by the Fokker-Planck equation:

$$\frac{\partial}{\partial t}P = \kappa\frac{\partial}{\partial v}\{(v-\theta)P\} + \frac{1}{2}\frac{\partial}{\partial x}(vP) + \rho\sigma\frac{\partial^2}{\partial x\partial v}(vP) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(vP) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial v^2}(vP). \qquad (8)$$

6. Option pricing in Heston's model:

In Foreign Exchange, volatilities are decomposed into a symmetric part of smile showing the convexity and a skew-symmetric part of the smile showing the skew. The explanation of this form is that the market quotes strangles, risk reversals or butterflies. In a volatility matrix the at-the-money column is the center of the symmetry.

Since we're using risk reversals and butterflies strategies, we need to write their formulas.[3]

The relationship between volatilities, risk reversals and butterflies is identical for all deltas and is given by

[3] Formulas are taken from John Hull's "Options, Futures and Derivatives" book.

$$RR = \sigma_{+} - \sigma_{-} \tag{9}$$

$$BF = \frac{\sigma_{+} + \sigma_{-}}{2} - \sigma_{ATM} \tag{10}$$

$$\sigma_{\pm} = \pm \frac{1}{2} RR + BF + \sigma_{ATM} \tag{11}$$

In these equations, σ_{ATM} denotes the at-the-money volatility of put and call options, σ_{+} the volatility of an out of the money call and σ_{-} the volatility of an out of the money put.

The value of any contingent claim or, the value of any asset from Merton (1973) and Black & Scholes (1973) is; $U\ (S,v,t)$ paying $g(S) = U(T,v,S)$ at time T and the model picks a functional form of $\lambda(t,v,S) = \lambda v$ which is the market price of volatility risk and it is independent of the particular contingent claim. This parameter can be obtained from an existing price and used to price all other claims.

We have two sources of uncertainty, the Wiener processes $W^{(1)}$ and $W^{(2)}$, the portfolio must include the possibility to trade in the money market, the underlying and another derivative security with value function $V(S,v,t)$.

Initial wealth X_0 which evolves according to:

$$dX = \Delta dS + \Gamma dV + r_d (X - \Gamma V)dt - (r_d - r_f)\Delta S dt \quad (12)$$

Δ is the number of units of the underlying held at time t and Γ is the number of derivative securities V held at time t.[4] We have to define r_d and r_f as the domestic and foreign interest rates respectively, because we apply this model in foreign exchange setting.

To price the European vanilla options with an efficient way, we have to find Δ and Γ as $X_t = U(S_t, v_t, t)$ for all $t \in [0,T]$. To find these parameters we can compare the differentials of U and X obtained with Ito's formula. The value of any contingent value U (S,v,t) must satisfy the following PDE from Heston (1993)

$$\frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma vS\frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + (r_d - r_f)S\frac{\partial U}{\partial S} + \{\kappa(\theta - v) - \lambda(S,v,t)\}\frac{\partial U}{\partial t} = 0$$

(13)

[4] Heston (1993) Carr&Madan (1999), Wystup (2003), they all used the same path.

Or it can be written as

$$U_t + \frac{1}{2}\sigma^2 vU_{vv} + \rho\sigma vSU_{vS} + \frac{1}{2}vS^2U_{SS} + \left[\kappa(\theta - v) - \lambda v\right]U_v + (r_d - r_f)SU_S - r_dU = 0 \quad (14)$$

We obtain a solution for equation (8) by expressing boundary conditions. For a European vanilla option these are:

$$U(0,v,t) = \frac{1-\theta}{2}Ke^{-r_d\tau}, \qquad (15)$$

$$\frac{\partial U}{\partial S}(\infty,v,t) = \frac{1+\phi}{2}e^{-r_f\tau}, \qquad (16)$$

$$(r_d - r_f)S\frac{\partial U}{\partial S}(S,0,t) + \kappa\theta\frac{\partial U}{\partial v}(S,0,t) + \frac{\partial U}{\partial t}(S,0,t) = r_dU(S,0,t) \qquad (14)$$

$$U(S,\infty,t) = Se^{-r_f\tau}, \quad \text{for} \quad \phi = +1 \qquad (15)$$

$$U(S,\infty,t) = Ke^{-r_d\tau}, \quad \text{for} \quad \phi = -1 \qquad (16)$$

where ϕ is a binary variable. In this study, since I focus on both European vanilla call and put options, I value this variable +1 and -1, respectively. K is the strike in units of the domestic currency, $\tau = T - t$, T is the expiration time in years, and t is the current time. This PDE equation could be solved using the method of characteristics functions from Heston (1993)

The Heston's solution for vanilla options and the value of a European call and put option is:

$$h(t) = HestonVanilla(\kappa, \theta, \sigma, \rho, \lambda, r_d, r_f, v_o, S_o, K, \tau, \phi) = \phi\left\{e^{-r_f T} S_t P + (\phi) - Ke^{-r_d T} P - (\phi)\right\},$$

(18)

We need to find characteristics functions to complete the pricing of the European vanilla options. To find this we need to guess a solution of

$$C(S, v, t) = SP_1 - KP(t, T)P_2, \qquad (19)$$

P_1 and P_2 are two probabilities defined by

$$P_j = P_j(\log S_t, v_t, t) \ (j = 1,2) \qquad (20)$$

$$P_1 = \mathrm{E}\left[\frac{S_T}{F_T} 1_{\{S_T \geq K\}} \middle| F_t\right], \qquad P_2 = P\left[S_T \geq K \middle| F_t\right] \qquad (21)$$

Setting x= log S, Probabilities P_j must also satisfy the PDE like our contingent claim.

$$\frac{1}{2}v\frac{\partial^2 P_j}{\partial x^2}+\rho\sigma v\frac{\partial^2 P_j}{\partial x\partial v}+\frac{1}{2}\sigma^2 v\frac{\partial^2 P_j}{\partial v^2}+(r-d+u_j v)\frac{\partial P_j}{\partial x}+(\alpha-b_j v_t)\frac{\partial P_j}{\partial v}+\frac{\partial P_j}{\partial t}=0 \quad (22)$$

Two important assumptions I made is that since I'm using derivative prices and calibrate our model to these prices, market price of volatility risk is zero ($\lambda=0$) . Crucial point is; I use no penalization on model parameters. This is an important assumption since previous studies used penalty function and calibrated the Heston's parameters. I'm not using a penalty function and we'll see that without any numerical restriction on model parameters, whether the observed volatility smile for short, medium and long term options can be fitted or not.

Probability function, P_j can be calculated from the Fourier transform f_j by finding the integral,

$$P_j(x,v,T)=\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-i\theta\log K}f_i(x,v,T,\theta)}{i\theta}\right)d\theta \qquad (23)$$

Before moving to the calibration to the smile we should emphasize the importance of the Greeks in the model.

7. Greeks

Greeks are important information for traders and have become standard information supplied by front-office implementations. Greeks are derivatives of the value function with respect to model and contract parameters.

In our model the spot delta is given by;

$$\Delta = \frac{\partial h(t)}{\partial S_t} = \phi e^{-r_f \tau} P + (\phi) , \tag{24}$$

Vega measures the first derivative with respective to volatility, is:

$$\frac{\partial h(t)}{\partial v(t)} = e^{-r_f \tau} S_t \frac{\partial}{\partial v_t} P_1(\log S_t, v_t, \tau, \log K) - K e^{-r_d \tau} \frac{\partial}{\partial v_t} P_2(\log S_t, v_t, \tau, \log K) \tag{25}$$

The formula for *volga* , which measures the second derivative with respect to volatility, is:

$$\frac{\partial^2 h(t)}{\partial v_t^2} = e^{-r_f \tau} S_t \frac{\partial^2}{\partial v_t^2} P_1(\log S_t, v_t, \tau, \log K) - Ke^{-r_d \tau} \frac{\partial^2}{\partial v_t^2} P_2(\log S_t, v_t, \tau, \log K), \tag{26}$$

Gamma, is the sensitivity of delta to the underlying which can be shown as:

$$\Gamma = \frac{\partial \Delta}{\partial S_t} = \frac{e^{-r_f \tau}}{S_t} p_1(\log S_t, v_{t,} \tau, \log K). \tag{27}$$

We have two interest rates, r_d and r_f so we have two *rho's* ,

The formulas for *rho* are :

$$\frac{\partial h(t)}{\partial r_d} = \phi K e^{-r_d \tau} \tau P - (\phi), \tag{28}$$

$$\frac{\partial h(t)}{\partial r_f} = -\phi S_t e^{-r_f \tau} \tau P + (\phi). \tag{29}$$

8. Calibration technique in the model

In Foreign Exchange markets, smile can be observed in the market for different currency pairs. I'm calibrating the parameters of the Heston's model to this smile.

