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STABILITY ANALYSIS OF MULTIDIMENSIONAL DIGITAL FILTERS

by

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Abstract

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Adviser: Prof. L. Roytman

This thesis presents two new results concerning the stability of multidimensional digital filters.

We considered an open problem concerning a class of N-D filters with nonessential singularities of the second kind, described by the real rational transfer function $G(z_1, z_2, ..., z_N) = P(z_1, z_2, ..., z_N) / Q(z_1, z_2, ..., z_N)$, in the region $\overline{U}^N - U^N$.

We applied transformations to reduce the transfer function to the 2-D case, and then utilized the 2-D stability conditions to obtain necessary conditions for stability.

We considered the problem of BIBO stability of three-dimensional linear shift-invariant filters, in the presence of nonessential singularities of the second kind. We derived necessary and sufficient conditions for boundedness, l_1 and l_2 stabilities of a function $G=P/Q^n$, where P/Q has simple NSSK of the second kind in the region $\overline{U}^3 - U^3$.

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Chapter 1.

Introduction.

Multidimensional digital filters find applications in geophysics, image processing, gravity and magnetic data as well as biomedical, sonar and radar data. The problem of bounded-input bounded-output (BIBO) stability is important to the design of practically useful recursive digital filters, since in an unstable filter any noise will propagate through to the output and be amplified.

This thesis presents two new fundamental results concerning the stability of multidimensional digital filters. In Chapter 2 we give an overview of the most important developments in the field over the last several decades.

One of the open problems is introduced in Chapter 3, where we apply a transformation of variables to obtain a necessary condition for the stability of N-dimensional filters in the presence of nonessential singularities of the second kind.

In Chapter 4 we consider the problem of finding necessary and sufficient conditions for the ℓ_2-stability of 3-dimensional filters with nonessential singularities. These conditions are then used in Chapter 5 to establish a necessary condition for the boundedness.

In Chapter 6 we derive two sufficient conditions for the ℓ_1-stability, one of which is also a necessary condition.

Finally, in Chapter 7 we present a sufficient condition for boundedness.

Throughout this thesis we use numerical examples to illustrate the computational aspect of the introduced methodology.

Chapter 2.

Stability of Multidimensional Digital Filters: An Overview.

2.1. Preliminaries and Definitions.

Consider the class of quarter-plane causal filters whose impulse responses have the first quadrant as their region of support. An N-dimensional (N-D) filter of such a class can be described by its real rational z-transform transfer function:

$$G(z_1,z_2,\dots,z_N)=\frac{P(z_1,z_2,\dots,z_N)}{Q(z_1,z_2,\dots,z_N)} \qquad (2.1.1)$$

where P and Q are mutually prime polynomials in N complex variables.

An N-tuple $\alpha=(\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_N)$ where $P(\alpha)\neq 0$ and $Q(\alpha)=0$ is called a pole, or nonessential singularity of a first kind, of $G=P/Q$. An N-tuple $\beta=(\beta_1,\beta_2,\beta_3,\dots,\beta_N)$ where $P(\beta)=Q(\beta)=0$ is called a nonessential singularity of a second kind (NSSK) of G. If an N-tuple β is a nonessential singularity of a second kind of G, then in any neighborhood of β there is a pole of $G(\beta)$.

We call a nonessential singularity of the second kind $(\alpha_1,\alpha_2,\dots,\alpha_N)$ simple if $(\partial Q/\partial z_i)_{(\alpha_1,\alpha_2,\dots,\alpha_N)}\neq 0$ for $i=1,\dots N$.

Assuming that $Q(z_1,z_2,\dots,z_N)\neq 0$ at the origin, by the continuity argument we can derive that $Q(z_1,z_2,\dots,z_N)\neq 0$ in some neighborhood around the origin. Therefore, G can be expanded into a power series in this neighborhood as

$$G(z_1,z_2,\dots,z_N)=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\dots\sum_{i_N=0}^{\infty}g(i_1,i_2,\dots,i_N)z_1^{i_1}z_2^{i_2}\dots z_N^{i_N} \qquad (2.1.2)$$

where $g(i_1,i_2,\dots,i_N)$ is the impulse response of G.

The filter is l_1 - stable, or BIBO stable, if and only if

$$\sum_{i_1,i_2,\ldots,i_N=0}^{\infty} |g(i_1,i_2,\ldots,i_N)| < \infty \qquad (2.1.3)$$

The filter is l_2 - stable, or the impulse response is square summable, if and only if

$$\sum_{i_1,i_2,\ldots,i_N=0}^{\infty} g^2(i_1,i_2,\ldots,i_N) < \infty \qquad (2.1.4)$$

We define

$$U^N = \{(z_1,z_2,\ldots,z_N):|z_i|<1, i=1,2,\ldots,N\}$$

to be the open unit polydisk,

$$\overline{U}^N = \{(z_1,z_2,\ldots,z_N):|z_i|\le 1, i=1,2,\ldots,N\}$$

to be the closed unit polydisk, and

$$T^N = \{(z_1,z_2,\ldots,z_N):|z_i|=1, i=1,2,\ldots,N\}$$

to be the distinguished boundary of the unit polydisk.

We denote by $\underline{A}(z_1,z_2,\ldots,z_N)$ the discrete paraconjugate of $A(z_1,z_2,\ldots,z_N)$:

$$\underline{A}(z_1,z_2,\ldots,z_N) = z_1^{\gamma_1} z_2^{\gamma_2} \ldots z_N^{\gamma_N} A\left(\frac{1}{z_1},\frac{1}{z_2},\ldots,\frac{1}{z_N}\right),$$

where $\gamma_1,\gamma_2,\ldots,\gamma_N$ are the highest degrees of $z_1,z_2,\ldots,z_N$, respectively, in $A(z_1,z_2,\ldots,z_N)$ as defined in [1].

Throughout this thesis we will use the definition and properties of resultants as described below [2]-[4].

The resultant of two polynomials in one variable

$$f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0, \text{ and}$$

$$g(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0$$

where $m, n \geq 1$ is denoted by $R(f,g)$ or $R_{m,n}(f,g)$. If $a_m \neq 0$ and $b_n \neq 0$ and the zeros of $f(x)$ and $g(x)$ are α_i, $i = 1,\dots,m$ and β_j, $j = 1,\dots,n$, respectively, then

$$R(f,g) = (a_m)^n (b_n)^m \prod_{\substack{i=1\\ j=1}}^{\substack{i=m\\ j=n}} (\alpha_i - \beta_j) \tag{2.1.5}$$

The resultant. therefore. is a polynomial in coefficients of $f(x)$ and $g(x)$. We note that according to (2.1.5) resultant $R(f,g) = 0$ when polynomials $f(x)$ and $g(x)$ have a common root or $a_m = b_n = 0$ (the latter means that the common root is at infinity).

The following are important properties of the resultants:

$$R_{m,n}(f,g) = (-1)^{mn} R_{m,n}(g,f) \tag{2.1.6}$$

$$R_{m+p,n}(fh,g) = R_{m,n}(f,g) R_{p,n}(h,g) \tag{2.1.7}$$

The Sylvester formula shows how to create a determinant for computing the resultant $R(f,g)$:

$$\begin{vmatrix}
a_0 & a_1 & a_2 & \dots & a_{m-1} & a_m & 0 & 0 & \dots & 0 \\
0 & a_0 & a_1 & \dots & a_{m-2} & a_{m-1} & a_m & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \dots & a_0 & a_1 & a_2 & a_3 & \dots & a_m \\
b_0 & b_1 & b_2 & \dots & b_{n-1} & b_n & 0 & 0 & \dots & 0 \\
0 & b_0 & b_1 & \dots & b_{n-2} & b_{n-1} & b_n & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \dots & b_0 & b_1 & b_2 & b_3 & \dots & b_n
\end{vmatrix}$$

First m rows reflect coefficients a, the next n rows reflect coefficients b; the order of the determinant is $m+n$. It can be shown that this Sylvester's determinant and resultant $R(f,g)$ are identical polynomials.

In a case of two-variable polynomials $f(z_1,z_2)$ and $g(z_1,z_2)$ we can think of z_1 as a constant, and then f and g may be considered one-variable polynomials in z_2:

$$f(z_1,z_2)=a_m(z_1)z_2^{\,m}+a_{m-1}(z_1)z_2^{\,m-1}+\cdots+a_1(z_1)z_2+a_0(z_1),\ \text{and}$$

$$g(z_1,z_2)=b_n(z_1)z_2^{\,n}+b_{n-1}(z_1)z_2^{\,n-1}+\cdots+b_1(z_1)z_2+b_0(z_1)$$

The resultant of these polynomials $R_{z_2}[f(z_1,z_2),g(z_1,z_2)]$ is a function of z_1. (Similarly, $R_{z_1}[f(z_1,z_2),g(z_1,z_2)]$ is defined as z_1-resultant of $f(z_1,z_2)$ and $g(z_1,z_2)$ and is a polynomial in z_2).

In this case $R_{z_2}[f(z_1,z_2),g(z_1,z_2)]$ can be also computed by the above Sylvester's determinant where $a_i=a_i(z_1)$ and $b_j=b_j(z_1)$.

The Cauchy-Schwartz's inequality

$$\sum_{i=1}^{n}a_ib_i\le\left(\sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}b_i^2\right)^{1/2} \tag{2.1.8}$$

can be obtained by noting that for any set of real numbers $a_i, i=1\ldots n$ and $b_i, i=1\ldots n$ the expression

$$\sum_{i=1}^{n}(a_ix+b_i)^2=\sum_{i=1}^{n}a_i^2x^2+2\sum_{i=1}^{n}a_ib_ix+\sum_{i=1}^{n}b_i^2$$

is always non-negative. Therefore, it cannot have real different roots, and its discriminant is non-negative:

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i \right)^2 \geq 0 \text{, or}$$

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \geq \left(\sum_{i=1}^{n} a_i b_i \right)^2 \qquad (2.1.9)$$

2.2. Recent Research.

The problem of bounded-input bounded-output (BIBO) stability, related to the design of practically useful recursive digital filters, has been the subject of intensive research over the past three decades.

Stability of digital filters depends on the existence of poles inside the unit polydisk and nonessential singularities of the second kind on its boundary. In the one-dimensional case, the filter is stable if all the roots of the denominator polynomial are outside the unit disk [5].

Shanks theorem [6], [7] (independently proven by C. Farmer) states that a causal recursive filter with the z-transform $H(z_1, z_2) = \dfrac{1}{B(z_1, z_2)}$, is stable if and only if there are no values of z_1 and z_2 such that $B(z_1, z_2) = 0$ and $|z_1| \leq 1, |z_2| \leq 1$.

It was shown several years later by Goodman [8] that Shanks' earlier theorem [9], considering the case $H(z_1, z_2) = \dfrac{A(z_1, z_2)}{B(z_1, z_2)}$, where $A(z_1, z_2)$ and $B(z_1, z_2)$ are mutually prime two-dimensional polynomials, has an important exception (special case). We will describe that case later in the chapter.

Huang [10] has derived a simplified version of Shanks' stability theorem and proved its equivalency to Ansell's [11] theorem.

It states that a 2-D polynomial $B(z_1,z_2) \neq 0$ in $\overline{U}^2$ if and only if the following holds:

1. $B(z_1,0) \neq 0 \qquad |z_1| \leq 1$

2. $B(z_1,z_2) \neq 0 \qquad |z_1| = 1, |z_2| \leq 1$

Anderson and Jury [12] considered a multidimensional case where $H(z_1,z_2,\ldots,z_N) = \dfrac{P(z_1,z_2,\ldots,z_N)}{Q(z_1,z_2,\ldots,z_N)}$ and $P(z_1,z_2,\ldots,z_N)$ and $Q(z_1,z_2,\ldots,z_N)$ do not have common zeros, and proved the following stability condition:

$$Q(z_1,z_2,\ldots,z_N) \neq 0 \qquad \bigcap_{i=1}^{N} |z_i| \leq 1 \tag{2.2.1}$$

They also showed that (2.2.1) is equivalent to

$$Q(z_1,z_2,\ldots,z_N) \neq 0 \qquad \left(\bigcap_{i=1}^{N-1} |z_i| = 1\right) \cap \left(|z_N| \leq 1\right)$$

$$Q(z_1,z_2,\ldots,z_{N-1},0) \neq 0 \qquad \bigcap_{i=1}^{N-1} |z_i| \leq 1$$

which, in turn, is equivalent to

$$Q(z_1,z_2,\ldots,z_N) \neq 0 \qquad \left(\bigcap_{i=1}^{N-1} |z_i| = 1\right) \cap \left(|z_N| \leq 1\right)$$

$$Q(z_1,z_2,\ldots,z_{N-1},0) \neq 0 \qquad \left(\bigcap_{i=1}^{N-2} |z_i| = 1\right) \cap \left(|z_{N-1}| \leq 1\right)$$

$$Q(z_1,z_2,\ldots,z_{N-2},0,0)\neq 0 \qquad \left(\bigcap_{i=1}^{N-3}|z_i|=1\right)\cap\left(|z_{N-2}|\leq 1\right)$$

$$\vdots$$

$$Q(z_1,z_2,0,\ldots,0)\neq 0 \qquad \left(|z_1|=1\right)\cap\left(|z_2|\leq 1\right)$$

$$Q(z_1,0,0,\ldots,0)\neq 0 \qquad |z_1|\leq 1$$

Goodman [8] considered the 2-D transfer function

$$G(z_1,z_2)=\frac{P(z_1,z_2)}{Q(z_1,z_2)}$$

where $P(z_1,z_2)$ and $Q(z_1,z_2)$ are mutually prime and $Q(0,0)\neq 0$,

and has proved that Shanks' direct extension of the 1-D criterion to the 2-D case [6] is sufficient for stability. It also has been shown in [8] that a 2-D digital filter can be BIBO stable even in the presence of a nonessential singularity of the second kind on the distinguished boundary of the unit bidisk. The results obtained in [8] can be summarized as follows:

$$\{g_{mn}\}\in\ell_1 \text{ or } \{g_{mn}\}\in\ell_2 \quad\Rightarrow\quad \lim g_{mn}=0$$

$$|g_{mn}|\leq M<\infty,\forall m,n \quad\Rightarrow\quad Q(z_1,z_2)\neq 0 \text{ in } U^2$$

$$|G(z_1,z_2)|\leq N<\infty \text{ in } U^2 \quad\Rightarrow\quad \{g_{mn}\}\in\ell_2$$

$$Q(z_1,0)\neq 0 \text{ in } \overline{U} \quad\Rightarrow\quad \sum_{m=0}^{\infty}|g_{mn}|<\infty,\forall n$$

$$\begin{tikzcd}
& \{g_{mn}\}\in\ell_2 & \\
Q(z_1,z_2)\neq 0 \text{ in } \overline{U}^2 \arrow[r, Rightarrow] & \textbf{BIBO stability} \arrow[u, Rightarrow] \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \{g_{mn}\}\in\ell_1 \\
& Q(z_1,z_2)\neq 0 \text{ in } \overline{U}^2-\overline{T}^2 &
\end{tikzcd}$$

O'Connor and Huang [13] proved that if $S_1[(M_1,N_1),(M_2,N_2)]$ and $S_2[(P_1,Q_1),(P_2,Q_2)]$ are two sectors with $D=M_1N_2-M_2N_1\neq 0$ and $E=P_1Q_2-P_2Q_1$, then there exists many linear mappings of the form $m=k_1m'+k_2n'$, $n=k_3m'+k_4n'$, $k_i\in Z$. This mapping helps to prove that if $b(m,n)$ is a recursive filter array with support β, then $b(m,n)$ is stable if and only if for any $k_i\in Z$ with $K=k_1k_4-k_2k_3\neq 0$, the array $g(m,n)=g(k_1m'+k_2n',k_3m'+k_4n')=b(m',n')$ with $(m',n')\in\beta$ is stable.

Alexander and Woods [15] presented two important results, the first is a necessary condition expressed in terms of tangents to the algebraic curve at a zero of the denominator polynomial on the distinguished boundary of the unit disk. The second result is a sufficient condition for the stability that is considerably weaker than the one proposed by Goodman in [8].

The difference between 2-D and higher dimensional cases as well as analyzing stability properties of 3-D filters, considered by Roytman et al in recent papers [15], [16], presents additional complexity, because, as shown in [16], a transfer function may have nonessential singularities of the second kind in the region $\overline{U}^3 - U^3$ and still be BIBO stable.

In a very important result, necessary and sufficient conditions for stability and boundedness of 2-D systems with nonessential singularities of the second kind on T^2 were later derived in [17]. Instead of computing Parseval's double integral, the residue approach was used to find whether it has a finite limit. The stability conditions were expressed in terms of the multiplicity of zeros of resultants of certain polynomials. The results were extended to the case when the transfer function has more then one pair of simple second kind singularities on the unit disk's boundary.

Fettweis and Basu in [18] presented various definitions and properties of discrete scattering Hurwitz polynomials that are important in the theory of multidimensional systems.

In [19] the same authors described and proved some stability-related properties of multidimensional polynomials. The paper formulated the continuity property of the zeros of a multivariable polynomial as function of its coefficients and presented simplified proofs of results of Strintzis [20], DeCarlo [21], Anderson and Jury [12], and Rudin [22]. This continuity property has been used to derive conditions for a polynomial *f(z)* in $z = (z_1, z_2, ..., z_k)$ to be devoid of zeroes in $|z| \leq 1$. The paper also shows how most of the related results can be proved in similar manner.

Other important results concerning the stability of multidimensional digital filters were presented in [23]-[27].

In this thesis, we consider open problems of BIBO stability of multidimensional digital filters with nonessential singularity of the second kind in the region $\overline{U}^N - U^N$ and derive sufficient and necessary conditions for stability and boundedness.

Chapter 3.

Necessary Conditions for the Stability of N-D Digital Filters with Nonessential Singularities of the Second Kind

We consider the open problem concerning the BIBO stability of N-D digital filters with nonessential singularities of the second kind in the region $\overline{U}^N - U^N$. We will apply transformations of variables to reduce the transfer function to the 2-D case, and then apply the 2-D stability theorems. We will illustrate this approach with several examples.

Let us begin this discussion by considering the real rational transfer function:

$$G(z_1, z_2, \ldots, z_N) = \frac{P(z_1, z_2, \ldots, z_N)}{Q(z_1, z_2, \ldots, z_N)}$$

where P and Q are relatively prime and $G(z_1, z_2, \ldots, z_N)$ has no polar singularities in $\overline{U}^N$ nor any nonessential singularities of the second kind anywhere except in the region $\overline{U}^N - U^N$. We further assume that these are simple.

Consider the following transformation of variables:

$$z_k = \omega_k z_2^{n_k} \qquad k=3,\ldots,N \qquad (3.1)$$

where n_k is a large positive integer. The values of the coefficients ω_k depend on the particular nonessential singularity and for $(z_1, z_2, z_3, \ldots, z_N) = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_N)$ will be determined from the following relationships:

$$\alpha_k = \omega_k \alpha_2^{n_k} \qquad k=3,\ldots,N$$

Substituting (3.1) into the expression for G (2.1.2)

$$G(z_1, z_2, \ldots, z_N) = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_N=0}^{\infty} g(i_1, i_2, \ldots, i_N) z_1^{i_1} z_2^{i_2} \ldots z_N^{i_N} \ ,$$

we obtain

$$\begin{aligned}
G(z_1,z_2,\dots,z_N) &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_N=0}^{\infty} g(i_1,i_2,\dots,i_N) z_1^{i_1} z_2^{i_2} (\omega_3 z_2^{n_3})^{i_3} (\omega_4 z_2^{n_4})^{i_4} \dots (\omega_N z_2^{n_N})^{i_N} \\
&= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_N=0}^{\infty} g(i_1,i_2,\dots,i_N) z_1^{i_1} z_2^{i_2+\sum_{k=3}^{N} n_k i_k} \prod_{k=3}^{N} \omega_k^{i_k} \\
&= \sum_{i_1=0}^{\infty}\sum_{l=0}^{\infty} h(i_1,l) z_1^{i_1} z_2^{l} \prod_{k=3}^{N} \omega_k^{i_k} \\
&= H(z_1,z_2)
\end{aligned}$$

where

$$H(z_1,z_2) = \underset{\substack{n_k \to \infty \\ z_k = \omega_k z_2^{n_k}}}{G} (z_1,z_2,\dots,z_N) \qquad k=3,\dots,N$$

and

$$\begin{aligned}
H(z_1,z_2) &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_N=0}^{\infty} g(i_1,i_2,\dots,i_N) z_1^{i_1} z_2^{i_2+\sum_{k=3}^{N} n_k i_k} \prod_{k=3}^{N} \omega_k^{i_k} \\
&= \sum_{i_1=0}^{\infty}\sum_{l=0}^{\infty} h(i_1,l) z_1^{i_1} z_2^{l} \prod_{k=3}^{N} \omega_k^{i_k}
\end{aligned} \tag{3.2}$$

Generally, this N-D to 2-D transformation does not preserve the form of *G(z)*. Specifically, as a result of the simplifications, the number of terms in (3.2) may be less than the one in (2.1.2).

Consider the expression $l = i_2 + \sum_{k=3}^{N} n_k i_k$ that determines which terms will be simplified.

We observe that terms of G in which the highest degree i_2 of z_2 is less than n_k will not be affected by the simplifications. These terms can be described as $\sum_{i_1=0}^{\infty} \sum_{l=0}^{\min\{n_k\}-1} h(i_1,l) z_1^{i_1} z_2^{l}$.

In addition, as all n_k grow larger, the number of these form-preserving terms increases. As all n_k approach infinity, the transformation becomes form-preserving.

This transformation will result in a 2-D series for which the following holds:

$$\sum_{i_1,l=0} |h(i_1,l)| \le \sum_{i_1,i_2,\dots,i_N=0} |g(i_1,i_2,\dots,i_N)|$$

$$\sum_{i_1,l=0} [h(i_1,l)]^2 \le \sum_{i_1,i_2,\dots,i_N=0} [g(i_1,i_2,\dots,i_N)]^2$$

If $G(z)$ is l_1 - stable then according to (2.1.3)

$$\sum_{i_1,i_2,\dots,i_N=0}^{\infty} |g(i_1,i_2,\dots,i_N)| < \infty$$

and therefore

$$\sum_{i_1,l=0}^{\infty} |h(i_1,l)| < \infty$$

which implies that $H(z_1,z_2)$ is also l_1 - stable, and we can apply the 2-D stability condition (b) stated in [27]:

The same argument can be repeated for l_2 - stability. If the filter is l_2 - stable, then according to (2.1.4)

$$\sum_{i_1,i_2,\dots,i_N=0}^{\infty} g^2(i_1,i_2,\dots,i_N) < \infty$$

and therefore

$$\sum_{i_1,l=0}^{\infty} [h(i_1,l)]^2 < \infty$$

which means that $H(z_1,z_2)$ is also l_2 - stable, and the 2-D stability condition (a) from [27] can be applied.

Using the methodology introduced in [17], which involves resultants and multiplicities, we can now restate it as the following necessary condition for stability.

Theorem 3.1.

(i) If $G(z_1, z_2, \ldots, z_N)$ is BIBO stable filter defined by (2.1.3), then

$$m_\alpha(R_{z_1}[\underline{Q}_1, Q_1]) < m_\alpha(R_{z_1}[P_1, Q_1])$$

(ii) If $G(z_1, z_2, \ldots, z_N)$ is l_2 - stable, then

$$m_\alpha(R_{z_1}[\underline{Q}_1, Q_1]) \leq 2m_\alpha(R_{z_1}[P_1, Q_1])$$

where

$$P_1(z_1, z_2) = P(z_1, z_2, \omega_3 z_2^{n_3}, \omega_4 z_2^{n_4}, \ldots, \omega_N z_2^{n_N})$$
$$Q_1(z_1, z_2) = Q(z_1, z_2, \omega_3 z_2^{n_3}, \omega_4 z_2^{n_4}, \ldots, \omega_N z_2^{n_N})$$

and m denotes the multiplicity and R the resultant.

The above property provides us with a tool for analyzing N-D digital filters. To illustrate this result consider the following examples.

Example 3.1.

$$G(z_1, z_2, z_3) = \frac{3 + z_1^2 z_2 + z_1^2 z_3 - 2z_2^2 + z_1}{3 + z_1^2 - z_2^2 + z_3^2}$$

This function has nonessential singularity of a second kind at (z_1, z_2, z_3)=(j,1,j), therefore, ω_3=j. We obtain:

$$P_1 = 3 + z_1^2 z_2 + jz_1^2 z_2^{n_3} - 2z_2^2 + z_1$$
$$Q_1 = 3 + z_1^2 - z_2^2 - z_2^{2n_3}$$
$$\underline{Q_1} = z_2^{2n_3}(3z_1^2 + 1) - z_1^2 z_2^{2n_3-2} - z_1^2$$

The corresponding resultants are:

$$
\begin{aligned}
R_{z_1}[\underline{Q}_1, Q_1] &= (10z_2^{2n_3} - 3z_2^{2n_3-2} - 3 - 3z_2^{2n_3+2} + z_2^2 - 3z_2^{4n_3} + z_2^{4n_3-2})^2 \\
R_{z_1}[P_1, Q_1] &= 12 + 6z_2^{2n_3+1} - 18z_2 - 4z_2^2 + 18z_2^3 - 4z_2^{2n_3+3} - 10z_2^{2n_3} - z_2^{6n_3} - 4jz_2^{3n_3+2} \\
&\quad + 2jz_2^{5n_3+1} + 4jz_2^{3n_3+3} - 12jz_2^{3n_3+1} + 6z_2^{4n_3} - 2z_2^4 + z_2^{2n_3+4} - z_2^{4n_3+2} - 18jz_2^{n_3} \\
&\quad - 4z_2^5 + z_2^6 + 18jz_2^{n_3+2} + 6jz_2^{3n_3} - 4jz_2^{n_3+4} - 12jz_2^{n_3+3} + 18jz_2^{n_3+1} + 2jz_2^{n_3+5}
\end{aligned}
$$

After lengthy calculations (the details shown in the Appendix A), we see that

$$m_1(R_{z_1}[\underline{Q}_1, Q_1]) = 4 \quad \text{and} \quad m_1(R_{z_1}[P_1, Q_1]) = 1 .$$

Therefore, the system is unstable.

<u>Example 3.2</u>. In this example we show how the above theorem can be applied to higher order cases.

$$G(z_1, z_2, z_3, z_4) = \frac{3z_1^2 z_2^2 z_3 z_4 - 2z_1 z_2^2 z_3^2 z_4 + 2z_2 z_3^3 z_4^2 - 5z_1 z_3 z_4^3 - z_1^3 z_3 + 3}{z_1^2 z_2^3 z_3 z_4 + z_1^3 z_2 z_3^2 z_4 + 2z_1 z_2 z_3 z_4^2 + 2z_2 z_3^3 z_4^2 + z_1 z_2 z_3 - 7}$$

This function has a nonessential singularity of a second kind at $(z_1, z_2, z_3, z_4) = (1,1,1,1)$, therefore, $\omega_3 = \omega_4 = 1$, and

$$
\begin{aligned}
P_1 &= -z_1^3 z_2^{n_3} + 3z_1^2 z_2^{2+n_3+n_4} - z_1(2z_2^{2n_3+n_4+2} + 5z_2^{n_3+3n_4}) + 2z_2^{3n_3+2n_4+1} + 3 \\
Q_1 &= z_1^2 z_2^{n_3+n_4+3} + z_1^3 z_2^{2n_3+n_4+1} + 2z_1 z_2^{n_3+2n_4+1} + 2z_2^{3n_3+2n_4+1} + z_1 z_2^{n_3+1} - 7 \\
\underline{Q}_1 &= z_1^3(2 - 7z_2^{3n_3+2n_4+1}) + z_1^2 z_2^{2n_3}(2 + z_2^{2n_4}) + z_1 z_2^{2n_3+n_4-2} + z_2^{n_3+n_4}
\end{aligned}
$$

After calculating the resultants and multiplicities of the root (see Appendix B for details) we obtain

$$m_1(R_{z_1}[\underline{Q}_1, Q_1]) = 2 \quad \text{and} \quad m_1(R_{z_1}[P_1, Q_1]) = 1$$

and therefore the system is unstable.

Finally, let us consider the 3-D transfer function with a nonessential singularity in the region $\overline{U}^3 - U^3 - T^3$.

