String Topology & Compactified Moduli Spaces

by

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Abstract

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The motivation behind this work is to solve the master equation $\partial X = X * X$ in $\oplus_{k,\ell} Hom(P_*^{\otimes k}, P_*^{\otimes \ell})$ where P_* is a chain complex computing $H_*^{S^1}(LM, M)$, the S^1-equivariant homology of the free loop space LM of a manifold M, relative to constant loops. Here, we solve a modification of this equation:

$$\partial X = X * X + A$$

and suggest an avenue for modifying the solution of the second equation to obtain a solution of the master equation. The solution of the second equation is constructed by building a pseudomanifold of string diagrams which has prescribed boundary. The string topology construction describes the action of cellular chains of the pseudomanifold on P_*. Further, the pseudomanifold is homeomorphic to a compactification of the moduli space of Riemann surfaces. A second smaller compactification is defined over which string topology operations conjecturally extend.

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Chapter 1

Background & Introduction

In 1999 Chas and Sullivan defined algebraic operations on the homology of the free loop space LM of an oriented manifold M [CS99]. They found relations among these operations which determined algebraic structures. The results of this thesis relate in particular to the involutive Lie bialgebra structure they discovered on $H_*^{S^1}(LM, M; \mathbb{Q})$, the S^1-equivariant homology of LM, relative to constant loops, with rational coefficients.

Chas and Sullivan defined these string topology operations by intersecting families of loops in M. For example, if two parametrized families of loops in M intersect transversally then at each point along the intersection locus two loops–one from each family–share a common point. The two loops can be cut at this point and reconnected to form one loop. See figure 1. This defines a two-to-one operation on transversally intersecting chains. Because it is not true that any two chains of loops intersect one another transversally, the operation on chains is only partially defined. However, Chas and Sullivan chose representative cycles that intersect transversally to obtain a fully-defined operation on homology. The operation is called the string bracket and it satisfies the Jacobi identity. The Chas-Sullivan bracket is similar to Goldman's bracket for homotopy classes of curves on surfaces [Gol85]. There is an analogous one-to-two operation called the string cobracket satisfying the coJacobi identity, which is similar to Turaev's cobracket [Tur91]. Together, the bracket and cobracket satisfy the relations of an involutive Lie bialgebra [CS04].

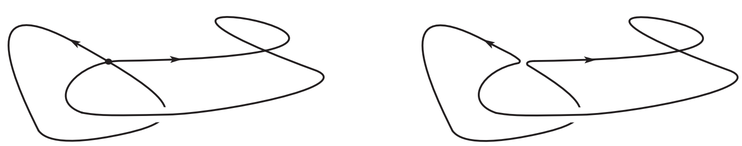


Figure 1.1: String bracket of two loops

Cohen and Jones subsequently defined the string bracket differently, using homotopy theory, and showed it was equivalent to Chas and Sullivan's [CJ02]. Many results since have used their homotopy-theoretic methods.

Cohen and Godin used the homotopy theoretic methods to define k-to-ℓ operations on $H_*(LM)$ [CG04]. These operations agree with those previously defined by Chas and Sullivan. Let Σ be an oriented surface of genus g and with two types of boundary components; k of them labeled as inputs and the remaining ℓ as outputs. A map $\rho : \Sigma \to M$ may be restricted to either inputs or outputs. The restriction to inputs is a map from k circles to M, i.e., a point in the cartesian product LM^k, while the restriction to outputs is a point in LM^ℓ.

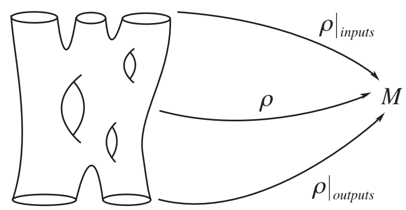


Figure 1.2: Restrictions to inputs and outputs

The maps in and out $LM^k \xleftarrow{in} Maps(\Sigma, M) \xrightarrow{out} LM^\ell$, restricting to inputs and outputs respectively, induce maps on homology

$$H_*(LM^k) \xleftarrow{in_*} H_*(Maps(\Sigma, M)) \xrightarrow{out_*} H_*(LM^\ell).$$

The problem of string topology is the following: given a map of k circles to M, when is it possible to map a surface Σ to M with the given map as its restriction to inputs? Both the Chas-Sullivan and Cohen-Godin constructions reverse the first arrow on homology. A string topology operation on homology is the composition $out_* \circ in_*^!$:

$$H_*(LM^k) \xrightarrow{in_*^!} H_*(Maps(\Sigma, M)) \xrightarrow{out_*} H_*(LM^\ell).$$

The Chas-Sullivan and Cohen-Godin constructions use a special class of graphs called string diagrams (sometimes called generalized chord diagrams or Sullivan chord diagrams). Briefly, a chord diagram is a graph constructed from a circle and intervals whose endpoints lie on the circle. A string diagram is a graph constructed from many circles and intervals whose endpoints lie on any of the circles. If, in a string diagram, the two endpoints of a single interval coincide on a circle, then the diagram is called degenerate. The Cohen-Godin construction uses only nondegenerate string diagrams while the Chas-Sullivan construction includes some degenerate string diagrams.

It seemed possible at first that the space SD of nondegenerate string diagrams was homotopy equivalent to the moduli space $\mathscr{M}$ of Riemann surfaces. This turned out not to be the case. There is an inclusion of SD in $\mathscr{M}$, but Godin proved that SD is too small–its dimension is smaller than the homological dimension of $\mathscr{M}$. She later constructed a larger space of graphs which contains SD, gives string topology operations and has the homotopy type of the full moduli space $\mathscr{M}$ [God07, God08]. This construction showed that the homology of $\mathscr{M}$ acts on the homology of the loop space LM. It has been unknown for some time whether this structure is an invariant of the homotopy type of the manifold or whether it is sensitive to some finer structure. Based on work of Costello [Cos07], Godin conjectured that it is a homotopy invariant.

Graphs and ribbon surfaces have been used to study moduli spaces of Riemann surfaces, independent of string topology for some time [Str84, Pen04]. Bödigheimer's harmonic compactification of moduli space is of particular interest for string topology [Böd06, Böd]. Harmonic functions on Riemann surfaces are used to construct a decomposition of the moduli space of Riemann surfaces into open cells. The harmonic compactification is defined by including all faces of the open cells.

This thesis is devoted to the equivariant setting for string topology. We present a fully-defined chain-level analogue of the action of the homology of the space of string diagrams SD on homology of the loop space, where SD is completed to $\overline{SD}$. This is the string topology construction. We make use of Sullivan's tool in outlining a chain-level string topology construction in [Sul07]. It is a representative of the Thom class of the diagonal which is local. The construction takes a family of k loops in M and describes how to extend it to a family of maps of string diagrams to M. The output of the construction is the restriction to the ℓ outputs.

We also present a fully-defined chain-level analogue of the extension of the action to the homology of moduli space $\mathscr{M}$, where $\mathscr{M}$ is compactified. In fact, $\mathscr{M}$ is enlarged to a slightly larger homeomorphic space and then compactified. The string topology construction is extended to to this new compact space, called $\overline{LD}/\!\!/\sim$. We also propose a smaller compactification $\bar{\mathscr{M}}$ over which we expect the string topology construction to extend.

For all (g, k, ℓ), $\overline{LD}(g, k, \ell)/\!\!/\sim$ is a pseudomanifold with boundary. The string topology construction for the boundary of $\overline{LD}(g, k, \ell)/\!\!/\sim$ reduces to compositions of operations arising from the string topology construction applied to lower-dimensional spaces $\overline{LD}(g', k', \ell')/\!\!/\sim$ and $\overline{LD}(g'', k'', \ell'')/\!\!/\sim$, along with a second term we call the anomaly, A. Therefore, if X is the sum over all (g, k, ℓ) of operations arising from fundamental chains of $\overline{LD}(g, k, \ell) /\!\!/ \sim$, then X satisfies

$$\partial X = X * X + A.$$

We expect the solution X to the equation with anomaly may be modified to one satisfying the master equation $\partial X = X * X$. In this setting, the induced operation on homology would satisfy $X * X = 0$. Again, X would be an infinite family of operations and $X * X = 0$ would describe an infinite list of quadratic relations in one package. The list of operations includes Chas and Sullivan's string bracket and cobracket and the list of operations includes the relations among the bracket and cobracket which would give $H^{S^1}(LM, M)$ the structure of an involutive Lie bialgebra structure. The algebraic structure provided by the master equation here would be some infinity version of the involutive Lie bialgebra structure. This kind of properad structure [Val07] has been studied independent of string topology, for example, as in [DCTT08].

In chapter 2 we discuss the spaces of string diagrams $\overline{SD}(g, k, \ell)$ and the space of string diagrams with levels $\overline{LD}(g, k, \ell)$, together with their respective cell decompositions. We also define an equivalence relation on each space $\sim$ called slide equivalence.

The string topology construction for $\overline{SD}(g, k, \ell)$ is defined in chapter 3. The specific chains on $\overline{SD}(g, k, \ell)$ which govern the action are introduced in section 3.1. An equivariant chain model P_* of the free loop space LM of a manifold M is introduced in section 3.2. Sections 3.3 and 3.4 describe further elements of the string topology construction, which is itself defined in section 3.5.

In chapter 4 we extend the string topology construction and build the solution to the equation with anomaly. In section 4.1, the pseudomanifolds $\overline{LD}(g, k, \ell)/\!\!/ \sim$ are defined. The long proof of proposition 4.3 of section 4.2 gives a step-by-step extension of the string topology construction to the top chain of $\overline{LD}(g, k, \ell)/\!\!/ \sim$. In section 4.3, its image is shown to solve the equation with anomaly.

In appendix A we describe how the solution to the equation with anomaly may be modified to solve the master equation and present an example where the solution is modified.

Appendix B describes the relationship between string diagrams and moduli spaces $\mathscr{M}(g, k, \ell)$ of Riemann surfaces. A compactification $\bar{\mathscr{M}}(g, k, \ell)$ is defined in section B.1 and $\overline{LD}(g, k, \ell)/ \sim$ is used in the proof of theorem B.1 in section B.2 that $\bar{\mathscr{M}}(g, k, \ell)$ is a pseudomanifold with boundary. Conjecture B.1 in section B.3 proposes the extension of string topology operations to the top chain of $\bar{\mathscr{M}}(g, k, \ell)$ or, rather, a space homeomorphic to it.

Chapter 2

String diagrams & string diagrams with levels

2.1 String diagrams

Definition 2.1. Let $g \geq 0$ and k and $\ell > 0$. A pre-string diagram of type (g, k, ℓ) is a connected graph Γ satisfying the following:

- The Euler characteristic χ of Γ is $2 - 2g - k - \ell$.
- Let V be the set of vertices of Γ. The set of edges of Γ is partitioned into E_0 and E. Elements e_0 of E_0 are called input edges and elements ch of E

are called chords.

- There are exactly two input pieces adjacent to each vertex such that the realization of $V \cup E_0$ is k disjoint labeled circles called input circles.
- There is at most one bivalent vertex on each input circle.
- There is one distinguished vertex on each input circle. If there is a bivalent vertex on an input circle, it is the distinguished vertex on that circle.
- An input edge $e_0 \in E_0$ has length L_i such that $0 \leq L_i \leq 1$ and if $e_0, e_1 \ldots e_n$ form the edges of an input circle then $\sum_{i=0}^{n} e_i = 1$. A chord ch has length 1.
- There exists a cyclic order of the half-edges adjacent to each vertex such that the k input circles form k of the boundary cycles of the resulting ribbon surface. The remaining ℓ boundary cycles are labeled and called output circles and the ribbon surface has genus g.

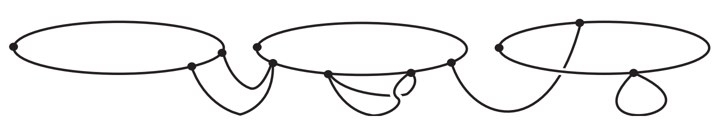


Figure 2.1: A pre-string diagram of type $(1, 3, 3)$.

Two pre-string diagrams of type (g, k, ℓ) are isomorphic if there is an isomorphism of graphs which preserves lengths, distinguished vertices and cyclic orders.

In order not to have edges of length 0, we impose the following: Let Γ and Γ' be pre-string diagrams. Let e be an input edge of Γ such that one half-edge, h_e is adjacent to the vertex v and its other half-edge h'_e is adjacent to the vertex v' and such that its length $L = 0$. Then Γ and Γ' are declared to be isomorphic, where Γ' is obtained from Γ by deleting the edge e and replacing v and v' by a vertex w. A remaining half-edge of Γ' is adjacent to w if it was adjacent to v or v' in Γ. If the cyclic order of half-edges of Γ adjacent to v was given by $\{h_e, h_1, \cdots, h_n\}$ and the cyclic order of half-edges of adjacent to v' was $\{h'_e, h'_1, \cdots, h'_m\}$, then the cyclic order of half-edges of Γ' adjacent to w is given by $\{h_1, \cdots, h_n, h'_1, \cdots, h'_m\}$. See figure 2.2.

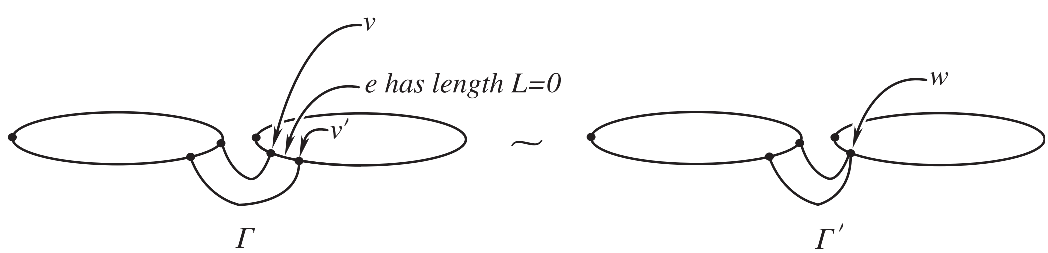


Figure 2.2: An input edge of length 0

Definition 2.2. A string diagram of type (g, k, ℓ) is an isomorphism class of pre-string diagrams.

Remark. A string diagram of type (g, k, ℓ) may be constructed from k disjoint (input) circles, each with total length 1 and a distinguished vertex, and a set of $2g - 2 + k + \ell$ intervals (chords), each of length 1. The string diagram is constructed by attaching chord endpoints anywhere on the input circles and prescribing cyclic orders of half-edges at vertices such that the resulting ribbon surface has genus g and $k+\ell$ boundary components, k of which are isotopic to the k input circles. See figure 2.3.

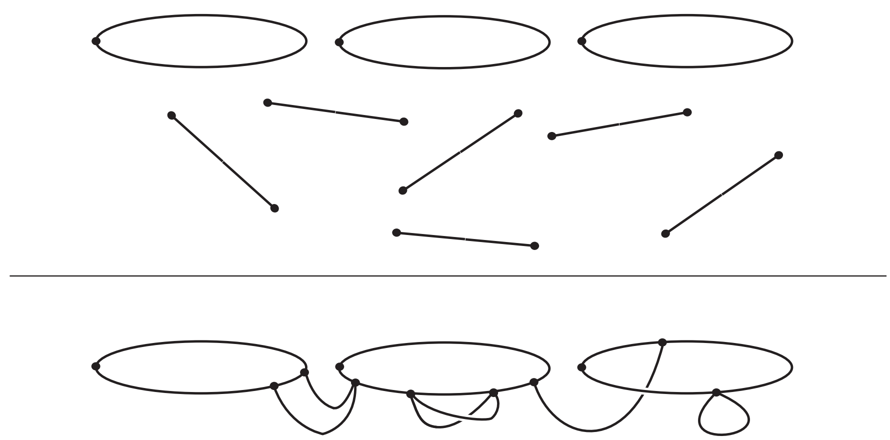


Figure 2.3: Input circles and intervals, attached to form a string diagram of type $(1, 3, 3)$.

Definition 2.3. An orientation of a string digram is an orientation of the corresponding ribbon surface.

Remark. An orientation of a string diagram induces an orientation on the boundary cycles of the corresponding ribbon surface. Such an orientation may be represented as a direction of each input edge which is compatible with the induced orientation on input circles. If input circles of an oriented string diagram are ordered, there is a canonical ordering of input edges. The first edge of the first input circle is the one directed away from the distinguished vertex. The edges of the first circle are then ordered by its cyclic order until the last edge, the one directed toward the input circle, is reached. The next edge is the first edge of the second input circle, and so on.

2.2 The space of string diagrams

Definition 2.4. Let $\overline{SD}(g,k,\ell)$ be the set of string diagrams of type (g,k,ℓ) and let $SD(g,k,\ell) \subset \overline{SD}(g,k,\ell)$ be the subset of diagrams whose subgraphs consisting of vertices and chords $V \cup E$ form a disjoint union of trees.

Two string diagrams Γ and Γ' have the same combinatorial type if they have the same underlying cyclically ordered graphs with distinguished vertices.

The set $\overline{SD}(g,k,\ell)$ is given the following topology. First, string diagrams of the same combinatorial type lie in the same connected component G of $\overline{SD}(g,k,\ell)$. Second, if Γ and $\Gamma' \in G$, then their sets of edges are in bijective correspondence and an ordering of input edges $\{e_1, e_2, \ldots, e_n\}$ of Γ induces such an order of input edges $\{e'_1, e'_2, \ldots, e'_n\}$ of Γ'. Let $f_G : G \to \mathbb{R}^n$ be defined as

$$f(\Gamma) = (L_1, L_2, \ldots, L_n).$$

Open sets in G are preimages of open sets in $\mathbb{R}^n$, where $\mathbb{R}^n$ has the usual topology.

Lemma 2.1. *The space $\overline{SD}(g,k,\ell)$ is a cell complex of dimension $4g-4+2k+2\ell$. The space $SD(g,k,\ell) \subset \overline{SD}(g,k,\ell)$ is a union of open cells whose faces may not be included.*

Proof. Consider the subspace G of $\overline{SD}(g, k, \ell)$ consisting of all string diagrams Γ of a fixed combinatorial type. That is, all string diagrams in G have the same underlying graph and their distinguished vertices and cyclic orders correspond to one another. We show that G is a closed cell.

A string diagram $\Gamma \in G$ is completely determined by the lengths L_i of its input pieces e_i. Therefore, the parameters in G are the L_i. Again, for any two Γ and $\Gamma' \in G$, the set of edges of Γ is in bijective correspondence with the set of edges of Γ'.

The input edges e_i in E_0 are partitioned into k subsets. Two input edges are in the same subset if they are part of the same input circle. Recall that

$$\sum_{e_i \in \text{ an input circle}} L_i = 1$$

so the parameter space for the lengths of the input pieces comprising a single input circle is a simplex. The dimension of the simplex is equal to the number of input edges making up the input circle. Let $\{e_1, e_2, \dots, e_n\}$ be an ordering of the edges making up the jth input circle and let $\Delta_j = \Delta^{n-1}$ be the corresponding simplex. Then

$$G = \Delta_1 \times \Delta_2 \times \cdots \times \Delta_k.$$

A string diagram Γ is at the boundary of a cell if the length L of one of its input edges e is 0. We picture moving from the interior to the boundary of a cell as collapsing an input edge. Here, either two chord endpoints come together or a chord endpoint and a distinguished vertex come together on an input circle.

A string diagram Γ is in the interior of a top-dimensional cell if it has the maximal number of input edges, that is, every vertex is trivalent except for the distinguished vertices, which are bivalent. As Γ has $2g-2+k+\ell$ chords attached to k input circles, the maximum number of input edges is $4g-4+2k+2\ell$ and $\overline{SD}(g,k,\ell)$ is a cell complex of dimension $4g-4+2k+2\ell$.

If $\Gamma \in SD(g,k,\ell)$ is in the interior of a cell G, then all Γ in the interior of G are in $SD(g,k,\ell)$. This is because all Γ in G have the same combinatorial type and no input edges have length 0. Therefore, for any string diagram in the interior of G, its subgraph of vertices and chords $V \cup E$ is a disjoint union of trees.

A face F of G contains all string diagrams Γ' of a given combinatorial type. The combinatorial type labeling F is obtained from the combinatorial type labeling G by the collapsing of an input edge. In collapsing such an edge, subgraphs $V' \cup E_1'$ of string diagrams Γ' in F consisting of vertices and chords are either all disjoint unions of trees or they all contain a cycle. If they are disjoint unions of trees, then F is in $SD(g,k,\ell)$. If they contain a cycle, then F is in

$$\overline{SD}(g,k,\ell) - SD(g,k,\ell).$$

If G is a cell of dimension $4g-4+2k+2\ell$ (i.e., G is top dimensional), then for all Γ in the interior of G, the subgraphs $V \cup E$ consist of disjoint chords and distinguished vertices. In particular, $SD(g, k, \ell)$ contains the interiors of all top-dimensional cells G.

Therefore, $SD(g, k, \ell)$ is a cell complex of dimension $4g - 4 + 2k + 2\ell$ where a cell may not include all its faces. The space $\overline{SD}(g, k, \ell)$ is the natural completion of $SD(g, k, \ell)$, that is, all faces are included. $\square$

By abuse of notation, we will sometimes refer to the combinatorial type labeling a cell G by G as well.

Remark. There is a canonical orientation of a cell G given by compatible orientations of all string diagrams Γ in G. We will use this orientation later.

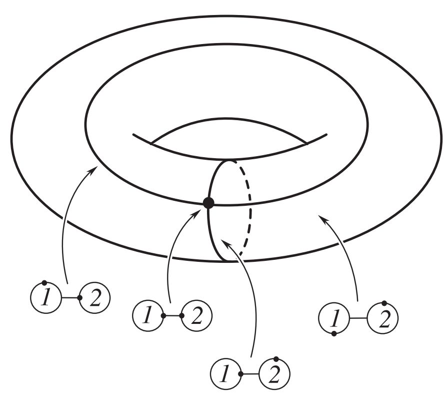


Figure 2.4: $\overline{SD}(0, 2, 1)$

Remark. The connected components of $\overline{SD}(g, k, \ell)$ are the closed $4g - 4 + 2k + 2\ell$-cells that are labeled by combinatorial types that are identical, except for the placement of their distinguished vertices. We can picture a path that goes from Γ in the interior of one top-dimensional cell, through a codimension 1 face and into the interior of another top-dimensional cell as follows. Let v be a bivalent distinguished vertex of Γ and fix all chord endpoints of Γ. Collapse one edge by sliding v along the input circle toward a second vertex w. When v and w coincide, the string diagram is in a codimension 1 face. Then slide v past w so it is again bivalent. See figure 2.5.

Example. The space $\overline{SD}(0,2,1)$ is the 2-dimensional torus, T^2. All string diagrams of type $(0,2,1)$ each have two input circles and one chord connecting them. The space of such diagrams is given by the configuration space of two points (the chord endpoints) one on each of two standard circles. This space, $S^1 \times S^1$, is decomposed into one 2-dimensional cell (a square), two 1-cells and one 0-cell. See figure 2.4.

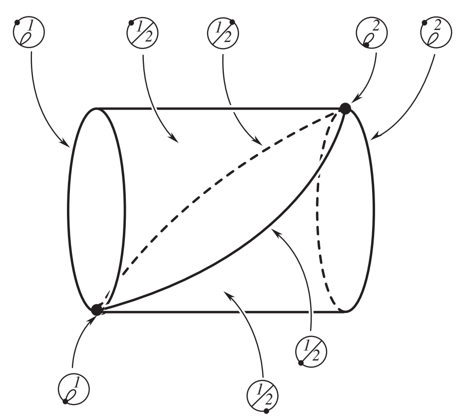


Figure 2.5: $\overline{SD}(0,1,2)$

Example. The space $\overline{SD}(0,1,2)$ is a closed cylinder. All string diagrams of type $(0,1,2)$ each have one input circle and one chord. This space, $S^1 \times I$, is decomposed into two 2-dimensional cells (each a 2-simplex), four 1-cells and two 0-cells. The space $SD(0,1,2)$ is the open cylinder. See figure 2.5.

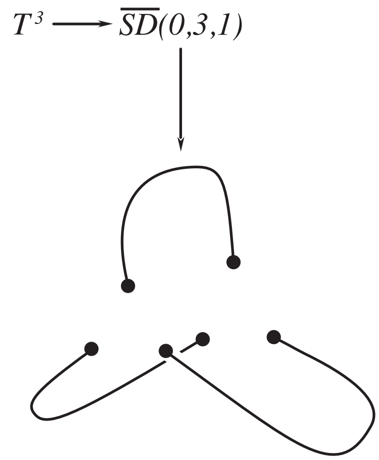


Figure 2.6: $\overline{SD}(0,3,1)$

Example. The space $\overline{SD}(0,3,1)$ is a 3-torus bundle over a space formed by three disjoint intervals. See figure 2.9. Over the interval endpoints, each S^1 factor in each T^3-fiber is decomposed into one 1-cell and one 0-cell. Over points in the interior, one S^1 factor in each T^3-fiber is decomposed into two 1-cells and two 0-cells while the other two S^1 factors are each decomposed into one 1-cell and one 0-cell. Figure 2.7 shows a selection of points in the base, as well as the decomposition of T^3-fibers over these points.

Remark. $SD(0,k,1) = \overline{SD}(0,k,1)$, though this is not true otherwise.

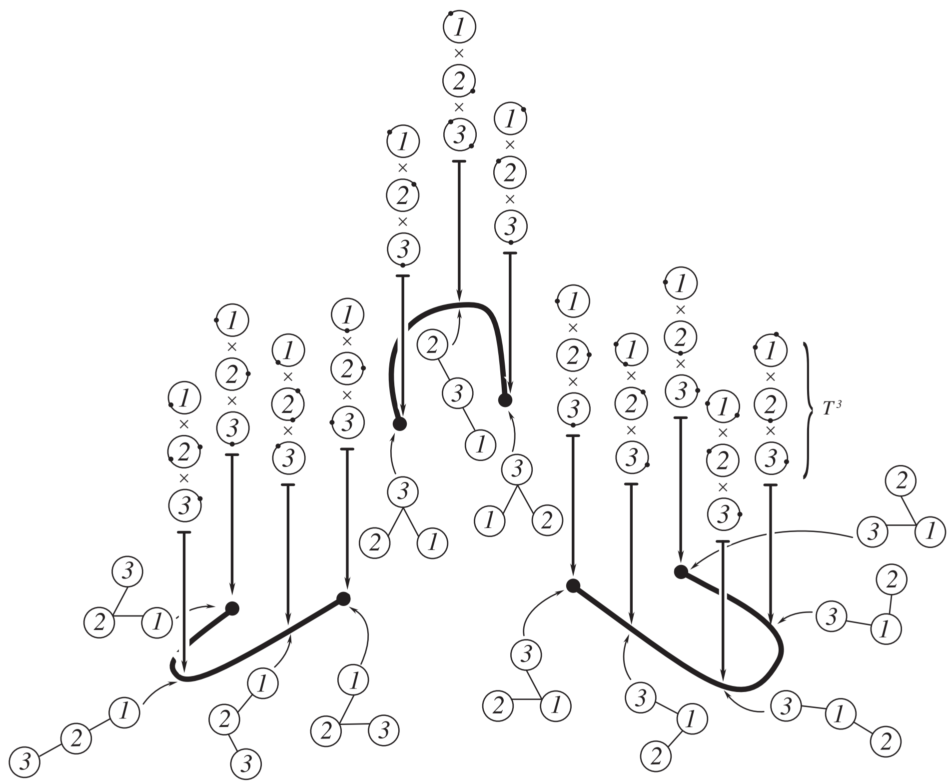


Figure 2.7: Fibers of $\overline{SD}(0,3,1)$

2.3 Slide equivalence

Definition 2.5. Two string diagrams of type (g,k,ℓ) Γ and Γ' differ by a chord slide if they are identical except for the placement of one endpoint of a single chord: ch in Γ and ch' in Γ'. In Γ if $\overline{ch}$ follows ch in the cyclic order at a vertex v, then in Γ', the corresponding chord ch' follows $\overline{ch}$ in the cyclic order at the other endpoint w of $\overline{ch}$. Chord slides generate an equivalence relation $\sim$ called slide equivalence. See figure 2.8.

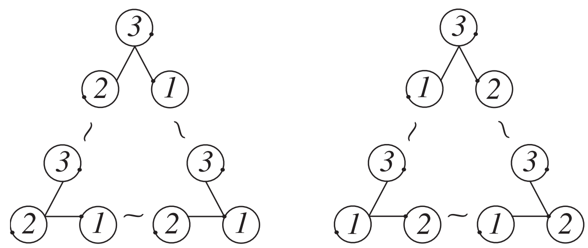


Figure 2.8: Two slide-equivalence classes of string diagrams of type $(0,3,1)$

Example. The space $\overline{SD}(0,3,1)/\sim$ is a 3-torus bundle over a space formed by three intervals and two vertices as in figure 2.9. The slide equivalence classes shown in figure 2.8 describe how fibers over interval endpoints in the base are identified.

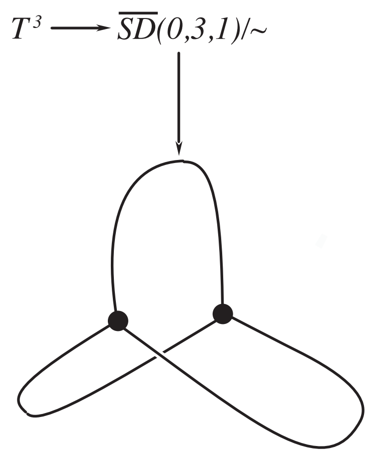


Figure 2.9: $\overline{SD}(0,3,1)/\sim$

2.4 String diagrams with levels

We make the following modifications to the definition of string diagrams:

Definition 2.6. A pre-string diagram of type (g, k, ℓ) with levels is a string diagram Γ of type (g, k, ℓ) satisfying the following:

- The set of chords E of Γ is partitioned into subsets $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_n$ and the set of subsets $\{E_1, E_2, \ldots, E_n\}$ is ordered. Chords in E_i are said to be at the ith level.
- To each pair of consecutive subsets of chords (E_i, E_{i+1}) a spacing parameter $s_i \in [0, 1]$ is assigned, $i = 1, 2, \ldots n - 1$.
- Instead of all chords having length 1, chords in E_1 have length 1, chords in E_i have length $1 + 2s_1 + 2s_2 + \cdots + 2s_{i-1}$, for $i \geq 2$.

In order not to have spacing parameters $s_i = 0$, we impose the following:

Let Γ be a pre-string diagram of type (g, k, ℓ) with levels and set of chords $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_n$ such that for some i the spacing parameter $s_i = 0$. Let Γ' be a pre-string diagram of type (g, k, ℓ) with levels and set of chords $E' = E'_1 \sqcup E'_2 \sqcup \cdots \sqcup E'_{n-1}$. Let Γ' be such that upon forgetting levels and spacing parameters, $\Gamma = \Gamma'$ and such that

- $E'_j = E_j$, for $j = 1, 2, \ldots i - 1$,
- $E'_i = E_i \sqcup E_{i+1}$,
- $E'_j = E_{j+1}$ for $j = i + 1, i + 2, \ldots, n - 1$,

- $s'_j = s_j$ for $j = 1, 2, \ldots, i-1$ and
- $s'_j = s_{j+1}$ for $j = i, i+1, \ldots, n-2$.

Then Γ and Γ' are declared to be isomorphic.

Definition 2.7. A string diagram of type (g, k, ℓ) with levels is an isomorphism class of pre-string diagrams of type (g, k, ℓ) with levels.

Remark. A string diagram of type (g, k, ℓ) with one level (equivalently, a string diagram with levels and all spacing parameters $s_i = 0$) is a string diagram of type (g, k, ℓ).

Remark. A string diagram with levels of type (g, k, ℓ) may be constructed from k disjoint (input) circles, each with total length 1 and a distinguished vertex, and a set of $2g - 2 + k + \ell$ intervals (chords). The set of chords is partitioned $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_n$. Chords in E_1 have length 1, chords in E_i have length $1 + 2s_1 + 2s_2 + \cdots + 2s_{i-1}$, for $i \geq 2$. As before, the string diagram is constructed by attaching chord endpoints anywhere on the input circles and prescribing cyclic orders of half-edges at vertices such that the resulting ribbon surface has genus g and $k + \ell$ boundary components, k of which are isotopic to the k input circles. We imagine attaching chords in stages. Chords in E_1 are attached first, chords in E_2 are attached second second and so on. See figure 2.10.

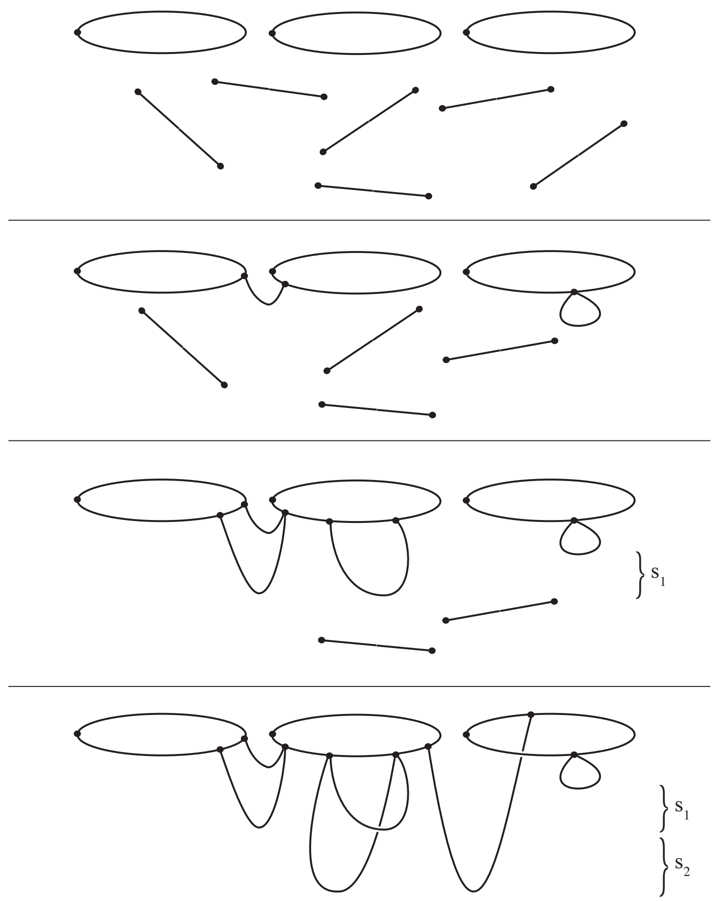


Figure 2.10: Construction of a string diagram with levels of type $(1, 3, 3)$

2.5 The space of string diagrams with levels

Definition 2.8. Let $\overline{LD}(g, k, \ell)$ be the set of string diagrams of type (g, k, ℓ) with levels.

The topology on the set $\overline{LD}(g, k, \ell)$ is analogous to the topology on the set

$\overline{SD}(g, k, \ell)$.

Two string diagrams with levels Γ and Γ' have the same combinatorial type if they have the same underlying cyclically ordered graphs with distinguished vertices and if the partitions $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_n$ and $E' = E'_1 \sqcup E'_2 \sqcup \cdots \sqcup E'_{n-1}$ of their sets of chords agree.

