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AN INVESTIGATION OF ALTERNATIVE DECISION RULE MODELS FOR  
PRODUCTION PLANNING UNDER CHANCE-CONSTRAINED SALES

*City University of New York*

PH.D.

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AN INVESTIGATION OF ALTERNATIVE DECISION RULE MODELS  
FOR PRODUCTION PLANNING UNDER CHANCE-CONSTRAINED SALES

by

JOHN FRANK AFFISCO

A dissertation submitted to the Graduate Faculty in  
Business in partial fulfillment of the requirements  
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1980

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This manuscript has been read and accepted for the Graduate Faculty in Business in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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## Abstract

### AN INVESTIGATION OF ALTERNATIVE DECISION RULE MODELS FOR PRODUCTION PLANNING UNDER CHANCE-CONSTRAINED SALES

by

John F. Affisco

Adviser: Professor Georghios P. Sphicas

This study deals with models to determine aggregate production plans when future demand is of a known stochastic nature. Specifically it proceeds along two directions. First, existing decision rule models, in which production depends solely upon previously realized sales feedback elements and an adjustment variable, which had been tested only under normal stationary demand are tested under exponential stationary demand. Second, two new forms of production decision rules are proposed and the corresponding decision rule models are formulated for the cases of normal and two parameter exponential demand. Both forms include, in addition to previous sales feedback elements, present forecasted demand elements.

The performance of these models under trend and seasonal demand patterns is investigated in a series of simulation studies. For each demand type two random sets of fifty replicates of demand over the respective planning horizon are generated. In each case the first set is used to select the best parameter value for the Type 1 quadratic programming

model. Then the second set is used to choose the best from the competing models with the previously determined parametric value. Finally, for each pattern of demand, production plans determined through the best stochastic decision rule model are compared to plans developed by the HMMS Linear Decision Rule. These comparisons are effected through the use of the Wilcoxon Matched-Pairs Signed-Ranks Test.

For both normal trend and seasonal demand with a 95% service level the stochastic decision rule models proved to be most competitive with the HMMS approach. In all three exponential demand cases the HMMS results significantly outperform the best decision rule models when the service level is 0.95. However when a less stringent service level of 75% was assumed in the stationary case the best decision rule model compared favorably to the HMMS approach.

Further research into the implications of service level for model performance should tell whether the stochastic decision rule approach is an acceptable technique for aggregate planning for exponential demand patterns. In addition, when demand is normally distributed future studies should concentrate on possible horizon effects in the best decision rule models. Ultimately these results should aid the systems analyst in selecting a model for aggregate production planning based upon the nature of the demand that is present.

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## CHAPTER I

### INTRODUCTION

#### The Problem

Many manufacturing systems are confronted with the critical problem of rationally determining production and work force levels in response to changing patterns in demand over a mid-range planning horizon. The view as to the best strategy to accomplish this depends on functional position in the organization. The production manager would like to have long uninterrupted production runs of items to lower production costs; marketing wishes to have large stocks of inventory so as to be able to satisfy all demand quickly; financial officers desire to hold minimal inventories to avoid the opportunity cost associated with holding higher levels; and finally, personnel wishes to have a stable labor force. It is evident that these goals are in conflict and that optimization of any one alone will lead to suboptimization of the overall problem. Therefore in order to deal with this fact the systems approach will be used in the analysis of this problem. The realization of the necessity for this approach to the aggregate production planning problem came as early as 1954 when it was discussed by Simon and Holt [6].

The strategies available to the decision maker are

constrained by the nature of the mid-range planning horizon. Over this period plant and equipment are considered fixed and thus fluctuations in demand may not be met by changes in physical capacity. Buffa and Taubert [1] suggest the following alternative pure strategies to adjust to fluctuations in demand: (1) adjust the size of the work force by hiring and firing in response to the fluctuations; (2) adjust the production rate by working overtime or undertime with the same work force; (3) absorb fluctuations through fluctuations in inventory, backlog of orders, and some lost sales; (4) increase or decrease the aggregate amount of subcontracting to absorb the fluctuations in demand; and (5) combinations of these four pure strategies. Each of these pure strategies have unique costs associated with them. Changing the size of the work force requires hiring and firing and the attendant costs involved. Adjusting the production rate with a fixed labor force may involve paying an overtime rate or the penalty of idle labor for undertime. Using inventories to absorb fluctuations involves the usual holding and backorder costs in addition to the cost of some lost sales. Finally subcontracting costs are quite complicated and include the costs of administering the contract.

In only very rare instances will the implementation of any one specific pure strategy result in a program which is economically optimal. Thus it is the nature of the aggregate planning problem that any successful model will determine some optimal combination of pure strategies. This optimal program

will consist of a set of dynamic, interdependent production and work force decisions usually for a six to a twelve month planning horizon. This research will be concerned with determining this optimal aggregate planning program when the probability distributions associated with demand in all future periods are known in advance. Specifically, alternative stochastic programming models will be developed and tested against existing models under various demand patterns in order to ascertain which models perform better in given situations.

These results will provide us with information as to the extent and limits of applicability of some existing models. Further they may give us some indication as to the relevance of demand characteristics as a criterion for model selection.

#### Review of the Literature

##### Linear Decision Rule Approach

The work presented by Holt, Modigliani, Muth, and Simon (HMMS) [4] remains even today a landmark contribution to the literature of aggregate production planning. HMMS formulated the aggregate production planning problem as a quadratic program which on solution generated a set of two Linear Decision Rules (LDRs), one each for production level and work force level, which guaranteed an optimal solution with respect to a quadratic cost function as the criterion. Their objective function was constructed on the assumption that a mixed strategy containing elements of production level changes, work

force changes, and inventory fluctuations would result in a policy superior to any one of these pure strategies executed independently. The technique was applied to several real industrial situations with some success and has, since its publication, become the standard of comparison for all subsequent work in the field.

### Model Formulation

HMMS define the variables whose scheduling constitute the decision problem as aggregate production rate and work force. Aggregate production rate is production per unit of time of a common unit for quantities of different products. Work force refers to the number of employees to whom a company is committed to supply regular work for one unit of time. These decisions are assumed to be made at regular time intervals, usually monthly, rather than continuously or intermittently, and the costs are expressed as costs per month. The following costs are related to the three alternative ways of absorbing order fluctuations and are hypothesized to be generally quadratic in nature.

Payroll, Hiring, and Layoff Costs. When order fluctuations are absorbed by monthly work force level adjustments a commitment to pay these employees at their regular time wages for a month is implied. This may be represented algebraically by the following linear function,

$$\text{Regular Payroll Cost} = c_1 W_t \quad (1.1)$$

where  $W_t$  represents the work force level in month  $t$  and  $c_1$  is a constant.

Now we must consider other labor costs associated with changes in the size of the work force. The cost of hiring and training rises with the number hired while the cost of laying off workers derives from terminal pay, reorganization etc., and rises with the number of workers laid off. The cost incurred each month depends on the change in the size of the work force between successive months and is a direct function of the size of the change. Because of this we may represent these costs algebraically by the following quadratic function,

$$\text{Cost of hiring and layoffs} = c_2(W_t - W_{t-1})^2 \quad (1.2)$$

where  $(W_t - W_{t-1})$  represents the monthly change in work force level and  $c_2$  is a constant.

Overtime Costs. When order fluctuations are absorbed by increasing and decreasing production without changing the work force, then overtime and idletime costs are incurred. Overtime involves wage payments at an hourly rate that is generally fifty percent higher than regular wages. Idle time cost is reflected in the fact that labor paid for at the regular rate is not being utilized for productive activities.

The cost of overtime depends on the scheduled production volume and the maximum capacity for production at straight time of a given work force. We may express this capacity as a linear function of the work force,  $kW_t$ , where  $k$  is a constant reflecting the average worker productivity. Any amount of production in excess of  $kW_t$  will incur overtime with the level of overtime required increasing directly with production.

This would indicate that overtime costs should be some linear function of the difference between scheduled production and capacity for a given month. However, this formulation neglects the possibilities of discontinuities and random disturbances in the production process. These situations may arise from nonhomogeneity of the work force, machine breakdowns, quality control problems, and bottleneck operations. The effect of these random disturbances is to smooth the proposed linear overtime cost to one that is quadratic in nature. The final algebraic form of these costs may be expressed as follows,

$$\text{Expected Overtime Cost} = c_3(P_t - c_4W_t)^2 + c_5P_t - c_6W_t \quad (1.3)$$

where as production,  $P_t$ , exceeds  $c_4W_t$ , a level set by the size of the work force, overtime costs increase. In addition,  $c_3$ ,  $c_4$ ,  $c_5$ , and  $c_6$  are constants while the linear terms  $c_5P_t$  and  $c_6W_t$  are added to improve the approximation.

Inventory, Backorder, and Machine Setup Costs. When order fluctuations are absorbed by adjustments in inventory and backorders, other costs are incurred. For instance, holding smaller aggregate inventories requires a decrease in the average production batch size to maintain a balanced inventory. One implication of this policy is a large number of smaller batches and the accompanying increase in machine setup costs. In addition there are the normal costs associated with holding inventory and stocking out. It is known from the literature on inventory control that both optimal batch size and optimal safety stock increase with the square root of demand. Thus, the optimal aggregate inventory must

increase with demand which HMMS express in terms of aggregate order rate (total shipments ordered per month). Further, the total expected back orders corresponding to any given size of inventory must also increase with aggregate order rate. Defining net inventory to be the difference between month ending aggregate inventory and back orders, it follows that the optimal net inventory level is a direct function of the aggregate order rate. This may be approximated by the following function,

$$\text{Optimal Net Inventory} = c_8 + c_9 S_t \quad (1.4)$$

where  $c_8$  and  $c_9$  are constants and  $S_t$  is the aggregate order rate.

When the actual net inventory held is greater than the optimal level, costs increase due to rising holding costs and other costs associated with high inventory levels. As net inventory falls below the optimal level, costs rise due to increased machine setups, increased backorders, and the decreased inventory. Over a range, these inventory-related costs may be approximated adequately by the following quadratic relationship,

$$\begin{aligned} \text{Expected Inventory, Backorder, and Setup Costs} = \\ c_7 (I_t - (c_8 + c_9 S_t))^2 \end{aligned} \quad (1.5)$$

where  $I_t$  is the month ending net inventory,  $(c_8 + c_9 S_t)$  is the optimal net inventory previously discussed and  $c_7$  is a constant.

Final Completed Model. A total cost function may now be constructed that adds together all the component costs

across the complete planning horizon. This function may be formally stated as

$$C_T = \sum_{t=1}^T C_t \quad (1.6)$$

where:

$$C_t = \left\{ \begin{array}{ll} c_1 W_t & \text{Regular Payroll Costs} \\ c_2 (W_t - W_{t-1})^2 & \text{Hiring and Layoff Costs} \\ c_3 (P_t - c_4 W_t)^2 + c_5 P_t - c_6 W_t & \text{Overtime Costs} \\ c_7 (I_t - (c_8 + c_9 S_t))^2 & \text{Inventory Connected Costs} \end{array} \right\} \quad (1.7)$$

The decision problem can now be stated as one of determining production and work force levels over the horizon T that minimize the cost function  $C_T$  subject to the following constraints,

$$I_{t-1} + P_t - S_t = I_t \quad t=1,2,\dots,T \quad (1.8)$$

which represent monthly inventory balances.

### The Paint Factory Application

The HMMS decision model can be applied to the aggregate planning decisions of various factories simply by determining the appropriate values of the cost parameters  $c_1, c_2, \dots, c_9$ . For the paint factory example discussed in detail in 4 the numerical values of these parameters were derived from statistical estimates based on accounting data together with subjective estimates of such intangible costs as delayed shipments to customers. Various techniques such as curve fitting and nonlinear regression were utilized to fit the hypothesized

quadratic functions to the cost relations. This resulted in the following values for the cost coefficients in the paint factory:

$$\begin{aligned} c_1=340.0 & \quad c_2=64.3 & \quad c_3=0.2 & \quad c_4=5.67 & \quad c_5=51.2 \\ c_6=281.0 & \quad c_7=0.0825 & \quad c_8=320.0 & \quad c_9=0.0 \end{aligned}$$

Thus the final form of the cost function for the paint factory is,

$$C_T = \sum_{t=1}^T \left[ 340W_t + 64.3(W_t - W_{t-1})^2 + 0.2(P_t - 5.67W_t)^2 + 51.2P_t - 281W_t + 0.0825(I_t - 320)^2 \right] \quad (1.9)$$

where  $C_T$  is the total cost for  $T$  months expressed in dollars,  $W_t$  is the work force for month  $t$  expressed in men,  $P_t$  is the production in gallons per month, and  $I_t$  is net inventory in gallons.

The solution of this decision problem for the paint factory was obtained by partially differentiating  $C_T$  with respect to each independent decision variable and solving the resulting set of linear equations by matrix inversion. The solution is in the form of the following two linear decision rules for determining production and work force levels,

$$\begin{aligned} P_t = & +0.458S_t + 0.233S_{t+1} + 0.111S_{t+2} + 0.014S_{t+4} - 0.001S_{t+5} \\ & - 0.007S_{t+6} - 0.008S_{t+7} - 0.008S_{t+8} - 0.007S_{t+9} - 0.005S_{t+10} \\ & - 0.004S_{t+11} + 1.005W_{t-1} + 153.0 - 0.464I_{t-1} \end{aligned} \quad (1.10)$$

$$\begin{aligned} W_t = & +0.0101S_t + 0.0088S_{t+1} + 0.0071S_{t+2} + 0.0055S_{t+3} \\ & + 0.0042S_{t+4} + 0.0031S_{t+5} + 0.0022S_{t+6} + 0.0016S_{t+7} \\ & + 0.0011S_{t+8} + 0.0008S_{t+9} + 0.0005S_{t+10} + 0.0004S_{t+11} \end{aligned}$$

$$+0.742W_{t-1}+2.09-0.010I_{t-1} \quad (1.11)$$

where:  $P_t$  = Production volume for the forthcoming month,  $t$ ,

$W_t$  = Number of employees required for the current month,  $t$ ,

$W_{t-1}$  = Number of employees in work force at end of previous month,  $t-1$ ,

$S_t$  = Forecast of number of units ordered for shipment during current month,  $t$ ,

$S_{t+1}$  = The same as above for the next month,  $t+1$ , etc.

There is a fairly complex interaction between these two decision rules. The production of one month affects the net inventory position at the end of the month. This in turn influences the employment decision in the second month which influences the production in the third month. From an overall perspective, the weights that are applied to the sales forecasts and feedback factors dictate that work force responds strongly to orders in the immediate future and to the inventory position. Thus, there is a continual complex dynamic between the two decisions.

In order to test the performance of the decision rules HMMS performed some experiments using sales forecast data for the period 1949-1953. The decision rules were applied to two sets of forecasts; the first a perfect forecast using actual values of demand; and the second, a moving average forecast. Cost performance based on these two forecasts were then compared to the actual performance of the decisions made by production executives for the period. The LDRs with the perfect forecast

outperformed the company decisions by 39 percent while the LDRs with the moving average forecast outperformed the company decisions by 29 percent. It is important to make note of the fact that the cost saving attributable to the decision rules (29%) is greater than the cost saving attributable to the complete elimination of forecast error(10%).

When faced with these results the authors felt that the magnitude of potential savings may have been affected by the large and unpredictable fluctuations in orders during the Korean War period. In order to control for this contingency the moving average experiment was repeated for the years 1952-1954, considered to be a more normal period. For this sample actual factory cost performance exceeded that of the decision rules with moving average forecasts by 8.5 percent or 51,000 dollars per year on average.

As in any mathematical modelling endeavor the LDR approach rests on a set of assumptions. These range from a constant underlying quadratic cost structure over the planning horizon to the ability to aggregate product in terms of a common unit. Two assumptions of interest to this research are those of deterministic demand and complete satisfaction of orders either through current stock or backordering. The HMMS formulation requires demand over time to be given as a set of point estimates. This seems to neglect the fact that in most applied situations demand contains stochastic elements. Further, by the nature of the uncertainty introduced, a choice to stockout without backordering may be a viable alternative

strategy for some time periods. Various approaches have been proposed in the literature to begin to deal with the ramifications of these shortcomings of the HMMS model. Of particular interest to this research are those which apply stochastic programming to the basic formulation of HMMS.

### Stochastic Programming Approaches

#### Reliability Programming With a Zero Order Rule

Sengupta and Portillo-Campbell (SPC) [5] developed a model based on a chance-constrained interpretation for the probability of supply falling short of demand. In addition to the HMMS decision variables their model allows for the optimal solution of the tolerance measures for the chance constraints and the resulting tradeoff between optimum cost and system reliability.<sup>1</sup>

The SPC approach begins with the HMMS formulation and replaces the material balance constraint with a chance constraint leading to the following structure,

$$\text{Min } \sum_{t=1}^T [(c_1 - c_6)W_t + c_2(W_t - W_{t-1})^2 + c_3(P_t - c_4W_t)^2]$$

---

<sup>1</sup>This formulation differs from the typical chance-constrained programming approach by (i) treating the tolerance measures for the chance constraints as decision variables which may be optimally solved for instead of as preassigned parameter values, (ii) including in the objective function a measure of the system reliability which depends on the tolerance levels for each chance constraint. The reliability measure relates the degree to which fluctuations in demand over time can be successfully met by the optimal solution. Formulation in this form further allows the determination of the cost of preassigning a set of tolerance measures and the cost of improving them.

$$+c_5 P_t + c_7 (I_t - c_8 - c_9 \bar{S}_t)^2 \} \quad (1.12)$$

subject to:

$$\text{Prob}(P_t + I_{t-1} \geq S_t) \geq u_t \quad t=1,2,\dots,T \quad (1.13)$$

$$0 \leq u_t \leq 1, I_0 \text{ (initial Inventory) given,}$$

where  $\bar{S}_t$  is the expected value of demand in period  $t$ . Here the chance constraint (1.13) has the following interpretation: supply in each period (i.e. production plus inventory from the last period) must be at least at the level of demand  $S_t$  with a very high probability, say  $u_t=0.95$ . Thus  $u_t$  serves as a measure of tolerance of the chance constraint which indicates the level of reliability of the production policy followed.

SPC assumed demand,  $S$ , to be a random variable with a two parameter exponential density function which is unchanged over time,

$$f(S) = F(S, \mu, S_0) = \mu \exp(-\mu(S - S_0)) \quad S_0 \leq S < \infty \quad (1.14)$$

and  $\mu = 1/(\bar{S}_t - S_0)$ . The authors explain that the use of this exponential distribution is motivated by the facts that: (1) it provides under existing conditions a limiting distribution in the class of distributions with a nonnegative domain<sup>2</sup> and (2) it retains the constraint linearity of the model after the

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<sup>2</sup>Since the authors treat the chance constraint for each period as a separate component of the total system they use a reliability measure which is a function of the tolerances of these constraints. Under these and some other general conditions it has been shown that the exponential distribution is the limiting distribution in the class of distributions with a nonnegative domain. A further explanation of this phenomena may be found in Weiss, G.H. "A Survey of Some Mathematical Models in the Theory of Reliability," Statistical Theory of Reliability, M. Zelen Ed., University of Wisconsin Press, 1963.

reliability measures are incorporated thus maintaining the quadratic programming structure.

The modelling proceeds by using the distribution assumption to develop deterministic equivalents for the chance constraint (1.13) and to develop a new objective function which reflects a penalty for the violation of the chance constraints. The particular form of the model is not critical to this research therefore it is not presented here in detail. It is sufficient to say that upon solution this model gives optimal production, work force, and tolerance levels for each month in the planning horizon. The model was solved for various initial condition and parameter combinations including the a priori setting of tolerance limits,  $u_t$ . Under this condition of preset service levels the SPC model essentially reduces to a stochastic generalization of the HMMS model. This complete procedure was repeated under the assumption of normally distributed sales. Results showed that when  $u_t$  was specified prior to solution, the normal assumption gave appreciably lower optimal costs than the exponential assumption.

The primary contribution to the literature on aggregate production planning of this work is that it showed stochastic programming to be a utilizable tool for dealing with stochastic demand in a production planning framework. Further, it points out that a blanket assumption of normality as an approximation for demand may be extremely crude at times if the true underlying distribution departs significantly from normality. Two other important limitations of the SPC research must be

mentioned. First, the assumption of identically distributed demand over time would seem to limit the discussion to stationary demand patterns and not explicitly consider patterns including trend and seasonal effects. Second, both the HMMS original material balance constraint (1.8) and the chance constraint based upon it (1.13) are zero-order decision rules. Thus the level of inventory  $I_t$  in (1.8) is defined only when the true value of sales,  $S_t$ , is observed.

### Reliability Programming with Induced Randomness

The assumption of a zero-order decision rule in previous research implies that the decision variables which are dependent on random sales variables have no characteristics of induced randomness. That is, the inventory variables behave independent of production and sales variables. Further, this assumption ignores the fact that variables such as inventory, production, and sales are dynamic and hence evolve over time. Thus, the chance constraint relationship must be validated over successive periods on a cumulative basis throughout the planning horizon.

Gupta and Sengupta (GS) [2] propose exact and approximate models for the aggregate production planning problem which relax the assumption of zero-order rules and allow for induced randomness. The exact model is achieved by beginning with the zero-order relation,

$$\text{Prob}(P_t + I_{t-1} \geq S_t) \geq u_t \quad t=1,2,\dots,T \quad (1.13)$$

and the following cumulative linear rule,

$$I_{t-1} = I_0 + \sum_{i=1}^{t-1} P_i - \sum_{i=1}^{t-1} S_i \quad (1.16)$$

and proceeds by substituting (1.16) into (1.13) for  $I_{t-1}$  giving,

$$\text{Prob}(I_0 + \sum_{i=1}^{t-1} P_i - \sum_{i=1}^{t-1} u_i \geq 0) \geq u_t \quad t=1,2,\dots,T. \quad (1.17)$$

The original chance constraint (1.13) is then replaced by (1.17) which allows for a more meaningful and realistic interpretation. The approximate model is obtained by replacing the random sales variable by mean sales, which is known a priori, and enforcing the additional restriction,

$$I_t = I_{t-1} + P_t - \bar{S} \quad (1.18)$$

which implies an average inventory at the end of period  $t$ .

Approximate and exact models are constructed for identical two parameter exponential and identical normal distribution cases. These models are transformed into reliability programming forms by developing deterministic equivalents for the chance constraint and including a reliability penalty function. The process here is much akin to the procedure in Sengupta and Portillo-Campbell [5]. The resulting models are solved for various combinations of parameters and initial conditions. A comparison between these results and the SPC solutions, with the zero-order rule, show the GS linear rules to be a more realistic and meaningful form. Specifically the SPC approach shows tendencies toward production at levels less than minimum demand and inventories that are much above levels directed by average sales conditions. The GS approach, however, shows more realistic production and inventory levels when compared to minimum and average sales parameters per period. The major explanation of this outcome is that the

lack of induced randomness in the zero-order rule made inventory independent of random sales.

The primary contribution to the field of this research is its recognition that the material balance constraints have their own dynamic which is critical in terms of the monthly production decision and derived inventory value. By substituting their cumulative linear rule for the zero-order rule, Gupta and Sengupta have created a more viable and realistic stochastic model for the aggregate production planning problem. However, this approach still is limited by the assumption of a stationary demand pattern.

#### Decision Rule Model Approach

Introduction. Gupta and Sengupta [3] propose an alternative stochastic programming formulation for the aggregate planning problem which provides for a feed back mechanism for updating of data. This is accomplished by using decision rules of the form

$$P_t = f(S_{t-1}, S_{t-1}, \theta) \quad (1.19)$$

$$W_t = f(P_t, W_{t-1}, \theta). \quad (1.20)$$

Here,  $\theta$  contains unknowns which may either vary from period to period or remain fixed. It is assumed that the realized sales data in periods  $t-1$  and  $t-2$  are known to the decision maker at time  $t$  when he decides on a production level. Likewise,  $P_t$  and  $W_{t-1}$  are known before he decides on the work force size at time  $t$ .

Formulation. The authors' preliminary statement of the problem is as follows

$$\begin{aligned} \text{Min } J = & \sum_{t=1}^T \left[ (c_1 - c_6)W_t + c_2(W_t - W_{t-1})^2 + c_3(P_t - c_4W_t)^2 \right. \\ & \left. + c_5P_t + c_7(I_0 + \sum_{i=1}^t P_i - (t+c_9)\bar{S} - c_8)^2 \right] \end{aligned} \quad (1.21)$$

subject to

$$\begin{aligned} \text{Prob}(I_0 + \sum_{i=1}^t P_i - \sum_{i=1}^t S_i \geq 0) \geq u_t \quad t=1, 2, \dots, T \\ 0 \leq u_t \leq 1 \end{aligned} \quad (1.17)$$

where  $u_t$  is a parameter preset by the decision maker and the objective function (1.21) is simply the SPC form (1.12) with (1.16) substituted for  $I_t$ . In order to provide for updating information on sales they consider the following decision rule for production levels in different time periods,

$$\begin{aligned} P_1 &= \bar{S} + \delta_1 && \text{for } t=1 \\ P_2 &= S_1 + (1-\alpha)\bar{S} + \delta_2 && \text{for } t=2 \\ P_t &= S_{t-1} + (1-\alpha)S_{t-2} + \delta_t && \text{for } t=3, 4, \dots, T \end{aligned} \quad (1.22)$$

where  $\bar{S}$  is expected demand and  $\alpha$  is a parameter satisfying  $0 \leq \alpha \leq 1$ .<sup>3</sup> Substituting (1.22) into (1.17) for the proper production variables gives the following set of chance constraints

$$\text{Prob}(I_0 + \bar{S} + \delta_1 - S_1 \geq 0) \geq u_t \quad \text{for } t=1 \quad (1.23)$$

$$\text{and } \text{Prob}(I_0 + (2-\alpha)\bar{S} + \sum_{i=1}^t \delta_i - (1-\alpha)S_{t-1} - S_t \geq 0) \geq u_t$$

$$\text{for } t=2, 3, \dots, T. \quad (1.24)$$

Next, sales are assumed to be independently and identically

<sup>3</sup>The parameter  $\alpha$  serves the purpose of weighting the impact of previous sales on the current production decision. As  $\alpha \rightarrow 1$  the relative importance of sales for the immediately preceding period rises in relationship to sales two periods ago. As  $\alpha \rightarrow 0$  the opposite reaction is observed. Thus  $\alpha$  serves to set the levels of feedback that would contribute to the present production decision.

normally distributed over time with parameters  $\bar{S}$  and  $\sigma^2$ . Therefore the chance constraints (1.23) and (1.24) may be replaced by their corresponding deterministic equivalents

$$I_0 + \delta_1 \geq F^{-1}(u_1) \quad \text{for } t=1 \quad (1.25)$$

$$\text{and } I_0 + \sum_{i=1}^t \delta_i \geq (1+(1-\alpha)^2)^{\frac{1}{2}} \sigma F^{-1}(u_t) \quad \text{for } t=2,2,\dots,T \quad (1.26)$$

where  $F$  is the cumulative distribution function of a unit normal variable with  $F^{-1}$  as its inverse.

A second set of structural constraints enforce a condition of nonnegative production in each time period. They are developed by using the production rule (1.22) as follows

$$\begin{aligned} P_1 &= \bar{S} + \delta_1 \geq 0 & \text{for } t=1 \\ P_2 &= S_1 + (1-\alpha)\bar{S} + \delta_2 \geq 0 & \text{for } t=2 \\ P_t &= S_{t-1} + (1-\alpha)S_{t-2} + \delta_t \geq 0 & \text{for } t=2,3,\dots,T \end{aligned} \quad (1.27)$$

In turn, these conditions are solved for  $\delta_1, \delta_2, \dots, \delta_T$  giving

$$\begin{aligned} \delta_1 &\geq -\bar{S} & \text{for } t=1 \\ \delta_2 &\geq -(1-\alpha)\bar{S} - \alpha \min(S_1) & \text{for } t=2 \\ \delta_t &\geq -\min(\alpha S_{t-1} + (1-\alpha)S_{t-2}) & \text{for } t=3,4,\dots,T. \end{aligned} \quad (1.28)$$

Thus, if there is, on empirical grounds, a lower bound on demand,  $S_t$ , it can be used here to stipulate a lower bound on the value of  $\delta_t$ . Otherwise, in the case of normal density

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<sup>4</sup>The variables  $\delta_1, \delta_2, \dots, \delta_T$  serve the purpose of providing an adjustment to the feedback elements in determining the final production decisions in each period. They are sign unrestricted and have lower bounds as determined by the conditions of nonnegativity of production in each time period. Further, by virtue of the substitution of the production rules into both the objective function and the constraints these adjustments are decision variables in the overall optimization and their values are obtained upon solution of this problem. Thus,

of demand, one can safely assume  $\min(S_t) = \bar{S} - 3\sigma$ .<sup>5</sup> Therefore, the following constraints must be taken into account:

$$\begin{aligned} \delta_1 &\geq -s && \text{for } t=1 \\ \delta_2 &\geq -(\bar{S} - 3\alpha\sigma) && \text{for } t=2 \\ \delta_t &\geq -(\bar{S} - 3\sigma) && \text{for } t=3, 4, \dots, T. \end{aligned} \quad (1.29)$$

Gupta and Sengupta propose three decision rule models for the aggregate planning problem which incorporate the above ideas with the original HMMS formulation.