In Heston's stochastic volatility model, to calibrate the whole volatility surface, we could try to fit empirical distributions of returns to the marginal distributions with a minimization scheme. Unfortunately, all historical approaches have one common imperfection-they do not allow for estimation of the market price of volatility risk $\lambda(t, V, S)$ [5] this implies in turn that one needs some extra input to make the transition from the physical to the risk neutral world.

Observing only the underlying spot price and estimating stochastic volatility models with this information will not give us correct derivative security prices.

I take the smile of the current vanilla options market as a given starting point. First, we need to find the strikes. The volatility skew in foreign exchange markets is described as a function of deltas.

This stage needs only the Fourier inversion of the cumulative normal distribution. We fit the five parameters: initial variance v_0, long-run variance θ, volatility of variance σ, mean reversion κ, and correlation ρ for a fixed time to maturity and a given

[5] However some studies found some evidence about nonzero volatility risk premium, see Bates (1996)

vector of market Black-Scholes implied volatilities $\{\hat{\sigma}_i\}_{i=1}^{n}$ for a given set of delta pillars $\{\Delta_i\}_{i=1}^{n}$.

In order to fit the model parameters to a given smile the following calibration process is used:

1. We're retrieving the strikes corresponding to given volatilities and deltas using Black-Scholes
2. We're calculating call option prices with Heston's stochastic volatility model
3. We're retrieving implied Black-Scholes volatilities σ_i from gained Heston's prices

The value function is:

$$h(t) = HestonVanilla(\kappa,\theta,\sigma,\rho,\lambda,r_d,r_f,v_o,S_o,K,\tau,\phi) = \phi\left\{e^{-r_f T} S_t P+(\phi) - Ke^{-r_d T} P-(\phi)\right\},$$

(30)

where $a = \kappa\theta,$ $u_1 = 1/2$, $u_2 = -1/2$, $\mathrm{b}_1 = \kappa + \lambda - \sigma\rho,$ $b_2 = \kappa + \lambda,$ $x = \log S_t$

satisfies;

$$HestonVanilla(\kappa,\theta,\sigma,\rho,\lambda,r_d,r_f,v_0,S_o,K,\tau,\phi) = HestonVanilla\left(\kappa+\lambda, \frac{\kappa}{\kappa+\lambda}\theta,\sigma,\rho,0,r_d,r_f,v_0,S_0,K,\tau,\phi\right)$$

(31)

Which means that we can set $\lambda = 0$ by default and just determine the remaining five parameters.

After fitting the parameters we calculate the option prices in Heston's model using equation (18) and retrieve the corresponding Black-Scholes model implied volatilities $\{\sigma_i\}_{i=1}^{n}$ with a standard bisection method.[6]

The next step is to define an objective function, which we select to be the Sum of Squared Errors (SSE):

$$SSE(\kappa, \theta, \sigma, \rho, v_0) = \sum_{i=1}^{n} \{\hat{\sigma}_i - \sigma_i(\kappa, \theta, \sigma, \rho, v_0)\}^2 \tag{32}$$

I observe implied volatilities, because they are all of comparable size. Finally, I minimize over this objective function to find the parameters.

9. Data

I consider an example of calibration to NZD/USD market data, strikes and implied volatilities, as of 25 January 2007, when the spot exchange rate was 0.6990. Our

[6] The Newton-Raphson method could also be used

relevant maturities are from one week (1W) to two years (2Y). I have ninety-seven observations (ninety-seven call and put deltas and corresponding ninety-seven implied volatilities). I use market volatilities for call and put options, 1Δ and 40Δ, Risk reversals and Butterflies.

I take the volatility smile on this date and the parameters in Heston's model are calibrated to this smile. The values of calibrated parameters are listed in Table I. During the calibration technique our model was fitted for different maturities independently. In other words, it can be observed that these parameters are not constant. In Table I , I show these parameters in average.

It can be seen in Figure III. that Heston's model fits quite well to the market for our time period. There are small fitting problems seen on the graph for short time periods such bas from 1 week or 1 month but then we see that it gives high performance for longer time periods.

10. The effects of changing the Heston parameters

The shape of the fitted smile curve will be influenced by changing the Heston parameters. When we change the value of the initial variance which is, the height of the smile curve can be adjusted but not the shape of it. Changing the correlation changes the symmetry.

Changing the volatility of variance to zero creates deterministic process and for the variance and volatility which doesn't take any smile. Increasing the volatility of volatility shifts the convexity.

When we change the long-run variance, we could see the similar effects on the shape of the skew like the results of changing the initial variance. Changing the mean reversion affects the center of the skew where it raises the center. This part is the at-the-money part.

The calibration calculates the parameter values that best fit the market but clearly exact fit is better than the best fit. It is difficult to find the exact fit. Exact fit makes it possible to get exact sensitivities with respect to market instruments. Instabilities are introduced when exact fit is not possible.

Market instruments used for hedging must be priced close to the market. Maturities from 1 week to 1 month don't have the best fit but from 2 months to 2 years, Heston's model does perform satisfactorily and according to our calculations and graphs, we might have the "exact fit" for these time periods.

In the case where the volatility is perfectly uncorrelated to the stock price $(\rho = -1)$, the dimension of the problem falls to one. The volatility is then non-stochastic but deterministic. We saw this structure when we tried to calibrate the model for very short term maturities. Correlation is –1 in this case.

This means that Heston's model does not produce enough skew (smile) for very short maturities.

11. Conclusion

Heston's model is one of the first models that were able to explain the skew and simultaneously allow a real life application and a good pricing of exotic options. According to our results, it can be seen that the model is particularly useful in explaining the volatility skew found in FX markets.

I perform the original work of Heston (1993) to a foreign exchange (FX) setting and then I calibrate Heston's model to market data. The fit is pretty good between the maturities between 1-month to 2- years; there were some fitting problems between the maturities of 1-week to 1-month. I could say that implied skew is often too weak for short maturities. The reason of this problem could be, during the calibration of the short-term smiles, the volatility of volatility appears to increase suddenly along with the speed of mean reversion. This problem could be fixed with adding jump-diffusion to the model but these jumps are not allowed in our model. These jump models result in incomplete markets since hedging is too difficult. Heston's model also results in an incomplete market but it can be made complete by the introduction of one or few traded options.

In parallel with our calculations and results, advantages of Heston's model over the other models are; non-negative and mean-reverting volatility process which is observed in the markets; the simplicity of calibration to market data.

I performed calibrations for different time slices of the volatility matrix and this leads us to have different values of the parameters. We might have a term structure of some parameters in Heston's model. The fit is very good for maturities between one and twenty-four months. Unfortunately, Heston's model does not perform satisfactorily for short maturities such as one to four weeks.

While only presenting the results for NZD/USD exchange rate, I observed similar results for other major liquid currency pairs. The calibration method presented very good fits for our maturities.

Tables

Table I. **Heston fit to the NZD/USD surface (in average)**

Initial variance	0.00926
Long-term variance	0.01066
Variance mean reversion	3.67077
Volatility of variance	0.38350
Correlation	-0.00035

Figures:

Figure I. Volatility smiles in different markets from 1987-2007 (provided by Bloomberg)