Example 3.3 [16].

$$G(z_1,z_2,z_3)=\frac{(1-z_1)(1-z_2)(1-z_3)}{(1-z_1)(1-z_2)+(1-z_3)(1-z_1z_2)}$$

This function has nonessential singularity of the second kind at $z_1=z_2=1$ and any z_3. We will choose the point (1, 1, 0.5) to illustrate how our approach works in the case when function possesses nonessential singularity of the second kind in the region $\overline{U}^3-U^3-T^3$.

We have (see Appendix C for details):

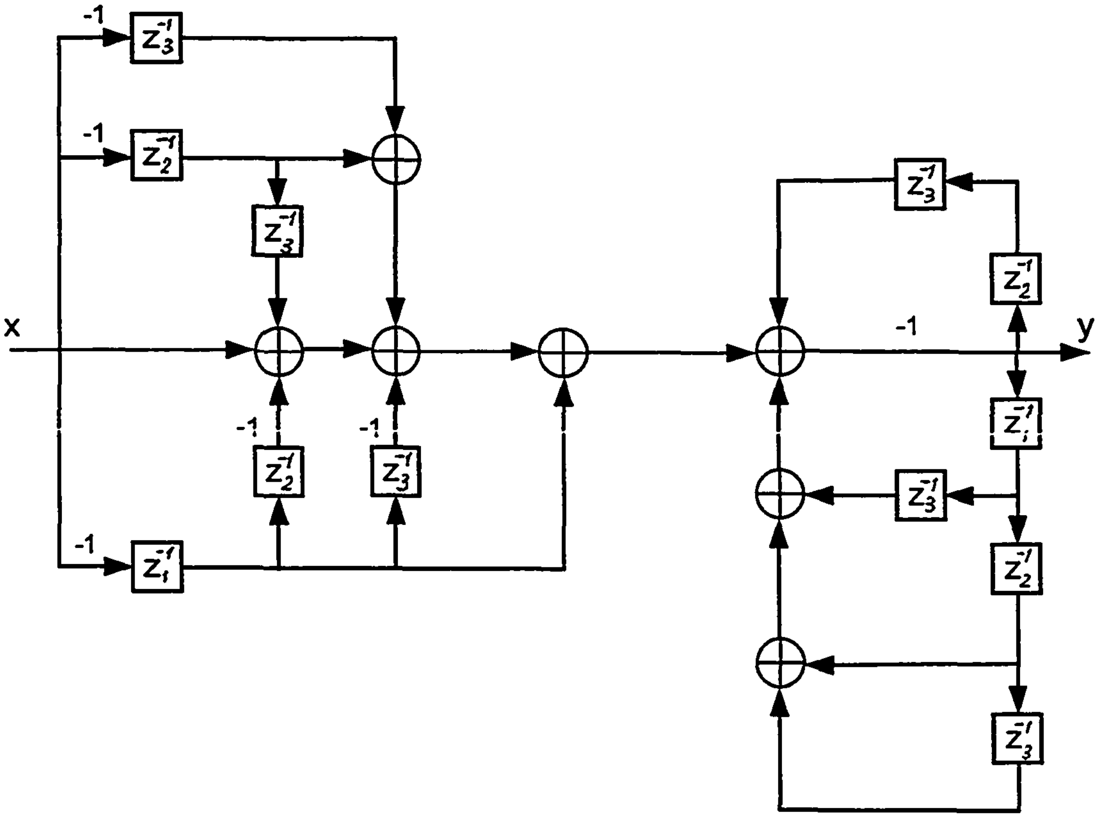
$$\omega_3=0.5$$
$$P_1=(1-z_1)(1-z_2)(1-0.5z_2^{n_3})$$
$$Q_1=(1-z_1)(1-z_2)+(1-0.5z_2^{n_3})(1-z_1z_2)$$
$$\underline{\underline{Q}}_1=(2z_1-1)z_2^{n_3+1}-z_1z_2^{n_3}-0.5z_1z_2+0.5$$
$$R_{z_1}[\underline{Q}_1,Q_1]=0.5-z_2+0.5z_2^2-2z_2^{n_3}+0.5z_2^{2n_3}+4z_2^{n_3+1}-2z_2^{n_3+2}-z_2^{2n_3+1}+0.5z_2^{2n_3+2}$$
$$R_{z_1}[P_1,Q_1]=-0.25(1-z_2)^2(2-z_2^{n_3})^2$$

The corresponding multiplicities are:

$$m_1(R_{z_1}[\underline{Q}_1,Q_1])=2 \text{ and } m_1(R_{z_1}[P_1,Q_1])=2,$$

which means that the system is unstable.

The system diagram is presented on the next page.



System diagram
for Example 3.3

In this chapter we have derived the necessary conditions for stability of N-D digital filters in the presence of nonessential singularities of the second kind in the region $\overline{U}^N - U^N$. We have demonstrated how the presented property is applied to several numerical examples.

Chapter 4.

l_2 - Stability Theorems.

Consider the real rational transfer function:

$$G(z_1,z_2,z_3)=\frac{P(z_1,z_2,z_3)}{Q(z_1,z_2,z_3)}$$

where P and Q are relatively prime and $G(z_1,z_2,z_3)$ has no polar singularities in $\overline{U}^3$ nor any nonessential singularities of the second kind anywhere except for the pair at $(\alpha_1,\alpha_2,\alpha_3)$ and $\left(\frac{1}{\alpha_1},\frac{1}{\alpha_2},\frac{1}{\alpha_3}\right)$ in the region $\overline{U}^3-U^3$.

We assume that these are simple.

In order to find whether $G\in l_2$, or $S=\sum_{i_1,i_2,i_3=0}^{\infty}g^2(i_1,i_2,i_3)<\infty$, we consider the following integral to evaluate it:

$$S_F=\left(\frac{1}{2\pi j}\right)^3\oint_{|z_1|=1}\oint_{|z_2|=1}\oint_{|z_3|=1}F(z_1,z_2,z_3)F\left(\frac{1}{z_1},\frac{1}{z_2},\frac{1}{z_3}\right)\frac{dz_1dz_2dz_3}{z_1z_2z_3}$$

Since $G(z_1,z_2,z_3)$ has a NSSK on T^3, the Parseval's integral cannot be used for deriving S. Therefore, we will use a different function obtained from $G(z_1,z_2,z_3)$ as a linear transform:

$$G(z_1,z_2,z_3)\to G(kz_1,kz_2,kz_3).$$

For $k<1$, $G(kz_1,kz_2,kz_3)$ has no singularities in $\overline{U}^3$, and the Parseval's integral can be applied to computing S:

$$S(k)=\left(\frac{1}{2\pi j}\right)^3 \oint\limits_{|z_1|=1}\oint\limits_{|z_2|=1}\oint\limits_{|z_3|=1} F(kz_1,kz_2,kz_3)F\left(\frac{k}{z_1},\frac{k}{z_2},\frac{k}{z_3}\right)\frac{dz_1dz_2dz_3}{z_1z_2z_3}$$

It can be shown that $S=\lim\limits_{k\to 1}S(k)$, and now the problem of l_2 stability is reduced to the one of finding if $S(k)$ has a finite limit for $k\to 1$.

We introduce the following transformations:

$$P_k(z_1,z_2,z_3)=P(kz_1,kz_2,kz_3)$$

$$Q_k(z_1,z_2,z_3)=Q(kz_1,kz_2,kz_3)$$

$$P_k\left(\frac{1}{z_1},\frac{1}{z_2},\frac{1}{z_3}\right)=\frac{1}{z_1^{m_1}}\frac{1}{z_2^{n_1}}\frac{1}{z_3^{p_1}}\underline{P}_k(z_1,z_2,z_3)$$

$$Q_k\left(\frac{1}{z_1},\frac{1}{z_2},\frac{1}{z_3}\right)=\frac{1}{z_1^{m_2}}\frac{1}{z_2^{n_2}}\frac{1}{z_3^{p_2}}\underline{Q}_k(z_1,z_2,z_3)$$

and rewrite the integral as

$$S(k)=\left(\frac{1}{2\pi j}\right)^3 \oint\limits_{|z_1|=1}\oint\limits_{|z_2|=1}\oint\limits_{|z_3|=1}\frac{P_k(z_1,z_2,z_3)\underline{P}_k(z_1,z_2,z_3)}{Q_k(z_1,z_2,z_3)\underline{Q}_k(z_1,z_2,z_3)}z_1^m z_2^n z_3^p dz_1dz_2dz_3$$

$$=\frac{1}{2\pi j}\oint\limits_{|z_1|=1} z_1^m J_1(z_1)dz_1$$

where

$$J_1(z_1)=\left(\frac{1}{2\pi j}\right)^2 \oint\limits_{|z_2|=1}\oint\limits_{|z_3|=1}\frac{P_k(z_1,z_2,z_3)\underline{P}_k(z_1,z_2,z_3)}{Q_k(z_1,z_2,z_3)\underline{Q}_k(z_1,z_2,z_3)}z_2^n z_3^p dz_2dz_3$$

Further, rewrite $J_1(z_1)$ as

$$J_1(z_1)=\frac{1}{2\pi j}\oint\limits_{|z_2|=1} z_2^n J_2(z_1,z_2)dz_2$$

where

$$J_2(z_1,z_2)=\frac{1}{2\pi j}\oint_{|z_3|=1}\frac{P_k(z_1,z_2,z_3)\underline{P}_k(z_1,z_2,z_3)}{Q_k(z_1,z_2,z_3)\underline{Q}_k(z_1,z_2,z_3)}z_3^p dz_3 \tag{4.1}$$

We recognize that $\underline{P}_k(z_1,z_2,z_3)$ and $\underline{Q}_k(z_1,z_2,z_3)$ are discrete paraconjugates of $P_k(z_1,z_2,z_3)$ and $Q_k(z_1,z_2,z_3)$, respectively [1], [18], [19].

We further note that $Q_k(z_1,z_2,z_3)\neq 0$ in $\overline{U}^3$, and therefore only zeros of $\underline{Q}_k(z_1,z_2,z_3)$ are important in evaluating J_2.

These zeros can be obtained from the equation $\underline{Q}_k(z_1,z_2,z_3)=0$ by expressing z_3 as function of z_1 and z_2 in a form

$$\underline{Q}_k(z_1,z_2,z_3)=a_{mk}(z_1,z_2)\prod_{i=1}^{m}\left[z_3-\phi_{ik}(z_1,z_2)\right]$$

In order to determine whether J_2 is finite by computing it we would need to find residues at $z_3=\phi_{ik}(z_1,z_2)$. That would require substituting $\phi_{ik}(z_1,z_2)$ for z_3 into the polynomials Q_k and $P_k\underline{P}_k$:

$$J_2(z_1,z_2)=res\left[\frac{P\underline{P}}{Q\underline{Q}}z_3^p\right]_{\underline{Q}=0,|z_3|\leq 1}+res\left[\frac{P\underline{P}}{Q\underline{Q}}z_3^p\right]_{z_3=0,p<0} \tag{4.2}$$

We note that substituting zeros of $\underline{Q}$ into $P\underline{P}$ and Q (first term in (4.2)) is the same as computing the resultants $R_{z_3}[\underline{Q},P\underline{P}]$ and $R_{z_3}[\underline{Q},Q]$

$$res\left[\frac{P\underline{P}}{Q\underline{Q}}z_3^p\right]_{\underline{Q}=0,|z_3|\leq 1}=\frac{R_{z_3}\left[\underline{Q},P\underline{P}\right]}{R_{z_3}\left[\underline{Q},Q\right]}\phi^{p}(z_1,z_2)$$

Using the properties of a resultant [4] and following the reasoning in [17] we can show that $R_{z_3}[\underline{Q},P\underline{P}]=(R_{z_3}[P,Q])^2$.

Let us denote $T(z_1,z_2)=R_{z_3}[\underline{Q},Q]$ and $M(z_1,z_2)=R_{z_3}[P,Q]$.

Obviously the integral $J_2(z_1,z_2)$ is finite if the term $\frac{M^2}{T}$ is finite, and consequently,

$J_1(z_1)$ is finite if the term $\oint\limits_{|z_2|=1}\frac{M^2}{T}z_2^n dz_2$ is finite.

Applying the residue approach to evaluate this term, we obtain

$$\oint\limits_{|z_2|=1}\frac{M^2}{T}z_2^n dz_2 = res\left[\frac{M^2}{T}z_2^n\right]_{T=0,|z_2|\le 1}+res\left[\frac{M^2}{T}z_2^n\right]_{z_3=0,n<0} \tag{4.3}$$

Again, only the first term in (4.3) is important for the stability. Note that since Q and $\underline{Q}$ are reciprocal to each other, therefore their resultant is a product of two reciprocal polynomials in two variables:

$$T(z_1,z_2)=\underline{V}(z_1,z_2)V(z_1,z_2) \tag{4.4}$$

In practice, however, this polynomial is generally not factorizable, therefore we cannot obtain the expression for $T(z_1,z_2)$ in closed form (4.4).

As it was shown in the 2-D case [17], only zeros of $\underline{V}(z_1,z_2)$ are located within the unit bidisk, therefore zeros of $V(z_1,z_2)$ do not affect the stability.

Let the order of zero $z_1=\alpha_1$ of the function $T(z_1,z_2)$ be t, and the order of the same zero of the function $M^2(z_1,z_2)$ be r. Obviously (4.3) is finite if and only if $t\le r$.

We can interpret orders of zero $z_1 = \alpha_1$ t and r to be the multiplicities of the factor $(z_1 - \alpha_1)$ in the resultants $R_{z_2}[\underline{V}, V]$ and $R_{z_2}[\underline{V}, M^2]$, respectively:

$$t = m_\alpha(R_{z_2}[\underline{V}, V]);$$

$$r = m_\alpha(R_{z_2}[\underline{V}, M^2])$$

Since we typically do not know expressions for $\underline{V}(z_1, z_2)$ and $V(z_1, z_2)$, we will need to represent multiplicities involving these functions in terms of $T(z_1, z_2)$. Indeed, the following holds:

$$r = m_\alpha(R_{z_2}[\underline{V}, M^2]) = 2m_\alpha(R_{z_2}[M, V]) = m_\alpha(R_{z_2}[M, \underline{V}V]) = m_\alpha(R_{z_2}[M, T])$$

In order to calculate t we observe that

$$\begin{aligned} R_{z_2}\left[T, \frac{\partial T}{\partial z_2}\right] &= R_{z_2}[V\underline{V}, V'\underline{V} + \underline{V}'V] = C_1 \prod [V'\underline{V} + \underline{V}'V]_{v=0} [V'\underline{V} + \underline{V}'V]_{\underline{v}=0} \\ &= C_1 \prod [V'\underline{V}]_{v=0} [\underline{V}'V]_{\underline{v}=0} = C_1 C_2 \prod [\underline{V}]_{v=0} [V]_{\underline{v}=0} = C_3 R_{z_2}[\underline{V}, V] R_{z_2}[V, \underline{V}] \\ &= C_3 (R_{z_2}[\underline{V}, V])^2 \end{aligned}$$

where C_1, C_2, C_3 are constants,

and thus

$$t = m_\alpha(R_{z_2}[\underline{V}, V]) = \frac{1}{2} m_\alpha\left(R_{z_2}\left[T, \frac{\partial T}{\partial z_2}\right]\right)$$

Therefore, for l_2-stability:

$$m_\alpha(R_{z_2}[\underline{V}, V]) \le m_\alpha(R_{z_2}[\underline{V}, M^2]), \text{ or}$$

$$m_\alpha\left(R_{z_2}\left[T, \frac{\partial T}{\partial z_2}\right]\right) \le 2m_\alpha(R_{z_2}[M, T])$$

We can now formulate the following l_2 - stability theorem.

Theorem 4.1.

Let $G(z_1, z_2, z_3) = \dfrac{P(z_1, z_2, z_3)}{Q(z_1, z_2, z_3)}$ have no polar singularities in $\overline{U}^3$ and no nonessential singularities of the second kind anywhere except for the pair at $(\alpha_1, \alpha_2, \alpha_3)$ and $\left(\dfrac{1}{\alpha_1}, \dfrac{1}{\alpha_2}, \dfrac{1}{\alpha_3}\right)$ in the region $\overline{U}^3 - U^3$. Let P and Q be relatively prime. Then G is l_2 stable if and only if

$$m_\alpha\left(R_{z_2}\left[T, \frac{\partial T}{\partial z_2}\right]\right) \leq 2 m_\alpha (R_{z_2}[M, T])$$

Consider the following example.

Example 4.1.

$P(z_1, z_2, z_3) = 1 - z_3$

$Q(z_1, z_2, z_3) = 3 - z_1 - z_2 - z_3$

$\underline{Q}(z_1, z_2, z_3) = 3 z_1 z_2 z_3 - z_1 z_2 - z_2 z_3 - z_1 z_3$

$T(z_1, z_2) = R_{z_3}[\underline{Q}, Q] = 3 z_1 z_2^2 + 3 z_1^2 z_2 - 10 z_1 z_2 - z_1^2 - z_2^2 + 3 z_1 + 3 z_2$

$M(z_1, z_2) = R_{z_3}[P, Q] = z_1 + z_2 - 2$

$R_{z_2}[M, T] = 2(z_1 - 1)^2$

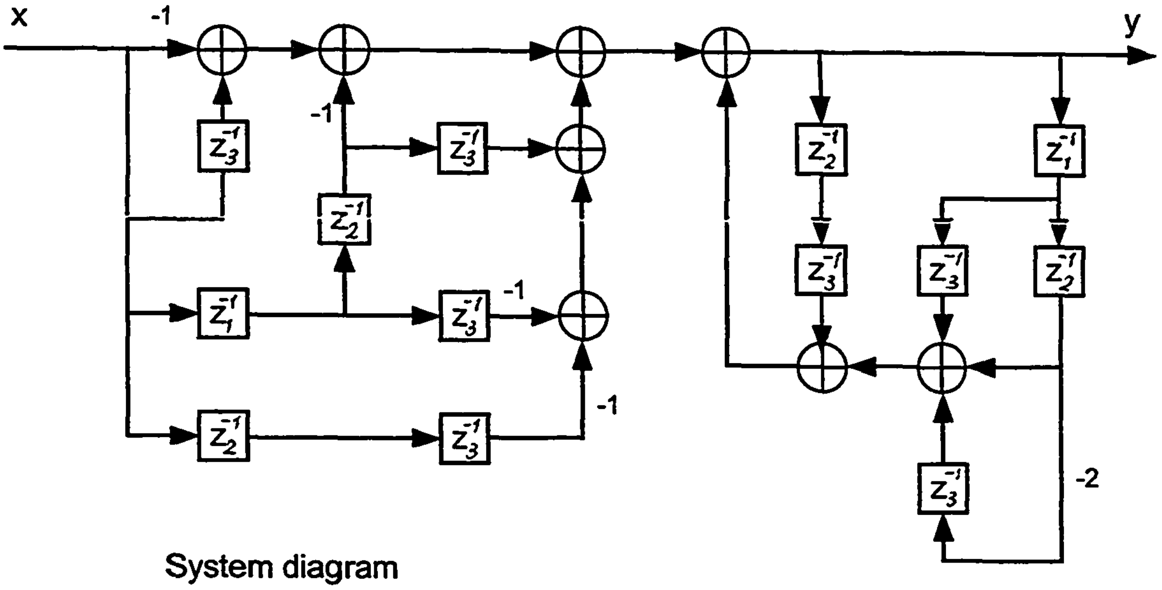
$m_{z_1 = 1}(R_{z_2}[M, T]) = 2$

$$R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]=-3(3z_1-1)^2(z_1-3)(z_1-1)^2$$

$$m_{z_1=1}\left(R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]\right)=2$$

Therefore, $m_a\left(R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]\right)\leq 2m_a(R_{z_2}[M,T])$, and the system is stable.

The system diagram for this example is presented on the next page.



System diagram
for Example 4.1

We will now consider the ℓ_2 stability of the function

$$G(z_1,z_2,z_3)=\frac{P(z_1,z_2,z_3)}{[Q(z_1,z_2,z_3)]^n}$$

where the function $\frac{P}{Q}$ is as defined in Theorem 4.1, and n is a positive integer. We will use the result in the discussion of ℓ_1 stability. In this case expression (4.1) becomes

$$J_2(z_1,z_2)=\frac{1}{2\pi j}\oint_{|z_3|=1}\frac{P_k\underline{P}_k}{{Q_k}^n{\underline{Q}_k}^n}z_3^p\,dz_3,$$

and in a process of evaluating it we need to compute the residue $res\left[\frac{P\underline{P}}{Q^n\underline{Q}^n}\right]_{\underline{Q}=0}$ which would require differentiating the function *(n-1)* times before substitution.

The relevant calculations are shown in Appendix D.

The most singular term in the resulting expression is $\frac{P\underline{P}}{\underline{Q}^{2n-1}}$; we shall analyze that term for stability.

Evaluating integral J_l for stability, and noting that

$$\frac{R_{z_3}\left[\underline{Q},P\underline{P}\right]}{R_{z_3}\left[\underline{Q},Q^{2n-1}\right]}=\frac{\left(R_{z_3}\left[P,Q\right]\right)^2}{\left(R_{z_3}\left[\underline{Q},Q\right]\right)^{2n-1}}=\frac{M^2}{T^{2n-1}}$$

we arrive at expression $res\left[\frac{M^2}{T^{2n-1}}\right]_{T=0,|z_2|\le 1}$ that determines the stability of integral J_l.

Therefore, in this case we will need to multiply the multiplicity *t*, which we calculated for the case of $G = \frac{P}{Q}$, by *(2n-1)*; the multiplicity *r* remains the same. We can now state the following theorem regarding the l_2 stability of $G = \frac{P}{Q^n}$.

Theorem 4.2.

Let $G(z_1, z_2, z_3) = \dfrac{P(z_1, z_2, z_3)}{[Q(z_1, z_2, z_3)]^n}$

where the function $P(z_1, z_2, z_3) / Q(z_1, z_2, z_3)$ is as defined in Theorem 4.1, and *n* is a positive integer.

Then *G* is l_2 stable if and only if

$$(2n-1)m_a\left(R_{z_2}\left[T, \frac{\partial T}{\partial z_2}\right]\right) \leq 2m_a(R_{z_2}[M, T])$$

We note that Theorem 4.2 reduces to Theorem 4.1 when $n=1$.

Corollary:

Let $G(z_1, z_2, z_3) = \left[\dfrac{P(z_1, z_2, z_3)}{Q(z_1, z_2, z_3)}\right]^n$, where the function $P(z_1, z_2, z_3) / Q(z_1, z_2, z_3)$ is as defined in Theorem 4.1, and *n* is a positive integer. Then *G* is l_2 stable if and only if

$$(2n-1)m_a\left(R_{z_2}\left[T, \frac{\partial T}{\partial z_2}\right]\right) \leq 2nm_a(R_{z_2}[M, T]).$$

This result can be obtained directly from Theorem 4.2 considering that $m_a(R_{z_2}[M^{2n}, T]) = 2nm_a(R_{z_2}[M, T])$.

Consider another example.

Example4.2.

Let $G(z_1,z_2,z_3)=(1-z_3)/(3-z_1-z_2-z_3)^n$. Note that P and Q are the same as in Example 4.1.

Then

$$(2n-1)m_{z_1=1}\left(R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]\right)=4n-2 \quad \text{and}$$

$$2m_{z_1=1}(R_{z_2}[M,T])=4$$

Therefore, the system is l_2 stable when $4n-2\le 4$, or $n=1$

Example 4.3.

Let $G(z_1,z_2,z_3)=\left[\frac{1-z_3}{3-z_1-z_2-z_3}\right]^n$, where $P(z_1,z_2,z_3)$ and $Q(z_1,z_2,z_3)$ are the same as in Example 4.1.

Then

$$(2n-1)m_{z_1=1}\left(R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]\right)=4n-2 \quad \text{and}$$

$$2nm_{z_1=1}(R_{z_2}[M,T])=4n$$

Therefore, the system is l_2 stable when $4n-2\le 4n$, or for all n.

In this chapter, we have derived necessary and sufficient conditions for the l_2 stability of 3-D digital filters with simple nonessential singularities of the second kind in the region $\overline{U}^3-U^3$. We have illustrated the results with numerical examples.

Chapter 5.

Necessary Conditions for Boundedness

Consider the function

$$G(z_1,z_2,z_3)=\frac{P(z_1,z_2,z_3)}{Q(z_1,z_2,z_3)}$$

where $G(z_1,z_2,z_3)$ defined in Theorem 4.1. Assuming that G is bounded in U^3, we can show that G^n is also bounded in U^3 for any positive integer n, and, therefore, G^n is l_2 stable for any n. Thus, from the corollary of Theorem 4.2, we have

$$(2n-1)m_\alpha\left(R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]\right)\le 2nm_\alpha(R_{z_2}[M,T])$$

for all n.

This condition will be satisfied for all n if and only if

$$m_\alpha\left(R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]\right)\le m_\alpha(R_{z_2}[M,T]) \tag{5.1}$$

Therefore, if G is bounded, then condition (5.1) is satisfied. By the same argument, using Theorem 4.2, we can show that if

$$G(z_1,z_2,z_3)=\frac{P(z_1,z_2,z_3)}{[Q(z_1,z_2,z_3)]^n}$$

is bounded, then

$$nm_\alpha\left(R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]\right)\le m_\alpha(R_{z_2}[M,T]) \tag{5.2}$$

Finally, if $G(z_1,z_2,z_3)=\left[\dfrac{P(z_1,z_2,z_3)}{Q(z_1,z_2,z_3)}\right]^n$ is bounded, then

$$m_\alpha\left(R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]\right)\le m_\alpha(R_{z_2}[M,T]) \tag{5.3}$$

Example 5.1.

Let $G(z_1,z_2,z_3)=\dfrac{(1-z_1z_2)(1-z_2z_3)}{(3-z_1z_2-z_2z_3-z_3)^3}$

Then (see Appendix E for details) $n=3$, $m_{z_1=1}\left(R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]\right)=2$, and

$m_{z_1=1}(R_{z_2}[M,T])=4$

Condition (5.2) does not hold, and we can conclude that G is not bounded.

We will later prove that conditions (5.1)-(5.3) are also sufficient for boundedness.

Chapter 6.

ℓ_1 - Stability Theorems.

Consider the function $G(z_1,z_2,z_3)=\dfrac{P(z_1,z_2,z_3)}{Q(z_1,z_2,z_3)}$ where G defined in Theorem 4.1. We will be considering the function G^*, the radial limit of G, and using the fact that its Fourier coefficients are the same as those of G [17]. Assuming that

$$G_1 = \frac{\partial^3}{\partial z_1 \partial z_2 \partial z_3}(z_1 z_2 z_3 G^*) \in \ell_2 \tag{6.1}$$

we find

$$\begin{aligned} G_1 &= G^* + z_1\frac{\partial G^*}{\partial z_1} + z_2\frac{\partial G^*}{\partial z_2} + z_3\frac{\partial G^*}{\partial z_3} + z_1 z_2\frac{\partial^2 G^*}{\partial z_1 \partial z_2} + z_1 z_3\frac{\partial^2 G^*}{\partial z_1 \partial z_3} \\ &+ z_2 z_3\frac{\partial^2 G^*}{\partial z_2 \partial z_3} + z_1 z_2 z_3\frac{\partial^3 G^*}{\partial z_1 \partial z_2 \partial z_3} \end{aligned} \tag{6.2}$$

Recall the Z-transform properties:

If $\quad G^*(z_1,z_2,z_3) \to g(i_1,i_2,i_3)$,

then

$$\begin{aligned} &z_1\frac{\partial G^*}{\partial z_1} \to i_1 g(i_1,i_2,i_3) \\ &z_1 z_2\frac{\partial^2 G^*}{\partial z_1 \partial z_2} \to i_1 i_2 g(i_1,i_2,i_3) \\ &z_1 z_2 z_3\frac{\partial^3 G^*}{\partial z_1 \partial z_2 \partial z_3} \to i_1 i_2 i_3 g(i_1,i_2,i_3) \end{aligned} \tag{6.3}$$

Therefore, from (6.2) and (6.3), the impulse response of $G_1(z_1,z_2,z_3)$ is

$$\begin{aligned} &g_1(i_1,i_2,i_3) = g + i_1 g + i_2 g + i_3 g + i_1 i_2 g + i_1 i_3 g + i_2 i_3 g + i_1 i_2 i_3 g \\ &= (i_1+1)(i_2+1)(i_3+1) g(i_1,i_2,i_3) \end{aligned} \tag{6.4}$$

Combining (6.1) with the l_2 stability condition (2.1.4),

we obtain:

$$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}(i_1+1)^2(i_2+1)^2(i_3+1)^2 g^2(i_1,i_2,i_3)<\infty. \tag{6.5}$$

Recall the Cauchy-Schwartz's inequality (2.9):

$$\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2\right)^{1/2}$$

and apply it to the expression $\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}|g(i_1,i_2,i_3)|$. Considering (6.5), after transformations we obtain:

$$\begin{aligned}
\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}|g(i_1,i_2,i_3)| &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}\frac{|(i_1+1)(i_2+1)(i_3+1)|}{|(i_1+1)(i_2+1)(i_3+1)|}|g(i_1,i_2,i_3)| \\
&\le \left[\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}\frac{1}{(i_1+1)^2(i_2+1)^2(i_3+1)^2}\right]^{1/2} \cdot \\
&\quad \cdot\left[\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}(i_1+1)^2(i_2+1)^2(i_3+1)^2|g(i_1,i_2,i_3)|^2\right]<\infty
\end{aligned}$$

or $\quad G\in\ell_1$

and therefore

$$\text{if } \frac{\partial^3}{\partial z_1\partial z_2\partial z_3}(z_1z_2z_3G^*)\in\ell_2,\ \text{then}\, G\in\ell_1. \tag{6.6}$$

To make sure that (6.1) holds, we will first find the term that exhibits the most singular behavior around the singularity and then test that term for the stability. We substitute $G=\frac{P}{Q}$ into (6.2) and after lengthy calculations we find that the term

$$\frac{z_1z_2z_3P}{Q^4}\frac{\partial Q}{\partial z_1}\frac{\partial Q}{\partial z_2}\frac{\partial Q}{\partial z_3} \tag{6.7}$$

is the most singular term. The calculation details are shown in the Appendix F.