First Decision Rule Model (DRM-1). If one incorporates (1.25), (1.26), and (1.29) in place of the chance constraint (1.17), substitutes the decision rule (1.27) for production in the objective function (1.21) and takes its expectation, the following quadratic programming model results

$$\begin{aligned} \text{Min} \quad & \sum_{t=1}^T \left[ (c_1 - c_6)W_t + c_2(W_t - W_{t-1})^2 + c_3(\bar{S} + \delta_t - c_4W_t)^2 \right. \\ & \left. + c_5\delta_t + c_7(I_0 + \sum_{i=1}^t \delta_i - c_8 - c_9\bar{S})^2 \right] \end{aligned} \quad (1.30)$$

subject to

$$\begin{aligned} I_0 + \delta_1 &\geq \sigma_F^{-1}(u_1) \\ I_0 + \sum_{i=1}^t \delta_i &\geq (1 + (1 - \alpha)^2)^{\frac{1}{2}} \sigma_F^{-1}(u_t) \quad \text{for } t=2, 3, \dots, T \\ \delta_1 &\geq -\bar{S} \\ \delta_2 &\geq -(\bar{S} - 3\alpha\sigma) \end{aligned}$$

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these variables are related to the nature of the demand distributions and the tolerance levels selected for the chance constraints.

<sup>5</sup>Care must be taken here when selecting parameter values so as to assure that  $\bar{S} - 3\sigma \geq 0$  since negative demand is not reasonable in an economic sense.

$$\delta_t \geq -(\bar{S}-3\sigma) \quad t=3,4,\dots,T.$$

For fixed values of  $u_t$  and  $\alpha$ , the above is a quadratic programming model with decision variables  $W_t$  and  $\delta_t$  which serves as an adjustment variable. To use this model in practice one must take the following steps:

- i) Solve DRM-1 for decision variables  $W_t$  and  $\delta_t$ .
- ii) Determine the production level,  $P_t$ , each period by the decision rule (1.22).
- iii) Determine finally the size of the work force,  $W_t$ , each period at the level actually obtained by the optimal solution of the quadratic program in step (i).

Second Decision Rule Model (DRM-2). For high variances of demand, production levels determined by solution of DRM-1 may fluctuate significantly over time. Also the size of the work force may show some instability in this sense. Therefore, in this second model, the problem is to find an improved solution. At time point  $t$ , the component of total expected cost affected by the level of the work force is

$$(c_1-c_6)W_t+c_2(W_t-W_{t-1})^2+c_3(P_t-c_4W_t)^2. \quad (1.31)$$

If one minimizes the cost component by classical methods, the result is the following linear decision rule

$$W_t = a_1P_t+a_2W_{t-1}+a_3 \quad (1.32)$$

where  $a_1=c_3c_4/D$ ,  $a_2=c_2/D$ ,  $a_3=(c_1-c_6)/2D$ ,  $D=c_2+c_3c_4^2$ .

To use DRM-2 in practice the following steps must be taken:

- i) Solve DRM-1 for the decision variables  $W_t$  and  $\delta_t$ .
- ii) Determine production levels by decision rule (1.22).
- iii) Determine finally the size of the work force by

the linear decision rule (1.32).

Third Decision Rule Model (DRM-3). DRM-3 takes the tack that optimal production levels may be determined by minimizing the fraction of costs at each time point,  $t$ , due to inventory carrying and shortage costs only. DRM-3 represents the inventory cost function linearly which leads to a linear programming formulation as follows

$$\text{Min } \sum_{t=1}^T (c_o I_t^+ + c_s I_t^-) \quad (1.33)$$

subject to

$$\begin{aligned} I_0 + \delta_1 &\geq \sigma F^{-1}(u_1) && \text{for } t=1 \\ I_0 + \sum_{i=1}^t \delta_i &\geq (1+(1-\alpha))^2 \sigma F^{-1}(u_t) && \text{for } t=2,3,\dots,T \\ \delta_1 &\geq -\bar{S} && \text{for } t=1 \\ \delta_2 &\geq -(\bar{S}-3\alpha\sigma) && \text{for } t=2 \\ \delta_t &\geq -(\bar{S}-3\sigma) && \text{for } t=3,4,\dots,T \\ I_t^+ - I_t^- &= I_0 + \sum_{i=1}^t \delta_i && \text{for } t=1,2,\dots,T \end{aligned} \quad (1.34)$$

where

$$\begin{aligned} I_t^+ &= I_t; \text{ if } I_t \geq 0 \text{ and zero otherwise} \\ I_t^- &= -I_t; \text{ if } I_t \leq 0 \text{ and zero otherwise} \\ c_o &= \text{cost of carrying inventory per unit} \\ c_s &= \text{cost of shortage per unit.} \end{aligned}$$

In this model a constraint (1.34) stipulating average inventory at the end of period  $t$  has been added. It is clear that for fixed values of  $u_t$  and  $\alpha$ , the above model is a linear programming model. To use DRM-3 in practice the following

steps must be taken:

- i) Solve DRM-3 for the decision variables  $I_t^+$ ,  $I_t^-$ , and  $\delta_t$ .
- ii) Determine the production level  $P_t$  in each period by decision rule (1.22).
- iii) Determine the size of the work force each period by decision rule (1.32).

Simulation Experiments. A series of simulation studies were performed to determine the operational efficiency of the different models. The authors simulated the costs of operating the models for a ten month horizon. The criterion for comparison was a linear cost function, constructed by the authors, comprising costs of regular payroll, hiring, layoff, overtime, idle time, inventory carrying, and shortages. Thus the cost of operating a model for T periods,  $OC_T$ , may be expressed as

$$\begin{aligned}
 OC_T = \sum_{t=1}^T & \left[ (c_1 - c_6)W_t + c_H(W_t - W_{t-1})^+ + c_L(W_t - W_{t-1})^- \right. \\
 & + c_V(P_t - c_4W_t)^+ + c_I(P_t - c_4W_t)^- + c_5P_t + c_OI_t^+ \\
 & \left. + c_S I_t^- \right]. \quad (1.35)
 \end{aligned}$$

The cost coefficients for hiring, layoff, overtime, idle time, inventory carrying and shortage are denoted respectively by  $c_H$ ,  $c_L$ ,  $c_V$ ,  $c_I$ ,  $c_O$ , and  $c_S$ . These cost coefficients are estimated as the slopes of the relevant quadratic cost functions of the HMMS model at appropriate points.

The first set of simulation experiments were conducted for the purpose of determining the optimal value of  $\alpha$ . A

series of 500 random numbers was generated and converted into a set of random samples from a normal distribution with mean 500 and standard deviation 100. These sample sets were used as random demand for ten periods. DRM-1 was then solved for each of these fifty samples with a constant  $u_t=0.95$  and values of  $\alpha$  from 0.4 to 0.9. The cost of operating the model for each trial was calculated according to (1.35). Operating cost means and variances calculated for each value of alpha showed  $\alpha=0.7, 0.8, 0.9$  to be admissible on the basis of smaller mean cost and variance. Next the Wilcoxon matched-pair signed-rank test was applied to determine if there was any significant difference in performance between alpha values. The results showed that while there was no significant difference in costs for  $\alpha=0.7$  and  $\alpha=0.8$ , the cost for DRM-1 for  $\alpha=0.7$  is significantly less than for  $\alpha=0.9$ .

A second set of simulation experiments were conducted to compare the operational efficiency of the three decision rule models and the original HMMS model. Fifty new samples were generated and solved for  $\alpha=0.7$  and initial conditions  $I_0=200$  and  $W_0=80$ . The sample mean cost was nearly equal for the three DRMs and the sample variances did not differ significantly. However, the sample mean and variance of costs for HMMS were high by comparison. This result can be traced to the fact that in some replications the cost of operating the HMMS model is abnormally high. Further analysis reveals that in all these cases initially demand was much higher than the average level of 500 units therefore early shortages occurred

which the feedback elements of the DRMs' picked up and corrected for by increases in production in subsequent periods. However, since the HMMS model has no feedback control, production continued to hover around average demand thus causing shortages in periods of high demand and sizable inventories being carried in slack periods. This seems to indicate that some feedback elements should be built in as stabilizers before attempting to apply optimization in multi-period decision making under uncertainty.

#### Significance of this Study

Stochastic programming approaches to the aggregate production planning problem have evolved from the early zero-order rule models to the superior linear rule with feedback models. Although this approach seems promising some major questions remain to be answered before it can be recommended as a general approach to the aggregate production planning problem. First, the viability of the DRM approach under a different demand distribution assumption must be investigated. This is especially important considering the previously discussed evidence dealing with the error of assuming normality when the underlying distribution is in fact not normal. Secondly, the DRM approach espoused by Gupta and Sengupta again limits the discussion to stationary demand patterns as expressed by identically and independently normally distributed demand over time. Therefore at this point in time the only mechanism that exists for handling trend and/or seasonal demand patterns in the aggregate planning problem

are the HMMS rules with the forecasted values of demand replaced by the expected value of demand in each period. There certainly exists a need here to determine whether the DRM approach can be used as a tool to better deal with this shortcoming. Finally, dealing directly with the production rules of the DRM approach it is pertinent to question whether production should depend so greatly solely on actual sales in previous periods, especially for changing demand patterns such as trend or seasonal. The weight assigned to forecasted demand for the current period in the HMMS production rule would seem to indicate that this might be a critical element to consider when setting a production rate. Thus some formulation considering both feedback and forecast elements would seem to be in order.

The significance of this study is two fold in nature. First it contributes to the development of the area of aggregate production planning, which is an important area of production management, by developing and testing new models which include both stochastic demand elements and feedback information. Second its results may be generalized to aid in the development of a taxonomy for model selection based on the nature of demand faced by the decision maker. The development of this taxonomy should lead to increased application of these methods to real world problems.

The formulation of the models to be tested is described in Chapter II, along with a presentation of the methodology that was used to compare the performance of different models

and a statement of the hypotheses of this study. Chapter III discusses the simulated studies of model performance when demand is assumed to follow a normal distribution. Results of this same nature when demand is assumed to follow a two parameter exponential density are discussed in Chapter IV. A comparison of the best decision rule models for each type of demand against the corresponding HMMS results is presented in Chapter V. The conclusions of this study are given in Chapter VI.

## CHAPTER II

### MODEL FORMULATION AND METHODOLOGY

#### Introduction

This research investigates the implications associated with the previously delineated questions by developing new models and testing their performance against existing models. Various models for differing demand assumptions are constructed using the Gupta and Sengupta DRM approach. In addition, for some cases the DRM production rule is replaced by rules containing both forecast and feedback elements. Simulation experiments are conducted and the results are used as a basis for comparing the performance of the competing models as against the Gupta and Sengupta criterion function (1.35). Nonparametric statistical tests are conducted where indicated to determine the statistical significance of the difference in performance of selected models.

#### Model Formulation

All the models constructed herein are built in accordance with the stochastic programming including feedback approach as presented by Gupta and Sengupta [3]. This approach essentially consists of the following steps common to all the models:

1. Develop the objective function based on the

- original HMMS form.
2. State the cumulative form of the inventory(material balance) constraint (1.16).
  3. Propose the form of the production rule to be used.
  4. Form the chance constraints by substituting the production rule into the inventory constraint and imposing the tolerance level  $u_t$ .
  5. Make a demand distribution assumption and develop the deterministic equivalents of the chance constraint by imposition of this demand assumption on them.
  6. Take the expectation of the objective function by substituting the expected value of the production rule for monthly production in it.
  7. Formulate the nonnegative production constraints by using the lower bound of demand in conjunction with the production rule.

### Models

Decision Rule Model Under Exponential Demand. This set of models is equivalent to the Gupta and Sengupta DRM-1, DRM-2, DRM-3 models with the exception that demand is assumed here to follow a two parameter exponential distribution.<sup>6</sup>

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<sup>6</sup>The two parameter exponential distribution is selected because of (i) its previous use in work of this nature, (ii) the mathematical tractability it offers for determining the distributions of sums of variables which follow this distribution, and (iii) the economic reality that demand is a continuous nonnegative concept.

DRM-1/EX is a quadratic programming model in decision variables  $W_t$  and  $\delta_t$  whose structure is as follows

$$\text{Min } \sum_{t=1}^T [(c_1 - c_6)W_t + c_2(W_t - W_{t-1})^2 + c_3(\bar{S} + \delta_t - c_4W_t)^2 + c_5\delta_t + c_7(I_0 + \sum_{i=1}^t \delta_i - c_8 - c_9\bar{S})^2]$$

subject to

$$I_0 + \bar{S} + \delta_1 \geq F^{-1}(u_1) \quad \text{for } t=1$$

$$I_0 + (2 - \alpha)\bar{S} + \sum_{i=1}^t \delta_i \geq F_Z^{-1}(u_t) \quad \text{for } t=2, 3, \dots, T$$

$$\delta_1 \geq -\bar{S} \quad \text{for } t=1$$

$$\delta_2 \geq -(1 - \alpha)\bar{S} - \alpha S_0 \quad \text{for } t=2$$

$$\delta_t \geq -S_0 \quad \text{for } t=3, 4, \dots, T$$

where

$\bar{S}$  is expected demand,

$F^{-1}(u_1)$  is the inverse of the demand distribution for period 1,

$F_Z^{-1}(u_t)$  is the inverse of  $Z = S_t + (1 - \alpha)S_{t-1}$  distribution for period  $t$ . This is a single distribution since we assume demand is identically distributed over time.

DRM-2/EX modifies the preceding model by the inclusion of an additional linear decision rule to finally determine work force level as follows

$$W_t = a_1 P_t + a_2 W_{t-1} + a_3$$

where  $a_1 = c_3 c_4 / D$ ,  $a_2 = c_2 / D$ ,  $a_3 = (c_1 - c_6) / 2D$ ,  $D = c_2 + c_3 c_4^2$ . The procedure is identical to that previously explained for DRM-2 in Chapter I.

DRM-3/EX is a linear programming model in decision

variables  $I_t^+$ ,  $I_t^-$ , and  $\delta_t$ . The model structure is as follows

$$\text{Min } \sum_{t=1}^T (c_o I_t^+ + c_s I_t^-)$$

subject to

$$I_0 + \bar{S} + \delta_1 \geq F^{-1}(u_1) \quad \text{for } t=1$$

$$I_0 + (2-\alpha)\bar{S} + \sum_{i=1}^t \delta_i \geq F_Z^{-1}(u_t) \quad \text{for } t=2,3,\dots,T$$

$$\delta_1 \geq -\bar{S} \quad \text{for } t=1$$

$$\delta_2 \geq -(1-\alpha)\bar{S} - \alpha S_0 \quad \text{for } t=2$$

$$\delta_t \geq -S_0 \quad \text{for } t=3,4,\dots,T$$

$$I_t^+ - I_t^- = I_0 + \sum_{i=1}^t \delta_i \quad \text{for } t=1,2,\dots,T$$

where

$$I_t^+ = I_t; \text{ if } I_t \geq 0 \text{ and zero otherwise}$$

$$I_t^- = -I_t; \text{ if } I_t \leq 0 \text{ and zero otherwise}$$

$c_o$  = cost of carrying inventory per unit

$c_s$  = cost of storage per unit

$F^{-1}(u_1)$  and  $F_Z^{-1}(u_t)$  are defined identically for this model as they are for DRM-1/EX.

The procedure for solving this model is similar to that previously explained for DRM-3 in Chapter I.

Alternative Production Rules. The production rule proposed by Gupta and Sengupta (1.22) contains feedback elements in terms of sales realized in previous periods but does not directly consider the impact of forecasted demand in the current period. This may be valid for the case of identical demand distributions over time since the forecast would be a

constant expected demand. However, for changing demand patterns forecasts of demand for the current period may be a factor more critical for the determination of production levels than previous sales. Further, in true application we know forecasts play a large part in decision making.

Considering these factors this research investigates alternative decision rules of the form

$$P_t = f(S_{t-1}, \bar{S}_t, \theta). \quad (2.1)$$

Here it is assumed that the decision maker knows in advance sales in the previous period  $t-1$  and has available to him a forecast of expected demand in the current period,  $\bar{S}_t$ . Further,  $\theta$  contains unknowns which may either vary from period to period or remain fixed. The first form of production rule is one based on sales in the immediately preceding period as follows

$$\begin{aligned} P_1 &= \bar{S}_1 + \epsilon_1 && \text{for } t=1 \\ P_t &= S_{t-1} + \alpha(\bar{S}_t - S_{t-1}) + \epsilon_t && \text{for } t=2,3,\dots,T \end{aligned} \quad (2.2)$$

where  $\epsilon_t$  is an adjustment variable and  $\alpha$  is a parameter satisfying  $0 \leq \alpha \leq 1$ . The rationale here is that production should be set at previous sales level plus some adjustment based on a comparison between forecasted demand for the current period and previously realized sales. Here the major emphasis remains on sales with forecasts as a moderating factor.

The second form of the production rule is one based on the forecast for the current period as follows

$$\begin{aligned}
 P_1 &= \bar{S}_1 + \epsilon_1 && \text{for } t=1 \\
 P_t &= \bar{S}_t + \alpha(S_{t-1} - \bar{S}_{t-1}) + \epsilon_t && \text{for } t=2,3,\dots,T
 \end{aligned}
 \tag{2.3}$$

where  $\bar{S}_{t-1}$  is the forecast for expected demand in the previous period,  $\epsilon_t$  is an adjustment variable, and  $\alpha$  is a parameter satisfying  $0 \leq \alpha \leq 1$ . The rationale here is that production should be set at the forecasted demand level plus some adjustment for the previous forecast error. Here the major emphasis is on the forecast with previous realized sales having a moderating effect.

Sales Based Decision Rule Models. The following two models use the sales based form of production rule (2.2) in their formulation. In addition the distribution assumption provides for independent though not identical distributions over time.

SBDRM-1/N is a quadratic programming model in decision variables  $W_t$  and  $\epsilon_t$ , under an assumption of independent normally distributed demand, whose structure is as follows

$$\begin{aligned}
 \text{Min } C_T &= (c_1 - c_6)W_1 + c_2(W_1 - W_0)^2 + c_3(\bar{S}_1 + \epsilon_1 - c_4W_1)^2 \\
 &+ c_7(I_0 + \epsilon_1 - c_8 - c_9\bar{S}_1)^2 + \sum_{t=2}^T [(c_1 - c_6)W_t \\
 &+ c_2(W_t - W_{t-1})^2 + c_3((1 - \alpha)\bar{S}_{t-1} + \alpha\bar{S}_t - c_4W_t)^2 + c_5\epsilon_t \\
 &+ c_7(I_0 + \sum_{i=1}^t \epsilon_i + (1 - \alpha)\bar{S}_1 + (\alpha - c_9 - 1)\bar{S}_t - c_8)^2]
 \end{aligned}
 \tag{2.4}$$

subject to

$$I_0 + \epsilon_1 \geq \sigma_1 F^{-1}(u_1) \quad \text{for } t=1 \tag{2.5}$$

$$I_0 + (1 - \alpha)\bar{S}_1 + (\alpha - 1)\bar{S}_t + \sum_{i=1}^T \epsilon_i \geq \text{Dev}_t F_{Z_t}^{-1}(u_t) \tag{2.6}$$

for  $t=2,3,\dots,T$

$$\epsilon_1 \geq -\bar{S}_1 \quad \text{for } t=1 \quad (2.7)$$

$$\epsilon_t \geq (\alpha-1)\bar{S}_{t-1} - \alpha\bar{S}_t + (1-\alpha)3\sigma_{t-1} \quad \text{for } t=2,3,\dots,T \quad (2.8)$$

where

$F^{-1}(u_1)$  is the inverse of the standardized distribution of sales in period 1.

$F_{Z_t}^{-1}(u_t)$  is the inverse of the  $Z_t = S_t + \alpha \sum_{i=1}^{t-1} S_i$  distribution for period t.

$\text{Dev}_t = (\sigma_t^2 + \alpha^2 \sum_{i=1}^{t-1} \sigma_i^2)^{\frac{1}{2}}$  for period t.

The procedure for using this model in practice is to:

- i) Solve SBDRM-1/N for decision variables  $W_t$  and  $\epsilon_t$ .
- ii) Determine the production levels by sales based decision rule (2.2).
- iii) Determine finally the size of the work force at the level actually obtained by the optimal solution obtained in (i).

SBDRM-2/N differs from SBDRM-1/N by allowing for the revision of work force levels using the linear decision rule (1.32). To use this model in practice steps (i) and (ii) above must be followed by a new step (iii) Determine finally the size of the work force by the linear decision rule (1.32).

SBDRM-3/N is a linear programming formulation of the problem which has as its objective function the linear inventory cost function (1.33). This function must be minimized subject to the above constraints, (2.5) through (2.8) and the additional average inventory relationship (1.34). The solution procedure for this model is as follows:

- i) Solve SBDRM-3/N for the decision variables  $I_t^+$ ,  $I_t^-$ , and  $\epsilon_t$ .
- ii) Determine the production level  $P_t$  in each period by decision rule (2.2).
- iii) Determine the size of the work force each period by decision rule (1.32).

SBDRM-1/EX is a quadratic programming model in decision variables  $W_t$  and  $\epsilon_t$  under an assumption of independent two parameter exponentially distributed demand whose structure is as follows

$$\begin{aligned} \text{Min } C_T = & (c_1 - c_6)W_1 + c_2(W_1 - W_0)^2 + c_3(\bar{S}_1 + \epsilon_1 - c_4W_1)^2 \\ & + c_7(I_0 + \epsilon_1 - c_8 - c_9\bar{S}_1)^2 + \sum_{t=2}^T \left\{ (c_1 - c_6)W_t \right. \\ & + c_2(W_t - W_{t-1})^2 + c_3((1-\alpha)\bar{S}_{t-1} + \alpha\bar{S}_t - c_4W_t)^2 + c_5\epsilon_t \\ & \left. + c_7(I_0 + \sum_{i=1}^t \epsilon_i + (1-\alpha)\bar{S}_1 + (\alpha - c_9 - 1)\bar{S}_t - c_8)^2 \right\} \end{aligned}$$

subject to

$$I_0 + \bar{S}_1 + \epsilon_1 \geq F^{-1}(u_1) \quad \text{for } t=1 \quad (2.9)$$

$$I_0 + \bar{S}_1 + \alpha \sum_{j=2}^t \bar{S}_j + \sum_{i=1}^t \epsilon_i \geq F_{Z_t}^{-1}(u_t) \quad \text{for } t=2, 3, \dots, T \quad (2.10)$$

$$\epsilon_1 \geq -\bar{S}_1 \quad \text{for } t=1 \quad (2.11)$$

$$\epsilon_t \geq -(1-\alpha)\bar{S}_0 - \alpha\bar{S}_t \quad \text{for } t=2, 3, \dots, T \quad (2.12)$$

where

$F^{-1}(u_1)$  is the inverse of the demand distribution for period 1.

$F_{Z_t}^{-1}(u_t)$  is the inverse of  $Z = S_t + \alpha \sum_{i=1}^{t-1} S_i$  distribution for  $t$ .

The procedure for using this model is the same as that explained for SBDRM-1/N in this chapter.

SBDRM-2/EX uses the linear decision rule for determination of final work force levels in a manner identical to that previously explained in conjunction with SBDRM-2/N.

SBDRM-3/EX is a linear programming formulation of the problem which has as its objective function the linear inventory cost function (1.33). This function must be minimized subject to the above constraints, (2.9) through (2.12) and the additional average inventory relationship (1.34). The solution procedure for this model is identical to that for SBDRM-3/N.

Forecast Based Decision Rule Models. The following two models use the forecast based form of production rule (2.3) in their formulation. In addition the distribution assumption provides for independent though not identical distributions of demand over time.

FBDRM-1/N is a quadratic programming model in decision variables  $W_t$  and  $\epsilon_t$ , under an assumption of independent normally distributed demand, whose structure is as follows

$$\begin{aligned} \text{Min } C_T = & \sum_{t=1}^T \left[ (c_1 - c_6)W_t + c_2(W_t - W_{t-1})^2 + c_3(\bar{S}_t + \epsilon_t - c_4W_t)^2 \right. \\ & \left. + c_5 \epsilon_t + c_7 \left( I_0 + \sum_{i=1}^t \epsilon_i - c_8 - c_9 \bar{S}_t \right)^2 \right] \end{aligned}$$

subject to

$$I_0 + \epsilon_1 \geq \sigma_1 F^{-1}(u_1) \quad \text{for } t=1 \quad (2.13)$$

$$I_0 + \sum_{i=1}^t \epsilon_i \geq \text{Dev}_t E_{Z_t}^{-1}(u_t) \quad \text{for } t=2,3,\dots,T \quad (2.14)$$

$$\epsilon_1 \geq -\bar{S}_1 \quad \text{for } t=1 \quad (2.15)$$

$$\epsilon_t \geq -(\bar{S}_t - 3\alpha\sigma_{t-1}) \quad \text{for } t=2,3,\dots,T \quad (2.16)$$

where

$F^{-1}(u_1)$  is the standardized distribution of sales in period 1 inverse.

$F_{Z_t}^{-1}(u_t)$  is the inverse of the standardized distribution of  $Z = S_t + (1-\alpha) \sum_{i=1}^{t-1} S_i$  for period  $t$ .

$\text{Dev}_t = (\sigma_1^2 + (1-\sigma)^2 \sum_{i=1}^{t-1} \sigma_i^2)^{\frac{1}{2}}$  for period  $t$ .

The procedure for using this model in practice is to:

- i) Solve FBDRM-1/N for decision variables  $W_t$  and  $\epsilon_t$ .
- ii) Determine production levels by decision rule (2.3).
- iii) Determine finally the size of the work force level at the value obtained by the optimal solution of the quadratic program in (i).

FBDRM-2/N differs from FBDRM-1/N by allowing for the revision of work force levels by using the linear decision rule (1.32). To use this model in practice steps (i) and (ii) above must be followed by a new step (iii) Determine finally the size of the work force by the linear decision rule (1.32).

FBDRM-3/N is a linear programming formulation of the problem which has as its objective function the linear inventory cost function (1.33). This function must be minimized subject to the above constraints, (2.13) through (2.16) and the additional average inventory relationship (1.34). The solution procedure for the model is as follows:

- i) Solve FBDRM-3/N for the decision variables  $I_t^+$ ,  $I_t^-$ , and  $\epsilon_t$ .
- ii) Determine the production level  $P_t$  in each period by decision rule (2.3).
- iii) Determine the size of the work force in each period by decision rule (1.32).

FBDRM-1/EX is a quadratic programming model in decision variables  $W_t$  and  $\epsilon_t$ , under an assumption of independent exponentially distributed demand, whose structure is as follows

$$\text{Min } C_T = \sum_{t=1}^T \left[ (c_1 - c_6)W_t + c_2(W_t - W_{t-1})^2 + c_3(\bar{S}_t + \epsilon_t - c_4W_t)^2 + c_5\epsilon_t + c_7(I_0^+ + \sum_{i=1}^t \epsilon_i - c_8 - c_9\bar{S}_t)^2 \right]$$

subject to

$$I_0^+ + \bar{S}_1 + \epsilon_1 \geq F^{-1}(u_1) \quad \text{for } t=1 \quad (2.17)$$

$$I_0^+ + \sum_{i=1}^t \bar{S}_i - \alpha \sum_{i=1}^{t-1} \bar{S}_i + \sum_{i=1}^t \epsilon_i \geq F_{Z_t}^{-1}(u_t) \quad \text{for } t=2,3,\dots,T \quad (2.18)$$

$$\epsilon_1 \geq -\bar{S}_1 \quad \text{for } t=1 \quad (2.19)$$

$$\epsilon_t \geq -\bar{S}_t + \alpha(\bar{S}_{t-1} - S_0) \quad \text{for } t=2,3,\dots,T \quad (2.20)$$

where

$F^{-1}(u_1)$  is the inverse of the sales distribution for period 1.

$F_{Z_t}^{-1}(u_t)$  is the inverse of the  $Z = S_t + (1-\alpha) \sum_{i=1}^{t-1} S_i$

distribution for period  $t$ .

The procedure for using this model is the same as that explained for FBDRM-1/N in this chapter.

FBDRM-2/EX differs from FBDRM-1/EX by allowing the

revision of work force levels by using the linear decision rule (1.32). To use this model in practice a procedure identical to that followed for FBDRM-2/N must be used.

FBDRM-3/EX is a linear programming formulation of the problem which has as its objective function the linear inventory cost function (1.33). This function must be minimized subject to the above constraints, (2.17) through (2.20), and the additional average inventory relationship (1.34). The solution procedure for this model is identical to that for FBDRM-3/N.

## Methodology

### Simulation Experiments

Parameter Testing. The first set of simulation studies are conducted for the purpose of determining optimal values of the parameter alpha for each quadratic programming model given a particular demand assumption. For each of the three demand patterns, stationary, increasing trend, and seasonal as expressed by expected demand in each period, a random sample from the appropriate distribution is generated to represent sales across the planning horizon. Each model type is solved for these random samples of demand with values of the parameter alpha ranging from 0.3 to 0.9. The preceding experiments are replicated fifty times with performance against the Gupta and Sengupta cost function (1.35) calculated. Table 1 gives the values of the cost coefficients for this function that are used in this study. In addition, means and standard deviations of cost for each model for a given

TABLE 1  
MODEL OPERATING COST ( $OC_T$ ) COEFFICIENTS

Coefficient	Related Cost	Value
$c_1$	Regular Payroll	340.0
$c_6$	Expected Overtime	281.0
$c_H$	Hiring	180.0
$c_L$	Layoff	360.0
$c_V$	Overtime	90.0
$c_I$	Idle Time	60.0
$c_5$	Expected Overtime	51.2
$c_o$	Inventory Carrying	20.0
$c_s$	Shortage	100.0
$c_4$	Expected Overtime	5.67

Source: Holt, Modigliani, Muth, and Simon [4]

parameter value are calculated. Finally, the parametric value for which each model operates most efficiently is determined by mean cost comparisons and statistical tests.