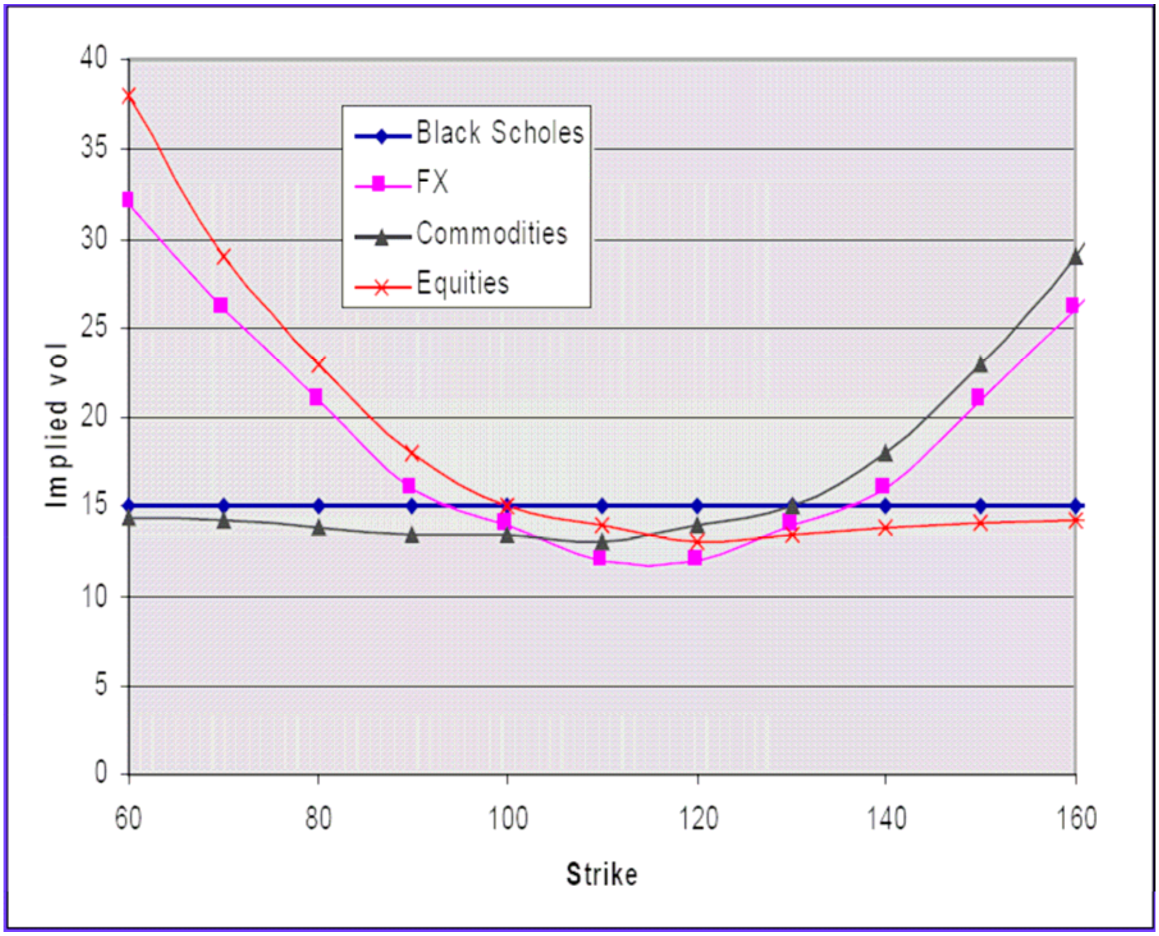


Figure II. Heston volatility typical path (negative correlation of asset and volatility from 1997-2007, provided by Bloomberg)

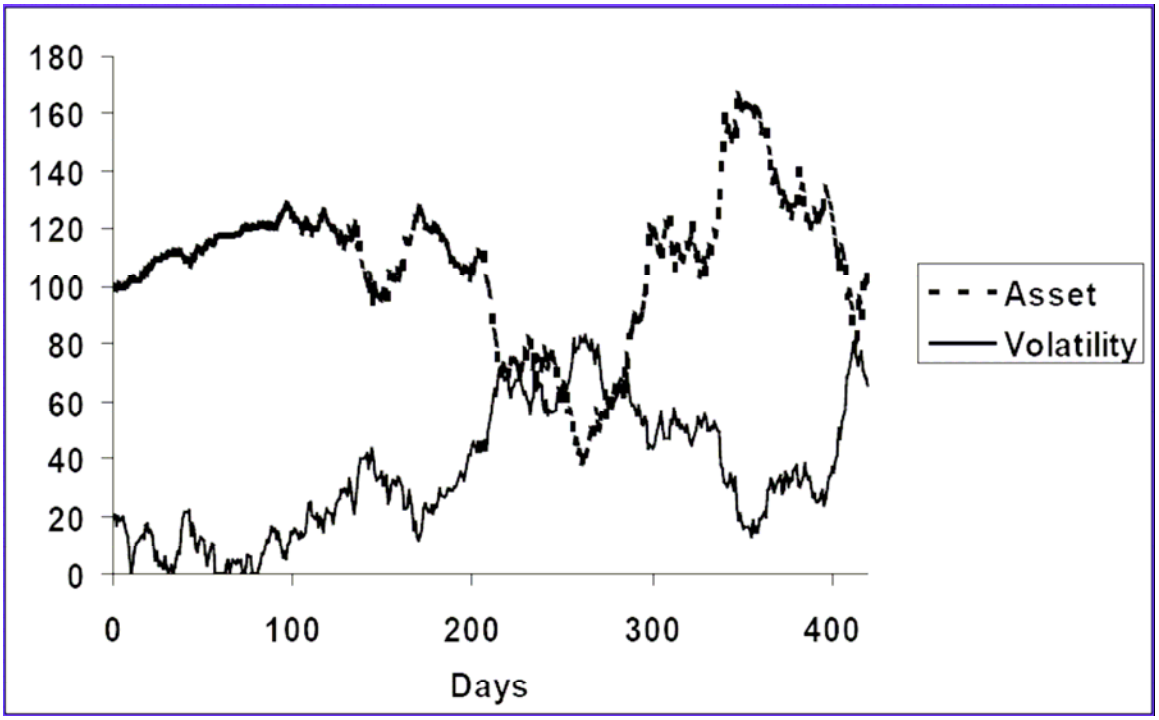
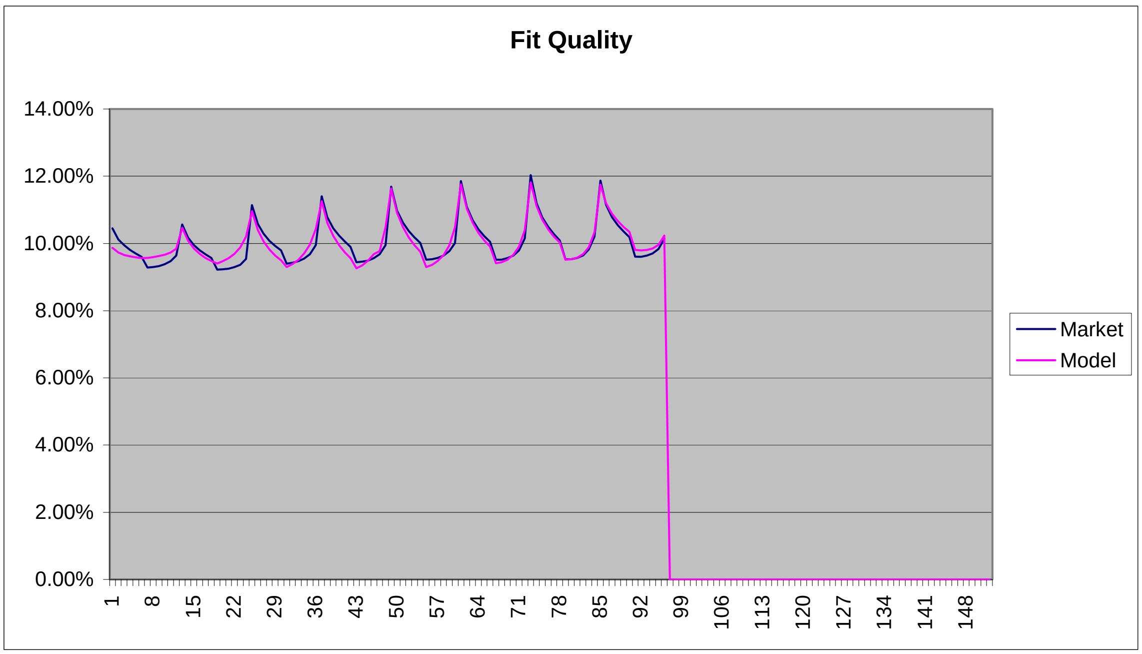


Figure III. My calibration results for NZD/USD exchange rate (Fit quality of our model to market volatility smile after calibration on January 25, 2007)



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Essay Three:

Pricing European options and calibration in equity-linked derivatives market

1. Introduction

European option prices are available in all markets with different strike prices and maturities. These options can be priced using the standard Black-Scholes model, which assumes that the volatility of the underlying process is a constant:

$$\frac{dS_t}{S_t} = (r_t - d_t)dt + \hat{\sigma} dW_t. \quad (1)$$

These options are often quoted in terms of the implied volatility. Black-Scholes model concludes that the underlying follows a lognormal process with this volatility, regardless of the asset level in the future. If the options are prices according to this model, then implied volatilities for traded options would be constant for all strikes and maturities but this behavior cannot be observed in the markets.