Lemma 2.2. *The space* $\overline{LD}(g, k, \ell)$ *is a cell complex of dimension* $6g+3k+3\ell-7$.

Proof. Again, a cell is labeled by a combinatorial type. The parameters within a cell are input edge lengths L_i and spacing parameters s_i. Therefore, a cell is a cartesian product of k simplices with a cube.

A string diagram with levels Γ is at the boundary of a cell if the length L of one of its input edges e is 0 or if one of its spacing parameters s_i is 0 or 1. We picture moving from the interior to the boundary of a cell as collapsing an input edge ($L = 0$) or as two chord levels coming together ($s_i = 0$) or being split apart ($s_i = 1$).

The maximum number of parameters is achieved when all nondistinguished vertices are trivalent and when each level E_i contains one chord. In this case, there are $4g-4+2k+2\ell$ input edge lengths L_i and $|\chi|-1 = 2g-2+k+\ell-1$ spacing parameters s_i, where $\chi = 2-2g-k-\ell$ is the Euler characteristic of all graphs $\Gamma \in \overline{LD}(g, k, \ell)$. Thus, $\overline{LD}(g, k, \ell)$ is a cell complex of dimension $6g + 3k + 3\ell - 7$.

If G is a cell of dimension $6g + 3k + 3\ell - 7$ (i.e, G is top dimensional), then for all Γ in the interior of G, all nondistinguished vertices are trivalent, all distinguished vertices are bivalent and each E_i contains just one chord. Therefore, each subgraph $V \cup E_i$ consists of disjoint vertices and a single edge with distinct endpoints.

Therefore, $\overline{LD}(g, k, \ell)$ is a cell complex of dimension $6g + 3k + 3\ell - 7$. □

2.6 Slide equivalence

We make the following modification to slide equivalence for string diagrams with levels:

Definition 2.9. Two string diagrams of type (g, k, ℓ) with levels Γ and Γ' differ by a chord slide if they are identical except for the placement of one endpoint of a single chord: ch in Γ and ch' in Γ'. In Γ, let $ch \in E_i$ and $\overline{ch} \in E_j$ where $j \leq i$. Similarly, in Γ', let $ch' \in E'_i$ and $\overline{ch} \in E'_j$ where $j \leq i$. In Γ if $\overline{ch}$ follows ch in the cyclic order at a vertex v, then in Γ', the corresponding chord ch' follows $\overline{ch}$ in the cyclic order at the other endpoint w of $\overline{ch}$. Let chord slides generate an equivalence relation $\sim$ called slide equivalence.

Proposition 2.1. *The space $\overline{LD}(g, k, \ell)/\sim$ is a pseudomanifold with boundary.*

Proof. We begin by showing that $\overline{LD}(g, k, \ell)$ is a (possibly disconnected) pseudomanifold with boundary. We show that every cell is the face of a top-dimensional cell and that a codimension 1 cell is the face of exactly 1 or 2 top-dimensional cells.

Recall that a cell of $\overline{LD}(g,k,\ell)$ is a cartesian product of k simplices with a cube. Any cell F of $\overline{LD}(g,k,\ell)$ is the face of some $6g+3k+3\ell-7$-dimensional cell G. We imagine moving from G to F by collapsing input edges and setting spacing parameters to 0 or 1.

Let C be a connected component of $\overline{LD}(g,k,\ell)$.

A codimension 1 face F of a top-dimensional cell G in $\overline{LD}(g,k,\ell)/\sim$ contains string diagrams with levels Γ which either have a single input edge e of length $L=0$ or have a spacing parameter s_i which is 0 or 1. There are four cases.

Case 1. If $s_i = 1$, then there is no other G' with F as its face, so F is at the boundary of C.

Case 2. If $s_i = 0$, then F is the face of exactly one other G'. As G is a top-dimensional cell, each Γ in the interior of G has one chord at each level. In particular, the combinatorial type labeling G has $E_i = \{ch_i\}$ and $E_{i+1} = \{ch_{i+1}\}$. Since $s_i = 0$ in the combinatorial type labeling F, $E_i = \{ch_i, ch_{i+1}\}$. Let G be labeled by the same combinatorial type as G except the chords ch_i and ch_{i+1} are interchanged, so $E'_j = E_j$ if $j \neq i, i+1$, $E'_i = \{ch_{i+1}\}$ and $E'_{j+1} = \{ch_i\}$. The top-dimensional cells G and G' are the only two containing F as a face.

Case 3. If e is an input edge joining a distinguished vertex v and a nondistinguished vertex w in the combinatorial type labeling G and e is collapsed in the combinatorial type labeling F, then F is the face of a second top-dimensional cell G. The combinatorial type labeling G' is the same as that labeling G, except the vertices v and w are interchanged. Again, when the input edge joining v and w in the combinatorial type labeling G is collapsed, the combinatorial type labeling F is obtained. The top-dimensional cells G and G' are the only two containing F as a face.

Case 4. If e is an input edge joining two nondistinguished vertices in the combinatorial type labeling G and e is collapsed in the combinatorial type labeling F, then there is no other G with F as its face, so F is at the boundary of C.

We now examine the quotient by slide equivalence.

Since G is a top-dimensional cell, no two chord endpoints in the combinatorial type labeling G may coincide, so G is the only cell in its slide-equivalence class.

In the first 3 cases, F is the only cell in its slide-equivalence class. In case 4, there are two sub-cases depending on whether the edge e that is being collapsed joins endpoints of distinct chords or the two endpoints of a single chord.

Case 4a. Let e be an input edge of the combinatorial type labeling G that is collapsed in the combinatorial type labeling F and assume e joins endpoints v_i and v_j of two distinct chords ch_i and ch_j. In the combinatorial type F, v_i

and v_j come together. There is one other codimension 1 face F' which is slide-equivalent to F. Assume that $j > i$. Then a string diagram with levels Γ in the interior of F is slide-equivalent to exactly one Γ'. The string diagram with levels Γ' is exactly the same as Γ, except v_j coincides with the other endpoint v_i' of ch_i. Since before identification, F is a face only of G and F' is a face only of G' and since F and F' are the only two cell in their slide-equivalence class, the codimension 1 cell $F \sim F'$ is the face of only G and G' after identification.

Case 4b. Let e be an input edge of the combinatorial type labeling G that is collapsed in the combinatorial type labeling F and assume e joins the two endpoints of a single chord. Then F is the only cell in its slide-equivalence class, so F remains at the boundary of $\overline{LD}(g,k,\ell)/\sim$.

□

Remark. We will see later that $\overline{LD}(g,k,\ell)/\sim$ is orientable.

Proposition 2.2. *A slide-equivalence class of string diagrams with levels is at the boundary of* $\overline{LD}(g,k,\ell)/\sim$ *if and only if a spacing parameter* $s_i = 1$ *or* Γ *has an output circle that is made up only of chords.*

Proof. In the proof of Proposition 2.1, we see that codimension 1 faces F in cases 1 and 4b are faces of only one top-dimensional cell G. Therefore, they are at the boundary of $\overline{LD}(g,k,\ell)/\sim$.

Clearly, if Γ has a spacing parameter $s_i = 1$, then Γ is in the closure of a face F

as in case 1, so its slide equivalence class is at the boundary of $\overline{LD}(g, k, \ell)/\sim$.

If Γ is a string diagram with levels that has an output circle that is made up only of chords, then there exists a slide-equivalent Γ' that has a chord whose endpoints coincide on an input circle. Then the slide-equivalence class of Γ is in the closure of a codimension 1 face F as in case 4b and is therefore at the boundary of $\overline{LD}(g, k, \ell)/\sim$.

If the slide-equivalence class of Γ is at the boundary of $\overline{LD}(g, k, \ell)/\sim$, then Γ is contained in the closure of a codimension 1 face F of the boundary. Again, by the proof of Proposition 2.1 Γ is as in case 1 or 4b.

□

Proposition 2.3. *The space* $\overline{LD}(g, k, \ell)/\sim$ *contains* $\overline{SD}(g, k, \ell)/\sim$ *as a deformation retract.*

Proof. Let $i : \overline{SD}(g, k, \ell)/\sim \longrightarrow \overline{LD}(g, k, \ell)/\sim$ be the inclusion map of slide-equivalence classes of string diagrams as slide-equivalence classes of string diagrams with one level. Let $r : \overline{LD}(g, k, \ell)/\sim \longrightarrow \overline{SD}(g, k, \ell)/\sim$ be the map that forgets levels and spacing parameters or, equivalently, sets them all to 0.

Then $r \circ i = id$ and $i \circ r$ is homotopic to id. The homotopy $\gamma : \overline{LD}(g, k, \ell)/\sim \times I \longrightarrow \overline{LD}(g, k, \ell)/\sim$ is given by the following. Let Γ be a slide-equivalence class of string diagram of type (g, k, ℓ) with levels and with spacing parameters $s_1, s_2, \ldots, s_n$. Then $\gamma(\Gamma, t)$ is identical to Γ except its spacing parameters are given by $(1-t)s_1, (1-t)s_2, \ldots, (1-t)s_n$. We have $\gamma(\Gamma, 0) = \Gamma$ and $\gamma(\Gamma, 1) = i(r(\Gamma))$. If Γ is a slide-equivalence class of string diagrams with levels such that all $s_i = 0$ then $\gamma(\Gamma, t) = \Gamma$ for all t.

$\square$

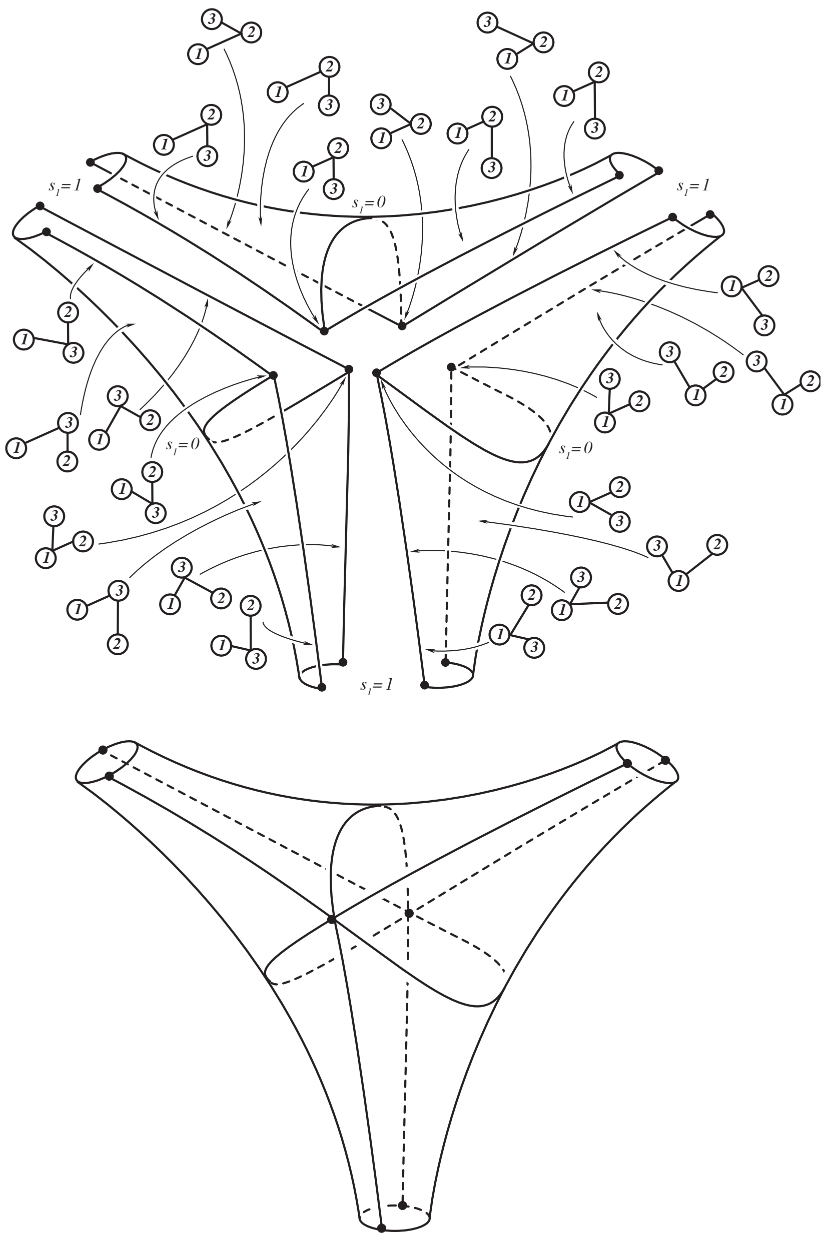


Figure 2.11: Base spaces for T^3-bundles $\overline{LD}(0,3,1)$ and $\overline{LD}(0,3,1)/\sim$.

Chapter 3

The string topology construction

We begin by describing a chain complex $C_*^{T^k}(\overline{SD}(g,k,\ell))$ and a chain complex $P_*(LM^k)$ computing the T^k-equivariant homology of the k-loop space of a manifold M. We then describe the string topology construction, which provides a chain map

$$ST_\mu : C_*^{T^k}(\overline{SD}(g,k,\ell)) \to Hom(P_*(kLM), P_*(\ell LM)).$$

Given $c \in C_*^{T^k}(\overline{SD}(g,k,\ell))$, $ST_\mu(c) \in Hom(P_*(kLM), P_*(\ell LM))$ factors through $Hom(P_*(kLM), P_*(Maps(X_c, M)))$, where X_c is a family of string diagrams of type (g,k,ℓ). The map $P_*(Maps(X_c, M)) \to P_*(\ell LM)$ is given by the restriction to output circles.

3.1 Saturated chains

When input circles are ordered and oriented compatibly for all string diagrams $\Gamma \in \overline{SD}(g,k,\ell)$, we obtain an ordering of input edges for each Γ or each combinatorial type G. Such an ordering induces a canonical identification of G with $[0,1,\dots n_1] \times [0,1,\dots n_2] \times \cdots \times [0,1,\dots n_k]$ where $[0,1,\dots n_j]$ is the standard n_j-simplex Δ^{n_j} representing the parameter space for the n_j+1 edges comprising the jth input circle. So for example, if the first edge e_0 of the second input circle has length L_0, this codimension 1 face of G corresponds to $[0,1,\dots n_1] \times [1,\dots n_2] \times \cdots \times [0,1,\dots n_k]$.

Definition 3.1. The chain complex $C_*(\overline{SD}(g,k,\ell))$ is generated by oriented cells G of $\overline{SD}(g,k,\ell)$. The boundary map

$$\partial : C_*(\overline{SD}(g,k,\ell)) \to C_{*-1}(\overline{SD}(g,k,\ell))$$

is induced by the boundary map on simplices,

$$\partial[0,1,\dots,n] = \sum_i (-1)^i [0,1,\dots,\hat{i},\dots,n]$$

and the Leibniz rule.

Remark. Let T^k be the k-dimensional torus. There is a free T^k-action on $\overline{SD}(g,k,\ell)$ given by rotating the distinguished vertices on the k input circles. The orbit of the T^k-action is contained in a connected component of $\overline{SD}(g,k,\ell)$.

The following definition is motivated by the notion of the T^k-orbit of a cell.

Definition 3.2. Fix an oriented cell G_i labeled by a combinatorial type with all distinguished vertices bivalent. The cell G_i gives rise to a chain $c = \sum a_j G_j$, as follows. For all j, the combinatorial type labeling G_j has bivalent distinguished vertices. If the combinatorial type labeling G_j is not the same as that labeling G_i, when distinguished vertices are forgotten, then $a_j = 0$. Otherwise, $a_i = \pm 1$. If the combinatorial type labeling G_j may be obtained from the combinatorial type labeling G_i by moving the distinguished vertex on the nth input circle past one chord endpoint, then $a_j = 1$ if the number of nondistinguished vertices on the jth circle is odd and $a_j = -1$ if the number of nondistinguished vertices on the jth circle is even. Given a_j, the other coefficients are generated analogously. Such a chain c is called a minimally saturated chain. A linear combination of minimally saturated chains is called a saturated chain.

Lemma 3.1. *The boundary of a saturated chain is a saturated chain.*

Proof. Let $c = \sum_i a_i G_i$ be a minimally saturated chain. It suffices to show that $\partial(c)$ is a saturated chain.

Let $\partial(G_i) = \sum_j (-1)^{\sigma_j} F^i_j$. For all F^i_j, in the combinatorial type labeling F^i_j, either the distinguished vertex coincides with a nondistinguished vertex or two nondistinguished vertices coincide. We first show that if in a combinatorial type labeling F^i_j where a distinguished vertex coincides with a nondistinguished vertex, then the coefficient of F^i_j in the expression for $\partial(c)$ is 0. We then show that if the combinatorial type labeling F_j^i has bivalent distinguished vertices, and the combinatorial type labeling $F_{j'}^{i'}$ differs from that of F_j^i except for placement of distinguished vertices on input circles, then the coefficients of $F_{j'}^{i'}$ and F_j^i in the expression for $\partial(c)$ satisfy the conditions of a minimally saturated chain.

If in the combinatorial type labeling F_j^i a distinguished vertex and nondistinguished vertex coincide, then F_j^i appears from collapsing either the first or last edge of an input circle in the combinatorial type labeling G_i. Assume that in combinatorial type labeling F_j^i the distinguished vertex v on the nth input circle coincides with the first nondistinguished vertex w, that is, the first edge e_o^n of the nth input circle of the combinatorial type labeling G_i is collapsed to obtain the combinatorial type labeling F_j^i. Then the coefficient of F_j^i in the expression for $\partial(G_i)$ is $(-1)^0 = 1$. Similarly, let the combinatorial type labeling $G_{i'}$ be obtained from the combinatorial type labeling G_i by interchanging the placement of the distinguished vertex v and the first nondistinguished vertex w on the nth input circle. Therefore, w becomes the last nondistinguished vertex on the nth input circle in the combinatorial type labeled by $G_{i'}$. The combinatorial type labeling F_j^i is again present in the expression for $\partial(G_{i'})$, obtained by collapsing the last edge of the nth input circle in the combinatorial type labeling $G_{i'}$.

Let $m_n + 1$ be the number of edges making up the nth input circle in the combinatorial type labeling G_i. Then there are also $m_n + 1$ edges making up the nth input circle in the combinatorial type labeling $G_{i'}$. If m_n is even, then the cells G_i and $G_{i'}$ have opposite signs in the expression for c. The coefficient for

F^i_j in the expression for $\partial(G_{i'})$ is $(-1)^{m_n} = 1$ so the coefficient on F^i_j in the expression for $\partial(c)$ is 0. If m_n is odd then the cells G_i and $G_{i'}$ have the same sign in the expression for c. The coefficient for F^i_j in the expression for $\partial(G_{i'})$ is $(-1)^{m_n} = -1$ so again the coefficient on F^i_j in the expression for $\partial(c)$ is 0.

Now assume the combinatorial type labeling F^i_j has bivalent distinguished vertices. That is, it is obtained by collapsing an input edge joining nondistinguished vertices in the nth input circle of the combinatorial type labeling G_i. Again, let the combinatorial type labeling $G_{i'}$ be obtained from that labeling G_i by interchanging the distinguished vertex and the first nondistinguished vertex. Similarly, let $F^{i'}_{j'}$ be obtained from F^i_j by interchanging the distinguished vertex and the first nondistinguished vertex. And let F^i_j be obtained from G_i by collapsing the m_pth edge on the nth input circle. Then $F^{i'}_{j'}$ is obtained from $G_{i'}$ by collapsing the $(m_p - 1)$st edge on the nth input circle. Therefore, the coefficient of F^i_j in the expression for $\partial(G_i)$ is $(-1)^{m_p}$ while the coefficient of $F^{i'}_{j'}$in the expression for $\partial(G_{i'})$ is $(-1)^{m_p-1}$.

Again, let $m_n + 1$ be the number of edges making up the nth input circle of G_i and $G_{i'}$. If $m_n + 1$ is even, then G_i and $G_{i'}$ have opposite signs in the expression for c and then F^i_j and $F^{i'}_{j'}$, which have an odd number of edges making up the nth input circle, have the same sign in the expression for $\partial(c)$. If $m_n + 1$ is odd, then G_i and $G_{i'}$ have the same sign in the expression for c and then F^i_j and $F^{i'}_{j'}$, which have an even number of edges making up the nth input circle, have opposite signs in the expression for $\partial(c)$. In either case, the coefficient condition for $\partial(c)$ to be saturated is satisfied.

□

Definition 3.3. Let $C_*^{T^k}(\overline{SD}(g,k,\ell))$ be the chain complex generated by minimally saturated chains with induced differential ∂.

Remark. We expect that the homology of $C_*^{T^k}(\overline{SD}(g,k,\ell))$ is the shifted T^k-equivariant homology $H_{*+k}^{T^k}(\overline{SD}(g,k,\ell))$ of $\overline{SD}(g,k,\ell)$. The space $\overline{SD}(g,k,\ell)$ is also a T^k-equivariant cell complex and there is a bijection between the set of generators c $C_*^{T^k}(\overline{SD}(g,k,\ell))$ and equivariant cells.

Definition 3.4. Let c be a generator of $C_n^{T^k}(\overline{SD}(g,k,\ell))$, that is, c is a minimally saturated n-chain. Let X_c be the full sub-cell complex of $\overline{SD}(g,k,\ell)$ whose top-dimensional cells are the n-cells appearing as terms with nonzero coefficients in the expression for c. The space X_c is called the sub-cell complex associated to the minimally saturated chain c.

Remark. Let X_c be the sub-cell complex of $\overline{SD}(g,k,\ell)$ associated to a minimally saturated chain c. The k-torus T^k acts freely on X_c by rotating each of the marked points in the k input circles.

Definition 3.5. The definitions for the complex of saturated chains $C_*^{T^k}(\overline{LD}(g,k,\ell))$ and the sub-cell complex X_c of $\overline{LD}(g,k,\ell)$ associated to a generator c are analogous to those for $\overline{SD}(g,k,\ell)$.

3.2 Equivariant chain model of the loop space

The following two definitions are similar to those appearing in [RS72] and [Sul77] respectively.

Definition 3.6. 1. A linear n-cell C is a subset of $\mathbb{R}^N$ which is a convex combination of points $v_1, \ldots, v_M$ in $\mathbb{R}^N$ which span an n-dimensional subspace. A face of C consists of points on the boundary of C which are a convex combination of some subset of $\{v_1, \ldots, v_M\}$.

2. A compact n-dimensional polyhedron B is a finite union of linear n-cells C identified along faces such that each point $a \in B$ has a neighborhood N which is a cone on a compact subspace $L \subset B$ and a is the cone point.

Definition 3.7. 1. Let $x_1, x_2, \ldots x_N$ be the usual coordinates for $\mathbb{R}^N$ and let $C \subset \mathbb{R}^N$ be a linear n-cell. A $\mathbb{Q}$-polynomial form of degree m on C is a differential form

$$\omega = \sum f_{i_1,\ldots,i_m}(x_1, \ldots, x_N) dx_{i_1} \wedge \cdots \wedge dx_{i_m}$$

on $\mathbb{R}^N$, restricted to B such that each $f_{i_1,\ldots,i_m}$ is a polynomial in $\{x_i, \ldots, x_N\}$ with rational ($\mathbb{Q}$) coefficients.

2. Let B be a compact n-dimensional polyhedron. A $\mathbb{Q}$-polynomial form on B is a collection of $\mathbb{Q}$-polynomial forms ω_C, one for each linear n-cell C

such that if $F \overset{i}{\hookrightarrow} C$ is a face, $i^*(\omega_C)$ and ω_F agree on multi-vectors tangent to F.

Definition 3.8. Let B be a compact polyhedron and let α be a $\mathbb{Q}$-polynomial form on B. The support of α $supp(\alpha)$ is the smallest closed subspace of B such that

$$\alpha|_{B-supp(\alpha)} \equiv 0.$$

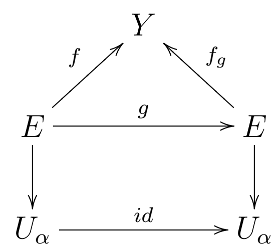
Definition 3.9. Let X and Y be topological spaces and G be a topological group which acts on X. A quadruple (B, α, E, f) consists of:

1. $B = B^n$, a compact polyhedron with orientations of linear n-cells;
2. α, a $\mathbb{Q}$-polynomial m-form on B;
3. U_α, an open set of B containing $supp(\alpha)$ and E, the total space of an X-bundle over U_α;
4. $f : E \to M$, a continuous map;

$$\begin{array}{ccccc} X & \longrightarrow & E & \xrightarrow{f} & Y \\ & & \downarrow{\scriptstyle \pi} & & \\ & & U_\alpha & & \end{array}$$

5. an action of G on E such that for all $g \in G$ the map induced by the action restricted to the element g on E is a bundle map over the identity on U_α

and there exists a map $f_g : E \to Y$ such that the diagram commutes.



If $B = \bigsqcup_i B_i/\sim$ where B_i are sub-polyhedra identified along codimension 1 faces by $\sim$, $\alpha_i = \alpha|_{B_i}$, $E_i = E|_{B_1}$ and $f_i = f|_{E_i}$ then we set

$$(B, \alpha, E, f) = \sum_i (B_i, \alpha_i, E_i, f_i).$$

Definition 3.10. Let quadruples (B, α, E, f) generate a vector space $Q_*(Maps(X, Y))$ over $\mathbb{Q}$. Let $D : Q_*(Maps(X, Y)) \to Q_*(Maps(X, Y))$ be a map defined on generators (B, α, E, f) by

$$\begin{aligned} D(B, \alpha, E, f) &= (\partial(B), \alpha|_{\partial(B)}, E|_{\partial(B)}, f|_{\partial(B)}) + (-1)^{n-m}(B, d(\alpha), E, f) \\ &= \sum_i (-1)^i (F_i, \alpha|_{F_i}, E|_{F_i}, f|_{F_i}) + (-1)^{n-m}(B, d(\alpha), E, f) \end{aligned}$$

and extended linearly, where

- $\partial(B) = \sum_{i=1}(-1)^i F_i$ where F_i is the ith oriented $(n-1)$-dimensional face of B;
- $d(\alpha)$ is the exterior derivative of the $\mathbb{Q}$-polynomial form α.

Remark. If $U_\alpha \cap F_i = \phi$ for some i, then $(F_i, \alpha|_{F_i}, E|_{F_i}, f|_{F_i}) = 0$.

Lemma 3.2. *The map* $D : Q_*(Maps(X,Y)) \to Q_*(Maps(X,Y))$ *is a differential, that is,* $D^2 = 0$.

Proof. We show that $D^2 = 0$ on generators (B, α, E, f) of $Q_*(Maps(X,Y))$.

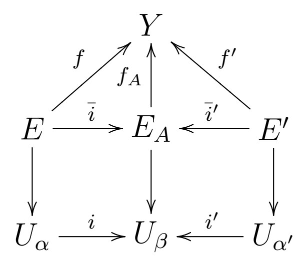
$$
\begin{aligned}
D^2(B,\alpha,E,f) &= D(\partial(B), \alpha|_{\partial(B)}, E|_{\partial(B)}, f|_{\partial(B)}) + (-1)^{n-m} D(B, d(\alpha), E, f) \\
&= (\partial^2(B), \alpha|_{\partial(B)}, E|_{\partial(B)}, f|_{\partial(B)}) + (-1)^{n-m-1}(\partial(B), d(\alpha)|_{\partial(B)}, E|_{\partial(B)}) \\
&\quad + (-1)^{n-m}(\partial(B), d(\alpha)|_{\partial(B)}, E|_{\partial(B)}) + (-1)^{2(n-m)}(B, d^2(\alpha), E, f) \\
&= 0
\end{aligned}
$$

∎

Definition 3.11. Let the following generate an equivalence relation. Two quadruples (B, α, E, f) and (B', α', E', f') are equivalent if

1. there exist a polyhedron A and maps $i : B \to A$ and $i' : B' \to A$;
2. there exists a $\mathbb{Q}$-polynomial form β on A such that $i^*(\beta) = \alpha$ and $(i')^*(\beta) = \alpha'$;
3. there exists an open neighborhood U_β of $supp(\beta)$ such that $U_\beta \subset i(U_\alpha) \cap i'(U_{\alpha'})$;
4. there exists an X-bundle over U_β with total space E_A such that the pullback bundles $i^*(E_A) = E$ and $(i')^*(E_A) = E'$;

5. there exists a map $f_A : E_A \to Y$ such that the diagram commutes



6. the maps of total spaces $\bar{i} : E \to E_A$ and $\bar{i}' : E' \to E_A$ are G-equivariant.

Remark. In particular, (B, α, E, f), (A, β, E_A, f_A) and $(im(i), \beta|_{im(i)}, E_A|_{im(i)}, f_a|_{im(i)})$ are all equivalent.

Lemma 3.3. *The differential D is well-defined on equivalence classes.*

Proof. It is enough to check that since (B, α, E, f) and $(im(i), \beta|_{im(i)}, E_A|_{im(i)}, f_a|_{im(i)})$ are equivalent, $D(B, \alpha, E, f)$ and $D(im(i), \beta|_{im(i)}, E_A|_{im(i)}, f_a|_{im(i)})$ are equivalent.

$$D(B, \alpha, E, f) = \sum_i (-1)^j (F_j, \alpha|_{F_j}, E|_{F_j}, f|_{F_j}) + (-1)^{n-m}(B, d(\alpha), E, f)$$

$$D(im(i), \beta|_{im(i)}, E_A|_{im(i)}, f_a|_{im(i)}) = \sum_j (-1)^j (i(F_j), \beta|_{i(F_j)}, E_A|_{i(F_j)}, f_A|_{i(F_j)}) + (-1)^{n-m}(A, d(\beta), E_A, f_A)$$

For all j, $(F_j, \alpha|_{F_j}, E|_{F_j}, f|_{F_j})$ and $(i(F_j), \beta|_{i(F_j)}, E_A|_{i(F_j)}, f_A|_{i(F_j)})$ are equivalent by restricting $i : B \to A$ to $i|_{F_j} : F_j \to A$.

Because $d(\alpha) = d(i^*(\beta)) = i^*(d(\beta))$, $(B, d(\alpha), E, f)$ and $(A, d(\beta), E_A, f_A)$ are also equivalent.

□

Definition 3.12. Let (B, α) denote the equivalence class $[(B, \alpha, E, f)]$ of the quadruple (B, α, E, f). Let D also denote the map induced on equivalence classes by the differential D. $P_*(Maps(X, Y))$ is the chain complex generated by (B, α) with induced differential D.

Remark. We will use the notation $D(B, \alpha) = (\partial(B), \alpha) + (-1)^{n-m}(B, d(\alpha))$.

Definition 3.13. The homology of the chain complex $(P_*(Maps(X, Y)), D)$ is called the G-equivariant homology of the mapping space $Maps(X, Y)$ with $\mathbb{Q}$ coefficients, $H_*^G(Maps(X, Y); \mathbb{Q})$.

Remark. We expect that $H_*^G(Maps(X, Y); \mathbb{Q})$ computes the singular homology of $H_*(Maps(X, Y) \times_G EG; \mathbb{Q})$, the usual G-equivariant homology of $Maps(X, Y)$. In the following diagram, $\pi \circ f$ determines a X-bundle over U_α up to G-equivalence and $p \circ f$ determines a map of the total space of this bundle to Y, again up to G-equivalence.

$$\begin{array}{ccccc} U_\alpha & \xrightarrow{f} & Maps(X,Y)\times_G EG & & \\ & & \downarrow \pi & \searrow^{p} & \\ & & BG & & Maps(X,Y)/G \end{array}$$

Definition 3.14. Let M be a d-dimensional manifold. The free k-loop space kLM of M is the space of continuous maps $\bigsqcup_k S^1 \to M$. In particular, $1LM = LM$, the free loop space of M and $kLM = LM^k$, the cartesian product of k copies of LM.

Remark. The k-dimensional torus T^k acts on kLM by rotating each of the k loops. Let $G = T^k$, $X = \bigsqcup_k S^1$ and M. The total space E^k of the $\bigsqcup_k$-bundle over U_α is disconnected; the k circles in each fiber have distinct labels so E^k determines k S^1-bundles over U_α.

Then the homology of $P_*(kLM)$ is $H_*^{T^k}(kLM) \cong H_*^{S^1}(LM)^{\otimes k}$. See figure 3.1.

3.3 The diagonal map

Fix a generator c of $C_*^{T^k}(\overline{SD}(g,k,\ell))$ and let X_c be its associated sub-cell complex of $\overline{SD}(g,k,\ell)$. All string diagrams in the interiors of $|c|$-dimensional cells of X_c have the same combinatorial type (except for the placement of the distinguished vertices on their k input circles). All string diagrams in X_c have the

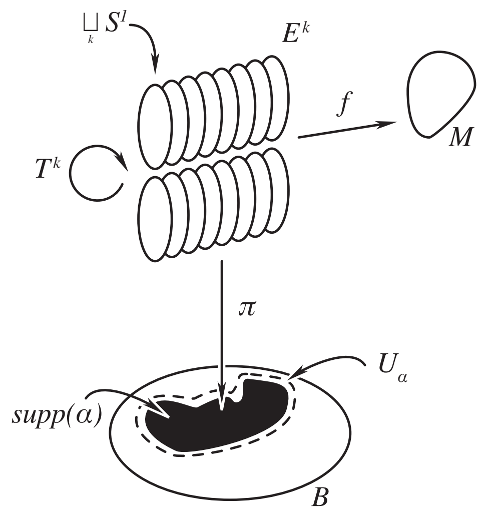


Figure 3.1: (B, α, E^k, f)

same combinatorial type (except for the placement of their distinguished vertices and for the placement of some chords' endpoints on input circles, which may coincide). Therefore, for any two $\Gamma, \Gamma' \in X_c$, there is a canonical bijection between their sets of chords. In particular, for any $\Gamma \in X_c$, there is a canonical bijection between its set of chords and a fixed finite set. Similarly, for all $\Gamma \in X_c$ there is a canonical bijection between its set of *half*-chords and a fixed finite set.

Notation.

1. Denote the finite set corresponding to the set of chords of Γ, for all Γ in X_c, by $\mathscr{C}$ and the finite set corresponding to the set of half-chords of Γ, for all $\Gamma \in X_c$, by $\mathscr{H}$. (By abuse of notation, we will sometimes refer to a chord of a string diagram Γ as an element ch of $\mathscr{C}$ and, similarly, a

half-chord of Γ as an element h of $\mathscr{H}$.)

2. For a fixed $\Gamma \in X_c$, each half-chord lies inside a unique chord. Denote the corresponding two-to-one map of finite sets taking an element $h \in \mathscr{H}$ to an element $ch \in \mathscr{C}$ by $R : \mathscr{H} \to \mathscr{C}$.

3. Let M be a topological space and let $\mathscr{S}$ be a finite set. With the discrete topology, $\mathscr{S}$ is a topological space. Denote the the space of maps $Hom(\mathscr{S}, M)$ by $M^{\mathscr{S}}$.