We will now test (6.7) for ℓ_2 by applying Theorem 4.2. Since the singularity at $(\alpha_1,\alpha_2,\alpha_3)$ is simple, we find that (6.1) holds if and only if

$$7m_\alpha\left(R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]\right)\le 2m_\alpha(R_{z_2}[M,T]) \tag{6.8}$$

Condition (6.8) is not a necessary condition for stability, and we will now derive another sufficient condition and then show that it is also necessary.

Consider the function $[G(z_1,z_2,z_3)]^n$, and determine if there exists a positive integer *n*, for which $[G(z_1,z_2,z_3)]^n\in\ell_1$. From (6.6) we can see that $G^n\in\ell_1$ if $\frac{\partial^3}{\partial z_1\partial z_2\partial z_3}[z_1z_2z_3(G^*)^n]\in\ell_2$. In order to test for this condition, we will first find the most singular term and then test that term for ℓ_2 stability. We will need to apply Theorem 4.2 to the term $\frac{z_1z_2z_3P^n}{Q^{n+3}}\frac{\partial Q}{\partial z_1}\frac{\partial Q}{\partial z_2}\frac{\partial Q}{\partial z_3}$. We obtain:

$$(2n+5)m_\alpha\left(R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]\right)\le 2nm_\alpha(R_{z_2}[M,T]) \tag{6.9}$$

since the singularity is simple. The condition (6.9) will hold for some *n* if and only if

$$m_\alpha(R_{z_2}[T,\frac{\partial T}{\partial z_2}])<m_\alpha(R_{z_2}[M,T]) \tag{6.10}$$

Therefore, if (6.10) holds, there exists some n for which $G^n \in \ell_1$. Thus, G^* converges uniformly everywhere in $\overline{U}^3$, and therefore, $G \in \ell_1$, making (6.6) a sufficient condition for ℓ_1 stability. Consider the following example.

Example 6.1.

Let $G(z_1,z_2,z_3) = \dfrac{(1-z_3)^n}{3-z_1-z_2-z_3}$

Then $m_{z_1=1}\left(R_{z_2}\left[T,\dfrac{\partial T}{\partial z_2}\right]\right) = 2$ and $nm_{z_1=1}(R_{z_2}[M,T]) = 2n$

(The details of calculations are presented in the Appendix G.)

Therefore, (6.10) holds when $2 < 2n$, or for n>1, while (6.8) which is a stricter condition holds when $7 \cdot 2 < 2n$, or for n>7.

We will now derive a necessary condition for ℓ_1 stability of $G(z_1,z_2,z_3)$. It can be shown that $\dfrac{\partial G^n}{\partial z_1} \in \ell_2$ in $\overline{U}^3 \forall n$. Hence, $\dfrac{\partial G^n}{\partial z_1}$ satisfies the condition of Theorem 4.2 for all n. It is sufficient to verify that the condition is satisfied for the term that exhibits the most singular behavior around the singularity: $\dfrac{P^n}{Q^{n+1}}\dfrac{\partial Q}{\partial z_1}$. Applying Theorem 4.2 results in $(2n+1)m_\alpha\left(R_{z_2}\left[T,\dfrac{\partial T}{\partial z_2}\right]\right) \le 2nm_\alpha(R_{z_2}[M,T])$, and therefore

$m_\alpha\left(R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]\right) < m_\alpha(R_{z_2}[M,T])$. This necessary condition is the same as sufficiency condition (6.9). Now we can state the following theorem regarding the ℓ_1 stability.

Theorem 6.1.

Let $G(z_1,z_2,z_3)=P(z_1,z_2,z_3)/Q(z_1,z_2,z_3)$ have no polar singularities in $\overline{U}^3$ and no nonessential singularities of the second kind anywhere except for the pair of simple ones $(\alpha_1,\alpha_2,\alpha_3)$ and $\left(\frac{1}{\alpha_1},\frac{1}{\alpha_2},\frac{1}{\alpha_3}\right)$ in the region $\overline{U}^3-U^3$. Further, let P and Q be relatively prime. Then G is l_1 stable if and only if

$$m_\alpha\left(R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]\right) < m_\alpha(R_{z_2}[M,T])$$

We will illustrate Theorem 6.1 by the following example.

Example 6.2.

Let $G(z_1,z_2,z_3)=\dfrac{3-z_1z_3-z_2z_3-z_1z_2}{(3-z_1-z_2-z_3)^n}$

Then $m_{z_1=1}\left(R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]\right)=2$ and $m_{z_1=1}(R_{z_2}[M,T])=6$

Therefore, the system is l_1 stable when $2n<6$, or for n<3; it's l_2 stable when $(2n-1)2\leq 12$, or $n\leq 3$. The calculations are presented in Appendix H.

Using similar arguments, we can prove the following:

Theorem 6.2.

Let $G(z_1,z_2,z_3) = P(z_1,z_2,z_3)/[Q(z_1,z_2,z_3)]^n$ where the function P/Q is as defined in Theorem 6.1, and n is a positive integer.

Then G is l_l stable if and only if

$$nm_a\left(R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]\right) < m_a(R_{z_2}[M,T])$$

We note that Theorem 6.2 reduces to Theorem 6.1 when n=1.

Corollary:

Let $G(z_1,z_2,z_3) = [P(z_1,z_2,z_3)/Q(z_1,z_2,z_3)]^n$ where the function P/Q is as defined in Theorem 6.1, and n is a positive integer.

Then G is l_l stable if and only if

$$m_a\left(R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]\right) < m_a(R_{z_2}[M,T]).$$

This follows from Theorem 6.2 and the fact that $m_a(R_{z_2}[M^n,T]) = nm_a(R_{z_2}[M,T])$.

Example 6.3.

Consider a generic function of the form:

$$G(z_1,z_2,z_3) = \frac{(1-z_1)^k(1-z_2)^r(1-z_3)^s}{(1-az_1-bz_2-cz_3)^n}$$

where $a+b+c=1,\ a>0, b>0, c>0,\ k+r+s>0$

Then (see Appendix I for details) $m_{z_1=1}\left(R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]\right)=2$ and

$m_{z_1=1}(R_{z_2}[M,T])=2(k+r+s)$

Thus, this system is l_1 stable when $2n<2(k+r+s)$, or for $n<k+r+s$; it's l_2 stable when $2(2n-1)\leq 2\cdot 2(k+r+s)$, or $n\leq 2(k+r+s)+1$.

In this chapter, we have determined necessary and sufficient conditions for l_l stability of 3-D digital filters with simple nonessential singularities of the second kind in the region $\overline{U}^3-U^3$. These conditions are illustrated with several examples.

Chapter 7.

Sufficient Conditions for Boundedness.

It can be proven that conditions (5.1)-(5.3) are also sufficient [17]. Indeed, if we consider a function

$$A(z_1,z_2,z_3) = (z_1-\alpha)G^n(z_1,z_2,z_3) = (z_1-\alpha)\frac{P^n}{Q^n} = \frac{P_1}{Q^n} \tag{7.1}$$

For (7.1)

$$M_1 = R_{z_3}[P_1,Q] = R_{z_3}[(z_1-\alpha),Q]\cdot R_{z_3}[P^n,Q] = BM^n$$

$$m(R_{z_2}[M_1,T]) = m(R_{z_2}[BM^n,T]) = m(R_{z_2}[B,T]) + m(R_{z_2}[M^n,T]) = m(R_{z_2}[B,T]) + nm(R_{z_2}[M,T])$$

Combining this with (5.1) $m_\alpha\left(R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]\right) \le m_\alpha(R_{z_2}[M,T])$

we obtain:

$$m(R_{z_2}[M_1,T]) \ge nm_\alpha\left(R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]\right) + m(R_{z_2}[B,T]),$$

or

$$m(R_{z_2}[M_1,T]) > nm_\alpha\left(R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]\right) \tag{7.2}$$

The expression (7.2) is ℓ_1-stability condition for (7.1). Thus, A is ℓ_1-stable for any n. Therefore, A is also bounded. Consequently, G is also bounded.

We proved that G is bounded provided (5.1) holds: $m_\alpha\left(R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]\right) \le m_\alpha(R_{z_2}[M,T])$

Summary and Conclusions.

This thesis presents two new results concerning the stability of multidimensional digital filters.

We considered an open problem concerning a class of N-D filters with nonessential singularities of the second kind, described by the real rational transfer function $G(z_1, z_2, ..., z_N) = P(z_1, z_2, ..., z_N) / Q(z_1, z_2, ..., z_N)$, in the region $\overline{U}^N - U^N$.

We applied transformations $z_k = \omega_k z_2^{n_k}$, $k=3,...,N$ to reduce the transfer function to the 2-D case, and then utilized the 2-D stability conditions to obtain the necessary conditions for stability in N-D case.

We considered the problem of BIBO stability of three-dimensional linear shift-invariant filters, in the presence of nonessential singularities of the second kind. We derived necessary and sufficient conditions for boundedness, l_1 and l_2 stabilities of a function $G=P/Q^n$, where P/Q has simple NSSK of the second kind in the region $\overline{U}^3 - U^3$.

The results of Chapter 3 have been published in [28].

The results of Chapters 4-7 were recently accepted for publication [29].

Future research will be focused on extending the latter result to an N-D case.

Appendix A.

$$\underline{Q} = 3 + z_1^2 - z_2^2 + z_3^2$$

$$z_3 = jz_2^n$$

$$\underline{Q}_1 = 3 + z_1^2 - z_2^2 - z_2^{2n}$$

$$\underline{Q}_1(j,1) = 0 \qquad \underline{Q}_1(-j,1) = 0$$

$$\underline{\underline{Q}}_1 = z_1^2 z_2^{2n}\left(3 + \frac{1}{z_1^2} - \frac{1}{z_2^2} - \frac{1}{z_2^{2n}}\right) = 3z_1^2 z_2^{2n} + z_2^{2n} - z_1^2 z_2^{2n-2} - z_1^2$$

$$\underline{\underline{Q}}_1(-j,1) = 0$$

$$R_{z_1}\left[\underline{\underline{Q}}_1, \underline{Q}_1\right] = \left(-10z_2^{2n+2} + 3z_2^{2n} + 3z_2^2 + 3z_2^{2n+4} - z_2^4 + 3z_2^{4n+2} - z_2^{4n}\right)^2 / z_2^4$$

$$R_{z_1}\left[\underline{\underline{Q}}_1, \underline{Q}_1\right]_{z_2=1} = 0$$

$$diff1 = \frac{\partial}{\partial z_2} R_{z_1}\left[\underline{\underline{Q}}_1, \underline{Q}_1\right]$$

$$\begin{aligned}
diff1 := -4\Big(& 9\left(z2^2\right)^n z2^2 + 3\,z2^6 - z2^8 - 60\left(z2^2\right)^{(2n)} z2^2 + 27\left(z2^4\right)^n z2^2 + 9\left(z2^2\right)^{(2n)} \\
& + 18\,z2^6\left(z2^2\right)^n\left(z2^4\right)^n - 66\left(z2^2\right)^n z2^4\left(z2^4\right)^n + 33\,z2^4\left(z2^2\right)^n n + 99\left(z2^6\right)^n z2^4\, n \\
& - 9\,z2^8\left(z2^4\right)^n n + 66\left(z2^4\right)^n n\, z2^2 - 27\,z2^6\left(z2^6\right)^n n + 12\left(z2^8\right)^n z2^2 n - 19\,z2^6\left(z2^2\right)^n n \\
& - 57\left(z2^6\right)^n z2^2 n - 138\left(z2^4\right)^n z2^4 n + 66\left(z2^4\right)^n z2^6 n + 3\,z2^8\left(z2^2\right)^n n - 9\left(z2^2\right)^n n\, z2^2 \\
& - 18\left(z2^8\right)^n z2^4 n + 38\left(z2^2\right)^n z2^2\left(z2^4\right)^n - 9\left(z2^4\right)^n n + 9\left(z2^6\right)^n n - 2\left(z2^8\right)^n n \\
& + 93\left(z2^4\right)^n z2^6 + 6\,z2^8\left(z2^2\right)^n + 3\left(z2^8\right)^n z2^2 - 27\,z2^6\left(z2^6\right)^n - 9\left(z2^8\right)^n z2^4 \\
& - 118\left(z2^4\right)^n z2^4 - 19\left(z2^2\right)^n z2^6 + 66\left(z2^6\right)^n z2^4 - 19\left(z2^6\right)^n z2^2 - 18\,z2^8\left(z2^4\right)^n \\
& - 6\left(z2^2\right)^n\left(z2^4\right)^n + \left(z2^4\right)^{(2n)} + 118\left(z2^2\right)^{(2n)} z2^4 - 60\left(z2^2\right)^{(2n)} z2^6 \\
& + 9\,z2^8\left(z2^2\right)^{(2n)} + 9\left(z2^4\right)^{(2n)} z2^4 - 6\left(z2^4\right)^{(2n)} z2^2\Big)\Big/ z2^5
\end{aligned}$$

$$diff1(z_2 = 1) = 0$$

$$diff2 = \frac{\partial^2}{\partial z_2^2} R_{z_1}\left[\underline{\underline{Q}}_1, \underline{Q}_1\right]$$

$$\begin{aligned}
diff2 := 4\Big(&-66\,z2^4\left(z2^2\right)^n n^2 - 264\left(z2^4\right)^n n^2 z2^2 + 144\left(z2^8\right)^n n^2 z2^4 + 18\left(z2^2\right)^n n^2 z2^2 \\
&- 6\,z2^8\left(z2^2\right)^n n^2 - 264\left(z2^4\right)^n n^2 z2^6 + 552\left(z2^4\right)^n n^2 z2^4 + 342\left(z2^6\right)^n n^2 z2^2 \\
&+ 38\,z2^6\left(z2^2\right)^n n^2 - 96\left(z2^8\right)^n n^2 z2^2 + 162\,z2^6\left(z2^6\right)^n n^2 + 36\,z2^8\left(z2^4\right)^n n^2 \\
&- 594\left(z2^6\right)^n n^2 z2^4 + 27\left(z2^2\right)^n z2^2 - 3\,z2^6 - 30\left(z2^6\right)^n + 3\,z2^8 + 5\left(z2^8\right)^n \\
&- 99\left(z2^4\right)^n z2^2 + 45\left(z2^4\right)^n + 36\left(z2^4\right)^n n^2 - 54\left(z2^6\right)^n n^2 + 16\left(z2^8\right)^n n^2 \\
&+ 33\,z2^4\left(z2^2\right)^n n + 99\left(z2^6\right)^n z2^4 n + 63\,z2^8\left(z2^4\right)^n n + 330\left(z2^4\right)^n n\,z2^2 \\
&+ 81\,z2^6\left(z2^6\right)^n n + 60\left(z2^8\right)^n z2^2 n + 57\,z2^6\left(z2^2\right)^n n - 285\left(z2^6\right)^n z2^2 n \\
&- 138\left(z2^4\right)^n z2^4 n - 198\left(z2^4\right)^n z2^6 n - 21\,z2^8\left(z2^2\right)^n n - 45\left(z2^2\right)^n n\,z2^2 \\
&- 18\left(z2^8\right)^n z2^4 n - 81\left(z2^4\right)^n n + 81\left(z2^6\right)^n n - 18\left(z2^8\right)^n n - 33\left(z2^4\right)^n z2^6 \\
&- 18\,z2^8\left(z2^2\right)^n - 9\left(z2^8\right)^n z2^2 + 9\,z2^6\left(z2^6\right)^n + 19\left(z2^2\right)^n z2^6 + 57\left(z2^6\right)^n z2^2 \\
&+ 27\,z2^8\left(z2^4\right)^n\Big)\Big/ z2^6
\end{aligned}$$

$$diff2(z_2 = 1) = 0$$

$$diff3 = \frac{\partial^3}{\partial z_2^3} R_{z_1}\left[\underline{Q}_1, Q_1\right]$$

$$\begin{aligned}
diff3 := 8\Big(&99\,z2^4\left(z2^2\right)^n n^2 + 1188\left(z2^4\right)^n n^2 z2^2 - 216\left(z2^8\right)^n n^2 z2^4 - 81\left(z2^2\right)^n n^2 z2^2 \\
&- 27\,z2^8\left(z2^2\right)^n n^2 - 396\left(z2^4\right)^n n^2 z2^6 - 828\left(z2^4\right)^n n^2 z2^4 - 1539\left(z2^6\right)^n n^2 z2^2 \\
&+ 57\,z2^6\left(z2^2\right)^n n^2 + 432\left(z2^8\right)^n n^2 z2^2 + 243\,z2^6\left(z2^6\right)^n n^2 + 162\,z2^8\left(z2^4\right)^n n^2 \\
&+ 891\left(z2^6\right)^n n^2 z2^4 - 54\left(z2^2\right)^n z2^2 + 90\left(z2^6\right)^n + 3\,z2^8 - 15\left(z2^8\right)^n + 72\left(z2^4\right)^n n^3 \\
&- 1782\left(z2^6\right)^n n^3 z2^4 + 576\left(z2^8\right)^n n^3 z2^4 - 162\left(z2^6\right)^n n^3 - 528\left(z2^4\right)^n n^3 z2^2 \\
&+ 1026\left(z2^6\right)^n n^3 z2^2 + 1104\left(z2^4\right)^n n^3 z2^4 - 528\left(z2^4\right)^n n^3 z2^6 + 64\left(z2^8\right)^n n^3 \\
&- 6\,z2^8\left(z2^2\right)^n n^3 + 18\left(z2^2\right)^n n^3 z2^2 + 486\,z2^6\left(z2^6\right)^n n^3 - 384\left(z2^8\right)^n n^3 z2^2 \\
&+ 38\,z2^6\left(z2^2\right)^n n^3 - 66\,z2^4\left(z2^2\right)^n n^3 + 72\,z2^8\left(z2^4\right)^n n^3 + 198\left(z2^4\right)^n z2^2 \\
&- 135\left(z2^4\right)^n - 270\left(z2^4\right)^n n^2 + 405\left(z2^6\right)^n n^2 - 120\left(z2^8\right)^n n^2 - 33\,z2^4\left(z2^2\right)^n n \\
&- 99\left(z2^6\right)^n z2^4 n + 117\,z2^8\left(z2^4\right)^n n - 858\left(z2^4\right)^n n\,z2^2 + 27\,z2^6\left(z2^6\right)^n n \\
&- 156\left(z2^8\right)^n z2^2 n + 19\,z2^6\left(z2^2\right)^n n + 741\left(z2^6\right)^n z2^2 n + 138\left(z2^4\right)^n z2^4 n \\
&- 66\left(z2^4\right)^n z2^6 n - 39\,z2^8\left(z2^2\right)^n n + 117\left(z2^2\right)^n n\,z2^2 + 18\left(z2^8\right)^n z2^4 n \\
&+ 333\left(z2^4\right)^n n - 333\left(z2^6\right)^n n + 74\left(z2^8\right)^n n - 18\,z2^8\left(z2^2\right)^n + 18\left(z2^8\right)^n z2^2 \\
&- 114\left(z2^6\right)^n z2^2 + 27\,z2^8\left(z2^4\right)^n\Big)\Big/ z2^7
\end{aligned}$$

$$diff3(z_2 = 1) = 0$$

$$diff4 = \frac{\partial^4}{\partial z_2^4} R_{z_1}\left[\underline{Q}_1, Q_1\right]$$

$$\begin{aligned}
diff4 := -8 \Big(& 363\, z2^4 \left(z2^2\right)^n n^2 + 9372 \left(z2^4\right)^n n^2 z2^2 - 792 \left(z2^8\right)^n n^2 z2^4 \\
& - 639 \left(z2^2\right)^n n^2 z2^2 + 105\, z2^8 \left(z2^2\right)^n n^2 - 132 \left(z2^4\right)^n n^2 z2^6 - 3036 \left(z2^4\right)^n n^2 z2^4 \\
& - 12141 \left(z2^6\right)^n n^2 z2^2 + 19\, z2^6 \left(z2^2\right)^n n^2 + 3408 \left(z2^8\right)^n n^2 z2^2 + 81\, z2^6 \left(z2^6\right)^n n^2 \\
& - 630\, z2^8 \left(z2^4\right)^n n^2 + 3267 \left(z2^6\right)^n n^2 z2^4 - 270 \left(z2^2\right)^n z2^2 + 630 \left(z2^6\right)^n - 3\, z2^8 \\
& - 105 \left(z2^8\right)^n + 1584 \left(z2^4\right)^n n^3 - 10692 \left(z2^6\right)^n n^3 z2^4 + 3456 \left(z2^8\right)^n n^3 z2^4 \\
& - 3564 \left(z2^6\right)^n n^3 - 7392 \left(z2^4\right)^n n^3 z2^2 + 14364 \left(z2^6\right)^n n^3 z2^2 + 6624 \left(z2^4\right)^n n^3 z2^4 \\
& + 1056 \left(z2^4\right)^n n^3 z2^6 + 1408 \left(z2^8\right)^n n^3 + 60\, z2^8 \left(z2^2\right)^n n^3 + 252 \left(z2^2\right)^n n^3 z2^2 \\
& - 972\, z2^6 \left(z2^6\right)^n n^3 - 5376 \left(z2^8\right)^n n^3 z2^2 - 76\, z2^6 \left(z2^2\right)^n n^3 - 396\, z2^4 \left(z2^2\right)^n n^3 \\
& - 720\, z2^8 \left(z2^4\right)^n n^3 + 990 \left(z2^4\right)^n z2^2 - 288 \left(z2^4\right)^n n^4 - 512 \left(z2^8\right)^n n^4 - 945 \left(z2^4\right)^n \\
& + 972 \left(z2^6\right)^n n^4 - 3222 \left(z2^4\right)^n n^2 + 4833 \left(z2^6\right)^n n^2 - 1432 \left(z2^8\right)^n n^2 \\
& - 99\, z2^4 \left(z2^2\right)^n n - 297 \left(z2^6\right)^n z2^4 n - 225\, z2^8 \left(z2^4\right)^n n - 5082 \left(z2^4\right)^n n\, z2^2 \\
& + 27\, z2^6 \left(z2^6\right)^n n - 924 \left(z2^8\right)^n z2^2 n + 19\, z2^6 \left(z2^2\right)^n n + 4389 \left(z2^6\right)^n z2^2 n \\
& + 414 \left(z2^4\right)^n z2^4 n - 66 \left(z2^4\right)^n z2^6 n + 75\, z2^8 \left(z2^2\right)^n n + 693 \left(z2^2\right)^n n\, z2^2 \\
& + 54 \left(z2^8\right)^n z2^4 n + 2871 \left(z2^4\right)^n n - 2871 \left(z2^6\right)^n n + 638 \left(z2^8\right)^n n + 18\, z2^8 \left(z2^2\right)^n \\
& + 90 \left(z2^8\right)^n z2^2 - 570 \left(z2^6\right)^n z2^2 - 27\, z2^8 \left(z2^4\right)^n - 4416 \left(z2^4\right)^n n^4 z2^4 \\
& - 6156 \left(z2^6\right)^n n^4 z2^2 + 2112 \left(z2^4\right)^n n^4 z2^2 + 3072 \left(z2^8\right)^n n^4 z2^2 \\
& - 2916\, z2^6 \left(z2^6\right)^n n^4 - 36 \left(z2^2\right)^n n^4 z2^2 + 12\, z2^8 \left(z2^2\right)^n n^4 + 2112 \left(z2^4\right)^n n^4 z2^6 \\
& - 4608 \left(z2^8\right)^n n^4 z2^4 + 10692 \left(z2^6\right)^n n^4 z2^4 - 288\, z2^8 \left(z2^4\right)^n n^4 + 132\, z2^4 \left(z2^2\right)^n n^4 \\
& - 76\, z2^6 \left(z2^2\right)^n n^4 \Big) \Big/ z2^8
\end{aligned}$$

$$diff4(z_2 = 1) = 3072\, n + 3072\, n^3 + 4608\, n^2 + 1536\, n^4 + 1536$$

Therefore, the multiplicity $m_{z_2=1}\left(R_{z_1}\left[Q_1, Q_1\right]\right) = 4$

$$z_3 = jz_2^n$$

$$P_1 = 3 + z_1^2 z_2 + jz_1^2 z_2^n + z_1 - 2z_2^2$$

$$R_{z_1}\left[P_1, Q_1\right] =$$

$$\begin{aligned}&-4\,z2^2 + 4\,z2^{(3+3\,n)}\,j + 2\,z2^{(5\,n+1)}\,j - 12\,z2^{(3\,n+1)}\,j - 4\,z2^{(3\,n+2)}\,j + 18\,z2^{(1+n)}\,j + 18\,z2^3 + 6\,z2^{(3\,n)}\,j\\ &\quad + 2\,z2^{(5+n)}\,j + 18\,z2^{(2+n)}\,j - z2^{(4\,n+2)} + 12 - 12\,z2^{(3+n)}\,j - 4\,z2^{(4+n)}\,j - 4\,z2^{(3+2\,n)} - z2^{(6\,n)}\\ &\quad - 2\,z2^4 - 4\,z2^5 + z2^6 + 6\,z2^{(2\,n+1)} - 18\,j\,z2^n - 18\,z2 - 10\,z2^{(2\,n)} + z2^{(4+2\,n)} + 6\,z2^{(4\,n)}\end{aligned}$$

$$diffp1 = \frac{\partial}{\partial z_2} R_{z_1}\left[P_1, Q_1\right] =$$

$$\begin{aligned}&-18 + 54\,z2^2 + 18\,z2^{(3\,n-1)}\,n\,j - 8\,z2^{(2\,n+2)}\,n + 12\,z2^{(2\,n)}\,n + 2\,z2^{(3+2\,n)}\,n - 2\,z2^{(4\,n+1)} - 18\,j\,z2^{(n-1)}\,n\\ &\quad - 6\,z2^{(6\,n-1)}\,n + 24\,z2^{(4\,n-1)}\,n - 8\,z2^{(3\,n+1)}\,j + 12\,z2^{(3\,n+2)}\,j + 36\,z2^{(1+n)}\,j - 8\,z2^3 - 12\,z2^{(3\,n)}\,j\\ &\quad - 36\,z2^{(2+n)}\,j - 12\,z2^{(3\,n+1)}\,j\,n + 2\,z2^{(4+n)}\,j\,n + 18\,z2^{(1+n)}\,j\,n + 10\,z2^{(5\,n)}\,j\,n + 12\,z2^{(3\,n+2)}\,j\,n\\ &\quad - 12\,z2^{(2+n)}\,j\,n + 18\,j\,z2^n\,n - 4\,z2^{(3+n)}\,j\,n - 36\,z2^{(3\,n)}\,j\,n - 16\,z2^{(3+n)}\,j + 10\,z2^{(4+n)}\,j + 4\,z2^{(3+2\,n)}\\ &\quad + 2\,z2^{(5\,n)}\,j - 4\,z2^{(4\,n+1)}\,n - 20\,z2^4 + 6\,z2^5 - 12\,z2^{(2\,n+2)} - 20\,z2^{(2\,n-1)}\,n + 18\,j\,z2^n - 8\,z2 + 6\,z2^{(2\,n)}\end{aligned}$$

$$diffp1(z_2 = 1) = 2 - 4\,n\,j + 6\,j$$

Therefore, the multiplicity $m_{z_2=1}\left(R_{z_1}\left[P_1, Q_1\right]\right) = 1$.