Merton (1973) introduced the jump-diffusion model to take into account the existence of discontinuities (jumps). In his model, the movements of the underlying are explained by a combination of a Brownian motion and an independent Poisson process a feature that is not covered in the standard Black-Scholes model.

Jumps are very important since they do exist in the markets so emphasizing them in a model is very essential. Even though this model explains the different implied volatility levels for different times of maturity, it still does not explain the smile shape for different strikes.

Another problem with Merton's model is that it leads to market incompleteness (hedging is difficult). Partial Different Equation (PDE) can be derived, simple Monte Carlo procedures can be used but the presence of jumps leads to integro-differential equations which are not easy to handle numerically.

Dupire (1994) and Derman and Kani (1994) presented important papers about local volatility models. According to these models, the volatility is assumed to be a deterministic function of time and spot. This local (deterministic) model is a natural extension of Black-Scholes model and gives a complete market model but it still couldn't explain the persistent smile shape seen in the markets. Moreover, local volatility models have generally instability problems and calibration is very sensitive to interpolation and extrapolation of market prices.

Heston (1993), Hull and White(1987) , Stein and Stein (1991), Ornstein-Uhlenbeck have proposed that the volatility smile can be explained by random fluctuations of the volatility, these are the stochastic volatility models, which are normal or lognormal processes possibly exhibiting mean reversion. The goal is to choose a model which takes into account the analytical tractability (can we price the European options accurately?) and also the implied volatility surface (does the model reflect the given market volatility surface for some set of model parameters?).

Hull-White model is not easily tractable especially when asset and volatility are correlated and volatility is not meaning reverting. The Ornstein-Uhlenbeck process can become negative. Heston is the most popular one among these models.

In this paper, I'm going to apply Heston's model with some differences from previous researches. First and foremost, I don't use penalty function on model parameters(no restriction on parameters) in this paper and another important point is I'm going to calibrate the Heston's model to derivative prices not the spot prices. I'm going to perform closed form solution which is the Fourier transforms (integral transforms method)[1] to obtain the valuation probabilities, then this leads us to the price of a European option analytically.

Another distinction in my paper is ; I use the volatilities of 1 Delta to 40 Delta of call and put options, in terms of Butterflies and Risk Reversals. To the best of our knowledge, European options haven't been priced and haven't been calibrated with these strategies , Butterflies and Risk reversals, in the literature. Our maximum maturities are longer than the previous studies. This is a good way to observe the fittings of calibration process in the equity options markets.

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[1] Fourier transform techniques also have been applied to option pricing by Bakshi and Chen (1997) and Carr and Madan (1999)

My goal is, with using Fourier transforms, to price European options efficiently and accurately to be able to make the calibration. My calibration will calculate the Heston's parameter values that best fit the market.

2. Characteristics of Equity Smile

Markets have a smile or smirk volatility structure with respect to strike, as well as a term structure. This shape is different for different markets and it tells us that the implied market distribution is no longer lognormal but has non-zero skew, excess kurtosis and higher order moments. We could say that kurtosis produces fatter tails at both ends of the distribution, where skew tends to increase one end and decrease the other.

There is asymmetry in risk assessment: A big down move is more risky than a big up move for a stock holder. There is negative correlation between stock and volatility. According to stochastic volatility models such as Hull and White model and Heston's model, the volatility smile could be explained by random fluctuations of the volatility. The volatility should be viewed as a random process exhibiting mean reversion. Furthermore, implied volatility seems to be correlated with asset returns.

Features of this implied volatility surface are: implied volatility decreases with increasing strike, volatility skew decays with time to maturity and its shape depends on the market. In the equity markets, the implied volatility is higher for lower strikes, with the skew becoming less pronounced at longer maturities. The distribution tail is therefore more pronounced (fatter) for lower spots, which is a reflection of the market's aversion to market crashes. The shape depends on the market.

3. The Heston model

Heston asset process has a variance σ_t^2 which follows a stochastic Cox, Ingersoll and Ross process:

$$\begin{aligned} \frac{dS_t}{S_t} &= (r_t - d_t)dt + \sigma_t dW_t^S, \\ d\sigma_t^2 &= \kappa(\theta^2 - \sigma_t^2)dt + \gamma\sigma_t dW_t^\sigma. \end{aligned} \tag{2}$$

Short vol: σ_t

Long vol: $\hat{\theta} = \theta\sqrt{\kappa/(\kappa+\lambda)}$ $\kappa > 0$

Vol of vol: γ $\gamma > 0$

Speed of mean reversion: $\hat{\kappa} = \kappa + \lambda$

Correlation: ρ

If we assume that $\sigma_t < 0$, above equation implies that at a later time t+Δt, the volatility will on average have moved up, because $\theta^2 - \sigma_t^2 \succ 0$ and the expectation of $\gamma\ \sigma_t\ dW_t^\sigma$ is zero. Similarly, the volatility will move down if σ_t is above the long volatility. The reversion speed determines the strength of the force pulling the volatility σ_t to the limit θ as time t increases.

We can see that how short and long volatility and mean reversion determine the term structure of implied volatility. The relationship between volatility smile and the correlation ρ between assets and volatility is another important concept.

Correlation ρ could be either negative or positive, assuming we have positive correlation that means positive asset returns will on average be coupled with an increase in volatility that results in fatter right tail of the asset distribution, the implied volatility increases with strike, commodities markets is a good example for this pattern.

Figure I shows the general strike dependence of the implied volatility surface in the other markets. Different markets have different volatility smiles and therefore deviate from the lognormal distribution in different ways. In the equity markets, it can be seen that there is negative skewness in terms of a negative correlation between assets and volatility.

The volatility parameter γ fattens both tails symmetrically and thus affects the kurtosis (smile) of the implied volatility. It also reduces the forcefulness of the reversion speed. These effects can be seen in Figure II, III and IV.

4. Different values of Heston's parameters

European options do not provide all information about forward volatilities. A term structure model is needed for stripping the forward volatilities from spot volatilities.

For calibration, we fit the model for different maturities. This means we have different values of Heston's parameters for each of our maturities. To fix these values, we're going to apply Cox- Ingersoll-Ross (CIR) process.[2]

Cox-Ingersoll-Ross process originally describes the evolution of interest rates. It is a type one factor model as describes interest rate movements as driven by only one source of market risk.

The model specifies that the instantaneous interest rate follows the stochastic differential equation:

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}\,dW_t \qquad (3)$$

where W_t is a Wiener process modeling the random market risk factor. The drift factor $a(b - r_t)$ assures mean reversion of interest rate to the long-run value b with speed of adjustment controlled by the strictly positive parameter a. The standard deviation factor $\sigma\sqrt{r_t}$ assures that the interest rate cannot become negative.

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[2] Vasicek (1977) performed similar derivation for short rate.

I’m going to apply Cox-Ingersoll-Ross (CIR) model which Heston (1993) and Stein (1991) used, to find a term structure of some parameters in Heston’s model. Applying equation (3) we could write

$$dv_t = \kappa(t)\{\theta(t) - v_t\}dt + \sigma(t)\sqrt{v_t}\,dW_t \tag{4}$$

for the nonnegative parameter functions $\sigma(t), \kappa(t), \theta(t)$.

After some algebra the variance is:

$$\text{var}(v_t) = \int_0^t \sigma^2(s) g(s) e^{2K(s)-2K(t)} ds \tag{5}$$

Given a volatility structure $\sigma(T)$, the forward volatility for the future period $[T_1, T_2]$ is given by:

$$\sigma^2_{T_1,T_2} = \frac{\sigma^2_{T_2} H(T_2) - \sigma^2_{T_1} H(T_1)}{H(T_2) - H(T_1)} \tag{6}$$

The implied forward volatility is explained to be the number $\hat{\sigma}_{[T_1,T_2]}$ to be plugged into Black-Scholes formula, to obtain the prices of forward start options

$$(\frac{S_{T_1}}{S_{T_2}} - k) \tag{7}$$

Assuming ρ_{T_1} and ρ_{T_2} are known values, to find the forward correlation coefficient:

$$\rho_{T_1,T_2} = \rho_{T_2} \tag{8}$$

The Heston model is an incomplete model which means that unlike in deterministic volatility models, option valuation depends on investor preferences. This dependence

is reflected in a market price of the risk function $\lambda(S_t, \sigma_t, t)$.