4. Let $T : \mathscr{S}_1 \to \mathscr{S}_2$ be a map of finite sets. Denote the induced map on spaces by $M^T : M^{\mathscr{S}_2} \to M^{\mathscr{S}_1}$ where $M^T(f) : \mathscr{S}_1 \to M$ is defined by $M^T(f)(s_1) = f(T(s_1))$ for $f \in M^{\mathscr{S}_2}$ and $s_1 \in \mathscr{S}_1$.

We arrive at the definition of the diagonal map.

Definition 3.15. Let c be a generator of $C_*^{T^k}(\overline{SD}(g, k, \ell))$. The diagonal map

$$\Delta_c : M^{\mathscr{C}} \to M^{\mathscr{H}}$$

is the map M^R induced on spaces by the map $R : \mathscr{H} \to \mathscr{C}$. The diagonal is the image of the diagonal map $\Delta_c(M^{\mathscr{C}}) \subset M^{\mathscr{H}}$.

3.4 The Thom class representative

Let M be a closed, oriented, complete Riemannian manifold of dimension d. Fix a generator c of $C_*^{T^k}(\overline{SD}(g,k,\ell))$.

Notation.

1. Fix $ch \in \mathscr{C}$ such that $R(h) = R(h') = ch$ for $h, h' \in \mathscr{H}$. By abuse of notation, ch also denotes the map $ch : M^{\mathscr{H}} \to M^{\{h,h'\}}$ which is the map induced on spaces by the inclusion of finite sets $\{h, h'\} \to \mathscr{H}$.

2. The map $\Delta_{ch} : M^{\{ch\}} \to M^{\{h,h'\}}$ is induced on spaces by $R|_{\{h,h'\}} : \{h, h'\} \to \{ch\}$. The map Δ_{ch} corresponds to the usual diagonal map $M \to M \times M$ and we refer to it as the diagonal map *corresponding to* ch.

3. Let $\varepsilon > 0$ and let N_ε be a tubular ε-neighborhood of $\Delta_{ch}(M^{\{ch\}}) \subset M^{\{h,h'\}}$. Let μ denote a closed $\mathbb{Q}$-polynomial differential form which represents the Thom class of the diagonal corresponding to ch and has support $supp(\mu) \subset N_\varepsilon$. (That is, $M^{\{ch\}} \cong M$ and the cohomology class $[\mu]$ of μ is Poincaré dual to the image of the fundamental class $[M]$ under the map $(\Delta_{ch})_* : H_*(M^{\{ch\}}) \to H_*(M^{\{h,h'\}})$ induced on homology by Δ_{ch}.)

We arrive at the representative of the Thom class of the diagonal.

Definition 3.16. Let

$$\mu_c = \bigwedge_{ch \in \mathscr{C}} ch^*(\mu)$$

be a closed $\mathbb{Q}$-polynomial form on $M^{\mathscr{H}}$ whose support $supp(\mu_c)$ lies in the neighborhood $N_\mu = \bigcap_{ch \in \mathscr{C}} ch^{-1}(N_\varepsilon)$ of the diagonal $\Delta_c(M^{\mathscr{C}})$.

Remark. The closed $\mathbb{Q}$-polynomial form μ_c represents the Thom class of the diagonal $\Delta_c(M^{\mathscr{C}}) \subset M^{\mathscr{H}}$.

3.5 The string topology construction

Let M be a d-dimensional closed, oriented, complete Riemannian manifold such that all balls of radius less than ε are contractible. In this section, we describe the string topology construction, which gives a chain map

$$ST_\mu : C_*^{T^k}(\overline{SD}(g,k,\ell)) \longrightarrow Hom(P_*(kLM), P_*(\ell LM)).$$

The map is first defined on generators c of $C_*^{T^k}(\overline{SD}(g,k,\ell))$ and extended as a chain map to all of $C_*^{T^k}(\overline{SD}(g,k,\ell))$.

Notation.

1. Let c be a generator of $C_n^{T^k}(\overline{SD}(g,k,\ell))$.

2. Let X_c be the sub-cell complex associated to c.

3. Let $\mathscr{C}$ and $\mathscr{H}$ denote the sets of chords and half-chords respectively.

4. Let μ_c be the representative of the Thom class of the diagonal

$$\Delta_c : M^{\mathscr{C}} \to M^{\mathscr{H}}$$

   which is supported in the neighborhood N_μ of $\Delta_c(M^{\mathscr{C}})$, as described above.

5. Let (B, α) be a generator of $P_{n-m}(kLM)$ with representative (B, α, E^k, f) where B is an n-dimensional polyhedron and α is a $\mathbb{Q}$-polynomial form of degree m.

We construct a representative of $ST_\mu(c)(B, \alpha)$ by constructing each component of the quadruple.

3.5.1 The base space

We begin with the $\bigsqcup_k S^1$-bundle over U_α. The k-torus T^k acts on $\bigsqcup_k S^1$ and on X_c, hence the fiber bundle

$$\begin{array}{ccc} \bigsqcup_k S^1 & \longrightarrow & E^k \\ & & \big\downarrow \pi \\ & & U_\alpha \end{array}$$

determines an associated X_c-bundle. Let

$$\begin{array}{ccc} X_c & \longrightarrow & E_c \\ & & \big\downarrow \pi_c \\ & & U_\alpha \end{array}$$

denote the associated bundle with fiber X_c. See figure 3.2.

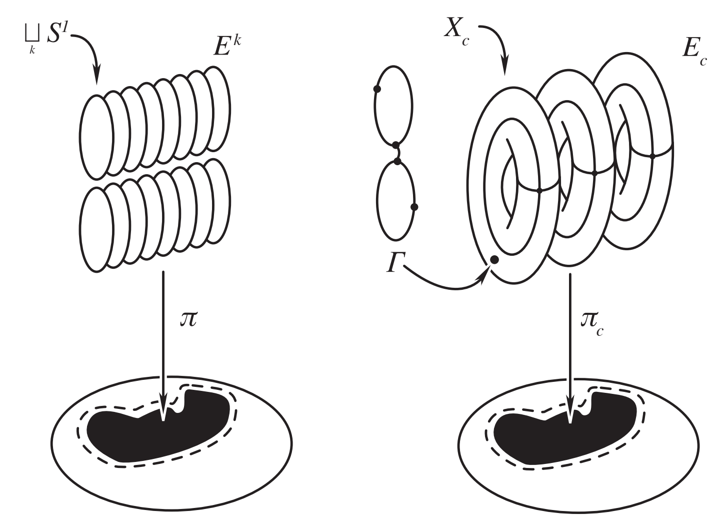


Figure 3.2: $\bigsqcup_k S^1$-bundle and associated X_c-bundle

Let W_α be a closed subset of U_α containing $supp(\alpha)$ such that under a decomposition of B, W_α is a closed regular sub-complex of dimension n and each codimension 1 cell f of W_α is the face of either 1 or 2 n-dimensional cells. As each cell w of W_α is contractible, the X_c-bundle over each cell is trivial and $\pi_c^{-1}(w) \sim X_c \times w$ is a cell complex. The top cells of $\pi_c^{-1}(W_\alpha)$ have dimension $n + dim(X_c) = n + |c|$. Denote the set of these cells by $\{B_i\}_{i \in I}$. The B_i form the base spaces of $ST_\mu(c)(B, \alpha)$. See figure 3.3. We choose a trivialization $\{U, \varphi\}$

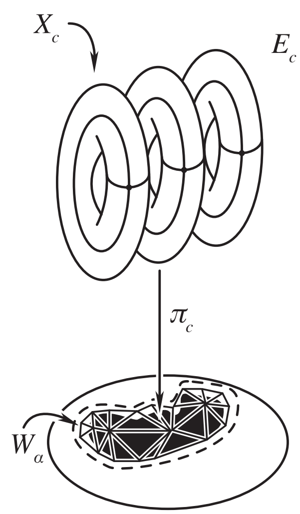


Figure 3.3: Decomposition of W_α

of $\pi_c^{-1}(W_\alpha)$ such that each n-cell w of W_α is contained in exactly one open set U_w and $\varphi_w : \pi_c^{-1}(w) \to w \times X_c$. A codimension 1 face f of w that is also a face of w' is contained in (at least) two open sets U_w and $U_{w'}$. The transition map $\varphi_{w'} \circ \varphi_w^{-1} : \varphi_w(\pi_c^{-1}(U_w \cap U_{w'})) \to \varphi_{w'}(\pi_c^{-1}(U_w \cap U_{w'}))$ need not be cellular. Recall that the transition map rotates distinguished vertices on input circles and that the fiber X_c is itself the total space of a T^k-bundle. Each map φ_w and $\varphi_{w'}$

induces a cell structure on $\pi_c^{-1}(f)$. We let their common subdivision be the cell complex structure on $\pi_c^{-1}(f)$. (Each S^1 factor of the T^k fiber is decomposed into intervals. The transition maps rotate these circles. The common subdivision of $\pi_c^{-1}(f)$ comes from the common subdivision of the circle given by rotation and identification.) This procedure gives a well-defined cell complex structure on all of $\pi_c^{-1}(W_\alpha)$ with top-dimensional cells B_i of the form $e \times w$ where w is an n-dimensional cell of W_α and e is a $|c|$-dimensional cell of X_c.

3.5.2 The $\mathbb{Q}$-polynomial form

We begin by choosing a point on each S^1 component of $\pi^{-1}(a)$ for all $a \in U_\alpha$. The choice is made consistent by specifying k sections of the $\bigsqcup_k S^1$-bundle, each of which has its image in a different component of E^k. Such a choice determines an identification of $\pi^{-1}(a)$ with the k labeled input circles of Γ, for all $\Gamma \in \pi_c^{-1}(a)$. Later we will see that the output of the string topology construction is independent of this choice.

With this identification, the evaluation map

$$ev : E_c \to M^{\mathscr{H}}$$

is defined component-wise as follows:

Let $\Gamma \in \pi_c^{-1}(a)$ for $a \in U_\alpha$ and let $p \in \Gamma$ be the endpoint of the half-chord h on an input circle of Γ. Let $\pi_h : M^{\mathscr{H}} \to M^{\{h\}}$ be the projection map induced by the inclusion of finite sets $\{h\} \to \mathscr{H}$. Then $\pi_h(ev(\Gamma)) = f(p)$. That is, the h-component of $ev(\Gamma)$ is $f(p)$. The components of $ev(\Gamma) \in M^{\mathscr{H}}$ for all other half-chords h' of Γ with endpoints p' are defined analogously. See figures 3.4 and 3.5.

Remark. For $ch \in \mathscr{C}$, the map $ch : M^{\mathscr{H}} \to M^{\{h,h'\}}$ is the same as the product of projection maps $\pi_h \times \pi_{h'} : M^{\mathscr{H}} \to M \times M$.

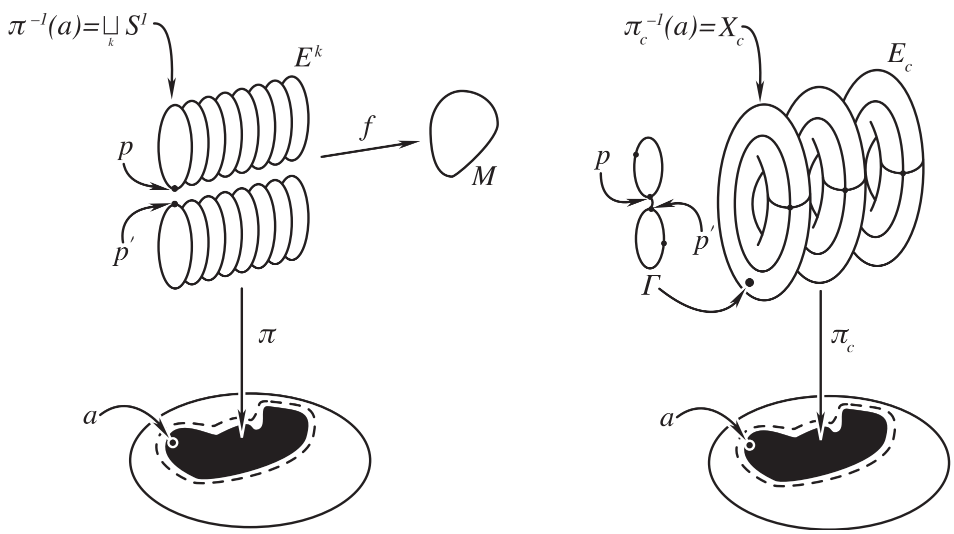


Figure 3.4: $\bigsqcup_k S^1$-bundle and associated X_c-bundle

The Thom class representative $\mu_c \in \Omega^{|\chi|d}(M^{\mathscr{H}})$, which has support $supp(\mu)$ in N_μ, pulls back to $ev^*(\mu_c) \in \Omega^{|\chi|d}(E_c)$ where χ is the Euler characteristic of all diagrams $\Gamma \in X_c$. Hence, $\pi_c^*(\alpha) \wedge ev^*(\mu_c)$ is a $\mathbb{Q}$-polynomial form in $\Omega^{m+|\chi|d}(E_c)$ whose support is contained in the open set $ev^{-1}(N_\mu)$.

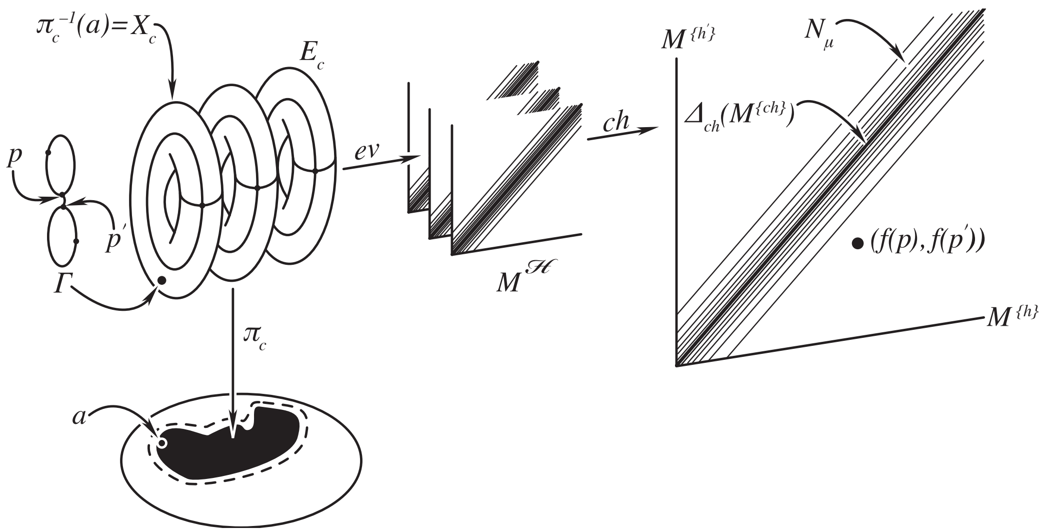


Figure 3.5: Evaluation map

3.5.3 The $\bigsqcup_\ell S^1$-bundle

There is a universal string-diagram bundle over E_c. That is, the fiber over the point $\Gamma \in E_c$ is the diagram Γ itself. We will be particularly interested in the restriction of this string-diagram bundle over E_c to a string-diagram bundle over $ev^{-1}(N_\mu)$. Denote its total space by $E_{\overline{SD}}$. See figure 3.6.

Recall that the half-edges adjacent to each vertex of Γ are cyclically ordered such that Γ has k input circles and ℓ output circles. Hence, the string-diagram bundle over $ev^{-1}(N_\mu)$ gives rise to a $\bigsqcup_\ell S^1$-bundle over $ev^{-1}(N_\mu)$; the fiber over Γ is the disjoint union of the ℓ output cirlces of Γ. Denote the total space of the $\bigsqcup_\ell S^1$-bundle over $ev^{-1}(N_\mu)$ by E^ℓ. See figure 3.6.

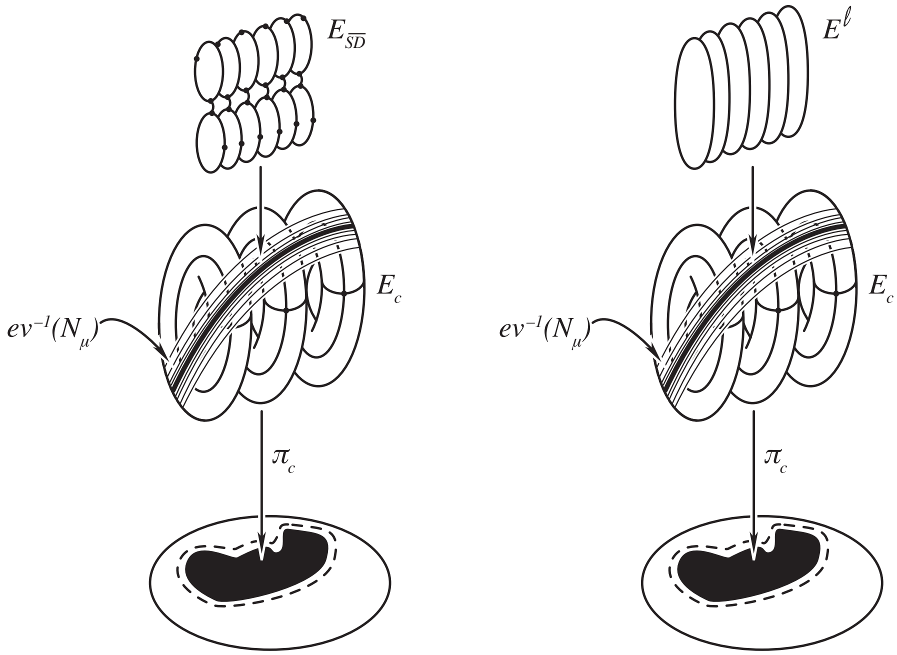


Figure 3.6: Pullback of N_μ, universal string-diagram bundle and $\bigsqcup_\ell S^1$-bundle

3.5.4 The map to M

If $\Gamma \in ev^{-1}(N_\mu)$, then for all chords ch in Γ, f maps the endpoints p and p' of ch to points $f(p)$ and $f(p')$ in M which lie in a ball of radius ε. Therefore, $f(p)$ and $f(p')$ can be joined by a unique geodesic segment of length less than 2ε in M. This determines a map $f_{\overline{SD}} : E_{\overline{SD}} \to M$ as follows. For a string diagram Γ, the fiber over the point $\Gamma \in \pi_c^{-1}(a)$, $f_{\overline{SD}}$ maps its k input circles via the original map $f|_{\pi^{-1}(a)}$ and it maps the chord joining p and p' to the unique geodesic segment joining $f(p)$ and $f(p')$. See figure 3.7.

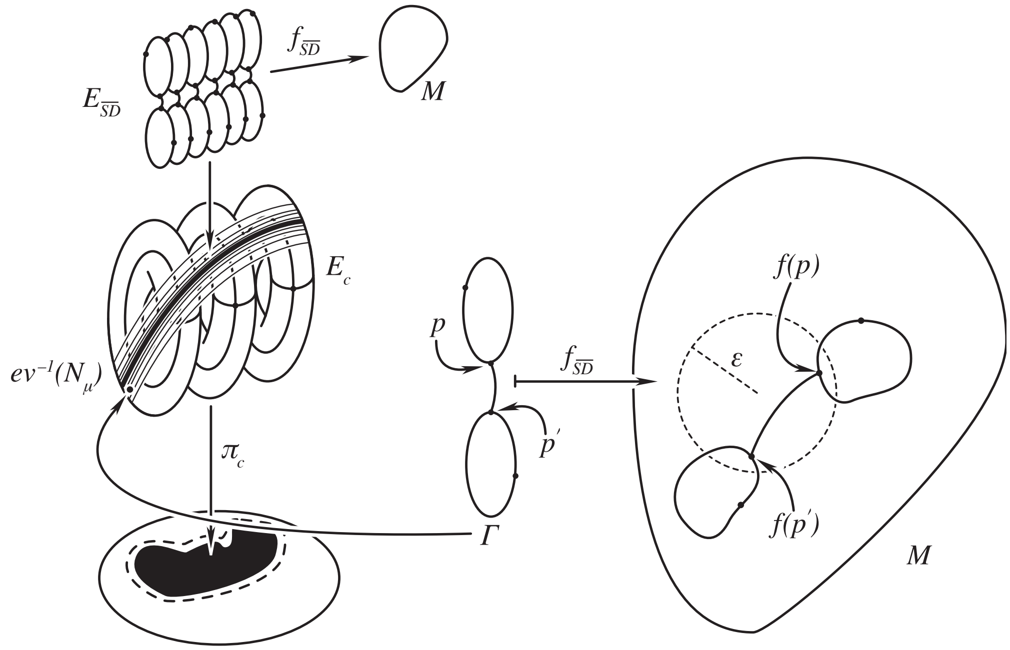


Figure 3.7: $f_{\overline{SD}} : E_{\overline{SD}} \to M$

For each $\Gamma \in ev^{-1}(N_\mu)$ the map $f_{\overline{SD}} : E_{\overline{SD}} \to M$ may be restricted either to the k input circles or ℓ output circles of Γ. For all $\Gamma \in \pi_c^{-1}(a)$, the restriction of $f_{\overline{SD}}$ to the inputs of the string diagram Γ (the fiber over the point $\Gamma \in \pi_c^{-1}(a)$ in $E_{\overline{SD}}$) agrees with f on $\pi^{-1}(a)$. Restriction of to output circles yields a new map $g : E^\ell \to M$. See figure 3.8.

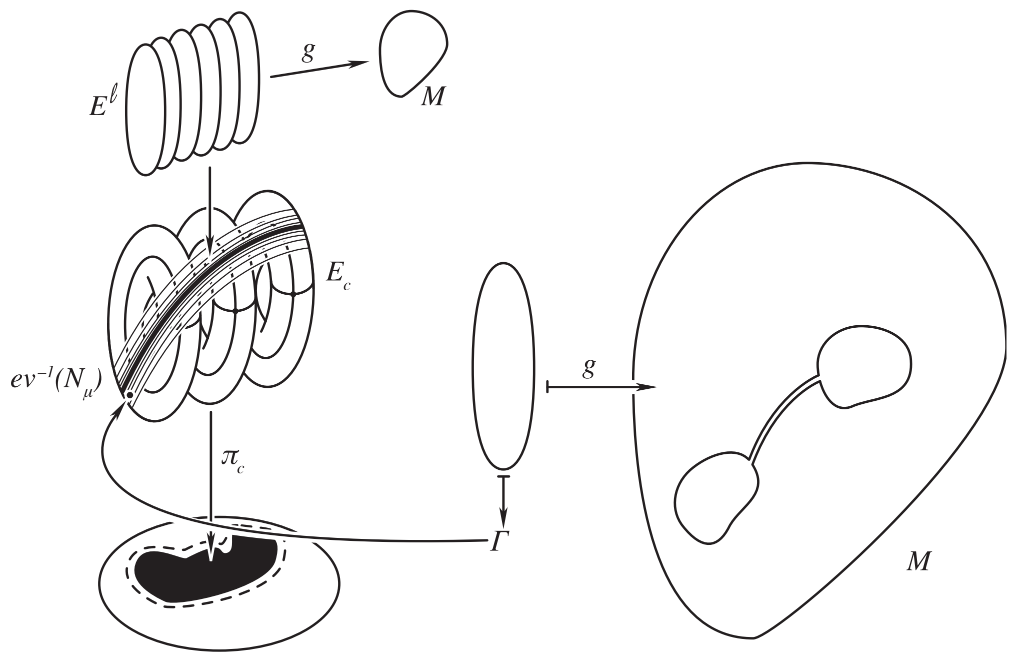


Figure 3.8: $g : E^\ell \to M$

3.5.5 The output of the string topology construction

Definition 3.17. Let

$$ST_\mu(c)(B,\alpha) = \left[\sum_{i\in I}(B_i, \alpha_i, E_i^\ell, g_i)\right] \in P_{m-n+|c|-|\chi|d}(\ell LM)$$

where

1. B_i are the top-dimensional cells $w \times e$ of $\pi_c^{-1}(W_\alpha)$;
2. $\alpha_i = (\pi_c^*(\alpha) \wedge ev^*(\mu_c))|_{B_i}$;

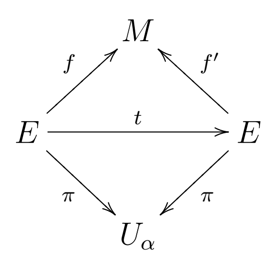
3. $E_i = E^\ell|_{B_i}$;

4. $g_i = g|_{B_i}$.

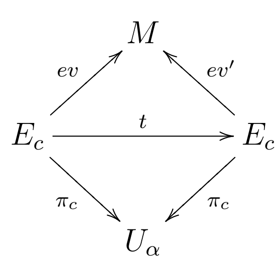
Lemma 3.4. *The output of the string topology construction is independent of the* k *sections* $U_\alpha \to E^k$ *chosen to define the evaluation map* ev.

The intuition is that since c is minimally saturated, X_c is closed under the T^k-action and the *family* of string diagrams with inputs identified with $\pi^{-1}(a)$ does not depend on the choices.

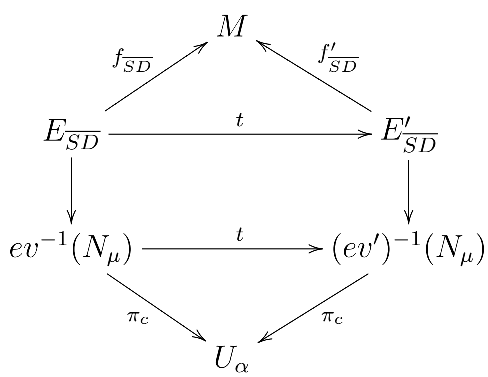
Proof. Let $t \in T^k$ correspond to the element that rotates the first choice to the second choice. Then



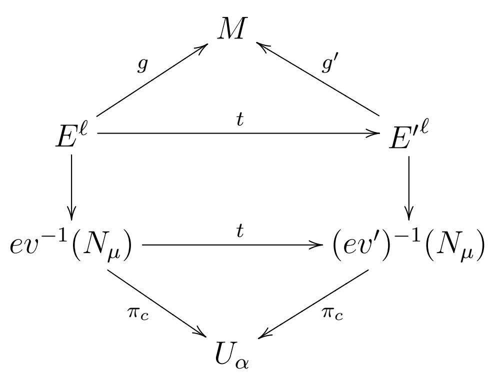
determines the commutative diagram



That is, $ev'(\Gamma) = ev(t^{-1}\Gamma)$. Then $E_{\overline{SD}}$ and $E'_{\overline{SD}}$ are universal string diagram bundles over $ev^{-1}(N_\mu)$ and $(ev')^{-1}(N_\mu)$ respectively, with $f_{\overline{SD}}$ and $f'_{\overline{SD}}$ the maps to M, respectively, then



commutes. We have $t^*(\pi^*(\alpha) \wedge (ev')^*(\mu_c)) = \pi^*(\alpha) \wedge ev'^*(\mu_c)$ and $t^*(E'_{\overline{SD}}) = E_{\overline{SD}}$. Likewise,

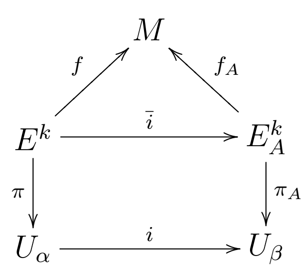


commutes and $t^*(E'^\ell) = E^\ell$. The action of T^k on $E_{\overline{SD}}$ given by rotating input marked points is trivial when restricted to outputs. As a result, $t : E^\ell \to E'^\ell$ is T^ℓ-equivariant.

Thus, the equivalence class of $ST_\mu(c)(B, \alpha)$ does not depend on the choice of marked points on input circles. □

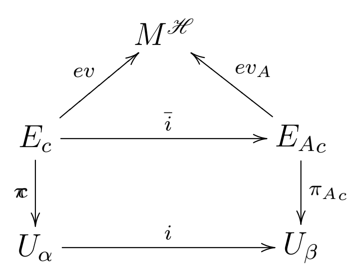
Lemma 3.5. *The map $ST_\mu(c)$ is well-defined on equivalence classes $(B, \alpha) \in P_*(kLM)$.*

Proof. If (A, β, E^k_A, f_A) is a different representative of (B, α) then

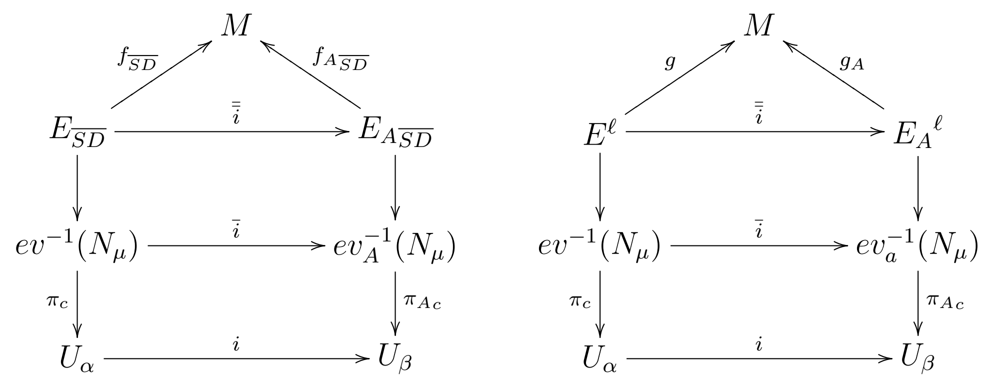


commutes.

Let the k sections of $E^k \to U_\alpha$ and $E^k_A \to U_\beta$ be chosen so all diagrams commute. Then



and



also commute. In particular, in the right-hand diagram $\bar{i}^*(E_A^\ell) = E^\ell$ and $\bar{i}^*(\pi_{A_c}^*(\beta) \wedge ev_A^*(\mu_c)) = \pi_c^*(\alpha) \wedge ev^*(\mu_c)$.

Therefore, $\sum(B_i, \alpha_i, E_i^\ell, g_i)$ is equivalent to $\sum(B_i', \alpha_i', {E'}_i^\ell, g_i')$ and ST_μ is well defined on generators (B, α). $\square$

Definition 3.18. The map $ST_\mu(c) : P_*(kLM) \to P_{*+|c|-|\chi|d}(\ell LM)$, where $\chi = 2 - 2g - k = \ell$ is the Euler characteristic of all $\Gamma \in X_c$, is defined for generators (B, α) as above and extended linearly to $P_*(kLM)$.

The map $ST_\mu : C_*^{T^k}(\overline{SD}(g, k, \ell)) \to Hom(P_*(kLM), P_*(\ell LM))$ is defined on generators c and extended linearly to $C_*^{T^k}(\overline{SD}(g, k, \ell))$.

Lemma 3.6. *The map*

$$ST_\mu : C_*^{T^k}(\overline{SD}(g, k, \ell)) \to Hom(P_*(kLM), P_*(\ell LM))$$

is a chain map of degree $(2-2g-k-\ell)d = |\chi|d$.

Proof. Let c be a generator of $C_*^{T^k}(\overline{SD}(g,k,\ell))$ and (B,α) be a generator of $P_*(kLM)$. In what follows, all $\mathbb{Q}$-polynomial forms are restricted appropriately.

Recall the top-dimensional cells B_i of $\pi_c^{-1}(W_\alpha)$ are of the form $e \times w$ where w is an oriented n-cell of the base W_α and e is an oriented $|c|$-cell of the fiber X_c. Recall also that if f is a face of w, then $\pi_c^{-1}(f)$ may be decomposed further.

We use the following formula for the boundary of any oriented cell $e \times w$:

$$\partial(e \times w) = \partial e \times w + (-1)^\delta e \times \partial w$$

where

$$\delta = \begin{cases} |e| & \text{if } |c| \text{ is odd} \\ |e|+1 & \text{if } |c| \text{ is even} \end{cases}$$

In both cases, $\partial^2 = 0$.

If A is a codimension 1 cell of $\pi_c^{-1}(W_\alpha)$, then it is either of the form $e \times x$ where x is a codimension 1 face of a top-dimensional cell w of W_α and e is a top-dimensional cell of X_c or it is of the form $f' \times w$ were f' is a codimension 1 face of the top-dimensional cell e of X_c, after subdivision and w is a top-dimensional cell of W_α.

Codimension 1 cells of the space X_c are also faces of either one or two

$|c|$-dimensional cells: one if two chord endpoints come together and two if a chord endpoint and distinguished vertex come together. Let $[\pi_c^{-1}(W_\alpha)] = \sum e \times w$, $[\pi_{\partial c}^{-1}(W_\alpha)] = \sum f' \times w$ and $[\pi_c^{-1}(\partial W_\alpha)] = \sum e \times x$. Then $\partial[\pi_c^{-1}(W_\alpha)] = [\pi_{\partial c}^{-1}(W_\alpha)] - [\pi_c^{-1}(\partial W_\alpha)]$. We reorient all cells if necessary so that

$$\partial[\pi_c^{-1}(W_\alpha)] = (-1)^{n+|c|-(m+|\chi|d)}([\pi_{\partial c}^{-1}(W_\alpha)] - [\pi_c^{-1}(\partial W_\alpha)])$$

where $\chi = 2 - 2g - k - \ell$ and $d = dim(M)$.

We use this to see that $ST_\mu(\partial(c))(B, \alpha) = D_{Hom}(ST_\mu(c))(B, \alpha)$.

With the above notation,

$$ST_\mu(\partial(c))(B, \alpha) = (\pi_{\partial(c)}^{-1}(W_\alpha), \pi_{\partial(c)}^*(\alpha) \wedge ev^*(\mu_c))$$

and

$$\begin{aligned}
D_{Hom}(ST_\mu(c))(B,\alpha) &= ST_\mu(c)(D(B,\alpha)) + (-1)^{|c|-|\chi|d+1} D(ST_\mu(c)(B,\alpha)) \\
&= (\pi_c^{-1}(\partial W_\alpha), \pi_c^*(\alpha) \wedge ev^*(\mu_c)) + (-1)^{n-m}(\pi_c^{-1}(W_\alpha), \pi_c^*(d\alpha) \wedge ev^*(\mu_c)) \\
&\quad + (-1)^{|c|-|\chi|d+1}\{(\partial(\pi_c^{-1}(W_\alpha), \pi_c^*(\alpha) \wedge ev^*(\mu_c)) \\
&\quad + (-1)^{n+|c|-(m+|\chi|d)}(\pi_c^{-1}(W_\alpha), d(\pi_c^*(\alpha) \wedge ev^*(\mu_c)))\} \\
&= (\pi_c^{-1}(\partial W_\alpha), \pi_c^*(\alpha) \wedge ev^*(\mu_c)) + (-1)^{n-m}(\pi_c^{-1}(W_\alpha), \pi_c^*(d\alpha) \wedge ev^*(\mu_c)) \\
&\quad + (-1)^{|c|-|\chi|d+1}(\partial(\pi_c^{-1}(W_\alpha)), \pi_c^*(\alpha) \wedge ev^*(\mu_c)) \\
&\quad + (-1)^{|c|-|\chi|d+1+n+|c|-n-|\chi|d}(\pi_c^{-1}(W_\alpha), \pi_c^*(d\alpha) \wedge ev^*(\mu_c)) \\
&= (\pi_c^{-1}(\partial W_\alpha), \pi_c^*(\alpha) \wedge ev^*(\mu_c)) + (-1)^{|c|-|\chi|d+1}(\partial(\pi_c^{-1}(W_\alpha)), \pi_c^*(\alpha) \wedge ev^*(\mu_c)) \\
&= (\pi_c^{-1}(\partial W_\alpha), \pi_c^*(\alpha) \wedge ev^*(\mu_c)) \\
&\quad + (-1)^{|c|-|\chi|d+1}\{(-1)^{|c|-|\chi|d+1}((\pi_{\partial(c)}^{-1}(W_\alpha), \pi_{\partial(c)}^*(\alpha) \wedge ev^*(\mu_c)) \\
&\qquad\qquad - (\partial W_\alpha), \pi_c^*(\alpha) \wedge ev^*(\mu_c))\} \\
&= (\pi_{\partial(c)}^{-1}(W_\alpha), \pi_{\partial(c)}^*(\alpha) \wedge ev^*(\mu_c))
\end{aligned}$$

Therefore, $ST_\mu \circ \partial = D_{Hom} \circ ST_\mu$, so ST_μ is a chain map.