Appendix B.

$$\underline{Q} = z_1^2 z_2^3 z_3 z_4 + z_1^3 z_2 z_3^2 z_4 + 2z_1 z_2 z_3 z_4^2 + 2z_2 z_3^2 z_4 + z_1 z_2 z_3 - 7$$

$$z_3 = z_2^{n_3}$$

$$z_4 = z_2^{n_4}$$

$$Q1 := z1^2\, z2^{(3+n3+n4)} + z1^3\, z2^{(1+2\,n3+n4)} + 2\, z1\, z2^{(1+n3+2\,n4)} + 2\, z2^{(1+3\,n3+n4)} + z1\, z2^{(1+n3)} - 7$$

$$\underline{Q}_1 = \quad z2^{n3} + z1\, z2^{(2\,n3-2)} + z1^2 \left(2\, z2^{(2\,n3-n4)} + z2^{(2\,n3+n4)}\right) + z1^3 \left(2 - 7\, z2^{(1+3\,n3+n4)}\right)$$

$$R_{z_1}\left[\underline{Q}_1, Q_1\right] =$$

$$832\, z2^{(7\,n3+3\,n4+1)} + 29204\, z2^{(1+3\,n3+n4)} - 588\, z2^{(1+8\,n3)} - 112\, z2^{(4\,n3+2\,n4)} + 3024\, z2^{(2+6\,n3)}$$
$$- 111300\, z2^{(4+12\,n3+4\,n4)} + 29204\, z2^{(5+15\,n3+5\,n4)} - 1652\, z2^{(10\,n3+4\,n4+2)}$$
$$- 24\, z2^{(3+4\,n3+2\,n4)} - 32\, z2^{(3+14\,n3)} - 6628\, z2^{(3+9\,n3+5\,n4)} + 294\, z2^{(-1+6\,n3+2\,n4)}$$
$$- 7\, z2^{(-4+10\,n3+2\,n4)} + 28\, z2^{(2+11\,n3+5\,n4)} - 294\, z2^{(1+8\,n3+2\,n4)} - 392\, z2^{(5+15\,n3+3\,n4)}$$
$$+ 28\, z2^{(-2+7\,n3+n4)} + 2\, z2^{(-3+13\,n3+3\,n4)} - 111300\, z2^{(2+6\,n3+2\,n4)} - 4\, z2^{(3+14\,n3+6\,n4)}$$
$$+ 588\, z2^{(-1+6\,n3)} - 6628\, z2^{(3+9\,n3+n4)} - 168\, z2^{(4+3\,n3+3\,n4)} + 3024\, z2^{(2+6\,n3+4\,n4)}$$
$$+ 56\, z2^{(-2+7\,n3+3\,n4)} + 224\, z2^{(2+11\,n3-n4)} - 2744 + 3024\, z2^{(4+12\,n3+2\,n4)}$$
$$+ 178912\, z2^{(3+9\,n3+3\,n4)} + 56\, z2^{(13\,n3+5\,n4)} - 214\, z2^{(-1+10\,n3+4\,n4)} + 28\, z2^{(13\,n3+3\,n4)}$$
$$- 107\, z2^{(-1+10\,n3+2\,n4)} + 8\, z2^{(5+6\,n3+6\,n4)} - 392\, z2^{(5+15\,n3+7\,n4)} - 84\, z2^{(4+3\,n3+n4)}$$
$$+ 3024\, z2^{(4+12\,n3+6\,n4)} + 2\, z2^{(9+5\,n3+3\,n4)} - 32\, z2^{(3+4\,n3+6\,n4)} - 28\, z2^{(4\,n3)}$$
$$+ 98\, z2^{(3+5\,n3+3\,n4)} + 392\, z2^{(3+5\,n3-n4)} - 392\, z2^{(1+8\,n3-2\,n4)} - 49\, z2^{(1+8\,n3+4\,n4)}$$
$$- 48\, z2^{(3+4\,n3+4\,n4)} + 224\, z2^{(4+7\,n3+7\,n4)} - 1652\, z2^{(10\,n3+6\,n4+2)} + 832\, z2^{(7\,n3+5\,n4+1)}$$
$$+ 28\, z2^{(4+7\,n3+n4)} - 413\, z2^{(10\,n3+2+2\,n4)} + 208\, z2^{(7\,n3+1+n4)} - 7\, z2^{(10+8\,n3+4\,n4)}$$
$$+ 28\, z2^{(6+5\,n3+3\,n4)} + 56\, z2^{(6+5\,n3+n4)} - 392\, z2^{(5+10\,n3+8\,n4)} + 392\, z2^{(3+13\,n3+7\,n4)}$$
$$- 49\, z2^{(5+10\,n3+2\,n4)} + 98\, z2^{(3+13\,n3+3\,n4)} - 1652\, z2^{(4+8\,n3)} + 392\, z2^{(3+5\,n3+n4)}$$
$$+ 832\, z2^{(5+11\,n3+n4)} - 112\, z2^{(6+14\,n3+2\,n4)} - 413\, z2^{(4+8\,n3+4\,n4)} + 208\, z2^{(5+11\,n3+5\,n4)}$$
$$- 28\, z2^{(6+14\,n3+6\,n4)} + 4\, z2^{(5+6\,n3)} - 1232\, z2^{(9\,n3+3\,n4)} - 2744\, z2^{(6+18\,n3+6\,n4)}$$
$$- 112\, z2^{(4\,n3+4\,n4)} - 4\, z2^{(3+4\,n3)} - 392\, z2^{(1+3\,n3+3\,n4)} - 392\, z2^{(1+3\,n3-n4)} - 2422\, z2^{(9\,n3+n4)}$$
$$+ 622\, z2^{(5+6\,n3+2\,n4)} + 1232\, z2^{(5+6\,n3+4\,n4)} - 2422\, z2^{(6+9\,n3+5\,n4)} + 622\, z2^{(1+12\,n3+4\,n4)}$$
$$- 1232\, z2^{(6+9\,n3+3\,n4)} + 1232\, z2^{(1+12\,n3+2\,n4)} + 588\, z2^{(7+12\,n3+6\,n4)} + 294\, z2^{(7+12\,n3+4\,n4)}$$
$$- 84\, z2^{(2+15\,n3+5\,n4)} - 168\, z2^{(2+15\,n3+3\,n4)} - 14\, z2^{(9\,n3+5\,n4)} - 4\, z2^{(3+9\,n3+7\,n4)}$$
$$- 28\, z2^{(6+9\,n3+7\,n4)} + 4\, z2^{(1+12\,n3+6\,n4)} - 28\, z2^{(9\,n3-n4)} - 4\, z2^{(3+9\,n3-n4)}$$
$$- 14\, z2^{(6+9\,n3+n4)} + 8\, z2^{(1+12\,n3)} + 168\, z2^{(2+11\,n3+3\,n4)} + 336\, z2^{(2+11\,n3+n4)}$$
$$- 48\, z2^{(3+14\,n3+2\,n4)} - 24\, z2^{(3+14\,n3+4\,n4)} + 336\, z2^{(4+7\,n3+5\,n4)} + 168\, z2^{(4+7\,n3+3\,n4)}$$
$$- 107\, z2^{(7+8\,n3+4\,n4)} - 214\, z2^{(7+8\,n3+2\,n4)} + 56\, z2^{(8+11\,n3+3\,n4)} + 28\, z2^{(8+11\,n3+5\,n4)}$$
$$- 588\, z2^{(5+10\,n3+6\,n4)} - 294\, z2^{(5+10\,n3+4\,n4)} + 392\, z2^{(3+13\,n3+5\,n4)} - 1652\, z2^{(4+8\,n3+2\,n4)}$$
$$+ 832\, z2^{(5+11\,n3+3\,n4)} - 112\, z2^{(6+14\,n3+4\,n4)}$$

$$dif1 = \frac{\partial}{\partial z_2} R_{z_1}\left[\underline{Q}_1, Q_1\right]$$

$$
\begin{aligned}
dif1 := {} & 2058\, z2^{(6+12\, n3+4\, n4)} + 832\, z2^{(7\, n3+3\, n4)} + 672\, z2^{(1+11\, n3+n4)} + 29204\, z2^{(3\, n3+n4)} \\
& - 144\, z2^{(2+14\, n3+2\, n4)} - 588\, z2^{(8\, n3)} - 3304\, z2^{(10\, n3+4\, n4+1)} + 146020\, z2^{(4+15\, n3+5\, n4)} \\
& - 445200\, z2^{(3+12\, n3+4\, n4)} - 72\, z2^{(2+14\, n3+4\, n4)} + 6048\, z2^{(1+6\, n3)} + 4116\, z2^{(6+12\, n3+6\, n4)} \\
& - 96\, z2^{(2+14\, n3)} - 72\, z2^{(2+4\, n3+2\, n4)} + 1232\, z2^{(12\, n3+2\, n4)} - 294\, z2^{(-2+6\, n3+2\, n4)} \\
& + 1344\, z2^{(3+7\, n3+5\, n4)} - 19884\, z2^{(2+9\, n3+5\, n4)} - 1498\, z2^{(6+8\, n3+2\, n4)} - 749\, z2^{(6+8\, n3+4\, n4)} \\
& + 672\, z2^{(3+7\, n3+3\, n4)} + 336\, z2^{(1+11\, n3+3\, n4)} - 1176\, z2^{(3\, n3+3\, n4)}\, n3 + 2464\, z2^{(1+11\, n3-n4)}\, n3 \\
& - 224\, z2^{(1+11\, n3-n4)}\, n4 + 1274\, z2^{(2+13\, n3+3\, n4)}\, n3 + 294\, z2^{(2+13\, n3+3\, n4)}\, n4 \\
& - 13216\, z2^{(3+8\, n3)}\, n3 + 728\, z2^{(13\, n3+5\, n4-1)}\, n3 + 280\, z2^{(13\, n3+5\, n4-1)}\, n4 \\
& - 128\, z2^{(2+4\, n3+6\, n4)}\, n3 - 192\, z2^{(2+4\, n3+6\, n4)}\, n4 + 196\, z2^{(-3+7\, n3+n4)}\, n3 \\
& + 28\, z2^{(-3+7\, n3+n4)}\, n4 - 1652\, z2^{(3+8\, n3+4\, n4)}\, n4 + 1960\, z2^{(2+5\, n3+n4)}\, n3 + 392\, z2^{(2+5\, n3+n4)}\, n4 \\
& + 9152\, z2^{(4+11\, n3+n4)}\, n3 + 832\, z2^{(4+11\, n3+n4)}\, n4 - 3136\, z2^{(8\, n3-2\, n4)}\, n3 - 392\, z2^{(8\, n3+4\, n4)}\, n3 \\
& - 196\, z2^{(8\, n3+4\, n4)}\, n4 - 192\, z2^{(2+4\, n3+4\, n4)}\, n3 - 192\, z2^{(2+4\, n3+4\, n4)}\, n4 \\
& - 59652\, z2^{(2+9\, n3+n4)}\, n3 + 26\, z2^{(-4+13\, n3+3\, n4)}\, n3 + 18144\, z2^{(3+12\, n3+6\, n4)}\, n4 \\
& + 10\, z2^{(8+5\, n3+3\, n4)}\, n3 + 6\, z2^{(8+5\, n3+3\, n4)}\, n4 + 490\, z2^{(2+5\, n3+3\, n4)}\, n3 + 294\, z2^{(2+5\, n3+3\, n4)}\, n4 \\
& + 1960\, z2^{(2+5\, n3-n4)}\, n3 - 392\, z2^{(2+5\, n3-n4)}\, n4 + 36288\, z2^{(3+12\, n3+6\, n4)}\, n3 \\
& - 1176\, z2^{(3\, n3+3\, n4)}\, n4 + 1456\, z2^{(7\, n3+n4)}\, n3 + 208\, z2^{(7\, n3+n4)}\, n4 + 784\, z2^{(8\, n3-2\, n4)}\, n4 \\
& + 1568\, z2^{(3+7\, n3+7\, n4)}\, n3 - 16520\, z2^{(10\, n3+6\, n4+1)}\, n3 - 9912\, z2^{(10\, n3+6\, n4+1)}\, n4 \\
& + 5824\, z2^{(7\, n3+5\, n4)}\, n3 + 4160\, z2^{(7\, n3+5\, n4)}\, n4 + 18144\, z2^{(1+6\, n3+4\, n4)}\, n3 \\
& + 12096\, z2^{(1+6\, n3+4\, n4)}\, n4 - 126\, z2^{(9\, n3+5\, n4-1)}\, n3 - 70\, z2^{(9\, n3+5\, n4-1)}\, n4 \\
& - 504\, z2^{(3+3\, n3+3\, n4)}\, n3 - 504\, z2^{(3+3\, n3+3\, n4)}\, n4 + 6\, z2^{(-4+13\, n3+3\, n4)}\, n4 \\
& - 667800\, z2^{(1+6\, n3+2\, n4)}\, n3 - 222600\, z2^{(1+6\, n3+2\, n4)}\, n4 - 56\, z2^{(2+14\, n3+6\, n4)}\, n3 \\
& - 24\, z2^{(2+14\, n3+6\, n4)}\, n4 - 2520\, z2^{(1+15\, n3+3\, n4)}\, n3 - 504\, z2^{(1+15\, n3+3\, n4)}\, n4 \\
& + 1568\, z2^{(3+7\, n3+7\, n4)}\, n4 + 5096\, z2^{(2+13\, n3+5\, n4)}\, n3 + 1960\, z2^{(2+13\, n3+5\, n4)}\, n4 \\
& - 13216\, z2^{(3+8\, n3+2\, n4)}\, n3 - 3304\, z2^{(3+8\, n3+2\, n4)}\, n4 - 6628\, z2^{(2+9\, n3+n4)}\, n4 \\
& + 392\, z2^{(-3+7\, n3+3\, n4)}\, n3 + 168\, z2^{(-3+7\, n3+3\, n4)}\, n4 - 6608\, z2^{(10\, n3+4\, n4+1)}\, n4 \\
& - 1335600\, z2^{(3+12\, n3+4\, n4)}\, n3 - 445200\, z2^{(3+12\, n3+4\, n4)}\, n4 - 59652\, z2^{(2+9\, n3+5\, n4)}\, n3 \\
& - 33140\, z2^{(2+9\, n3+5\, n4)}\, n4 - 448\, z2^{(2+14\, n3)}\, n3 - 96\, z2^{(2+4\, n3+2\, n4)}\, n3 - 48\, z2^{(2+4\, n3+2\, n4)}\, n4 \\
& + 2352\, z2^{(3+7\, n3+5\, n4)}\, n3 + 1680\, z2^{(3+7\, n3+5\, n4)}\, n4 + 1764\, z2^{(-2+6\, n3+2\, n4)}\, n3 \\
& + 588\, z2^{(-2+6\, n3+2\, n4)}\, n4 + 438060\, z2^{(4+15\, n3+5\, n4)}\, n3 + 146020\, z2^{(4+15\, n3+5\, n4)}\, n4 \\
& + 1244\, z2^{(4+6\, n3+2\, n4)}\, n4 + 87612\, z2^{(3\, n3+n4)}\, n3 - 672\, z2^{(2+14\, n3+2\, n4)}\, n3 \\
& + 448\, z2^{(7+11\, n3+3\, n4)} + 28\, z2^{(-5+10\, n3+2\, n4)} - 1712\, z2^{(6+8\, n3+2\, n4)}\, n3 \\
& - 428\, z2^{(6+8\, n3+2\, n4)}\, n4 + 616\, z2^{(7+11\, n3+3\, n4)}\, n3 + 168\, z2^{(7+11\, n3+3\, n4)}\, n4 \\
& + 1176\, z2^{(3+7\, n3+3\, n4)}\, n3 + 504\, z2^{(3+7\, n3+3\, n4)}\, n4 + 336\, z2^{(1+11\, n3+n4)}\, n4 - 4704\, z2^{(8\, n3)}\, n3
\end{aligned}
$$

$$-16520\, z2^{(10 n3+4 n4+1)} n3-11088\, z2^{(5+9 n3+3 n4)} n3-3696\, z2^{(5+9 n3+3 n4)} n4$$
$$-252\, z2^{(5+9 n3+7 n4)} n3-196\, z2^{(5+9 n3+7 n4)} n4-1568\, z2^{(5+14 n3+2 n4)} n3$$
$$-224\, z2^{(5+14 n3+2 n4)} n4-252\, z2^{(9 n3-n4-1)} n3+48\, z2^{(12 n3+6 n4)} n3+24\, z2^{(12 n3+6 n4)} n4$$
$$-392\, z2^{(5+14 n3+6 n4)} n3-168\, z2^{(5+14 n3+6 n4)} n4-36\, z2^{(2+9 n3+7 n4)} n3$$
$$-1176\, z2^{(4+10 n3+4 n4)} n4+224\, z2^{(7+11 n3+5 n4)}+4\, z2^{(2+9 n3-n4)} n4-126\, z2^{(5+9 n3+n4)} n3$$
$$-14\, z2^{(5+9 n3+n4)} n4+96\, z2^{(12 n3)} n3+5824\, z2^{(7 n3+3 n4)} n3+2496\, z2^{(7 n3+3 n4)} n4$$
$$-28\, z2^{(2+9 n3+7 n4)} n4-2140\, z2^{(-2+10 n3+4 n4)} n3-856\, z2^{(-2+10 n3+4 n4)} n4$$
$$+7464\, z2^{(12 n3+4 n4)} n3+2488\, z2^{(12 n3+4 n4)} n4+4928\, z2^{(4+6 n3+4 n4)} n4$$
$$-448\, z2^{(4 n3+4 n4-1)} n3-448\, z2^{(4 n3+4 n4-1)} n4+3528\, z2^{(6+12 n3+4 n4)} n3$$
$$+1176\, z2^{(6+12 n3+4 n4)} n4+28\, z2^{(9 n3-n4-1)} n4-36\, z2^{(2+9 n3-n4)} n3$$
$$+2288\, z2^{(4+11 n3+5 n4)} n3+1040\, z2^{(4+11 n3+5 n4)} n4+5096\, z2^{(2+13 n3+7 n4)} n3$$
$$+2744\, z2^{(2+13 n3+7 n4)} n4-16\, z2^{(2+4 n3)} n3-49392\, z2^{(5+18 n3+6 n4)} n3$$
$$-16464\, z2^{(5+18 n3+6 n4)} n4+24\, z2^{(4+6 n3)} n3-1176\, z2^{(3 n3-n4)} n3-96\, z2^{(2+14 n3+2 n4)} n4$$
$$-336\, z2^{(2+14 n3+4 n4)} n3-96\, z2^{(2+14 n3+4 n4)} n4-21798\, z2^{(5+9 n3+5 n4)} n3$$
$$-12110\, z2^{(5+9 n3+5 n4)} n4+392\, z2^{(3 n3-n4)} n4-21798\, z2^{(9 n3+n4-1)} n3$$
$$-2422\, z2^{(9 n3+n4-1)} n4+7392\, z2^{(4+6 n3+4 n4)} n3+1610208\, z2^{(2+9 n3+3 n4)} n3$$
$$-3304\, z2^{(3+8 n3+4 n4)} n3-490\, z2^{(4+10 n3+2 n4)} n3-98\, z2^{(4+10 n3+2 n4)} n4$$
$$+196\, z2^{(3+7 n3+n4)} n3+28\, z2^{(3+7 n3+n4)} n4-4130\, z2^{(10 n3+1+2 n4)} n3$$
$$-826\, z2^{(10 n3+1+2 n4)} n4+6048\, z2^{(3+12 n3+2 n4)} n4-56\, z2^{(9+8 n3+4 n4)} n3$$
$$-28\, z2^{(9+8 n3+4 n4)} n4+140\, z2^{(5+5 n3+3 n4)} n3+84\, z2^{(5+5 n3+3 n4)} n4+280\, z2^{(5+5 n3+n4)} n3$$
$$+56\, z2^{(5+5 n3+n4)} n4-3920\, z2^{(4+10 n3+8 n4)} n3-11088\, z2^{(9 n3+3 n4-1)} n3$$
$$-3696\, z2^{(9 n3+3 n4-1)} n4+536736\, z2^{(2+9 n3+3 n4)} n4-2940\, z2^{(4+10 n3+6 n4)}$$
$$-168\, z2^{(1+15 n3+5 n4)}-1176\, z2^{(4+15 n3+3 n4)} n4-3136\, z2^{(4+10 n3+8 n4)} n4$$
$$+1848\, z2^{(1+11 n3+3 n4)} n3+504\, z2^{(1+11 n3+3 n4)} n4+3696\, z2^{(1+11 n3+n4)} n3$$
$$+18144\, z2^{(1+6 n3)} n3+36288\, z2^{(3+12 n3+2 n4)} n3-70\, z2^{(-5+10 n3+2 n4)} n3$$
$$-14\, z2^{(-5+10 n3+2 n4)} n4+7056\, z2^{(6+12 n3+6 n4)} n3+3528\, z2^{(6+12 n3+6 n4)} n4$$
$$-856\, z2^{(6+8 n3+4 n4)} n3-428\, z2^{(6+8 n3+4 n4)} n4+308\, z2^{(7+11 n3+5 n4)} n3$$
$$+140\, z2^{(7+11 n3+5 n4)} n4-5880\, z2^{(4+10 n3+6 n4)} n3-3528\, z2^{(4+10 n3+6 n4)} n4$$
$$-2940\, z2^{(4+10 n3+4 n4)} n3+3528\, z2^{(-2+6 n3)} n3+9152\, z2^{(4+11 n3+3 n4)} n3$$
$$+2496\, z2^{(4+11 n3+3 n4)} n4+308\, z2^{(1+11 n3+5 n4)} n3+140\, z2^{(1+11 n3+5 n4)} n4$$
$$-5880\, z2^{(4+15 n3+3 n4)} n3+56\, z2^{(1+11 n3+5 n4)}+4160\, z2^{(4+11 n3+3 n4)}-294\, z2^{(8 n3+2 n4)}$$
$$-6608\, z2^{(3+8 n3+2 n4)}+1176\, z2^{(2+13 n3+5 n4)}-1470\, z2^{(4+10 n3+4 n4)}-672\, z2^{(5+14 n3+4 n4)}$$
$$-1960\, z2^{(4+15 n3+3 n4)}+40\, z2^{(4+6 n3+6 n4)}+107\, z2^{(-2+10 n3+2 n4)}-56\, z2^{(-3+7 n3+n4)}$$
$$+12096\, z2^{(3+12 n3+6 n4)}-112\, z2^{(4 n3-1)} n3-222600\, z2^{(1+6 n3+2 n4)}-336\, z2^{(3+3 n3+n4)}$$

$$-6\,z2^{(-4+13\,n3+3\,n4)}-1960\,z2^{(4+15\,n3+7\,n4)}-336\,z2^{(1+15\,n3+3\,n4)}-12\,z2^{(2+14\,n3+6\,n4)}$$
$$-96\,z2^{(2+4\,n3+6\,n4)}+18\,z2^{(8+5\,n3+3\,n4)}-392\,z2^{(8\,n3-2\,n4)}+1176\,z2^{(2+5\,n3-n4)}$$
$$+294\,z2^{(2+5\,n3+3\,n4)}-19884\,z2^{(2+9\,n3+n4)}-588\,z2^{(-2+6\,n3)}-112\,z2^{(-3+7\,n3+3\,n4)}$$
$$+6048\,z2^{(1+6\,n3+4\,n4)}+896\,z2^{(3+7\,n3+7\,n4)}-144\,z2^{(2+4\,n3+4\,n4)}-672\,z2^{(3+3\,n3+3\,n4)}$$
$$-49\,z2^{(8\,n3+4\,n4)}+208\,z2^{(7\,n3+n4)}-826\,z2^{(10\,n3+1+2\,n4)}-70\,z2^{(9+8\,n3+4\,n4)}$$
$$+112\,z2^{(3+7\,n3+n4)}+448\,z2^{(1+11\,n3-n4)}+832\,z2^{(7\,n3+5\,n4)}-3304\,z2^{(10\,n3+6\,n4+1)}$$
$$-245\,z2^{(4+10\,n3+2\,n4)}-1960\,z2^{(4+10\,n3+8\,n4)}+1176\,z2^{(2+13\,n3+7\,n4)}+336\,z2^{(5+5\,n3+n4)}$$
$$+536736\,z2^{(2+9\,n3+3\,n4)}+168\,z2^{(5+5\,n3+3\,n4)}+12096\,z2^{(3+12\,n3+2\,n4)}-168\,z2^{(5+14\,n3+6\,n4)}$$
$$-1652\,z2^{(3+8\,n3+4\,n4)}-672\,z2^{(5+14\,n3+2\,n4)}+4160\,z2^{(4+11\,n3+n4)}+1176\,z2^{(2+5\,n3+n4)}$$
$$-6608\,z2^{(3+8\,n3)}+294\,z2^{(2+13\,n3+3\,n4)}-12\,z2^{(2+4\,n3)}-16464\,z2^{(5+18\,n3+6\,n4)}$$
$$+20\,z2^{(4+6\,n3)}+1040\,z2^{(4+11\,n3+5\,n4)}+3110\,z2^{(4+6\,n3+2\,n4)}-12\,z2^{(2+9\,n3+7\,n4)}$$
$$+214\,z2^{(-2+10\,n3+4\,n4)}-392\,z2^{(3\,n3-n4)}-392\,z2^{(3\,n3+3\,n4)}-7392\,z2^{(5+9\,n3+3\,n4)}$$
$$+622\,z2^{(12\,n3+4\,n4)}-168\,z2^{(5+9\,n3+7\,n4)}-14532\,z2^{(5+9\,n3+5\,n4)}+4\,z2^{(12\,n3+6\,n4)}$$
$$+29204\,z2^{(3\,n3+n4)}\,n4+14784\,z2^{(12\,n3+2\,n4)}\,n3+2464\,z2^{(12\,n3+2\,n4)}\,n4-448\,z2^{(4\,n3+2\,n4-1)}\,n3$$
$$-224\,z2^{(4\,n3+2\,n4-1)}\,n4+48\,z2^{(4+6\,n3+6\,n4)}\,n3-448\,z2^{(5+14\,n3+4\,n4)}\,n4$$
$$+364\,z2^{(13\,n3+3\,n4-1)}\,n3+84\,z2^{(13\,n3+3\,n4-1)}\,n4+48\,z2^{(4+6\,n3+6\,n4)}\,n4$$
$$-5880\,z2^{(4+15\,n3+7\,n4)}\,n3-2744\,z2^{(4+15\,n3+7\,n4)}\,n4-252\,z2^{(3+3\,n3+n4)}\,n3$$
$$-84\,z2^{(3+3\,n3+n4)}\,n4+3732\,z2^{(4+6\,n3+2\,n4)}\,n3-1568\,z2^{(5+14\,n3+4\,n4)}\,n3$$
$$-1260\,z2^{(1+15\,n3+5\,n4)}\,n3-420\,z2^{(1+15\,n3+5\,n4)}\,n4-1070\,z2^{(-2+10\,n3+2\,n4)}\,n3$$
$$-214\,z2^{(-2+10\,n3+2\,n4)}\,n4-2352\,z2^{(8\,n3+2\,n4)}\,n3-588\,z2^{(8\,n3+2\,n4)}\,n4-12\,z2^{(2+9\,n3-n4)}$$
$$-84\,z2^{(5+9\,n3+n4)}+8\,z2^{(12\,n3)}+6160\,z2^{(4+6\,n3+4\,n4)}$$

$$\mathit{dif}\,1(z_2=1)=0$$

$$\mathit{dif}\,2=\frac{\partial^2}{\partial z_2^2}R_{z_1}\left[\underline{Q}_1,Q_1\right]$$