According to Heston's model, market price has the form $\lambda\sigma_t^2$, then we have additional drift term but one important point that we want to emphasize is that in our calibration, instead of using the spot data, we're using the derivative prices and we calibrate the model to derivative prices. I assume that $\lambda = 0$.[3]

Why I didn't use the market price of risk to price derivatives? Because I estimated the market price of risk using one security, then it must apply to our contingent claim as

[3] The market price of risk might depend in complex ways on the state of the world as in the case of the Vasicek (1977) or Cox- Ingersoll- Ross (1985) interest rate models on such as short-rate.

well. In other words, we had to make some other assumptions about the quantities the market price of risk can depend on.

Figure V shows the historical shape of the volatility surface for equity indices. Implied volatility decreases with increasing strike and volatility skew decays with time to maturity.

Figure VI shows the shape of Heston implied volatilities. Implied volatility surface is not bad for long maturity and there is little skew in short maturity.

5. Calibration:

Calibration to market data can be done in an efficient fashion. In practice, European options are used to calibrate the volatility model and, the models are designed to reprice the market options consistently. We need to have an accurate implementation of a European option pricer. This implementation is required for the calibration of a stochastic volatility model.

My purpose is to find a set of parameters for the stochastic volatility model so that the difference between the given market prices $(B_n, n = 1,2,..., N)$ and the model prices $((H_n, n = 1,2,..., N)$ is small.

Our utility function U which should be minimized is;

$$U(\alpha,\kappa,\sigma,\rho)=\sum_{n=1}^{N}w_n(B_n-H_n)^2 \qquad (3)$$

The number of options and the choice of weights w_n, are very important for our calibration process. It is important to use a model that is adapted to the product and quite a few products depend strongly on the volatility structure such as variance swaps, volatility swaps and cliquets.

We will use $w_n=\dfrac{Vega_n}{B_n^2}$ for the weights.

$Vega_n$ is the vega sensitivity of option n which measures the first derivative with respective to volatility and the number of market prices has to be bigger than or equal to the model parameters.

6. Pricing options

I use Fourier inversion, theory which is based Heston's paper[4] to obtain the probabilities and to calculate the price of a European option.

[4] Carr and Madan (1999) also used this method

Analytically if we price European options fast, this allows calibration so the Fourier method is very efficient and a good choice for calibration. Standard arbitrage arguments [Black and Scholes (1973), Merton (1973)] demonstrate that the value of any asset $U(S,v,t)$ must satisfy the partial different equation (PDE):

$$\frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2}+\rho\sigma vS\frac{\partial^2 U}{\partial S\partial v}+\frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2}+(r\ -d\ )S\frac{\partial U}{\partial S}+\{\kappa(\theta-v_t)\}\frac{\partial U}{\partial v}-rU+\frac{\partial U}{\partial t}=0$$

(4)

Heston assumed that a European call option with strike price K and maturing at time T satisfies the PDE (4) subject to the following boundary conditions:

$$U(S,v,T)=Max(0,S-K), \tag{5}$$

$$U(0,v,T)=0, \tag{6}$$

$$\frac{\partial U}{\partial S}=(\infty,v,t), \tag{7}$$

$$rS\frac{\partial U}{\partial S}(S,0,t)+\kappa\theta\frac{\partial U}{\partial v}(S,0,t)-rU(S,0,t)+U_t(S,0,t)=0 \tag{8}$$

$$U(S,\infty,t)=S. \tag{9}$$

In parallel with the Black-Scholes formula, we guess a solution of the form

$$C(S,v,t)=SP_1-KP(t,T)P_2, \tag{10}$$

where the first term is the present value of the spot asset upon optimal exercise, and the second term is the present value of the strike –price payment.

The value C of a call (or any other derivative of S) satisfies the following PDE:[5]

$$\frac{1}{2}vS^2\frac{\partial^2 C}{\partial S^2}+\rho\sigma vS\frac{\partial^2 C}{\partial S\partial v}+\frac{1}{2}\sigma^2 v\frac{\partial^2 C}{\partial v^2}+(r\ -d\ )S\frac{\partial C}{\partial S}+\{\kappa(\theta-v_t)\}\frac{\partial C}{\partial v}-rC+\frac{\partial C}{\partial t}=0$$

(11)

Since I'm using the derivative prices and calibrate the model to derivative prices we assume that $\lambda=0$ and d is the dividend yield for the Euro Stoxx 50 index.

The call price at time t=0 with payoff;

$$\text{Max }(S_T-K,0)$$

at maturity T has the form

$$C=e^{-rT}(F_T P_1-KP_2), \tag{12}$$

K is the strike price, $F_T=S_0e^{(r-d)T}$ is the forward price at maturity T and

$P_j = P_j(\log S_t, v_t, t)$ $(j=1,2)$ are two probabilities defined by

[5] With changing binary variable, +1 for call options, -1 for put options.

$$P_1 = \mathrm{E}\left[\frac{S_T}{F_T} 1_{\{S_T \geq K\}} \middle| F_t\right], \qquad P_2 = P\left[S_T \geq K \middle| F_t\right] \tag{13}$$

setting x=log S and we substitute(4) into (3) . Probabilities P_j have to satisfy the PDE.

$$\frac{1}{2}v\frac{\partial^2 P_j}{\partial x^2} + \rho\sigma v\frac{\partial^2 P_j}{\partial x \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 P_j}{\partial v^2} + (r - d + u_j v)\frac{\partial P_j}{\partial x} + (\alpha - b_j v_t)\frac{\partial P_j}{\partial v} + \frac{\partial P_j}{\partial t} = 0 \tag{14}$$

and the terminal condition $P_j(x, v, T) = 1_{\{x \geq \log K\}}$, where

$$u_1 = \frac{1}{2} \ , \ u_2 = -\frac{1}{2} \ , \ b_1 = \kappa - \rho\sigma \ , \ b_2 = \kappa \ . \tag{15}$$

Heston and Carr & Madan used a closed form solution for their characteristics functions

$f_j = f_j(x, v, t, \theta)$. This function satisfy (5) subject to the terminal condition

$$f_j(x, v, T, \theta) = e^{i\theta x} \tag{16}$$

and the characteristic functions are given by

$$f_j(x,v,t,\theta) = \exp\left[C(T-t,\theta)D(T-t,\theta)v + i\theta x\right] \tag{17}$$

$$C(t,\theta) = r\theta\ it + \frac{a}{\sigma^2}\left[(b_j - \rho\sigma\theta i + d)t - 2\log\left(\frac{1-ge^{dt}}{1-g}\right)\right] \tag{18}$$

$$D(t,\theta) = \frac{b_j - \rho\sigma\theta i + d}{\sigma^2}\left(\frac{1-e^{dt}}{1-ge^{dt}}\right), \tag{19}$$

and g and d are defined as

$$g = \frac{b_j - \rho\sigma\theta i + d}{b_j - \rho\sigma\theta i - d}, \tag{20}$$

$$d^2 = (\rho\sigma\theta i - b_j)^2 - \sigma^2(2u_j\theta i - \theta^2), \tag{21}$$

P_j can be calculated from the Fourier transform f_j by finding the integral

$$P_j(x,v,T) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-i\theta\log K} f_i(x,v,T,\theta)}{i\theta}\right)d\theta \tag{22}$$

7. Data

I consider an example of calibration to Euro Stoxx 50 market data as of 29 January 2007, when the index was 4211. Our relevant maturities are from one week (1W) to three years (3Y). I use market implied volatilities for call and put options between 1Δ and 40Δ, Risk reversals and Butterflies. Interest rates and dividend rates were provided by Bloomberg.

8. Calibration results

In my paper, I allowed all the parameters to find the best possible values. For our expiries, sets of parameters were used to produce fits to the whole available market smile surface, which covered expiries from one week to approximately 3 years.

Calibration of Heston's model gives good results but it has some problems with very short-term and very long-term, in our paper short maturities are from one week to one month. Long maturity is approximately three years.

The best way to see how well the Heston model fits the implied volatility surface is to compare the two volatility surfaces graphically. Comparison of the Euro Stoxx 50 implied volatility surface (provided by Bloomberg) with the Heston fit as of January 29, 2007 are shown in Figure V. and Figure VI.

Our fit quality can be seen in Figure VII. From the figure, it can be said that the fit for short maturities is not that great. It has a nice fit between 1 month to 30 months.