Model Testing. Second, the above experiments are repeated for each of the three model types (Type 1, Type 2, and Type 3), under their respective demand assumptions, with the optimal parametric value for alpha as determined above. This procedure has been followed in previous research and seems to make intuitive sense based on the fact that identical production values are exhibited by all three models in these studies. Indeed Type 2 and Type 3 are shown to result in identical solutions in the normal stationary case. In

addition some preliminary studies conducted for both the Type 2 and Type 1 models indicated that no bias in favor of the Type 1 models exists since generally the alpha value that is best for the quadratic programming model is also best for the other models. Further, optimal plans as determined by the HMMS LDRs are derived for each demand assumption. Statistical tests are performed to discern significant differences between operating efficiency of competing models for a given demand assumption.

### Statistical Testing

All statistical tests in this research are performed on the cost of operating competing models as determined from the linear cost function (1.35). Even for the cases where normal density of demand is assumed it is not likely that this linear cost function will generate values that are normal in nature. Therefore the conventional parametric tests are not applicable here. If we consider only two competing models at any one time the Wilcoxon Matched-Pairs Signed-Ranks Test may be applied to discern any significant differences in performance between them. This nonparametric test is indicated when observations can be paired and the relative magnitude of performance is the critical issue.

In this research the Wilcoxon test is applied to determine if differences in performance exist between different models used to solve the aggregate planning problem for a given demand assumption. In the simulation experiments, the random sample of demand acts as the matched pair of the

Wilcoxon signed-rank test, and the two models to be compared are the two different treatments to be tested. The cost of operating is equivalent to scores of treatment in the Wilcoxon signed-rank test. The hypothesis to be tested may be stated as follows

$H_0$ : There is no difference between the operating cost of Model A and Model B.

$H_1$ : Model A costs more to operate than Model B for the planning horizon.

This is clearly a one-tailed test and it is tested at the 0.95 significance level.

In the Wilcoxon test the difference of score between each matched pair (in this case for operating cost for Model A and Model B) due to different treatments is noted with a sign and ranked. If  $T$  is the smaller sum of the like-signed ranks, then it is known that for sample sizes  $N \geq 25$ , the quantity

$$Z = \frac{T - N(N+1)/4}{\left[ N(N+1)(2N+1)/24 \right]^{1/2}} \quad (2.21)$$

is approximately normally distributed with zero mean and variance unity. This allows the use of the standard normal distribution to construct rejection and acceptance regions to the hypothesis under consideration.

The Wilcoxon test is applied to pairs of models in a type of successive scheme. For a given demand assumption the test is applied to determine which model in the following classes performs the best

i) Sales Based Decision Rule Models.

ii) Forecast Based Decision Rule Models.

Then the best model from each class is compared against each other to determine an overall best. The same procedure is followed for the case of exponential stationary demand to determine whether DRM-1/EX, DRM-2/EX, or DRM-3/EX is most efficient. Then the overall best decision rule models are compared to the corresponding HMMS solutions to determine which is the most efficient. This establishes some relationship between demand and the most efficiently operating models.

### Hypotheses

The major issue under investigation in this research is whether the character of demand is a critical factor to be considered when choosing a model for the aggregate production planning problem. In the light of this the following hypotheses are proposed for investigation.

#### Hypotheses 1

Null Hypothesis  $H_0$ : For two parameter exponential stationary demand pattern there will be no significant difference in performance between the best DRM model and the HMMS model as measured by operating cost.

Alternative Hypothesis  $H_1$ : For two parameter exponential stationary demand pattern the operating cost for the best DRM model will be significantly less than that of the HMMS model.

#### Hypothesis 2

Null Hypothesis  $H_0$ : For both normal and two parameter exponential increasing demand patterns there will be no significant difference in performance between the best decision rule model and the HMMS model as measured by operating costs.

Alternative Hypothesis  $H_1$ : For both normal and two

parameter exponential increasing trend demand patterns the operating cost for the best decision rule model will be significantly less than that of the HMMS model.

### Hypothesis 3

Null Hypothesis  $H_0$ : For both normal and two parameter exponential seasonal demand there will be no significant difference in performance between the best decision rule model and the HMMS model as measured by operating cost.

Alternative Hypothesis  $H_1$ : For both normal and two parameter exponential seasonal demand patterns the operating cost for the best decision rule model will be significantly less than that of the HMMS model.

## CHAPTER III

### SIMULATION TESTING OF MODELS UNDER NORMAL DEMAND

#### Increasing Trend Demand

In this study a five percent increasing trend demand pattern for the expected values of normally distributed demand over a ten period planning horizon was used. These expected values were calculated according to the following exponential equation

$$\bar{S}_t = 500(1.05)^{t-1} \quad (3.1)$$

where 500 serves essentially as a base value. In addition, a constant standard deviation of 100 is assumed. Thus, for the ten period planning horizon the following values for parameters of the normal distribution were used:

<u>Period(t)</u>	<u>Mean(<math>\bar{S}_t</math>)</u>	<u>Standard Deviation(<math>\sigma_t</math>)</u>
1	500	100
2	525	100
3	551	100
4	579	100
5	608	100
6	638	100
7	670	100
8	704	100
9	739	100
10	776	100

### Parameter Testing

Both the FBDRM-1/N and SBDRM-1/N quadratic programming models were solved for values of the parameter alpha from 0.3 to 0.9 by tenths. The resulting solutions are presented in Tables 2 and 3. Next fifty replicate random samples of demand across the planning horizon were generated by utilizing the central limit theorem method. Production plans for these samples were determined by the appropriate rules for each model and each plan's cost was determined according to (1.35). The mean cost and standard deviation for each value of alpha for each model was calculated and is presented in Tables 4 and 5.

Based on the preceding data FBDRM-1/N with values of alpha of 0.5 and 0.6 were selected for further investigation since they exhibited the lowest mean costs and standard deviations. An attempt to determine definitively which parametric value was most efficient was made through the use of the Wilcoxon signed-rank test. However, the calculated values of  $T=623$  and  $Z=-0.14$  led to the conclusion that there was not sufficient evidence to reject the hypothesis that no significant difference existed in the performance of the model with  $\alpha=0.5$ , and  $\alpha=0.6$ . In view of this situation the model with  $\alpha=0.6$  was selected for further investigation since it exhibited the smallest mean cost.

For the SBDRM-1/N model the parametric value of  $\alpha=0.4$  exhibited both the smallest mean and standard deviation. In addition, a statistical test comparing this value with 0.5

TABLE 2  
OPTIMAL VALUES OF  $(W_t, \epsilon_t)$  for FBDRM-1/N TREND

$\alpha$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$t$	<u>Values of <math>W_t</math></u>						
1	85.52	85.52	85.47	85.41	85.37	85.32	85.29
2	90.35	90.22	90.10	89.98	89.88	89.79	89.73
3	94.73	94.51	94.30	94.10	93.92	93.78	93.68
4	98.96	98.62	98.30	98.00	97.72	97.50	97.35
5	103.10	102.61	102.15	101.71	101.31	100.99	100.77
6	107.10	106.43	105.78	105.17	104.61	104.17	103.86
7	110.84	109.94	109.06	108.24	107.50	106.90	106.48
8	114.13	112.95	111.81	110.75	109.78	109.00	108.46
9	116.69	115.23	113.82	112.50	111.30	110.33	109.66
10	118.18	116.49	114.88	113.36	111.98	110.86	110.09
$t$	<u>Values of <math>\epsilon_t</math></u>						
1	56.74	57.26	57.76	58.24	58.66	58.05	58.29
2	35.96	36.32	36.66	36.99	37.29	37.99	38.15
3	20.27	20.44	20.61	20.77	20.91	21.26	21.34
4	13.37	13.32	13.27	13.22	13.18	13.28	13.25
5	10.45	9.97	9.51	9.07	8.68	8.43	8.21
6	10.07	8.73	7.44	6.23	5.14	4.28	3.67
7	10.36	7.36	4.48	1.77	-0.68	-2.66	-4.04
8	10.34	4.22	-1.65	-7.16	-12.16	-16.22	-19.02
9	9.47	-2.34	-13.68	-24.33	-34.00	-41.85	-47.26
10	4.42	-17.59	-38.75	-58.62	-76.64	-91.05	-101.37

$W_0=80, I_0=200$

TABLE 3  
OPTIMAL VALUES OF  $(W_t, \epsilon_t)$  for SBDRM-1/N TREND

$\alpha$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$t$	<u>Values of <math>W_t</math></u>						
1	88.07	88.15	85.25	85.35	85.43	85.59	85.74
2	89.34	89.50	89.71	89.90	90.21	90.41	90.73
3	93.22	93.43	93.65	94.07	94.42	94.87	95.36
4	97.03	97.29	97.83	98.16	98.81	99.26	99.92
5	100.82	101.13	101.64	102.24	103.17	103.63	104.44
6	104.57	104.91	105.48	106.27	107.21	107.93	108.84
7	108.14	108.52	109.33	110.13	111.59	112.01	112.89
8	111.34	111.74	112.67	113.62	114.57	115.55	116.36
9	113.86	114.29	115.32	116.42	117.25	118.28	119.05
10	115.36	115.81	116.92	118.09	119.01	119.89	120.63
$t$	<u>Values of <math>\epsilon_t</math></u>						
1	53.86	54.59	54.67	54.83	54.97	55.09	55.42
2	47.57	46.42	44.59	42.61	41.13	39.18	37.98
3	25.08	25.16	24.31	23.39	22.95	22.46	28.89
4	17.19	16.33	15.83	15.43	15.86	16.22	18.29
5	12.95	12.05	14.10	12.05	14.42	15.23	19.87
6	11.88	11.19	13.57	12.20	13.97	19.06	27.91
7	13.14	12.28	14.77	14.65	14.89	27.57	31.66
8	15.07	13.75	19.93	18.67	23.52	29.91	29.95
9	14.66	15.03	22.13	24.80	26.71	28.80	28.48
10	15.44	14.96	22.31	29.50	26.82	28.89	27.36

$W_0=80, I_0=200$

TABLE 4  
FBDRM-1/N TREND OPERATING COST STATISTICS

$\alpha$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Mean	521707	518222	516156	515798	516412	518226	521372
S.D.	17775	13513	10408	10211	13285	17789	23111

TABLE 5  
SBDRM-1/N TREND OPERATING COST STATISTICS

$\alpha$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Mean	518376	509882	512671	515360	520683	526019	533533
S.D.	14389	11220	12581	13379	17153	23106	29546

showed it to be more efficient. Thus  $\alpha=0.4$  was chosen for further work.

#### Model Testing

A new set of fifty replicates of demand was generated to serve as an input for this series of experiments. All three forecast based decision rule models(i.e. FBDRM-1/N, FBDRM-2/N, and FBDRM-3/N) were solved using a value of the parameter  $\alpha$  of 0.6. In addition, solutions were obtained for the three sales based decision rule models(i.e. SBDRM-1/N, SBDRM-2/N, and SBDRM-3/N) with  $\alpha=0.4$ . The development of a production plan by the Type 3 models involves the solution of a linear programming problem. These results are presented

in Table 6. Again each plan generated is costed according to the linear cost function (1.35). Means and standard deviations of cost were calculated and are presented in Table 7.

Comparisons of models within classes were done with the aid of the Wilcoxon test. First, for the forecast based rules Table 7 shows FBDRM-1/N to have the lowest mean cost by an appreciable margin. This prompted, according to our sequential pairwise testing scheme, a test to determine whether FBDRM-1/N did indeed exhibit a significantly lower cost. This resulted in values of  $T=0$  and  $z=-6.15$  indicating that the Type 1 model achieved a significantly lower cost than its competitor as explained by values of  $T=0$  and  $Z=-6.15$ . The identical procedure was followed for the sales based rules. SBDRM-1/N outperformed both the SBDRM-2/N and SBDRM-3/N models as indicated by identical values for  $T$  and  $Z$  of 0.0 and -6.15 respectively. Finally, to determine the best overall model for the increasing trend demand case FBDRM-1/N was compared with SBDRM-1/N. The sales based model outperformed the forecast based model with  $T=0$  and  $Z=-6.15$ . The results of all the preceding statistical tests are summarized in Table 8.

#### Seasonal Demand

A quarterly seasonal pattern of expected demand over a ten period planning horizon was utilized in this study. The following values for the parameters of the normal distribution were used:

TABLE 6  
OPTIMAL VALUES OF  $(I_t^+, I_t^-, \epsilon_t)$  FOR LINEAR PROGRAMMING  
MODELS UNDER TREND DEMAND

MODEL	FBDRM-3/N			SBDRM-3/N		
DV t	$I_t^+$	$I_t^-$	$\epsilon_t$	$I_t^+$	$I_t^-$	$\epsilon_t$
1	164.00	0.0	-36.00	164.00	0.0	-36.00
2	176.63	0.0	12.63	191.63	0.0	27.63
3	188.42	0.0	11.79	246.91	0.0	55.28
4	199.52	0.0	11.09	219.02	0.0	-27.89
5	210.02	0.0	10.06	274.82	0.0	55.80
6	220.03	0.0	10.01	302.83	0.0	28.01
7	229.60	0.0	9.57	331.60	0.0	28.77
8	238.79	0.0	9.19	361.19	0.0	29.59
9	247.64	0.0	8.85	391.03	0.0	29.84
10	256.18	0.0	8.54	421.78	0.0	30.75

$$W_0=80, I_0=200.$$

TABLE 7  
OPERATING COST STATISTICS FOR BEST DECISION RULE MODELS UNDER  
NORMAL TREND DEMAND

MODEL	FBDRM-1 $\alpha=0.6$	FBDRM-2 $\alpha=0.6$	FBDRM-3 $\alpha=0.6$	SBDRM-1 $\alpha=0.4$	SBDRM-2 $\alpha=0.4$	SBDRM-3 $\alpha=0.4$
Mean	513629	546706	532274	505732	541756	533201
S.D.	10879	9972	9806	11500	10414	10643

TABLE 8  
RESULTS OF WILCOXON TESTS TO DETERMINE MOST EFFICIENT MODEL  
UNDER NORMAL TREND DEMAND

MODELS	T	Z*	P(X≤Z) <sup>#</sup>	CONCLUSION
FBDRM-1/N vs FBDRM-2/N	0.0	-6.15	0.0	FBDRM-1/N outperforms FBDRM-2/N.
FBDRM-1/N vs FBDRM-3/N	0.0	-6.15	0.0	FBDRM-1/N outperforms FBDRM-3/N.
SBDRM-1/N vs SBDRM-2/N	0.0	-6.15	0.0	SBDRM-1/N outperforms SBDRM-2/N.
SBDRM-1/N vs SBDRM-3/N	0.0	-6.15	0.0	SBDRM-1/N outperforms SBDRM-3/N.
FBDRM-1/N vs SBDRM-1/N	0.0	-6.15	0.0	SBDRM-1/N outperforms FBDRM-1/N.

$$*Z = \frac{T - N(N+1)/4}{[N(N+1)(2N+1)/24]^{1/2}} = \frac{T - 50(51)/4}{[50(51)(101)/24]^{1/2}} = \frac{T - 637.5}{103.59}$$

<sup>#</sup>All results are highly significant since the rejection region consists of all values such that  $P(X \leq Z) \leq 0.05$  (i.e. A 0.05 significance level.).

<u>Period(t)</u>	<u>Mean(<math>\bar{S}_t</math>)</u>	<u>Standard Deviation(<math>\sigma_t</math>)</u>
1	500	100
2	500	100
3	500	100
4	625	100
5	625	100
6	625	100
7	550	100
8	550	100
9	550	100
10	500	100

As can be seen demand is forecasted at a base level in the first quarter, peaks in the second quarter, decreases in the third quarter, and trails off to the base level in the fourth quarter.

#### Parameter Testing

As in the trend case, both FBDRM-1/N and SBDRM-1/N, the quadratic programming models, were solved for alpha values ranging from 0.3 to 0.9. The resulting solutions are presented in Tables 9 and 10. Next fifty replicate random samples of demand across the planning horizon were generated based on the seasonal normal parameters. Production plans for these samples were determined by the appropriate rules for each model and each plan's cost was determined according to (1.35). The mean cost and standard deviation for each value of alpha for each model were calculated and appear in Tables 11 and 12.

Based on the data in Table 11 FBDRM-1/N with values of alpha of 0.6 and 0.7 were chosen for further investigation.

TABLE 9  
OPTIMAL VALUES OF  $(W_t, \epsilon_t)$  for FBDRM-1/N SEASONAL

$\alpha$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$t$	<u>Values of <math>W_t</math></u>						
1	84.18	84.05	83.98	83.92	83.87	83.84	83.81
2	87.43	87.15	86.99	86.88	86.76	86.68	86.62
3	90.13	89.69	89.42	89.21	89.03	88.88	88.78
4	92.44	91.83	91.43	91.11	90.84	90.61	90.46
5	94.01	93.24	92.68	92.22	91.83	91.50	91.28
6	94.71	93.85	93.12	92.49	91.93	91.48	91.17
7	94.57	93.80	92.88	92.03	91.29	90.68	90.27
8	94.28	93.45	92.32	91.24	90.28	89.49	88.95
9	93.82	92.94	91.67	90.38	89.19	88.21	87.54
10	93.11	93.32	91.12	89.73	88.35	87.23	86.46
$t$	<u>Values of <math>\epsilon_t</math></u>						
1	56.20	56.23	56.70	57.17	57.79	57.94	58.17
2	53.10	52.21	52.40	52.69	52.99	53.23	53.39
3	59.21	57.13	56.97	57.06	57.20	57.32	57.40
4	-32.69	-36.94	-37.64	-37.82	-37.86	-37.89	-37.92
5	-17.30	-25.39	-27.06	-27.75	-28.14	-28.47	-28.69
6	-13.90	-28.21	-32.28	-33.96	-35.06	-35.95	-36.56
7	20.91	25.00	20.32	16.76	14.31	12.31	10.94
8	19.65	14.50	4.53	-2.49	-7.50	-11.57	-14.36
9	22.59	9.12	-9.88	-23.19	-32.85	-40.71	-46.12
10	13.68	14.65	11.60	-2.29	-20.31	-34.96	-45.04

$W_0=80, I_0=200$

TABLE 10  
OPTIMAL VALUES OF  $(W_t, \epsilon_t)$  for SBDRM-1/N SEASONAL

$\alpha$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$t$	<u>Values of <math>W_t</math></u>						
1	83.65	83.70	83.71	83.75	83.78	83.81	83.84
2	86.34	86.43	86.45	86.52	86.58	86.64	86.70
3	88.43	88.58	88.60	88.70	88.77	88.86	88.94
4	90.13	90.34	90.33	90.46	90.54	90.63	90.73
5	91.21	91.46	91.39	91.51	91.57	91.65	91.74
6	91.80	92.06	91.84	91.91	91.88	91.90	91.91
7	91.79	92.06	91.65	91.65	91.51	91.44	91.36
8	91.40	91.72	91.09	91.02	90.77	90.62	90.45
9	90.73	91.26	90.29	90.20	89.85	89.64	89.42
10	90.14	90.78	89.63	89.55	89.13	88.87	88.64
$t$	<u>Values of <math>\epsilon_t</math></u>						
1	54.96	55.59	56.10	56.35	56.69	57.02	57.30
2	49.87	48.89	49.74	50.35	51.04	51.73	52.33
3	48.94	50.28	51.59	52.68	53.83	55.03	56.06
4	35.04	24.62	14.37	3.89	-6.52	-16.84	-27.44
5	-54.04	-50.47	-46.56	-42.92	-39.19	-36.05	-31.88
6	-44.60	-43.43	-41.82	-40.52	-39.11	-37.64	-36.50
7	-34.44	-27.66	-20.48	-13.56	-6.56	0.47	7.20
8	11.00	8.32	5.74	3.09	0.46	-2.14	-5.12
9	-14.83	-17.05	-19.95	-22.45	-25.08	-27.63	-30.73
10	-31.26	-28.45	-28.09	-26.27	-24.92	-23.32	-20.59

$$W_0=80, I_0=200$$

TABLE 11  
FBDRM-1/N SEASONAL OPERATING COST STATISTICS

$\alpha$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Mean	470865	464265	460241	459118	459303	460400	462620
S.D.	21797	17288	13236	11541	12050	15139	19667

TABLE 12  
SBDRM-1/N SEASONAL OPERATING COST STATISTICS

$\alpha$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Mean	451441	449544	450070	450615	452634	455583	459216
S.D.	14822	15305	18618	23317	28673	34457	40424

The hypothesis that  $\alpha=0.6$  outperforms  $\alpha=0.7$  was subjected to the Wilcoxon test. No support was found to indicate this as the calculated values of T and Z respectively were 599 and -0.372. However,  $\alpha=0.6$  was found to outperform  $\alpha=0.5$  and based on this result and the fact that the model with this parametric value exhibited both the smallest mean and standard deviation, it was decided to use this value for the model testing work.

For the SBDRM-1/N model  $\alpha=0.3$  and  $\alpha=0.4$  were admissible based on the data from Table 12. The Wilcoxon test indicated support for the hypothesis that the parametric value  $\alpha=0.4$  led to a more efficient model than that of  $\alpha=0.3$ . The relevant statistics for this test were  $T=315.5$  and  $Z=-3.11$ . Thus,  $\alpha=0.4$  was selected for use in the

model testing work.

### Model Testing

A new set of fifty replicates of seasonally normally distributed demand was generated to serve as a basis for these studies. Again, as in the trend case, all three decision rule models of both classes (i.e. forecast based and sales based) are solved using the associated parametric values derived from the parameter testing stage. The results of the solution of the linear programming problems necessary for the implementation of the Type 3 models (i.e. FBDRM-3/N and SBDRM-3/N) are presented in Table 13. Means and standard deviations of costs as calculated according to the linear cost function (1.35) appear in Table 14.

Within class comparisons of models led to the conclusion that the Type 3 models were the most efficient. Further, when SBDRM-3/N was compared with FBDRM-3/N the results indicated that the former significantly outperformed the latter. The statistics and conclusions of this complete series of Wilcoxon tests are summarized in Table 15.

TABLE 13  
OPTIMAL VALUES OF  $(I_t^+, I_t^-, \epsilon_t)$  FOR LINEAR PROGRAMMING  
MODELS UNDER SEASONAL DEMAND

MODEL	FBDRM-3/N			SBDRM-3/N		
DV	$I_t^+$	$I_t^-$	$\epsilon_t$	$I_t^+$	$I_t^-$	$\epsilon_t$
t						
1	164.00	0.0	-36.00	164.00	0.0	-36.00
2	176.63	0.0	12.62	171.22	0.0	7.22
3	188.42	0.0	11.79	178.15	0.0	6.93
4	199.52	0.0	11.09	259.82	0.0	81.67
5	210.02	0.0	10.51	266.26	0.0	6.44
6	220.03	0.0	10.01	272.48	0.0	6.22
7	229.60	0.0	9.57	233.52	0.0	-38.96
8	238.79	0.0	9.19	239.38	0.0	5.86
9	247.64	0.0	8.85	245.08	0.0	5.70
10	256.18	0.0	8.54	220.64	0.0	-24.44

$$W_0=80, I_0=200$$

TABLE 14  
OPERATING COST STATISTICS FOR BEST DECISION RULE MODELS UNDER  
NORMAL SEASONAL DEMAND

MODEL	FBDRM-1 $\alpha=0.6$	FBDRM-2 $\alpha=0.6$	FBDRM-3 $\alpha=0.6$	SBDRM-1 $\alpha=0.4$	SBDRM-2 $\alpha=0.4$	SBDRM-3 $\alpha=0.4$
Mean	471446	475188	452119	484684	485571	447167
S.D.	9328	8132	13306	9853	8843	16180

TABLE 15  
RESULTS OF WILCOXON TESTS TO DETERMINE MOST EFFICIENT MODEL  
UNDER NORMAL SEASONAL DEMAND

MODELS	T	Z*	P(X≤Z) <sup>#</sup>	CONCLUSION
FBDRM-1/N vs FBDRM-2/N	76.0	-5.42	0.0000	FBDRM-1/N outperforms FBDRM-2/N.
FBDRM-1/N vs FBDRM-3/N	7.0	-6.09	0.0000	FBDRM-3/N outperforms FBDRM-1/N.
SBDRM-1/N vs SBDRM-2/N	373.0	-2.55	0.0054	SBDRM-1/N outperforms SBDRM-2/N.
SBDRM-1/N vs SBDRM-3/N	4.0	-6.12	0.0000	SBDRM-3/N outperforms SBDRM-1/N.
FBDRM-3/N vs SBDRM-3/N	104.0	-5.15	0.0000	SBDRM-3/N outperforms FBDRM-3/N.

$$*Z = \frac{T-637.5}{103.59}$$

<sup>#</sup>All results are significant at the 0.01 level.

## CHAPTER IV

### SIMULATION TESTING OF MODELS UNDER TWO PARAMETER EXPONENTIAL DEMAND

#### Stationary Demand

This study investigates the performance of the Gupta and Sengupta Decision Rule Model when demand follows a two parameter exponential distribution of the form (1.14). A constant value of expected demand of 500 and a constant lower bound on demand of 100 are assumed for each of the ten periods in the planning horizon.

#### Parameter Testing

The DRM-1/EX model was solved for values of the parameter alpha from 0.3 to 0.9 by tenths. The resulting solutions are presented in Table 16. Next fifty replicate random samples of demand across the planning horizon were generated by the inverse function approach. Production plans were developed for each sample using the Gupta and Sengupta decision rules. Each plan was costed according to (1.35) and means and standard deviations for each value of alpha were calculated and are presented in Table 17.

Based on the data in Table 17 DRM-1/EX with values of alpha of 0.6, 0.7, and 0.8 were selected for further investigation since although alpha=0.6 exhibited lower mean cost, both alpha=0.7 and alpha=0.8 had lower standard deviations.

The Wilcoxon signed-rank test was used to determine which value was most efficient. Comparing  $\alpha=0.6$  to  $\alpha=0.7$  led to the conclusion that 0.6 outperformed 0.7 at the 0.05 level of significance. The calculated values for T and Z were 455 and -1.762 respectively. Further, a test of  $\alpha=0.6$  against  $\alpha=0.8$  showed 0.6 to outperform 0.8 at a highly significant level (better than 0.01) with values of T=248 and Z=-3.76. Thus, 0.6 was chosen for further work.

### Model Testing

A new set of fifty replicates of stationary demand was generated for use in this study. All three Gupta and Sengupta decision rule models (i.e. DRM-1/EX; DRM-2/EX, and DRM-3/EX) were solved using the value of 0.6 for the parameter alpha. The development of a production plan using the Type 3 model involves the solution of a linear programming problem. These results are presented in Table 18. It is of interest to note that the values of  $\delta_t$  obtained from the solution of DRM-3/EX are identical to those that result from the solution of DRM-1/EX with a value of  $\alpha=0.6$ . This means that all three models will generate the same values for production across the planning horizon. Further, since both DRM-2/EX and DRM-3/EX use the same rule for determining work force, their overall plans will be identical. Each plan generated is costed according to (1.35). Means and standard deviations of cost were calculated and are presented in Table 19.

Comparison of models was done with the aid of the Wilcoxon test. DRM-2/EX and DRM-3/EX show the lower mean cost

TABLE 16  
OPTIMAL VALUES OF  $(W_t, \delta_t)$  FOR DRM-1/EX STATIONARY

$\alpha$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
<u>t</u>	<u>Values of <math>W_t</math></u>						
1	90.10	89.70	89.36	89.09	88.89	88.76	88.70
2	90.31	89.46	88.74	88.17	87.76	87.50	87.35
3	88.51	87.89	87.36	86.95	86.65	86.45	86.35
4	87.21	86.76	86.37	86.06	85.84	85.70	85.62
5	86.27	85.94	85.65	85.42	85.26	85.15	85.09
6	85.60	85.35	85.13	84.96	84.84	85.70	85.62
7	85.12	84.94	84.77	84.64	84.55	84.49	84.45
8	84.81	84.66	84.53	84.42	84.35	84.30	84.28
9	84.62	84.49	84.38	84.29	84.23	84.19	84.17
10	84.52	84.41	84.31	84.23	84.17	84.13	84.11
<u>t</u>	<u>Values of <math>\delta_t</math></u>						
1	598.29	598.29	598.29	598.29	598.29	598.29	598.29
2	151.34	108.66	72.10	43.51	22.60	9.26	2.14
3	0.00	0.00	0.00	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5	0.00	0.00	0.00	0.00	0.00	0.00	0.00
6	0.00	0.00	0.00	0.00	0.00	0.00	0.00
7	0.00	0.00	0.00	0.00	0.00	0.00	0.00
8	0.00	0.00	0.00	0.00	0.00	0.00	0.00
9	0.00	0.00	0.00	0.00	0.00	0.00	0.00
10	0.00	0.00	0.00	0.00	0.00	0.00	0.00

$W_0=80, I_0=200$

TABLE 17  
 DRM-1/EX STATIONARY OPERATING COST STATISTICS

$\alpha$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Mean	762683	744831	736456	733964	736711	743960	756970
S.D.	180032	177477	174885	172635	171535	170456	171924

TABLE 18  
 OPTIMAL VALUES OF  $(I_t^+, I_t^-, \delta_t)$  FOR DRM-3/EX LINEAR PROGRAMMING  
 MODEL UNDER STATIONARY DEMAND

$t$	DV	$I_t^+$	$I_t^-$	$\delta_t$
1		798.29	0.0	598.29
2		841.80	0.0	43.51
3		841.80	0.0	0.00
4		841.80	0.0	0.00
5		841.80	0.0	0.00
6		841.80	0.0	0.00
7		841.80	0.0	0.00
8		841.80	0.0	0.00
9		841.80	0.0	0.00
10		841.80	0.0	0.00

$$W_0=80, I_0=200$$

and thus a test to determine whether these models significantly outperform DRM-1/EX was conducted. Values of  $T=142$  and  $Z=-4.78$  led to the conclusion that DRM-2/EX and DRM-3/EX both outperform DRM-1/EX at a high (better than 0.01) level of significance.