$$\mathit{dif2}:=-214\,\%84+80\,z2^{(3+6\,n3)}-2016\,\%75-144\,\%26+6048\,\%87-222600\,\%8-3360\,\%32-24\,\%17$$
$$-324\,z2^{(1+9\,n3-n4)}\,n3^2-3304\,\%41-288\,\%54-1335600\,\%15+588\,\%9+4032\,\%29-39768\,\%4$$
$$+584080\,\%28+96\,z2^{(12\,n3-1)}\,n3-288\,\%60-8988\,\%38+16640\,\%50+2016\,\%37-19824\,\%78$$
$$-1008\,\%88+24\,\%74+672\,z2^{(11\,n3+n4)}+336\,\%10+36288\,\%21+3136\,\%61+588\,\%76$$
$$-4704\,z2^{(8\,n3-1)}\,n3+12440\,\%27+12348\,\%13-37632\,z2^{(8\,n3-1)}\,n3^2-5880\,\%71-24\,\%57-840\,\%70$$
$$+336\,\%2-420\,\%86-24\,z2^{(1+9\,n3-n4)}+1568\,\%3+4160\,\%69+24640\,\%62-428\,\%64$$
$$+1152\,z2^{(12\,n3-1)}\,n3^2-82320\,\%20-36960\,\%82-24\,z2^{(1+4\,n3)}+2352\,\%66+336\,\%22-980\,\%79$$
$$+1073472\,\%19+36288\,\%77-826\,\%47+24696\,\%46+6048\,z2^{(6\,n3)}+1680\,\%43-960\,\%65\,n4-840\,\%59$$
$$-630\,\%31-11760\,\%24-7840\,\%35-144\,\%45-144\,\%17\,n4^2-19824\,z2^{(2+8\,n3)}-168\,\%39-7840\,\%11$$
$$-3304\,\%40+56\,\%33-3360\,\%6+16640\,\%7+1176\,z2^{(-3+6\,n3)}-39768\,z2^{(1+9\,n3+n4)}+160\,\%52$$

$$- 7840 \%49 + 2352\, z2^{(1 + 5\, n3 - n4)} + 2688 \%23 + 2352 \%68 + 588 \%67 - 192 \%65 + 144 \%80$$

$$+ 448\, z2^{(11\, n3 - n4)} - 336 \%18 - 4494 \%5 + 840 \%42 + 2352\, z2^{(1 + 5\, n3 + n4)} - 72660 \%48 - 4956 \%55$$

$$+ 168 \%34 - 23128 \%55\, n3 - 4410 \%79\, n3 - 882 \%79\, n4 + 1372 \%22\, n3 - 40656 \%82\, n4 - 2772 \%59\, n3$$

$$+ 20\, z2^{(1 + 9\, n3 - n4)}\, n4 - 1386 \%86\, n3 + 22392 \%27\, n3^2 - 11088 \%85\, n4^2 - 99792 \%85\, n3^2$$

$$+ 224\, z2^{(11\, n3 - n4)}\, n4^2 + 28 \%34\, n4^2 - 428 \%84\, n4^2 - 1176 \%30\, n4^2 - 18900 \%39\, n3^2 - 768 \%54\, n4^2$$

$$+ 20800 \%83\, n4^2 - 350 \%25\, n4^2 + 6776 \%61\, n3^2 - 445200 \%8\, n4^2 - 756 \%88\, n3^2 - 3424 \%64\, n4^2$$

$$+ 9952 \%63\, n4^2 - 384 \%26\, n4^2 - 25088 \%35\, n4^2 - 1512 \%18\, n4^2 - 165200 \%41\, n3^2 - 99792 \%82\, n3^2$$

$$- 18816 \%30\, n3^2 - 392\, z2^{(3\, n3 - n4 - 1)}\, n4^2 - 196182 \%81\, n3^2 - 2422 \%81\, n4^2 + 44352 \%62\, n3^2$$

$$- 3528 \%36\, n3^2 - 1792 \%32\, n4^2 + 50 \%80\, n3^2 + 4928 \%44\, n4^2 + 177408 \%44\, n3^2 - 26432 \%55\, n3^2$$

$$- 196 \%79\, n4^2 + 1372 \%22\, n3^2 - 11088 \%82\, n4^2 - 2268 \%59\, n3^2 - 4\, z2^{(1 + 9\, n3 - n4)}\, n4^2 - 1134 \%86\, n3^2$$

$$+ 252 \%42\, n4^2 + 66248 \%66\, n3^2 - 41300 \%47\, n3^2 - 1652 \%47\, n4^2 - 448 \%31\, n3^2 - 6608 \%78\, n4^2$$

$$+ 21168 \%46\, n4^2 + 435456 \%77\, n3^2 + 84672 \%46\, n3^2 - 112 \%31\, n4^2 + 108864\, z2^{(6\, n3)}\, n3^2$$

$$- 64\, z2^{(1 + 4\, n3)}\, n3^2 - 6628\, z2^{(1 + 9\, n3 + n4)}\, n4^2 + 2744 \%10\, n3^2 - 14 \%86\, n4^2 + 1400 \%43\, n3^2$$

$$+ 56 \%43\, n4^2 + 882 \%76\, n4^2 - 1512 \%75\, n4^2 + 18 \%74\, n4^2 + 144 \%73\, n4^2 - 23128 \%78\, n4 - 92512 \%78\, n3$$

$$+ 254016 \%77\, n3 - 33140\, z2^{(1 + 9\, n3 + n4)}\, n4 - 1960 \%10\, n3 - 154 \%86\, n4 + 1470 \%76\, n4 - 3528 \%75\, n3$$

$$- 3528 \%75\, n4 - 42 \%74\, n4 - 180 \%57\, n3 + 24 \%73\, n4 + 48 \%73\, n3 + 42336 \%77\, n4 - 728 \%72\, n3$$

$$- 10584 \%71\, n4 - 4312 \%70\, n3 - 26460 \%71\, n3 - 10584 \%11\, n4 + 630 \%1\, n3 + 2450 \%76\, n3 + 20592 \%69\, n3$$

$$+ 9360 \%69\, n4 - 1848 \%70\, n4 + 102 \%80\, n4 - 280 \%72\, n4 + 5824 \%56\, n3 - 182 \%74\, n3 + 12096 \%77\, n4^2$$

$$- 105728\, z2^{(2 + 8\, n3)}\, n3^2 + 9464 \%72\, n3^2 - 6272\, z2^{(1 + 14\, n3)}\, n3^2 - 384 \%45\, n3^2 - 4704 \%71\, n4^2$$

$$- 5488 \%70\, n3^2 - 29400 \%71\, n3^2 - 165700 \%4\, n4^2 - 96 \%45\, n4^2 - 3528 \%11\, n4^2 - 700 \%1\, n3^2$$

$$+ 2450 \%76\, n3^2 + 25168 \%69\, n3^2 + 5200 \%69\, n4^2 - 1008 \%70\, n4^2 - 2100 \%39\, n4^2 + 18 \%80\, n4^2$$

$$+ 1400 \%72\, n4^2 + 40768 \%56\, n3^2 - 10700 \%84\, n3^2 + 338 \%74\, n3^2 + 392\, z2^{(1 + 5\, n3 - n4)}\, n4^2 - 192 \%60\, n4^2$$

$$+ 9800\, z2^{(1 + 5\, n3 - n4)}\, n3^2 + 108864 \%21\, n4^2 + 208\, z2^{(7\, n3 + n4 - 1)}\, n4^2 - 196182 \%48\, n3^2$$

$$- 3528 \%36\, n4^2 + 14491872 \%19\, n3^2 - 165200 \%40\, n3^2 - 21952 \%32\, n3^2 - 784 \%17\, n3^2 + 10976 \%23\, n3^2$$

$$+ 40768 \%83\, n3^2 + 28 \%22\, n4^2 + 10192\, z2^{(7\, n3 + n4 - 1)}\, n3^2 + 435456 \%21\, n3^2 - 889056 \%20\, n3^2$$

$$- 98784 \%20\, n4^2 - 59472 \%40\, n4^2 + 1610208 \%19\, n4^2 + 20328 \%2\, n3^2 - 37800 \%18\, n3^2 + 576 \%73\, n3^2$$

$$+ 25480 \%68\, n3 + 9800 \%68\, n4 + 126 \%1\, n4 + 6370 \%67\, n3 + 1470 \%67\, n4 + 29204\, z2^{(-1 + 3\, n3 + n4)}\, n4$$

$$- 298260\, z2^{(1 + 9\, n3 + n4)}\, n3 - 640 \%65\, n3 + 6420 \%64\, n3 + 7464 \%63\, n3 + 44352 \%62\, n4 + 448 \%14\, n3$$

$$+ 2520 \%61\, n4 - 3360 \%60\, n3 - 2156 \%59\, n4 - 364 \%58\, n3 - 84 \%58\, n4 - 140 \%57\, n4 + 2496 \%56\, n4$$

$$+ 1008\, z2^{(11\, n3 + n4)}\, n4 + 45864 \%13\, n3 + 82368 \%50\, n3 + 1960\, z2^{(1 + 5\, n3 + n4)}\, n4$$

$$+ 9800\, z2^{(1 + 5\, n3 + n4)}\, n3 - 11564 \%55\, n4 - 960 \%54\, n3 - 196 \%51\, n4 - 28 \%53\, n4 - 392 \%51\, n3$$

$$- 3136 \%16\, n3 + 7488 \%50\, n4 + 48384 \%87\, n4^2 + 144\, z2^{(3 + 6\, n3)}\, n3^2 + 108864 \%87\, n3^2$$

$$- 448\, z2^{(4\, n3 - 2)}\, n3^2 - 448 \%12\, n4^2 + 25480 \%66\, n3 + 13720 \%66\, n4 - 239778 \%48\, n3 - 133210 \%48\, n4$$

$$+ 54432\, z2^{(6\, n3)}\, n3 + 5824 \%83\, n3 + 4160 \%83\, n4 + 10976 \%23\, n4^2 + 66248 \%68\, n3^2 + 9800 \%68\, n4^2$$

$$- 28 \%1\, n4^2 + 16562 \%67\, n3^2 + 882 \%67\, n4^2 + 19208 \%66\, n4^2 + 700 \%42\, n3^2 + 29204\, z2^{(-1 + 3\, n3 + n4)}\, n4^2$$

$$- 6848 \%5\, n3^2 - 536868\, z2^{(1 + 9\, n3 + n4)}\, n3^2 - 512 \%65\, n3^2 - 1152 \%65\, n4^2 + 1372 \%34\, n3^2$$

$$- 21400 \%64\, n3^2 + 89568 \%63\, n3^2 + 19712 \%62\, n4^2 + 7488 \%7\, n4^2 - 13696 \%38\, n3^2 + 700 \%33\, n4^2$$

$$+ 288 \%52\, n3^2 - 1792 \%14\, n3^2 + 504 \%61\, n4^2 - 9408 \%60\, n3^2 - 856 \%38\, n4^2 - 1372 \%59\, n4^2$$

$$
\begin{aligned}
&+4732\,\%58\,n3^2+252\,\%58\,n4^2-196\,\%57\,n4^2+7488\,\%56\,n4^2+336\,z2^{(11\,n3+n4)}\,n4^2-39200\,\%35\,n3^2\\
&+9800\,z2^{(1+5\,n3-n4)}\,n3+6570900\,\%28\,n3^2-21952\,\%6\,n3^2-1568\,\%16\,n4^2+8232\,\%37\,n3^2\\
&+262836\,z2^{(-1+3\,n3+n4)}\,n3^2-26432\,\%41\,n4^2-17248\,\%6\,n3+2100\,\%3\,n4-1176\,z2^{(3\,n3-n4-1)}\,n3\\
&-84\,\%88\,n4^2-4900\,\%79\,n3^2-1512\,\%75\,n3^2+27104\,z2^{(11\,n3-n4)}\,n3^2-1960\,z2^{(1+5\,n3-n4)}\,n4\\
&-11128\,\%5\,n3-105728\,\%78\,n3^2-324\,\%57\,n3^2+3388\,\%33\,n3^2-2268\,\%53\,n3^2+36288\,\%87\,n4\\
&+216\,z2^{(3+6\,n3)}\,n3+1512\,\%2\,n4^2-16027200\,\%15\,n3^2-1780800\,\%15\,n4^2-1680\,\%26\,n3+126\,\%25\,n3\\
&-980\,\%34\,n3+3210\,\%84\,n3+642\,\%84\,n4-1764\,\%88\,n3-588\,\%88\,n4+432\,\%52\,n3+432\,\%52\,n4\\
&-52920\,\%49\,n3-24696\,\%49\,n4-480\,\%60\,n4+54432\,\%87\,n3+112\,z2^{(4\,n3-2)}\,n3+224\,\%12\,n4\\
&+33588\,\%27\,n3+3696\,\%85\,n4+11088\,\%85\,n3-672\,z2^{(11\,n3-n4)}\,n4-960\,\%54\,n4+70\,\%25\,n4\\
&+252\,\%53\,n3+9240\,\%61\,n3-667800\,\%8\,n4+3942540\,\%28\,n3+2568\,\%64\,n4+2488\,\%63\,n4-480\,\%26\,n4\\
&-1512\,\%18\,n4-121968\,\%82\,n3+392\,z2^{(3\,n3-n4-1)}\,n4+21798\,\%81\,n3+2422\,\%81\,n4+66528\,\%62\,n3\\
&+7392\,z2^{(11\,n3-n4)}\,n3+170\,\%80\,n3-504\,\%88\,n4\,n3+21952\,\%23\,n4\,n3+2352\,z2^{(3\,n3-n4-1)}\,n3\,n4\\
&+145152\,\%87\,n4\,n3-1260\,\%25\,n3\,n4-2671200\,\%8\,n3\,n4-2688\,\%26\,n3\,n4-3920\,z2^{(1+5\,n3-n4)}\,n4\,n3\\
&-252\,\%86\,n3\,n4-1792\,\%12\,n4\,n3-66528\,\%85\,n4\,n3-4928\,z2^{(11\,n3-n4)}\,n4\,n3-4280\,\%84\,n4\,n3\\
&-9408\,\%30\,n4\,n3-12600\,\%39\,n3\,n4+58240\,\%83\,n4\,n3-6272\,\%6\,n3\,n4-62720\,\%35\,n4\,n3\\
&-15120\,\%18\,n4\,n3-132160\,\%41\,n3\,n4-66528\,\%82\,n3\,n4-43596\,\%81\,n3\,n4-7056\,\%36\,n3\,n4\\
&-12544\,\%32\,n4\,n3+60\,\%80\,n3\,n4+59136\,\%44\,n4\,n3-1960\,\%79\,n3\,n4+392\,\%22\,n3\,n4\\
&+72\,z2^{(1+9\,n3-n4)}\,n4\,n3-16520\,\%47\,n3\,n4-448\,\%31\,n3\,n4-52864\,\%78\,n4\,n3+84672\,\%46\,n4\,n3\\
&+145152\,\%77\,n3\,n4+2352\,\%10\,n3\,n4+560\,\%43\,n3\,n4+2940\,\%76\,n4\,n3-3024\,\%75\,n3\,n4+156\,\%74\,n4\,n3\\
&+576\,\%73\,n4\,n3+7280\,\%72\,n3\,n4-384\,\%45\,n3\,n4-23520\,\%71\,n4\,n3-4704\,\%70\,n3\,n4-596520\,\%4\,n4\,n3\\
&-35280\,\%11\,n4\,n3+22880\,\%69\,n3\,n4+50960\,\%68\,n3\,n4-280\,\%1\,n4\,n3+7644\,\%67\,n3\,n4\\
&+71344\,\%66\,n4\,n3+840\,\%42\,n3\,n4+175224\,z2^{(-1+3\,n3+n4)}\,n4\,n3-6848\,\%5\,n3\,n4\\
&-119304\,z2^{(1+9\,n3+n4)}\,n3\,n4-1536\,\%65\,n3\,n4+392\,\%34\,n3\,n4-17120\,\%64\,n3\,n4+59712\,\%63\,n3\,n4\\
&+59136\,\%62\,n4\,n3+54912\,\%7\,n4\,n3-6848\,\%38\,n3\,n4+3080\,\%33\,n4\,n3+576\,\%52\,n3\,n4-3584\,\%14\,n3\,n4\\
&+3696\,\%61\,n4\,n3-2688\,\%60\,n3\,n4-3528\,\%59\,n4\,n3+2184\,\%58\,n3\,n4-504\,\%57\,n4\,n3+34944\,\%56\,n4\,n3\\
&+7392\,z2^{(11\,n3+n4)}\,n4\,n3+7056\,\%37\,n4\,n3+28224\,\%13\,n3\,n4+18304\,\%50\,n3\,n4\\
&+3920\,z2^{(1+5\,n3+n4)}\,n4\,n3-26432\,\%55\,n4\,n3-1536\,\%54\,n3\,n4-3136\,\%51\,n4\,n3+504\,\%53\,n4\,n3\\
&-82320\,\%49\,n3\,n4+12544\,\%16\,n3\,n4-217980\,\%48\,n4\,n3+23520\,\%29\,n4\,n3-70560\,\%24\,n3\,n4\\
&+4380600\,\%28\,n4\,n3+14928\,\%27\,n4\,n3+435456\,\%21\,n4\,n3+2912\,z2^{(7\,n3+n4-1)}\,n4\,n3\\
&+9661248\,\%19\,n3\,n4-198240\,\%40\,n3\,n4-672\,\%17\,n3\,n4-592704\,\%20\,n3\,n4+11088\,\%2\,n3\,n4\\
&+7056\,\%9\,n4\,n3-10684800\,\%15\,n3\,n4+3080\,\%3\,n3\,n4+1512\,\%37\,n4^2+42336\,\%13\,n3^2+100672\,\%50\,n3^2\\
&+392\,z2^{(1+5\,n3+n4)}\,n4^2+9800\,z2^{(1+5\,n3+n4)}\,n3^2-6608\,\%55\,n4^2-768\,\%54\,n3^2-784\,\%51\,n4^2\\
&-28\,\%53\,n4^2+288\,\%52\,n4^2-88200\,\%49\,n3^2-3136\,\%51\,n3^2-25088\,\%16\,n3^2+832\,\%50\,n4^2\\
&-19208\,\%49\,n4^2-60550\,\%48\,n4^2-12390\,\%47\,n3-2478\,\%47\,n4+91728\,\%46\,n3+45864\,\%46\,n4\\
&-2240\,z2^{(1+14\,n3)}\,n3-480\,\%45\,n3-240\,\%45\,n4+14784\,\%44\,n3+2464\,\%44\,n4-5292\,\%9\,n3\\
&-1764\,\%9\,n4+3080\,\%43\,n3+616\,\%43\,n4+1540\,\%42\,n3+924\,\%42\,n4+8400\,\%29\,n4^2-58800\,\%24\,n3^2\\
&+16464\,\%29\,n3^2+730100\,\%28\,n4^2+2488\,\%27\,n4^2-31752\,\%24\,n4-49560\,\%41\,n3-19824\,\%41\,n4\\
&-49560\,\%40\,n3-29736\,\%40\,n4-3780\,\%39\,n3-1260\,\%39\,n4-10584\,z2^{(-3+6\,n3)}\,n3-298260\,\%4\,n3\\
&-165700\,\%4\,n4-22256\,\%38\,n3-5564\,\%38\,n4-5564\,\%5\,n4+8232\,\%37\,n3+3528\,\%37\,n4+1512\,\%2\,n4\\
&-1176\,\%36\,n3-1176\,\%36\,n4-92512\,z2^{(2+8\,n3)}\,n3-35280\,\%35\,n3-28224\,\%35\,n4-140\,\%34\,n4
\end{aligned}
$$

$$- 2464 \%6\, n4 + 82368 \%7\, n3 + 22464 \%7\, n4 + 924 \%33\, n3 + 420 \%33\, n4 - 17248 \%32\, n3 - 4928 \%32\, n4$$

$$- 80\, z2^{(1 + 4\, n3)}\, n3 - 1064 \%31\, n3 - 532 \%31\, n4 - 2352 \%30\, n3 - 588 \%30\, n4 + 11760 \%29\, n4$$

$$- 52920 \%24\, n3 + 16464 \%29\, n3 + 1314180 \%28\, n4 + 11196 \%27\, n4 + 10976 \%23\, n4 - 4006800 \%8\, n3^2$$

$$- 4704 \%26\, n3^2 - 1134 \%25\, n3^2 - 3528\, z2^{(3\, n3 - n4 - 1)}\, n3^2 - 1792 \%14\, n4^2 + 4704 \%13\, n4^2$$

$$- 1792 \%12\, n3^2 - 88200 \%11\, n3^2 + 10584 \%9\, n3^2 - 21168 \%24\, n4^2 + 21168\, z2^{(-3 + 6\, n3)}\, n3^2$$

$$+ 127008 \%21\, n4 + 208\, z2^{(7\, n3 + n4 - 1)}\, n4 + 8051040 \%19\, n3 - 280 \%17\, n3 + 10976 \%23\, n3 + 196 \%22\, n4$$

$$+ 1456\, z2^{(7\, n3 + n4 - 1)}\, n3 + 254016 \%21\, n3 - 543312 \%20\, n3 - 181104 \%20\, n4 + 2683680 \%19\, n4$$

$$- 7560 \%18\, n3 - 120 \%17\, n4 - 180\, z2^{(1 + 9\, n3 - n4)}\, n3 - 840 \%10\, n4 + 784 \%16\, n4$$

$$+ 87612\, z2^{(-1 + 3\, n3 + n4)}\, n3 - 9349200 \%15\, n3 - 3116400 \%15\, n4 + 11088\, z2^{(11\, n3 + n4)}\, n3 + 4620 \%3\, n3$$

$$+ 448 \%14\, n4 + 15288 \%13\, n4 + 448 \%12\, n3 - 52920 \%11\, n3 + 504 \%10\, n4^2 + 1176 \%9\, n4^2 - 2003400 \%8\, n3$$

$$+ 100672 \%7\, n3^2 - 448 \%6\, n4^2 + 40656\, z2^{(11\, n3 + n4)}\, n3^2 - 1712 \%5\, n4^2 - 536868 \%4\, n3^2 + 3388 \%3\, n3^2$$

$$+ 700 \%3\, n4^2 + 5544 \%2\, n3 - 192\, z2^{(1 + 14\, n3)} - 140 \%1$$

$$\%1 := z2^{(-6 + 10\, n3 + 2\, n4)}$$

$$\%2 := z2^{(11\, n3 + 3\, n4)}$$

$$\%3 := z2^{(6 + 11\, n3 + 5\, n4)}$$

$$\%4 := z2^{(1 + 9\, n3 + 5\, n4)}$$

$$\%5 := z2^{(5 + 8\, n3 + 4\, n4)}$$

$$\%6 := z2^{(4 + 14\, n3 + 2\, n4)}$$

$$\%7 := z2^{(3 + 11\, n3 + 3\, n4)}$$

$$\%8 := z2^{(6\, n3 + 2\, n4)}$$

$$\%9 := z2^{(-3 + 6\, n3 + 2\, n4)}$$

$$\%10 := z2^{(-4 + 7\, n3 + 3\, n4)}$$

$$\%11 := z2^{(3 + 15\, n3 + 3\, n4)}$$

$$\%12 := z2^{(4\, n3 + 2\, n4 - 2)}$$

$$\%13 := z2^{(5 + 12\, n3 + 4\, n4)}$$

$$\%14 := z2^{(4\, n3 + 4\, n4 - 2)}$$

$$\%15 := z2^{(2 + 12\, n3 + 4\, n4)}$$

$$\%16 := z2^{(8\, n3 - 2\, n4 - 1)}$$

$$\%17 := z2^{(1 + 14\, n3 + 6\, n4)}$$

$$\%18 := z2^{(15\, n3 + 3\, n4)}$$

$$\%19 := z2^{(1 + 9\, n3 + 3\, n4)}$$

$$\%20 := z2^{(4 + 18\, n3 + 6\, n4)}$$

$$\%21 := z2^{(2 + 12\, n3 + 6\, n4)}$$

$$\%22 := z2^{(2 + 7\, n3 + n4)}$$

$$\%23 := z2^{(2 + 7\, n3 + 7\, n4)}$$

$$\%24 := z2^{(3 + 10\, n3 + 6\, n4)}$$

$$\%25 := z2^{(9\, n3 + 5\, n4 - 2)}$$

$$\%26 := z2^{(1 + 14\, n3 + 4\, n4)}$$

$$\%27 := z2^{(3 + 6\, n3 + 2\, n4)}$$

$$\%28 := z2^{(3 + 15\, n3 + 5\, n4)}$$

$$\%29 := z2^{(2 + 7\, n3 + 5\, n4)}$$

$$\%30 := z2^{(8\, n3 + 2\, n4 - 1)}$$

$$\%31 := z2^{(8 + 8\, n3 + 4\, n4)}$$

$$\%32 := z2^{(4 + 14\, n3 + 4\, n4)}$$

$$\%33 := z2^{(11\, n3 + 5\, n4)}$$

$$\%34 := z2^{(-4 + 7\, n3 + n4)}$$

$$\%35 := z2^{(3 + 10\, n3 + 8\, n4)}$$

$$\%36 := z2^{(3\, n3 + 3\, n4 - 1)}$$

$\%37 := z2^{(2+7\,n3+3\,n4)}$

$\%38 := z2^{(5+8\,n3+2\,n4)}$

$\%39 := z2^{(15\,n3+5\,n4)}$

$\%40 := z2^{(10\,n3+6\,n4)}$

$\%41 := z2^{(10\,n3+4\,n4)}$

$\%42 := z2^{(4+5\,n3+3\,n4)}$

$\%43 := z2^{(4+5\,n3+n4)}$

$\%44 := z2^{(12\,n3+2\,n4-1)}$

$\%45 := z2^{(1+4\,n3+2\,n4)}$

$\%46 := z2^{(5+12\,n3+6\,n4)}$

$\%47 := z2^{(10\,n3+2\,n4)}$

$\%48 := z2^{(4+9\,n3+5\,n4)}$

$\%49 := z2^{(3+15\,n3+7\,n4)}$

$\%50 := z2^{(3+11\,n3+n4)}$

$\%51 := z2^{(8\,n3+4\,n4-1)}$

$\%52 := z2^{(3+6\,n3+6\,n4)}$

$\%53 := z2^{(9\,n3-n4-2)}$

$\%54 := z2^{(1+4\,n3+4\,n4)}$

$\%55 := z2^{(2+8\,n3+4\,n4)}$

$\%56 := z2^{(7\,n3+3\,n4-1)}$

$\%57 := z2^{(1+9\,n3+7\,n4)}$

$\%58 := z2^{(13\,n3+3\,n4-2)}$

$\%59 := z2^{(4+9\,n3+7\,n4)}$

$\%60 := z2^{(1+14\,n3+2\,n4)}$

$\%61 := z2^{(6+11\,n3+3\,n4)}$

$\%62 := z2^{(3+6\,n3+4\,n4)}$

$\%63 := z2^{(12\,n3+4\,n4-1)}$

$\%64 := z2^{(-3+10\,n3+4\,n4)}$

$\%65 := z2^{(1+4\,n3+6\,n4)}$

$\%66 := z2^{(1+13\,n3+7\,n4)}$

$\%67 := z2^{(1+13\,n3+3\,n4)}$

$\%68 := z2^{(1+13\,n3+5\,n4)}$

$\%69 := z2^{(3+11\,n3+5\,n4)}$

$\%70 := z2^{(4+14\,n3+6\,n4)}$

$\%71 := z2^{(3+10\,n3+4\,n4)}$

$\%72 := z2^{(13\,n3+5\,n4-2)}$

$\%73 := z2^{(12\,n3+6\,n4-1)}$

$\%74 := z2^{(-5+13\,n3+3\,n4)}$

$\%75 := z2^{(2+3\,n3+3\,n4)}$

$\%76 := z2^{(1+5\,n3+3\,n4)}$

$\%77 := z2^{(2+12\,n3+2\,n4)}$

$\%78 := z2^{(2+8\,n3+2\,n4)}$

$\%79 := z2^{(3+10\,n3+2\,n4)}$

$\%80 := z2^{(7+5\,n3+3\,n4)}$

$\%81 := z2^{(9\,n3+n4-2)}$

$\%82 := z2^{(4+9\,n3+3\,n4)}$

$\%83 := z2^{(7\,n3+5\,n4-1)}$

$\%84 := z2^{(-3+10\,n3+2\,n4)}$

$\%85 := z2^{(9\,n3+3\,n4-2)}$

$\%86 := z2^{(4+9\,n3+n4)}$

$\%87 := z2^{(6\,n3+4\,n4)}$

$\%88 := z2^{(2+3\,n3+n4)}$

$dif2(z_2 = 1) =$

$$-174080\,n3 - 96256\,n4 - 204800\,n3\,n4 - 237568\,n3^2 - 81920\,n4^2 - 67584$$