9. Conclusion

I examined the calibration of Heston's model under no penalization and I also presented this calibration not with spot prices but derivative prices. For calibration, I needed to compute plenty of option prices. The volatilities are in terms of Risk Reversals and Butterflies. To perform the calibrations, from 1Δ to 40Δ for calls and puts (wide range of calls and puts with different vanilla options strategies) were firstly used in this paper in the literature, to the best of our knowledge.

I used Fourier transforms to calculate the price of call and put options. Following Heston, the idea is that once we know the characteristic function of the call price as a function of its log strike, the Fourier inversion could be used to compute the prices.

The Heston model is a mean reverting model that allows for a correlation between volatility and asset level. Heston's model is a robust model and has nice properties but from our results it can be seen that it has some drawbacks. It can be seen that Heston's parameters are not independent. Changing the reversion speed in Heston has a smaller effect than changing the long volatility.

Eventhough the quality of the fit is good, Heston’s model doesn’t fit perfectly to the market volatility surface for very short maturity but it can fit the smile for vanilla options very well from one month (1M) to two and a half years (2,5Y)

I calibrated Heston’s model in Euro Stoxx 50 index. Other than Euro Stoxx 50 index, I also applied this model for FTSE 100 index and Nikkei 225 index and my applications gave similar results and close fits for these indexes.

Other stochastic volatility models could give better fits than Heston model. However, calibration is very difficult numerically. Heston model answers the calibration question well but calibration becomes more and more complex. Markets need more complicated models.

Figures:

Figure I. Negative skewness of equity markets from 1987-2007 (provided by Reuters)

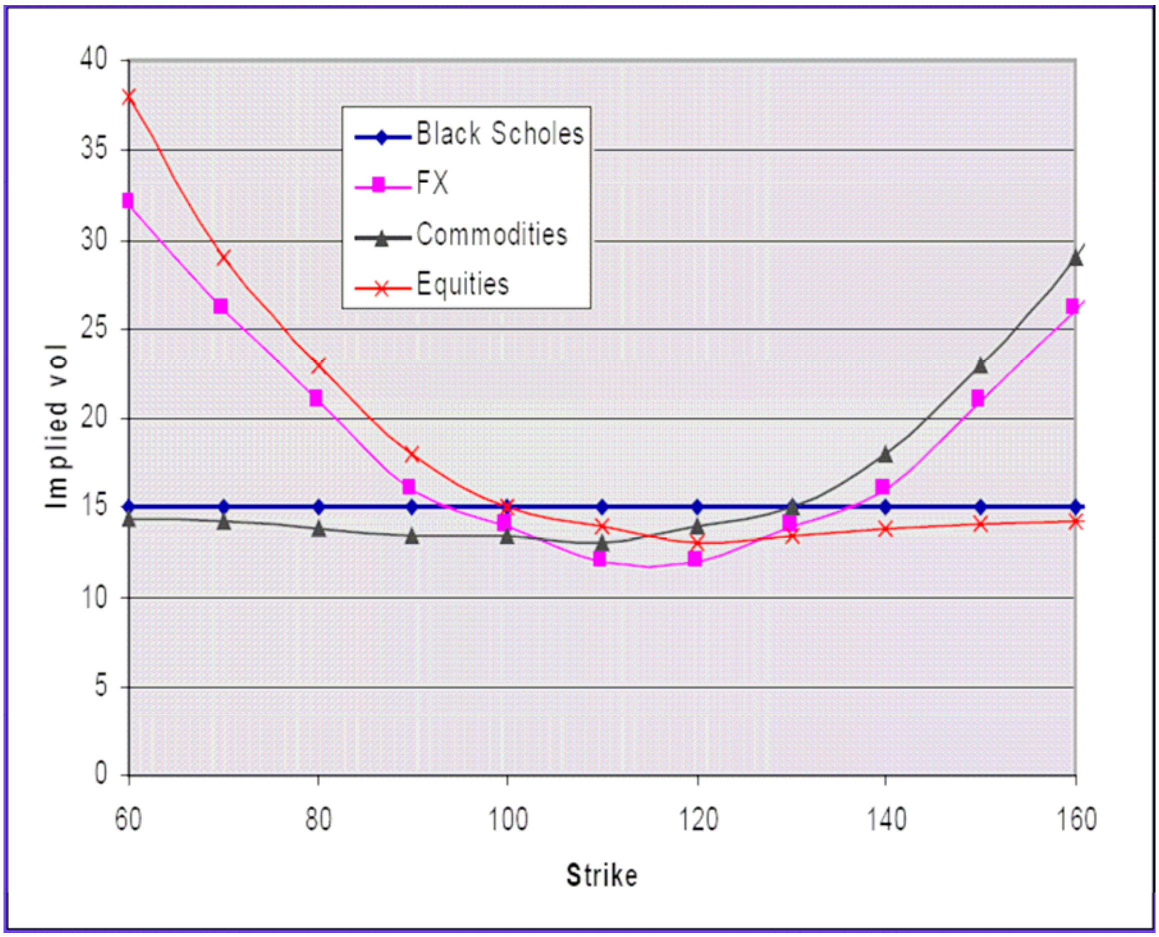


Figure II. Heston implied volatility for various correlations (Figure was provided by applying Bloomberg volatility surface index)

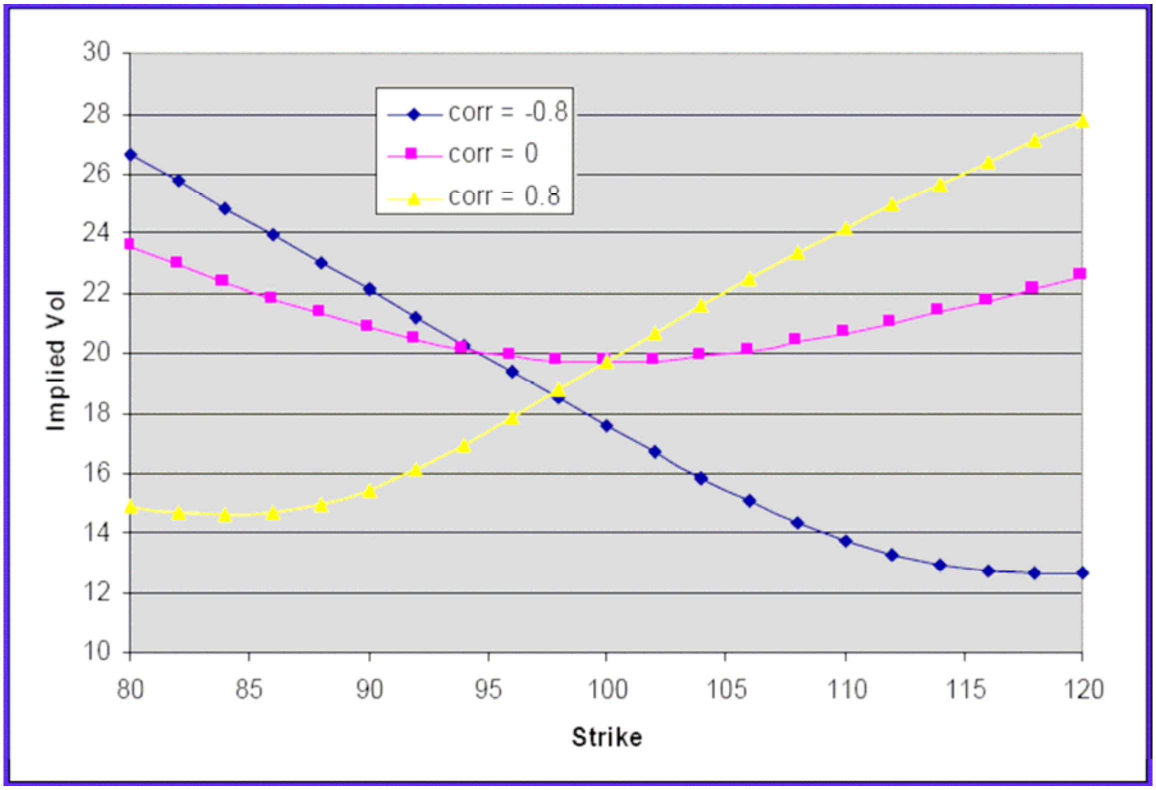


Figure III. Heston implied volatility for various volatility of volatility. (Figure was provided with applying Bloomberg volatility surface index)