TABLE 19  
OPERATING COST STATISTICS FOR BEST DECISION RULE MODELS  
UNDER STATIONARY EXPONENTIAL DEMAND

MODEL	DRM-1 $\alpha=0.6$	DRM-2 $\alpha=0.6$	DRM-3 $\alpha=0.6$
Mean	725400	715927	715927
S.D.	151577	147771	147771

#### Increasing Trend Demand

In this study a five percent increasing trend demand pattern for the expected values of two parameter exponentially distributed demand over a six period planning horizon was used. Again a lower bound on demand in each period of 100 is assumed. Thus for the six period planning horizon the following values for parameters of the exponential distribution were used:

Period(t)	Mean( $\bar{S}_t$ )	$\mu=1/(\bar{S}_t-S_0)$	$S_0$
1	500	0.0025	100
2	525	0.0024	100
3	551	0.0022	100
4	579	0.0021	100
5	608	0.0020	100
6	638	0.0019	100

#### Parameter Testing

Both the FBDRM-1/EX and SBDRM-1/EX models were solved for values of the parameter alpha from 0.3 to 0.9 by tenths.

TABLE 20  
OPTIMAL VALUES OF  $(W_t, \epsilon_t)$  FOR FBDRM-1/EX TREND

$\alpha$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$t$	<u>Values of <math>W_t</math></u>						
1	110.43	109.63	109.52	109.63	109.94	110.15	110.41
2	132.99	131.31	131.07	131.32	131.96	132.40	132.95
3	157.04	155.07	155.29	156.17	157.57	157.97	159.51
4	181.76	182.45	183.82	185.80	188.10	188.61	190.91
5	209.76	211.46	213.41	216.10	219.10	219.77	222.68
6	231.81	233.75	235.98	239.08	242.56	243.37	246.72
$t$	<u>Values of <math>\epsilon_t</math></u>						
1	598.29	598.29	598.29	598.29	598.29	598.29	598.29
2	170.01	128.13	93.45	66.17	46.39	63.87	26.96
3	328.06	148.19	111.27	89.78	88.86	83.03	105.56
4	291.83	389.93	428.73	462.61	487.07	486.64	507.51
5	752.08	740.15	743.15	752.53	766.37	771.80	785.04
6	1952.52	1977.50	2005.96	2046.99	2093.58	2106.29	2149.57

$$W_0=80, I_0=200$$

These results are presented in Tables 20 and 21. Next fifty replicate random samples of demand across the planning horizon were generated utilizing the inverse function method. Production plans for these samples were determined by the appropriate rules for each model and each plan's cost was determined according to (1.35). The mean cost and standard deviation for each value of alpha for each model was calculated and is presented in Tables 22 and 23.

Based on the preceding data FBDRM-1/EX with values of

TABLE 21  
OPTIMAL VALUES OF ( $W_t$ ,  $\epsilon_t$ ) FOR SBDRM-1/EX TREND

$\alpha$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$t$	<u>Values of <math>W_t</math></u>						
1	109.90	109.62	109.50	109.62	110.53	108.83	114.71
2	131.87	131.29	131.05	131.29	133.20	129.62	141.98
3	157.42	156.12	155.24	155.03	156.55	150.93	169.79
4	188.08	185.72	183.75	182.40	181.35	173.82	201.95
5	219.33	216.17	213.47	211.51	209.49	197.64	236.52
6	243.05	239.36	236.21	233.94	231.67	212.82	263.30
$t$	<u>Values of <math>\epsilon_t</math></u>						
1	598.29	598.29	598.29	598.29	598.29	598.29	598.29
2	63.89	81.17	105.95	138.13	225.01	212.20	278.35
3	96.05	105.38	124.27	158.59	288.36	245.94	193.49
4	499.03	468.35	433.51	393.75	294.48	385.61	457.58
5	779.62	764.03	752.75	747.83	758.03	780.40	902.27
6	2132.39	2078.16	2031.93	1998.28	1968.26	2060.98	2402.46

$$W_0=80, I_0=200$$

alpha of 0.4 and 0.5 were selected for further work since they exhibited the lowest mean costs. A Wilcoxon test comparing these two models showed 0.4 to outperform 0.5 at the 0.05 level of significance. The calculated values of T and Z were 437 and -1.936 respectively. Thus, the model with alpha=0.4 was selected for further work.

For SBDRM-1/EX model the parametric values of 0.5, 0.6, and 0.7 were chosen as candidates for statistical testing. A comparison of 0.5 and 0.6 resulted in values of T=425 and

TABLE 22  
FBDRM-1/EX TREND OPERATING COST STATISTICS

$\alpha$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Mean	901465	890957	897422	910493	927200	939517	955453
S.D.	35283	34464	42859	54511	68871	84875	100863

TABLE 23  
SBDRM-1/EX TREND OPERATING COST STATISTICS

$\alpha$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Mean	932658	915814	902839	895480	904968	929172	995726
S.D.	85066	69416	55852	45685	35704	38295	49965

Z=-2.051 leading to the conclusion that alpha=0.6 outperforms alpha=0.5 at the 0.05 level. Also alpha=0.6 was found to outperform alpha=0.7 at the 0.01 level of significance as indicated by values of T=288 and Z=-3.374. Thus SBDRM-1/EX with alpha=0.6 was selected to advance to the model testing stage.

#### Model Testing

A new set of fifty replicates of demand were generated to serve as input for this series of experiments. All three forecast based decision rule models (i.e. FBDRM-1/EX, FBDRM-2/EX, and FBDRM-3/EX) were solved using a value of the parameter alpha of 0.4. In addition, solutions were obtained for the

three sales based decision rule models (i.e. SBDRM-1/EX, SBDRM-2/EX, and SBDRM-3/EX) with  $\alpha=0.6$ . The development of a production plan by the Type 3 models involves the solution of a linear programming problem. These results are presented in Table 24. Again each plan generated is costed according to the linear cost function (1.35). Means and Standard deviations of cost were calculated and are presented in Table 25.

Comparisons of models within classes were done with the aid of the Wilcoxon test. First, for the forecast based rules Table 25 shows FBDRM-1/EX to have the lowest mean cost by a wide margin. This prompted, according to our sequential pairwise testing scheme, a test to determine whether FBDRM-1/EX did indeed exhibit a significantly lower cost. This resulted in values of  $T=0$  and  $Z=-6.15$  indicating that the Type 1 model significantly outperformed the Type 2 model. Next the performance of FBDRM-1/EX was compared to that of FBDRM-3/EX. Again the Type 1 model achieved a significantly lower cost than its competitor as explained by values of  $T=0$  and  $Z=-6.15$ . The identical procedure was followed for the sales based rules. SBDRM-1/EX outperformed both SBDRM-2/EX and SBDRM-3/EX models as indicated by identical values for  $T$  and  $Z$  of 0.0 and -6.15 respectively. Finally, to determine the best overall model for exponentially distributed increasing trend demand FBDRM-1/EX was compared to SBDRM-1/EX. No evidence was found that would indicate a significant difference in performance between the two models as explained by values of  $T=542$  and  $Z=-0.922$ . The results of all the preceding statistical tests are

TABLE 24  
OPTIMAL VALUES OF  $(I_t^+, I_t^-, \epsilon_t)$  FOR LINEAR PROGRAMMING  
MODELS UNDER EXPONENTIAL TREND DEMAND

MODEL	FBDRM-3/EX			SBDRM-3/EX		
DV t	$I_t^+$	$I_t^-$	$\epsilon_t$	$I_t^+$	$I_t^-$	$\epsilon_t$
1	798.29	0.0	598.29	798.29	0.0	598.29
2	926.42	0.0	128.13	936.42	0.0	138.13
3	1074.61	0.0	148.19	1095.01	0.0	158.59
4	1085.92	0.0	11.31	1117.52	0.0	22.51
5	1389.06	0.0	303.14	1432.26	0.0	314.74
6	4439.54	0.0	3050.48	4494.74	0.0	3062.48

$$W_0=80, I_0=200$$

TABLE 25  
OPERATING COST STATISTICS FOR BEST DECISION RULE MODELS UNDER  
EXPONENTIAL TREND DEMAND

MODEL	FBDRM-1 $\alpha=0.4$	FBDRM-2 $\alpha=0.4$	FBDRM-3 $\alpha=0.4$	SBDRM-1 $\alpha=0.6$	SBDRM-2 $\alpha=0.6$	SBDRM-3 $\alpha=0.6$
Mean	898704	988686	1024929	898757	988112	1025096
S.D.	44148	36959	55639	44425	37191	55592

summarized in Table 26.

#### Seasonal Demand

A quarterly seasonal pattern of expected demand over a

TABLE 26  
RESULTS OF WILCOXON TESTS TO DETERMINE MOST EFFICIENT MODEL  
UNDER EXPONENTIAL TREND DEMAND

MODELS	T	Z	P(X≤Z)	CONCLUSION
FBDRM-1/EX vs FBDRM-2/EX	0	-6.15	0.00	FBDRM-1/EX outperforms FBDRM-2/EX.
FBDRM-1/EX vs FBDRM-3/EX	0	-6.15	0.00	FBDRM-1/EX outperforms FBDRM-3/EX.
SBDRM-1/EX vs SBDRM-2/EX	0	-6.15	0.00	SBDRM-1/EX outperforms SBDRM-2/EX.
SBDRM-1/EX vs SBDRM-3/EX	0	-6.15	0.00	SBDRM-1/EX outperforms SBDRM-3/EX.
FBDRM-1/EX vs SBDRM-1/EX	542	-0.922	0.1785	No evidence to conclude either model is superior.

six period planning horizon was utilized in this study. The following values for parameters of the two parameter exponential distribution were used:

Period(t)	Mean( $\bar{S}_t$ )	$\mu=1/(\bar{S}_t-S_0)$	$S_0$
1	500	0.0025	100
2	500	0.0025	100

3	500	0.0025	100
4	625	0.0019	100
5	625	0.0019	100
6	625	0.0019	100

### Parameter Testing

Both the FBDRM-1/EX and SBDRM-1/EX models were solved for values of the parameter alpha from 0.3 to 0.9 by tenths. These results are presented in Tables 27 and 28. Then fifty replicate random samples of demand across the planning horizon were generated utilizing the inverse function method. Production plans for these samples were determined by the appropriate rules for each model and each plan's cost was determined according to (1.35). The mean cost and standard deviation for each value of alpha for each model was calculated and is presented in Tables 29 and 30.

Based on the preceding data FBDRM-1/EX with values of alpha of 0.4 and 0.5, which exhibited the lowest mean cost, were subjected to the Wilcoxon test. The result was that 0.5 significantly outperformed 0.4 with values of  $T=69$  and  $Z=-5.49$ . Thus, the model with alpha=0.5 was chosen for further work.

For the SBDRM-1/EX model the parametric values of alpha of 0.5 and 0.6 exhibited the lowest mean cost. The Wilcoxon test comparing their performance indicated that 0.5 significantly outperformed 0.6 as explained by values of  $T=57$  and  $Z=-5.60$ . Thus, the model with alpha=0.5 was chosen for further work.

TABLE 27  
OPTIMAL VALUES OF  $(W_t, \epsilon_t)$  FOR FBDRM-1/EX SEASONAL

$\alpha$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$t$	<u>Values of <math>W_t</math></u>						
1	109.99	106.32	105.18	106.28	106.76	107.46	108.24
2	132.07	124.35	121.96	124.28	125.29	126.76	128.40
3	156.33	144.55	141.31	145.58	147.59	150.22	152.99
4	186.70	169.04	165.24	171.66	174.57	178.31	182.25
5	216.13	194.61	189.86	198.15	201.87	206.67	211.72
6	239.29	214.08	208.47	218.12	222.42	228.01	233.90
$t$	<u>Values of <math>\epsilon_t</math></u>						
1	598.29	598.29	598.29	598.29	598.29	598.29	598.29
2	151.34	108.66	72.10	43.51	22.60	9.26	2.14
3	122.65	102.34	66.94	80.06	97.68	114.65	129.55
4	393.56	298.11	299.72	351.23	372.56	397.35	422.03
5	779.02	653.66	619.74	662.79	682.05	708.19	736.30
6	2071.26	1718.90	1638.62	1769.99	1827.86	1904.08	1984.93

$$W_0=80, I_0=200$$

### Model Testing

A new set of fifty replicates of demand were generated to serve as an input for this series of experiments. All three forecast based decision rule models (i.e. FBDRM-1/EX, FBDRM-2/EX, and FBDRM-3/EX) were solved using a value of the parameter alpha of 0.5. In addition, solutions were obtained for the three sales based decision rules models (i.e. SBDRM-1/EX, SBDRM-2/EX, and SBDRM-3/EX) with alpha=0.5. The development of a production plan by the Type 3 models involves

TABLE 28  
OPTIMAL VALUES OF ( $W_t$ ,  $\epsilon_t$ ) FOR SBDRM-1/EX SEASONAL

$\alpha$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$t$	Values of $W_t$						
1	106.74	106.26	105.16	106.29	109.98	115.77	125.97
2	125.24	124.23	121.93	124.30	132.06	144.20	165.62
3	147.50	145.50	141.27	144.46	156.31	175.25	209.11
4	174.52	171.62	165.18	169.05	185.66	212.99	260.02
5	201.95	198.23	189.87	194.73	216.14	251.52	311.91
6	222.82	218.46	208.71	214.39	239.44	280.80	351.39
$t$	Values of $\epsilon_t$						
1	598.29	598.29	598.29	598.29	598.29	598.29	598.29
2	22.60	43.51	72.10	108.66	151.34	195.21	248.03
3	93.27	76.28	66.94	94.34	122.65	139.94	290.14
4	454.02	421.06	356.39	347.29	427.56	589.50	833.23
5	618.71	608.49	574.55	617.37	751.91	962.27	1322.82
6	1847.89	1787.16	1652.23	1731.56	2079.42	2653.50	3632.11

$$W_0=80, I_0=200$$

the solution of a linear programming problem. These results are presented in Table 31. Again each plan generated is costed according to a linear cost function (1.35). Means and standard deviations of cost were calculated and are presented in Table 32.

Comparisons of models within classes were done with the aid of the Wilcoxon test. First, for the forecast based rules Table 32 shows FBDRM-1/EX to have the lowest mean cost by a wide margin. Based on this fact a sequential testing scheme

TABLE 29  
FBDRM-1/EX SEASONAL OPERATING COST STATISTICS

$\alpha$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Mean	909531	828463	810175	844155	862794	886396	911853
S.D.	35866	33000	37461	45403	55484	66939	79278

TABLE 30  
SBDRM-1/EX SEASONAL OPERATING COST STATISTICS

$\alpha$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Mean	842708	839069	805688	824193	905590	1040719	1279649
S.D.	122569	44650	37106	33215	35898	43621	53375

was followed to determine whether FBDRM-1/EX did indeed exhibit a significantly lower cost. The results indicated that FBDRM-1/EX significantly outperformed both the Type 2 and Type 3 models. The relevant statistics in both cases are an identical  $T=0$  and  $Z=-6.15$ . The same procedure was followed for the sales based rules. Sbdm-1/EX outperformed both the SBDRM-2/EX and SBDRM-3/EX models as indicated by identical values for  $T$  and  $Z$  of 0.0 and -6.15 respectively. Finally, to determine the best overall model for exponentially distributed seasonal demand FBDRM-1/EX was compared to SBDRM-1/EX. The result of this test showed SBDRM-1/EX to significantly outperform FBDRM-1/EX with values of  $T=103$  and  $Z=-5.16$ . The results of all the preceding statistical tests are

TABLE 31  
OPTIMAL VALUES OF  $(I_t^+, I_t^-, \epsilon_t)$  FOR LINEAR PROGRAMMING  
MODELS UNDER EXPONENTIAL SEASONAL DEMAND

MODEL	FBDRM-3/EX			SBDRM-3/EX		
	DV $I_t^+$	$I_t^-$	$\epsilon_t$	$I_t^+$	$I_t^-$	$\epsilon_t$
1	798.29	0.0	598.29	798.29	0.0	598.29
2	870.39	0.0	72.10	870.39	0.0	72.10
3	937.33	0.0	66.94	937.33	0.0	66.94
4	1199.85	0.0	265.52	1262.35	0.0	325.02
5	1836.23	0.0	636.38	1898.73	0.0	636.38
6	3693.22	0.0	1856.99	3755.72	0.0	1856.99

$$W_0=80, I_0=200$$

TABLE 32  
OPERATING COST STATISTICS FOR BEST DECISION RULE MODELS UNDER  
UNDER EXPONENTIAL SEASONAL DEMAND

MODEL	FBDRM-1 $\alpha=0.5$	FBDRM-2 $\alpha=0.5$	FBDRM-3 $\alpha=0.5$	SBDRM-1 $\alpha=0.5$	SBDRM-2 $\alpha=0.5$	SBDRM-3 $\alpha=0.5$
Mean	817132	880193	910665	813460	873724	910665
S.D.	87891	85213	85889	88395	85417	85889

summarized in Table 33.

TABLE 33  
RESULTS OF WILCOXON TESTS TO DETERMINE MOST EFFICIENT MODEL  
UNDER EXPONENTIAL SEASONAL DEMAND

MODELS	T	Z	$P(X \leq Z)$	CONCLUSION
FBDRM-1/EX vs FBDRM-2/EX	0	-6.15	0.00	FBDRM-1/EX outperforms FBDRM-2/EX
FBDRM-1/EX vs FBDRM-3/EX	0	-6.15	0.00	FBDRM-1/EX outperforms FBDRM-3/EX.
SBDRM-1/EX vs SBDRM-2/EX	0	-6.15	0.00	SBDRM-1/EX outperforms SBDRM-2/EX.
SBDRM-1/EX vs SBDRM-3/EX	0	-6.15	0.00	SBDRM-1/EX outperforms SBDRM-3/EX.
FBDRM-1/EX vs SBDRM-1/EX	103	-5.16	0.00	SBDRM-1/EX outperforms FBDRM-1/EX.

## CHAPTER V

### COMPARISON OF DECISION RULE MODEL AND HMMS LDR DERIVED PRODUCTION PLANS

The purpose of this chapter is to compare the performance of the best decision rule models under given demand assumptions, as determined in Chapters III and IV, with that of the HMMS Linear Decision Rules. To accomplish this, production plans were determined by the use of the LDRs, (1.10) and (1.11), with respect to the identical random samples of demand that were previously used in the decision rule model testing phases of this research. Further, the Wilcoxon test was again utilized to determine whether significant differences existed in the performance of competing models.

#### Development of HMMS Production Plans

The HMMS model consists of two linear decision rules, one each for production and work force, of the form,

$$P_t = f_1(W_{t-1}, I_{t-1}, S_t, \dots, S_{t+11})$$

$$W_t = f_2(W_{t-1}, I_{t-1}, S_t, \dots, S_{t+11})$$

where  $W_t$ ,  $W_{t-1}$  are the work force values for periods  $t$  and  $t-1$ ,  
 $P_t$  is the production for period  $t$ ,  
 $S_t, \dots, S_{t+11}$  are the forecasted number of units ordered for shipment during the respective period,  $t, \dots, t+11$ .

Production plans are generated by the use of these rules in conjunction with the typical material balance inventory constraint (1.8) in a rolling horizon scheme. Thus, for the first period production and work force values are calculated based on period zero work force and inventory values in addition to forecasts of demand through period twelve. For the second period the information required is now the work force and inventory values for period one and demand forecasts through period thirteen. This rolling procedure continues throughout the complete planning horizon with the imputed benefit of allowing for revision of the deterministic demand forecasts.

In the case of stochastic demand with a quadratic criterion function it has been shown (HMMS [4], Chapter 6) that by substituting the expected value of the demand distribution of the proper period,  $\bar{S}_t$ , for the demand forecast,  $S_t$ , these rules will still generate optimal production plans. This procedure was used throughout this research to generate HMMS production plans. In the seasonal and trend cases this required an assumption to be made about the nature of demand beyond the specific planning horizon. In the seasonal case the assumption was made that the quarterly seasonal pattern repeated itself thus generating expected values for twenty four periods which was sufficient for our purposes. In the trend case demand was assumed to continue to grow at a rate of five percent per period as explained by (3.1).

## Model Comparisons Under Normal Demand

Production plans for both the trend and seasonal cases were developed by the HMMS approach and costed utilizing (1.35). Means and standard deviations of cost across samples were calculated and appear in Table 34 along with the corresponding data for the best decision rule models. In both cases the decision rule models exhibit lower mean costs and standard deviations. For the trend case the Wilcoxon test led to the conclusion that no significant difference existed between the performance of SBDRM-1/N and HMMS as explained by values of  $T=555$  and  $Z=-0.796$ . For the seasonal case the Wilcoxon test led to the conclusion that SBDRM-3/N significantly outperformed HMMS as explained by values of  $T=115$  and  $Z=-5.04$ .

A closer inspection of the results for the trend case shows that SBDRM-1/N led to lower cost production plans than HMMS in twenty eight of the samples while the converse was true in the remaining twenty two samples. For those samples where SBDRM-1/N was superior the average cost difference was \$11,638 while for those where HMMS was superior it was \$10,819. In an attempt to gain further insight into the workings of these models the two samples which exhibited the greatest cost differences were investigated. Overall cost comparisons for the production plans for each of these two cases are presented in Tables 35 and 36. Figures 1, 2, and 3 compare production, work force, and inventory values for Case 1 as determined by the SBDRM-1/N and HMMS models. Figures 4, 5, and 6 serve the same function for Case 2.

TABLE 34  
 COMPARISON OF COST STATISTICS FOR DECISION RULE MODEL  
 AND HMMS LDR PRODUCTION PLANS UNDER NORMAL DEMAND

PATTERN	TREND		SEASONAL		
	MODEL	SBDRM-1/N	HMMS	SBDRM-3/N	HMMS
MEAN	505732	507289	447167	462564	
S.D.	11500	12105	16180	19581	

For Case 1 HMMS exhibits a 5.34 percent cost savings over SBDRM-1/N. Figure 1 shows that for the first three periods production values determined by both models exceeds actual demand while in the seventh through ninth periods production values trail demand. Both sets of work force values show an increasing trend with the HMMS values growing at a greater rate from periods six through ten. This explains the significantly lower overtime cost exhibited by the HMMS model. In the case of inventory values both models are quite close for the first four periods with slightly larger amounts being held by the HMMS model. However, in the last six periods SBDRM-1/N results in higher inventory levels being held than HMMS. To recapitulate, while SBDRM-1/N seems to lead to quicker adaptation to actual demand by the production level it seems to suffer from lower work force and higher overtime levels. This leads to appreciably higher overtime levels and costs to meet its production plan and of course to higher inventory costs.

TABLE 35  
 COST COMPARISONS NORMAL TREND DEMAND - CASE 1

COST	SBDRM-1/N	HMMS
Regular Payroll	60285	62621
Hiring & Layoff	6446	9548
Overtime & Idletime	373288	346281
Inventory	76772	72134
Total Operating Cost	516791	490584
	105.34%	100.00%

TABLE 36  
 COST COMPARISONS NORMAL TREND DEMAND - CASE 2

COST	SBDRM-1/N	HMMS
Regular Payroll	60285	65119
Hiring & Layoff	6446	9885
Overtime & Idletime	397334	402114
Inventory	32188	52089
Total Operating Cost	496253	529207
	100.00%	106.64%

For Case 2 SBDRM-1/N exhibits a 6.64 percent cost savings over HMMS. The most striking feature of the production values of Figure 4 is that the HMMS rule tends to react more slowly and to a lesser degree to abrupt changes in demand. This is an expected result due to the fact that previous