The map of chains $ST_\mu(c)$ has degree $|c| - (2g-2+k+\ell)d$ so ST_μ has degree $(2-2g-k-\ell)d = |\chi|d$.

$\square$

3.5.6 Composition

Remark. If c is a generator of $C_*^{T^k}(\overline{SD}(g,k,\ell))$ it determines a map

$$ST_\mu(c) \in Hom(P_*(kLM), P_*(\ell LM))$$

but it also determines a map

$$ST_\mu(c) \in Hom(P_*((k+m)LM), P_*((\ell+m)LM))$$

were $m > 0$. If (B, α) is a generator of $P_*((k+m')LM)$, the fiber over a point $a \in supp(\alpha)$ is $k+m$ disjoint circles. For c to act, k of the $k+m$ circles are chosen (consistently over $supp(\alpha)$) and each is identified with one of the k input circles of string diagrams $\Gamma \in X_c$. The generator c acts as the identity on the remaining m circles. That is, if $\Gamma \in ev^{-1}(N_\mu)$ is a string diagram of type (g,k,ℓ) and $\pi_c(\Gamma) = a \in W_\alpha$, then $f_{\overline{SD}}$ maps Γ to M and g is the restriction of $f_{\overline{SD}}$ to the output circles of Γ. The map g may be extended to the remaining m circles via the initial map f; if p is a point on one of the remaining m circles, then $g(p) = f(p)$.

This allows for composition of string topology operations $ST_\mu(c)$ which is similar to that in a PROP or a properad.

Definition 3.19. Let $c \in C_*^{T^k}(\overline{SD}(g,k,\ell))$ and $c' \in C_*^{T^{k'}}(\overline{SD}(g',k',\ell'))$ with corresponding string topology operations $ST_\mu(c)$ and $ST_\mu(c')$ respectively. Select n outputs of c and n inputs of c' and let ε be a pairing of outputs with inputs. Then composition $ST_\mu(c') *_\varepsilon ST_\mu(c)$ is a map in

$$Hom(P_*((k + k' - n)LM), P_*((\ell + \ell' - n)LM))$$

where $ST_\mu(c)$ acts as the identity on $k' - n$ of the circles and $ST_\mu(c')$ acts as the identity on $k' - n$ of the circles.

We will return to composition later in the proof of propositon 4.3. In particular, there we will be interested in configurations of chords that arise from composition. For the time being, consider a single $\Gamma \in ev^{-1}(N_\mu)$, a single $\Gamma' \in (ev')^{-1}(N'_\mu)$ and their images in M under $f_{\overline{SD}}$ and $f'_{\overline{SD}}$ respectively. An output circle of Γ which is paired with an input circle of Γ' is made up of images of input edges and chords. Effectively, the string topology construction identifies the input circle of Γ' with this output circle in all possible ways (an S^1 of ways in the figure) to produce a *family* of output circles of the composition. The endpoint of a chord of Γ' may land on the image of an input edge of Γ or it may land on the image of a chord. In figure 3.9, one member of the S^1 family is shown, where the image of the endpoint of a chord of Γ' lies on the image of a chord of Γ. (Parallel curves in the figure should in fact be common curves in M.) In general, there is a T^n family.

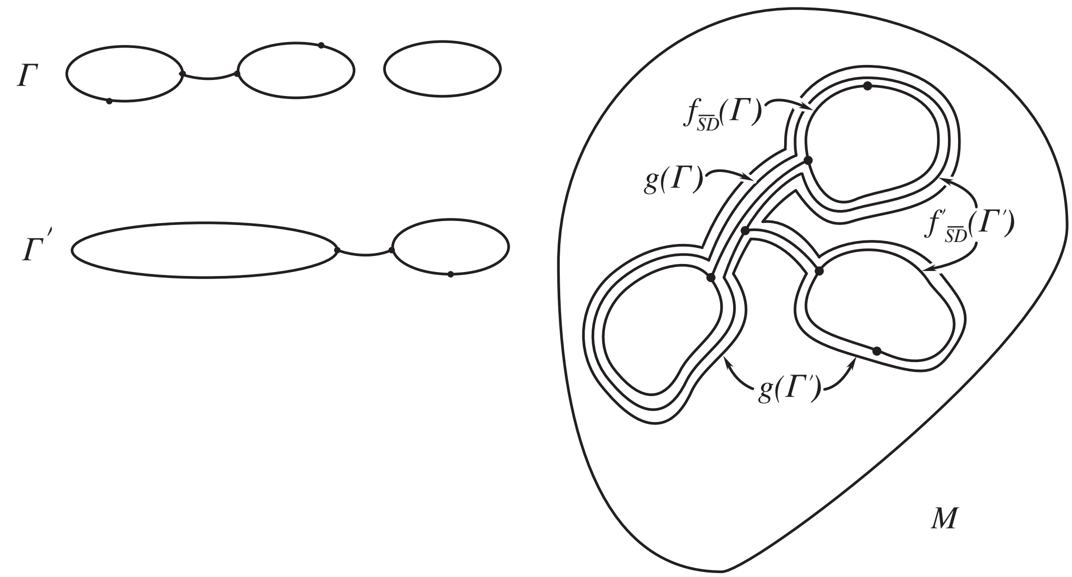


Figure 3.9: Composition of string topology operations

3.5.7 Choice of Thom class representative

The following general lemma will be used in several proofs below.

Lemma 3.7. *Let X be a polyhedrom and η and η' be two closed $\mathbb{Q}$-polynomial forms on X such that $\eta' - \eta = d\zeta$. Let t be the I parameter for the polyhedron $X \times I$ and let $\rho : X \times I \to X$ be the projection map $(p, t) \mapsto p$. Then*

$$\tilde{\eta} = (1-t)\rho^*(\eta) + t\rho^*(\eta') + dt \wedge \rho^*(\zeta)$$

is a closed form which restricts to η on $X \times \{0\}$ and to η' on $X \times \{1\}$.

Proof. The form $\tilde{\eta}$ is closed:

$$\begin{aligned} d(\tilde{\eta}) &= d((1-t)\rho^*(\eta)) + d(t\rho^*(\eta')) + d(dt \wedge \rho^*(\zeta)) \\ &= -dt \wedge \rho^*(\eta) + dt \wedge \rho^*(\eta') - dt \wedge \rho^*(d\zeta) \\ &= dt \wedge (\eta' - \eta - d\zeta) \\ &= 0 \end{aligned}$$

The form $\tilde{\eta}$ restricts to η on $X \times \{0\}$. Let $i_0 : X \to X \times I$ be defined by $i(p) = (p, 0)$. Then $i_0^*(dt) = 0$ and $i_0^*(\tilde{\eta}) = \eta + 0 + 0 = \eta$.

Similarly, $\tilde{\eta}$ restricts to η' on $X \times \{1\}$. Let $i_1 : X \to X \times I$ be defined by $i(p) = (p, 1)$. Then $i_1^*(dt) = 0$ and $i_1^*(\tilde{\eta}) = 0 + \eta' + 0 = \eta'$.

□

In fact, there is a generalization of this lemma which will be used as well.

Lemma 3.8. *Let $X \subset Y$ be a sub-polyhedron and polyhedron, respectively and let η be a closed $\mathbb{Q}$-polynomial form on X. Then there exists a closed $\mathbb{Q}$-polynomial form $\tilde{\eta}$ on Y that restricts to η on X if and only if the cohomology class $[\eta]$ extends to a cohomology class in $H_*(Y)$.*

Remark. Given a form ζ on X, there exist extensions of ζ to Y, but if ζ is closed, a given extension may not be. In what follows, we will require the extension to be closed.

Proof. Assume $[\eta]$ extends to a cohomology class in $H_*(Y)$ and let $\bar{\eta}$ be any representative of the extension. Then $[\bar{\eta}]|_X = [\eta]$ so $\bar{\eta}|_X = \eta + d\zeta$. Let $\tilde{\zeta}$ be any extension of ζ to Y. Then $(\tilde{\eta} - d\tilde{\zeta})|_X = \eta + d\zeta - d\zeta$. □

Proposition 3.1. *Let μ' be a different choice of representative of the Thom class of the diagonal with $supp(\mu') \subset N_\varepsilon$. The chain maps ST_μ and ST'_μ are chain homotopic.*

Proof. We define a chain homotopy

$$H : C_*^{T^k}(\overline{SD}(g,k,\ell)) \to Hom(P_*(kLM), P_*(\ell LM))$$

first on generators c of $C_*^{T^k}(\overline{SD}(g,k,\ell))$ for generators (B,α) of $P_*(kLM)$ by choosing a representative quadruple (B,α,E^k,f) and constructing each component of a quadruple representing $H(c)(B,\alpha)$.

The base space.

For both $ST_\mu(c)(B,\alpha)$ and $ST_{\mu'}(c)$, the base space is $\pi_c^{-1}(W_\alpha)$. For $H(c)(B,\alpha)$, the base space is $I \times \pi_c^{-1}(W_\alpha)$.

The $\mathbb{Q}$-differential form.

Let $\mu'_c - \mu_c = d\nu$ on $M^{\mathscr{H}}$. On $\pi_c^{-1}(W_\alpha)$, the forms $ev^*(\mu_c)$ and $ev^*(\mu'_c)$ are closed and satisfy $ev^*(\mu'_c) - ev^*(\mu_c) = d(ev^*(\nu))$. Applying Lemma 3.7 to $ev^*(\mu_c)$ and $ev^*(\mu'_c)$, $\tilde{\mu}$ is a closed form on $\pi_c^{-1}(W_\alpha) \times I$ which restricts to $ev^*(\mu_c)$ on $\{0\} \times \pi_c^{-1}(W_\alpha)$ and to $ev^*(\mu'_c)$ on $\{1\} \times \pi_c^{-1}(W_\alpha)$.

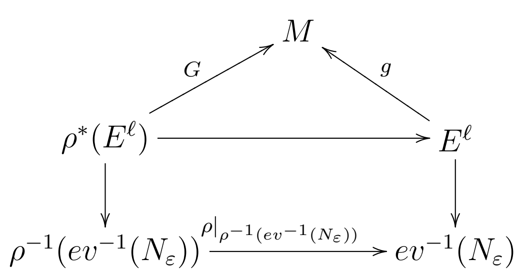
Therefore, $\rho^*(\pi_c^*(\alpha)) \wedge \tilde{\mu}$ restricts to $\pi_c^*(\alpha) \wedge ev^*(\mu_c)$ on $\{0\} \times \pi_c^{-1}(W_\alpha)$ and to $\pi_c^*(\alpha) \wedge ev^*(\mu_c')$ on $\{1\} \times \pi_c^{-1}(W_\alpha)$.

The $\bigsqcup_\ell S^1$-bundle.

The $\bigsqcup_\ell S^1$-bundle over $\rho^{-1}(ev^{-1}(N_\mu))$ is the pull-back of the $\bigsqcup_\ell S^1$-bundle over $ev^{-1}(N_\mu)$ by ρ^*.

The map to M.

The support $supp(\rho^*(\pi_c^*(\alpha)) \wedge \tilde{\mu})$ is contained in $\rho^{-1}(ev^{-1}(N_\mu))$. Let $(p,t) \in \pi_c^{-1}(W_\alpha) \times I$. Then $G : \rho^*(E^\ell) \to M$ is defined by $G(p,t) = g(p)$.



We set $H(c)(B,\alpha) = (\pi_c^{-1}(W_\alpha) \times I, \rho^*(\pi_c^*(\alpha)) \wedge \tilde{\mu})$.

Recall the cell decomposition above of $\pi_c^{-1}(W_\alpha)$ given by the triangulation of W_α and trivialization of $\pi_c^{-1}(W_\alpha)$ such that each top-dimensional cell w of W_α is contained in exactly one open set U_w. The oriented top-dimensional cells B_i of $\pi_c^{-1}(W_\alpha)$ are given by products of oriented top-dimensional cells $e \times w$. Here, we reorient all the cells of X_c if necessary so that $c = \sum(-1)^{|\chi|d+1}e$ and

$\pi_c^{-1}(w) = \sum(-1)^{|\chi|d+1} e \times w.$

Again, codimension 1 cells of $I \times \pi_c^{-1}(W_\alpha)$ are faces of either one or two top-dimensional cells and with this orientation,

$$\partial[I \times \pi_c^{-1}(W_\alpha)] = [\partial I \times \pi_c^{-1}(W_\alpha)] - [I \times \pi_{\partial c}^{-1}(W_\alpha)] + (-1)^{|\chi|d+1}[I \times \pi_c^{-1}(\partial W_\alpha)].$$

We reorient all cells if necessary so that

$$\partial[I \times \pi_c^{-1}(W_\alpha)] = (-1)^{|c|+1}([\partial I \times \pi_c^{-1}(W_\alpha)] - [I \times \pi_{\partial c}^{-1}(W_\alpha)] + (-1)^{|\chi|d+1}[I \times \pi_c^{-1}(\partial W_\alpha)]).$$

The degree of $H(c) \in Hom(P_*(kLM), P_*(\ell LM))$ is $|c|+1-|\chi|d$ and the degree of $H \in Hom(C_*^{T^k}(\overline{SD}(g,k,\ell)), Hom(P_*(kLM), P_*(\ell LM)))$ is $1 - |\chi|d$.

Below, all $\mathbb{Q}$-polynomial forms are restricted appropriately.

$$\begin{aligned} &ST_{\mu'}(c)(B,\alpha) - ST_\mu(c)(B,\alpha) \\ &\qquad = (\pi_c^{-1}(W_\alpha), \pi_c^*(\alpha) \wedge ev^*(\mu'_c)) - (\pi_c^{-1}(W_\alpha), \pi_c^*(\alpha) \wedge ev^*(\mu_c)) \end{aligned}$$

and

$$
\begin{aligned}
&(H\circ\partial+(-1)^{-|\chi|d}D_{Hom}\circ H)(c)(B,\alpha)\\
&\quad=H(\partial(c))(B,\alpha)+(-1)^{-|\chi|d}(H(c)(D(B,\alpha))+(-1)^{|c|-|\chi|d}D(H(c)(B,\alpha))\\
&\quad=(I\times\pi^{-1}_{\partial(c)}(W_\alpha),\rho^*(\pi^*_{\partial(c)}(\alpha)\wedge\tilde{\mu})+(-1)^{-|\chi|d}\{H(c)(\partial B,\alpha)\\
&\qquad+(-1)^{n-m}H(c)(B,d\alpha)+(-1)^{|c|-|\chi|d}D(I\times\pi_c^{-1}(W_\alpha),\rho^*(\pi_c^*(\alpha)\wedge\tilde{\mu})\}\\
&\quad=(I\times\pi^{-1}_{\partial(c)}(W_\alpha),\rho^*(\pi^*_{\partial(c)}(\alpha)\wedge\tilde{\mu})+(-1)^{-|\chi|d}\{(I\times\pi_c^{-1}(\partial W_\alpha),\rho^*(\pi_c^*(\alpha)\wedge\tilde{\mu})\\
&\qquad+(-1)^{n-m}(I\times\pi_c^{-1}(W_\alpha),\rho^*(\pi_c^*(d\alpha)\wedge\tilde{\mu}))\\
&\qquad+(-1)^{|c|-|\chi|d}[(\partial(I\times\pi_c^{-1}(W_\alpha)),\rho^*(\pi_c^*(\alpha)\wedge\tilde{\mu})\\
&\qquad+(-1)^{n+|c|+1-(m+|\chi|d)}(I\times\pi_c^{-1}(W_\alpha),\rho^*(\pi_c^*(d\alpha)\wedge\tilde{\mu})]\}\\
&\quad=(I\times\pi^{-1}_{\partial(c)}(W_\alpha),\rho^*(\pi^*_{\partial(c)}(\alpha)\wedge\tilde{\mu}))+(-1)^{|\chi|d}(I\times\pi_c^{-1}(\partial W_\alpha),\rho^*(\pi_c^*(\alpha)\wedge\tilde{\mu}))\\
&\qquad+(-1)^{n-m+|\chi|d}(I\times\pi_c^{-1}(W_\alpha),\rho^*(\pi_c^*(d\alpha)\wedge\tilde{\mu}))\\
&\qquad+(-1)^{|\chi|d+|c|+1-|\chi|d}(\partial(I\times\pi_c^{-1}(W_\alpha)),\rho^*(\pi_c^*(\alpha)\wedge\tilde{\mu}))\\
&\qquad+(-1)^{|\chi|d+|c|-|\chi|d+n+1-m-|\chi|d}(I\times\pi_c^{-1}(W_\alpha),\rho^*(\pi_c^*(d\alpha)\wedge\tilde{\mu}))\\
&\quad=(I\times\pi^{-1}_{\partial(c)}(W_\alpha),\rho^*(\pi^*_{\partial(c)}(\alpha)\wedge\tilde{\mu}))+(-1)^{|\chi|d}(I\times\pi_c^{-1}(\partial W_\alpha),\rho^*(\pi_c^*(\alpha)\wedge\tilde{\mu}))\\
&\qquad+(-1)^{|c|+1}(\partial(I\times\pi_c^{-1}(W_\alpha)),\rho^*(\pi_c^*(\alpha)\wedge\tilde{\mu}))\\
&\quad=(I\times\pi^{-1}_{\partial(c)}(W_\alpha),\rho^*(\pi^*_{\partial(c)}(\alpha)\wedge\tilde{\mu})+(-1)^{|\chi|d}(I\times\pi_c^{-1}(\partial W_\alpha),\rho^*(\pi_c^*(\alpha)\wedge\tilde{\mu})\\
&\qquad+(\partial I\times\pi_c^{-1}(W_\alpha),\rho^*(\pi_c^*(\alpha)\wedge\tilde{\mu})-(I\times\pi^{-1}_{\partial(c)}(W_\alpha),\rho^*(\pi_c^*(\alpha)\wedge\tilde{\mu})\\
&\qquad+(-1)^{|\chi|d+1}(I\times\pi_c^{-1}(\partial W_\alpha,\rho^*(\pi_c^*(\alpha)\wedge\tilde{\mu})\\
&\quad=(\{1\}\times\pi_c^{-1}(W_\alpha),\rho^*(\pi_c^*(\alpha)\wedge\tilde{\mu})-(\{0\}\times\pi_c^{-1}(W_\alpha),\rho^*(\pi_c^*(\alpha)\wedge\tilde{\mu})\\
&\quad=(\pi_c^{-1}(W_\alpha),\pi_c^*(\alpha)\wedge ev^*(\mu_c'))-(\pi_c^{-1}(W_\alpha),\pi_c^*(\alpha)\wedge ev^*(\mu_c))
\end{aligned}
$$

Therefore, for a generator c of $C_*^{T^k}(\overline{SD}(g,k,\ell))$,

$$ST_\mu(c) - ST_{\mu'}(c) = (H \circ \partial + (-1)^{-|\chi|d} D_{Hom} \circ H)(c).$$

Extending H linearly over $C_*^{T^k}(\overline{SD}(g,k,\ell))$,

$$ST_\mu - ST'_\mu = H \circ \partial + (-1)^{-|\chi|d} D_{Hom} \circ H,$$

so ST_μ and $ST_{\mu'}$ are chain homotopic.

□

$H(c)$ is well defined on equivalence classes (B, α) because $ST_\mu(c)$ and $ST_{\mu'}(c)$ are well defined.

Example (The string bracket)**.** Let c be a generator of $C_2^{T^2}(\overline{SD}(0,2,1))$. The string topology construction determines a two-to-one map called the string bracket

$$ST_\mu(c) : P_*(2LM) \to P_{*+2-d}(LM).$$

Example (The string cobracket)**.** Let c be a generator of $C_2^{T^2}(\overline{SD}(0,1,2))$. The string topology construction determines a one-to-two map called the string cobracket

$$ST_\mu(c) : P_*(LM) \to P_{*+2-d}(2LM).$$

Chapter 4

Extension of construction & solution to equation with anomaly

4.1 The space $\overline{LD}(g,k,\ell)/\!\!/ \sim$

Recall the space $\overline{LD}(g,k,\ell)$ of string diagrams with levels and slide equivalence $\sim$ for string diagrams with levels. In this section we build a space $\overline{LD}(g,k,\ell)/\!\!/ \sim$ and extend the string topology construction over it. The inspiration for $\overline{LD}(g,k,\ell)/\!\!/ \sim$ is $\overline{LD}(g,k,\ell)/\sim$. Modifications must be made because the string topology construction is not well-defined on slide-equivalence classes. However, it is well-defined up to a contractible set of homotopies.

4.1.1 The slide complex of a slide-equivalence class

Let e be a cell of $\overline{LD}(g, k, \ell)$. Recall that a cell e is labeled by a combinatorial type of string diagram with levels. Its parameter space X_e is a product of k simplices and a cube, given by the placement of vertices on input circles and spacing parameters.

If $\Gamma_0 \in e$ and $\Gamma_0' \in e'$ are slide equivalent, then they have different combinatorial types but the placement of their vertices on input circles and their spacing parameters are the same. Therefore, if Γ is any other point in e and $\Gamma' \in e'$ has the same such parameters, then Γ and Γ' are also slide equivalent.

Definition 4.1. Two cells e and e' are called slide equivalent if for all $\Gamma \in e$ there is a unique $\Gamma' \in e'$ such that Γ and Γ' are slide equivalent.

Here we build a cell complex $P_{\mathfrak{E}}$ for each slide-equivalence class $\mathfrak{E}$ of cells e of $\overline{LD}(g, k, \ell)$. Later, we describe how to adjoin $P_{\mathfrak{E}}$ to $\overline{LD}(g, k, \ell)$. There is one vertex v_e of $P_{\mathfrak{E}}$ for each representative e of $\mathfrak{E}$.

Slide cells

The main picture here is that if Γ and Γ' are two string diagrams with levels that differ by a single chord slide, then there is an interval of graphs parametrizing the slide. Let Γ and Γ' differ by a slide of an endpoint of the chord ch over the chord $\overline{ch}$ which has length L. The chord $\overline{ch}$ itself parametrizes the chord slide. We identify $\overline{ch}$ with the interval $[0, L]$ such that $\Gamma = \Gamma_0$ and $\Gamma' = \Gamma_L$. For $t \in [0, L]$ we construct a graph Γ_t whose half-edges at each vertex are cyclically ordered (a cyclically-ordered graph) as follows. By abuse of notation, we also refer to the sliding endpoint of ch as t.

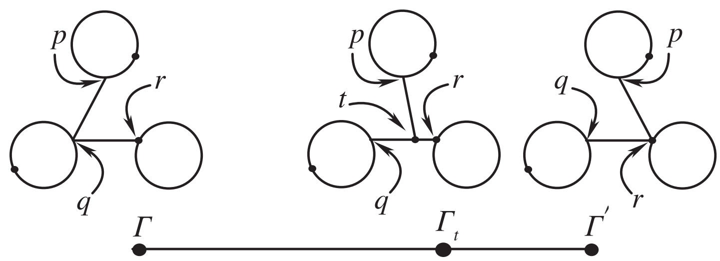


Figure 4.1: Interval parametrizing a chord slide

Except for t, Γ_t is identical to Γ_0 and Γ_L. In Γ_t, the endpoint t of ch coincides with the point t along the chord $\overline{ch} \sim [0, L]$. If $t \in (0, L)$, the endpoint t of ch is in the interior of the chord $\overline{ch}$. We see in figure 4.1 if $t = 0$, then $\Gamma_t = \Gamma$ and if $t = L$, $\Gamma_t = \Gamma'$. The cyclic order of half-edges at this new vertex is induced by that in Γ_0 and Γ_L.

In fact, several chords $ch_1, \ldots, ch_n$ which are adjacent at one endpoint $\overline{ch}$ may slide along $\overline{ch}$ simultaneously and there is a corresponding interval of cyclically-ordered graphs parametrizing the slide.

Remark. If $t \in (0, L)$, then all Γ_t have the same combinatorial type.

Definition 4.2. Let e and e' be two cells of $\overline{LD}(g, k, \ell)$ that differ either by a single chord slide over or by a sequence of chord slides where the chords sliding are adjacent in the cyclic order of half-edges at one one vertex. There is a 1-cell $s_{e,e'}$ of $P_{\mathfrak{E}}$ for the interval associated to e' and e' described above. The 1-cells $s_{e,e'}$ are called slide cells. Loops $s_{e,e}$ at vertices v_e are not included.

Given $\mathfrak{E}$, denote the complex of slide cells by $S_{\mathfrak{E}}$.

Remark. If e is a cell of $\overline{LD}(g,k,\ell)$ which is labeled by a combinatorial type where no two chords' endpoints coincide on an input circle, then e is the only cell in its slide-equivalence class and $S_{\mathfrak{E}}$ is a point. If e is a codimension 1 face of $\overline{LD}(g,k,\ell)$ where two chords' endpoints coincide on an input circle, then there are two representatives e and e' representing $\mathfrak{E}$ and $S_{\mathfrak{E}}$ is an interval.

Multi-slide cells

The notion of slide cells generalizes to many chords sliding simultaneously (though their endpoints need not coincide throughout the slide, as they do above). For example, two chords ch and ch' may slide over a chord $\overline{ch}$, ch and ch' may slide over different chords $\overline{ch}$ and $\overline{ch}'$ or the chord ch' may slide over the chord ch which is itself sliding over $\overline{ch}$. In this case, there is a cell parametrizing such a multiple chord slide and each cell is labeled by a combinatorial type. The 0-dimensional faces of such cells correspond to a collection of representatives e of a slide-equivalence class $\mathfrak{E}$ of cells of $\overline{LD}(g,k,\ell)$ and the 1-dimensional faces correspond to the intervals described above.

Definition 4.3. Let $e_1, \ldots, e_n$ be a collection of slide-equivalent cells of $\overline{LD}(g,k,\ell)$ corresponding to vertices $v_{e_1}, \ldots, v_{e_n}$ of an n cell $s_{e_1,\ldots,e_n}$ parametrizing a simultaneous chord slide as described above. Such n-cells are called multi-slide cells.

Faces of multi-slide cells are identified to form a cell complex as in figure 4.2. Denote the complex of slide cells and multi-slide cells by $S_{\mathfrak{E}} \cup SS_{\mathfrak{E}}$.

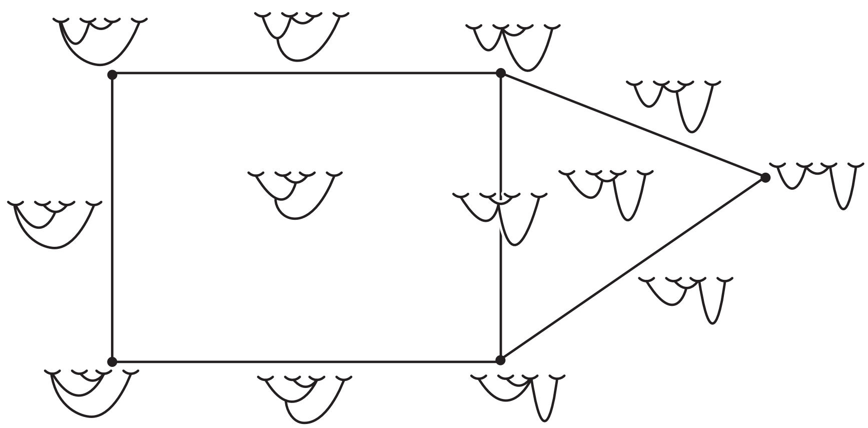


Figure 4.2: Slide cells and multi-slide cells

Triangles

Given a slide-equivalence class $\mathfrak{E}$ such that two adjacent chords are at the same level, then there is a triangle $T_{e,e',e''}$ in S with vertices e, e', e'' that is not the boundary of a 2-cell. This is the case when the chords slide over one another. See figure 2.8. We fill in $T_{e,e',e''}$ with a 2-simplex $\Delta_{e,e',e''}$ whose points are identified with cyclically-ordered graph constructed similarly to graphs in the slide graph but whose edge lengths differ. The barycenter is identified with a graph where the chords have been replaced with a tripod whose edge lengths are each half the length of the chords. Lines joining $T_{e,e',e''}$ with the barycenter of $\Delta_{e,e',e''}$ parametrize a deformation of the pair of chords to the tripod. A line from a vertex of $T_{e,e',e''}$ to the barycenter of describes the shrinking of each chord and the growth of a third edge such that the rates of shrinking and growing are correlated. A line from the interior of an edge of $T_{e,e',e''}$ to the barycenter of $\Delta_{e,e',e''}$ describes the sliding of one chord's endpoint to the interior of the other and its shrinking such that the rates of sliding and shrinking are correlated. Figure 4.3 shows the cyclically-ordered graph labeling the barycenter, along with the cyclically-ordered graphs labeling two other interior points.

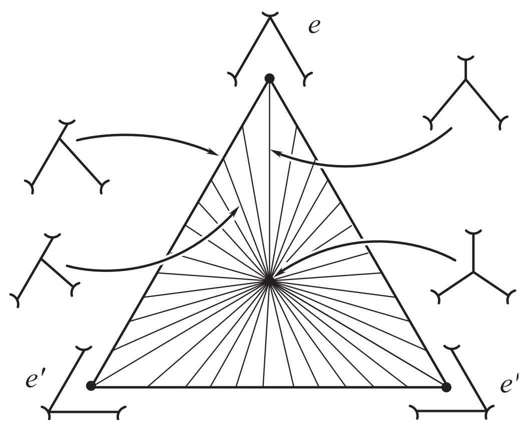


Figure 4.3: Triangle

There may be degenerate triangles. See figure 4.10 for two examples that occur when $(g, k, \ell) = (0, 2, 2)$. The triangle on the left is nondegenerate while the triangle on the right degenerates to a bigon because there are only two slide-equivalent representatives.

Denote the complex of slide cells, multi-slide cells and triangles by $S_{\mathfrak{c}} \cup SS_{\mathfrak{c}} \cup T_{\mathfrak{c}}$.

Combination cells

The tripods added in $T_{\mathfrak{c}}$ replace pairs of distinct chords at the same level whose endpoints coincide and that are adjacent at the common vertex. New graphs are created by chord slides, parametrized by cells in $S_{\mathfrak{c}}$. The triangle parametrizes a deformation of such a configuration to a tripod.

There are generalizations of these when more than two chords endpoints coincide and the chords are at the same level, or when the endpoint of a tripod and the endpoint of a chord at the same level coincide, or when endpoints of two tripods coincide and the tripods are at the same level, et cetera. Corollae with higher-valence central vertices play the role of tripods when chord endpoints coincide.

Chord and corolla endpoints may slide over chords and corollae at the same level or lower levels. Therefore, some combination cells are analogues of slide cells and multi-slide cells where in addition to chord endpoints, corolla endpoints may slide. There are generalizations of triangles in this setting. A generalized triangle generalizes such a deformation when rather than two chords, there are two corollae or one chord and one corolla.

We stop adding cells when all chords and corollae which are at the same level, share an endpoint and are adjacent have been deformed to one corollae. Note that the endpoints of one corolla need not be distinct. This is the analogue of one chord's endpoints coinciding on an input circle.

Denote the complex of side cells, multi-slide cells, triangles and combination cells by $S_{\mathfrak{E}} \cup SS_{\mathfrak{E}} \cup T_{\mathfrak{E}} \cup SST_{\mathfrak{E}}$.

Definition 4.4. Given a slide-equivalence class $\mathfrak{E}$ of cells of $\overline{LD}(g, k, \ell)$, the slide complex $P_{\mathfrak{E}}$ of $\mathfrak{E}$ is $SS_{\mathfrak{E}} \cup SS_{\mathfrak{E}} \cup T_{\mathfrak{E}} \cup SST_{\mathfrak{E}}$.

Remark. Given $\mathfrak{E}$, if for all representatives e of $\mathfrak{E}$ the subgraph consisting just of chords has Euler characteristic $\chi > 0$, then $P_{\mathfrak{E}}$ is contractible.

Lemma 4.1. *Given $\mathfrak{E}$, a codimension 1 cell $p_{\mathfrak{E}}$ of $P_{\mathfrak{E}}$ is the face of exactly zero, one or two top-dimensional cells.*

Proof. In what follows, "chord" is used to refer to a chord, corolla or cyclically-ordered graph representing a deformation.

A codimension 1 face $p_{\mathfrak{E}}$ of a top-dimensional cell either comes from a corolla breaking into two pieces, where one is sliding over the other (for example, a 1-dimensional face of a triangle in $T_{\mathfrak{E}}$) or it comes from two chord endpoints coinciding.

If a corolla breaks, then $p_{\mathfrak{E}}$ is not the face of any other top-dimensional cell of

$P_{\mathfrak{C}}$.

If two chord endpoints coincide, either two sliding endpoints come together in the interior of the chord they are sliding across or one chord endpoint that is sliding reaches the end of the chord it is sliding across. If the endpoints coinciding belong to the same chord, then $p_{\mathfrak{C}}$ is not the face of any other top-dimensional cell. Likewise, if the chord endpoints coincide on an input circle, then $p_{\mathfrak{C}}$ is not the face of any other top-dimensional cell. However, if two chord endpoints coincide in the interior of another chord, then they are necessarily at different levels. Otherwise, $p_{\mathfrak{C}}$ would have higher codimension. Therefore, the chord at the higher level may continue sliding and $p_{\mathfrak{C}}$ is the face of only these two top-dimensional cells of $P_{\mathfrak{C}}$.

A codimension 1 cell $p_{\mathfrak{C}}$ need not be the face of any top-dimensional cell of $P_{\mathfrak{C}}$. In this case, there is a vertex v_e where a representative has two chords at the same level that are not adjacent, though there is also a representative $v_{e'}$ where they are adjacent. Intuitively, if the two chords were adjacent, then they could slide over one another and deform to a tripod or generalized tripod. By finding themselves not adjacent, at either end of a chord at a lower level, for example, they are missing out on a potential tripod.

□

Definition 4.5. Let $Q_{\mathfrak{C}} \subset P_{\mathfrak{C}}$ be the subcomplex where cells that are not faces of top-dimensional cells are deleted.