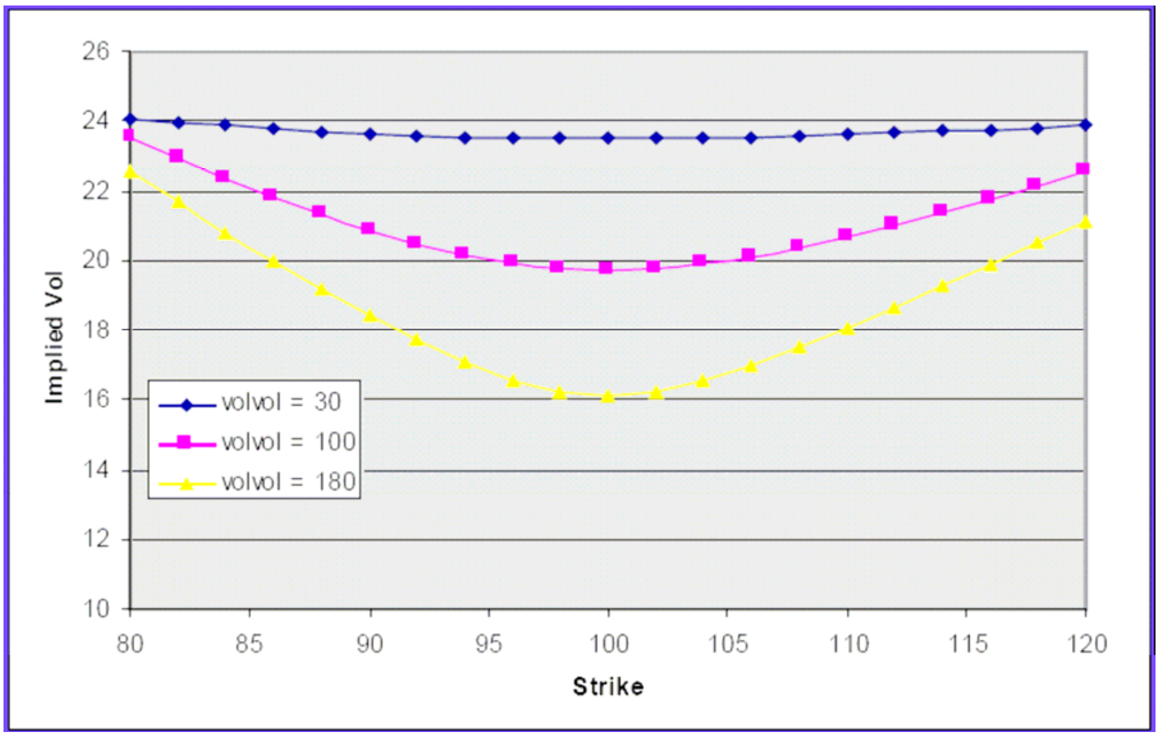


Figure IV. Heston implied volatility for various mean reversions (Figure was provided with applying Bloomberg volatility surface index)

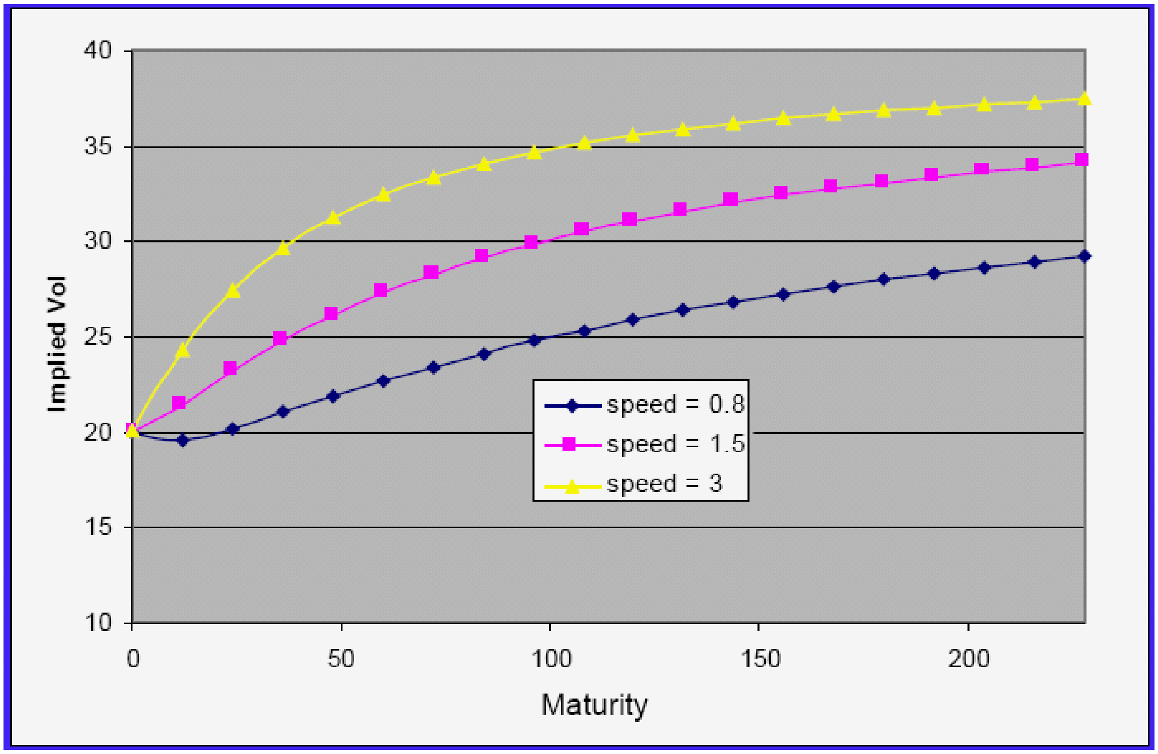


Figure V. Implied volatility surface for equity index (Euro Stoxx 50) on January 29, 2007, (provided by Bloomberg) The shape for previous days and years Heston implied volatilities were observed by the author and Heston's stochastic volatility model generates roughly the same shape of volatility surface.