Figure 1. Sample Production Values Under Normal Trend Demand - Case 1

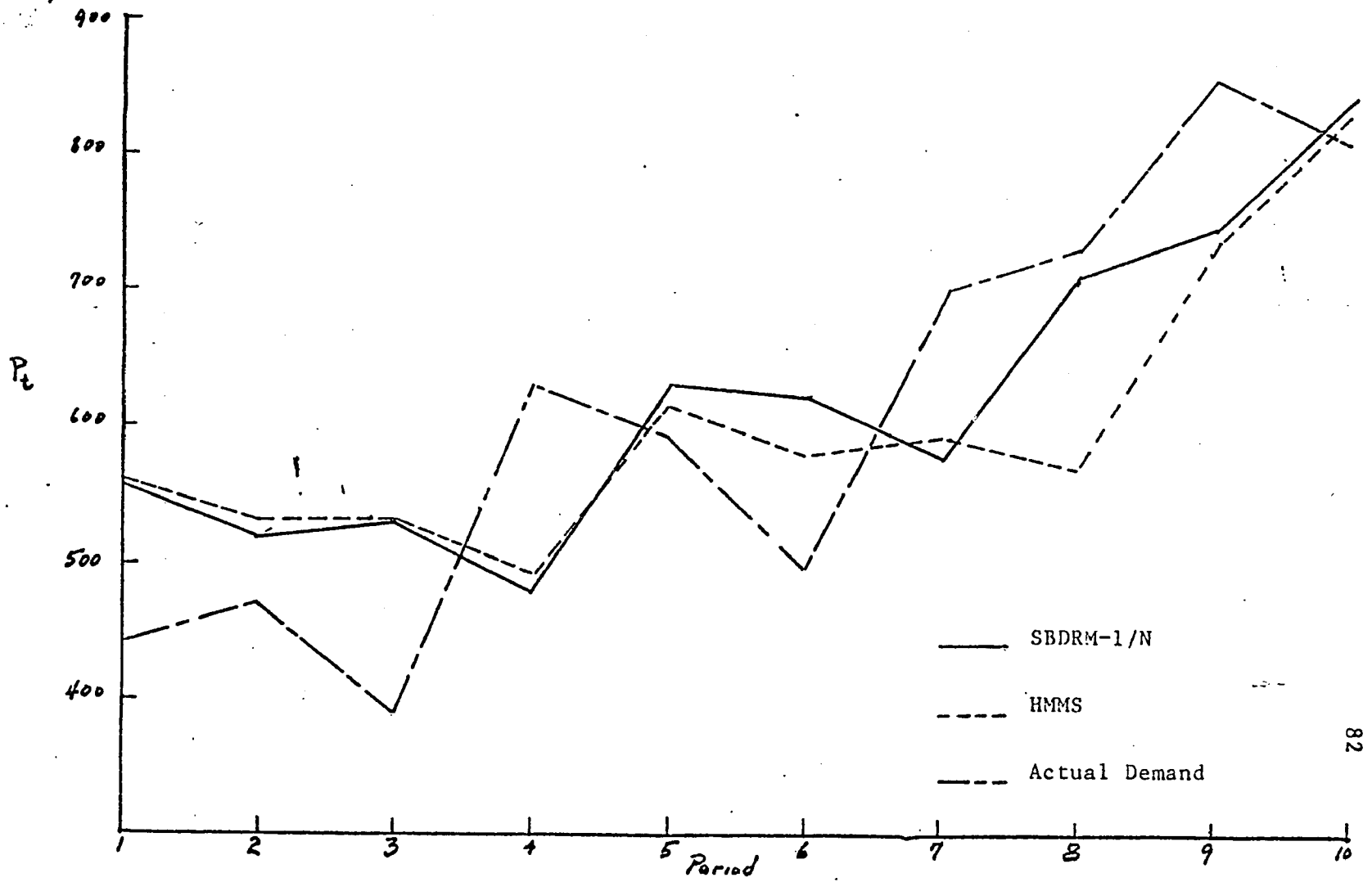


Figure 2. Sample Work Force Values Under Normal Trend Demand - Case 1

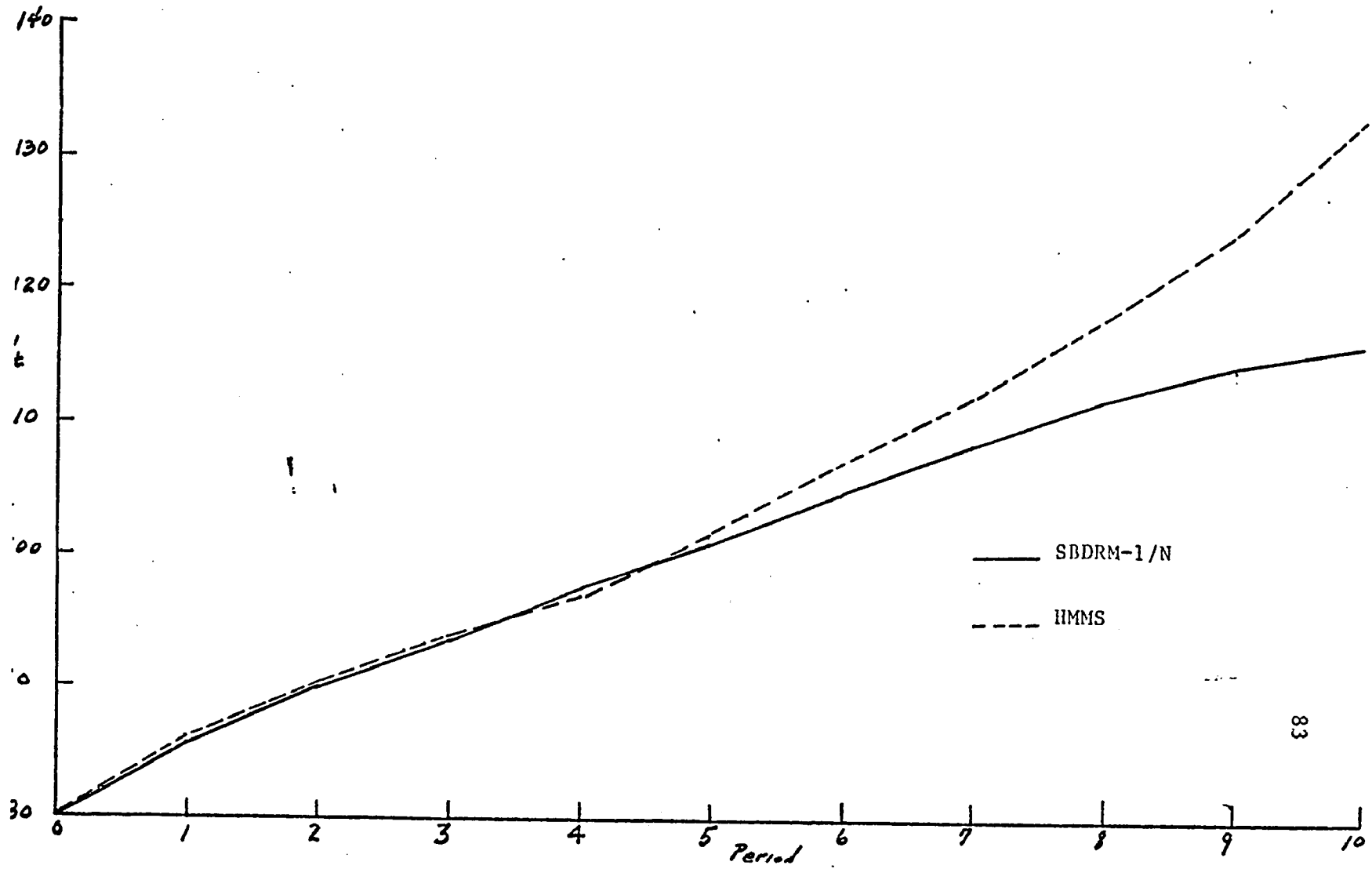
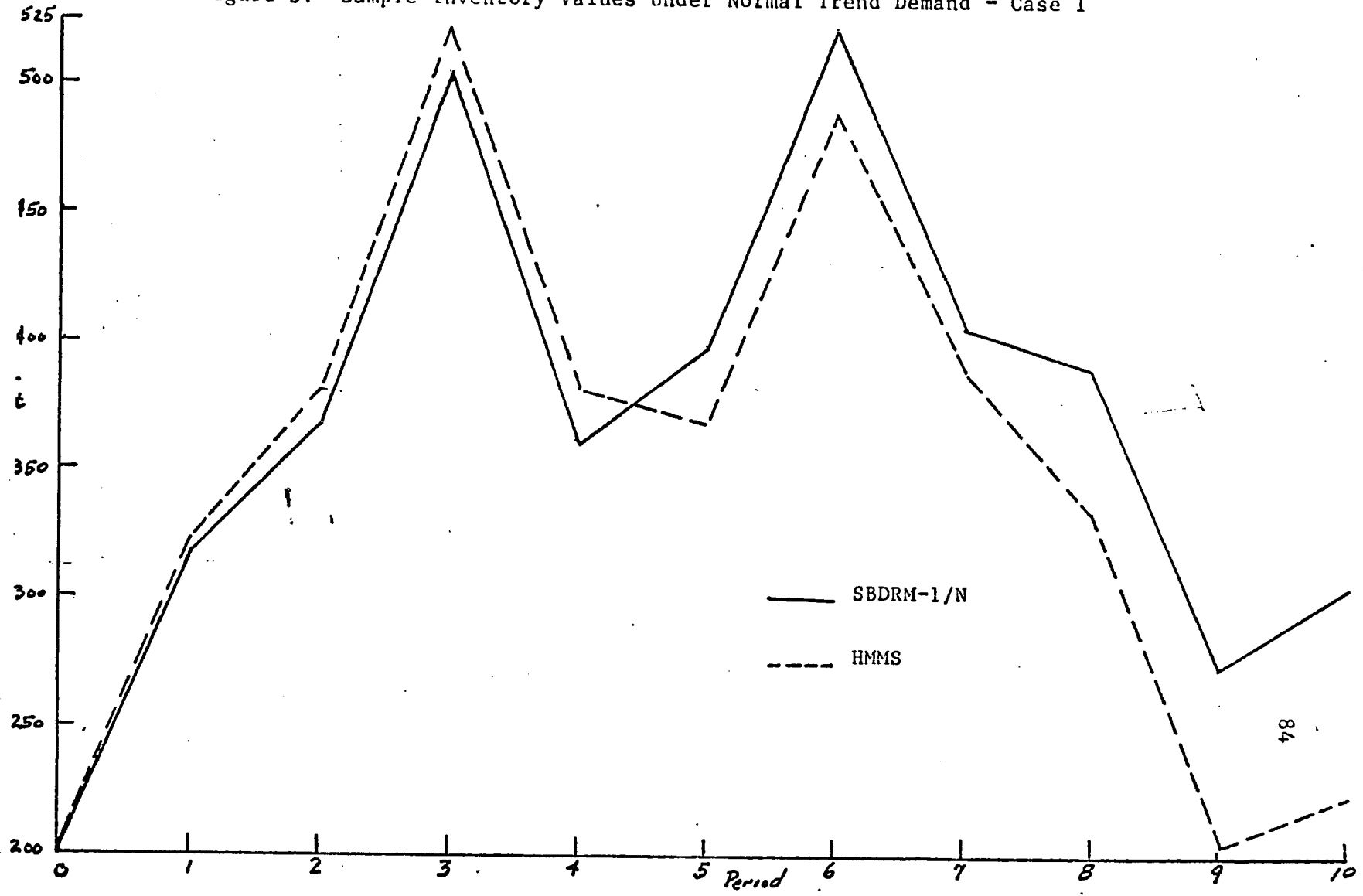


Figure 3. Sample Inventory Values Under Normal Trend Demand - Case 1



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Figure 4. Sample Production Values Under Normal Trend Demand - Case 2

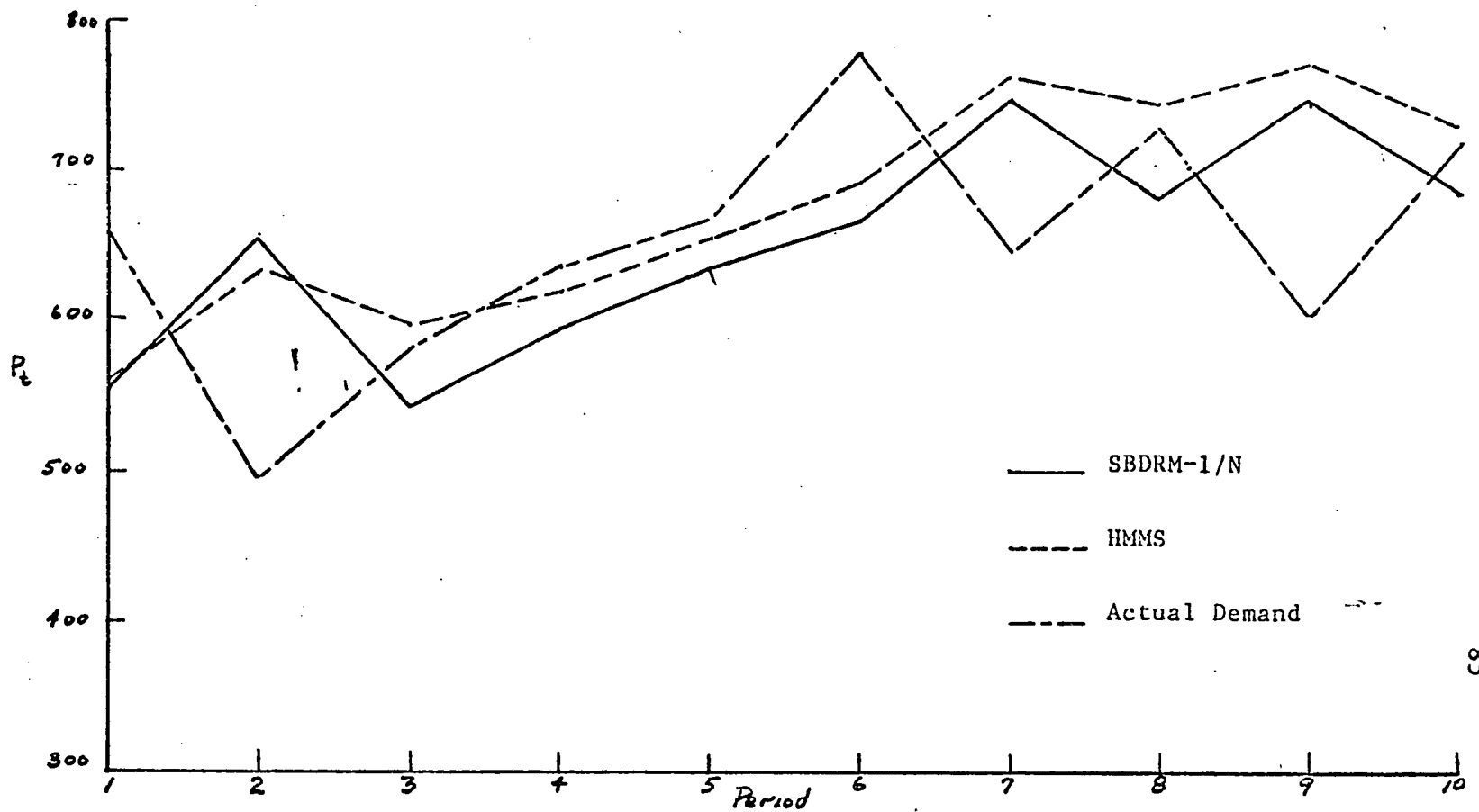


Figure 5. Sample Work Force Values Under Normal Trend Demand - Case 2

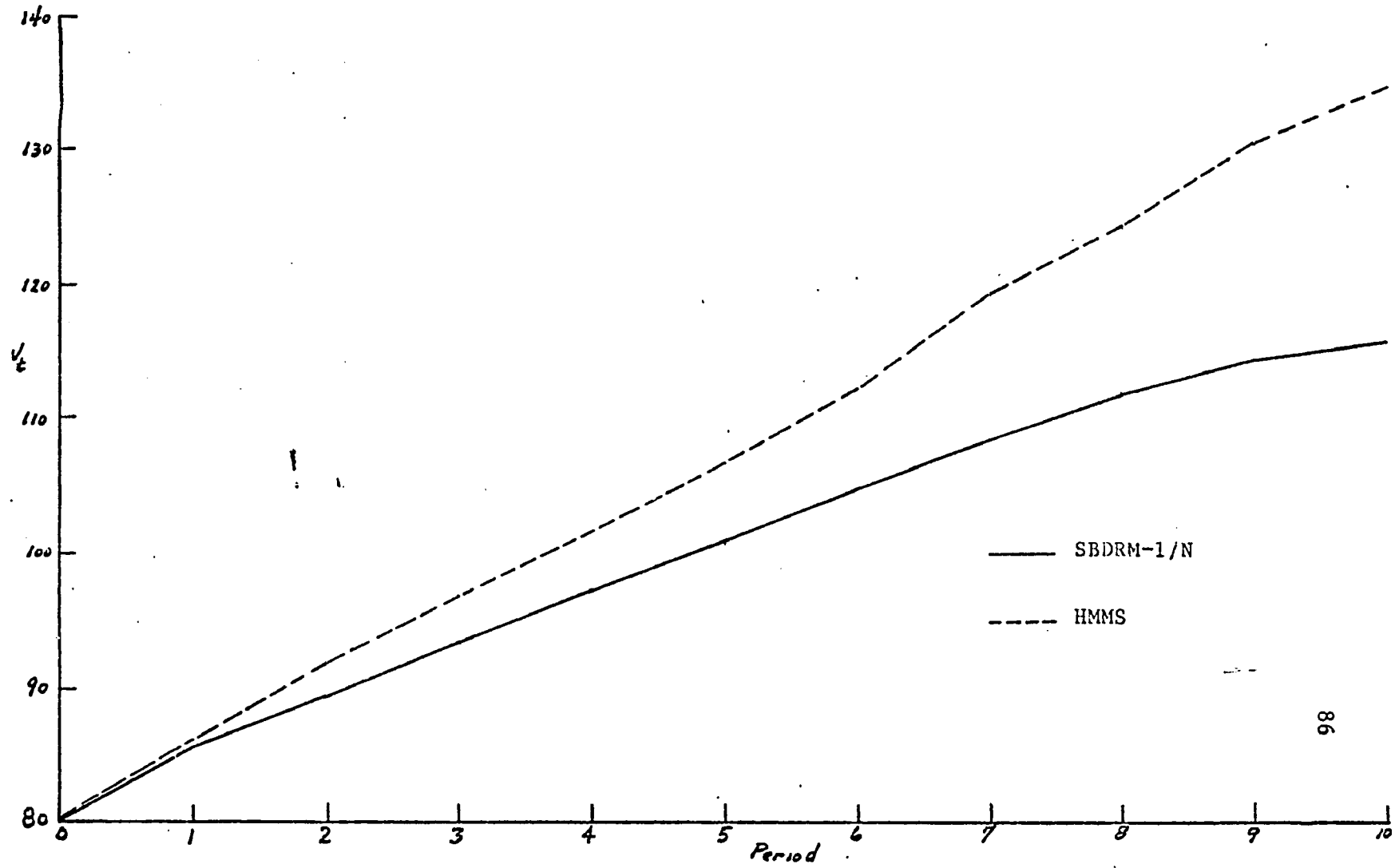
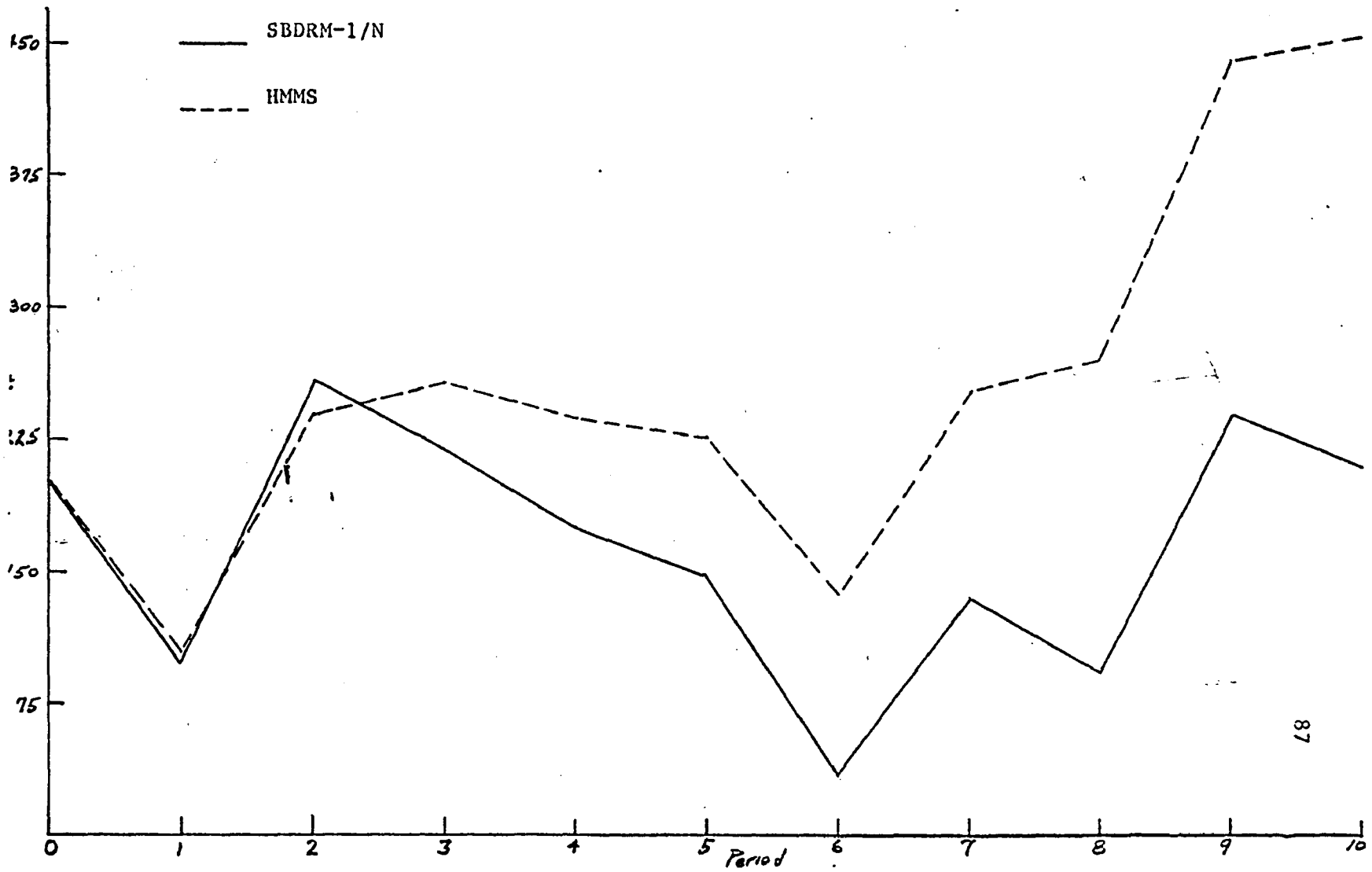


Figure 6. Sample Inventory Values Under Normal Trend Demand - Case 2



demand is a factor directly built into the SBDRM-1/N model. The work force values again exhibit an increasing trend, however in this case the HMMS values increase at a greater rate beginning with period two. The inventory patterns show a striking similarity beginning with period three. The HMMS pattern seems to be just a vertical translation of the SBDRM-1/N values. For Case 2 the major factor contributing to the poorer performance of the HMMS model is the higher inventory levels which lead to an almost twenty thousand dollar cost difference. If one compares the actual demand experienced in each case in order to gain a feel for model performance we find that Case 1 demand is appreciably lower than that of Case 2 for the first six periods while the converse is true for three of the last four periods. Further, demand for Case 1 ranges from a low of 387.69 in period three to a high of 862.26 in period nine with a mean across the ten period planning horizon of 613.99. Case 2 demand ranges from a low of 494.51 to a high of 777.88 in periods two and six respectively with a mean across the ten period planning horizon of 649.69.

In the case of seasonal demand SBDRM-3/N led to lower cost production plans in forty-one of the fifty samples with an average cost difference of \$20,956. Figures 7, 8, and 9 depict production, work force, and inventory values for the demand sample exhibiting the greatest cost difference in favor of SBDRM-3/N. Cost comparisons for these two models for this sample are presented in Table 37. The major savings are

TABLE 37  
COST COMPARISONS NORMAL SEASONAL DEMAND

COST	SBDRM-3/N	HMMS
Regular Payroll	52802	54126
Hiring & Layoff	3379	4524
Overtime & Idletime	357165	367760
Inventory	25422	52394
Total Operating Cost	438768	478804
	100.00%	109.12%

realized in inventory costs as SBDRM-3/N consistently holds lower inventory positions as illustrated by Figure 9. Once again the decision rule model reacts more quickly to changes in actual demand while making more efficient use of inventories to maintain the accepted service level. Further the work force level, which for SBDRM-3/N is calculated by the linear work force rule (1.32), remains high in the last three periods while the HMMS levels decrease in anticipation of the next two periods of base level expected demand.

#### Model Comparisons Under Exponential Demand

Production plans for all three exponential cases, stationary, trend, and seasonal, were developed by the HMMS approach and costed according to (1.35). Means and standard deviations of cost across the samples were calculated and appear in Table 38 along with the corresponding data for the best decision rule models. In all cases the HMMS model

Figure 7. Sample Production Values Under Normal Seasonal Demand

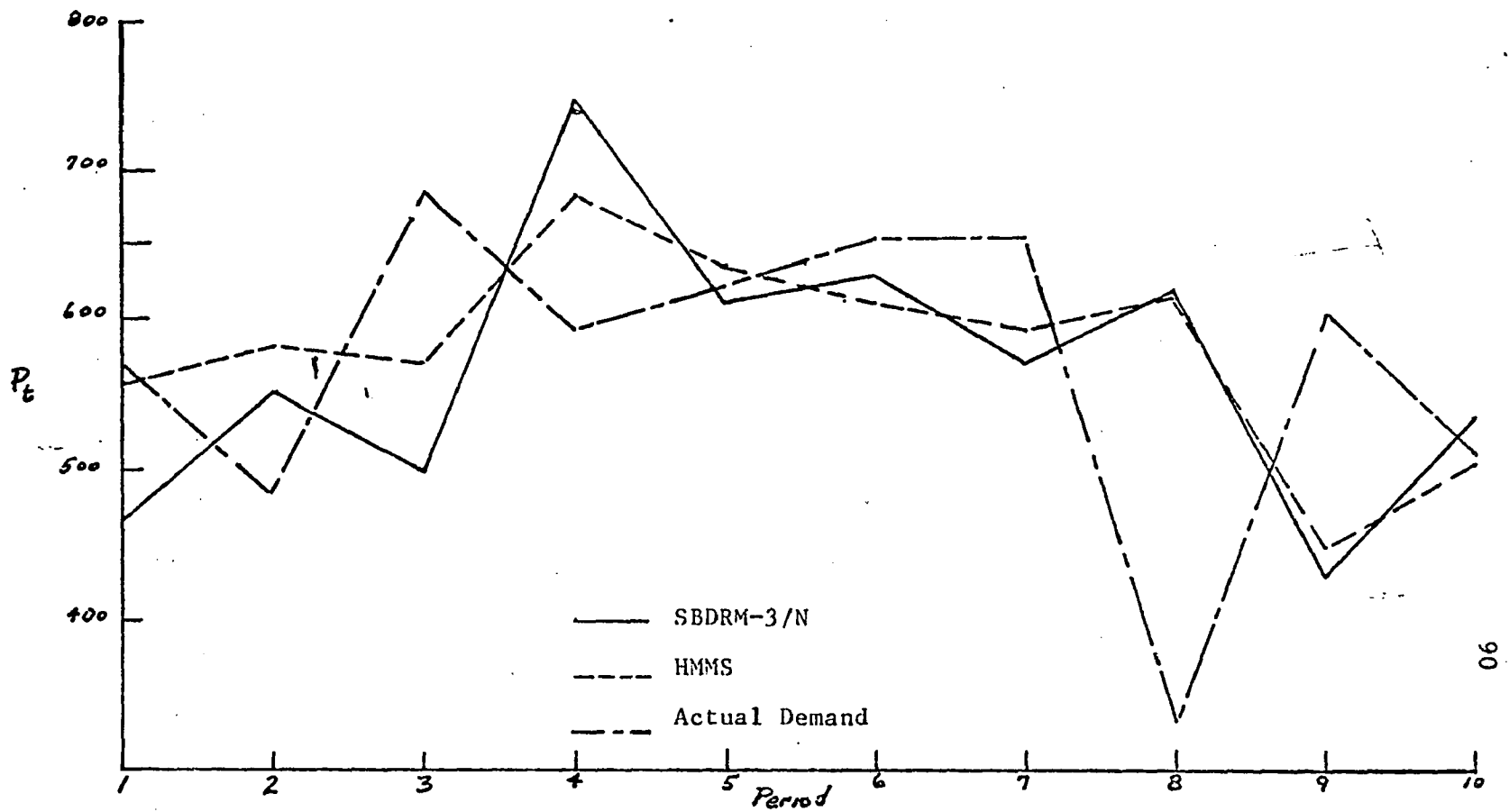


Figure 8. Sample Work Force Values Under Normal Seasonal Demand

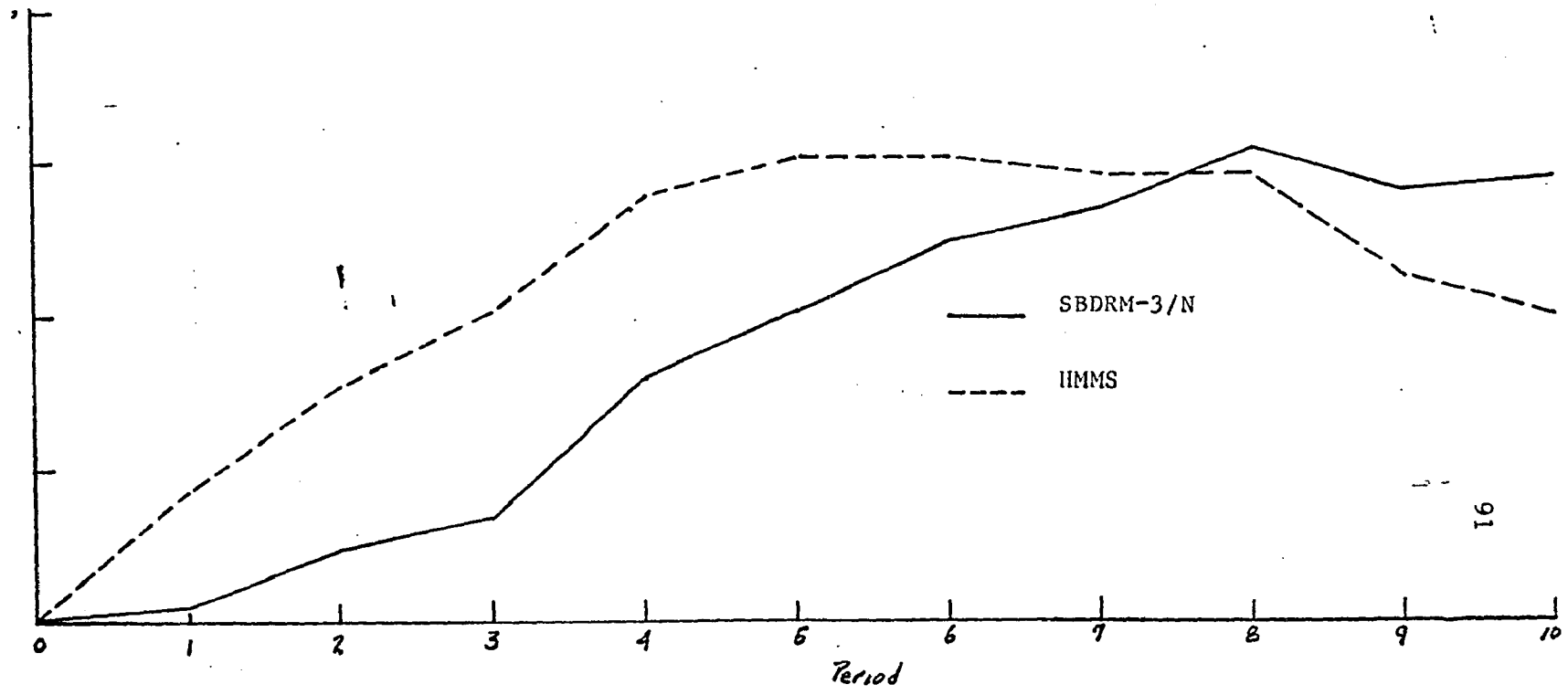
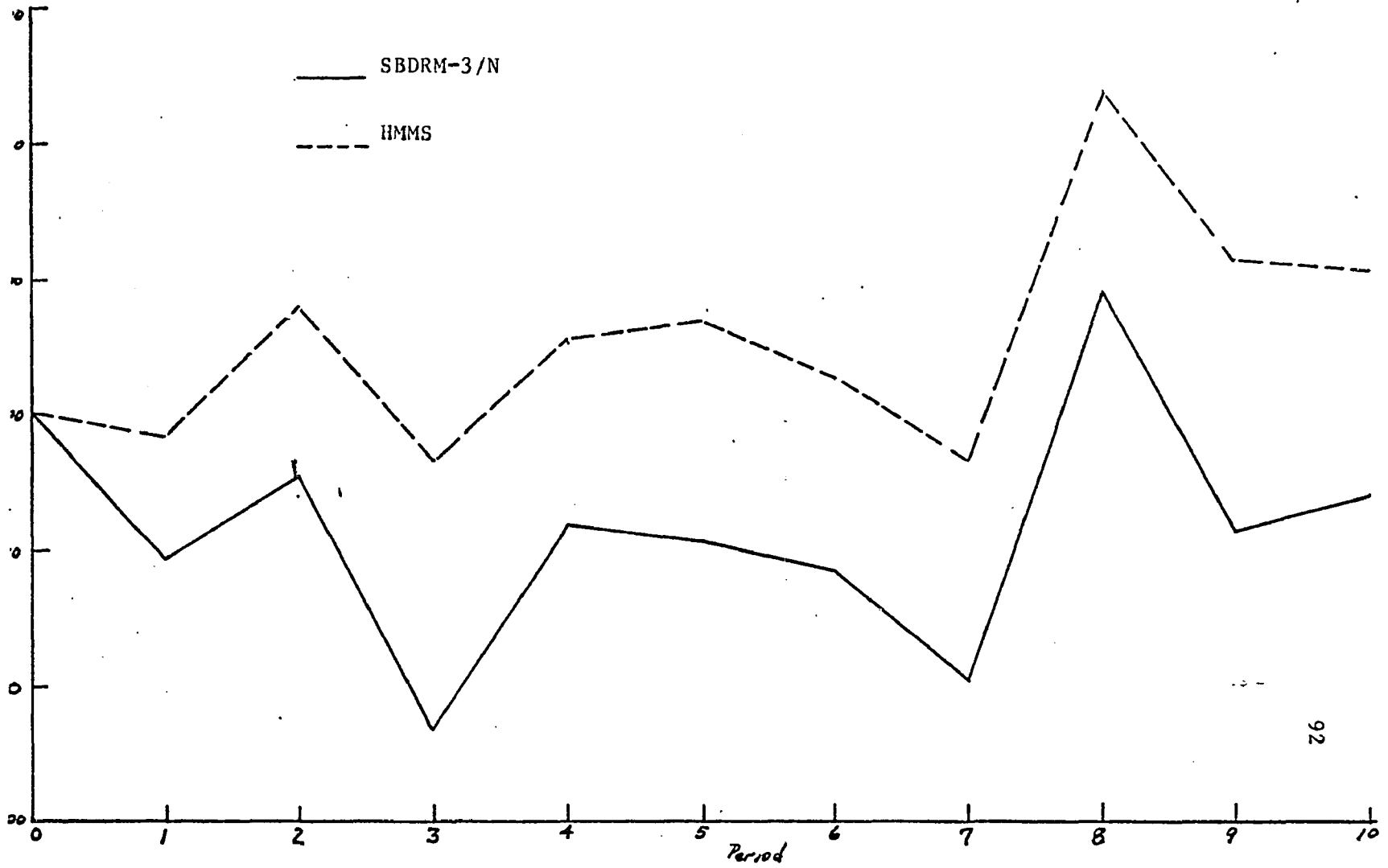


Figure 9. Sample Inventory Values Under Normal Seasonal Demand



exhibits a much lower mean cost and greater standard deviation. For all three demand types the Wilcoxon test resulted in the conclusion that the HMMS model outperformed the particular decision rule model under consideration. For the stationary case  $T=45$  and  $Z=-5.72$  while in both the trend and seasonal cases  $T=0$  and  $Z=-6.15$ .