Corollary 4.1. *$Q_{\mathfrak{E}}$ is a pseudomanifold with boundary.*

Lemma 4.2. *Let e be a cell of $\overline{LD}(g,k,\ell)$ with slide-equivalence class $\mathfrak{E}$ and let f be a face of e with slide-equivalence class $\mathfrak{F}$. Then there is a canonical map $\iota : P_{\mathfrak{E}} \to P_{\mathfrak{F}}$.*

Proof. It is enough to analyze the case where f is a codimension 1 face of e. Recall that a codimension 1 face of a cell corresponds to one of the following four cases:

- A chord endpoint and a distinguished vertex coincide on an input circle.
- A spacing parameter $s_i = 1$.
- A spacing parameter $s_i = 0$.
- Two chord endpoints coincide on an input circle.

If f is a face of e then chords whose endpoints coincide in slide diagrams with levels in e also coincide in slide diagrams in f. That is, if two open cells are slide-equivalent, their closures are slide-equivalent. This provides a map $\mathfrak{E} \to \mathfrak{F}$ of vertices of slide complexes which extends to $\iota : P_{\mathfrak{E}} \to P_{\mathfrak{F}}$. □

Remark. In the first two cases the map is an isomorphism of cell complexes. Generically, in the second two cases $P_{\mathfrak{E}}$ is either a codimension 1 subcomplex of $P_{\mathfrak{F}}$ or $P_{\mathfrak{E}}$ and $P_{\mathfrak{F}}$ are isomorphic cell complexes. However if the connected component of the subgraph of chords consisting just of the chords that are sliding and the chords over which they are sliding has nonpositive Euler characteristic, the map $\mathfrak{E} \to \mathfrak{F}$ induced by setting $s_i = 0$ may identify two distinct representatives of $\mathfrak{E}$. Higher dimensional cells of $P_{\mathfrak{F}}$ are also identified under ι in this case.

4.1.2 $\overline{LD}(g, k, \ell)/\!\!/ \sim$

The space $\overline{LD}(g, k, \ell)/\!\!/ \sim$ is built by adjoining cells to $\overline{LD}(g, k, \ell)$.

Definition 4.6. Given a slide-equivalence class $\mathfrak{E}$ of cells of $\overline{LD}(g, k, \ell)$ let $P_{\mathfrak{E}}$ be the slide complex of $\mathfrak{E}$ with vertices v_e labeled by representatives e of $\mathfrak{E}$ and let $X_{\mathfrak{E}}$ be the common parameter space for cells $e \in \mathfrak{E}$. The space $\overline{LD}(g, k, \ell)/\!\!/ \sim$ is constructed from $\overline{LD}(g, k, \ell)$ by adjoining the complex $X_{\mathfrak{E}} \times P_{\mathfrak{E}}$ to $\overline{LD}(g, k, \ell)$ under the identification of $X_{\mathfrak{E}} \times v_e$ with e for all slide-equivalence classes $\mathfrak{E}$. The adjunction is compatible with faces: if f is a face of e and $\iota : P_{\mathfrak{E}} \to P_{\mathfrak{F}}$ is the canonical map of slide complexes, then $X_{\mathfrak{F}} \subset X_{\mathfrak{E}}$ and $X_{\mathfrak{F}} \times P_{\mathfrak{F}}$ is identified with $X_{\mathfrak{F}} \times \iota(P_{\mathfrak{E}})$.

Example. The space $\overline{LD}(0, 3, 1)$ is a three-torus bundle over a space with six 2-cells, each a square. There are three components, each formed by two squares identified along one edge where $s = 0$. In step 1, slide cells are added. In step 2, triangles are added. There are no multi-slide cells and no combination cells. The space $\overline{LD}(0, 3, 1)/\!\!/ \sim$ is homeomorphic to a 3-torus-bundle over a 2-sphere with three boundary components. See figure 4.4.

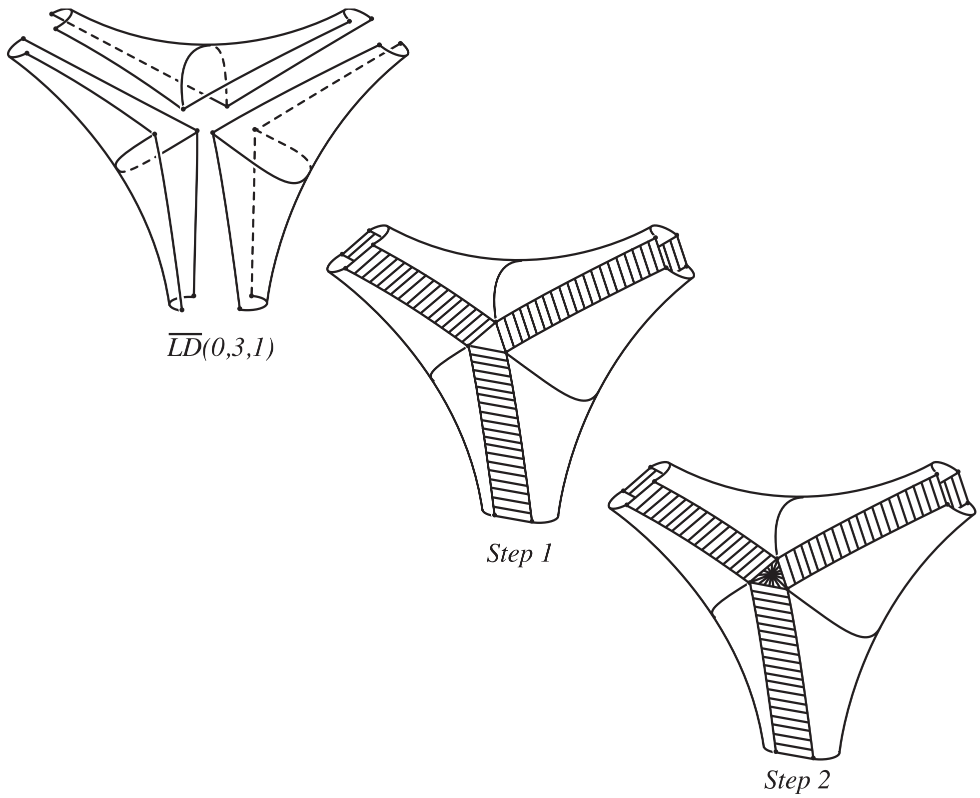


Figure 4.4: Base space for $\overline{LD}(0,3,1)/\!\!/\sim$

Proposition 4.1. *The space $\overline{LD}(g,k,\ell)/\!\!/\sim$ is a pseudomanifold of dimension $6g+3k+3\ell-7$ with boundary. The set of codimension 1 cells at the boundary is partitioned into:*

- *cells labeled by diagrams where one spacing parameter $s_i = 1$;*
- *cells labeled by diagrams where a connected component of the subgraph consisting of the complement of the input circle edges has Euler characteristic $\chi = 0$.*

Proof. We first check that a cell A of $\overline{LD}(g,k,\ell)/\!\!/\sim$ is a face of some top-dimensional cell. Let $A = X_{\mathfrak{E}} \times p_{\mathfrak{E}}$ where $p_{\mathfrak{E}}$ is a cell of the slide complex $P_{\mathfrak{E}}$. If $p_{\mathfrak{E}}$ is the face of a top-dimensional cell of $P_{\mathfrak{E}}$ and $X_{\mathfrak{E}} \times P_{\mathfrak{E}}$ is top dimensional, then A is the face of $X_{\mathfrak{E}} \times P_{\mathfrak{E}}$. If $X_{\mathfrak{E}} \times P_{\mathfrak{E}}$ is not top dimensional, then there exists a cell d of $\overline{LD}(g,k,\ell)$ such that $X_{\mathfrak{D}} \times P_{\mathfrak{D}}$ is top-dimensional and there exists a top-dimensional cell $p_{\mathfrak{D}}$ of $P_{\mathfrak{E}}$ such that $\iota(p_{\mathfrak{D}}) = p_{\mathfrak{E}}$. (Such a d need not be unique; there may be many ways to separate chords into different levels so that $X_{\mathfrak{D}} \times P_{\mathfrak{D}}$ is top-dimensional.) Therefore, the cell A is a face of $X_{\mathfrak{D}} \times P_{\mathfrak{D}}$.

We now examine codimension 1 cells of $\overline{LD}(g,k,\ell)/\!\!/\sim$. If a codimension 1 cell A is the face of exactly one top-dimensional cell, then it is at the boundary.

The space $\overline{LD}(g,k,\ell)$ is a pseudomanifold with boundary. The codimension 1 cells e at the boundary are labeled by combinatorial types where a spacing parameter $s_i = 1$ or two chord endpoints coincide on an input circle. If the endpoints coinciding do not belong to the same chord, then the slide complex $P_{\mathfrak{E}}$ of the slide-equivalence class $\mathfrak{E}$ of e is an interval $s_{e,e'}$. New top-dimensional cells $X_{\mathfrak{E}} \times s_{e,e'}$ are adjoined to $\overline{LD}(g,k,\ell)$ so that the slide-equivalent cells e and e' are then codimension 1 faces of exactly two top-dimensional cells: one from $\overline{LD}(g,k,\ell)$ and one from $X_{\mathfrak{E}} \times s_{e,e'}$.

The space $\overline{LD}(g,k,\ell) \cup \bigcup_{\mathfrak{E}} (X_{\mathfrak{E}} \times S_{\mathfrak{E}})$ is a pseudomanifold with boundary. However, if f is a codimension 1 face of e then there may be new codimension 1 boundary, if $S_{\mathfrak{F}}$ is not contractible. Using slide complexes, we shall see that the new codimension 1 boundary is filled in in $\overline{LD}(g,k,\ell)/\!\!/\sim$ unless it corresponds to one of the exceptions described above.

A top-dimensional cell of $P_{\mathfrak{E}}$ has maximum dimension equal to $codim(e)$, the codimension of the representatives e of $\mathfrak{E}$ in $\overline{LD}(g,k,\ell)$. Therefore, no cells of $\overline{LD}(g,k,\ell)/\!\!/\sim$ have dimension greater than $6g+3k+3\ell-7$. A top-dimensional cell of $P_{\mathfrak{E}}$ has dimension $codim(e)$ if and only if for all representatives e labeling its vertices v_e, the chords at each level form a connected subgraph and they are adjacent in the cyclic orders at common vertices.

Let A be a codimension 1 cell of $\overline{LD}(g,k,\ell)/\!\!/\sim$ that is not also a codimension 1 cell of the sub-cell complex $\overline{LD}(g,k,\ell)$ The cases are as follows:

1. A is codimension 1 cell of $X_{\mathfrak{E}} \times P_{\mathfrak{E}}$ where $dim(P_{\mathfrak{E}}) = codim(e)$.
   (a) $A = X_{\mathfrak{E}} \times p_{\mathfrak{E}}$ where $p_{\mathfrak{E}}$ is a codimension 1 cell of $P_{\mathfrak{E}}$.
      i. $p_{\mathfrak{E}}$ is a codimension 1 cell that is the face of exactly two top-dimensional cells of $P_{\mathfrak{E}}$.
      ii. $p_{\mathfrak{E}}$ is the codimension 1 face of exactly one top-dimensional cell of $P_{\mathfrak{E}}$.
      iii. $p_{\mathfrak{E}}$ is a codimension 1 cell that is not the face of any top-dimensional cell of $P_{\mathfrak{E}}$.
   (b) $A = X_{\mathfrak{F}} \times p_{\mathfrak{E}}$ where f is a codimension 1 face of e and $p_{\mathfrak{E}}$ is a top-dimensional cell of $P_{\mathfrak{E}}$.

i. $\iota : P_{\mathfrak{E}} \to P_{\mathfrak{F}}$ is an isomorphism on $p_{\mathfrak{E}}$.

ii. $\iota : P_{\mathfrak{E}} \to P_{\mathfrak{F}}$ is injective but not surjective on $p_{\mathfrak{E}}$.

iii. $\iota : P_{\mathfrak{E}} \to P_{\mathfrak{F}}$ is not injective on $p_{\mathfrak{E}}$.

2. A is a top-dimensional cell of $X_{\mathfrak{E}} \times P_{\mathfrak{E}}$ and $dim(P_{\mathfrak{E}}) = codim(e) - 1$.

   (a) The subgraph consisting of the complement of input circles has $\chi > 0$.

   (b) The subgraph consisting of the complement of input circles has $\chi = 0$.

In what follows, “chord” refers either to a chord or to a corolla or to a cyclically-ordered graph representing a deformation (minus input circles), as in $T_{\mathfrak{E}}$.

Note first that if the complement of the input circles has a connected component with Euler characteristic $\chi \leq 0$, then A cannot be a cell of type 1. Intuitively, there is a potential tripod that is not possible when the endpoints coinciding belong to the same chord; a chord's endpoint does not slide along the chord itself.

In case 1(a)(i), there are two top-dimensional cells of $P_{\mathfrak{E}}$ with $p_{\mathfrak{E}}$ as a face, so there are two top-dimensional cells of $\overline{LD}/\!\!/\sim$ with $X_{\mathfrak{E}} \times p_{\mathfrak{E}}$ as a face.

To approach a codimension 1 face $p_{\mathfrak{E}}$ of $P_{\mathfrak{E}}$ as in case 1(a)(ii), either a corolla breaks into two corollae where one is sliding over the other or a chord ch which is sliding over another chord $\overline{ch}$, which is at a lower level, approaches an endpoint of $\overline{ch}$, which is on an input circle. In either of these cases, $p_{\mathfrak{E}} = \iota(p_{\mathfrak{D}})$ for exactly one $\mathfrak{D}$ where e is the codimension 1 face of some cell d such that $X_{\mathfrak{D}} \times p_{\mathfrak{D}}$ is top-dimensional. Passing from d to e either brings two chords which are adjacent at some endpoint to the same level and creates a triangle, or passing from d to e brings endpoints of two chords at different levels together on an input circle. Thus, A is a codimension 1 face of one top-dimensional cell in $X_{\mathfrak{E}} \times P_{\mathfrak{E}}$ and of one top-dimensional cell in $X_{\mathfrak{D}} \times P_{\mathfrak{D}}$. See case 1(b)(ii).

In case 1(a)(iii), at lest one of the vertices v_e is labeled by a cell e where two chords at the same level, ch and ch' are not adjacent in the cyclic order at some vertex (though they must be adjacent in another representative in order for $X_{\mathfrak{E}} \times P_{\mathfrak{E}}$ to be top-dimensional. Either their endpoints coincide on an input circle, but are not adjacent or their endpoints do not coincide at all. In the case that their endpoints do not coincide, there is a sequence of chords separating them; that is a string of chords $\overline{ch}_1, \overline{ch}_2, \ldots, \overline{ch}_N$ which are all below the level of ch and ch', such that $\overline{ch}_i$ and $\overline{ch}_{i+1}$ have a common endpoint and are adjacent in the cyclic order at that vertex and such that ch and ch' may slide over all of them. In this case, there are two ways to pull ch and ch' apart into two levels. We write $p_{\mathfrak{E}} = \iota(p_{\mathfrak{D}})$ where $p_{\mathfrak{D}}$ is a top-dimensional cell of $P_{\mathfrak{D}}$ and e is a codimension 1 face of d *and* $p_{\mathfrak{E}} = \iota(p_{\mathfrak{D}'})$ where $p_{\mathfrak{D}'}$ is a top-dimensional cell of $P_{\mathfrak{D}'}$ and e is a codimension 1 face of d'. The cells d and d' correspond to $ch \in E_i, ch' \in E_{i+1}$ and $ch' \in E_i, ch \in E_{i+1}$ respectively.

Case 1(b)(i) is similar to case 1(a)(iii) if f is obtained from e by setting a spacing parameter $s_i = 0$. If $dim(P_{\mathfrak{E}}) = codim(e)$ for all representatives e of $\mathfrak{E}$ and a spacing parameter $s_i = 0$, then the subgraph of chords in E_i is connected and chords are adjacent at their common endpoints. If $P_{\mathfrak{E}}$ and $P_{\mathfrak{F}}$ are isomorphic, then f is obtained from e by setting $s_j = 0$ such that no chords in E_j and E_{j+1} are adjacent at any common endpoint. This remains true when considering not just vertices of $P_{\mathfrak{E}}$ but any interior point as well. Therefore, A is the face of exactly two top-dimensional cells: $X_{\mathfrak{E}} \times p_{\mathfrak{E}}$ and $X_{\mathfrak{E}'} \times p_{\mathfrak{E}'}$ where representatives e' of $\mathfrak{E}'$ are obtained from representatives e' of $\mathfrak{E}'$ by interchanging E_j and E_{j+1}. In this case, A is the face of exactly two top-dimensional cells.

If f is obtained from e by setting a spacing parameter $s_i = 1$ or if two endpoints of a chord coincide on an input circle, then $\iota : P_{\mathfrak{E}} \to P_{\mathfrak{F}}$ is an isomorphism. Such a cell a is the face of exactly one top-dimensional cell, that is $X_{\mathfrak{E}} \times P_{\mathfrak{E}}$.

In case 1(b)(ii), $X_{\mathfrak{F}} \times P_{\mathfrak{F}}$ is top-dimensional in $\overline{LD}(g,k,\ell)/\!\!/ \sim$. A is a codimension 1 face of $X_{\mathfrak{E}} \times p_{\mathfrak{E}}$ and a codimension 1 face of a top-dimensional cell in $X_{\mathfrak{F}} \times P_{\mathfrak{F}}$. See case 1(a)(i).

In case 1(b)(iii), $dim(\iota(p_{\mathfrak{E}})) < dim(p_{\mathfrak{E}})$ so in fact,

$$dim(A) = dim(f) + dim(\iota(p_{\mathfrak{E}})) < dim(E) + dim(P_{\mathfrak{E}}) - 2.$$

Therefore, there is no such codimension 1 cell A.

Case 2(a) is virtually identical to case 1(b)(i) where the cell A is the boundary of two chords, whose levels are interchanged.

In case 2(b), such a cell is at the boundary. If $\chi = 0$ then a top-dimensional cell consists either of a corolla with a pair of endpoints coinciding on an input circle, or two corollae at the same level, where two endpoints of one are sliding over the other. A cell of the first type is the face of exactly one top-dimensional cell where the no corolla endpoints coincide. A cell of the second type is also the face of exactly one top-dimensional cell: one where the two corollae are at different levels.

Note that if $\chi < 0$ then $P_{\mathfrak{e}}$ has dimension at most $codim(e) - 2$ so can never give rise to a codimension 1 cell A of $\overline{LD}(g, k, \ell)/\!\!/ \sim$.

□

Remark. All cells of $\overline{LD}(g, k, \ell)/\!\!/ \sim$ are labeled by combinatorial types of cyclically-ordered graphs with distinguished vertices on input circles. There is a T^k-action which rotates the distinguished vertices and a complex $C_*^{T^k}(\overline{LD}(g, k, \ell)/\!\!/ \sim)$ of saturated chains which is completely analogous to $C_*^{T^k}(\overline{LD}(g, k, \ell))$.

Definition 4.7. Let $\overline{SD}(g, k, \ell)/\!\!/ \sim \subset \overline{LD}(g, k, \ell)/\!\!/ \sim$ be the complex of cells e of $\overline{SD}(g, k, \ell)$ with $X_e \times P_{\mathfrak{e}}$ adjoined.

Proposition 4.2. *The space* $\overline{LD}(g, k, \ell)/\!\!/ \sim$ *contains* $\overline{SD}(g, k, \ell)/\!\!/ \sim$ *as a defor-*

mation retract.

Proof. The proof is analogous to the proof of proposition 2.3. Again, let

$$i : \overline{SD}(g,k,\ell)/\!\!/\sim \longrightarrow \overline{LD}(g,k,\ell)/\!\!/\sim$$

be the inclusion map and let

$$r : \overline{LD}(g,k,\ell)/\!\!/\sim \longrightarrow \overline{SD}(g,k,\ell)/\!\!/\sim$$

be the map that forgets levels and spacing parameters or, equivalently, sets them all to 0. Again, $r \circ i = id$ and $i \circ r$ is homotopic to the identity, where the homotopy shrinks spacing parameters.

□

4.1.3 Slide complexes for generators of $C_*^k(\overline{LD}(g,k,\ell))$

Let c be a generator of $C_*^k(\overline{LD}(g,k,\ell))$ such that $c = \sum \pm e_i$ for e_i a cell of $\overline{LD}(g,k,\ell)$. Recall that the combinatorial types labeling the e_i are identical except for the placement of the distinguished vertex. If c' is another generator of $C_*^k(\overline{LD}(g,k,\ell))$ such that $c' = \sum \pm e_i'$ and such that for all i, e_i and e_i' each differ by a *compatible* chord slide then we say that c and c' are slide-equivalent generators and denote their slide-equivalence class by $\mathfrak{C}$. The slide complexes $P_{\mathfrak{C}_i}$ are canonically isomorphic to one another and the canonical isomorphism preserves the decomposition $P_{\mathfrak{C}_i} = S_{\mathfrak{C}_i} \cup SS_{\mathfrak{C}_i} \cup T_{\mathfrak{C}_i} \cup SST_{\mathfrak{C}_i}$.

Definition 4.8. For a generator c of $C_*^k(\overline{LD}(g,k,\ell))$ the slide complex $P_{\mathfrak{C}}$ is the isomorphism class of $P_{\mathfrak{C}_i}$.

Remark. The sub-cell complex X_c of $\overline{LD}(g,k,\ell)$ associated to c is canonically isomorphic to $X_{c'}$ where c' is a slide-equivalent generator. We denote the isomorphism class by $X_{\mathfrak{C}}$ and note that $X_{\mathfrak{C}} = \bigcup_i X_{\mathfrak{C}_i}$. Then

$$\overline{LD}(g,k,\ell)/\!\!/ \sim = \overline{LD}(g,k,\ell) \cup \bigcup_{\mathfrak{C}} (X_{\mathfrak{C}} \times P_{\mathfrak{C}})$$

where the adjunction of $X_{\mathfrak{C}} \times P_{\mathfrak{C}}$ is compatible with the adjunction of cells above.

4.2 Extension of ST_μ to $\overline{LD}(g,k,\ell)/\!\!/ \sim$

Proposition 4.3. *The string topology construction ST_μ extends to the top chain of $\overline{LD}(g,k,\ell)/\!\!/ \sim$ and satisfies*

$$ST_\mu(\partial[\overline{LD}(g,k,\ell)/\!\!/ \sim]) = D_{Hom}(ST_\mu([\overline{LD}(g,k,\ell)/\!\!/ \sim])).$$

Most of this section will be devoted to the proof, which is long and will be given in several parts. In part 1(a) the string topology construction is extended to cells adjoined to $\overline{SD}(g,k,\ell)$ which correspond to slide cells in $S_{\mathfrak{C}}$ for cells e of $\overline{SD}(g,k,\ell)$. In part 1(b) the extension from part 1 is generalized to cells adjoined to $\overline{SD}(g,k,\ell)$ which correspond to multi-slide cells in $SS_{\mathfrak{C}}$ for cells e of $\overline{SD}(g,k,\ell)$. In part 2(a) the construction is extended to certain chains of $\overline{LD}(g,k,\ell)/\!\!/\sim$ called split chains. In part 2(b) the extension from part 2 is generalized to certain chains of $\overline{LD}(g,k,\ell)/\!\!/\sim$ called higher split chains. Finally, in part 3, the construction is extended to all remaining cells of $\overline{LD}(g,k,\ell)/\!\!/\sim$.

Proof. **Part 1(a).**

Let c and c' be generators of $C_*^{T^k}(\overline{SD}(g,k,\ell))$ that differ by a chord slide. Let c and c' have corresponding evaluation maps $ev : \pi_c^{-1}(W_\alpha) \to M^{\mathscr{H}}$ and $ev' : \pi_{c'}^{-1}(W_\alpha) \to M^{\mathscr{H}}$. By Proposition 3.1, without loss of generality, we may assume that μ is supported in a $\frac{\varepsilon}{2}$-tubular neighborhood of the diagonal $\Delta_{ch}(M^{\{ch\}}) \subset M^{\{h,h'\}}$.

All $\Gamma \in X_c$ has a corresponding $\Gamma' \in X_{c'}$ which differs by a chord slide. There must be a configuration of chords as in figure 4.1 and we use the labeling of chord endpoints provided there.

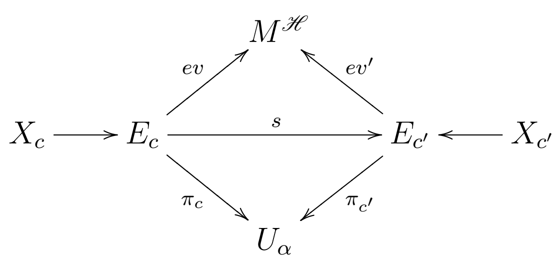
Let $b_{c,c'} = X_{\mathfrak{C}} \times s_{c,c'}$ be the complex of cells adjoined to $\overline{SD}(g,k,\ell)$ corresponding to slide cells $s_{c,c'} \subset S_{\mathfrak{C}}$. We define $K_{c,c'} \in Hom(P_*(kLM), P_*(\ell LM))$ below. We will then set $ST_\mu(b_{c,c'}) = K_{c,c'}$ and extend ST_μ linearly over

$C_*^{T^k}(\overline{SD}(g,k,\ell) \cup \bigcup_{\mathfrak{C}}(X_C \times S_{\mathfrak{C}}))$.

For a generator (B, α) of $P_*(kLM)$, we choose a representative (B, α, E^k, f) and construct each component of a quadruple representing $K_{c,c'}(B, \alpha)$.

The base space.

For $ST_\mu(c)(B, \alpha)$ the base space is $\pi_c^{-1}(W_\alpha)$ and for $ST_{\mu'}(c')$, the base space is $\pi_{c'}^{-1}(W_\alpha)$. As the terms of c and c' are pairwise slide-equivalent, there is a canonical isomorphism $X_c \to X_{c'}$ of complexes. This induces a canonical isomorphism $s : E_c \to E_{c'}$ of total spaces of the corresponding X_c- and $X_{c'}$- bundles, which restricts to an isomorphism $\pi_c^{-1}(W_\alpha) \to \pi_{c'}^{-1}(W_\alpha)$. The following diagram commutes.



Let $\pi_c^{-1}(W_\alpha) \cong \pi_{c'}^{-1}(W_\alpha)$ be denoted by $\pi_{\mathfrak{C}}^{-1}(W_\alpha)$ and have fiber denoted by $X_{\mathfrak{C}}$. The base space for $K_{c,c'}(B, \alpha)$ is $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c,c'}$ where $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times v_c$ is identified with $\pi_c^{-1}(W_\alpha)$ and $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times v_c$ is identified with $\pi_{c'}^{-1}(W_\alpha)$.

A point $\Gamma_t \in \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c,c'}$ corresponds to a cyclically-ordered graph as in figure 4.1.

The $\mathbb{Q}$-polynomial form.

The $\mathbb{Q}$-polynomial form on $\pi_c^{-1}(W_\alpha)$ is given by $\pi_c^*(\alpha) \wedge ev^*(\mu_c)$ and $\mathbb{Q}$-polynomial form on $\pi_c^{-1}(W_\alpha)$ is given by $\pi_c^*(\alpha) \wedge ev^*(\mu_c)$.

Let $\Gamma, \Gamma', \Gamma_t$ be as in figure 4.1. For exposition, we choose an ordering of half-chords so that for $\Gamma \in \pi_c^{-1}(a) \subset \pi_c^{-1}(W_\alpha)$, $ev(\Gamma) = (f(q), f(r), f(p), f(q))$ and $ev'(\Gamma') = (f(q), f(r), f(p), f(r))$ where f refers to the restriction of $f : E^k \to M$ to $\pi^{-1}(a)$. Let $N \subset ev^{-1}(N_{\mu_c}) \cap (ev')^{-1}(N_{\mu_{c'}})$. Note that the chords of Γ_t may be ordered: the first chord $\overline{ch}$ is attached to input circles and one end of the second chord ch is attached to an input circle while the other endpoint is attached to the first chord. We define the $\mathbb{Q}$-polynomial form on $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c,c'}$ in two steps.

Define $ev_1 : \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c,c'} \to M^2$ by $ev_1(\Gamma_t) = (f(q), f(r))$. The $\mathbb{Q}$-polynomial form μ on M^2 is the representative of the Thom class of the diagonal which is supported in an $\frac{\varepsilon}{2}$ neighborhood N_μ of the diagonal. If Γ_t is in $ev_1^{-1}(N_\mu)$ then the points $f(q)$ and $f(r)$ may be joined by a unique geodesic.

The sub-chord diagram consisting of ch and the input circles it touches is mapped to M by a map $f_{\overline{SD}}$ as in the usual string topology construction. In particular, the endpoint t on the chord $\overline{ch}$ is mapped to a point along the geodesic segment. The ratio between the lengths of geodesic segments joining $f(q)$ to $f(t)$ and $f(t)$ to $f(r)$ corresponds to $t \in [0, L]$. (If $f(q) = f(r)$, then the geodesic segment is just this point and $f_{\overline{SD}}$ also sends t to this point.)

Define $ev_2 : ev_1^{-1}(N_\mu) \to M^2$ by $(f(p), f_{\overline{SD}}(t))$.

The $\mathbb{Q}$-polynomial form $ev_1^*(\mu)$ is defined on all of $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c,c'}$. The form $ev_2^*(\mu)$ is defined on $ev_1^{-1}(N_\mu)$ at its support is a closed subset of this open subset, so it is extended by 0 to a form, also called $ev_2^*(\mu)$ on all of $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c,c'}$.

Let $\rho : \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c,c'} \to \pi_{\mathfrak{C}}^{-1}(W_\alpha)$ be the projection map.

The $\mathbb{Q}$-polynomial form on $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c,c'}$ is given by

$$\rho^*(\pi_{\mathfrak{C}}^*(\alpha)) \wedge ev_1^*(\mu) \wedge ev_1^*(\mu).$$

This form restricts to forms $\pi_c^*(\alpha) \wedge ev^*(\mu_c)$ and $\pi_{c'}^*(\alpha) \wedge (ev')^*(\mu_{c'})$ on $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times v_c$ and $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times v_{c'}$ respectively.

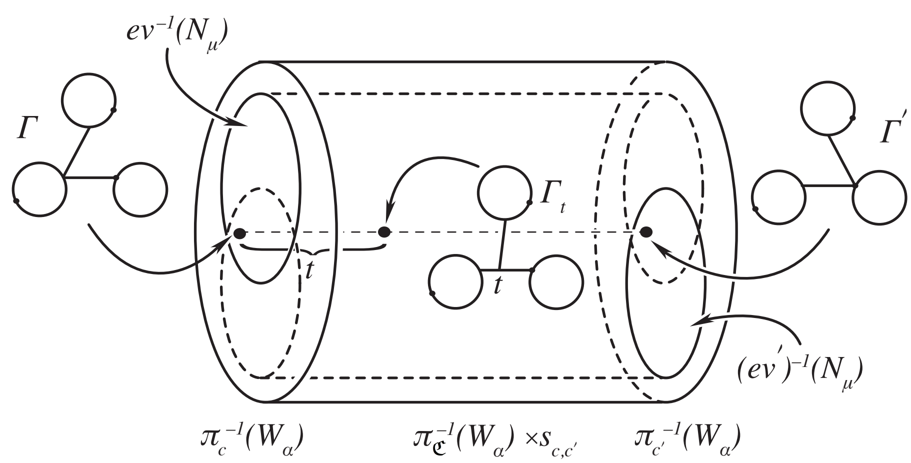


Figure 4.5: Base space for slide cell adjoined to $\overline{SD}(g, k, \ell)$

The $\bigsqcup_\ell S^1$-bundle.

Recall the $\bigsqcup_\ell S^1$-bundle with total space E^ℓ over $ev^{-1}(N_\mu) \subset \pi_c^{-1}(W_\alpha)$ given in the string topology construction by restricting the universal string-diagram bundle to output circles. Likewise, there is a universal cyclically-ordered-graph bundle over $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c,c'}$ whose restriction to output circles gives the $\bigsqcup_\ell S^1$-bundle. Denote the total space by of the universal cyclically-ordered-graph bundle over $ev_2^{-1}(N_\mu)$ by $E_{\overline{SD}}$ and the total space of the corresponding bundle of output circles by $E^\ell_{c,c'}$.

The map to M.

As in the string topology construction, there is a map $f_{\overline{SD}} : E_{\overline{SD}} \to M$ which sends chords to geodesic segments. Let $g : E^\ell_{c,c'} \to M$ be the restriction of $f_{\overline{SD}}$ to output circles.

(Note, if many chords are sliding together, the construction is completely analogous to the one above, but there is more than one component for t under the evaluation maps.)

Part 1(a) Summary.

Let

$$\begin{aligned} K_{c,c'}(B,\alpha) &= [(\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c,c'}, \rho^*(\pi_{\mathfrak{C}}^*(\alpha)) \wedge ev_1^*(\mu) \wedge ev_1^*(\mu), E^\ell_{c,c'}, g)] \\ &= (\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c,c'}, \rho^*(\pi_{\mathfrak{C}}^*(\alpha)) \wedge ev_1^*(\mu) \wedge ev_1^*(\mu)) \end{aligned}$$

and extend linearly to $P_*(kLM)$.

Part 1(b).

Let $s_{c_1,\ldots,c_n}$ be a multi-slide cell in $SS_{\mathfrak{C}}$ for slide-equivalent generators $c_1, \ldots, c_n$ of $C_*^{T^k}(\overline{SD}(g,k,\ell))$. Let $b_{c_1,\ldots,c_n}$ be the complex $X_{\mathfrak{C}} \times s_{c_1,\ldots,c_n}$ adjoined to $\overline{SD}(g,k,\ell)$.

The extension of ST_μ to $b_{c_1,\ldots,c_n}$ is a direct generalization of the extension in part 1(a). The base space for $ST_\mu(b_{c_1,\ldots,c_n})(B,\alpha)$ is $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c_1,\ldots,c_n}$. For a cyclically-ordered graph $\Gamma_t \in \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c_1,\ldots,c_n}$, chords are partitioned into sets and ordered as in part 1(a) according to where their endpoints lie. This determines a sequence of evaluation maps ev_i. The $\mathbb{Q}$-polynomial form on $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c_1,\ldots,c_n}$ is given by $\rho^*(\pi_{\mathfrak{C}}^*(\alpha)) \wedge ev_1^*(\mu) \wedge \cdots \wedge ev_m^*(\mu)$. Again, there is a universal cyclically-ordered-graph bundle over a neighborhood of $supp(\rho^*(\pi_{\mathfrak{C}}^*(\alpha)) \wedge ev_1^*(\mu) \wedge \cdots \wedge ev_m^*(\mu))$ and the string topology construction defines a map of its total space into M. Again, the universal cyclically-ordered graph bundle restricts to a $\bigsqcup_\ell S^1$-bundle and the map to M restricts to output circles.

The map $ST_\mu(b_{c_1,\ldots,c_n})(B,\alpha)$ is extended linearly to $P_*(kLM)$.

Part 2(a).

We now extend ST_μ to particular chains of $\overline{LD}(g,k,\ell)/\!\!/ \sim$ called split chains.

We use composition of string topology operations $ST_\mu(c)$ both to locate particular chains in $C_*^{T^k}(\overline{LD}(g,k,\ell)/\!\!/\sim)$ and form the extension.