Therefore, the multiplicity $m_{z_2=1}\left(R_{z_1}\left[\underline{Q}_1, Q_1\right]\right)=2$

$$P = 3z_1^2 z_2^2 z_3 z_4 - 2z_1 z_2^2 z_3^2 z_4 + 2z_2 z_3^3 z_4^2 - 5z_1 z_3 z_4^3 - z_1^3 z_3 + 3$$

$$z_3 = z_2^{n_3}$$

$$z_4 = z_2^{n_4}$$

$$P1 := 3\,z1^2\,z2^{(2+n3+n4)} - 2\,z1\,z2^{(2+2\,n3+n4)} + 2\,z2^{(1+3\,n3+2\,n4)} - 5\,z1\,z2^{(n3+3\,n4)} - z1^3\,z2^{n3} + 3$$

$R_{z_1}\left[P_1, Q_1\right]=$

$$36\,z2^{(5+8\,n3+5\,n4)} + 1050\,z2^{(4+8\,n3+9\,n4)} - 6\,z2^{(8+8\,n3+4\,n4)} - 147\,z2^{(3+4\,n3+n4)}$$
$$- 250\,z2^{(3+10\,n3+12\,n4)} + 72\,z2^{(6+8\,n3+6\,n4)} + 120\,z2^{(3+6\,n3+8\,n4)} - 560\,z2^{(4+7\,n3+7\,n4)}$$
$$+ 14\,z2^{(4+6\,n3+n4)} + 120\,z2^{(5+10\,n3+6\,n4)} - 45\,z2^{(7+6\,n3+6\,n4)} - 45\,z2^{(5+6\,n3+6\,n4)}$$
$$+ 936\,z2^{(5+6\,n3+3\,n4)} + 84\,z2^{(5+7\,n3+4\,n4)} - 441\,z2^{(7+4\,n3+3\,n4)} + 56\,z2^{(4+6\,n3+5\,n4)}$$
$$+ 175\,z2^{(6+5\,n3+8\,n4)} - 252\,z2^{(7+10\,n3+6\,n4)} - 75\,z2^{(3+7\,n3+8\,n4)} - 3\,z2^{(5\,n3+3)}$$
$$- 150\,z2^{(3+7\,n3+10\,n4)} - 1359\,z2^{(7+5\,n3+4\,n4)} - 150\,z2^{(7+9\,n3+10\,n4)} + 140\,z2^{(2+5\,n3+7\,n4)}$$
$$+ 525\,z2^{(6+6\,n3+9\,n4)} - 10\,z2^{(8+8\,n3+7\,n4)} + 120\,z2^{(3+6\,n3+6\,n4)} + 378\,z2^{(8+7\,n3+4\,n4)}$$
$$- 945\,z2^{(4+6\,n3+6\,n4)} - 126\,z2^{(8+4\,n3+3\,n4)} + 56\,z2^{(4+6\,n3+3\,n4)} - 4\,z2^{(11+10\,n3+7\,n4)}$$
$$+ 140\,z2^{(2+5\,n3+5\,n4)} - 27\,z2^{(7+6\,n3+3\,n4)} - 8\,z2^{(10+9\,n3+7\,n4)} - 16\,z2^{(8\,n3+4+8\,n4)}$$
$$+ 24\,z2^{(7+11\,n3+8\,n4)} - 24\,z2^{(11+11\,n3+6\,n4)} - 1284\,z2^{(4+8\,n3+6\,n4)} - 84\,z2^{(9+7\,n3+5\,n4)}$$
$$+ 30\,z2^{(3+6\,n3+4\,n4)} + 18\,z2^{(9+7\,n3+4\,n4)} - 12\,z2^{(10+7\,n3+5\,n4)} - 105\,z2^{(4+5\,n3+5\,n4)}$$
$$- 40\,z2^{(3+8\,n3+8\,n4)} + 84\,z2^{(4+11\,n3+6\,n4)} - 63\,z2^{(4+4\,n3+n4)} - 63\,z2^{(4+5\,n3+2\,n4)}$$
$$- 126\,z2^{(4+5\,n3+4\,n4)} + 28\,z2^{(10+7\,n3+4\,n4)} + 108\,z2^{(7+10\,n3+5\,n4)} - 294\,z2^{(2+7\,n3+3\,n4)}$$
$$+ 18\,z2^{(5+8\,n3+3\,n4)} + 2205\,z2^{(3+5\,n3+5\,n4)} + 30\,z2^{(5+7\,n3+2\,n4)} + 223\,z2^{(5+5\,n3+2\,n4)}$$
$$+ 420\,z2^{(8+7\,n3+7\,n4)} - 114\,z2^{(5+8\,n3+6\,n4)} - 8\,z2^{(9+13\,n3+8\,n4)} + 35\,z2^{(2+5\,n3+3\,n4)}$$
$$- 36\,z2^{(3+10\,n3+3\,n4)} + 108\,z2^{(5+6\,n3+5\,n4)} + 72\,z2^{(10+11\,n3+7\,n4)} - 108\,z2^{(7+6\,n3+7\,n4)}$$
$$- 24\,z2^{(10+10\,n3+8\,n4)} + 168\,z2^{(3+10\,n3+4\,n4)} - 42\,z2^{(6+6\,n3+3\,n4)} + 44\,z2^{(7+11\,n3+6\,n4)}$$
$$- 16\,z2^{(5+9\,n3+6\,n4)} - 42\,z2^{(5+8\,n3+4\,n4)} + 756\,z2^{(8+8\,n3+5\,n4)} - 4\,z2^{(5+9\,n3+2\,n4)}$$
$$- 294\,z2^{(3+4\,n3+3\,n4)} + 875\,z2^{(2+7\,n3+11\,n4)} - 126\,z2^{(4+4\,n3+3\,n4)} + 32\,z2^{(7+11\,n3+4\,n4)}$$
$$+ 252\,z2^{(2+7\,n3+2\,n4)} - 441\,z2^{(1+4\,n3+n4)} + 36\,z2^{(6+8\,n3+4\,n4)} - 112\,z2^{(6+8\,n3+5\,n4)}$$
$$+ 56\,z2^{(6+10\,n3+6\,n4)} - 36\,z2^{(5+12\,n3+7\,n4)} - 16\,z2^{(5+9\,n3+4\,n4)} - 20\,z2^{(7+12\,n3+10\,n4)}$$
$$- 24\,z2^{(5\,n3+3+6\,n4)} + 242\,z2^{(3+8\,n3+4\,n4)} - 108\,z2^{(9+11\,n3+6\,n4)} + 200\,z2^{(3+9\,n3+10\,n4)}$$
$$- 378\,z2^{(8+5\,n3+4\,n4)} - 630\,z2^{(5+9\,n3+8\,n4)} - 27\,z2^{(9+5\,n3+4\,n4)} + 84\,z2^{(9\,n3+2+2\,n4)}$$
$$+ 160\,z2^{(5+10\,n3+8\,n4)} - 40\,z2^{(3+8\,n3+6\,n4)} - 8\,z2^{(12\,n3+3+3\,n4)} + 270\,z2^{(5+9\,n3+7\,n4)}$$
$$+ 343\,z2^{(3\,n3)} + 189\,z2^{(2+5\,n3+2\,n4)} - 6\,z2^{(9+6\,n3+3\,n4)} - 18\,z2^{(3+5\,n3+2\,n4)}$$
$$- 54\,z2^{(4+9\,n3+5\,n4)} - 90\,z2^{(7+9\,n3+7\,n4)} - 60\,z2^{(6+9\,n3+8\,n4)} + 84\,z2^{(4+7\,n3+2\,n4)}$$
$$- 378\,z2^{(6+7\,n3+4\,n4)} + 80\,z2^{(4+9\,n3+10\,n4)} - 80\,z2^{(6+11\,n3+10\,n4)} - 12\,z2^{(9+6\,n3+5\,n4)}$$

$$+12\,z2^{(6+11\,n3+5\,n4)}-20\,z2^{(8+8\,n3+9\,n4)}-30\,z2^{(7+5\,n3+7\,n4)}-168\,z2^{(6+8\,n3+3\,n4)}$$
$$+490\,z2^{(3+4\,n3+4\,n4)}-15\,z2^{(7+5\,n3+5\,n4)}-12\,z2^{(4+8\,n3+4\,n4)}+32\,z2^{(6+10\,n3+8\,n4)}$$
$$-50\,z2^{(7+8\,n3+9\,n4)}-36\,z2^{(9+7\,n3+6\,n4)}-9\,z2^{(7+5\,n3+2\,n4)}+24\,z2^{(6+11\,n3+7\,n4)}$$
$$+56\,z2^{(8+10\,n3+5\,n4)}+54\,z2^{(5+5\,n3+4\,n4)}-18\,z2^{(8+9\,n3+5\,n4)}-50\,z2^{(4+10\,n3+10\,n4)}$$
$$-120\,z2^{(5+8\,n3+8\,n4)}-60\,z2^{(8+9\,n3+10\,n4)}-700\,z2^{(2+6\,n3+9\,n4)}-90\,z2^{(7+6\,n3+8\,n4)}$$
$$+72\,z2^{(6+9\,n3+5\,n4)}-12\,z2^{(5+10\,n3+3\,n4)}-16\,z2^{(9+13\,n3+6\,n4)}-36\,z2^{(3+5\,n3+4\,n4)}$$
$$-27\,z2^{(3+6\,n3+3\,n4)}+72\,z2^{(7+12\,n3+5\,n4)}-12\,z2^{(11+11\,n3+8\,n4)}-300\,z2^{(5+11\,n3+10\,n4)}$$
$$+48\,z2^{(5+7\,n3+6\,n4)}-12\,z2^{(9+9\,n3+4\,n4)}-54\,z2^{(3+8\,n3+3\,n4)}-18\,z2^{(7+7\,n3+4\,n4)}$$
$$-24\,z2^{(7+9\,n3+6\,n4)}+20\,z2^{(4+9\,n3+6\,n4)}+42\,z2^{(8+6\,n3+3\,n4)}+180\,z2^{(5+11\,n3+7\,n4)}$$
$$-24\,z2^{(8+8\,n3+8\,n4)}-8\,z2^{(6+15\,n3+9\,n4)}-504\,z2^{(6+9\,n3+4\,n4)}-180\,z2^{(7+9\,n3+9\,n4)}$$
$$-72\,z2^{(8+9\,n3+9\,n4)}-108\,z2^{(7+6\,n3+5\,n4)}-30\,z2^{(7+8\,n3+6\,n4)}-60\,z2^{(7+8\,n3+8\,n4)}$$
$$+84\,z2^{(8+6\,n3+5\,n4)}+168\,z2^{(4+7\,n3+4\,n4)}+105\,z2^{(6+5\,n3+5\,n4)}-24\,z2^{(5+10\,n3+5\,n4)}$$
$$+210\,z2^{(6+5\,n3+7\,n4)}-560\,z2^{(4+7\,n3+5\,n4)}+180\,z2^{(6+12\,n3+9\,n4)}-20\,z2^{(6+11\,n3+8\,n4)}$$
$$-2\,z2^{(4+8\,n3+2\,n4)}+20\,z2^{(6+10\,n3+4\,n4)}+24\,z2^{(7+12\,n3+7\,n4)}+564\,z2^{(6+9\,n3+7\,n4)}$$
$$-24\,z2^{(4+13\,n3+5\,n4)}-84\,z2^{(5+7\,n3+5\,n4)}-24\,z2^{(8+8\,n3+6\,n4)}-72\,z2^{(4+11\,n3+5\,n4)}$$
$$+100\,z2^{(3+9\,n3+8\,n4)}-120\,z2^{(7+12\,n3+8\,n4)}-24\,z2^{(8+10\,n3+6\,n4)}-16\,z2^{(8+12\,n3+8\,n4)}$$
$$-72\,z2^{(9+10\,n3+7\,n4)}-24\,z2^{(9+9\,n3+6\,n4)}+252\,z2^{(8+7\,n3+6\,n4)}-120\,z2^{(9+10\,n3+8\,n4)}$$
$$-36\,z2^{(7+5\,n3+6\,n4)}-72\,z2^{(8+9\,n3+7\,n4)}+630\,z2^{(6+6\,n3+8\,n4)}-42\,z2^{(5+7\,n3+3\,n4)}$$
$$-24\,z2^{(5+14\,n3+7\,n4)}+72\,z2^{(8+13\,n3+7\,n4)}-70\,z2^{(5+8\,n3+7\,n4)}-9\,z2^{(9+4\,n3+3\,n4)}$$
$$-28\,z2^{(7+9\,n3+5\,n4)}-8\,z2^{(11+10\,n3+5\,n4)}+84\,z2^{(10+8\,n3+5\,n4)}-294\,z2^{(6\,n3+1+n4)}$$
$$-4\,z2^{(10+9\,n3+5\,n4)}+48\,z2^{(7+12\,n3+9\,n4)}-40\,z2^{(9+9\,n3+7\,n4)}+140\,z2^{(8+6\,n3+6\,n4)}$$
$$-100\,z2^{(4+10\,n3+12\,n4)}-72\,z2^{(9+10\,n3+5\,n4)}-30\,z2^{(8+9\,n3+8\,n4)}+80\,z2^{(4+9\,n3+8\,n4)}$$
$$+315\,z2^{(6+6\,n3+6\,n4)}+12\,z2^{(10+10\,n3+6\,n4)}-36\,z2^{(10+8\,n3+6\,n4)}-252\,z2^{(9+8\,n3+6\,n4)}$$
$$-350\,z2^{(2+6\,n3+7\,n4)}+108\,z2^{(9+8\,n3+5\,n4)}$$

$$R_{z_1}\left[P_1,Q_1\right]_{z_2=1}=0$$

$$dif1=\frac{\partial}{\partial z_2}R_{z_1}\left[P_1,Q_1\right]$$

$$dif1:=-672\,z2^{(5+8\,n3+5\,n4)}-3087\,z2^{(6+4\,n3+3\,n4)}-48\,z2^{(7+8\,n3+4\,n4)}-252\,z2^{(3+4\,n3+n4)}$$
$$+504\,z2^{(2+10\,n3+4\,n4)}-400\,z2^{(3+10\,n3+12\,n4)}+1080\,z2^{(5+12\,n3+9\,n4)}-210\,z2^{(6+8\,n3+6\,n4)}$$
$$+1260\,z2^{(5+5\,n3+7\,n4)}+336\,z2^{(5+10\,n3+6\,n4)}+1120\,z2^{(7+6\,n3+6\,n4)}+1890\,z2^{(5+6\,n3+6\,n4)}$$
$$+432\,z2^{(5+9\,n3+5\,n4)}+1936\,z2^{(2+8\,n3+4\,n4)}\,n3-252\,z2^{(5+6\,n3+3\,n4)}-2268\,z2^{(5+7\,n3+4\,n4)}$$
$$-44\,z2^{(10+10\,n3+7\,n4)}-216\,z2^{(3+9\,n3+5\,n4)}-1008\,z2^{(7+4\,n3+3\,n4)}+540\,z2^{(4+6\,n3+5\,n4)}$$
$$-120\,z2^{(2+8\,n3+8\,n4)}-1050\,z2^{(6+9\,n3+10\,n4)}-192\,z2^{(7+10\,n3+6\,n4)}-120\,z2^{(5+11\,n3+8\,n4)}$$
$$-3024\,z2^{(7+5\,n3+4\,n4)}-480\,z2^{(7+9\,n3+10\,n4)}+360\,z2^{(2+6\,n3+6\,n4)}-1400\,z2^{(1+6\,n3+9\,n4)}$$

$$- 3780\, z2^{(3 + 6\, n3 + 6\, n4)} + 800\, z2^{(4 + 10\, n3 + 8\, n4)} + 162\, z2^{(8 + 7\, n3 + 4\, n4)} - 80\, z2^{(7 + 8\, n3 + 7\, n4)}$$

$$- 225\, z2^{(4 + 6\, n3 + 6\, n4)} - 81\, z2^{(8 + 4\, n3 + 3\, n4)} + 4680\, z2^{(4 + 6\, n3 + 3\, n4)} + 6615\, z2^{(2 + 5\, n3 + 5\, n4)}$$

$$+ 672\, z2^{(3 + 7\, n3 + 4\, n4)} + 336\, z2^{(7 + 6\, n3 + 3\, n4)} + 300\, z2^{(2 + 9\, n3 + 8\, n4)} - 600\, z2^{(8\, n3 + 4 + 8\, n4)}$$

$$+ 280\, z2^{(1 + 5\, n3 + 7\, n4)} - 750\, z2^{(2 + 10\, n3 + 12\, n4)} - 240\, z2^{(7 + 9\, n3 + 8\, n4)} - 756\, z2^{(8 + 7\, n3 + 5\, n4)}$$

$$- 570\, z2^{(4 + 8\, n3 + 6\, n4)} + 630\, z2^{(5 + 5\, n3 + 5\, n4)} - 120\, z2^{(9 + 7\, n3 + 5\, n4)} - 120\, z2^{(4 + 10\, n3 + 5\, n4)}$$

$$+ 280\, z2^{(9 + 7\, n3 + 4\, n4)} - 64\, z2^{(3 + 8\, n3 + 8\, n4)} - 9513\, z2^{(6 + 5\, n3 + 4\, n4)} + 1115\, z2^{(4 + 5\, n3 + 2\, n4)}$$

$$- 264\, z2^{(10 + 11\, n3 + 6\, n4)} + 270\, z2^{(4 + 5\, n3 + 4\, n4)} + 448\, z2^{(7 + 10\, n3 + 5\, n4)} - 700\, z2^{(1 + 6\, n3 + 7\, n4)}$$

$$- 1008\, z2^{(5 + 8\, n3 + 3\, n4)} + 120\, z2^{(9 + 10\, n3 + 6\, n4)} + 4200\, z2^{(3 + 8\, n3 + 9\, n4)} - 756\, z2^{(6 + 6\, n3 + 5\, n4)}$$

$$- 420\, z2^{(3 + 5\, n3 + 5\, n4)} + 504\, z2^{(6 + 12\, n3 + 5\, n4)} + 432\, z2^{(5 + 8\, n3 + 6\, n4)} - 450\, z2^{(2 + 7\, n3 + 10\, n4)}$$

$$- 420\, z2^{(6 + 8\, n3 + 8\, n4)} + 1470\, z2^{(2 + 4\, n3 + 4\, n4)} + 224\, z2^{(3 + 6\, n3 + 5\, n4)} + 120\, z2^{(5 + 10\, n3 + 4\, n4)}$$

$$- 189\, z2^{(6 + 6\, n3 + 3\, n4)} - 1260\, z2^{(6 + 9\, n3 + 9\, n4)} + 280\, z2^{(1 + 5\, n3 + 5\, n4)} + 216\, z2^{(5 + 8\, n3 + 4\, n4)}$$

$$+ 972\, z2^{(8 + 8\, n3 + 5\, n4)} - 96\, z2^{(3 + 13\, n3 + 5\, n4)} - 504\, z2^{(3 + 4\, n3 + 3\, n4)} - 588\, z2^{(1 + 7\, n3 + 3\, n4)}$$

$$+ 308\, z2^{(6 + 11\, n3 + 6\, n4)} - 81\, z2^{(2 + 6\, n3 + 3\, n4)} - 294\, z2^{(6\, n3 + n4)} - 40\, z2^{(9 + 9\, n3 + 5\, n4)}$$

$$- 72\, z2^{(8 + 13\, n3 + 8\, n4)} - 1764\, z2^{(6 + 10\, n3 + 6\, n4)} - 48\, z2^{(5 + 15\, n3 + 9\, n4)} - 3024\, z2^{(5 + 9\, n3 + 4\, n4)}$$

$$+ 360\, z2^{(2 + 6\, n3 + 8\, n4)} - 168\, z2^{(6 + 9\, n3 + 6\, n4)} + 56\, z2^{(3 + 6\, n3 + n4)} - 48\, z2^{(3 + 8\, n3 + 4\, n4)}$$

$$+ 320\, z2^{(3 + 9\, n3 + 10\, n4)} - 63\, z2^{(6 + 5\, n3 + 2\, n4)} + 3360\, z2^{(7 + 7\, n3 + 7\, n4)} + 180\, z2^{(4 + 8\, n3 + 5\, n4)}$$

$$- 162\, z2^{(2 + 8\, n3 + 3\, n4)} - 243\, z2^{(8 + 5\, n3 + 4\, n4)} - 360\, z2^{(5 + 9\, n3 + 8\, n4)} + 192\, z2^{(5 + 10\, n3 + 8\, n4)}$$

$$- 5136\, z2^{(3 + 8\, n3 + 6\, n4)} + 900\, z2^{(4 + 11\, n3 + 7\, n4)} + 3384\, z2^{(5 + 9\, n3 + 7\, n4)} - 54\, z2^{(2 + 5\, n3 + 2\, n4)}$$

$$- 108\, z2^{(8 + 9\, n3 + 4\, n4)} - 252\, z2^{(3 + 5\, n3 + 2\, n4)} - 576\, z2^{(7 + 9\, n3 + 7\, n4)} + 1350\, z2^{(4 + 9\, n3 + 7\, n4)}$$

$$+ 72\, z2^{(5 + 11\, n3 + 5\, n4)} - 9\, z2^{(5\, n3 + 2)} + 150\, z2^{(4 + 7\, n3 + 2\, n4)} - 126\, z2^{(6 + 7\, n3 + 4\, n4)}$$

$$- 108\, z2^{(2 + 5\, n3 + 4\, n4)} + 80\, z2^{(3 + 9\, n3 + 6\, n4)} + 720\, z2^{(9 + 11\, n3 + 7\, n4)} - 144\, z2^{(8 + 13\, n3 + 6\, n4)}$$

$$- 756\, z2^{(6 + 6\, n3 + 7\, n4)} - 88\, z2^{(10 + 10\, n3 + 5\, n4)} - 350\, z2^{(4 + 8\, n3 + 7\, n4)} - 108\, z2^{(2 + 10\, n3 + 3\, n4)}$$

$$+ 90\, z2^{(4 + 8\, n3 + 3\, n4)} + 726\, z2^{(2 + 8\, n3 + 4\, n4)} + 70\, z2^{(1 + 5\, n3 + 3\, n4)} - 210\, z2^{(4 + 8\, n3 + 4\, n4)}$$

$$+ 576\, z2^{(7 + 13\, n3 + 7\, n4)} + 3150\, z2^{(5 + 6\, n3 + 9\, n4)} - 160\, z2^{(7 + 8\, n3 + 9\, n4)} - 60\, z2^{(4 + 10\, n3 + 3\, n4)}$$

$$- 648\, z2^{(8 + 10\, n3 + 5\, n4)} + 224\, z2^{(6 + 11\, n3 + 4\, n4)} - 225\, z2^{(2 + 7\, n3 + 8\, n4)} - 288\, z2^{(3 + 11\, n3 + 5\, n4)}$$

$$- 441\, z2^{(2 + 4\, n3 + n4)} + 90\, z2^{(2 + 6\, n3 + 4\, n4)} - 196\, z2^{(6 + 9\, n3 + 5\, n4)} + 336\, z2^{(3 + 11\, n3 + 6\, n4)}$$

$$- 2240\, z2^{(3 + 7\, n3 + 7\, n4)} + 6048\, z2^{(7 + 8\, n3 + 5\, n4)} - 20\, z2^{(4 + 9\, n3 + 2\, n4)} - 882\, z2^{(2 + 4\, n3 + 3\, n4)}$$

$$- 504\, z2^{(3 + 5\, n3 + 4\, n4)} + 600\, z2^{(4 + 10\, n3 + 6\, n4)} + 224\, z2^{(3 + 6\, n3 + 3\, n4)} + 1750\, z2^{(1 + 7\, n3 + 11\, n4)}$$

$$- 480\, z2^{(5 + 11\, n3 + 10\, n4)} - 120\, z2^{(4 + 14\, n3 + 7\, n4)} + 504\, z2^{(1 + 7\, n3 + 2\, n4)} + 756\, z2^{(6 + 10\, n3 + 5\, n4)}$$

$$+ 3780\, z2^{(5 + 6\, n3 + 8\, n4)} + 3024\, z2^{(7 + 7\, n3 + 4\, n4)} - 180\, z2^{(4 + 12\, n3 + 7\, n4)} - 80\, z2^{(4 + 9\, n3 + 6\, n4)}$$

$$- 3920\, z2^{(3 + 7\, n3 + 7\, n4)}\, n3 - 3920\, z2^{(3 + 7\, n3 + 7\, n4)}\, n4 - 270\, z2^{(4 + 6\, n3 + 6\, n4)}\, n3$$

$$- 270\, z2^{(4 + 6\, n3 + 6\, n4)}\, n4 - 972\, z2^{(8 + 11\, n3 + 6\, n4)} + 1600\, z2^{(4 + 10\, n3 + 8\, n4)}\, n3$$

$$+ 1280\, z2^{(4 + 10\, n3 + 8\, n4)}\, n4 + 1200\, z2^{(4 + 10\, n3 + 6\, n4)}\, n3 + 720\, z2^{(4 + 10\, n3 + 6\, n4)}\, n4$$

$$- 96\, z2^{(3 + 8\, n3 + 4\, n4)}\, n3 - 3920\, z2^{(3 + 7\, n3 + 5\, n4)}\, n3 - 2800\, z2^{(3 + 7\, n3 + 5\, n4)}\, n4$$

$$+ 1960\, z2^{(2 + 4\, n3 + 4\, n4)}\, n3 + 1960\, z2^{(2 + 4\, n3 + 4\, n4)}\, n4 - 54\, z2^{(8 + 6\, n3 + 3\, n4)}$$