The HMMS model led to lower cost solutions than the DRM-2,3/EX models in forty-four of the fifty replicates. Over these forty-four samples an average cost savings of \$149,237 was realized. Table 39 provides a cost comparison for the sample for which the cost savings is the greatest. Figures 10, 11, and 12 depict the production, work force, and inventory values as determined by each model in response to this sample of demand. After the first period the production values calculated by the models are compatible. The production value of 1098.29 for the first period as determined by the DRM-2,3/EX models is quite high especially when it is compared to the actual demand of 179.27. This results from the fact that the production value for the first period is simply the sum of the expected demand for that period, which is 500, and the value of the adjustment variable  $\delta_1$ . In both the quadratic and linear optimization problems solved as part of the decision rule model approach, the value of  $\delta_1$  turns out to be a relatively high 598.29. This large production early has two adverse effects on the overall performance of these models. First, high levels of overtime and their concomitant costs are required to achieve this level

TABLE 38  
 COMPARISON OF COST STATISTICS FOR DECISION RULE MODEL AND  
 HMMS LDR PRODUCTION PLANS UNDER EXPONENTIAL DEMAND

PATTERN	STATIONARY			TREND		SEASONAL	
	MODEL	DRM-2,3	HMMS	FBDRM-1	SBDRM-1	HMMS	SBDRM-1
MEAN	715927	591526	898704	898757	422880	813460	404058
S.D.	147771	212320	44148	44425	125426	88395	164872

TABLE 39  
 COST COMPARISONS EXPONENTIAL STATIONARY DEMAND

COST	DRM-2,3/EX	HMMS
Regular Payroll	47955	44609
Hiring & Layoff	9865	7574
Overtime & Idletime	392666	281245
Inventory	206560	114305
Total Operating Cost	657047	447733
	146.75%	100.00%

of production in period one. Second, a net increase in inventory of 912.02 units is experienced in the first period and induces high levels of inventory to be held throughout the planning horizon. The few cases in which DRM-2,3/EX performed acceptably were characterized by levels of actual demand in excess of 1000 units early in the planning horizon which forced higher backorder costs by the HMMS approach.

Figure 10. Sample Production Values Under Exponential Stationary Demand

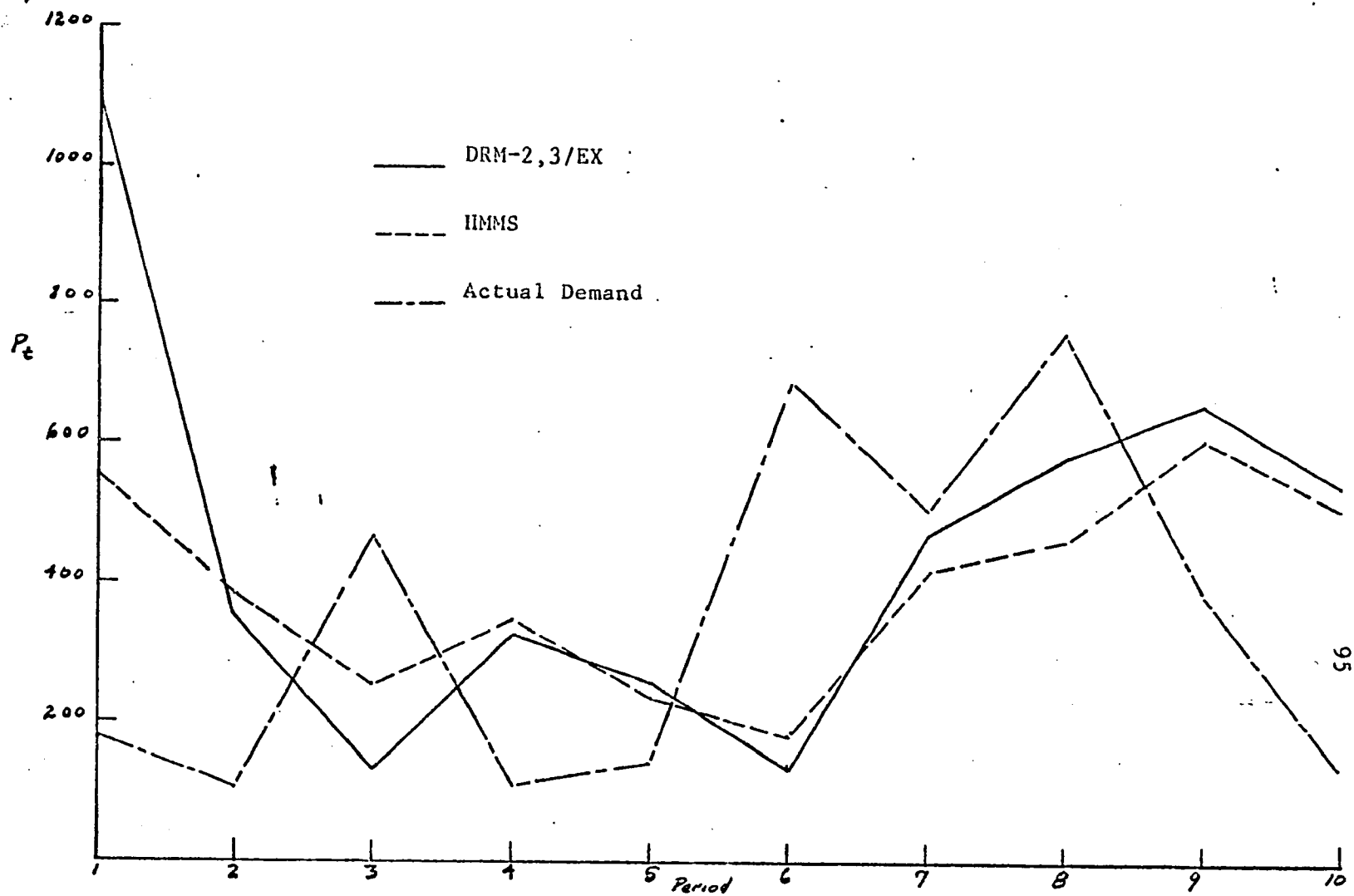


Figure 11. Sample Work Force Values Under Exponential Stationary Demand

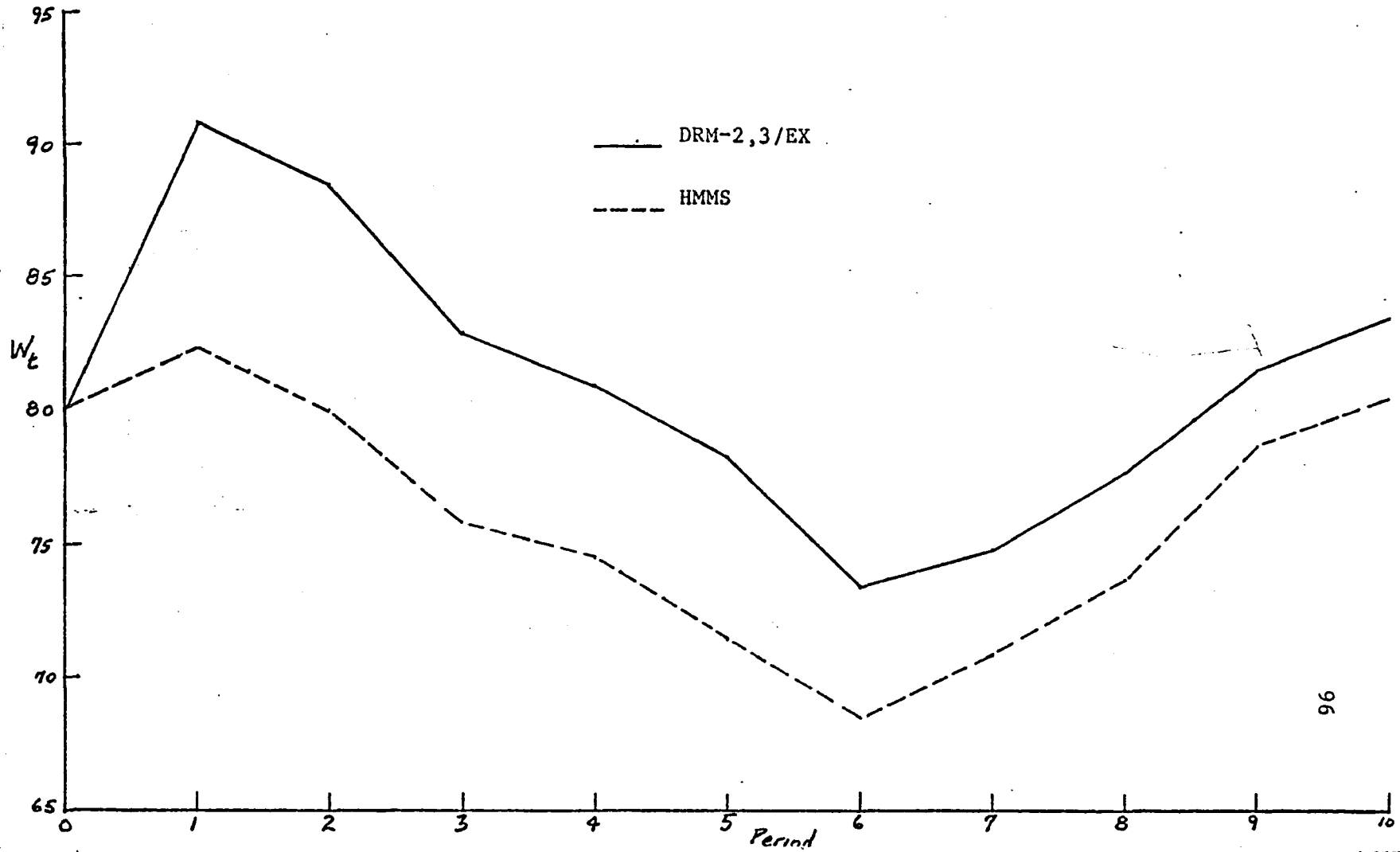


Figure 12. Sample Inventory Values Under Exponential Stationary Demand

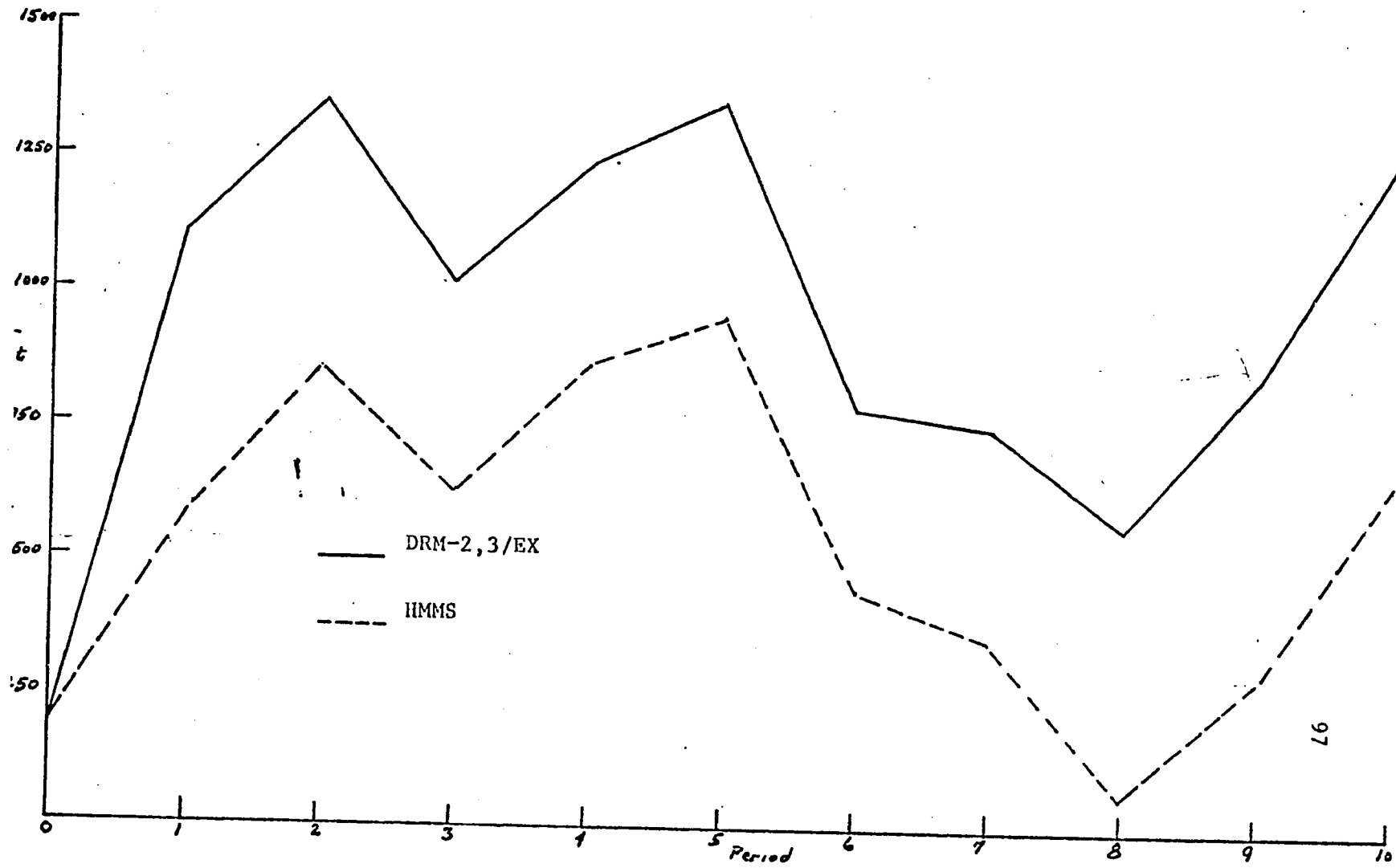


TABLE 40  
COST COMPARISONS EXPONENTIAL TREND DEMAND

COST	SBDRM-1/EX	HMMS
Regular Payroll	60404	39533
Hiring & Layoff	27710	9369
Overtime & Idletime	583335	210497
Inventory	284359	84205
Total Operating Cost	955808	343604
	278.17%	100.00%

Under the assumption of increasing trend demand HMMS led to lower cost production plans in all fifty samples when compared to either the FBDRM-1/EX or the SBDRM-1/EX models. On average HMMS outperformed the former by \$475,885 and the latter by \$475,877 over the six period planning horizon. Table 40 compares overall costs for the sample for which cost savings over SBDRM-1/EX are the greatest. Figures 13, 14, and 15 depict production, work force, and inventory values for this sample. The production values supplied by both decision rule models are quite high in periods 1, 4, 5, and 6 due to high values of the epsilon adjustment variables for those periods as determined by the solution of the respective quadratic programming models. This, in effect, serves to mask the ability of the production rules to respond accurately to changes in actual demand. In addition this has the consequences of maintaining inventory at extremely high levels.

Figure 13. Sample Production Values Under Exponential Trend Demand

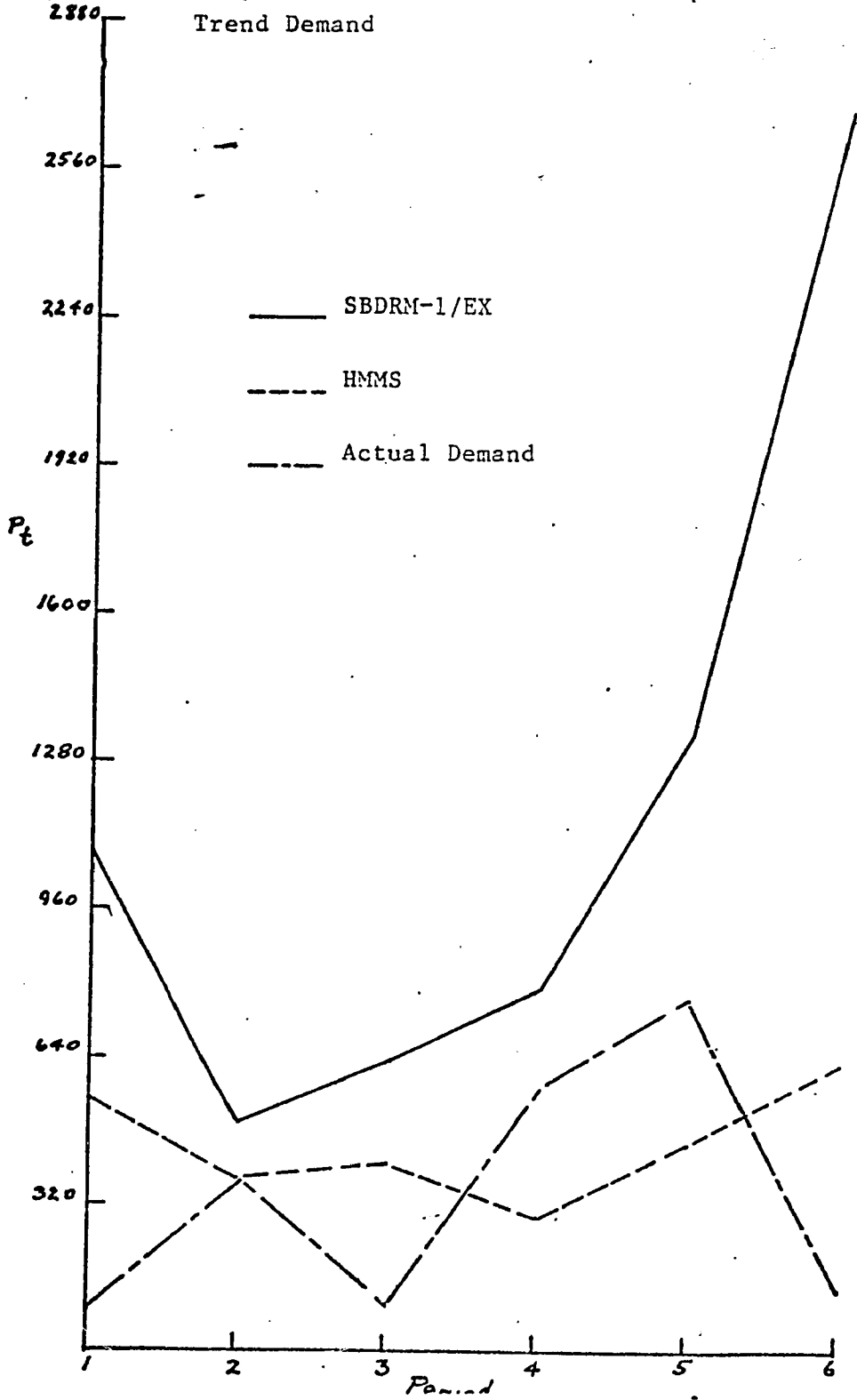
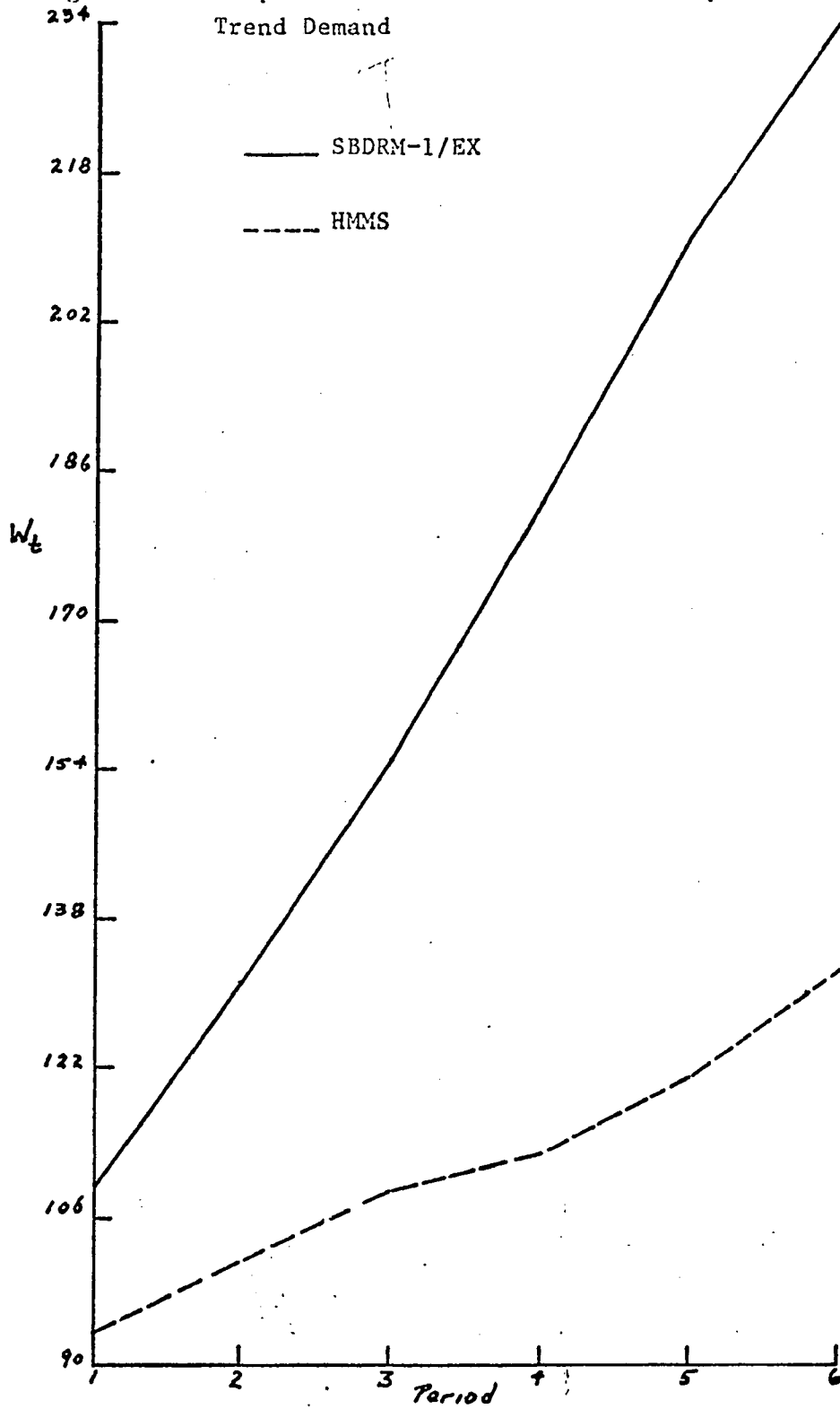


Figure 14. Sample Work Force Values Under Exponential Trend Demand



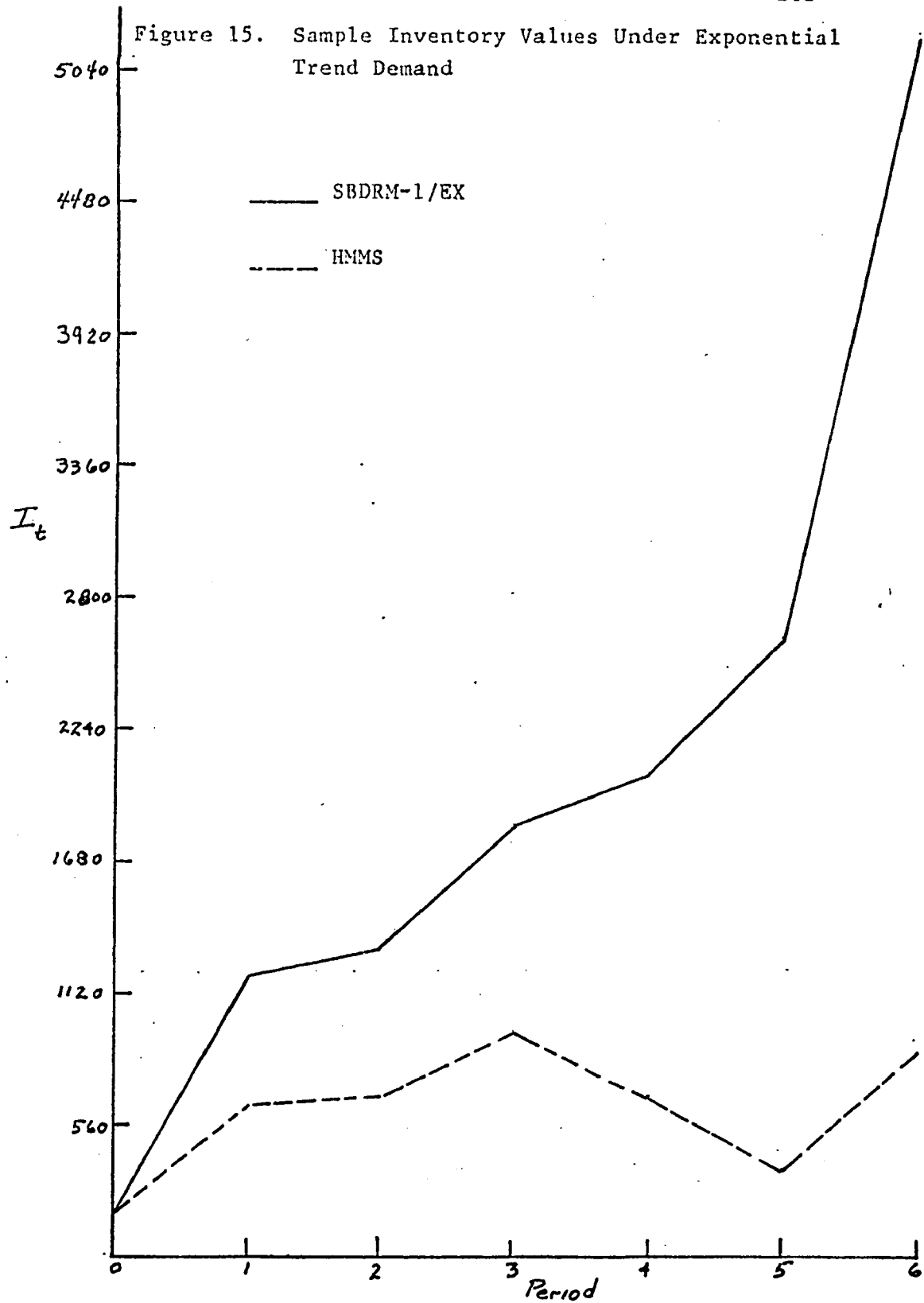


TABLE 41  
COST COMPARISONS EXPONENTIAL SEASONAL DEMAND

COST	SBDRM-1/EX	HMMS
Regular Payroll	54995	33990
Hiring & Layoff	23168	4361
Overtime & Idletime	539683	186990
Inventory	219122	59077
Total Operating Cost	836968	284418
	294.27%	100.00%

As a result inordinately high costs are experienced especially in the last two periods.