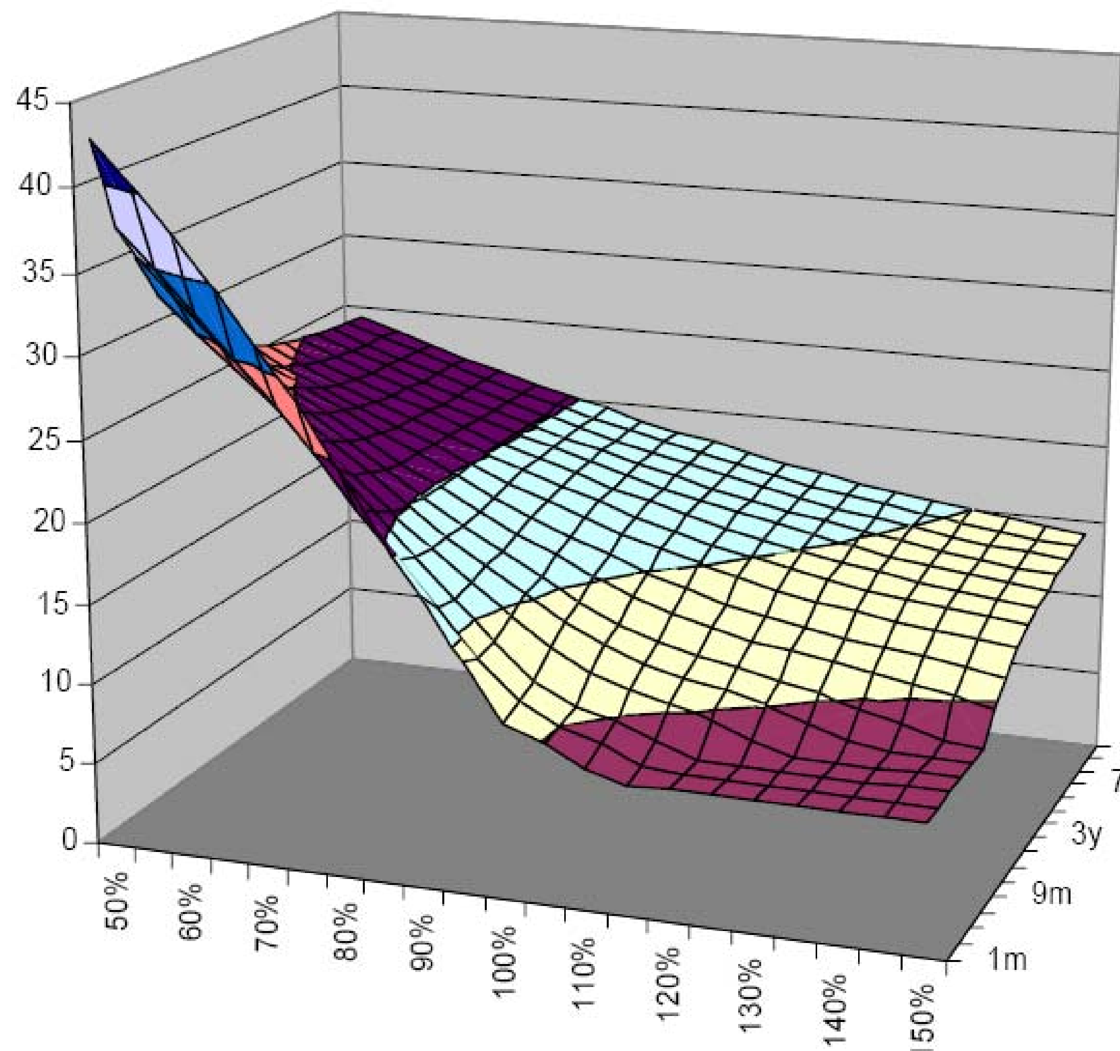


Figure VI. The shape of Heston implied volatilities for Euro Stoxx 50 on January 29, 2007, The shape of the previous date's Heston implied volatilities was observed by the author and Heston's stochastic volatility model generates roughly the same shape of volatility surface.

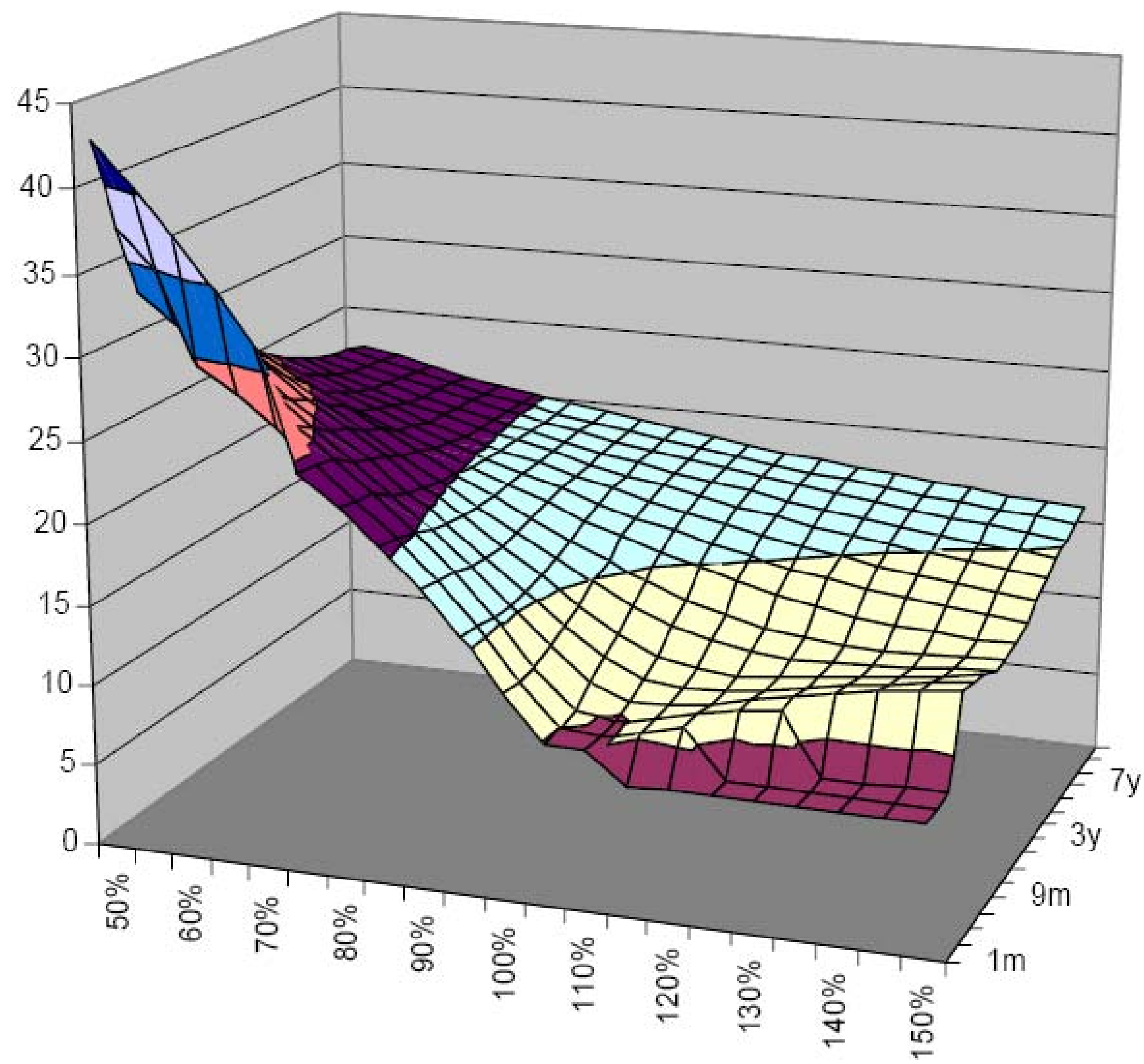
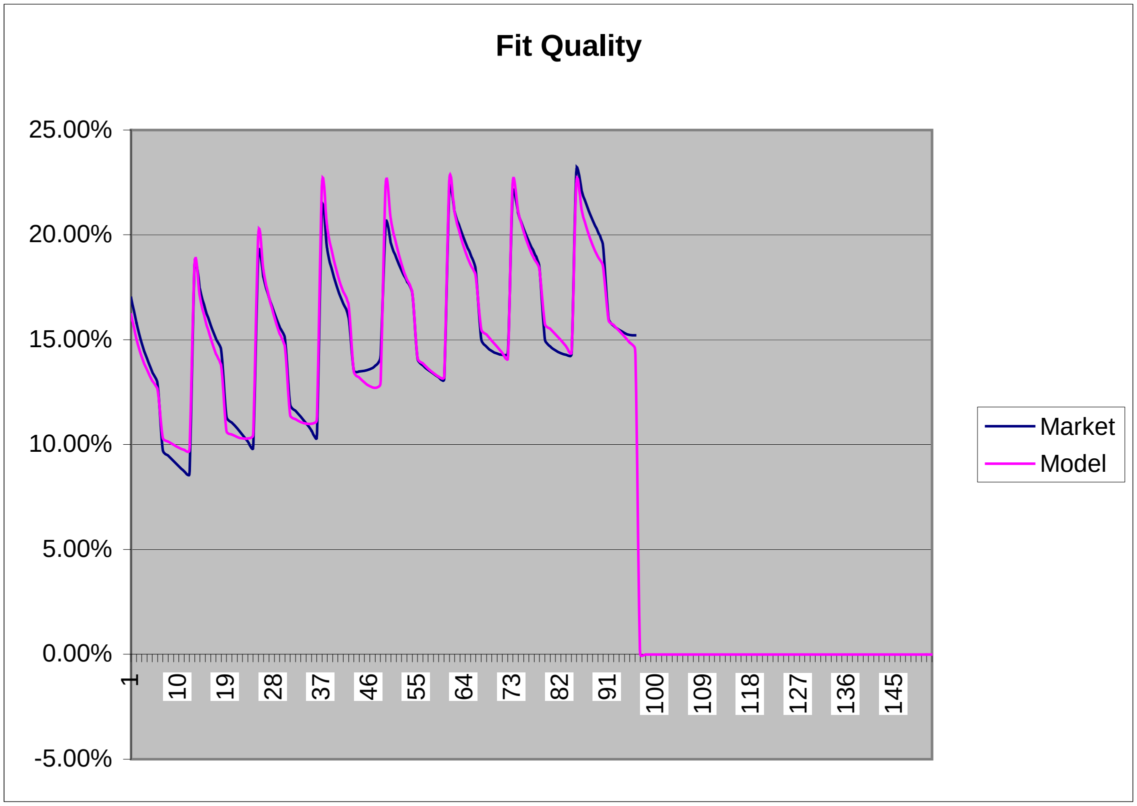


Figure VII. My calibration results for Euro Stoxx 50 index (Fit quality of our model to market volatility smile after calibration on January 29, 2007)



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