$$
\begin{aligned}
&+3948\,z2^{(5+9\,n3+7\,n4)}\,n4-16\,z2^{(3+8\,n3+2\,n4)}\,n3-4\,z2^{(3+8\,n3+2\,n4)}\,n4-48\,z2^{(3+8\,n3+4\,n4)}\,n4\\
&+5076\,z2^{(5+9\,n3+7\,n4)}\,n3+600\,z2^{(2+9\,n3+10\,n4)}+648\,z2^{(5+9\,n3+5\,n4)}\,n3\\
&+360\,z2^{(5+9\,n3+5\,n4)}\,n4+144\,z2^{(5+11\,n3+7\,n4)}-80\,z2^{(4+9\,n3+4\,n4)}-576\,z2^{(7+9\,n3+9\,n4)}\\
&-441\,z2^{(4\,n3+n4)}+672\,z2^{(7+6\,n3+5\,n4)}-192\,z2^{(7+8\,n3+6\,n4)}-192\,z2^{(7+8\,n3+8\,n4)}\\
&+168\,z2^{(9\,n3+1+2\,n4)}-120\,z2^{(2+8\,n3+6\,n4)}-108\,z2^{(8+6\,n3+5\,n4)}-24\,z2^{(12\,n3+2+3\,n4)}\\
&+420\,z2^{(4+7\,n3+4\,n4)}-252\,z2^{(6+5\,n3+6\,n4)}-105\,z2^{(6+5\,n3+5\,n4)}-140\,z2^{(6+12\,n3+10\,n4)}\\
&-210\,z2^{(6+5\,n3+7\,n4)}-420\,z2^{(4+7\,n3+5\,n4)}+336\,z2^{(6+12\,n3+9\,n4)}-72\,z2^{(5\,n3+2+6\,n4)}\\
&-132\,z2^{(10+11\,n3+8\,n4)}+168\,z2^{(6+11\,n3+8\,n4)}-2240\,z2^{(3+7\,n3+5\,n4)}-630\,z2^{(6+9\,n3+7\,n4)}\\
&+1029\,z2^{(3\,n3-1)}\,n3-2268\,z2^{(8+8\,n3+6\,n4)}+320\,z2^{(3+9\,n3+8\,n4)}-128\,z2^{(7+12\,n3+8\,n4)}\\
&-1500\,z2^{(4+11\,n3+10\,n4)}-1080\,z2^{(8+10\,n3+8\,n4)}+378\,z2^{(1+5\,n3+2\,n4)}+336\,z2^{(3+7\,n3+2\,n4)}\\
&-324\,z2^{(8+7\,n3+6\,n4)}-240\,z2^{(9+10\,n3+8\,n4)}-350\,z2^{(6+8\,n3+9\,n4)}-360\,z2^{(8+9\,n3+7\,n4)}\\
&-216\,z2^{(8+9\,n3+6\,n4)}-630\,z2^{(6+6\,n3+8\,n4)}+240\,z2^{(4+7\,n3+6\,n4)}-210\,z2^{(4+7\,n3+3\,n4)}\\
&+2016\,z2^{(7+7\,n3+6\,n4)}-144\,z2^{(7+9\,n3+5\,n4)}-648\,z2^{(8+10\,n3+7\,n4)}-80\,z2^{(9+9\,n3+7\,n4)}\\
&-140\,z2^{(6+9\,n3+5\,n4)}\,n4-144\,z2^{(5\,n3+2+6\,n4)}\,n4-294\,z2^{(4+7\,n3+3\,n4)}\,n3\\
&-126\,z2^{(4+7\,n3+3\,n4)}\,n4-1890\,z2^{(7+5\,n3+4\,n4)}\,n3+1764\,z2^{(1+7\,n3+2\,n4)}\,n3\\
&+504\,z2^{(1+7\,n3+2\,n4)}\,n4-180\,z2^{(6+5\,n3+6\,n4)}\,n3-216\,z2^{(6+5\,n3+6\,n4)}\,n4\\
&-120\,z2^{(5\,n3+2+6\,n4)}\,n3-320\,z2^{(2+8\,n3+6\,n4)}\,n3-240\,z2^{(2+8\,n3+6\,n4)}\,n4\\
&+756\,z2^{(9\,n3+1+2\,n4)}\,n3-315\,z2^{(3+5\,n3+2\,n4)}\,n3-126\,z2^{(3+5\,n3+2\,n4)}\,n4\\
&+288\,z2^{(6+12\,n3+7\,n4)}\,n3+168\,z2^{(6+12\,n3+7\,n4)}\,n4+336\,z2^{(3+6\,n3+5\,n4)}\,n3\\
&+280\,z2^{(3+6\,n3+5\,n4)}\,n4-960\,z2^{(6+12\,n3+8\,n4)}\,n4-1512\,z2^{(7+5\,n3+4\,n4)}\,n4\\
&-84\,z2^{(9+7\,n3+5\,n4)}\,n3-60\,z2^{(9+7\,n3+5\,n4)}\,n4-312\,z2^{(3+13\,n3+5\,n4)}\,n3\\
&-120\,z2^{(3+13\,n3+5\,n4)}\,n4+1176\,z2^{(3+7\,n3+4\,n4)}\,n3+672\,z2^{(3+7\,n3+4\,n4)}\,n4\\
&-270\,z2^{(7+9\,n3+8\,n4)}\,n3-240\,z2^{(7+9\,n3+8\,n4)}\,n4+588\,z2^{(4+7\,n3+4\,n4)}\,n3\\
&+336\,z2^{(4+7\,n3+4\,n4)}\,n4+2160\,z2^{(5+12\,n3+9\,n4)}\,n3+1620\,z2^{(5+12\,n3+9\,n4)}\,n4\\
&-1764\,z2^{(6+4\,n3+3\,n4)}\,n3-1323\,z2^{(6+4\,n3+3\,n4)}\,n4-320\,z2^{(2+8\,n3+8\,n4)}\,n3\\
&-320\,z2^{(2+8\,n3+8\,n4)}\,n4-525\,z2^{(3+5\,n3+5\,n4)}\,n3-525\,z2^{(3+5\,n3+5\,n4)}\,n4\\
&+320\,z2^{(5+10\,n3+8\,n4)}\,n3+256\,z2^{(5+10\,n3+8\,n4)}\,n4-960\,z2^{(8\,n3+4+8\,n4)}\,n3\\
&-960\,z2^{(8\,n3+4+8\,n4)}\,n4-1350\,z2^{(6+9\,n3+10\,n4)}\,n3-1500\,z2^{(6+9\,n3+10\,n4)}\,n4\\
&+5616\,z2^{(4+6\,n3+3\,n4)}\,n3+2808\,z2^{(4+6\,n3+3\,n4)}\,n4-504\,z2^{(7+4\,n3+3\,n4)}\,n3\\
&-378\,z2^{(7+4\,n3+3\,n4)}\,n4+200\,z2^{(5+10\,n3+4\,n4)}\,n3+80\,z2^{(5+10\,n3+4\,n4)}\,n4\\
&-132\,z2^{(10+11\,n3+8\,n4)}\,n3-96\,z2^{(10+11\,n3+8\,n4)}\,n4-144\,z2^{(4+9\,n3+4\,n4)}\,n3\\
&-64\,z2^{(4+9\,n3+4\,n4)}\,n4+132\,z2^{(5+11\,n3+5\,n4)}\,n3+60\,z2^{(5+11\,n3+5\,n4)}\,n4\\
&-150\,z2^{(6+5\,n3+7\,n4)}\,n3-210\,z2^{(6+5\,n3+7\,n4)}\,n4-500\,z2^{(3+10\,n3+10\,n4)}\,n3\\
&-500\,z2^{(3+10\,n3+10\,n4)}\,n4+264\,z2^{(5+11\,n3+7\,n4)}\,n3+168\,z2^{(5+11\,n3+7\,n4)}\,n4\\
&+1764\,z2^{(7+7\,n3+6\,n4)}\,n3+1512\,z2^{(7+7\,n3+6\,n4)}\,n4-240\,z2^{(7+10\,n3+6\,n4)}\,n3\\
&-144\,z2^{(7+10\,n3+6\,n4)}\,n4-720\,z2^{(8+10\,n3+7\,n4)}\,n3-504\,z2^{(8+10\,n3+7\,n4)}\,n4
\end{aligned}
$$

$$- 630\, z2^{(3 + 5\, n3 + 4\, n4)}\, n3 - 504\, z2^{(3 + 5\, n3 + 4\, n4)}\, n4 - 192\, z2^{(7 + 12\, n3 + 8\, n4)}\, n3$$

$$- 128\, z2^{(7 + 12\, n3 + 8\, n4)}\, n4 + 875\, z2^{(5 + 5\, n3 + 8\, n4)}\, n3 + 1400\, z2^{(5 + 5\, n3 + 8\, n4)}\, n4$$

$$+ 900\, z2^{(2 + 9\, n3 + 8\, n4)}\, n3 + 800\, z2^{(2 + 9\, n3 + 8\, n4)}\, n4 - 1440\, z2^{(6 + 12\, n3 + 8\, n4)}\, n3$$

$$- 75\, z2^{(6 + 5\, n3 + 5\, n4)}\, n3 - 240\, z2^{(4 + 10\, n3 + 5\, n4)}\, n3 - 120\, z2^{(4 + 10\, n3 + 5\, n4)}\, n4$$

$$- 216\, z2^{(6 + 9\, n3 + 6\, n4)}\, n3 - 144\, z2^{(6 + 9\, n3 + 6\, n4)}\, n4 + 336\, z2^{(3 + 6\, n3 + 3\, n4)}\, n3$$

$$+ 168\, z2^{(3 + 6\, n3 + 3\, n4)}\, n4 - 2500\, z2^{(2 + 10\, n3 + 12\, n4)}\, n3 - 3000\, z2^{(2 + 10\, n3 + 12\, n4)}\, n4$$

$$- 40\, z2^{(10 + 10\, n3 + 7\, n4)}\, n3 - 28\, z2^{(10 + 10\, n3 + 7\, n4)}\, n4 - 80\, z2^{(7 + 8\, n3 + 7\, n4)}\, n3$$

$$- 70\, z2^{(7 + 8\, n3 + 7\, n4)}\, n4 + 1050\, z2^{(5 + 5\, n3 + 7\, n4)}\, n3 + 1470\, z2^{(5 + 5\, n3 + 7\, n4)}\, n4$$

$$+ 2646\, z2^{(7 + 7\, n3 + 4\, n4)}\, n3 + 1512\, z2^{(7 + 7\, n3 + 4\, n4)}\, n4 - 4200\, z2^{(1 + 6\, n3 + 9\, n4)}\, n3$$

$$- 6300\, z2^{(1 + 6\, n3 + 9\, n4)}\, n4 + 936\, z2^{(7 + 13\, n3 + 7\, n4)}\, n3 + 504\, z2^{(7 + 13\, n3 + 7\, n4)}\, n4$$

$$- 432\, z2^{(2 + 8\, n3 + 3\, n4)}\, n3 - 162\, z2^{(2 + 8\, n3 + 3\, n4)}\, n4 + 720\, z2^{(2 + 6\, n3 + 6\, n4)}\, n3$$

$$+ 720\, z2^{(2 + 6\, n3 + 6\, n4)}\, n4 - 48\, z2^{(7 + 8\, n3 + 4\, n4)}\, n3 - 24\, z2^{(7 + 8\, n3 + 4\, n4)}\, n4$$

$$+ 3150\, z2^{(5 + 6\, n3 + 9\, n4)}\, n3 + 4725\, z2^{(5 + 6\, n3 + 9\, n4)}\, n4 + 525\, z2^{(5 + 5\, n3 + 5\, n4)}\, n3$$

$$+ 525\, z2^{(5 + 5\, n3 + 5\, n4)}\, n4 + 504\, z2^{(7 + 6\, n3 + 5\, n4)}\, n3 + 420\, z2^{(7 + 6\, n3 + 5\, n4)}\, n4$$

$$- 220\, z2^{(5 + 11\, n3 + 8\, n4)}\, n3 - 160\, z2^{(5 + 11\, n3 + 8\, n4)}\, n4 - 540\, z2^{(7 + 9\, n3 + 10\, n4)}\, n3$$

$$- 600\, z2^{(7 + 9\, n3 + 10\, n4)}\, n4 + 2940\, z2^{(7 + 7\, n3 + 7\, n4)}\, n4 + 288\, z2^{(4 + 8\, n3 + 5\, n4)}\, n3$$

$$+ 180\, z2^{(4 + 8\, n3 + 5\, n4)}\, n4 - 162\, z2^{(7 + 9\, n3 + 5\, n4)}\, n3 - 90\, z2^{(7 + 9\, n3 + 5\, n4)}\, n4$$

$$+ 1980\, z2^{(4 + 11\, n3 + 7\, n4)}\, n3 + 1260\, z2^{(4 + 11\, n3 + 7\, n4)}\, n4 - 108\, z2^{(8 + 9\, n3 + 4\, n4)}\, n3$$

$$- 48\, z2^{(8 + 9\, n3 + 4\, n4)}\, n4 - 126\, z2^{(6 + 7\, n3 + 4\, n4)}\, n3 - 72\, z2^{(6 + 7\, n3 + 4\, n4)}\, n4$$

$$+ 210\, z2^{(4 + 7\, n3 + 2\, n4)}\, n3 + 60\, z2^{(4 + 7\, n3 + 2\, n4)}\, n4 + 576\, z2^{(5 + 8\, n3 + 6\, n4)}\, n3$$

$$+ 432\, z2^{(5 + 8\, n3 + 6\, n4)}\, n4 - 15\, z2^{(5\, n3 + 2)}\, n3 - 192\, z2^{(7 + 8\, n3 + 8\, n4)}\, n3 - 192\, z2^{(7 + 8\, n3 + 8\, n4)}\, n4$$

$$+ 484\, z2^{(6 + 11\, n3 + 6\, n4)}\, n3 + 264\, z2^{(6 + 11\, n3 + 6\, n4)}\, n4 - 252\, z2^{(5 + 6\, n3 + 3\, n4)}\, n3$$

$$- 126\, z2^{(5 + 6\, n3 + 3\, n4)}\, n4 + 1680\, z2^{(2 + 10\, n3 + 4\, n4)}\, n3 + 672\, z2^{(2 + 10\, n3 + 4\, n4)}\, n4$$

$$- 360\, z2^{(8 + 10\, n3 + 5\, n4)}\, n4 - 912\, z2^{(4 + 8\, n3 + 6\, n4)}\, n3 - 684\, z2^{(4 + 8\, n3 + 6\, n4)}\, n4$$

$$+ 672\, z2^{(9 + 8\, n3 + 5\, n4)}\, n3 + 420\, z2^{(9 + 8\, n3 + 5\, n4)}\, n4 - 1764\, z2^{(6\, n3 + n4)}\, n3 - 294\, z2^{(6\, n3 + n4)}\, n4$$

$$- 270\, z2^{(6 + 6\, n3 + 6\, n4)}\, n3 - 270\, z2^{(6 + 6\, n3 + 6\, n4)}\, n4 - 360\, z2^{(8 + 9\, n3 + 7\, n4)}\, n3$$

$$- 280\, z2^{(8 + 9\, n3 + 7\, n4)}\, n4 - 120\, z2^{(5 + 15\, n3 + 9\, n4)}\, n3 - 720\, z2^{(8 + 10\, n3 + 5\, n4)}\, n3$$

$$- 216\, z2^{(9 + 8\, n3 + 6\, n4)}\, n4 + 1890\, z2^{(5 + 6\, n3 + 6\, n4)}\, n3 + 1890\, z2^{(5 + 6\, n3 + 6\, n4)}\, n4$$

$$+ 720\, z2^{(3 + 9\, n3 + 8\, n4)}\, n3 + 640\, z2^{(3 + 9\, n3 + 8\, n4)}\, n4 - 10272\, z2^{(3 + 8\, n3 + 6\, n4)}\, n3$$

$$- 7704\, z2^{(3 + 8\, n3 + 6\, n4)}\, n4 + 120\, z2^{(9 + 10\, n3 + 6\, n4)}\, n3 + 72\, z2^{(9 + 10\, n3 + 6\, n4)}\, n4$$

$$- 588\, z2^{(2 + 4\, n3 + n4)}\, n3 - 147\, z2^{(2 + 4\, n3 + n4)}\, n4 - 648\, z2^{(6 + 6\, n3 + 5\, n4)}\, n3 - 540\, z2^{(6 + 6\, n3 + 5\, n4)}\, n4$$

$$- 4536\, z2^{(5 + 9\, n3 + 4\, n4)}\, n3 - 2016\, z2^{(5 + 9\, n3 + 4\, n4)}\, n4 - 128\, z2^{(3 + 8\, n3 + 8\, n4)}\, n3$$

$$- 128\, z2^{(3 + 8\, n3 + 8\, n4)}\, n4 - 480\, z2^{(6 + 8\, n3 + 8\, n4)}\, n3 - 480\, z2^{(6 + 8\, n3 + 8\, n4)}\, n4$$

$$- 72\, z2^{(9 + 9\, n3 + 7\, n4)}\, n3 - 56\, z2^{(9 + 9\, n3 + 7\, n4)}\, n4 - 1050\, z2^{(2 + 7\, n3 + 10\, n4)}\, n3$$

$$- 1500\, z2^{(2 + 7\, n3 + 10\, n4)}\, n4 - 1620\, z2^{(6 + 9\, n3 + 9\, n4)}\, n3 - 1620\, z2^{(6 + 9\, n3 + 9\, n4)}\, n4$$

$$+ 840\, z2^{(7 + 6\, n3 + 6\, n4)}\, n3 + 840\, z2^{(7 + 6\, n3 + 6\, n4)}\, n4 + 8400\, z2^{(3 + 8\, n3 + 9\, n4)}\, n3$$

$$+9450\, z2^{(3+8\, n3+9\, n4)}{}_{n4}-36\, z2^{(9+9\, n3+5\, n4)}{}_{n3}-20\, z2^{(9+9\, n3+5\, n4)}{}_{n4}$$

$$-648\, z2^{(7+9\, n3+9\, n4)}{}_{n3}-648\, z2^{(7+9\, n3+9\, n4)}{}_{n4}+700\, z2^{(1+5\, n3+5\, n4)}{}_{n3}$$

$$+700\, z2^{(1+5\, n3+5\, n4)}{}_{n4}-162\, z2^{(6+6\, n3+3\, n4)}{}_{n3}-81\, z2^{(6+6\, n3+3\, n4)}{}_{n4}$$

$$-162\, z2^{(2+6\, n3+3\, n4)}{}_{n3}-81\, z2^{(2+6\, n3+3\, n4)}{}_{n4}-1200\, z2^{(3+10\, n3+12\, n4)}{}_{n4}$$

$$+576\, z2^{(6+12\, n3+9\, n4)}{}_{n3}+432\, z2^{(6+12\, n3+9\, n4)}{}_{n4}+700\, z2^{(1+5\, n3+7\, n4)}{}_{n3}$$

$$+980\, z2^{(1+5\, n3+7\, n4)}{}_{n4}-588\, z2^{(8+7\, n3+5\, n4)}{}_{n3}-240\, z2^{(6+12\, n3+10\, n4)}{}_{n3}$$

$$-240\, z2^{(9+10\, n3+8\, n4)}{}_{n3}-288\, z2^{(9+8\, n3+6\, n4)}{}_{n3}-96\, z2^{(12\, n3+2+3\, n4)}{}_{n3}$$

$$-24\, z2^{(12\, n3+2+3\, n4)}{}_{n4}-80\, z2^{(10+10\, n3+5\, n4)}{}_{n3}-40\, z2^{(10+10\, n3+5\, n4)}{}_{n4}$$

$$-1000\, z2^{(3+10\, n3+12\, n4)}{}_{n3}-200\, z2^{(6+12\, n3+10\, n4)}{}_{n4}-72\, z2^{(5+15\, n3+9\, n4)}{}_{n4}$$

$$-104\, z2^{(8+13\, n3+8\, n4)}{}_{n3}-64\, z2^{(8+13\, n3+8\, n4)}{}_{n4}+720\, z2^{(2+6\, n3+8\, n4)}{}_{n3}$$

$$+960\, z2^{(2+6\, n3+8\, n4)}{}_{n4}+84\, z2^{(3+6\, n3+n4)}{}_{n3}+14\, z2^{(3+6\, n3+n4)}{}_{n4}-45\, z2^{(6+5\, n3+2\, n4)}{}_{n3}$$

$$-18\, z2^{(6+5\, n3+2\, n4)}{}_{n4}+2940\, z2^{(7+7\, n3+7\, n4)}{}_{n3}-420\, z2^{(8+7\, n3+5\, n4)}{}_{n4}$$

$$+864\, z2^{(8+8\, n3+5\, n4)}{}_{n3}+540\, z2^{(8+8\, n3+5\, n4)}{}_{n4}-2100\, z2^{(1+6\, n3+7\, n4)}{}_{n3}$$

$$-2450\, z2^{(1+6\, n3+7\, n4)}{}_{n4}+180\, z2^{(2+6\, n3+4\, n4)}{}_{n3}+120\, z2^{(2+6\, n3+4\, n4)}{}_{n4}$$

$$+126\, z2^{(8+7\, n3+4\, n4)}{}_{n3}+72\, z2^{(8+7\, n3+4\, n4)}{}_{n4}-240\, z2^{(6+8\, n3+6\, n4)}{}_{n3}$$

$$-180\, z2^{(6+8\, n3+6\, n4)}{}_{n4}+264\, z2^{(6+11\, n3+8\, n4)}{}_{n3}+192\, z2^{(6+11\, n3+8\, n4)}{}_{n4}$$

$$-6795\, z2^{(6+5\, n3+4\, n4)}{}_{n3}-5436\, z2^{(6+5\, n3+4\, n4)}{}_{n4}+864\, z2^{(6+12\, n3+5\, n4)}{}_{n3}$$

$$+360\, z2^{(6+12\, n3+5\, n4)}{}_{n4}-264\, z2^{(10+11\, n3+6\, n4)}{}_{n3}-144\, z2^{(10+11\, n3+6\, n4)}{}_{n4}$$

$$-2016\, z2^{(8+8\, n3+6\, n4)}{}_{n3}-1512\, z2^{(8+8\, n3+6\, n4)}{}_{n4}-36\, z2^{(4+9\, n3+2\, n4)}{}_{n3}$$

$$-8\, z2^{(4+9\, n3+2\, n4)}{}_{n4}+6048\, z2^{(7+8\, n3+5\, n4)}{}_{n3}+3780\, z2^{(7+8\, n3+5\, n4)}{}_{n4}$$

$$-336\, z2^{(4+8\, n3+4\, n4)}{}_{n3}-168\, z2^{(4+8\, n3+4\, n4)}{}_{n4}-144\, z2^{(4+9\, n3+6\, n4)}{}_{n3}$$

$$-96\, z2^{(4+9\, n3+6\, n4)}{}_{n4}-432\, z2^{(4+12\, n3+7\, n4)}{}_{n3}-252\, z2^{(4+12\, n3+7\, n4)}{}_{n4}$$

$$+560\, z2^{(5+10\, n3+6\, n4)}{}_{n3}+336\, z2^{(5+10\, n3+6\, n4)}{}_{n4}-896\, z2^{(5+8\, n3+5\, n4)}{}_{n3}$$

$$-560\, z2^{(5+8\, n3+5\, n4)}{}_{n4}+1080\, z2^{(6+10\, n3+5\, n4)}{}_{n3}+540\, z2^{(6+10\, n3+5\, n4)}{}_{n4}$$

$$-1764\, z2^{(4\, n3+n4)}{}_{n3}-441\, z2^{(4\, n3+n4)}{}_{n4}-540\, z2^{(6+6\, n3+8\, n4)}{}_{n3}-720\, z2^{(6+6\, n3+8\, n4)}{}_{n4}$$

$$+3780\, z2^{(5+6\, n3+8\, n4)}{}_{n3}+5040\, z2^{(5+6\, n3+8\, n4)}{}_{n4}+1800\, z2^{(2+9\, n3+10\, n4)}{}_{n3}$$

$$+2000\, z2^{(2+9\, n3+10\, n4)}{}_{n4}-1188\, z2^{(8+11\, n3+6\, n4)}{}_{n3}-648\, z2^{(8+11\, n3+6\, n4)}{}_{n4}$$

$$+144\, z2^{(4+8\, n3+3\, n4)}{}_{n3}+54\, z2^{(4+8\, n3+3\, n4)}{}_{n4}-560\, z2^{(4+8\, n3+7\, n4)}{}_{n3}$$

$$-490\, z2^{(4+8\, n3+7\, n4)}{}_{n4}-120\, z2^{(4+10\, n3+3\, n4)}{}_{n3}-36\, z2^{(4+10\, n3+3\, n4)}{}_{n4}$$

$$+560\, z2^{(7+10\, n3+5\, n4)}{}_{n3}+280\, z2^{(7+10\, n3+5\, n4)}{}_{n4}-525\, z2^{(2+7\, n3+8\, n4)}{}_{n3}$$

$$-600\, z2^{(2+7\, n3+8\, n4)}{}_{n4}-192\, z2^{(7+8\, n3+6\, n4)}{}_{n3}-144\, z2^{(7+8\, n3+6\, n4)}{}_{n4}$$

$$+270\, z2^{(4+5\, n3+4\, n4)}{}_{n3}+216\, z2^{(4+5\, n3+4\, n4)}{}_{n4}-336\, z2^{(4+14\, n3+7\, n4)}{}_{n3}$$

$$-168\, z2^{(4+14\, n3+7\, n4)}{}_{n4}-648\, z2^{(7+9\, n3+7\, n4)}{}_{n3}-504\, z2^{(7+9\, n3+7\, n4)}{}_{n4}$$

$$-648\, z2^{(6+6\, n3+7\, n4)}{}_{n3}-756\, z2^{(6+6\, n3+7\, n4)}{}_{n4}+648\, z2^{(4+6\, n3+5\, n4)}{}_{n3}$$

$$+540\, z2^{(4+6\, n3+5\, n4)}{}_{n4}+352\, z2^{(6+11\, n3+4\, n4)}{}_{n3}+128\, z2^{(6+11\, n3+4\, n4)}{}_{n4}$$

$$-504\, z2^{(3+4\, n3+3\, n4)}{}_{n3}-378\, z2^{(3+4\, n3+3\, n4)}{}_{n4}+6125\, z2^{(1+7\, n3+11\, n4)}{}_{n3}$$

$$+9625\, z2^{(1+7\, n3+11\, n4)}\, n4 - 1176\, z2^{(2+4\, n3+3\, n4)}\, n3 - 882\, z2^{(2+4\, n3+3\, n4)}\, n4$$
$$-192\, z2^{(9+10\, n3+8\, n4)}\, n4 - 588\, z2^{(4+7\, n3+5\, n4)}\, n3 - 420\, z2^{(4+7\, n3+5\, n4)}\, n4$$
$$-252\, z2^{(3+4\, n3+n4)}\, n3 - 63\, z2^{(3+4\, n3+n4)}\, n4 + 924\, z2^{(3+11\, n3+6\, n4)}\, n3$$
$$+504\, z2^{(3+11\, n3+6\, n4)}\, n4 - 360\, z2^{(2+10\, n3+3\, n4)}\, n3 - 108\, z2^{(2+10\, n3+3\, n4)}\, n4$$
$$-2058\, z2^{(1+7\, n3+3\, n4)}\, n3 - 882\, z2^{(1+7\, n3+3\, n4)}\, n4 + 168\, z2^{(9\, n3+1+2\, n4)}\, n4$$
$$+196\, z2^{(9+7\, n3+4\, n4)}\, n3 + 112\, z2^{(9+7\, n3+4\, n4)}\, n4 - 252\, z2^{(8+7\, n3+6\, n4)}\, n3$$
$$-216\, z2^{(8+7\, n3+6\, n4)}\, n4 - 1200\, z2^{(8+10\, n3+8\, n4)}\, n3 - 960\, z2^{(8+10\, n3+8\, n4)}\, n4$$
$$-2520\, z2^{(6+10\, n3+6\, n4)}\, n3 - 1512\, z2^{(6+10\, n3+6\, n4)}\, n4 - 135\, z2^{(8+5\, n3+4\, n4)}\, n3$$
$$-108\, z2^{(8+5\, n3+4\, n4)}\, n4 - 5670\, z2^{(4+9\, n3+8\, n4)}\, n3 - 5040\, z2^{(4+9\, n3+8\, n4)}\, n4$$
$$-90\, z2^{(2+5\, n3+2\, n4)}\, n3 - 36\, z2^{(2+5\, n3+2\, n4)}\, n4 - 36\, z2^{(8+6\, n3+3\, n4)}\, n3 - 18\, z2^{(8+6\, n3+3\, n4)}\, n4$$
$$+945\, z2^{(1+5\, n3+2\, n4)}\, n3 + 378\, z2^{(1+5\, n3+2\, n4)}\, n4 - 486\, z2^{(3+9\, n3+5\, n4)}\, n3$$
$$-270\, z2^{(3+9\, n3+5\, n4)}\, n4 - 540\, z2^{(5+9\, n3+8\, n4)}\, n3 - 480\, z2^{(5+9\, n3+8\, n4)}\, n4$$
$$-810\, z2^{(6+9\, n3+7\, n4)}\, n3 - 630\, z2^{(6+9\, n3+7\, n4)}\, n4 - 3300\, z2^{(4+11\, n3+10\, n4)}\, n3$$
$$+1115\, z2^{(4+5\, n3+2\, n4)}\, n3 + 446\, z2^{(4+5\, n3+2\, n4)}\, n4 + 252\, z2^{(7+6\, n3+3\, n4)}\, n3$$
$$+126\, z2^{(7+6\, n3+3\, n4)}\, n4 - 180\, z2^{(2+5\, n3+4\, n4)}\, n3 - 144\, z2^{(2+5\, n3+4\, n4)}\, n4$$
$$+180\, z2^{(3+9\, n3+6\, n4)}\, n3 + 120\, z2^{(3+9\, n3+6\, n4)}\, n4 + 11025\, z2^{(2+5\, n3+5\, n4)}\, n3$$
$$+11025\, z2^{(2+5\, n3+5\, n4)}\, n4 - 36\, z2^{(8+4\, n3+3\, n4)}\, n3 - 27\, z2^{(8+4\, n3+3\, n4)}\, n4$$
$$+792\, z2^{(9+11\, n3+7\, n4)}\, n3 + 504\, z2^{(9+11\, n3+7\, n4)}\, n4 - 208\, z2^{(8+13\, n3+6\, n4)}\, n3$$
$$-96\, z2^{(8+13\, n3+6\, n4)}\, n4 - 252\, z2^{(6+9\, n3+5\, n4)}\, n3 - 3000\, z2^{(4+11\, n3+10\, n4)}\, n4$$
$$-75\, z2^{(6+5\, n3+5\, n4)}\, n4 - 1344\, z2^{(5+8\, n3+3\, n4)}\, n3 - 504\, z2^{(5+8\, n3+3\, n4)}\, n4$$
$$-400\, z2^{(6+8\, n3+9\, n4)}\, n3 - 450\, z2^{(6+8\, n3+9\, n4)}\, n4 - 880\, z2^{(5+11\, n3+10\, n4)}\, n3$$
$$-800\, z2^{(5+11\, n3+10\, n4)}\, n4 + 336\, z2^{(4+7\, n3+6\, n4)}\, n3 + 288\, z2^{(4+7\, n3+6\, n4)}\, n4$$
$$-2646\, z2^{(5+7\, n3+4\, n4)}\, n3 - 1512\, z2^{(5+7\, n3+4\, n4)}\, n4 + 588\, z2^{(3+7\, n3+2\, n4)}\, n3$$
$$+168\, z2^{(3+7\, n3+2\, n4)}\, n4 + 720\, z2^{(3+9\, n3+10\, n4)}\, n3 + 800\, z2^{(3+9\, n3+10\, n4)}\, n4$$
$$-72\, z2^{(8+6\, n3+5\, n4)}\, n3 - 60\, z2^{(8+6\, n3+5\, n4)}\, n4 - 216\, z2^{(8+9\, n3+6\, n4)}\, n3$$
$$-144\, z2^{(8+9\, n3+6\, n4)}\, n4 - 160\, z2^{(7+8\, n3+9\, n4)}\, n3 - 180\, z2^{(7+8\, n3+9\, n4)}\, n4$$
$$+1050\, z2^{(5+5\, n3+8\, n4)} - 840\, z2^{(6+12\, n3+8\, n4)} - 8\, z2^{(3+8\, n3+2\, n4)} - 3150\, z2^{(4+9\, n3+8\, n4)}$$
$$-200\, z2^{(3+10\, n3+10\, n4)} - 315\, z2^{(6+6\, n3+6\, n4)} + 288\, z2^{(5+8\, n3+4\, n4)}\, n3 + 144\, z2^{(5+8\, n3+4\, n4)}\, n4$$
$$+1890\, z2^{(4+9\, n3+7\, n4)}\, n4 + 968\, z2^{(2+8\, n3+4\, n4)}\, n4 - 5670\, z2^{(3+6\, n3+6\, n4)}\, n3$$
$$-5670\, z2^{(3+6\, n3+6\, n4)}\, n4 + 2430\, z2^{(4+9\, n3+7\, n4)}\, n3 + 168\, z2^{(6+12\, n3+7\, n4)}$$
$$-360\, z2^{(3+11\, n3+5\, n4)}\, n4 + 105\, z2^{(1+5\, n3+3\, n4)}\, n4 + 175\, z2^{(1+5\, n3+3\, n4)}\, n3$$
$$-792\, z2^{(3+11\, n3+5\, n4)}\, n3 - 360\, z2^{(9+8\, n3+6\, n4)} + 840\, z2^{(9+8\, n3+5\, n4)}$$

$dif1(z_2 = 1) =$

$$-3200\, n3 + 3840\, n4 - 5440$$

The multiplicity $m_{z_2=1}\left(R_{z_1}\left[P_1, Q_1\right]\right) = 1$

Recalling that for stability

$$m_{z_2=1}\left(R_{z_1}\left[\underset{\sim}{Q}_1, \underset{\sim}{Q}_1\right]\right) < m_{z_2=1}\left(R_{z_1}\left[P_1, \underset{\sim}{Q}_1\right]\right),$$

we see that it does not hold, and therefore the system is unstable.