We begin by locating a generator of $C_*^{T^k}(\overline{LD}(g,k,\ell)/\!\!/\sim)$ called a split generator.

Recall that slide cells and multi-slide cells in $S_\mathfrak{c} \cup SS_\mathfrak{c}$ parametrize chord slides and multiple chord slides for cells of in $\overline{LD}(g,k,\ell)$ and that points Γ in $S_\mathfrak{c} \cup SS_\mathfrak{c}$ correspond to cyclically-ordered graphs where chords' endpoints lie in the interiors of other chords. A split generator d satisfies the following:

- For all Γ in the interior of a top-dimensional cell of X_d, the set of chords is partitioned into two subsets $E = E_1 \sqcup E_2$ and the spacing parameter $s_1 = 1$.
- For all Γ in the interior of a top-dimensional cell of X_d, the subgraph Γ' consisting of chords in E_1, together with the input circles they touch, forms a string diagram of type (g', k', ℓ').
- For all Γ in the interior of a top-dimensional cell of X_d, the subgraph Γ'' consisting of all remaining input circles and chords, together with the n *output* circles of Γ' that they touch, (which are each rescaled to have length 1) is a string diagram of type (g'', k'', ℓ'') up to a choice of distinguished vertices on the n output circles of Γ' which form n of the input circles of Γ''.

Note that for all Γ in the interior of a top-dimensional cell of X_d where d is a split generator, endpoints of chords in Γ' must lie on input circles of Γ and endpoints of chords in Γ'' must lie either on an input circle of Γ or on a chord of Γ'. Additionally, Γ determines a pairing ε of n output circles of Γ' with n input circles of Γ.

Note also that the combinatorial types of Γ' and Γ'' do not depend on the choice of Γ in the interior of a top-dimensional cell of X_d. We say that $\Gamma' \in X_{d'}$ and $\Gamma'' \in X_{d''}$ where d' is a generator of $C_*^{T^{k'}}(\overline{SD}(g',k',\ell'))$ and d'' is a generator of $C_*^{T^{k''}}(\overline{SD}(g'',k'',\ell''))$. Therefore d determines d', d'' and the pairing ε of output circles to input circles. We write $split(d) = (d', d'', \varepsilon)$.

However from d', d'' and ε we cannot reconstruct d. This information alone does not determine exactly how output circles and input circles are identified. In general there are several split generators d_i such that $split(d_i) = (d', d'', \varepsilon)$. A split chain $d = \sum d_i$ such that $split(d_i) = (d', d'', \varepsilon)$ for fixed d' and d''.

We denote such a split chain by $d'' *_\varepsilon d'$ and its corresponding sub cell complex of $\overline{LD}(g,k,\ell)/\!\!/\sim$ by $X_{d'' *_\varepsilon d}$, which is a union of sub cell complexes X_{d_i} of split generators.

Let $r : \overline{LD}(g,k,\ell)/\!\!/ \sim \longrightarrow \overline{SD}(g,k,\ell)/\!\!/ \sim$ be the map which sets all spacing parameters to 0. Let r_* be the map induced on $C_*^{T^k}(\overline{LD}(g,k,\ell)/\!\!/ \sim)$ and let $r_*(\sum d_i) = \sum r_*(d_i) = \sum \bar{d}_i$.

Generators d_i and $\bar{d}_i$ appear in the expression for $\partial(\tilde{d}_i) \in C_*^{T^k}(\overline{LD}(g,k,\ell)/\!\!/ \sim)$; d_i when $s_1 = 1$ and $\bar{d}_i$ when $s_1 = 0$.

Set $ST_\mu(\sum d_i) = ST_\mu(d'') *_\varepsilon ST_\mu(d')$ where $*_\varepsilon$ is composition of string topology operations corresponding to the pairing ε.

We defne $L_{d',d''} \in Hom(P_*(kLM), P_*(\ell LM))$ and then set $ST_\mu(\sum \tilde{d}_i) = L_{d',d''}$.

For a generator (B,α) of $P_*(kLM)$, we choose a representative (B,α,E^k,f) and construct each component of a quadruple representing $L_{d',d''}(B,\alpha)$.

The base space.

We construct a map $\circ$ from the base space of $ST_\mu(\sum d_i)(B,\alpha)$ to the base space of $ST_\mu(\sum \bar{d}_i)$ below. The base space of $L_{d',d''}(B,\alpha)$ will be the mapping cylinder of $\circ$.

The base spaces for $ST_\mu(\overline{d}_i)(B,\alpha)$ are $\pi_{\overline{d}_i}^{-1}(W_\alpha)$. Let $\overline{d} = \sum \overline{d}_i$. We identify codimension 1 faces of the fibers $X_{\overline{d}_i}$ with one another appropriately to form the fiber $X_{\overline{d}}$ for a bundle $\pi_{\overline{d}} : E_{\overline{d}} \to U_\alpha$. The base space for $ST_\mu(\sum \overline{d}_i)(B,\alpha) = ST_\mu(d)(B,\alpha)$ is then $\pi_{\overline{d}}^{-1}(W_\alpha)$.

The base space for $ST_\mu(d')(B,\alpha)$ is $\pi_{d'}^{-1}(W_\alpha)$ and $ev_{d'} : \pi_{d'}^{-1}(W_\alpha) \to M^{\mathscr{H}_d}$. The base space for $ST_\mu(d)(B,\alpha) = ST_\mu(d'')(ST_\mu(d')(B,\alpha))$ is $\pi_{d''}^{-1}(ev_{d'}^{-1}(N_{\mu_{d'}}))$. A point Γ'' in $\pi_{d''}^{-1}(ev_{d'}^{-1}(N_{\mu_{d'}}))$ is a string diagram of type (g'',k'',ℓ'') and $\pi_{d''}(\Gamma'') = \Gamma'$ is a string diagram of type (g',k',ℓ').

For the evaluation map $ev_{d''} : \pi_{d''}^{-1}(ev_{d'}^{-1}(N_{\mu_{d'}})) \to M^{\mathscr{H}_{d''}}$, one marked point must be chosen on each S^1 component of $\pi_{d''}^{-1}(\Gamma')$. If an S^1 component is the restriction of $E'_{\overline{SD}}$ to outputs $\Gamma' \in E'_{\overline{SD}}$, the choice of marked point amounts to the choice of a marked point on an output circle of Γ' which is identified with an input circle of $\Gamma'' \in \pi_{d''}^{-1}(\Gamma')$.

The map $\circ : \pi_{d''}^{-1}(ev_{d'}^{-1}(N_\mu)) \to \pi_{\overline{d}}^{-1}(W_\alpha)$ is given by the identification. That is, if $\pi_{d'}(\pi_{d''}(\Gamma'')) = a \in W_\alpha$ and the and the identification of outputs of Γ' with inputs of Γ'' prescribed by the map $\pi_{d''}$ yields a graph Γ', then
$\circ(\Gamma'') = \Gamma \in \pi_{\overline{d}}^{-1}(a)$.

The base space for $L_{d',d''}(B, \alpha)$ is the mapping cylinder for $\circ$:

$$M(\circ) = (\pi_{d''}^{-1}(ev_{d'}^{-1}(N_\mu)) \times I \sqcup \pi_{\overline{d}}^{-1}(W_\alpha))/((\Gamma'', 1) \sim \circ(\Gamma'')).$$

The $\mathbb{Q}$-polynomial form.

The $\mathbb{Q}$-polynomial form on $\pi_{d''}^{-1}(ev_{d'}^{-1}(N_\mu))$ is $\pi_{d''}^*(\pi_{d'}^*(\alpha) \wedge ev_{d'}^*(\mu_{d'})) \wedge ev_{d''}(\mu_{d''})$. The $\mathbb{Q}$-polynomial form on $\pi_{\overline{d}}^{-1}(W_\alpha)$ restricts to $\pi_{\overline{d}_i}^*(\alpha) \wedge ev^*(\mu_{\overline{d}_i})$ on $\pi_{\overline{d}_i}^{-1}(W_\alpha)$.

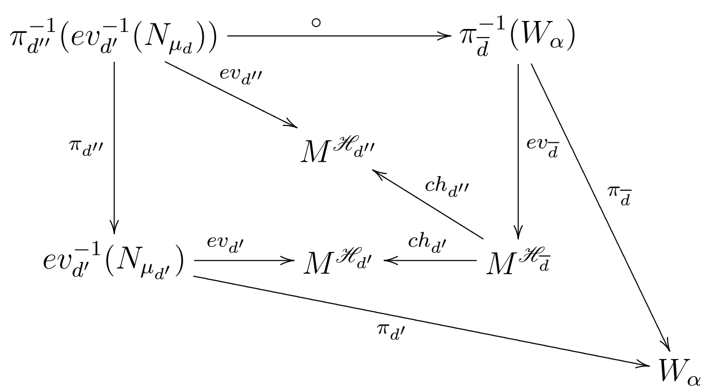
Note that $supp(\pi_{\overline{d}}^*(\alpha) \wedge ev_{\overline{d}}^*(\mu_{\overline{d}})) \subset im(\circ) \subset \pi_{\overline{d}}(W_\alpha)$.

The chains d' and d'' determine a partition of the set of half-chords $\mathscr{H}$ of all graphs $\Gamma \in \pi_{\overline{d}}^{-1}(W_\alpha)$; if a chord ch is a chord of the string diagram Γ', then its half-chords are in $\mathscr{H}_{d'}$ and if a chord ch is a chord of the string diagram Γ'', then its half-chords are in $\mathscr{H}_{d''}$. The inclusions of sets $\mathscr{H}_{d'} \hookrightarrow \mathscr{H}_{\overline{d}}$ and $\mathscr{H}_{d''} \hookrightarrow \mathscr{H}_{\overline{d}}$ induce maps $ch_{d'} : M^{\mathscr{H}_{\overline{d}}} \to M^{\mathscr{H}_{d''}}$ and $ch_{d'} : M^{\mathscr{H}_{\overline{d}}} \to M^{\mathscr{H}_{d''}}$ respectively.

With this notation, $\mu_{\overline{d}} = ch^*_{d'}(\mu_{d'}) \wedge ch^*_{d''}(\mu_{d''})$ so

$$\pi^*_{\overline{d}}(\alpha) \wedge ev^*_{\overline{d}}(\mu_{\overline{d}}) = \pi^*_{\overline{d}}(\alpha) \wedge ev^*_{\overline{d}}(ch^*_{d'}(\mu_{d'})) \wedge ev^*_{\overline{d}}(ch^*_{d''}(\mu_{d''})).$$

The following diagram commutes.



Therefore,

$$\circ^* \left(\pi^*_{\overline{d}}(\alpha)\right) = \pi^*_{d''}(\pi^*_{d'}(\alpha))$$

$$\circ^* \left(ev^*_{\overline{d}}(ch^*_{d'}(\mu_{d'}))\right) = \pi^*_{d''}(ev_{d'}(\mu_{d'}))$$

$$\circ^* \left(ev_{\overline{d}}(ch_{d''}(\mu_{d''}))\right) = ev^*_{d''}(\mu_{d''})$$

and

$$\circ^*(\pi_{\overline{d}}(\alpha) \wedge ev^*_{\overline{d}}(\mu_{\overline{d}})) = \pi^*_{d''}(\pi_{d'}(\alpha) \wedge ev^*_{d'}(\mu_{d'})) \wedge ev^*_{d''}(\mu_{d''}).$$

That is, the pullback of the $\mathbb{Q}$-polynomial form on one space is the $\mathbb{Q}$-form on the other.

Let $\rho : M(\circ) \to \pi^{-1}_{\overline{d}}(W_\alpha)$ be given by $(\Gamma'', s) \mapsto \circ(\Gamma'')$. Then the $\mathbb{Q}$-polynomial form on $M(\circ)$ is

$$\rho^*(\pi^*_{\overline{d}}(\alpha) \wedge ev^*_{\overline{d}}(\mu_{\overline{d}})).$$

The restriction of this form to either end of the mapping cylinder is the form already defined there:

$$\begin{aligned}\rho^*(\pi^*_{\overline{d}}(\alpha) \wedge ev^*_{\overline{d}}(\mu_{\overline{d}}))|_{\pi^{-1}_{d''}(ev^{-1}_{d'}(N_\mu))} &= \circ^*(\pi^*_{\overline{d}}(\alpha) \wedge ev^*_{\overline{d}}(\mu_{\overline{d}})) \\ &= \pi^*_{d''}(\pi^*_{d'}(\alpha) \wedge ev^*_{d'}(\mu_{d'})) \wedge ev_{d''}(\mu_{d''})\end{aligned}$$

and

$$\rho^*(\pi^*_{\overline{d}}(\alpha) \wedge ev^*_{\overline{d}}(\mu_{\overline{d}}))|_{\pi^{-1}_{\overline{d}}(W_\alpha)} = \pi^*_{\overline{d}}(\alpha) \wedge ev^*_{\overline{d}}(\mu_{\overline{d}})$$

The $\bigsqcup_\ell S^1$-bundle.

Recall there is a universal cyclically-ordered-graph bundle $E_{\overline{SD}}$ over $\pi^{-1}_{\overline{d}}(W_\alpha)$ such that the fiber over the point Γ is the cyclically-ordered graph Γ in $\overline{SD}(g,k,\ell) \cup \bigcup_{\mathfrak{E}}(S_{\mathfrak{E}} \cup SS_{\mathfrak{E}})$. Recall also that the fibers of the $\bigsqcup_\ell S^1$-bundle are given by restrictions to output circles of Γ. Denote the total space of this

$\bigsqcup_\ell S^1$-bundle by E^ℓ.

Let $\rho : M(\circ) \to \pi_{\overline{d}}^{-1}(W_\alpha)$ be the map above. The $\bigsqcup_\ell S^1$-bundle over $supp(\rho^*(\pi_{\overline{d}}^*(\alpha)) \wedge \tilde{\mu})$ is the pullback bundle $\rho^*(E^\ell)|_{supp(\rho^*(\pi_{\overline{d}}^*(\alpha))\wedge\tilde{\mu})}$.

Note that the restriction of $\rho^*(E^\ell)$ to $ev_{d''}^{-1}(N_\mu)$ agrees with the $\bigsqcup_\ell S^1$-bundle defined there by $ST_\mu(d'') * (ST_\mu(d')(B, \alpha))$.

The map to M.

The string topology construction for d defines a map $g : E^\ell \to M$ which is the restriction of $f_{\overline{SD}} : E_{\overline{SD}} \to M$ to output circles. Let $\overline{\rho} : \rho^*(E^\ell) \to E^\ell$ be the map of total spaces of bundles induced by $\rho : M(\circ) \to \pi_{\overline{d}}^{-1}(W_\alpha)$.

The map $G : \rho^*(E^\ell)|_{supp(\rho^*(\pi_{\overline{d}}^*(\alpha)\wedge ev_{\overline{d}}^*(\mu_{\overline{d}})))} \to M$ is given by $g \circ \overline{\rho}$ (where here $\circ$ means the usual composition of maps, not to be confused with

$$\circ : \pi_{d''}^{-1}(ev_{d'}^{-1}(N_\mu)) \to \pi_{\overline{d}}^{-1}(W_\alpha)$$

above).

Note that the restriction of G to $\circ^*(E^\ell)$ agrees with the map defined there by $ST_\mu(d'') * (ST_\mu(d')(B, \alpha))$.

Part 2(a) Summary.

Let

$$L_{d',d''}(B,\alpha) = [(M(\circ), \rho^*(\pi^*_{\overline{d}}(\alpha) \wedge ev^*_{\overline{d}}(\mu_{\overline{d}})), \rho^*(E^\ell), G)]$$
$$= (M(\circ), \rho^*(\pi^*_{\overline{d}}(\alpha) \wedge ev^*_{\overline{d}}(\mu_{\overline{d}}))).$$

Set $ST_\mu(\sum \tilde{d}_i)(B,\alpha) = L_{d',d''}(B,\alpha)$ and extend linearly to all of $P_*(kLM)$.

Part 2(b).

Here, we locate higher order split chains and inductively extend ST_μ as in part 2(a).

Given pairings ε and ε' of outputs with inputs, composition of string topology operations $ST_\mu(d''') *_{\varepsilon'} ST_\mu(d'') *_\varepsilon ST_\mu(d')$ is associative by definition.

A split generator of order m is a generator d of $C^k(\overline{LD}(g,k,\ell)/\!\!/ \sim)$ with associated sub-cell complex X_d and satisfies the following:

- For all Γ in the interior of a top-dimensional cell of X_d, the set of chords is partitioned into m subsets $E = E_1 \sqcup \cdots \sqcup E_m$ and the spacing parameters $s_1, s_2, \ldots, s_m = 1$.
- For all Γ in the interior of a top-dimensional cell of X_d, the subgraph Γ^1 consisting of chords in E_1, together with the input circles they touch, is a string diagram of type (g_1, k_1, ℓ_1).

- For all Γ in the interior of a top-dimensional cell of X_d, the subgraph Γ^2 consisting of chords in E_2, together with the input circles they touch and the n_1 output circles of Γ^1 that they touch (which are each rescaled to have length 1) is a string diagram of type (g_2, k_2, ℓ_2) up to a choice of distinguished vertices on the n_i output circles of Γ^1 which form n_1 of the input circles of Γ^2.

- Analogously, for $j \geq 2$, for all Γ in the interior of a top-dimensional cell of X_d, the subgraph Γ^j consisting of chords in E_j, together with the input circles they touch and the output circles of $\Gamma^1, \Gamma^2, \ldots, \Gamma^{j-1}$ that they touch (which are each rescaled to have length 1) is a string diagram of type (g_j, k_j, ℓ_j) up to a choice of distinguished vertices on the output circles of $\Gamma^1, \Gamma^2, \ldots, \Gamma^{j-1}$ which form input circles of Γ^j.

We say that $\Gamma^j \in X_{d^j}$ where d^j is a generator of $C_*^{T^{k_j}}(\overline{SD}(g_j, k_j, \ell_j))$. Therefore, d determines $d^1, d^2, \ldots, d^m$ and pairings ε_j of output circles of Γ^j with input circles of Γ^{j+1}. We write $split(d) = (d^1, \ldots, d^m, \varepsilon_1, \ldots, \varepsilon_{m-1})$.

Again, from $d^1, \ldots, d^m, \varepsilon_1, \ldots, \varepsilon_{m-1}$ we cannot reconstruct d. In general, several split generators d_i satisfy $split(d_i) = (d^1, \ldots, d^m, \varepsilon_1, \ldots, \varepsilon_{m-1})$. Again, a split chain of order m is $d = \sum d_i$ such that $split(d_i) = (d^1, \ldots, d^m, \varepsilon_1, \ldots, \varepsilon_{m-1})$. Denote the associated sub-cell complex by $X_{d^m *_{\varepsilon_{m-1}} \cdots *_{\varepsilon_1} d^1}$.

Let $\tilde{d}_i$ be the generator of $C_*^{T^k}(\overline{LD}(g, k, \ell) /\!\!/ \sim)$ that has $0 < s_j < 1$ for

$j = 1, 2, \ldots, m$ and such that the sub-cell complex X_{d_i} consists of codimension m faces of the sub-cell complex $X_{\tilde{d}_i}$.

Assume by induction that $ST_\mu(d))$ has been defined for order $m-1$ split chains d. We define $L_{d^1,\ldots,d^m} \in Hom(P_*(kLM), P_*(\ell LM))$ such that

$$ST_\mu(\partial(\sum \tilde{d}_i)) = D_{Hom} L_{d^1,\ldots,d^m}.$$

We will then set $ST_\mu(\sum \tilde{d}_i) = L_{d^1,\ldots,d^m}$.

We demonstrate the inductive step for $m = 3$.

The base space.

Let $ST_\mu(\sum d_i)(B, \alpha) = ST_\mu(d^3)(ST_\mu(d^2)(ST_\mu(d^1)(B, \alpha)))$. There is now a square I^2 of compositions $\circ$. See figure 4.6. Place the base space $\pi_{d^3}^{-1}(ev_{d^2}(N_{\mu_{d^2}}))$ of $ST_\mu(d^3)(ST_\mu(d^2)(ST_\mu(d^1)(B, \alpha)))$ at the vertex $(1, 1)$ and place the base space $\pi_{\overline{d}}^{-1}(W_\alpha)$ at the vertex $(0, 0)$.

The other two vertices correspond to two different splittings of cyclically-ordered graphs in $\pi_{\overline{d}}$ into two levels. The maps from the base spaces at $(0, 1)$ and $(1, 0)$ to the base space at $(0, 0)$ and the maps from the base space at $(1, 1)$ to the base spaces at $(0, 1)$ and $(1, 0)$ are each analogous to the map $\circ$ described in part 1(a). In particular, the diagram commutes: $\circ_3 \circ_1 = \circ_4 \circ_2$.

For the moment, we refer to the base space at a vertex just by its coordinates.

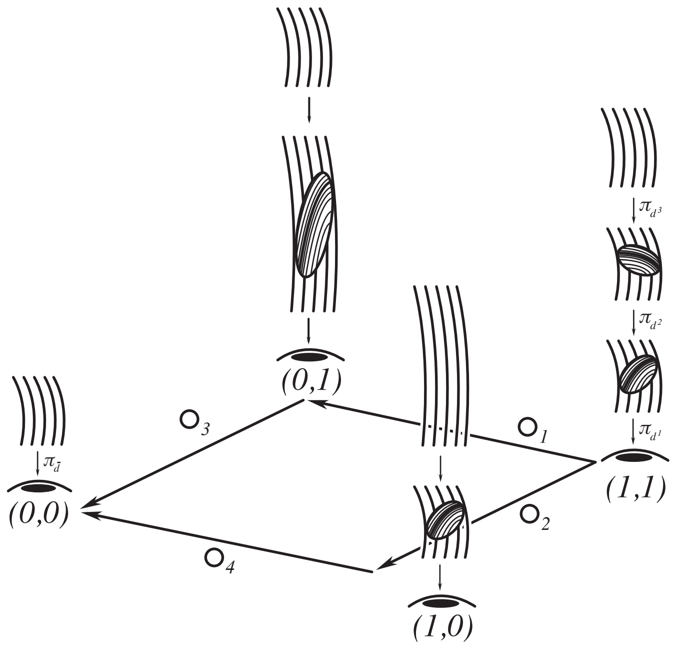


Figure 4.6: Cube of compositions

Let $M(\circ_1 \cup \circ_2) = M(\circ_1) \sqcup M(\circ_2)$ where $(\Gamma^3, 0) \in M(\circ_1) \sim (\Gamma^3, 0) \in M(\circ_2)$ be the union of mapping cylinders of $\circ_1$ and $\circ_2$ along their domains $(1, 1)$. Define

$$\bigcirc : M(\circ_1 \cup \circ_2) \to (0, 0)$$

by

$$\bigcirc(\Gamma^3, s) = \begin{cases} \circ_3(\circ_1(\Gamma^3)) = \circ_4(\circ_2(\Gamma^3)) & \text{if } 0 \leq s < 1 \\ \circ_3(\Gamma^3_1) & \text{if } \Gamma^3_1 \in (0, 1) \\ \circ_4(\Gamma^3_2) & \text{if } \Gamma^3_2 \in (1, 0) \end{cases}$$

See figure 4.7.

Then the base space for $L_{d^1,d^2,d^3}(B, \alpha)$ is the mapping cylinder $M(\bigcirc)$.

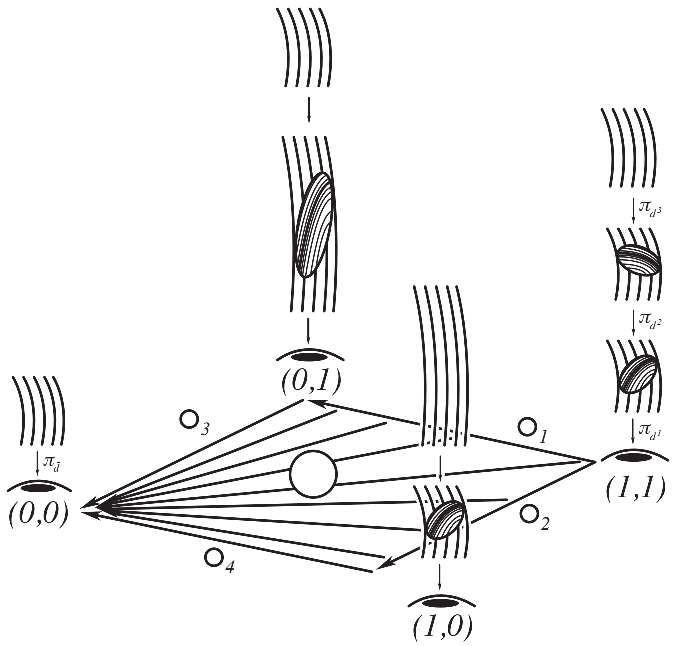


Figure 4.7: The map $\bigcirc$

The $\mathbb{Q}$-polynomial form, $\bigsqcup_\ell S^1$-bundle and map to M.

Let $\rho : M(\bigcirc) \to \pi_{\bar{d}}^{-1}(W_\alpha)$ be the map $(\Gamma, s) \mapsto \bigcirc(\Gamma)$, analogous to the map ρ in part 1(a). Then as in part 1(a), the $\mathbb{Q}$-polynomial form and the $\bigsqcup_\ell S^1$-bundle over its support are the pull-backs of those over $\pi_{\bar{d}}^{-1}(W_\alpha)$. The map to M factors through $\pi_{\bar{d}}^{-1}(W_\alpha)$ as in part 1(a).

In general, there is a cube I^m of compositions and we extend the definition of $L_{d^1,\ldots,d^m}$ inductively. At each stage, the map $\bigcirc$ goes from faces of the cube that have $(1, 1, \ldots, 1)$ as a vertex to $(0, 0, \ldots 0)$.

Set $ST_\mu(\sum \tilde{d}_i)(B, \alpha) = L_{d^1,\ldots,d^m}(B, \alpha)$ and extend linearly to all of $P_*(kLM)$.

Before moving to part 3 we remark that a generator c of $C_*^{T^k}(\overline{LD}(g,k,\ell))$ may be a term in more than one chain $\tilde{d}$ of $\overline{LD}(g,k,\ell)/\!\!/\sim$. However, if X_c has codimension 0, then it is a term in exactly one chain $\tilde{d}$. The same is true of all codimension 0 cells adjoined to $\overline{LD}(g,k,\ell)$ corresponding to slide cells and multi-slide cells in $P_{\mathfrak{F}}$ where f is a codimension 1 face of a cell $e \in X_c$ and f is obtained from e where chord endpoints coincide on an input circle.

Part 3(a).

Let $T_{c,c'',c'''}$ be a triangle cell of the slide complex of slide-equivalent generators c of $C_*^{T^k}(\overline{SD}(g,k,\ell))$. Assume for the moment that for all string diagrams Γ in the sub-cell complexes X_c, X_c' and X_c'' of $\overline{SD}(g,k,\ell)$ the subgraph consisting of chords has Euler characteristic $\chi > 0$.

The extension of the string topology construction to generators of $C_*^{T^k}(\overline{LD}(g,k,\ell))$ corresponding to triangle cells in $T_{\mathfrak{C}}$ generalizes to all remaining generators.

Let $a_{c,c',c''} = X_{\mathfrak{C}} \times T_{c,c',c''}$ be the complex of cells corresponding to the *interior* triangle $T_{c,c',c''}$.

For a generator (B,α) of $P_*(kLM)$, we choose a representative (B,α,E^k,f) and construct each component of a quadruple representing $ST_\mu(a_{c,c',c''})(B,\alpha)$.

The base space.

The base space for $ST_\mu(a_{c,c',c''})(B,\alpha)$ is

$$\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times T_{c,c',c''}.$$

The $\mathbb{Q}$-polynomial form.

For convenience, we choose an ordering of the endpoints of chords that coincide labeled as in figure 4.8. The evaluation map

$$EV : \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times T_{c,c',c''} \to M^3$$

is defined similarly as ev in the string topology construction:

$$EV(\Gamma) = (f(p), f(q), f(r)).$$

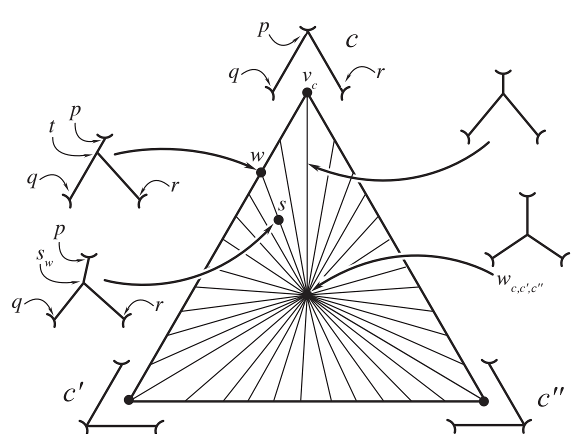


Figure 4.8: Triangle

Let $\mu_\Delta \in \Omega^{2d}(M^3)$ be a form representing the Thom class of the diagonal $\Delta : M \to M \times M \times M, x \mapsto (x, x, x)$ which is supported in an $\frac{\varepsilon}{2}$-neighborhood N_{μ_Δ} of the diagonal.

Let $\rho : \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times T_{c,c',c''} \to \pi_{\mathfrak{C}}^{-1}(W_\alpha)$ be the projection map.

The $\mathbb{Q}$-polynomial form on $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times T_{c,c',c''}$ is

$$\rho^*(\pi_{\mathfrak{C}}^*(\alpha)) \wedge EV^*(\mu_\Delta).$$

The $\bigsqcup_\ell S^1$-bundle.

As usual, there is a universal cyclically-ordered graph bundle over $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times T_{c,c',c''}$ with total space $E_{\overline{SD}}$. Restricting to output circles gives a $\bigsqcup_\ell S^1$-bundle with total space E^ℓ.

The map to M.

If $\Gamma \in EV^{-1}(N_{\mu_\Delta})$ then $f(p), f(q)$ and $f(r)$ all lie in an ε ball in M. We use this to map the total space of the universal cyclically-ordered graph bundle over $EV^{-1}(N_{\mu_\Delta})$ analogously to the map in the string topology construction. We examine four cases separately.

1. $\Gamma \in \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times v_c$;

2. $\Gamma \in \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c,c'}$ where $s_{c,c'}$ corresponds to the face of $T_{c,c',c''}$ which is

the slide cell joining v_c to v_c';

3. $\Gamma \in \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times w_{c,c',c''}$ where $w_{c,c',c''}$ is the barycenter of the triangle $T_{c,c',c''}$; and

4. all other Γ.

Recall that $\pi_{\mathfrak{C}}^{-1}$ describes the configuration of points p, q, r on input circles while $T_{c,c',c''}$ describes how they are connected.

If $\Gamma \in \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times v_c$ then $f_{\overline{SD}}$ maps the chords to geodesic segments exactly as in $ST_\mu(c)(B, \alpha)$, the usual string topology construction.

If $\Gamma \in \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c,c'}$ then $f_{\overline{SD}}$ maps the chords to geodesic segments exactly as in $ST_\mu(b_{c,c'})(B, \alpha)$, the string topology construction extended as in part 1(a).

If $\Gamma \in \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times w_{c,c',c''}$, then $f_{\overline{SD}}$ maps $w_{c,c',c''}$ to the barycenter of the triangle in M with vertices $f(p), f(q)$ and $f(r)$ and the edges of the tripod to the geodesic segments joining $f(p), f(q)$ and $f(r)$ to $f_{\overline{SD}}(w_{c,c',c''})$.

We have defined the map $f_{\overline{SD}}$ over

$$EV^{-1}(N_{\mu_\Delta}) \cap (\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times \partial(T_{c,c',c''})) \cup EV^{-1}(N_{\mu_\Delta}) \cap (\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times w_{c,c',c''}).$$

Any other Γ has its $T_{c,c',c''}$ component on a line segment $I_w \subset T_{c,c',c''}$ joining a point on w the boundary to the barycenter. The line segment I_w is identified with the interval $[0,1]$ such that $0 \sim w$ and $1 \sim w_{c,c',c''}$.

Fix a point $w \in \partial(T_{c,c',c''})$. Let $\Gamma_0 \in EV^{-1}(N_{\mu_\Delta})$ be a cyclically-ordered graph with $T_{c,c',c''}$ component w. Let $\Gamma_1 \in EV^{-1}(N_{\mu_\Delta})$ with $T_{c,c',c''}$ component $w_{c,c',c''}$ such that $\rho(\Gamma_0) = \rho(\Gamma_1)$. Then if $s \in [0,1]$, let Γ_s be the point corresponding to $s \in I_w$. Let s_w be the central vertex of the tripod in Γ_s. We describe $f_{\overline{SD}}(s_w)$. Let $t \in \Gamma_0$ be the endpoint of the chord ch which in the interior of the chord $\overline{ch}$ as in figure 4.8 and let $f_{\overline{SD}}(t)$ be its image in M. Let $\gamma : [0,1] \to M$ be path whose image is the geodesic segment such that $\gamma(0) = f_{\overline{SD}}(t)$ and $\gamma(1)$ is the barycenter of the triangle with vertices $f(p), f(q)$ and $f(r)$ and that is parametrized by arc length. Then $f_{\overline{SD}}(s)_w = \gamma(s)$. The edges of Γ_s joining p, q and r to s are mapped by $f_{\overline{SD}}$ to geodesic segments joining $f(p), f(q)$ and $f(r)$ to $f_{\overline{SD}}(s)$.

Now we have defined $f_{\overline{SD}}$ over all $EV^{-1}(N_{\mu_\Delta})$ the map

$$g : E^\ell|_{supp(\rho^*(\pi^*_{\mathfrak{C}}(\alpha)) \wedge EV^*(\mu_\Delta))} \to M$$

is again the restriction of $f_{\overline{SD}}$ to output circles.

However, the definition of $ST_\mu(a_{c,c',c''})(B,\alpha)$ is not yet complete. In order for this extension of ST_μ to commute with differentials, we must modify the definition above. The issue is that the $\mathbb{Q}$-polynomial forms on $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c,c'}$ defined in part 1(a) and in part 3(a), where $s_{c,c'}$ is a slide cell which is a face of the trianlge $T_{c,c'',c'''}$, need not agree. However, they are cohomologous so we may again apply lemma 3.7.

Let $T_{c,c'',c'''}$ be a triangle cell of the slide complex of slide-equivalent generators c of $C_*^{T^k}(\overline{LD}(g,k,\ell))$. Adjoin $\partial(T_{c,c'',c'''}) \times I$ to $T_{c,c'',c'''}$ by identifying $\partial(T_{c,c'',c'''}) \times \{0\}$ to $\partial(T_{c,c'',c'''})$. We will refer to the complex of new cells as the collar. By abuse of notation, we call this new space $T_{c,c'',c'''}$ as well. Points in this modified $T_{c,c'',c'''}$ still correspond to cyclically-ordered graphs Γ. If Γ is in $\partial(T_{c,c'',c'''})$, then $(\Gamma,t) = \Gamma$ for $(\Gamma,t) \in \partial(T_{c,c'',c'''}) \times I$. With $T_{c,c'',c'''}$ representing the new triangles, the adjunction of $X_{\mathfrak{C}} \times T_{c,c'',c'''}$ to $\overline{LD}(g,k,\ell)$ is unchanged in the construction of $\overline{LD}(g,k,\ell)/\!\!/\sim$.