Much the same result was experienced under seasonal demand as was under trend demand. The HMMS model outperformed SBDRM-1/EX for all fifty samples with an average cost differential of \$409,771 over six periods. Table 41 compares overall costs for the sample which exhibited the greatest cost savings. Figures 16, 17, and 18 depict production, work force, and inventory for this sample. Again the high values for epsilon in periods one, four, five, and six dictate high production values in those periods and the accompanying high inventories. As previously noted costs in the final two periods were inordinately high.

#### An Experiment With a Decreased Service Level

In view of the poor overall performance of the stochastic

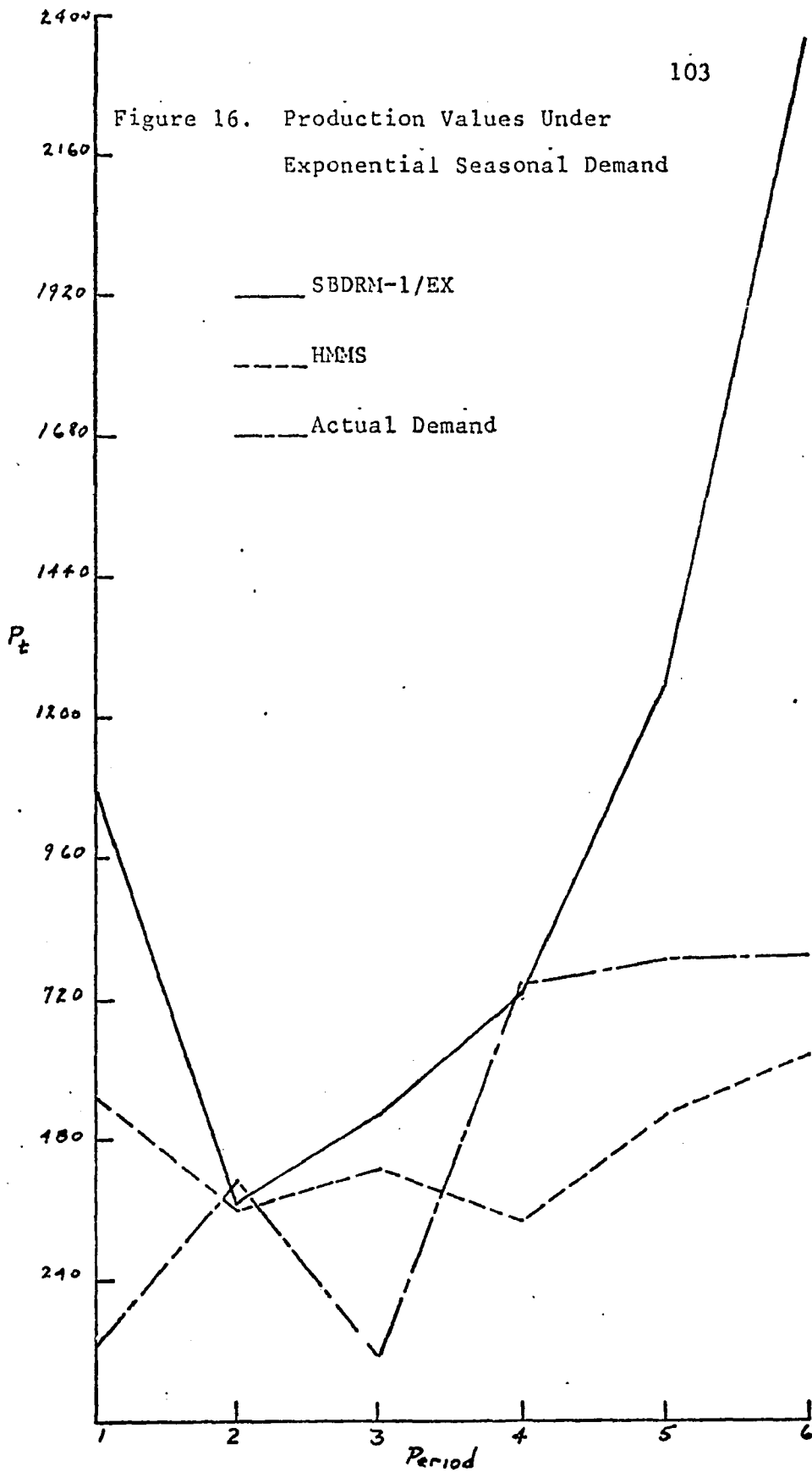


Figure 17. Sample Work Force Values Under Exponential Seasonal Demand

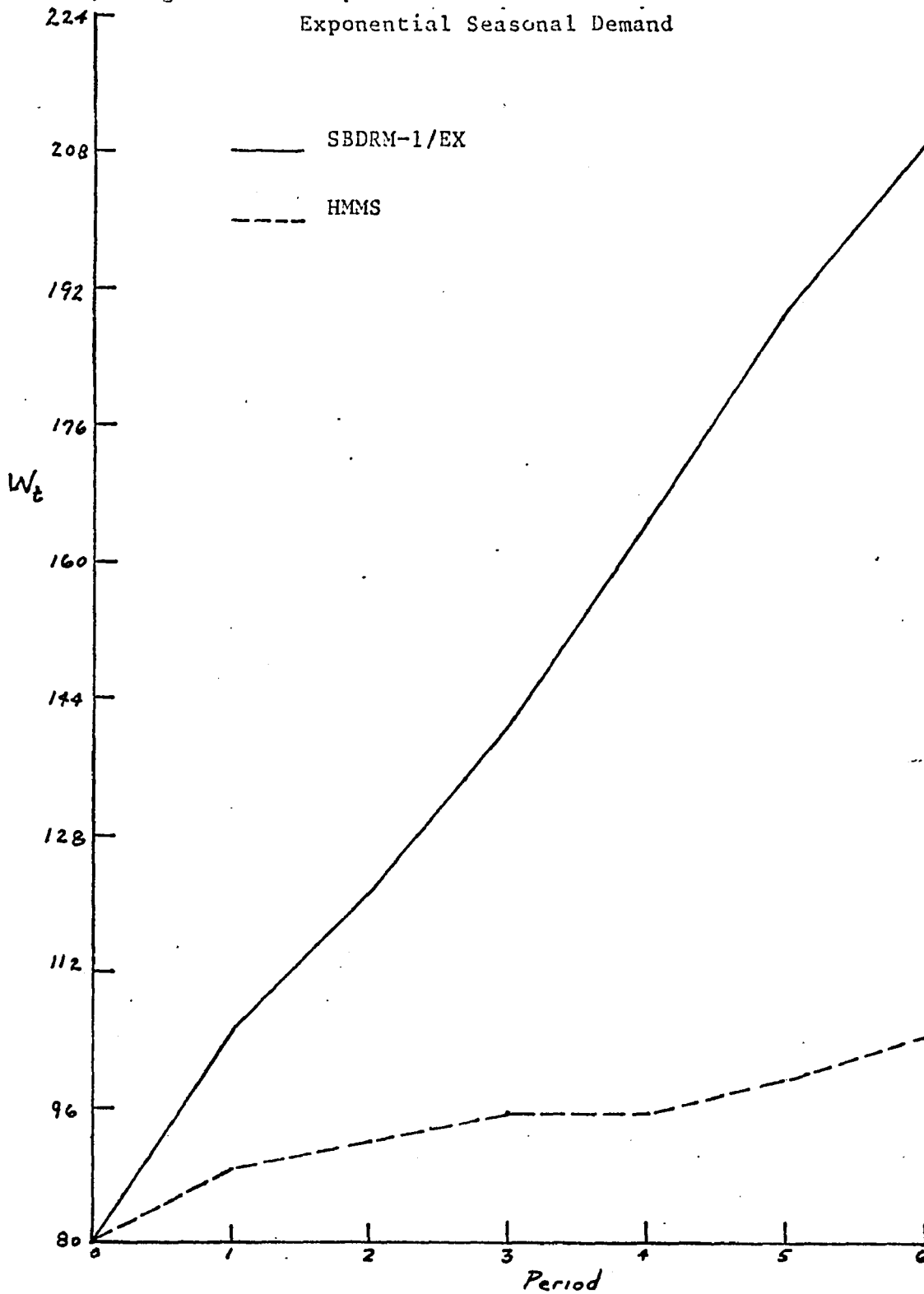


Figure 18. Sample Inventory Values Under Exponential Seasonal Demand

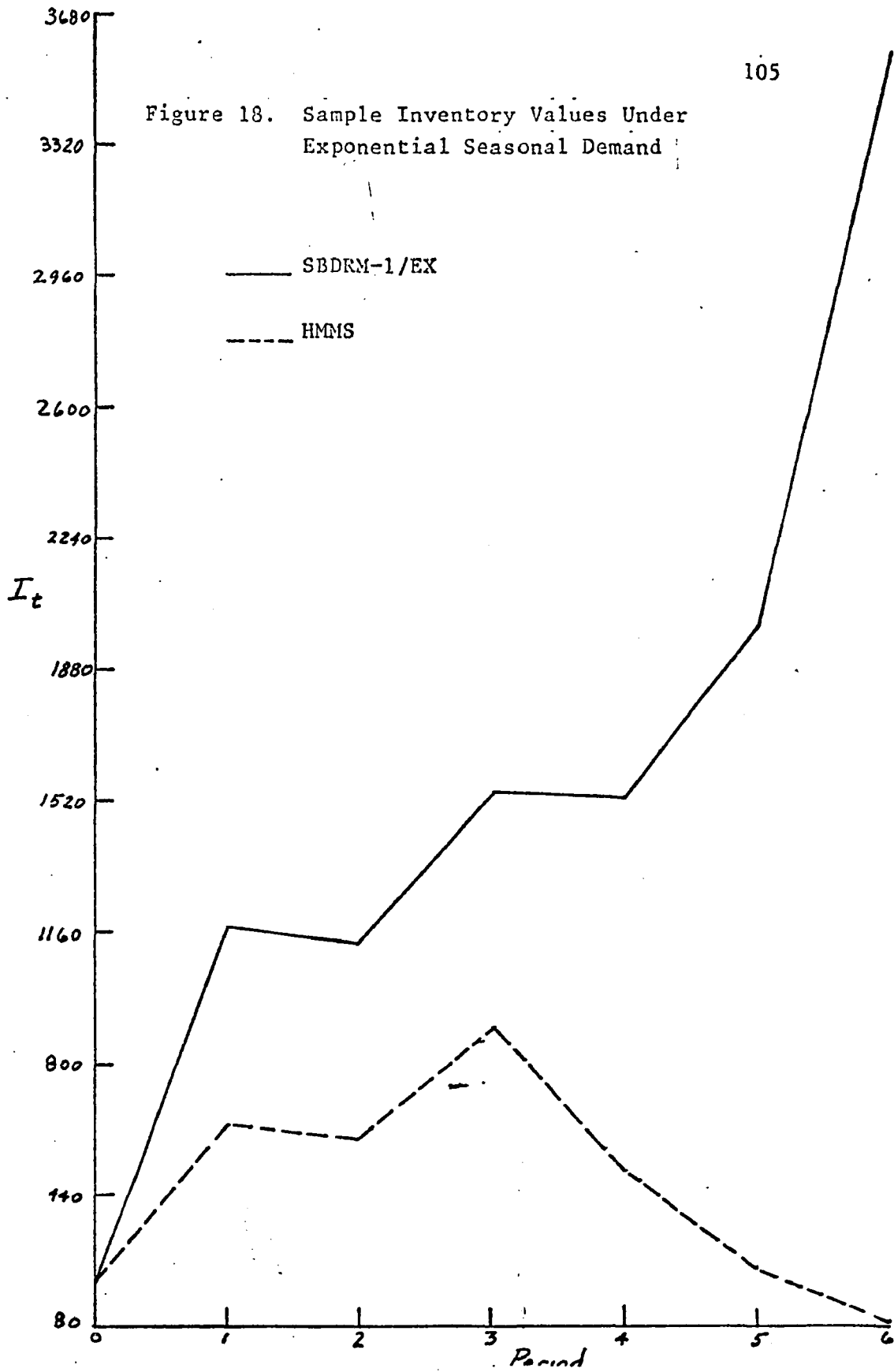


TABLE 42  
 OPTIMAL VALUES OF  $(I_t^+, I_t^-, \delta_t)$  FOR DRM-3/EX LINEAR PROGRAMMING  
 MODEL UNDER EXPONENTIAL STATIONARY DEMAND ( $u_t=0.75$ )

t	DV	$I_t^+$	$I_t^-$	$\delta_t$
1		154.20	0.0	-45.80
2		189.18	0.0	34.98
3		189.18	0.0	0.00
4		189.18	0.0	0.00
5		189.18	0.0	0.00
6		189.18	0.0	0.00
7		189.18	0.0	0.00
8		189.18	0.0	0.00
9		189.18	0.0	0.00
10		189.18	0.0	0.00

$W_0=80, I_0=200$

decision rule models vis-a-vis the HMMS model for the exponential demand cases one further experiment was conducted. For all previous experiments the constraint tolerance level, or system service level,  $u_t$ , was set at a constant 0.95 for all time periods. In this experiment DRM-3/EX was resolved for the case of stationary exponential demand with a constant 0.75 value for  $u_t$ , the service level, across the planning horizon. These results are presented in Table 42. Production plans were calculated based on these results and the random sample of fifty replicates previously utilized in the model testing phase of the stationary exponential research. Costs were calculated according to (1.35). A comparison of the mean cost of the production plans generated with service levels

of 0.95 and 0.75 shows the latter to be lower by \$121,027. Further the mean cost of \$594,900 for DRM-3/EX with  $u_t=0.75$  compares favorably with the \$591,526 exhibited by the HMMS model. A Wilcoxon test was performed to determine whether a significant difference between the performance of these models existed. This resulted in values of  $T=585$  and  $Z=-0.507$  indicating that no significant difference between the performance of the competing models existed.

The performance of DRM-3/EX with a seventy-five percent service level is encouraging especially when one considers the fact that this model, while the best for  $u_t=0.95$ , is not necessarily the best decision rule model for  $u_t=0.75$ . It was arbitrarily selected to determine whether decreasing the service level could lead to an improvement in performance for a particular model sufficient enough to compete effectively with HMMS. The findings suggest that a complete analysis of all cases of exponential demand with decreased service levels is in order for the future.

## CHAPTER VI

### SUMMARY CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER STUDIES

The purpose of this research has been the development and testing of models for the solution of the aggregate production planning problem under different types of stochastic demand distributions. The approach focused on chance-constrained programming decision rule models as a technique to deal with stationary, trend, and seasonal stochastic demand patterns. The HMMS LDR approach was used as a criterion for comparison to judge the relative performance of the proposed models.

The first set of studies concerned normally distributed trend and seasonal demand patterns. In both cases the best decision rule models performed credibly. Under trend demand no significant difference in performance was found between the SBDRM-1/N and HMMS models while for the seasonal demand situation SBDRM-3/N significantly outperformed HMMS. It is interesting to note that for both demand types the sales based rules emerged as the better of the decision rule types. Further, the alpha value of 0.4 indicates that forecasted demand for the present period is an important element that should be included in the decision rule formulation. These

findings provide a basic foundation for future, more extensive, work on the use of the sales based rules models under these demand conditions. Of particular interest is the success of the linear programming SBDRM-3/N model in the case of seasonal demand since LP finds wide acceptance in industry as a technique for production planning. Future studies should include investigations into the impact of allowing variances to fluctuate and become greater as we move further out in the planning horizon, into possible horizon effects that are inherent in multiple period optimization models, and the utilization of the models in a rolling horizon scheme.

The set of studies conducted under the assumption of exponentially distributed demand were initially focused on two aims. The first was to test the efficiency of the Gupta and Sengupta decision rule models under a different demand assumption. Here it was found that when demand was exponential stationary and a ninety-five percent service level was selected, the HMMS model outperformed DRM-2,3/EX. The second was to test the forecast based and sales based decision rule models under trend and seasonal demand patterns. In both these cases, when a service level of 0.95 was assumed, the decision rule models were greatly outperformed by the HMMS approach. For all three demand types the decision rule models suffered from the fact that the results of the solutions to the optimization problems included extremely high values for the adjustment variables delta (in the stationary case) and epsilon (in the trend and seasonal cases). In view of this

a final experiment was conducted to gain some insight on the effect of a less stringent service level requirement on the performance of the best decision rule model, DRM-3/EX, in the stationary case. This resulted in values of delta of a much closer order of magnitude to those presented by Gupta and Sengupta [3] for the normal stationary case. The improvement in the performance of DRM-3/EX for  $u_t=0.75$  lends some support to the idea that expecting the system to perform up to a 95% service level when faced with exponentially distributed demand may be impractical in view of the wider possible range of realized values for demand when compared to the normal distribution. Future research in this area should be directed toward investigating the proposed models under less stringent service levels for each of the demand types.

The contributions of this research to the area of aggregate production planning are significant. Perhaps the most important is the fact that its results present prima-facie evidence for the position that demand characteristics must be considered when selecting an aggregate production planning model. Further it extends stochastic programming approaches to the problem by investigating nonstationary demand patterns including both trend and seasonal. Specifically the newly proposed models perform well under the normal demand assumption indicating that knowledge of the complete distribution of demand in each future period can lead to results superior to the HMMS approach which requires knowledge only of the expected value of demand. Finally it points out the

difficulty associated with the decision rule model approach when confronted with the exponential distribution and suggests that future research consider the efficacy of less stringent service level requirements when utilizing these models for demand of this nature. Together these major studies should aid the systems analyst and planner in developing a taxonomy of model selection based on the demand characteristics he faces.

APPENDIX A  
MATHEMATICAL MODEL DEVELOPMENT

## Exponential Stationary Demand

Objective Function - DRM-1/EX

For this case the objective function is identical to the Gupta and Sengupta formulation presented herein as equation (1.30).

Chance Constraints

We begin with the following chance constraints,

$$\text{Prob} \left[ I_0 + \bar{S} + \delta_1 - S_1 \geq 0 \right] \geq u_1 \quad \text{for } t=1 \quad (1.23)$$

$$\text{Prob} \left[ I_0 + (2 - \alpha) \bar{S} + \sum_{i=1}^t \delta_i - (1 - \alpha) S_{t-1} \right] \geq u_t \quad \text{for } t=2, 3, \dots, T. \quad (1.24)$$

If we now assume a two-parameter exponential distribution for demand that is identical across time then

$$f(S) = \mu e^{-\mu(S - S_0)} \quad S_0 \leq S < \infty.$$

The problem now becomes one of determining the deterministic equivalents of (1.23) and (1.24). Isolating the random variables on the right hand side of the inequalities gives

$$\text{Prob} \left[ I_0 + \bar{S} + \delta_1 \geq S_1 \right] \geq u_1 \quad \text{for } t=1$$

$$\text{Prob} \left[ I_0 + (2 - \alpha) \bar{S} + \sum_{i=1}^t \delta_i \geq S_t + (1 - \alpha) S_{t-1} \right] \geq u_t \quad \text{for } t=2, 3, \dots, T.$$

Utilizing the quantile rule the following deterministic equivalents emerge

$$I_0 + \bar{S} + \delta_1 \geq F_S^{-1}(u_1) \quad \text{for } t=1$$

$$I_0 + (2 - \alpha) \bar{S} + \sum_{i=1}^t \delta_i \geq F_Z^{-1}(u_t) \quad \text{for } t=2, 3, \dots, T.$$

where  $F_S^{-1}(u_1)$  is the inverse of the c.d.f. for  $S$ .

$F_Z^{-1}(u_t)$  is the inverse of the convoluted c.d.f. of

$$Z = S_t + (1 - \alpha)S_{t-1}.$$

### Production Nonnegativity Constraints

We begin with the Gupta and Sengupta production rules and constrain them to be nonnegative as follows

$$P_1 = \bar{S} + \delta_1 \geq 0 \quad t=1$$

$$P_2 = \alpha S_1 + (1 - \alpha)\bar{S} + \delta_2 \geq 0 \quad t=2$$

$$P_t = \alpha S_{t-1} + (1 - \alpha)S_{t-2} + \delta_t \quad t=3, \dots, T.$$

Solving for  $\delta_t$  gives

$$\delta_1 \geq -\bar{S} \quad t=1$$

$$\delta_2 \geq -(1 - \alpha)\bar{S} - \min \alpha(S_1) \quad t=2$$

$$\delta_t \geq -\min(\alpha S_{t-1} + (1 - \alpha)S_{t-2}) \quad t=3, \dots, T.$$

Under the two-parameter exponential distribution assumption  $\min(S_t) = S_0$  for  $t=1, 2, \dots, T$ , therefore

$$\delta_1 \geq -\bar{S} \quad t=1$$

$$\delta_2 \geq -(1 - \alpha)\bar{S} - \alpha S_0 \quad t=2$$

$$\delta_t \geq -S_0 \quad t=3, \dots, T.$$

### Forecast Based Exponential Models

#### Objective Function - FBDRM-1/EX

We begin with equation (1.12) as follows

$$\sum_{t=1}^T \left[ (c_1 - c_6)W_t + c_2(W_t - W_{t-1})^2 + c_3(P_t - c_4W_t)^2 + c_5P_t + c_7(I_t - c_8 - c_9\bar{S}_t)^2 \right]$$

Substituting for  $I_t$ :  $I_t = I_0 + \sum_{i=1}^t P_i - \sum_{i=1}^t S_i$  gives

$$\sum_{t=1}^T \left[ (c_1 - c_6)W_t + c_2(W_t - W_{t-1})^2 + c_3(P_t - c_4W_t)^2 + c_5P_t + c_7 \left( I_0 + \sum_{i=1}^t P_i - \sum_{i=1}^t S_i - c_8 - c_9\bar{S}_t \right)^2 \right]$$

Taking the expected value with respect to S gives

$$\sum_{t=1}^T \left[ (c_1 - c_6)W_t + c_2(W_t - W_{t-1})^2 + c_3(P_t - c_4W_t)^2 + c_5P_t + c_7(I_0 + \sum_{i=1}^t P_i - \sum_{i=1}^t \bar{S}_i - c_8 - c_9\bar{S}_t)^2 \right] \quad (\text{A-1})$$

Next we take the expectation of the forecast based decision rules giving

$$\begin{aligned} P_1 &= \bar{S}_1 + \epsilon_1 & t=1 \\ P_t &= \bar{S}_t + \alpha(\bar{S}_{t-1} - \bar{S}_{t-1}) + \epsilon_t = \bar{S}_t + \epsilon_t & t=2, 3, \dots, T. \end{aligned} \quad (\text{A-2})$$

Substituting (A-2) into (A-1) gives

$$\sum_{t=1}^T \left[ (c_1 - c_6)W_t + c_2(W_t - W_{t-1})^2 + c_3(\bar{S}_t + \epsilon_t - c_4W_t)^2 + c_5(\bar{S}_t + \epsilon_t) + c_7(I_0 + \sum_{i=1}^t \epsilon_i - c_8 - c_9\bar{S}_t)^2 \right]$$

Since  $c_5\bar{S}_t$  is a constant for any solution we may drop it from the objective function giving

$$\sum_{t=1}^T C_t = \sum_{t=1}^T \left[ (c_1 - c_6)W_t + c_2(W_t - W_{t-1})^2 + c_3(\bar{S}_t + \epsilon_t - c_4W_t)^2 + c_5\epsilon_t + c_7(I_0 + \sum_{i=1}^t \epsilon_i - c_8 - c_9\bar{S}_t)^2 \right]$$

### Chance Constraints

We begin with the inventory balance relationships

$$\begin{aligned} I_1 &= I_0 + P_1 - S_1 & t=1 \\ I_t &= I_0 + \sum_{i=1}^t P_i - \sum_{i=1}^t S_i & t=2, 3, \dots, T. \end{aligned} \quad (\text{A-3})$$

Next we state the forecast based rules

$$\begin{aligned} P_1 &= \bar{S}_1 + \epsilon_1 & t=1 \\ P_t &= \bar{S}_t + \alpha(S_{t-1} - \bar{S}_{t-1}) & t=2, 3, \dots, T. \end{aligned} \quad (\text{A-4})$$

Substituting (A-4) into (A-3) gives

$$I_1 = I_0 + \bar{S}_1 - S_1 + \epsilon_1 \quad t=1$$

$$I_t = I_0 + \sum_{i=1}^t \bar{S}_i - \sum_{i=1}^t S_i - \alpha \sum_{i=1}^{t-1} \bar{S}_i + \alpha \sum_{i=1}^{t-1} S_i + \sum_{i=1}^t \epsilon_i \quad t=2,3,\dots,T. \quad (A-5)$$

Converting (A-5) into chance constraints gives

$$\text{Prob} [I_0 + \bar{S}_1 + S_1 + \epsilon_1 \geq 0] \geq u_1 \quad t=1$$

$$\text{Prob} [I_0 + \sum_{i=1}^t \bar{S}_i - \sum_{i=1}^t S_i - \alpha \sum_{i=1}^{t-1} \bar{S}_i + \alpha \sum_{i=1}^{t-1} S_i + \sum_{i=1}^t \epsilon_i \geq 0] \geq u_t \quad t=2,3,\dots,T.$$

Isolating the random variables on the right hand side gives

$$\text{Prob} [I_0 + \bar{S}_1 + \epsilon_1 \geq S_1] \geq u_1 \quad t=1$$

$$\text{Prob} [I_0 + \sum_{i=1}^t \bar{S}_i - \alpha \sum_{i=1}^{t-1} \bar{S}_i + \sum_{i=1}^t \epsilon_i \geq S_t + (1-\alpha) \sum_{i=1}^{t-1} S_i] \geq u_t \quad t=2,3,\dots,T.$$

Assuming that  $S_i \sim f(S_i) = \mu_i e^{-\mu_i(S_i - S_0)}$   $S_0 \leq S_i \leq \infty$

we may write the following deterministic equivalents

$$I_0 + \bar{S}_1 + \epsilon_1 \geq F_{S_1}^{-1}(u_1) \quad t=1$$

$$I_0 + \sum_{i=1}^t \bar{S}_i - \alpha \sum_{i=1}^{t-1} \bar{S}_i + \sum_{i=1}^t \epsilon_i \geq F_{Z_t}^{-1}(u_t) \quad t=2,3,\dots,T.$$

where  $F_{S_1}^{-1}(u_1)$  is the inverse of the c.d.f. for  $S_1$ .

$F_{Z_t}^{-1}(u_t)$  is the inverse of the convoluted c.d.f. of

$$Z = S_t + (1-\alpha) \sum_{i=1}^{t-1} S_i \quad \text{for } t=2,3,\dots,T.$$

### Production Nonnegativity Constraints

We begin with the Forecast Based production rules and constrain them to be nonnegative

$$P_1 = \bar{S}_1 + \epsilon_1 \geq 0 \quad t=1$$

$$P_t = \bar{S}_t + \alpha(S_{t-1} - \bar{S}_{t-1}) + \epsilon_t \geq 0 \quad t=2,3,\dots,T.$$

Solving for  $\epsilon_t$  gives

$$\epsilon_1 \geq -\bar{S}_1 \quad t=1$$

$$\epsilon_t \geq -\bar{S}_t - \min \alpha(S_{t-1}) + \alpha \bar{S}_{t-1} \quad t=2,3,\dots,T.$$

Since  $\min(S_{t-1}) = S_0$  this becomes

$$\epsilon_t \geq -\bar{S}_t + \alpha(\bar{S}_{t-1} - S_0) \quad t=2,3,\dots,T.$$

### Sales Based Exponential Models

#### Objective Function - SBDRM-1/EX

The analysis is the same as for the forecast based model previously discussed through equation (A-1). Hence we begin with a restatement of this relationship,

$$\sum_{t=1}^T \left[ (c_1 - c_6)W_t + c_2(W_t - W_{t-1})^2 + c_3(P_t - c_4W_t)^2 + c_5P_t + c_7(I_0 + \sum_{i=1}^t P_i - \sum_{i=1}^t \bar{S}_i - c_8 - c_9\bar{S}_t)^2 \right]. \quad (A-1)$$

Next we take the expectation of the sales based decision rules giving

$$P_1 = \bar{S}_1 + \epsilon_1 \quad t=1$$

$$P_2 = \bar{S}_{t-1} + \alpha(\bar{S}_t - \bar{S}_{t-1}) + \epsilon_t$$

t=1

(A-6)

t=2,3,\dots,T.