To check whether $(1,1,1,1)$ is a simple singularity:

$$\underset{\sim}{Q} = z_1^2 z_2^3 z_3 z_4 + z_1^3 z_2 z_3^2 z_4 + 2 z_1 z_2 z_3 z_4^2 + 2 z_2 z_3^2 z_4 + z_1 z_2 z_3 - 7$$

$$\left.\frac{\partial \underset{\sim}{Q}}{\partial z_1}\right|_{z_1=1, z_2=1, z_3=1, z_4=1} = 8$$

$$\left.\frac{\partial \underset{\sim}{Q}}{\partial z_2}\right|_{z_1=1, z_2=1, z_3=1, z_4=1} = 9$$

$$\left.\frac{\partial \underset{\sim}{Q}}{\partial z_3}\right|_{z_1=1, z_2=1, z_3=1, z_4=1} = 12$$

$$\left.\frac{\partial \underset{\sim}{Q}}{\partial z_4}\right|_{z_1=1, z_2=1, z_3=1, z_4=1} = 8$$

Thus, $(1,1,1,1)$ is a simple singularity.

Appendix C.

$$Q = (1 - z_1)(1 - z_2) + (1 - z_3)(1 - z_1 z_2)$$

$$z_3 = 0.5 z_2^n$$

$$Q_1 = (1 - z_1)(1 - z_2) + (1 - 0.5 z_2^{n_3})(1 - z_1 z_2)$$

$$\underline{Q}_1 = (2z_1 - 1) z_2^{n_3+1} - z_1 z_2^{n_3} - 0.5 z_1 z_2 + 0.5$$

$$R_{z_1}[\underline{Q}_1, Q_1] = 0.5 - z_2 + 0.5 z_2^2 - 2 z_2^{n_3} + 0.5 z_2^{2n_3} + 4 z_2^{n_3+1} - 2 z_2^{n_3+2} - z_2^{2n_3+1} + 0.5 z_2^{2n_3+2}$$

$$difa = \frac{\partial}{\partial z_2} R_{z_1}[\underline{Q}_1, Q_1] =$$

$$4\,z2^n\,n + 4\,z2^n - 2\,z2^{(n-1)}\,n - 1 - 4\,z2^{(n+1)} - 2\,z2^{(n+1)}\,n + z2 - 2\,z2^{(2\,n)}\,n - z2^{(2\,n)} + z2^{(2\,n-1)}\,n$$
$$+ z2^{(2\,n+1)}\,n + z2^{(2\,n+1)}$$

$$difa(z_2 = 1) = 0$$

$$difb = \frac{\partial^2}{\partial z_2^2} R_{z_1}[\underline{Q}_1, Q_1] =$$

$$4\,z2^{(n-1)}\,n^2 + 4\,z2^{(n-1)}\,n - 2\,z2^{(n-2)}\,n^2 + 2\,z2^{(n-2)}\,n - 6\,z2^n\,n - 4\,z2^n - 2\,z2^n\,n^2 + 1 - 4\,z2^{(2\,n-1)}\,n^2$$
$$- 2\,z2^{(2\,n-1)}\,n + 2\,z2^{(2\,n-2)}\,n^2 - z2^{(2\,n-2)}\,n + 2\,z2^{(2\,n)}\,n^2 + 3\,z2^{(2\,n)}\,n + z2^{(2\,n)}$$

$$difb(z_2 = 1) = -2$$

Therefore the multiplicity $m_{z_2=1}\left(R_{z_1}\left[\underline{Q}_1, Q_1\right]\right) = 2$

$$P = (1 - z_1)(1 - z_2)(1 - z_3)$$

$$z_3 = 0.5 z_2^n$$

$$P_1 = (1 - z_1)(1 - z_2)(1 - 0.5 z_2^{n_3})$$

$$R_{z_1}[P_1, Q_1] = -0.25(1 - z_2)^2 (2 - z_2^{n_3})^2$$

$$R_{z_1}\left[P, Q_1\right]_{z_2=1} = 0$$

$$dif1 = \frac{\partial}{\partial z_2} R_{z_1}\left[P_1, Q_1\right] =$$

$$2 - 2\,z2^{n} - 2\,z2 + 2\,z2^{(n+1)} + \frac{1}{2} z2^{(2\,n)} - \frac{1}{2} z2^{(2\,n+1)} + z2^{(n-1)}\,n - 2\,z2^{n}\,n - \frac{1}{2} z2^{(2\,n-1)}\,n + z2^{(2\,n)}\,n$$
$$+ z2^{(n+1)}\,n - \frac{1}{2} z2^{(2\,n+1)}\,n$$

$$dif1(z_2 = 1) = 0$$

$$dif2 = \frac{\partial^2}{\partial z_2^2} R_{z_1}\left[P, Q_1\right]$$

$$-2\,z2^{(n-1)}\,n - 2 + 3\,z2^{n}\,n + 2\,z2^{n} + z2^{(2\,n-1)}\,n - \frac{3}{2} z2^{(2\,n)}\,n - \frac{1}{2} z2^{(2\,n)} + z2^{(n-2)}\,n^2 - z2^{(n-2)}\,n$$
$$- 2\,z2^{(n-1)}\,n^2 - z2^{(2\,n-2)}\,n^2 + \frac{1}{2} z2^{(2\,n-2)}\,n + 2\,z2^{(2\,n-1)}\,n^2 + z2^{n}\,n^2 - z2^{(2\,n)}\,n^2$$

$$dif2(z_2 = 1) = -0.5$$

Therefore, $m_{z_2=1}\left(R_{z_1}\left[P, Q_1\right]\right) = 2$

Also, as we have seen earlier, $m_{z_2=1}\left(R_{z_1}\left[\underline{Q}_1, Q_1\right]\right) = 2$

Thus, the condition for stability $m_{z_2=1}\left(R_{z_1}\left[P, Q_1\right]\right) > m_{z_2=1}\left(R_{z_1}\left[\underline{Q}_1, Q_1\right]\right)$ does not hold, and the filter is unstable.

Appendix D.

Calculate the following expression (Chapter 4):

$$res\left[\frac{P\underline{P}}{Q^n\underline{Q}^n}\right]_{\underline{Q}=0} = \frac{1}{(n-1)!}\lim_{Q\to 0}\frac{\partial^{n-1}}{\partial Q^{n-1}}\left(\frac{P\underline{P}z_3^p}{Q^n}\right)$$

We will denote $L = P\underline{P}z_3^p$ and $F = \dfrac{L}{Q^n}$

Then

$$\frac{\partial F}{\partial Q} = (-n)LQ^{-n-1}$$

$$\frac{\partial^2 F}{\partial Q^2} = (-n)LQ^{-n-1}$$

$$\frac{\partial^3 F}{\partial Q^3} = (-n)(-n-1)LQ^{-n-2}$$

$$\vdots$$

$$\frac{\partial^{n-1} F}{\partial Q^{n-1}} = (-n)(-n-1)(-n-2)\cdots(-n-(n-2))LQ^{-n-(n-1)}$$

$$= (-1)^{n-1}\frac{(2n-2)!}{(n-1)!}LQ^{-2n+1}$$

Finally,

$$res\left[\frac{P\underline{P}}{Q^n\underline{Q}^n}\right]_{\underline{Q}=0} = (-1)^{n-1}\frac{(2n-2)!}{(n-1)!(n-1)!}\lim_{Q\to 0}\frac{P\underline{P}z_3^p}{Q^{2n-1}}$$

Appendix E.

$$Q = 3 - z_1 z_2 - z_2 z_3 - z_3$$

$$\underline{Q} = 3 z_1 z_2 z_3 - z_3 - z_1 - z_1 z_2$$

$$T(z_1, z_2) = R_{z_3}[\underline{Q}, Q] = 8 z_1 z_2 - 3 - 3 z_1^2 z_2^2 - z_1 - z_1 z_2^2$$

$$\frac{\partial T}{\partial z_2} = 8 z_1 - 6 z_1^2 z_2 - 2 z_1 z_2$$

$$R_{z_2}\left[T, \frac{\partial T}{\partial z_2}\right] = -12 z_1^2 (3 z_1 + 1)(z_1 - 1)^2$$

The multiplicity $m_\alpha\left(R_{z_2}\left[T, \frac{\partial T}{\partial z_2}\right]\right) = 2$

$$P = (1 - z_1 z_2)(1 - z_2 z_3)$$

$$M(z_1, z_2) = R_{z_3}[P, Q] = (1 - z_1 z_2)(-2 z_2 + z_1 z_2^2 + 1)$$

$$R_{z_2}[M, T] = -4 z_1^2 (z_1 - 1)^4 (z_1 + 3)$$

Therefore, in this case multiplicity $m_\alpha(R_{z_2}[M, T])$ =4

Appendix F.

In order to find the most singular term of G_1 (6.1) we substitute $G^* = \dfrac{P}{Q}$ into

$$\begin{aligned}G_1 &= G^* + z_1\frac{\partial G^*}{\partial z_1} + z_2\frac{\partial G^*}{\partial z_2} + z_3\frac{\partial G^*}{\partial z_3} + z_1z_2\frac{\partial^2 G^*}{\partial z_1\partial z_2} + z_1z_3\frac{\partial^2 G^*}{\partial z_1\partial z_3}\\ &+ z_2z_3\frac{\partial^2 G^*}{\partial z_2\partial z_3} + z_1z_2z_3\frac{\partial^3 G^*}{\partial z_1\partial z_2\partial z_3}\end{aligned}$$

and calculate G_1:

$$\begin{aligned}
\mathit{G1} :=\ & \frac{\mathrm{P}(\mathit{z1},\mathit{z2},\mathit{z3})}{\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})} + \frac{\mathit{z3}\left(\frac{\partial}{\partial \mathit{z3}}\mathrm{P}(\mathit{z1},\mathit{z2},\mathit{z3})\right)}{\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})} - \frac{\mathit{z3}\,\mathrm{P}(\mathit{z1},\mathit{z2},\mathit{z3})\left(\frac{\partial}{\partial \mathit{z3}}\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})\right)}{\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})^2}\\
&+ \frac{\mathit{z2}\left(\frac{\partial}{\partial \mathit{z2}}\mathrm{P}(\mathit{z1},\mathit{z2},\mathit{z3})\right)}{\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})} + \frac{\mathit{z2}\,\mathit{z3}\left(\frac{\partial^2}{\partial \mathit{z2}\,\partial \mathit{z3}}\mathrm{P}(\mathit{z1},\mathit{z2},\mathit{z3})\right)}{\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})}\\
&- \frac{\mathit{z2}\,\mathit{z3}\left(\frac{\partial}{\partial \mathit{z2}}\mathrm{P}(\mathit{z1},\mathit{z2},\mathit{z3})\right)\left(\frac{\partial}{\partial \mathit{z3}}\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})\right)}{\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})^2} - \frac{\mathit{z2}\,\mathrm{P}(\mathit{z1},\mathit{z2},\mathit{z3})\left(\frac{\partial}{\partial \mathit{z2}}\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})\right)}{\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})^2}\\
&- \frac{\mathit{z2}\,\mathit{z3}\left(\frac{\partial}{\partial \mathit{z3}}\mathrm{P}(\mathit{z1},\mathit{z2},\mathit{z3})\right)\left(\frac{\partial}{\partial \mathit{z2}}\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})\right)}{\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})^2}\\
&+ 2\,\frac{\mathit{z2}\,\mathit{z3}\,\mathrm{P}(\mathit{z1},\mathit{z2},\mathit{z3})\left(\frac{\partial}{\partial \mathit{z2}}\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})\right)\left(\frac{\partial}{\partial \mathit{z3}}\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})\right)}{\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})^3}\\
&- \frac{\mathit{z2}\,\mathit{z3}\,\mathrm{P}(\mathit{z1},\mathit{z2},\mathit{z3})\left(\frac{\partial^2}{\partial \mathit{z2}\,\partial \mathit{z3}}\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})\right)}{\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})^2} + \frac{\mathit{z1}\left(\frac{\partial}{\partial \mathit{z1}}\mathrm{P}(\mathit{z1},\mathit{z2},\mathit{z3})\right)}{\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})}\\
&+ \frac{\mathit{z1}\,\mathit{z3}\left(\frac{\partial^2}{\partial \mathit{z1}\,\partial \mathit{z3}}\mathrm{P}(\mathit{z1},\mathit{z2},\mathit{z3})\right)}{\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})} - \frac{\mathit{z1}\,\mathit{z3}\left(\frac{\partial}{\partial \mathit{z1}}\mathrm{P}(\mathit{z1},\mathit{z2},\mathit{z3})\right)\left(\frac{\partial}{\partial \mathit{z3}}\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})\right)}{\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})^2}\\
&+ \frac{\mathit{z1}\,\mathit{z2}\left(\frac{\partial^2}{\partial \mathit{z2}\,\partial \mathit{z1}}\mathrm{P}(\mathit{z1},\mathit{z2},\mathit{z3})\right)}{\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})} + \frac{\mathit{z1}\,\mathit{z2}\,\mathit{z3}\left(\frac{\partial^3}{\partial \mathit{z2}\,\partial \mathit{z1}\,\partial \mathit{z3}}\mathrm{P}(\mathit{z1},\mathit{z2},\mathit{z3})\right)}{\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})}\\
&- \frac{\mathit{z1}\,\mathit{z2}\,\mathit{z3}\left(\frac{\partial^2}{\partial \mathit{z2}\,\partial \mathit{z1}}\mathrm{P}(\mathit{z1},\mathit{z2},\mathit{z3})\right)\left(\frac{\partial}{\partial \mathit{z3}}\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})\right)}{\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})^2}\\
&- \frac{\mathit{z1}\,\mathit{z2}\left(\frac{\partial}{\partial \mathit{z1}}\mathrm{P}(\mathit{z1},\mathit{z2},\mathit{z3})\right)\left(\frac{\partial}{\partial \mathit{z2}}\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})\right)}{\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})^2}\\
&- \frac{\mathit{z1}\,\mathit{z2}\,\mathit{z3}\left(\frac{\partial^2}{\partial \mathit{z1}\,\partial \mathit{z3}}\mathrm{P}(\mathit{z1},\mathit{z2},\mathit{z3})\right)\left(\frac{\partial}{\partial \mathit{z2}}\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})\right)}{\mathrm{Q}(\mathit{z1},\mathit{z2},\mathit{z3})^2}
\end{aligned}$$

$$+2\,\frac{z1\,z2\,z3\left(\frac{\partial}{\partial z1}\mathrm{P}(z1,z2,z3)\right)\left(\frac{\partial}{\partial z2}\mathrm{Q}(z1,z2,z3)\right)\left(\frac{\partial}{\partial z3}\mathrm{Q}(z1,z2,z3)\right)}{\mathrm{Q}(z1,z2,z3)^3}$$

$$-\frac{z1\,z2\,z3\left(\frac{\partial}{\partial z1}\mathrm{P}(z1,z2,z3)\right)\left(\frac{\partial^2}{\partial z2\,\partial z3}\mathrm{Q}(z1,z2,z3)\right)}{\mathrm{Q}(z1,z2,z3)^2}-\frac{z1\,\mathrm{P}(z1,z2,z3)\left(\frac{\partial}{\partial z1}\mathrm{Q}(z1,z2,z3)\right)}{\mathrm{Q}(z1,z2,z3)^2}$$

$$-\frac{z1\,z3\left(\frac{\partial}{\partial z3}\mathrm{P}(z1,z2,z3)\right)\left(\frac{\partial}{\partial z1}\mathrm{Q}(z1,z2,z3)\right)}{\mathrm{Q}(z1,z2,z3)^2}$$

$$+2\,\frac{z1\,z3\,\mathrm{P}(z1,z2,z3)\left(\frac{\partial}{\partial z1}\mathrm{Q}(z1,z2,z3)\right)\left(\frac{\partial}{\partial z3}\mathrm{Q}(z1,z2,z3)\right)}{\mathrm{Q}(z1,z2,z3)^3}-\frac{z1\,z3\,\mathrm{P}(z1,z2,z3)\,\%1}{\mathrm{Q}(z1,z2,z3)^2}$$

$$-\frac{z1\,z2\left(\frac{\partial}{\partial z2}\mathrm{P}(z1,z2,z3)\right)\left(\frac{\partial}{\partial z1}\mathrm{Q}(z1,z2,z3)\right)}{\mathrm{Q}(z1,z2,z3)^2}$$

$$-\frac{z1\,z2\,z3\left(\frac{\partial^2}{\partial z2\,\partial z3}\mathrm{P}(z1,z2,z3)\right)\left(\frac{\partial}{\partial z1}\mathrm{Q}(z1,z2,z3)\right)}{\mathrm{Q}(z1,z2,z3)^2}$$

$$+2\,\frac{z1\,z2\,z3\left(\frac{\partial}{\partial z2}\mathrm{P}(z1,z2,z3)\right)\left(\frac{\partial}{\partial z1}\mathrm{Q}(z1,z2,z3)\right)\left(\frac{\partial}{\partial z3}\mathrm{Q}(z1,z2,z3)\right)}{\mathrm{Q}(z1,z2,z3)^3}$$

$$-\frac{z1\,z2\,z3\left(\frac{\partial}{\partial z2}\mathrm{P}(z1,z2,z3)\right)\%1}{\mathrm{Q}(z1,z2,z3)^2}$$

$$+2\,\frac{z1\,z2\,\mathrm{P}(z1,z2,z3)\left(\frac{\partial}{\partial z1}\mathrm{Q}(z1,z2,z3)\right)\left(\frac{\partial}{\partial z2}\mathrm{Q}(z1,z2,z3)\right)}{\mathrm{Q}(z1,z2,z3)^3}$$

$$+2\,\frac{z1\,z2\,z3\left(\frac{\partial}{\partial z3}\mathrm{P}(z1,z2,z3)\right)\left(\frac{\partial}{\partial z1}\mathrm{Q}(z1,z2,z3)\right)\left(\frac{\partial}{\partial z2}\mathrm{Q}(z1,z2,z3)\right)}{\mathrm{Q}(z1,z2,z3)^3}$$

$$-6\,\frac{z1\,z2\,z3\,\mathrm{P}(z1,z2,z3)\left(\frac{\partial}{\partial z1}\mathrm{Q}(z1,z2,z3)\right)\left(\frac{\partial}{\partial z2}\mathrm{Q}(z1,z2,z3)\right)\left(\frac{\partial}{\partial z3}\mathrm{Q}(z1,z2,z3)\right)}{\mathrm{Q}(z1,z2,z3)^4}$$

$$+2\,\frac{z1\,z2\,z3\,\mathrm{P}(z1,z2,z3)\,\%1\left(\frac{\partial}{\partial z2}\mathrm{Q}(z1,z2,z3)\right)}{\mathrm{Q}(z1,z2,z3)^3}$$

$$+2\,\frac{z1\,z2\,z3\,\mathrm{P}(z1,z2,z3)\left(\frac{\partial}{\partial z1}\mathrm{Q}(z1,z2,z3)\right)\left(\frac{\partial^2}{\partial z2\,\partial z3}\mathrm{Q}(z1,z2,z3)\right)}{\mathrm{Q}(z1,z2,z3)^3}$$

$$-\frac{z1\,z2\,\mathrm{P}(z1,z2,z3)\left(\frac{\partial^2}{\partial z2\,\partial z1}\mathrm{Q}(z1,z2,z3)\right)}{\mathrm{Q}(z1,z2,z3)^2}$$

$$-\frac{z1\,z2\,z3\left(\frac{\partial}{\partial z3}\mathrm{P}(z1,z2,z3)\right)\left(\frac{\partial^2}{\partial z2\,\partial z1}\mathrm{Q}(z1,z2,z3)\right)}{\mathrm{Q}(z1,z2,z3)^2}$$

$$+2\,\frac{z1\,z2\,z3\,\mathrm{P}(z1,z2,z3)\left(\frac{\partial^2}{\partial z2\,\partial z1}\mathrm{Q}(z1,z2,z3)\right)\left(\frac{\partial}{\partial z3}\mathrm{Q}(z1,z2,z3)\right)}{\mathrm{Q}(z1,z2,z3)^3}$$

$$-\frac{z1\,z2\,z3\,\mathrm{P}(z1,z2,z3)\left(\frac{\partial^3}{\partial z2\,\partial z1\,\partial z3}\mathrm{Q}(z1,z2,z3)\right)}{\mathrm{Q}(z1,z2,z3)^2}$$

$$\%1 := \frac{\partial^2}{\partial z1\,\partial z3}\mathrm{Q}(z1,z2,z3)$$

We observe that the term

$$\frac{z_1 z_2 z_3 P(z_1,z_2,z_3)}{Q(z_1,z_2,z_3)^4}\frac{\partial Q(z_1,z_2,z_3)}{\partial z_1}\frac{\partial Q(z_1,z_2,z_3)}{\partial z_2}\frac{\partial Q(z_1,z_2,z_3)}{\partial z_3}$$

contains Q in the highest degree in the denominator, therefore it can be considered the most singular term of the expression.

Appendix G.

$$G(z_1, z_2, z_3) = \frac{(1-z_3)^n}{3-z_1-z_2-z_3} = \frac{P^n}{Q}$$

$$P = 1 - z_3$$

$$\underline{Q} = 3 - z_1 - z_2 - z_3$$

$$\underline{\underline{Q}} = 3z_1z_2z_3 - z_2z_3 - z_1z_2 - z_1z_3$$

$$T = R_{z_3}[\underline{Q}, Q] = 10z_1z_2 - 3z_2 - 3z_1 - 3z_1^2z_2 + z_1^2 - 3z_1z_2^2 + z_2^2$$

$$\frac{\partial T}{\partial z_2} = 10z_1 - 3 - 3z_1^2 - 6z_1z_2 + 2z_2$$

$$R_{z_2}[T, \frac{\partial T}{\partial z_2}] = 3(3z_1 - 1)^2(z_1 - 3)(z_1 - 1)^2$$

Then $m_{z_1=1}\left(R_{z_2}\left[T, \frac{\partial T}{\partial z_2}\right]\right) = 2$

$$M = R_{z_3}[P, Q] = R_{z_3}[1 - z_3, 3 - z_1 - z_2 - z_3] = z_1 + z_2 - 2$$

$$R_{z_2}[M, T] = -2(z_1 - 1)^2$$

and $nm_{z_1=1}(R_{z_2}[M, T]) = 2n$

Appendix H.

$$G(z_1,z_2,z_3)=\frac{3-z_1z_3-z_2z_3-z_1z_2}{(3-z_1-z_2-z_3)^n}$$

$$P=3-z_1z_3-z_2z_3-z_1z_2$$

$$Q=3-z_1-z_2-z_3$$

As in Example 6.1,

$$\underline{Q}=3z_1z_2z_3-z_2z_3-z_1z_2-z_1z_3$$

$$T=R_{z_3}[\underline{Q},Q]=10z_1z_2-3z_2-3z_1-3z_1^2z_2+z_1^2-3z_1z_2^2+z_2^2$$

$$\frac{\partial T}{\partial z_2}=10z_1-3-3z_1^2-6z_1z_2+2z_2$$

$$R_{z_2}[T,\frac{\partial T}{\partial z_2}]=3(3z_1-1)^2(z_1-3)(z_1-1)^2$$

Then $m_{z_1=1}\left(R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]\right)=2$

$$M=R_{z_3}[P,Q]=3+z_1z_2-3z_1-3z_2+z_1^2+z_2^2$$

$$R_{z_2}[M,T]=9(z_1-1)^6$$

$$m_{z_1=1}(R_{z_2}[M,T])=6$$

Appendix I.

Let

$$G(z_1,z_2,z_3)=\frac{(1-z_1)^k(1-z_2)^r(1-z_3)^s}{(1-az_1-bz_2-cz_3)^n}$$

where $a+b+c=1,\ \ a>0, b>0, c>0,\ \ k+r+s>0$

$$Q=1-az_1-bz_2-(1-a-b)z_3$$

$$\underline{Q}=z_1z_2z_3-az_2z_3-bz_1z_3-z_1z_2+az_1z_2+bz_1z_2$$

$$T=R_{z_3}[\underline{Q},Q]=-az_2-bz_1-az_1^2z_2+abz_1^2-bz_1z_2^2+abz_2^2+2az_1z_2+2bz_1z_2-2abz_1z_2$$

$$\frac{\partial T}{\partial z_2}=-a-az_1^2-2bz_1z_2+2abz_2+2az_1+2bz_1-2abz_1$$

$$R_{z_2}[T,\frac{\partial T}{\partial z_2}]=ab(z_1-1)^2(z_1-a)(az_1^2-4bz_1-2az_1+4b^2z_1+4abz_1+a)$$

Then $m_{z_1=1}\left(R_{z_2}\left[T,\frac{\partial T}{\partial z_2}\right]\right)=2$

$$P=(1-z_1)^k(1-z_2)^r(1-z_3)^s=P_1^kP_2^rP_3^s$$

$$M=R_{z_3}[P,Q]=R_{z_3}[P_1^k,Q]R_{z_3}[P_2^r,Q]R_{z_3}[P_3^s,Q]=M_1^kM_2^rM_3^s$$

$$m_{z_1=1}(R_{z_2}[M,T])=m_{z_1=1}(R_{z_2}[M_1^k,T])+m_{z_1=1}(R_{z_2}[M_2^r,T])+m_{z_1=1}(R_{z_2}[M_3^s,T])$$
$$=kR_{z_2}[M_1,T]+rR_{z_2}[M_2,T]+sR_{z_2}[M_3,T]$$

$$M_1=R_{z_3}[P_1,Q]=1-z_1$$

$$M_2=R_{z_3}[P_2,Q]=1-z_2$$

$$M_3=R_{z_3}[P_3,Q]=az_1+bz_2-a-b$$

$R_{z_2}[M_1,T]=(z_1-1)^2$

$R_{z_2}[M_2,T]=a(b-1)(z_1-1)^2$

$R_{z_2}[M_3,T]=ab(a+b)(a+b-1)(z_1-1)^2$

Therefore,

$$m_{z_1=1}(R_{z_2}[M,T])=2(k+r+s)$$

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