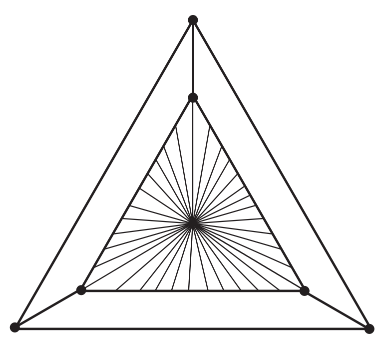


Figure 4.9: Triangle and collar

Assume $\Gamma \in EV^{-1}(N_{\mu_\Delta}) \cap \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c,c'} \subset \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times T_{c,c',c''}$, so that $f(p), f(q)$ and $f(r)$ all lie in an ε-ball in M. Then construction of part 1(a) is defined and the construction of part 3(a) restricts to the face $s_{c,c'} \subset T_{c,c',c''}$. In both cases, the base space is $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c,c'}$. The cyclically-ordered graphs $\Gamma \in \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c,c'}$ are the same in both cases and so their $\bigsqcup_\ell S^1$ bundles are also the same. Where they are both defined, the maps from the total spaces of these bundles to M also agree, by definition.

However, the closed $\mathbb{Q}$-polynomial forms defined in the different parts need not agree. They are, however, cohomologous and we again apply lemma 3.7 to extend the form over the new cells.

First recall the evaluation maps

$$ev : \pi_c^{-1}(W_\alpha) \to M^4, ev(\Gamma) = (f(p), f(q), f(r), f(p))$$

used in the string topology construction for $ST_\mu(c)(B,\alpha)$ and

$$EV : \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times T_{c,c',c''} \to M^3, EV(\Gamma) = (f(p), f(q), f(r))$$

used in the extension to $ST_\mu(a_{c,c',c''})(B,\alpha)$. In particular EV restricts to $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times v_c \sim \pi_c^{-1}(W_\alpha)$ and we compare forms here.

The map ev factors through M^3. Let $i : M^3 \to M^4$ be defined by

$$(x, y, z) \mapsto (x, y, z, x).$$

Let $ch_1 : M^4 \to M^2$ be defined by $(x, y, z, w) \mapsto (x, y)$ and $ch_2 : M^4 \to M^2$ be defined by $(x, y, z, w) \mapsto (z, w)$. Then recall the form $\mu_c = ch_1^*(\mu) \wedge ch_2^*(\mu)$. The form $i^*(\mu_c)$ represents the Thom class of the diagonal $\Delta : M \to M^3$ and we have that $ev^*(\mu_c)$ and $EV^*(\mu_\Delta)$ are cohomologous forms on $\pi_c^{-1}(W_\alpha)$. Let $\tilde{\mu}$ be the form guaranteed by lemma 3.7 on $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times I$.

Let $\rho : \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times I \to \pi_{\mathfrak{C}}^{-1}(W_\alpha)$ be the projection map. Then $\rho^*(\pi_{\mathfrak{C}}(\alpha)) \wedge \tilde{\mu}$ restricts to the forms on $\pi_c^{-1}(W_\alpha)$ and $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times v_c$ in defined for $ST_\mu(c)(B, \alpha)$ and $ST_\mu(a_{c,c',c''})(B, \alpha)$ respectively.

On $EV^{-1}(N_{\mu_\Delta}) \cap \pi_C^{-1}(W_\alpha) \times s_{c,c'}$, the maps $ev_1, ev_2 : \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c,c'} \to M^2$ are both defined. Let $\tilde{ev}(\Gamma) = (ev_1(\Gamma), ev_2(\Gamma)) \in M^4$. The map $\tilde{ev}$ need not factor through M^3 but it is homotopic to one that does, namely ev or ev'. Therefore, $ev_1^*(\mu) \wedge ev_2^*(\mu)$ and $EV^*(\mu_\Delta)$ are cohomologous on $EV^{-1}(N_{\mu_\Delta})$ and they extend by 0 to $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c,c'}$. Let

$$\rho_I : \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c,c'} \times I \to \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c,c'}$$

and

$$\rho : \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c,c'} \to \pi_{\mathfrak{C}}^{-1}(W_\alpha)$$

be projection maps. Then the $\mathbb{Q}$ polynomial form on $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times s_{c,c'} \times I$ corresponding to the cell $s_{c,c'} \times I$ in the collar is

$$\rho_I^*(\rho^*(\pi_{\mathfrak{C}}^*(\alpha))) \wedge \tilde{\mu}.$$

The operation $ST_\mu(a_{c,c',c''})$ generalizes to generators c of $C_*^{T^k}(\overline{SD}(g, k, \ell) /\!\!/ \sim)$

where more than two chord endpoints coincide on an input circle if the sub-graph of chords has positive Euler characteristic χ.

If the generator c of $C_*^{T^k}(\overline{SD}(g,k,\ell)/\!\!/ \sim)$ is labeled by a combinatorial type such that the sub-graph obtained by deleting input circle edges and isolated distinguished vertices has many components, then the appropriate operation is applied to each component. In X_c there is a face corresponding to endpoints coming together on input circles such that disjoint components are now connected, then a collar is added to such cells and the construction of part 3(a) using lemma 3.7 is applied provided the subgraphs consisting of chords continue to have $\chi > 0$.

Part 3(b).

Let c be a generator of $C_*^{T^k}(\overline{SD}(g,k,\ell))$ such that the slide complex $P_{\mathfrak{c}}$ has a triangle $T_{c,c,c''}$ and such that a face of the sub-cell complexes $X_c, X_{c'}$ and $X_{c''}$ contains string diagrams Γ whose sub-graph consisting of chords has Euler characteristic $\chi \leq 0$. Then the evaluation map EV is not consistent on all of $\pi_{\mathfrak{c}}^{-1}(W_\alpha) \times T_{c,c',c''}$; the number of vertices on input circles in a component of the sub-graph of chords and corollae is not constant so we proceed as follows:

If the subgraph of chords labeling a face has $\chi = 0$, then there is either an output circle consisting entirely of chords or there is a representative diagram with sub-configuration of chords ch and ch' such that endpoints coincide at either end, but the cyclic order does does not produce an output consisting entirely of ch and ch'. The evaluation map EV may be extended to such cells in both cases. See figure 4.10.

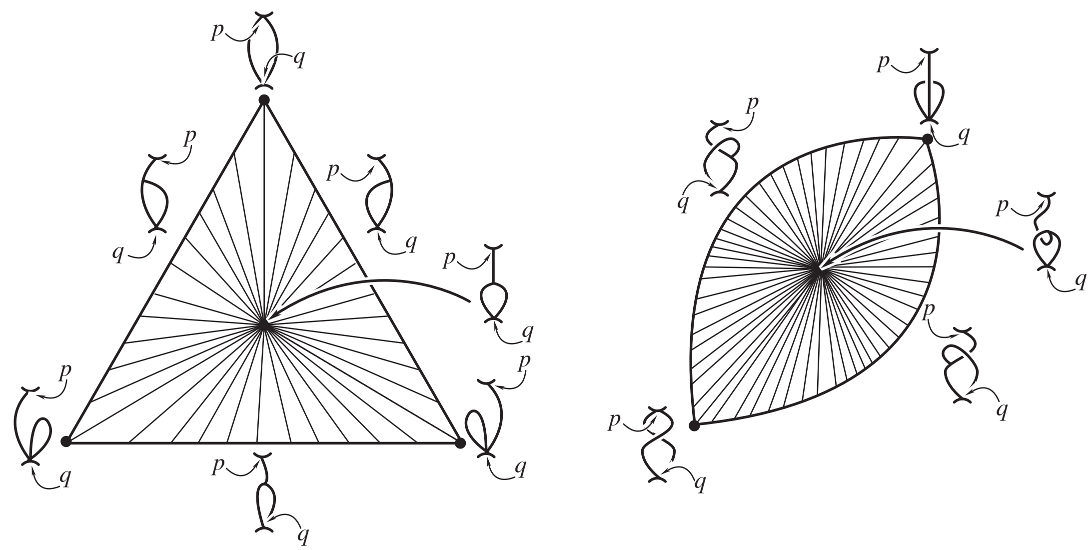


Figure 4.10: Triangles for $\chi = 0$

In both of these cases, there are only 2 vertices on input circles but we define $EV : \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times T_{c,c',c''} \to M^3$ (which factors through M^2) by

$$EV(\Gamma) = (f(p), f(q), f(q)).$$

In both cases, the map $M^2 \to M^3$ is unambiguous. Otherwise, the construction of $ST_\mu(a_{c,c',c''})$ proceeds as in the positive Euler characteristic case above.

There is an analogous construction if $\chi < 0$. Notice that in this case there is a representative with a subgraph is in figure 4.11. Here, the evaluation map

$\pi_c^{-1}(W_\alpha) \to M^6$ factors through M^2, whose dimension is $2d$. However, the $\mathbb{Q}$-polynomial form μ_c that is pulled back to $\pi_c^{-1}(W_\alpha)$ via ev^* has degree $3d$. Since ev factors through M^2, $ev^*(\mu_c)$ is 0. This turns out to be true for all representatives and also so for any cell in $P_{\mathfrak{C}}$, the corresponding string topology operation is 0.

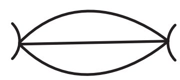


Figure 4.11: $\chi < 0$

Part 3(c).

For a slide-equivalence class $\mathfrak{C}$ of generators c of $C_*^{T^k}(\overline{LD}(g,k,\ell))$ such that the slide complex $P_{\mathfrak{C}}$ contains triangle cells $T_{\mathfrak{C}}$ and combination cells $SST_{\mathfrak{C}}$, the extension to cells with a spacing parameter $s > 0$ is analogous to the extensions in part 2(a) and 2(b).

At each stage, the extension of ST_μ is well-defined on equivalence classes because $ST_\mu : \overline{SD}(g,k,\ell) \to Hom(P_*(kLM), P_*(\ell LM))$ is well-defined.

Let $\tilde{d}_i \in C^{T^k}_{6g+3k+2\ell-7}(\overline{LD}(g,k,\ell)/\!\!/ \sim)$ correspond to the split chain d_i of order $2g-2+k+\ell-1$ (as in part 2) and let a_j be a generator of $C^{T^k}_{6g+3k+2\ell-7}(\overline{LD}(g,k,\ell)/\!\!/ \sim)$ corresponding to a cell in $T_{\mathfrak{C}} \cup SST_{\mathfrak{C}} \subset P_{\mathfrak{C}}$ for some generator c of $C_*^{T^k}(\overline{LD}(g,k,\ell))$. Then the fundamental chain $[\overline{LD}(g,k,\ell)/\!\!/ \sim]$ of $\overline{LD}(g,k,\ell)/\!\!/ \sim$ is $\sum_i \tilde{d}_i + \sum_j a_j$.

Recall that base spaces for $ST_\mu(\tilde{d}_i)$ and $ST_\mu(a_j)$ are each cartesian products if $\pi_{\mathfrak{E}}^{-1}(W_\alpha)$ with some cell P. Because $\overline{LD}(g,k,\ell)/\!\!/\sim$ is a pseudomanifold with boundary and because the canonical map $\iota : P_{\mathfrak{E}} \to P_{\mathfrak{F}}$ exists for a face f of a cell e, the base spaces fit together for all $\tilde{d}_i$ and a_j so that $ST_\mu([\overline{LD}(g,k,\ell)/\!\!/\sim])$ has base space $\pi_{\overline{LD}}^{-1}(W_\alpha)$ where $\pi_{\overline{LD}} : E_{\overline{LD}} \to W_\alpha$ is a fiber bundle with fiber $\overline{LD}(g,k,\ell)/\!\!/\sim$. At each stage of the extension, the $\mathbb{Q}$-polynomial forms, the $\bigsqcup_\ell S^1$-bundle and map of its total space to M agree on codimension 1 faces agree. Therefore, the extension ST_μ of the string topology construction to $[\overline{LD}(g,k,\ell)/\!\!/\sim]$ satisfies

$$ST_\mu(\partial[\overline{LD}(g,k,\ell)/\!\!/\sim]) = D_{Hom}(ST_\mu([\overline{LD}(g,k,\ell)/\!\!/\sim])).$$

□

Corollary 4.2. *Let $[\overline{LD}(g',k',\ell')/\!\!/\sim]$ and $[\overline{LD}(g'',k'',\ell'')/\!\!/\sim]$ be fundamental chains of $\overline{LD}(g',k',\ell')/\!\!/\sim$ and $\overline{LD}(g'',k'',\ell'')/\!\!/\sim$ respectively. Let ϵ be a pairing of N outputs of $ST_\mu([\overline{LD}(g',k',\ell')/\!\!/\sim])$ with N inputs of $ST_\mu([\overline{LD}(g'',k'',\ell'')/\!\!/\sim])$ so that $g'+g''+N-1=g, k'+k''-N=k$ and $\ell'+\ell''-N=\ell$ so that their composition according to the pairing ε is $*_\varepsilon$. Then $ST_\mu([\overline{LD}(g'',k'',\ell'')/\!\!/\sim]) *_\varepsilon ST_\mu([\overline{LD}(g',k',\ell')/\!\!/\sim])$ is in $Hom(P_*(kLM), P_*(\ell LM))$ and corresponds to a term in the expression for $ST_\mu(\partial([\overline{LD}(g,k,\ell)/\!\!/\sim]))$.*

Proof. Recall that the construction as in part 2 of the proof of proposition 4.3, defines string topology operations for cells in $\overline{LD}(g,k,\ell)/\!\!/\sim$ where $s_i = 1$ as compositions of string topology operations Recall also that such cells are at the boundary of $\overline{LD}(g,k,\ell)/\!\!/\sim$ As a direct consequence,

$$ST_\mu([\overline{LD}(g'',k'',\ell'')/\!\!/\sim]) *_\varepsilon ST_\mu([\overline{LD}(g',k',\ell')/\!\!/\sim])$$

is a term in the expression for $ST_\mu(\partial([\overline{LD}(g,k,\ell)/\!\!/\sim]))$ corresponding to a chain in $C_*^{T^k}(\overline{LD}(g,k,\ell)/\!\!/\sim)$ where one spacing parameter $s_i = 1$ and
$0 < s_j < 1$ for $j \neq i$.

□

4.3 Solution to the equation with anomaly

Lemma 4.3. *Let $kM \subset kLM$ be the subspace of maps $\bigsqcup_k S^1 \to M$ such that the restriction to one S^1 component has image in M whose image lies in a ball of radius ε. The map*

$$ST_\mu(c) : P_*(kLM) \to P_*(\ell LM)$$

induces a map of relative chains

$$ST_\mu(c) : P_*(kLM, kM) \to P_*(\ell LM, \ell M)$$

where $P_*(kLM, kM) = P_*(kLM)/P_*(kM)$.

Proof. The space $1M \subset LM$ consisting of loops in M lying in a ball of radius ε (small loops) has the homotopy type of M, the space of constant loops. Let (B, α, E^k, f) represent a generator (B, α) of $P_*(kM)$ and let $c \in C_*^{T^k}(\overline{SD}(g, k, \ell))$ be a generator with associated sub-cell complex X_c. We describe $ST_\mu(c)(B, \alpha)$ in this setting.

We may replace the fiber $\bigsqcup_k S^1$ of E^k by a disjoint union of circles and points; one point for each component whose image in M is small. Denote the total space of this fiber bundle by $E^k_\circ$.

The k-torus T^k still acts on the fiber of $E^k_\circ$ and we form the principal T^k-bundle and associated X_c-bundle as before. Recall that the associated bundle construction effectively identifies the input circles of all $\Gamma \in X_c$ with the fiber of $E^k_\circ$ in all possible ways. Now, because S^1 acts trivially on components which are points, the fiber is no longer truly the sub-cell complex X_c, but a quotient of such. Recall that each cell e of $\overline{SD}(g, k, \ell)$ is a cartesian product of k simplices, one simplex for each input circle. Here, because some input circles have been replaced by points, the cell e has the corresponding simplex factors collapsed to points. The same simplex factors are crushed for all cells e with nonzero coefficient in the expression for c. Therefore, the fiber of the associated X_c bundle is really a quotient of X_c. In particular, the new fiber has dimension $< |c|$.

The rest of the string topology construction is essentially unchanged, but the degree of $ST_\mu(c) < |c| - (2g - 2 + k + \ell)d$. That is, the degree of $ST_\mu(c)$ is too low and we declare its output to be the zero chain in $P_*(\ell LM)$.

Therefore, there is an induced string topology operaton on relative chains

$$ST_\mu(c) : P_*(kLM, kM) \to P_*(\ell LM, \ell M).$$

The induced string topology construction on relative chains extends to $\overline{LD}(g, k, \ell)/\!\!/ \sim$ as before.

□

Theorem 4.1. *There exist $X(g, k, \ell) \in Hom(P_*(kLM, kM), P_*(\ell LM, \ell M))$ such that*

$$D_{Hom}(X(g, k, \ell)) = \sum_\varepsilon X(g'', k'', \ell'') *_\varepsilon X(g', k', \ell') + A(g, k, \ell)$$

where ε is any pairing of N outputs of $X(g', k', \ell')$ with N inputs of $X(g'', k'', \ell'')$ such that $g = g' + g'' + N - 1, k = k' + k'' - N, \ell = \ell' + \ell'' - N$ and $*_\varepsilon$ is composition of operations according to the pairing ε.

Proof. By prop 4.1, $\partial[\overline{LD}(g, k, \ell)/\!\!/ \sim]$ has two types of terms: cells labeled by diagrams where one spacing parameter $s_i = 1$ and cells labeled by diagrams where a connected component of the subgraph consisting of the complement of the input circle edges has Euler characteristic $\chi = 0$. Here we separate the case where $\chi = 0$ into two situations: one which leads to an output consisting of chords and one which does not.

If a generator c of $C_*^{T^k}(\overline{LD}(g,k,\ell)/\!\!/ \sim)$ corresponds to diagrams that each have a small output then the image of $ST_\mu(c)$ is contained in $P_*(kM)$ and the map induced by $ST_\mu(c)$ on relative chains $P_*(kLM, kM)$ is identically zero.

Therefore, the only terms c of $\partial([\overline{LD}(g,k,\ell]/\!\!/ \sim)$ that potentially act nontrivially on relative chains are those with a spacing parameter $s_i = 1$ or with $\chi = 0$ not leading to an output of chords.

We write $\partial([\overline{LD}(g,k,\ell)/\!\!/ \sim] = \sum_i \alpha_i + \sum_j \beta_j + \sum_k \gamma_k$ where α_i correspond to $\chi = 0$ codimension 1 cells not giving an output of chords, β_j correspond to $s = 1$ cells and γ_k correspond to $\chi = 0$ cells giving an output of chords.

If

$$\begin{aligned} X(g,k,\ell) &= ST_\mu([\overline{LD}(g,k,\ell)/\!\!/ \sim]), \\ \sum X(g'',k'',\ell'') *_\varepsilon X(g',k',\ell') &= ST_\mu(\sum_j \beta_j) \text{ and} \\ A(g,k,\ell) &= ST_\mu(\sum_i \alpha_i). \end{aligned}$$

then

$$D_{Hom}(X(g,k,\ell)) = \sum X(g'',k'',\ell'') *_\varepsilon X(g',k',\ell') + A(g,k,\ell).$$

□

Let $X = \sum_{g,k,\ell} X(g,k,\ell)$ and $A = \sum_{g,k,\ell} A(g,k,\ell)$ in $\oplus_{k,\ell} Hom(P_*(kLM), P_*(\ell LM))$. Then

$$D_{Hom}(X) = X * X + A$$

where $*$ is a sum of all possible compositions.

Appendix A

Conjectured solution to the master equation

We expect that the solution X of the equation with anomaly A in the equation $D_{Hom}(X) = X * X + A$ may be modified so that the modified X satisfies the master equation $D_{Hom}(X) = X * X$.

A.1 Idea of proof

The proof would require a careful examination of the codimension 1 boundary of $\overline{LD}(g, k, \ell) /\!\!/ \sim$ corresponding to cells where there is subgraph of chords has

Euler characteristic $\chi = 0$. Recall the subcomplex $Q_{\mathfrak{C}}$ of the slide complex $P_{\mathfrak{C}}$ is a pseudomanifold with boundary. If $Q_{\mathfrak{C}}$ were closed, then adjoining $X_{\mathfrak{C}} \times cone(Q_{\mathfrak{C}})$ to $\overline{LD}(g,k,\ell)$ along $X_{\mathfrak{C}}$ would kill this boundary.

We could then extend the string topology construction over such cells by proceeding as follows:

If $\chi = 0$, then there is codimension 1 boundary created by $X_{\mathfrak{C}} \times Q_{\mathfrak{C}}$ if $Q_{\mathfrak{C}}$ is not contractible; $X_{\mathfrak{C}} \times Q_{\mathfrak{C}} \subset \partial(\overline{LD}(g,k,\ell) /\!\!/ \sim)$. Denote the corresponding chain in $C_*^{T^k}(\overline{LD}(g,k,\ell) /\!\!/ \sim)$ by a.

The operation $ST_\mu(a) \in Hom(P_*(kLM), P_*(\ell LM))$ is already defined. Base spaces, $\mathbb{Q}$-polynomial forms, $\bigsqcup_\ell S^1$-bundles and maps to M all agree along faces so $ST_\mu(a)(B,\alpha)$ has base space $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times Q_{\mathfrak{C}}$, $\mathbb{Q}$-polynomial form $\rho^*(\pi_{\mathfrak{C}}^*(\alpha)) \wedge \bar{\mu}$ (where $\bar{\mu}$ restricts to the appropriate closed form on each cell), $\bigsqcup_\ell S^1$-bundle with total space E^ℓ and map to M $g : E^\ell \to M$. Recall that g is constructed by restricting $f_{\overline{SD}} : E_{\overline{SD}} \to M$ to output circles, where $E_{\overline{SD}}$ is a universal cyclically-ordered graph bundle and $f_{\overline{SD}}$ is the map defined by the string topology construction which maps chords to short geodesic segments.

We construct an operation $Y_a(g,k,\ell) \in Hom(P_*(kLM), P_*(\ell LM))$ such that $D_{Hom}(Y_a(g,k,\ell)) = ST_\mu(a)$. We construct each component of $Y_a(g,k,\ell)(B,\alpha)$. The definition is very similar to that for the extension of ST_μ to cells corresponding to triangles $T_{\mathfrak{C}}$ and generalized deformations in $SST_{\mathfrak{C}}$.

The base space.

The base space for $Y_a(g,k,\ell)(B,\alpha)$ is $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times cone(Q_{\mathfrak{C}})$, where $cone(Q_{\mathfrak{C}})$ is the cone on the sub-complex $Q_{\mathfrak{C}}$ of $P_{\mathfrak{C}}$.

The $\mathbb{Q}$-polynomial form.

The $\mathbb{Q}$-polynomial form on $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times cone(Q_{\mathfrak{C}})$ is constructed in three steps.

Fix $a \in W_\alpha$ and $p \in \pi_{\mathfrak{C}}^{-1}(a)$. Then for different cells of $Q_{\mathfrak{C}}$, the evaluation map may be defined differently, but for all $\Gamma \in \{p\} \times Q_{\mathfrak{C}}$, the components consisting only of input circles and the vertices on input circles are the same. In particular, $f_{\overline{SD}}$ restricted to input circles is constant. (The difference comes in the configuration of chords.)

Let $p_1, p_2, \ldots p_N$ denote an ordering of the vertices on input circles of Γ corresponding to the component of the complement of Γ's input circle edges with $\chi = 0$. The first step is the map $ve : \pi_{\mathfrak{C}}^{-1} \times cone(Q_{\mathfrak{C}}) \to M^N$. Let

$$ve(\Gamma) = (f(p_1), f(p_2), \ldots f(p_N)).$$

Indeed, ve is an evaluation map, but we are not pulling back a Thom class representative via this map. Let N_Δ be an ε-neighborhood of the image of the diagonal $\Delta : M \to M^N$.

Given $\Gamma \in ve^{-1}(N_\Delta)$, the points $f(p_1), f(p_2), \ldots f(p_N)$ all lie in some ε-ball in M. Let t be the barycenter of the (possibly degenerate) simplex in the ε-ball with vertices $f(p_1), f(p_2), \ldots f(p_N)$.

The second step is the definition of the evaluation map

$$ev : ve^{-1}(N_\Delta) \to M^{N+1},$$

which is given by

$$ev(\Gamma) = (f(p_1), f(p_2), \ldots f(p_N), t).$$

Let μ be the representative of the Thom class of the diagonal $M \to M^{N+1}$ which is supported in an ε-neighborhood of the diagonal.

As usual, let $\rho : \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times cone(Q_{\mathfrak{C}}) \to \pi_{\mathfrak{C}}^{-1}(W_\alpha)$ be the projection map.

Consider the form $\rho^*(\pi_{\mathfrak{C}}^*(\alpha)) \wedge ev^*(\mu)$. This is almost the form we want, though, it may not restrict to $\rho^*(\pi_{\mathfrak{C}}^*(\alpha)) \wedge \bar{\mu}$ on $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times Q_{\mathfrak{C}}$. However, $\rho^*(\pi_{\mathfrak{C}}^*(\alpha)) \wedge ev^*(\mu)$, restricted and $\rho^*(\pi_{\mathfrak{C}}^*(\alpha)) \wedge \bar{\mu}$ are cohomologous.

The third step is the construction of a collar on $Q_{\mathfrak{C}}$ so that lemma 3.7 may be applied as in part 3(a) of the proof of proposition 4.3. This gives a $\mathbb{Q}$-polynomial form on $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times cone(Q_{\mathfrak{C}})$ which restricts to the one already defined on $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times Q_{\mathfrak{C}}$.

The $\bigsqcup_\ell S^1$-bundle and map to M.

Fix $a \in W_\alpha$ and $p \in \pi_{\mathfrak{C}}^{-1}(a)$ If

$$\Gamma \in (\{p\} \times Q_{\mathfrak{C}}) \cap ve^{-1}(N_\Delta) \subset \pi_{\mathfrak{C}}^{-1}(W_\alpha) \times cone(Q_{\mathfrak{C}})$$

then $f_{\overline{SD}}$ restricted to input circles is constant. But for any such Γ the images of all chords under $f_{\overline{SD}}$ lie in a single ε-ball in M. By assumption, ε-balls are contractible. Therefore, for any such Γ, $g|_\Gamma$ is homotopic, any two homotopies are homotopic and so on. The cells of $Q_{\mathfrak{C}}$ describe $6g-8+3k+3\ell$-fold homotopies. We fix a $6g-8+3k+3\ell$-fold homotopy H_p. We do this consistently for all $p \in \pi_{\mathfrak{C}}^{-1}(a)$ and for all $a \in W_\alpha$. This gives an extension of the $\bigsqcup_\ell S^1$-bundle over $(\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times Q_{\mathfrak{C}}) \cap ve^{-1}(N_\Delta)$ to a $\bigsqcup_\ell S^1$-bundle over all of $ve^{-1}(N_\Delta)$ as well as a map H from its total space to M which restricts to g over $\pi_{\mathfrak{C}}^{-1}(W_\alpha) \times Q_{\mathfrak{C}}) \cap ve^{-1}(N_\Delta)$.

However, $Q_{\mathfrak{C}}$ need not be closed. We expect that the extra pseudomanifold boundary arising from $X_{\mathfrak{C}} \times cone(Q_{\mathfrak{C}})$ should be realizable as $s_i = 1$ boundary. That is, the fundamental chain of $\overline{LD}(g,k,\ell)/\!\!/ \sim \cup \bigcup_{\mathfrak{C}} X_{\mathfrak{C}} \times cone(Q_{\mathfrak{C}})$ has two types of boundary terms: those arising from codimension 1 cells with a spacing parameter $s_i = 1$ or an output formed by a single chord. Then the operation induced on relative chains by $ST_\mu(\partial[\overline{LD}(g,k,\ell)/\!\!/ \sim \cup \bigcup_{\mathfrak{C}} X_{\mathfrak{C}} \times cone(Q_{\mathfrak{C}})])$

would corespond only to compositions

$$ST_\mu([\overline{LD}(g'', k'', \ell'') /\!\!/ \sim \cup \bigcup_{\mathfrak{E}} X_{\mathfrak{E}} \times cone(Q_{\mathfrak{E}})])$$

$$*_\varepsilon$$

$$ST_\mu([\overline{LD}(g', k', \ell') /\!\!/ \sim \cup \bigcup_{\mathfrak{E}} X_{\mathfrak{E}} \times cone(Q_{\mathfrak{E}})]).$$

That is, if $X(g, k, \ell) = ST_\mu([\overline{LD}(g, k, \ell) /\!\!/ \sim \cup \bigcup_{\mathfrak{E}} X_{\mathfrak{E}} \times cone(Q_{\mathfrak{E}})])$, then

$$D_{Hom} X(g, k, \ell) = \sum X(g'', k'', \ell'') *_\varepsilon X(g', k', \ell')$$

and

$$D_{Hom} X = X * X.$$

A.2 Example

One example where the relevant $Q_{\mathfrak{E}}$ is closed is $(g, k, \ell) = (0, 2, 2)$.

Recall that the inspiration for $\overline{LD}(g, k, \ell) /\!\!/ \sim$ is $\overline{LD}(g, k, \ell) / \sim$ which is built from cells of $\overline{LD}(g, k, \ell)$ by identifying slide equivalent ones. The space $\overline{LD}(0, 2, 2)$ is a T^2-bundle over a space where there are ten 3-dimensional cells as in figure A.1.

Some of the codimension 1 identifications are shown in figure A.2 (the others

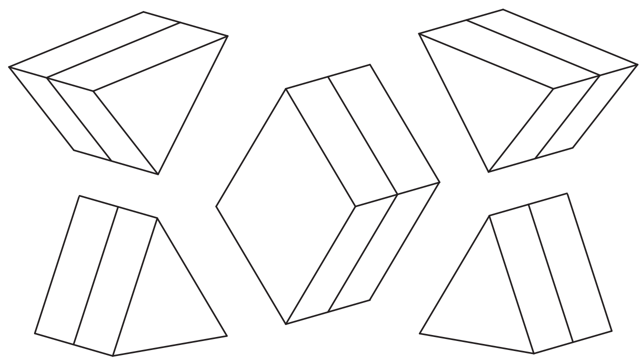


Figure A.1: Base space for $\overline{LD}(0, 2, 2)$

are analogous) and the result is a T^2-bundle over the space shown in figure A.3. Note that the base space is contractible.

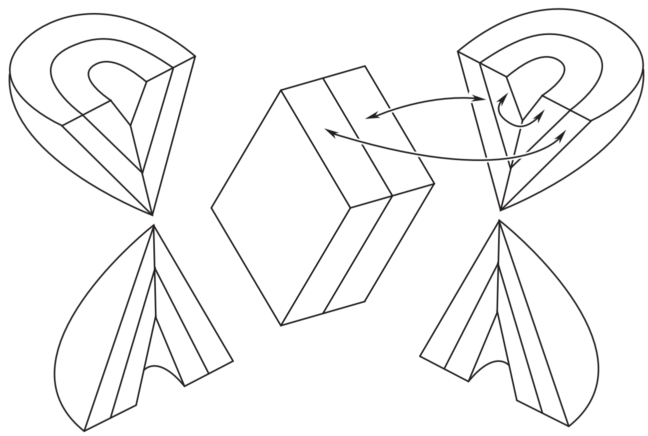


Figure A.2: Identifications of slide-equivalent cells

Consider the slide-equivalence class $\mathfrak{C}$ of cells e labeled by string diagrams where $s = 0$, where each input circle has only one vertex. There are three slide-equivalent representatives. There are three top-dimensional cells of the slide complex $P_{\mathfrak{C}}$: one multi-slide cell which is a square and two triangles, each with one edge collapsed (see figure 4.10) as in figure A.4.

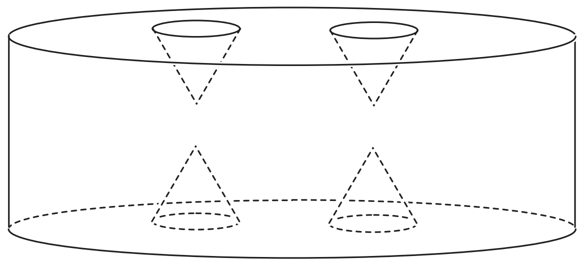


Figure A.3: Base space for $\overline{LD}(0,2,2)/\sim$

The space $\overline{LD}(0,2,2)/\!\!/\sim$ has several slide cells, multi-slide cells and triangles added to the cells of $\overline{LD}(0,2,2)$. The base space is as in figure A.5. In particular, there are two slide-equivalence classes with noncontractible slide complex, which keep $\overline{LD}(0,2,2)/\!\!/\sim$ from having the homotopy type of $\overline{LD}(0,2,2)/\sim$.

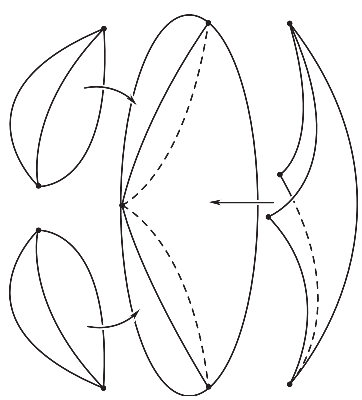


Figure A.4: Slide complex for $\chi = 0$

The space $\overline{LD}(0,2,2)/\!\!/\sim \cup \bigcup_{\mathfrak{C}} cone(Q_{\mathfrak{C}})$ is a pseudomanifold with boundary. Codimension 1 cells at the boundary either have an output consisting of one chord, or have a spacing parameter $s = 1$.

From the construction for cones on $Q_{\mathfrak{C}}$ described above, we obtain the follow-

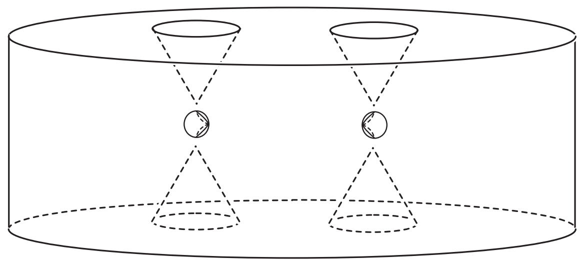


Figure A.5: $\overline{LD}(0,2,2)/\!\!/\sim$

ing:

Corollary A.1. *1. The string topology construction* ST_μ *extends over the fundamental chain* $[\overline{LD}(0,2,2)/\!\!/\sim \cup \bigcup_{\mathfrak{E}} cone(Q_{\mathfrak{E}})]$ *of* $\overline{LD}(0,2,2)/\!\!/\sim \cup \bigcup_{\mathfrak{E}} cone(Q_{\mathfrak{E}})$.