Substituting (A-6) into (A-1) gives

$$\begin{aligned} & [(c_1 - c_6)W_1 + c_2(W_1 - W_0)^2 + c_3(\bar{S}_1 + \epsilon_1 - c_4W_1)^2 + c_5(\bar{S}_1 + \epsilon_1) \\ & + c_7(I_0 + \epsilon_1 - c_8 - c_9\bar{S}_1)^2] + \sum_{t=2}^T [(c_1 - c_6)W_t + c_2(W_t - W_{t-1})^2 \\ & + c_3((1 - \alpha)\bar{S}_{t-1} + \alpha\bar{S}_t + \epsilon_t - c_4W_t)^2 + c_5((1 - \alpha)\bar{S}_{t-1} + \alpha\bar{S}_t + \epsilon_t) \\ & + c_7(I_0 + \sum_{i=1}^t \epsilon_i + (1 - \alpha)\bar{S}_1 + (\alpha - c_9 - 1)\bar{S}_t - c_8)^2] \end{aligned}$$

Since  $c_5((1-\alpha)\bar{S}_{t-1} + \alpha\bar{S}_t)$  is a constant for any solution we may drop it from the objective function giving

$$\begin{aligned} \sum_{t=1}^T C_t = & [(c_1 - c_6)W_1 + c_2(W_1 - W_0)]^2 + c_3(\bar{S}_1 + \epsilon_1 - c_4W_1)^2 + c_5\epsilon_1 \\ & + c_7(I_0 + \epsilon_1 - c_8 - c_9\bar{S}_1)^2 + \sum_{t=2}^T [(c_1 - c_6)W_t + c_2(W_t - W_{t-1})^2 \\ & + c_3((1-\alpha)\bar{S}_{t-1} + \alpha\bar{S}_t + \epsilon_t - c_4W_t)^2 + c_5\epsilon_t \\ & + c_7(I_0 + \sum_{i=1}^t \epsilon_i + (1-\alpha)\bar{S}_1 + (\alpha - c_9 - 1)\bar{S}_t - c_8)^2] \end{aligned}$$

### Chance Constraints

We begin with the inventory balance relationships

$$\begin{aligned} I_1 &= I_0 + P_1 - S_1 & t=1 \\ I_t &= I_0 + \sum_{i=1}^t P_i - \sum_{i=1}^t S_i & t=2, 3, \dots, T. \end{aligned} \tag{A-3}$$

Next we state the sales based rules

$$\begin{aligned} P_1 &= \bar{S}_1 + \epsilon_1 & t=1 \\ P_t &= S_{t-1} + \alpha(\bar{S}_t - S_{t-1}) + \epsilon_t & t=2, 3, \dots, T. \end{aligned} \tag{A-7}$$

Substituting (A-7) into (A-3) gives

$$\begin{aligned} I_1 &= I_0 + \bar{S}_1 - S_1 + \epsilon_1 & t=1 \\ I_t &= I_0 + \bar{S}_1 + \alpha \left( \sum_{j=2}^t \bar{S}_j - \sum_{i=1}^{t-1} S_i \right) - S_t + \sum_{i=1}^t \epsilon_i & t=2, 3, \dots, T. \end{aligned} \tag{A-8}$$

Isolating the random variables on the right hand side gives

$$\begin{aligned} \text{Prob} [I_0 + \bar{S}_1 + \epsilon_1 \geq S_1] &\geq u_1 & t=1 \\ \text{Prob} [I_0 + \bar{S}_1 + \alpha \sum_{j=2}^t \bar{S}_j + \sum_{i=1}^t \epsilon_i \geq S_t + \alpha \sum_{i=1}^{t-1} S_i] &\geq u_t & t=2, 3, \dots, T. \end{aligned}$$

Assuming that  $S_i \sim f(S_i) = \mu_i e^{-\mu_i(S_i - S_0)} \quad S_0 \leq S_i \leq \infty$

then we may write the following deterministic equivalents

$$I_0 + \bar{S}_1 + \epsilon_1 \geq F_{S_1}^{-1}(u_1) \quad t=1$$

$$I_0 + \bar{S}_1 + \alpha \sum_{j=2}^t \bar{S}_j + \sum_{i=1}^t \epsilon_i \geq F_{Z_t}^{-1}(u_t) \quad t=2,3,\dots,T.$$

where  $F_{S_1}^{-1}(u_1)$  is the inverse of the c.d.f. for  $S_1$ .

$F_{Z_t}^{-1}(u_t)$  is the inverse of the convoluted c.d.f. of

$$Z_t = S_t + \alpha \sum_{i=1}^{t-1} S_i \quad \text{for } t=2,3,\dots,T.$$

### Production Nonnegativity Constraints

We begin with the sales based production rules and constrain them to be nonnegative

$$P_1 = \bar{S}_1 + \epsilon_1 \geq 0 \quad t=1$$

$$P_t = S_{t-1} + \alpha(\bar{S}_t - S_{t-1}) + \epsilon_t \geq 0 \quad t=2,3,\dots,T.$$

Solving for  $\epsilon_t$  gives

$$\epsilon_1 \geq -\bar{S}_1 \quad t=1$$

$$\epsilon_t \geq -\min(S_{t-1}) - \alpha \bar{S}_t + \min \alpha(S_{t-1})$$

$$\epsilon_t \geq -S_0 - \alpha \bar{S}_t + \alpha S_0$$

$$\epsilon_t \geq -(1-\alpha)S_0 - \alpha \bar{S}_t \quad t=2,3,\dots,T.$$

### Forecast Based Normal Models

#### Objective Function - FBDRM-1/N

For this case the form of the objective function is identical to the forecast based exponential previously derived in this appendix.

#### Chance Constraints

The analysis is the same as for the forecast based

exponential case through the chance constraints with the random variables isolated, which we now restate

$$\text{Prob} \left[ I_0 + \bar{S}_1 + \epsilon_1 \geq S_1 \right] \geq u_1 \quad t=1$$

$$\text{Prob} \left[ I_0 + \sum_{i=1}^t \bar{S}_i - \alpha \sum_{i=1}^{t-1} \bar{S}_i + \sum_{i=1}^t \epsilon_i \geq S_t + (1-\alpha) \sum_{i=1}^{t-1} S_i \right] \geq u_t$$

t=2,3,...,T.

Assuming that  $S_i \sim N(\bar{S}_i, \sigma_i^2)$  and  $S_1, S_2, \dots, S_T$  are independent, the deterministic equivalent for  $t=1$  based on the standard normal distribution is simply

$$I_0 + \epsilon_1 \geq \sigma_1 F_{S_1}^{-1}(u_1).$$

For  $t=2,3,\dots,T$  the distribution of  $Z_t = S_t + (1-\alpha) \sum_{i=1}^{t-1} S_i$  is known to be  $N(\bar{S}_t + (1-\alpha) \sum_{i=1}^{t-1} \bar{S}_i, \sigma_t^2 + (1-\alpha)^2 \sum_{i=1}^{t-1} \sigma_i^2)$ . Standardizing

the chance constraints gives the following where the parameters of the convoluted distributions are represented by  $\text{Mean}_t$  and  $\text{Dev}_t$  respectively,

$$\text{Prob} \left\{ \frac{I_0 + \sum_{i=1}^t \epsilon_i}{\text{Dev}_t} \geq \frac{Z - \text{Mean}_t}{\text{Dev}_t} \right\} \geq u_t \quad t=2,3,\dots,T.$$

After cross multiplying the deterministic equivalents as determined by the quantile rule are

$$I_0 + \sum_{i=1}^t \epsilon_i \geq \text{Dev}_t F_{Z_t}^{-1}(u_t) \quad t=2,3,\dots,T.$$

where  $F_{S_1}^{-1}(u_1)$  is the inverse of the c.d.f. for  $S_1$ .

$F_{Z_t}^{-1}(u_t)$  is the inverse of the c.d.f. for a standard normal random variable.

Production Nonnegativity Constraints

We begin with the forecast based rules and constrain them to be nonnegative

$$P_1 = \bar{S}_1 + \epsilon_1 \geq 0 \quad t=1$$

$$P_t = \bar{S}_t + \alpha(S_{t-1} - \bar{S}_{t-1}) + \epsilon_t \geq 0 \quad t=2,3,\dots,T.$$

Solving for  $\epsilon_t$  gives

$$\epsilon_1 \geq -\bar{S}_1 \quad t=1$$

$$\epsilon_t \geq -\bar{S}_t - \alpha \min(S_{t-1}) + \alpha \bar{S}_{t-1} \quad t=2,3,\dots,T.$$

Under the normal distribution assumption  $\min(S_{t-1}) = (\bar{S}_{t-1} - 3\sigma_{t-1})$ .

Then

$$\epsilon_t \geq -\bar{S}_t - \alpha \bar{S}_{t-1} + 3\alpha \sigma_{t-1} + \alpha \bar{S}_{t-1}$$

$$\epsilon_t \geq -(\bar{S}_t - 3\alpha \sigma_{t-1}) \quad t=2,3,\dots,T.$$

Sales Based Normal Models

Objective Function - SBDRM-1/N

For this case the form of the objective function is identical to the sales based exponential previously derived in this appendix.

Chance Constraints

The analysis is the same as for the sales based exponential case through the chance constraints with the random variables isolated, which are now restated

$$\text{Prob} [I_0 + \bar{S}_1 + \epsilon_1 \geq S_1] \geq u_1 \quad t=1$$

$$\text{Prob} [I_0 + \bar{S}_1 + \alpha \sum_{j=2}^t \bar{S}_j + \sum_{i=1}^t \epsilon_i \geq S_t + \alpha \sum_{i=1}^{t-1} S_i] \geq u_t \quad t=2,3,\dots,T.$$

Assuming that  $S_i \sim N(\bar{S}_i, \sigma_i^2)$  and  $S_1, S_2, \dots, S_T$  are independent

the deterministic equivalent for  $t=1$  based on the standard normal distribution is simply

$$I_0 + \epsilon_1 \geq \sigma_1 F_{S_1}^{-1}(u_1)$$

For  $t=2,3,\dots,T$  the distribution of  $Z_t = S_t + \alpha \sum_{i=1}^{t-1} S_i$  is known

to be  $N(\bar{S}_t + \alpha \sum_{i=1}^{t-1} \bar{S}_i, \sigma_t^2 + \alpha^2 \sum_{i=1}^{t-1} \sigma_i^2)$ . Standardizing the

chance constraints gives the following where the parameters of the convoluted distributions are represented by  $\text{Mean}_t$  and  $\text{Dev}_t$  respectively,

$$\text{Prob} \left[ \frac{I_0 + (1-\alpha)\bar{S}_1 + (\alpha-1)\bar{S}_t + \sum_{i=1}^t \epsilon_i}{\text{Dev}_t} \geq \frac{Z - \text{Mean}_t}{\text{Dev}_t} \right] \geq u_t$$

$t=2,3,\dots,T.$

After cross multiplying the deterministic equivalents as determined by the quantile rule are

$$I_0 + (1-\alpha)\bar{S}_1 + (\alpha-1)\bar{S}_t + \sum_{i=1}^t \epsilon_i \geq \text{Dev}_t F_{Z_t}^{-1}(u_t) \quad t=2,3,\dots,T.$$

where  $F_{S_1}^{-1}(u_1)$  is the inverse of the c.d.f. for  $S_1$ .

$F_{Z_t}^{-1}(u_t)$  is the inverse of the c.d.f. for a standard normal random variable.

### Production Nonnegativity Constraints

We begin with the sales based rules and constrain them to be nonnegative

$$P_1 = \bar{S}_1 + \epsilon_1 \geq 0 \quad t=1$$

$$P_t = S_{t-1} + \alpha(\bar{S}_t - S_{t-1}) + \epsilon_t \geq 0 \quad t=2,3,\dots,T.$$

solving for  $\epsilon_t$  gives

$$\epsilon_1 \geq -\bar{S}_1 \quad t=1$$

$$\epsilon_t \geq -\min(S_{t-1}) - \alpha \bar{S}_t + \alpha \min(S_{t-1})$$

Under the normal distribution assumption  $\min(S_{t-1}) = (\bar{S}_{t-1} - 3\sigma_{t-1})$ . Then,

$$\epsilon_t \geq -(\bar{S}_{t-1} - 3\sigma_{t-1}) - \alpha \bar{S}_t + \alpha (\bar{S}_{t-1} - 3\sigma_{t-1})$$

Rearranging gives

$$\epsilon_t \geq (\alpha - 1)\bar{S}_{t-1} - \alpha \bar{S}_t + (1 - \alpha)3\sigma_{t-1} \quad t=2,3,\dots,T.$$

APPENDIX B  
DEVELOPMENT OF NORMAL DISTRIBUTION PARAMETERS

FBDRM-1/N

Given:  $S_t \sim N(\bar{S}_t, \sigma_t^2)$

Find: Distribution of  $Z = S_t + (1 - \alpha) \sum_{i=1}^{t-1} S_i$

If  $S_i \sim N(\bar{S}_i, \sigma_i^2)$  and  $S_1, S_2, \dots, S_n$  are independent,

Then  $a_i S_i \sim N(a_i \bar{S}_i, a_i^2 \sigma_i^2)$

And  $\sum_{i=1}^n a_i S_i \sim N\left(\sum_{i=1}^n a_i \bar{S}_i, \sum_{i=1}^n a_i^2 \sigma_i^2\right)$

Therefore  $\sum_{i=1}^{t-1} S_i \sim N\left(\sum_{i=1}^{t-1} \bar{S}_i, \sum_{i=1}^{t-1} \sigma_i^2\right)$

And  $Z = S_t + (1 - \alpha) \sum_{i=1}^{t-1} S_i \sim N\left(\bar{S}_t + (1 - \alpha) \sum_{i=1}^{t-1} \bar{S}_i, \sigma_t^2 + (1 - \alpha)^2 \sum_{i=1}^{t-1} \sigma_i^2\right)$

SBDRM-1/N

Given:  $S_t \sim N(\bar{S}_t, \sigma_t^2)$

Find: Distribution of  $Z = S_t + \alpha \sum_{i=1}^{t-1} S_i$

For same reasons as above

$\sum_{i=1}^{t-1} S_i \sim N\left(\sum_{i=1}^{t-1} \bar{S}_i, \sum_{i=1}^{t-1} \sigma_i^2\right)$

And  $Z = S_t + \alpha \sum_{i=1}^{t-1} S_i \sim N\left(\bar{S}_t + \alpha \sum_{i=1}^{t-1} \bar{S}_i, \sigma_t^2 + \alpha^2 \sum_{i=1}^{t-1} \sigma_i^2\right)$

TABLE 43  
VALUES OF  $Dev_t$  FOR FBDRM-1/N TREND AND SEASONAL

$\alpha$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
t							
1	100.00	100.00	100.00	100.00	100.00	100.00	100.00
2	122.07	116.62	111.80	107.70	104.40	101.98	100.50
3	140.71	131.15	122.47	114.89	108.63	103.92	101.00
4	157.16	144.22	132.29	121.66	112.69	105.83	101.49
5	172.05	156.20	141.42	128.06	116.62	107.70	101.98
6	185.74	167.33	150.00	134.16	120.42	109.54	102.47
7	198.49	177.76	158.11	140.00	124.10	111.36	102.96
8	210.48	187.62	165.83	145.60	127.67	113.14	103.44
9	221.81	196.98	173.21	151.00	131.15	114.89	103.92
10	232.59	205.91	180.28	156.20	134.54	116.62	104.40

TABLE 44  
VALUES OF  $Dev_t$  FOR SBDRM-1/N TREND AND SEASONAL

$\alpha$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
t							
1	100.00	100.00	100.00	100.00	100.00	100.00	100.00
2	104.40	107.70	111.80	116.62	122.07	128.06	134.54
3	108.63	114.89	122.47	131.15	140.71	151.00	161.86
4	112.69	121.66	132.29	144.22	157.16	170.88	185.20
5	116.62	128.06	141.42	156.20	172.05	188.70	205.91
6	120.42	134.16	150.00	167.33	185.74	204.94	224.72
7	124.10	140.00	158.11	177.76	198.49	220.00	242.07
8	127.67	145.60	165.83	187.62	210.48	234.09	258.26
9	131.15	151.00	173.21	196.98	221.81	247.39	273.50
10	134.54	156.20	180.28	205.91	232.59	260.00	287.92

APPENDIX C

SAMPLE CONVOLUTIONS FOR EXPONENTIAL MODELS

CONVOLUTION FOR STATIONARY DEMAND

Problem: To determine the density function for  $Z=S_t+(1-\alpha)S_{t-1}$ .

Assume demand is identically distributed over time, therefore

$$S_t \sim f_S(s) = \mu \exp(-\mu(S-S_0)) \quad S_0 \leq S \leq \infty$$

$$S_{t-1} \sim f_S(s) = \mu \exp(-\mu(S-S_0)) \quad S_0 \leq S \leq \infty$$

Define

$$Y=(1-\alpha)S_{t-1}, \text{ then,}$$

$$Y \sim f_Y(y) = (\mu/(1-\alpha)) \exp(-\mu(y/(1-\alpha)-S_0)) \quad (1-\alpha)S_0 \leq Y < \infty$$

Therefore now the problem becomes find the density function for  $Z=S_t+Y$ . Since we assume  $S_t$  and  $S_{t-1}$  are independent we may use the following convolution formula

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{S_t}(z-y) f_Y(y) dy$$

$$f_Z(z) = \int_{(1-\alpha)S_0}^{z-S_0} \mu \exp(-\mu(z-y-S_0)) \cdot (\mu/(1-\alpha)) \exp(-\mu(y/(1-\alpha)-S_0)) dy$$

$$f_Z(z) = (\mu^2/(1-\alpha)) \int_{(1-\alpha)S_0}^{z-S_0} \mu \exp(-\mu(z-y-S_0) - \mu(y/(1-\alpha)-S_0)) dy$$

Integration by parts yields the following density function:

$$f_Z(z) = -\mu/\alpha \left[ \exp(-\mu/(1-\alpha)(z-(2-\alpha)S_0)) - \exp(-\mu(z-(2-\alpha)S_0)) \right]$$

$$(2-\alpha)S_0 \leq Z < \infty$$

CONVOLUTIONS FOR SBDRM EXPONENTIAL TRENDPeriod 2

Problem: To determine the density function for  $Z=S_2+\alpha S_1$

$$\text{where, } S_1 \sim f_{S_1}(s_1) = \mu_1 \exp(-\mu_1(S_1 - S_0)) \quad S_0 \leq S_1 < \infty$$

$$S_2 \sim f_{S_2}(s_2) = \mu_2 \exp(-\mu_2(S_2 - S_0)) \quad S_0 \leq S_2 < \infty$$

Define  $Y = \alpha S_1$ , then,

$$Y \sim f_Y(y) = (\mu_1 / \alpha) \exp(-\mu_1(y/\alpha - S_0)) \quad \alpha S_0 \leq Y < \infty$$

Therefore the problem now becomes find the density function for  $Z=S_2+Y$ . Since we assume  $S_1$  and  $S_2$  are independent we may use the following convolution formula

$$f_Z(z) = \int_{\alpha S_0}^{z-S_0} \mu_2 \exp(-\mu_2(z-y-S_0)) \cdot (\mu_1 / \alpha) \exp(-\mu_1(y/\alpha - S_0)) dy$$

$$f_Z(z) = (\mu_1 \mu_2 / \alpha) \int_{\alpha S_0}^{z-S_0} \exp(-\mu_2(z-y-S_0) - \mu_1(y/\alpha - S_0)) dy$$

Integration by parts yields the following density function:

$$f_Z(z) = (\mu_1 \mu_2 / \alpha \mu_2 - \mu_1) \left[ \exp(-\mu_1 / \alpha (z - (1 + \alpha) S_0)) - \exp(-\mu_2 (z - (1 + \alpha) S_0)) \right] \quad (1 + \alpha) S_0 \leq Z < \infty$$

Period 3

Problem: To determine the density function for  $Z=S_3+\alpha(S_1+S_2)$

$$\text{where, } S_3 \sim f_{S_3}(s_3) = \mu_3 \exp(-\mu_3(S_3 - S_0)) \quad S_0 \leq S_3 < \infty$$

$$X = (S_1 + S_2) \sim f_X(x) = (\mu_1 \mu_2 / \mu_2 - \mu_1) \left[ \exp(-\mu_1(x - 2S_0)) - \exp(-\mu_2(x - 2S_0)) \right] \quad 2S_0 \leq X < \infty$$

Define  $Y = \alpha(S_1 + S_2) = \alpha X$ , then,

$$Y \sim f_Y(y) = (\mu_1 \mu_2 / \alpha (\mu_2 - \mu_1)) \left[ \exp(-\mu_1 (y/\alpha - 2S_0)) - \exp(-\mu_2 (y/\alpha - 2S_0)) \right] \\ 2\alpha S_0 \leq Y < \infty$$

Therefore the problem becomes find the density function for  $Z = S_3 + Y$ . Since we assume  $S_1, S_2, S_3$  are independent we may use the following convolution formula

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{S_3}(z-y) f_Y(y) dy$$

$$f_Z(z) = \int_{2\alpha S_0}^{z-S_0} \mu_3 \exp(-\mu_3 (z-y-S_0)) \cdot (\mu_1 \mu_2 / \alpha (\mu_2 - \mu_1)) \left[ \exp(-\mu_1 (y/\alpha - 2S_0)) - \exp(-\mu_2 (y/\alpha - 2S_0)) \right] dy$$

$$f_Z(z) = (\mu_1 \mu_2 \mu_3 / \alpha (\mu_2 - \mu_1)) \int_{2\alpha S_0}^{z-S_0} \left[ \exp(-\mu_1 (y/\alpha - 2S_0) - \mu_3 (z-y-S_0)) - \exp(-\mu_2 (y/\alpha - 2S_0) - \mu_3 (z-y-S_0)) \right] dy$$

Integration by parts yields the following density function:

$$f_Z(z) = (\mu_1 \mu_2 \mu_3 / \alpha (\mu_2 - \mu_1)) \left[ \alpha / (\alpha \mu_3 - \mu_1) \left( \exp(-\mu_1 (z - (2\alpha + 1)S_0)) - \exp(-\mu_3 (z - (2\alpha + 1)S_0)) \right) - (\alpha / (\alpha \mu_3 - \mu_2)) \left( \exp(-\mu_2 / \alpha (z - (2\alpha + 1)S_0)) - \exp(-\mu_3 (z - (2\alpha + 1)S_0)) \right) \right] \\ (2\alpha + 1)S_0 \leq Z < \infty$$

CONVOLUTIONS FOR SBDRM EXPONENTIAL SEASONAL

Period 3

Problem: To determine the density function for  $Z=S_3+\alpha(S_1+S_2)$

where,  $S_3 \sim f_{S_3}(s_3) = \mu_I \exp(-\mu_I(S_3-S_0)) \quad S_0 \leq S_3 < \infty$

$X=(S_1+S_2) \sim \mu_I^2(X-2S_0) \exp(-\mu_I(X-2S_0)) \quad 2S_0 \leq X < \infty$

Define  $Y = \alpha(S_1+S_2) = \alpha X$ , then,

$Y \sim f_Y(y) = \frac{\mu_I^2}{\alpha} (y/\alpha - 2S_0) \exp(-\mu_I(y/\alpha - 2S_0)) \quad 2\alpha S_0 \leq Y < \infty$

Therefore the problem now becomes find the density function for  $Z=S_3+Y$ . Since we assume  $S_1, S_2, S_3$  are independent we may use the following convolution formula

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{S_3}(z-y) f_Y(y) dy$$

$$f_Z(z) = \int_{2\alpha S_0}^{z-S_0} (\mu_I^3/\alpha) \exp(-\mu_I(z-y-S_0)) (y/\alpha - 2S_0) \exp(-\mu_I(y/\alpha - 2S_0)) dy$$

Integration by parts yields the following density function:

$$f_Z(z) = \left( \frac{\mu_I^3}{\alpha} \right) \left[ \begin{aligned} & ((z-S_0)/(\alpha-1)\mu_I) \exp(-\mu_I/\alpha(z-(2\alpha+1)S_0)) \\ & - (2\alpha S_0/(\alpha-1)\mu_I) \exp(-\mu_I(z-(2\alpha+1)S_0)) - (\alpha/(\alpha-1)^2 \mu_I^2) \\ & \exp(-\mu_I/\alpha(z-(2\alpha+1)S_0)) + (\alpha/(\alpha-1)^2 \mu_I^2) \exp(-\mu_I(z-(2\alpha+1)S_0)) \\ & - (2S_0\alpha/(\alpha-1)\mu_I) \exp(-\mu_I/\alpha(z-(2\alpha+1)S_0)) \\ & + (2S_0\alpha/(\alpha-1)^2 \mu_I^2) \exp(-\mu_I(z-(2\alpha+1)S_0)) \end{aligned} \right]$$

$(2\alpha+1)S_0 \leq Z < \infty$

CONVOLUTIONS FOR FORECAST BASED MODELS

For all forecast based models the correct convoluted density function may be derived from the corresponding sales based function by substituting the constant  $(1 - \alpha)$  for  $\alpha$  in these functions.

APPENDIX B

VALUES OF  $F_{Z_t}^{-1}(u_t)$  FOR EXPONENTIAL MODELS

DETERMINING VALUES OF  $F_{Z_t}^{-1}(u_t)$  USING NEWTON'S METHOD

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In order to construct the deterministic equivalents of the chance constraints in all of the exponential models the inverses of the convoluted distribution functions for each time period are required. Since the form of all these distributions contain sums of exponential functions the required inverses may not be solved for analytically. Since this is the case a numerical method is required. In this research Newton's method for approximating roots of equations is used.

Newton's method proceeds by constructing successive tangents to the function of interest beginning with an initial guess. This procedure leads to the following recursive relationship

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (D-1)$$

For the application of interest we are required to find the value of  $z$  such that

$$F_{Z_t}(z) = u_t \quad (D-2)$$

where  $F_{Z_t}(z)$  is the correct convoluted distribution function for the period  $t$ . Rearranging (D-2) into the required form gives

$$F_{Z_t}(z) - u_t = 0. \quad (D-3)$$

The final required element is the derivative of the function of interest. In this case it is known that the derivative of the distribution function is simply the corresponding density

function. Thus we may rewrite (D-1) as follows,

$$z_{n+1} = z_n - \frac{F_{Z_t}(z_n) - u_t}{f_{Z_t}(z_n)} \quad (D-4)$$

In reality,  $z_{n+1}$ , the root determined in this manner is nothing more than the inverse of the distribution corresponding to a  $u_t$  probability level. Thus,  $z_{n+1} = F_{Z_t}^{-1}(u_t)$  which is the value that we seek.

Although convergence generally is rather quick computations can be very extensive depending upon the form of the function in question. Therefore to aid in calculations an IBM supplied subroutine, RTNI, was used in conjunction with user prepared mainline and subroutine subprograms. All roots were calculated within an error of  $\pm 0.01$ . The resulting values are presented in this appendix in Tables 45 through 49 inclusive.

TABLE 45  
VALUES OF  $F_Z^{-1}(u_t)$  FOR STATIONARY DEMAND

$\alpha$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$t$	<u><math>u_t=0.95</math></u>						
1	1298.29	1298.29	1298.29	1298.29	1298.29	1298.29	1298.29
2-10	1799.63	1706.95	1620.39	1541.80	1470.89	1407.55	1350.43
$t$	<u><math>u_t=0.85</math></u>						
1	858.85	858.85	858.85	858.85	858.85	858.85	858.85
2-10	1322.28	1245.53	1170.19	1098.85	1030.87	968.09	910.99
$t$	<u><math>u_t=0.75</math></u>						
1	654.52	654.52	654.52	654.52	654.52	654.52	654.52
2-10	1088.02	1021.28	954.04	889.18	825.07	763.65	706.66

TABLE 46  
VALUES OF  $F_{Z_t}^{-1}(u_t)$  FOR FBDRM EXPONENTIAL TREND

$\alpha$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$t$							
1	1298.29	1298.29	1298.29	1298.29	1298.29	1298.29	1298.29
2	1843.30	1751.42	1666.74	1589.46	1519.68	1457.16	1400.25
3	2564.86	2240.61	2066.51	1915.24	1772.27	1686.51	1575.79
4	2808.21	2610.52	2444.06	2279.42	2047.78	1823.61	1715.11
5	3698.39	3290.06	2936.76	2801.92	2327.83	2113.67	1956.38
6	7057.50	6735.34	6483.31	6264.89	6068.93	5821.71	5622.19

$u_t=0.95$

TABLE 47  
VALUES OF  $F_{Z_t}^{-1}(u_t)$  FOR SBDRM EXPONENTIAL TREND

$\alpha$ $t$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	1298.29	1298.29	1298.29	1298.29	1298.29	1298.29	1298.29
2	1519.68	1589.46	1666.74	1751.42	1843.30	1930.49	2049.18
3	1772.27	1915.24	2066.51	2240.61	2564.86	2617.23	2738.53
4	2047.78	2279.42	2444.06	2610.52	2808.21	3071.45	3310.13
5	2327.83	2801.92	2936.76	3290.06	3698.39	4214.17	4793.64
6	6068.93	6264.89	6483.31	6735.34	7057.50	7558.72	8445.36

$u_t=0.95$

TABLE 48  
VALUES OF  $F_{Z_t}^{-1}(u_t)$  FOR FBDRM EXPONENTIAL SEASONAL

$\alpha$ $t$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	1298.29	1298.29	1298.29	1298.29	1298.29	1298.29	1298.29
2	1799.63	1706.95	1620.39	1541.80	1470.89	1407.55	1350.43
3	2272.28	2109.29	1937.33	1784.27	1643.24	1516.81	1402.59
4	3025.54	2796.10	2574.85	2366.98	2174.41	1997.07	1832.50
5	4097.41	3796.83	3523.73	3278.81	2937.51	2864.84	2682.31
6	7125.87	6151.71	5693.22	5662.69	5496.93	5369.58	5251.79

$u_t=0.95$

TABLE 49

VALUES OF  $F_{Z_t}^{-1}(u_t)$  FOR SBDRM EXPONENTIAL SEASONAL

$\alpha$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	1298.29	1298.29	1298.29	1298.29	1298.29	1298.29	1298.29
2	1470.89	1541.80	1620.39	1706.95	1799.63	1893.95	1996.32
3	1643.24	1784.27	1937.33	2101.29	2272.28	2418.29	2736.46
4	2174.41	2366.98	2574.85	2796.10	3025.54	3251.36	3535.89
5	2937.51	3278.81	3523.73	3796.83	4097.41	4409.11	4747.00
6	5496.93	5662.69	5693.22	6151.71	7125.87	8527.77	10762.05

 $u_t=0.95$

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