2. Let $X(0,2,2)$ *be the operation induced on relative chains by* $ST_\mu([\overline{LD}(0,2,2)/\!\!/\sim \cup \bigcup_{\mathfrak{E}} cone(Q_{\mathfrak{E}})])$. *Let* $X(0,2,1)$ *be the operation induced on relative chains by the string bracket and* $X(0,1,2)$ *be the operation induced by the string cobracket. Then*

$$D_{Hom}X(0,2,2) = \sum_{\varepsilon} X(0,1,2) *_\varepsilon X(0,2,1) + X(0,2,1) * X(0,1,2).$$

Indeed, because for codimension 1 cells where there is an output consisting of just one chord, the operation induced on relative chains is identically zero, we may visualize this relation by collapsing such cells as in figure A.6. The corresponding space is a pseudomanifold with five boundary components; ST_μ sends each one to a different composition of bracket and cobracket. As such, the new pseudomanifold itself shows the relation.

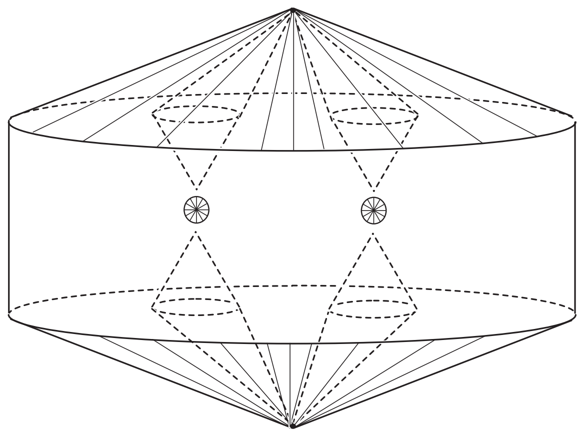


Figure A.6: $\overline{LD}(0,2,2)/\!\!/ \sim$ with cones

Remark. By homotopical algebra, the master equation solution passes from chains to homology [Sul09]. On homology, where D_{Hom} is identically zero, we see a five-term relation among these compositions of the string bracket and cobracket giving $H^{S^1}(LM, M; \mathbb{Q})$ corresponding to the Drinfeld compatibility of bracket and cobracket of a Lie bialgebra.

Appendix B

String diagrams and moduli spaces

B.1 The compactification of the moduli space of Riemann surfaces

Definition B.1.
- Let $\mathscr{M}(g, k+\ell)$ be the moduli space of Riemann surfaces of genus g, with $k+\ell$ labeled punctures.
- Let $\mathscr{M}(g, k, \ell)$ be the moduli space of Riemann surfaces of genus g, with $k+\ell$ labeled punctures, k of which (input punctures) are decorated by real tangent directions and the remaining ℓ of which (output punctures) are decorated by nonnegative numbers, called weights, whose sum is k.

Remark. This moduli space $\mathscr{M}(g, k, \ell)$ of surfaces with decorated punctures is homeomorphic to the total space of a principal T^k-bundle over $\mathscr{M}(g, k+\ell) \times \Delta^{\ell-1}$, where T^k is the k-torus of tangent directions and $\Delta^{\ell-1}$ is the $(\ell-1)$-dimensional simplex of output weights.

Recall that stable nodal surfaces are added to $\mathscr{M}(g, k+\ell)$ to obtain the Deligne-Mumford compactification $\bar{\mathscr{M}}(g, k+\ell)$. An essential simple closed curve α on a topological surface of genus g with $k+\ell$ punctures corresponds to a real codimension 2 subspace $S_\alpha \subset \bar{\mathscr{M}}(g, k+\ell)$. That is, S_α is the moduli space of genus g nodal surfaces with $k+\ell$ punctures and where the curve α has been shrunk to a point and replaced by a node. If α and β are two nonhomotopic, nonintersecting essential simple closed curves, then $S_\alpha \cap S_\beta$ corresponds to the moduli space of nodal surfaces where both α and β have been shrunk to a point and replaced by a node. The space $S_\alpha \cap S_\beta$ has real codimension 2 in both S_α and S_β, so it has real codimension 4 in $\bar{\mathscr{M}}(g, k+\ell)$. In general, the subspace $S_A = \bigcap_{\alpha \in A} S_\alpha$ of nodal surfaces corresponding to a collection A of n collapsing curves is a codimension $2n$ subspace of $\bar{\mathscr{M}}(g, k+\ell)$. The subspace S_A is the union of components of strata of $\bar{\mathscr{M}}(g, k+\ell)$.

Over each such codimension 2 subspace S_α there is an S^1-bundle of smooth surfaces corresponding to the unit normal bundle of $S_\alpha \subset \bar{\mathscr{M}}(g, k+\ell)$. Points in the total space of such a bundle correspond to smooth Riemann surfaces where the curve α is small in the hyperbolic metric. The circle acts on the S^1-bundle by a Dehn twist about α. Such a circle action is trivial when α has been collapsed to a node. A Deligne-Mumford stratum corresponds to such a trivialization of the circle action; the S^1-bundle has been "filled in" to give a D^2-bundle. The Deligne-Mumford stratum is the image of the zero section of this D^2-bundle. There is a T^n-bundle over S_A which is filled in in $\bar{\mathscr{M}}(g, k+\ell)$.

Definition B.2. Let E_A be the total space of the T^n-bundle over
$S_A \subset \bar{\mathscr{M}}(g, k+\ell)$. Replace S_A by E_A and denote the resulting space by $\bar{\mathscr{M}}^*_A(g, k+\ell)$. We say that $\bar{\mathscr{M}}(g, k+\ell)$ has been blown up along S_A and E_A is the blow-up of S_A.

Remark. Blowing up $\bar{\mathscr{M}}(g, k+\ell)$ along S_A has the effect of carving out a tubular neighborhood of S_A from $\bar{\mathscr{M}}(g, k+\ell)$. The resulting space has codimension 1 boundary, namely, E_A. It is analogous to compacifying the open disk with the closed disk, rather than with its one-point compactification, the sphere. If $\bar{\mathscr{M}}(g, k+\ell)$ is blown up along all of its Deligne-Mumford stratum, a compactification of $\mathscr{M}(g, k+\ell)$ is obtained that has codimension 1 boundary and the same homotopy type as $\mathscr{M}(g, k+\ell)$.

Definition B.3. Let $(\Sigma, w_1, w_2, \ldots, w_\ell) \in \bar{\mathscr{M}}(g, k+\ell) \times \Delta^{l-1}$ such that ℓ of the punctures are decorated by the w_i. The complement of the subspace of nodes in Σ consists of connected components, each of which is a smooth punctured Riemann surface. If, for a given component, the sum of the weights decorating its punctures is equal to the number of undecorated punctures not corresponding to nodes, the component is called balanced. Otherwise it is called unbalanced.

Remark. If Σ minus its nodes consists of exactly two connected components, then they are either both balanced or both unbalanced. If a component has no punctures, then it is balanced. If a component has no undecorated punctures and each of its decorations is weight 0, then it is balanced.

The compactification $\bar{\mathscr{M}}(g,k,\ell)$ of $\mathscr{M}(g,k,\ell)$ is defined by first compactifying the base of the T^k-bundle over $\mathscr{M}(g,k+\ell)\times\Delta^{l-1}$ and then extending the T^k-bundle over the compact base.

Definition B.4. Let $\overline{\mathscr{M}(g,k+\ell)\times\Delta^{l-1}}$ be defined by modifying Deligne-Mumford strata of $\bar{\mathscr{M}}(g,k+\ell)\times\Delta^{l-1}$ as follows:

- If, for all Σ in a Deligne-Mumford stratum S_A in $\bar{\mathscr{M}}(g,k+\ell)$, Σ minus its nodes is connected then, for all $(\Sigma,w_1,w_2,\ldots,w_\ell)\in\Delta^{\ell-1}$, the S^1-bundle over S_A is filled in by the usual Deligne-Mumford stratum S_A.
- If, for all Σ of a Deligne-Mumford stratum S_A, Σ minus its nodes consists of two smooth punctured Riemann surfaces Σ_1 and Σ_2 then, for all $(\Sigma,w_1,w_2,\ldots,w_\ell)$ such that the components Σ_1 and Σ_2 are balanced and each Σ_i has at least one decorated puncture which does not correspond to a node, the S^1-bundle over S_A is filled in by the usual Deligne-Mumford stratum $S_A=\mathscr{M}_1(g_1,k_1,\ell_1)\times\mathscr{M}_2(g_2,k_2,\ell_2)$.
- If, for all Σ of a Deligne-Mumford stratum S_A, Σ minus its nodes consists of two smooth punctured Riemann surfaces Σ_1 and Σ_2 then, for all $(\Sigma,w_1,w_2,\ldots,w_\ell)$ such that the components Σ_1 and Σ_2 are balanced and

one component, say Σ_2, has no undecorated puncture which does not correspond to a node, the S^1-bundle over S_A is filled in by collapsing the factor $\mathscr{M}_2(g_2, k_2, \ell_2)$ of the usual Deligne-Mumford stratum $S_A = \mathscr{M}_1(g_1, k_1, \ell_1) \times \mathscr{M}_2(g_2, k_2, \ell_2)$ to a point.

- If, for all Σ of a Deligne-Mumford stratum S_A, Σ minus its nodes consists of two smooth punctured Riemann surfaces then, for all $(\Sigma, w_1, w_2, \ldots, w_\ell)$ such that the components are unbalanced, the S^1-bundle over S_A is not filled in by the usual Deligne-Mumford stratum S_A but instead S_A is replaced by its blow-up E_A.

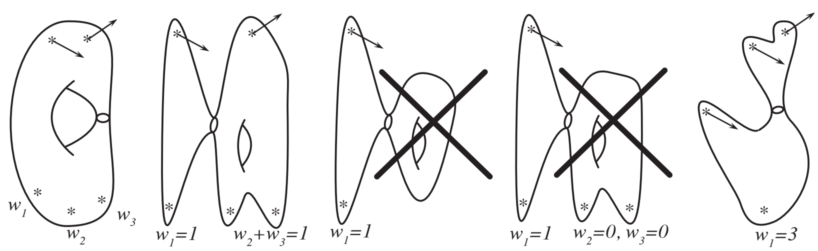


Figure B.1: Elements of $\bar{\mathscr{M}}(g, k, \ell)$

Remark. Higher codimension strata of $\bar{\mathscr{M}}(g, k+\ell)$ are given by intersections of lower codimension strata. Similarly, higher codimension strata of $\overline{\mathscr{M}(g, k+\ell) \times \Delta^{l-1}}$ are determined by intersections of strata described above. In particular, Deligne-Mumford strata and modified Deligne-Mumford strata are included unless components become unbalanced, in which case a blow-up stratum is inlcuded.

If a point $\Sigma \in \bar{\mathscr{M}}(g, k, \ell)$ is on the boundary, then either Σ is in a blow-up stratum or Σ has an output puncture decorated by a weight 0.

Lemma B.1. *The T^k-bundle of tangent directions over $\mathscr{M}(g, k+\ell) \times \Delta^{l-1}$ extends canonically over $\overline{\mathscr{M}(g, k+\ell) \times \Delta^{l-1}}$.*

Proof. We first show the T^k-bundle over $\mathscr{M}(g, k+\ell) \times \Delta^{l-1}$ extends canonically over $\bar{\mathscr{M}}(g, k+\ell) \times \Delta^{l-1}$.

Let S_A be a Deligne-Mumford stratum in $\mathscr{M}(g, k+\ell)$ with tubular neighborhood N_A. The T^k bundle over $\mathscr{M}(g, k+\ell) \times \Delta^{l-1}$ restricts to $N_A - S_A \times \Delta^{l-1}$. The space $N_A - S_A$ is a punctured D^2-bundle over S_A, so the T^k-bundle extends canonically (up to bundle isomorphism) over the puncture. Therefore, the T^k-bundle over $\mathscr{M}(g, k+\ell) \times \Delta^{l-1}$ extends canonically over $\bar{\mathscr{M}}(g, k+\ell) \times \Delta^{l-1}$.

A stratum in $\overline{\mathscr{M}(g, k+\ell) \times \Delta^{l-1}}$ corresponds either a usual Deligne-Mumford stratum S_A, a Deligne-Mumford stratum where a $\mathscr{M}(g_i, k_i, \ell_i)$ factor has been collapsed to a point or to a blow-up stratum E_A. In the first case the T^k-bundle over $\bar{\mathscr{M}}(g, k+\ell) \times \Delta^{l-1}$ is unchanged. In the second case the T^k-bundle over the collapsed factor is replaced by the unique T^k-bundle over a point. As points in blow-up strata E_A correspond to smooth surfaces, the T^k-bundle of tangent directions over such E_A is obtained from the T^k-bundle $\mathscr{M}(g, k, \ell)$ over the open moduli space $\mathscr{M}(g, k+\ell) \times \Delta^{l-1}$.

□

Definition B.5. The compactification $\bar{\mathscr{M}}(g, k, \ell)$ of $\mathscr{M}(g, k, \ell)$ is given by the total space of the extended T^k-bundle over $\overline{\mathscr{M}(g, k+\ell) \times \Delta^{l-1}}$.

Example. The open moduli space $\mathscr{M}(0,3,1)$ is a 3-torus bundle over $\mathscr{M}(0,3+1)$, which is a sphere with three punctures. While in the Deligne-Mumford compactification $\bar{\mathscr{M}}(0,3+1)$ these punctures would be filled in with points, in $\overline{\mathscr{M}}(0,3+1)$, these points are replaced with circles. Each of the three blow-up strata is labeled by a shrinking curve which separates two input punctures from the third and the output puncture. See figure B.2.

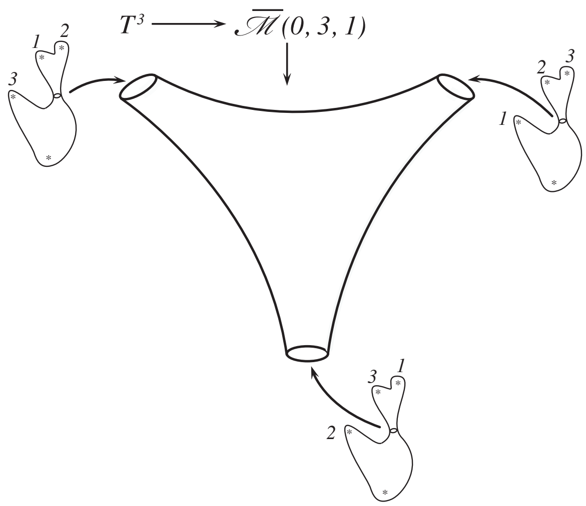


Figure B.2: $\bar{\mathscr{M}}(0,3,1)$

Example. $\mathscr{M}(0,2,2)$ is a 2-torus bundle over a 3-dimensional space with the homotopy type of the figure 8, while $\bar{\mathscr{M}}(0,2,2)$ is a 2-torus bundle over a 3-dimensional contractible space. See figure B.3.

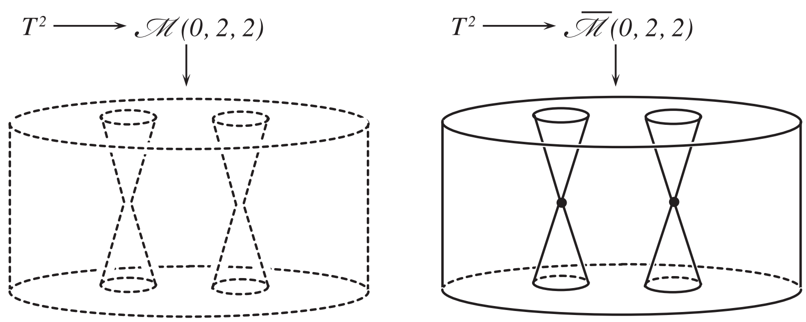


Figure B.3: $\bar{\mathscr{M}}(0,2,2)$

B.2 String diagrams with levels and moduli space

Theorem B.1. *The space* $\bar{\mathscr{M}}(g,k,\ell)$ *is an orientable pseudomanifold of dimension* $6g+3k+3\ell-7$ *with boundary.*

Proof. The proof has four steps.

1. We describe a construction which produces a space $CS(g,k,\ell)$ homeomorphic to $\overline{LD}(g,k,\ell)/\sim$.

2. We define an equivalence relation $\approx$ on $CS(g,k,\ell)$ and use it to impose an equivalence relation $\approx$ on $\overline{LD}(g,k,\ell)/\sim$.

3. We show that $(\overline{LD}(g,k,\ell)/\sim)/\approx$ is an orientable pseudomanifold of dimension $6g+3k+3\ell-7$ with boundary.

4. We show that $\overline{LD}(g,k,\ell)/\approx$ and $\bar{\mathscr{M}}(g,k,\ell)$ are homeomorphic.

Step 1.

This step uses ideas appearing in [Böd06].

Let $\Gamma \in \overline{LD}(g, k, \ell)$. From Γ we construct a metric space.

Select a representative of the isomorphism class of Γ whose nondistinguished vertices are trivalent and with no spacing parameter $s_i = 0$. This means that there may be input edges e that have length $L = 0$ and if two chords have the same length they are at the same level. By abuse of notation, refer to the representative by Γ as well.

Augment Γ by inserting a vertex at the midpoint of every chord and two infinite half-edges at the vertex so that the cyclic order of half-edges at each of the new vertex alternates between half-edges from chords and infinite half-edges. Form the infinite ribbon surface Σ associated to the augmented cyclically ordered metric graph.

Each input circle γ of Γ gives Σ a boundary component of length 1. For each γ, attach a half-infinite cylinder $\gamma \times \mathbb{R}_{\leq 0}$ to Σ by identifying $\gamma \times \{0\}$ with $\gamma \subset \Sigma$.

The remaining (infinite) boundary components of Σ each contain one input edge e, two half-chords h and h' corresponding to chords ch and ch' respectively and two half-infinite edges. Assume that h follows e in the cyclic order at the vertex v and e follows h' in the cyclic order at the vertex v' and assume that

$ch \in E_i$ and $ch' \in E_j$. Therefore, h has length $\frac{1}{2} + s_1 + s_2 + \cdots + s_{i-1}$ and h has length $\frac{1}{2} + s_1 + s_2 + \cdots + s_{j-1}$ Let $R = e \times \mathbb{R}_{\geq 0}$ be half-infinite rectangular strip. R is attached to an infinite boundary component of Σ in the following way:

The end $e \times \{0\}$ of R is identified with $e \subset \Sigma$. The side $v \times \mathbb{R}_{\geq 0}$ of R is attached to Σ by attaching $v \times [0, \frac{1}{2} + s_1 + s_2 + \cdots + s_{i-1}]$ to h and $v \times [\frac{1}{2} + s_1 + s_2 + \cdots + s_{i-1}, \infty)$ to the infinite edge following h in the cyclic order at the midpoint of ch. The side $v' \times \mathbb{R}_{\geq 0}$ of R is attached to Σ by attaching $v' \times [0, \frac{1}{2} + s_1 + s_2 + \cdots + s_{j-1}]$ to h' and $v' \times [\frac{1}{2} + s_1 + s_2 + \cdots + s_{j-1}, \infty)$ to the infinite edge preceding h' in the cyclic order at the midpoint of ch'. See figure B.4.

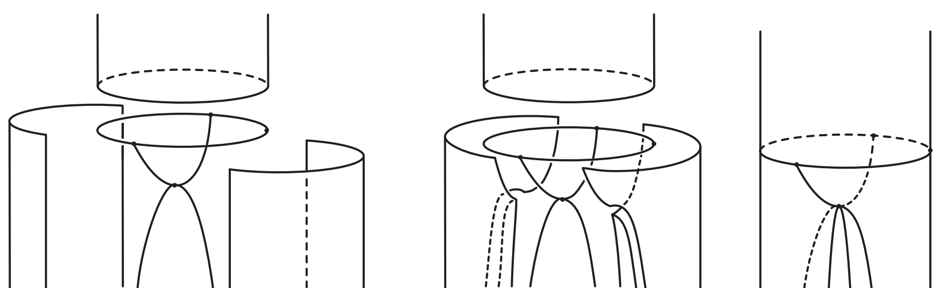


Figure B.4: Metric space construction for a string diagram with one level of type $(0, 1, 2)$.

If v is a bivalent distinguished vertex of Γ such that input edges e and e' are adjacent to v, then the construction is performed by considering $e \cup v \cup e'$ as one edge and attaching the half-infinite strip $R = e \cup v \cup e' \times \mathbb{R}_{\geq 0}$ to Σ as above.

Generically, this construction produces a metric space which is a topological surface S and the graph Γ decomposes S into its cylindrical ends and rectangular strips. In particular, S and Γ are homotopy equivalent. If S is a smooth surface, it has a canonical Riemannian metric that is flat except at points corresponding to midpoints of chords. If there are input edges e of length $L = 0$, S may or may not be a surface. Regardless, we say that S is a cylindrical surface as it is formed from k disjoint infinite cylinders of circumference 1 by cutting and pasting as above.

The construction is well-defined on slide-equivalence classes. If in Γ, ch and ch' are adjacent to the vertex w, the construction is performed on the isomorphism class representative that has an edge e of length $L = 0$ whose endpoints v and v' are trivalent and coincide with endpoints of chords ch and ch' respectively. Assume that $ch \in E_i$ and $ch' \in E_j$ and $i \leq j$.

The rectangular strip $R = e \times \mathbb{R}_{\geq 0}$ is identified with a half-infinite interval $w \times \mathbb{R}_{\geq 0}$. According to the construction, $w \times [0, \frac{1}{2} + s_1 + s_2 + \cdots + s_{i-1}]$ is identified with h and $w \times [0, \frac{1}{2} + s_1 + s_2 + \cdots + s_{j-1}]$ is identified with h'. So here, h is identified with a subinterval $[0, \frac{1}{2} + s_1 + s_2 + \cdots + s_{i-1}]$ of h. The remaining subinterval $[s_i + s_{i+1} + \cdots + s_{j-1}]$ of h is identified with a subinterval of a half-infinite edge added to the midpoint of ch.

Now let Γ' be a string diagram with levels that is obtained from Γ by sliding the endpoint v' of ch' to the other endpoint w' of ch. The identifications are analogous, except the $w' \times [0, \frac{1}{2} + s_1 + s_2 + \cdots + s_{j-1}]$ is identified with the other half-chord of ch. The metric space S' is the same as S and the decomposition into half-infinite cylinders and rectangular strips is also the same.

The construction produces a map $F : \overline{LD}(g,k,\ell)/\sim \longrightarrow CS(g,k,\ell)$ which is continuous and invertible by definition. If S is the output of the construction, then a string diagram with levels Γ decomposes S into half-infinite cylinders and rectangular strips. Here, Γ is determined only up to slide-equivalence. Therefore, $CS(g,k,\ell)$ and $\overline{LD}(g,k,\ell)/\sim$ are homeomorphic.

As $\overline{LD}(g,k,\ell)/\sim$ is a pseudomanifold with boundary, so is $CS(g,k,\ell)$.

Step 2.

The construction of step 1 may produce a cylindrical surface that is homeomorphic to a smooth or possibly nodal surface with a metric graph attached to the surface at some of its vertices. The edges of the graph correspond to rectangular strips R_e where the length of E is 0 and its sides are identified by the construction. Figure B.5 shows one string diagram with levels which produces a smooth surface and a second nearby string diagram with levels which produces a surface with a graph attached.

A cylindrical surface has a decomposition into connected components, each of which is a (possibly nodal) surface or a graph, some of whose vertices are attached to the surface components. The graph may have edges of half-infinite length. These correspond to output circles of string diagrams with levels with

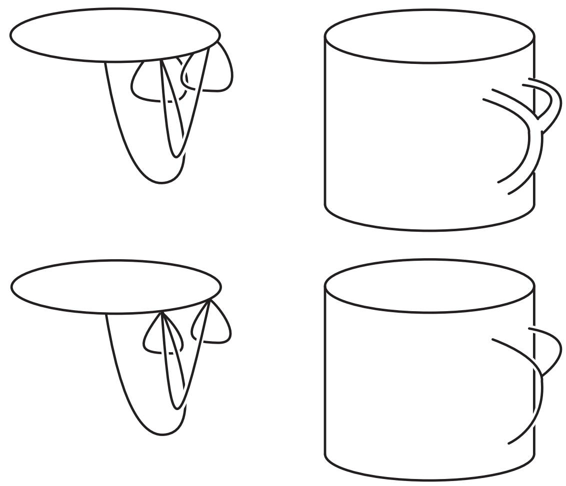


Figure B.5: Smooth cylindrical surface and cylindrical surface with graph component

outputs made entirely of chords.

Let $q : CS(g,k,\ell) \to CS(g,k,\ell)$ be a map which collapses each graph component to a point.

Two cylindrical surfaces S and S' are equivalent under the equivalence relation $\approx$ if

- $q(S) = q(S')$ and
- the decompositions of S and S' have corresponding graph and nodal surface components and their corresponding graph components have the same Euler characteristic and have the same number of half-infinite edges.

Step 3.

There may be codimension 1 cells of $\overline{LD}(g, k, \ell)$ which give rise to cylindrical surfaces with graph components: if two endpoints of a single chord coincide on an input circle. In this case, the graph is a half-infinite interval with its end attached to a smooth surface. (The other end corresponds to an output circle that is made entirely of the chord.) Such a cylindrical surface is the only one in its $\approx$ equivalence class.

The induced equivalence relation $\approx$ on $\overline{LD}(g, k, \ell)/\sim$ affects cells of codimension at least 2. With the exception of the case above, a slide-equivalence class of string diagrams with levels Γ does not give rise to a cylindrical surface with graph components unless it is in a cell of codimension at least 2.

A cell of high codimension may be identified with one of its faces and/or two high codimension cells e and e' with isomorphic parameter spaces X_e and $X_{e'}$ may be identified. Therefore $(\overline{LD}(g, k, \ell)/\sim)/\approx$ remains a pseudomanifold with boundary.

Step 4.

This step also uses ideas appearing in [Böd06].

There is a forgetful map $F : CS(g, k, \ell)/\approx \to \bar{\mathscr{M}}(g, k, \ell)$ which ignores the decomposition of a class of cylindrical surfaces S into rectangular strips. If c_i is the circumference of the cylinder corresponding to the ith output circle, the weight w_i at the ith output puncture is equal to $\frac{c_i}{k}$. The collection of distinguished vertices on the k input circles corresponds to the torus of tangent directions at the k input punctures.

We construct a map $H : \mathscr{M}(g,k,\ell) \to CS(g,k,\ell)/\approx$ and an extension of it to $\bar{\mathscr{M}}(g,k,\ell)$ which is inverse of F.

Let Σ be a Riemann surface in $\mathscr{M}(g,k,\ell)$ with weights $w_1 > 0, \dots w_\ell > 0$ decorating output punctures. There is a harmonic function $u : \Sigma \to \mathbb{R}$ with singularities at punctures. The residue about an input puncture p_i is -1 and the residue about an output puncture q_i is kw_i. The function u is unique up to an additive constant.

Let $crit(u) \subset \Sigma$ be the set of critical points of u and $\{a_1 < a_2 < \cdots < a_N\} \subset \mathbb{R}$ be the set of critical values of u. There is a riemannian metric on $\Sigma - crit(u)$ such that $||grad(u)|| = 1$. Let a be a regular value of u. The level set $u^{-1}(a)$ is a disjoint collection of circles. In this metric, $||u^{-1}(a)|| = k$ is constant and $u^{-1}(a_i, a_{i+1})$ is a disjoint collection of (open) cylinders.

A component of the level set $u^{-1}(a_i)$ includes a wedge of circles, that is, a graph together with a cyclic order of the half-edges at the vertex that describes how cylinders in $u^{-1}(a_{i-1}, a_i)$ are attached to cylinders in $u^{-1}(a_i, a_{i+1})$. This gives a cylindrical structure on the surface Σ. We decompose Σ into rectangular strips using the gradient flow lines of u.

The critical graph $G_u \subset \Sigma$ of the function u has $crit(u)$ as its vertices and gradient flow lines flowing toward and away from critical points as its edges. The edges may be half-infinite: they may flow away from input punctures or toward output punctures. The complement $\Sigma - G_u$ is a disjoint collection of infinite rectangular strips whose sides are sequences of gradient flow lines.

From G_u we construct a string diagram with levels by deleting edges flowing toward output punctures and cutting edges flowing from input punctures by a $u^{-1}(a)$ where $a < a_1$. Each component of $u^{-1}(a)$ gives an input circle. Let $h : (0, \infty) \to (0, 1)$ be given by retscaling. The spacing parameter s_i is given by $h(a_{i+1} - a_i)$. The tangent direction at an input puncture corresponds to a distinguished vertex on the corresponding input circle.

So far, the image of the map H is a subset of $CS(g, k, \ell)/\approx$ where each $\approx$-equivalence class has just one representative; $H(\Sigma)$ has no nodes and no graph components. There are two steps of the extension of the map

$$\mathscr{M}(g, k, \ell) \to CS(g, k, \ell)/\approx$$

to $\bar{\mathscr{M}}(g, k, \ell)$.

First, recall that the weight w_i decorating the output puncture q_i gives a cylindrical end of a cylindrical surface in $CS(g, k, \ell)$ that has circumference kw_i.

Let $\Sigma \in \bar{\mathscr{M}}(g,k,\ell)$ be a smooth Riemann surface such that the decoration on the output q_i is $w_i = 0$. Then $\Sigma \in \partial(\bar{\mathscr{M}}(g,k,\ell))$. Let $\{\Sigma_1, \Sigma_2, \dots\}$ be a sequence of points in $\mathscr{M}(g,k,\ell)$ converging to Σ. We define $H(\Sigma)$ so that H takes convergent sequences to convergent sequences. As w_i tends to 0 in Σ_j, the circumference of corresponding cylindrical end of $S_j = H(\Sigma_j)$ tends to 0. Therefore, the circumference of the corresponding cylindrical end of $S = H(\Sigma)$ is 0.

Next, let $\Sigma \in \bar{\mathscr{M}}(g,k,\ell)$ be a decorated surface with nodes. Then the complement of the nodes in Σ is a disjoint collection of smooth surfaces. In each smooth component, fill in the puncture created by the deletion of a node by a distinguished point. By definition, each component Σ' of Σ is balanced. If the component Σ' has an input puncture, then there is a harmonic function $u' : \Sigma' \to \mathbb{R}$ as above. The cylindrical surface $H(\Sigma)$ is assembled by identifying cylindrical surfaces $H(\Sigma')$ at the images of marked points corresponding to nodes of Σ. If a component Σ' has no input puncture then it has no output puncture or its output punctures are all decorated by weight 0. In the case that Σ' has no output puncture, then it corresponds to a collapsed graph component whose Euler characteristic label is the Euler characteristic of the handle-body with boundary Σ'. The same is true if Σ has output punctures, but a corolla is attached by some of its univalent vertices to images of nodes in other components of $H(\Sigma'')$. There is one free univalent vertex of the corolla for each output puncture q_i of Σ'. These correspond to cylindrical ends of $H(\Sigma)$ that have circumference 0 and we extend such edges to be half-infinite accordingly.

If Σ is in a blow-up stratum E_A of $\bar{\mathscr{M}}(g,k,\ell)$ corresponding to a collection A of curves α which are small in the hyperbolic metric. Consider the components of $\Sigma - A$. If a_i is a critical value corresponding to a critical point on one component and a_{i+1} is a critical value corresponding to a critical point on an adjacent component, then $a_{i+1} - a_i$ is arbitrarily large. In this case, the spacing parameter s_i is defined to be 1.

We have not explicitly mentioned tangent directions decorating input punctures or distinguished vertices on input circles. Here, we make an abstract identification which depends on a choice. Indeed, for the map F, fix a cylindrical surface S and choose an assignment of tangent directions to input punctures of $F(S)$. We then declare the map F to be T^k-equivariant. For such a choice for F there is a corresponding choice for H such that F and H are inverse. So we have that $CS(g,k,\ell)/\approx$ and $\bar{\mathscr{M}}(g,k,\ell)$ are homeomorphic, though the homeomorphism is not necessarily canonical.

Summary.

1. $CS(g,k,\ell)$ and $\overline{LD}(g,k,\ell)$ are homeomorphic.

2. $CS(g,k,\ell)/\approx$ is a pseudomanifold with boundary.

3. $CS(g,k,\ell)/\approx$ and $\bar{\mathscr{M}}(g,k,\ell)$ are homeomorphic (the homeomorphism

induces a cell complex structure on $\bar{\mathscr{M}}(g,k,\ell))$ so $\bar{\mathscr{M}}(g,k,\ell)$ is a pseudo-manifold with boundary.

□

B.3 Compactified moduli spaces and string topology operations

Recall that certain chains in $C_*^{T^k}(\overline{LD}(g,k,\ell)/\!\!/ \sim)$ determine string topology operations $ST_\mu(c) \in Hom(P_*(kLM), P_*(\ell LM))$. Let

$$q : \overline{LD}(g,k,\ell)/\!\!/ \sim \longrightarrow \overline{LD}(g,k,\ell)/\sim$$

be the map which collapses slide-complex factors of adjoined cells (and possibly cones on $\chi = 0$ slide complexes) to points. The corresponding chains in $\overline{LD}(g,k,\ell)/\sim$ need not determine string topology operations, but they do up to chain homotopy.

We expect that there is a space $(\overline{LD}(g,k,\ell)/\!\!/ \sim)/\!\!/ \approx$ which has the same homotopy type as $(\overline{LD}(g,k,\ell)/\sim)/\approx$ such that certain chains on it determine string topology operations. If the space $\overline{LD}(g,k,\ell)/\!\!/ \sim$ is a "thick" version of $\overline{LD}(g,k,\ell)/\sim$ then the space $(\overline{LD}(g,k,\ell)/\!\!/ \sim)/\!\!/ \approx$ will be a "thick" version of

$\overline{\mathscr{M}}(g,k,\ell)$.

The reason for this expectation is the following. Given $\Sigma \in \bar{\mathscr{M}}(g,k,\ell)$, a level set $u^{-1}(a)$ for a regular value a of the harmonic function $u : \Sigma \to \mathbb{R}$ is a disjoint union of circles. We imagine pushing these circles through a critical point, which describes how they are cut and reconnected to form another disjoint union of circles corresponding to a level set for a higher regular value. The cutting and pasting is similar to the cutting and pasting of loops in M described by the string topology construction, when the loops intersect one another *transversally*. In this setting, a graph component of a cylindrical surface describes a trivial interaction of constant loops, so the string topology construction is insensitive to it.

Conjecture B.1. *There is a space $(\overline{LD}(g,k,\ell)/\!\!/ \sim)/\!\!/ \approx$ which is built from $\overline{LD}(g,k,\ell)/\!\!/ \sim$ by adjoining cells such that the following hold:*

1. *The space $(\overline{LD}(g,k,\ell)/\!\!/ \sim)/\!\!/ \approx$ is a pseudomanifold of dimension $6g+3k+3\ell-7$ with boundary and has the homotopy type of $\bar{\mathscr{M}}(g,k,\ell)$.*

2. *The string topology construction ST_μ may be extended to some chains in $C_*^{T^k}((\overline{LD}(g,k,\ell)/\!\!/ \sim)/\!\!/ \approx)$.*

3. *If $[(\overline{LD}(g,k,\ell)/\!\!/ \sim)/\!\!/ \approx]$ is the fundamental chain, $ST_\mu([(\overline{LD}(g,k,\ell)/\!\!/ \sim)/\!\!/ \approx]) = X(g,k,\ell)$ and $X = \sum X(g,k,\ell)$, then X solves the master equation.*

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