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CONTRIBUTIONS TO THE THEORY OF BLOCK DESIGNS

by

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FOREWORD

This dissertation studies the relation of balanced incomplete block designs to two distinct areas of combinatorial mathematics, game theory and graph theory. In its relation to game theory, this study is devoted to an unsolved problem stated in reference [21] by Richardson and again in [14] by Hoffman and Richardson. The problem seeks the minimum number of elements in a blocking coalition of a block design game. A solution is found for block design games which are dual to Steiner triple systems. In the course of solving this problem for this restricted class of block design games a number of properties of Steiner triple system are developed.

In considering the relation of balanced incomplete block designs to graph theory, the writer has introduced a particular kind of graph called here a graph on the binomial coefficient $\binom{v}{k}$ with adjacency parameter λ. It is shown that for $\lambda = 1$ this graph has the property that if there exists a balanced incomplete block design with parameters the given v, k, and with $\lambda = 1$ then every maximum internally stable set of vertices is such a design and, moreover, every balanced incomplete block design with $\lambda = 1$ is a maximum internally stable set for some suitably defined graph. The introduction of this particular graph is a formalization of a suggestion made by Berge in his informal discussion of the Kirkman schoolgirl problem in [2]. As a result of this innovation it becomes possible to formulate in graph theoretic terms some of the unsolved problems arising in block design theory. As an interesting by-product we note that the long known proof of the non-existence of a finite affine plane on 36 points solves one problem and raises another. We are now faced with a new problem: What is the internal stability number of the graph on the binomial coefficient $\binom{36}{6}$ with $\lambda = 1$? It is surprising that the answer is unknown. (See Example 1, part II of this dissertation.) This is an instance of a general problem which is not solved here: What is the nature of maximum internally stable sets of vertices in graphs on binomial coefficients for which balanced incomplete block designs do not exist?

In a related context, a balanced incomplete block design with $\lambda = 1$ is shown to be a solution of a suitably defined irreflexive relation. This characterization of balanced incomplete block designs with $\lambda = 1$ enables us to generate an infinite set of non-trivial examples of relativizations and extensions of solutions of irreflexive relations. We thus have an application of some of the results developed by Richardson in [18], [19], and [20].

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CHAPTER I. ON A RESTRICTED CLASS OF BLOCK DESIGN GAMES

1. INTRODUCTION

Block design games have been developed by Richardson [21] and by Hoffman and Richardson [14], who proved a number of theorems concerning such games by studying the number of elements in a blocking coalition. Hoffman and Richardson listed as unsolved (except for PG(2,3)) the following problem: What is the minimum number of elements in a blocking coalition of a block design game?

In this section we consider blocking coalitions in those games that are dual to block designs having $\lambda = 1$ and $r - k > 0$. For such games certain blocking coalitions are shown to be related to sets of mutually disjoint blocks in the design to which the game is dual. In particular, for Steiner triple systems the largest odd-numbered set of mutually disjoint triples is shown to yield a minimum blocking coalition in the dual. A lower bound for the number of elements in the largest set of mutually disjoint triples is found, which results in a smaller upper bound for a minimum blocking coalition than that heretofore known for the duals of Steiner triple systems.

Some extensions of these results show that some of the games that are dual to designs having $\lambda = 1$, $r - k > 0$, and $k > 3$ have easily obtained minimum blocking coalitions and have no equitable main simple solutions.

2. NOTATION

In general, we use v, b, k, r, λ as parameters of the balanced incomplete block design (written BIBD) consisting of v elements arranged in b blocks of k elements each with each element occuring r times in the design and any pair of distinct elements appearing together in the same block exactly λ times. Following Hoffman and Richardson [14] we use v^*, b^*, etc. as parameters of the block design and unstarred parameters in the dual. $|B|$ will denote the number of elements in a blocking coalition. Elements belonging to the same block will be called collinear. The usual set-theoretic notation will be used.

3. PRELIMINARY RESULTS

The parameters of a BIBD satisfy

$$vr = bk, \tag{1}$$

$$r(k - 1) = \lambda(v - 1). \tag{2}$$

If we postulate the existence of at least two distinct blocks so that $v > k$, then

$$r \geq k. \tag{3}$$

This is equivalent to Fisher's inequality:

$$b \geq v; \tag{4}$$

cf. [7, 15, or 22]. The designs having $r = k$ are called symmetric designs. Finite projective planes of order n are symmetric, balanced incomplete block designs with $k = n + 1$ and $\lambda = 1$. Finite Euclidean planes are balanced incomplete block designs with $\lambda = 1$, $k = n$, $r = n + 1$ so that $r - k = 1$. It has long been known in connection with the search for finite models of a Bolyai-Lobachevsky plane that designs are impossible for $\lambda = 1$ and $r - k = 2$. A general necessary relation between r and k is given by the following: A BIBD with $\lambda = 1$, and $k = p^m q$, where p is a prime, $(p,q) = 1$, satisfies $r \equiv 0, 1 \pmod{p^m}$. This follows from (1) and (2). The necessary conditions for the existence of a BIBD are sometimes stated in the form [11]:

(5) $$\lambda(v - 1) \equiv 0 \pmod{k - 1},$$

(6) $$\lambda v(v - 1) \equiv 0 \pmod{k(k - 1)}.$$

If k is a power of a prime, then a BIBD with $\lambda = 1$ has one of the following sets of parameters:

(7) Case I: $v = (k - 1)kt + 1$, $b = t((k - 1)kt + 1)$, $r = kt$.

(8) Case II: $v = (k - 1)kt + k$, $b = ((k - 1)t + 1)(kt + 1)$, $r = kt+1$.

If k is composite, v,b,k,r, with $\lambda = 1$, can sometimes be found so as to satisfy (1) and (2) with $r \equiv 0, 1 \pmod{p^m}$ and $r \not\equiv 0, 1 \pmod{k}$; but such designs have not been constructed [15,p.127]. In general (1) and (2) and consequently (5) and (6) are not sufficient for the existence of a BIBD.

BIBD with $\lambda = 1$ and $k = 3$ are known as Steiner triple systems (written STS). Owing to the work of Reiss [17] and Moore [16], it is possible to construct an STS for every $t \geq 1$ in Cases I and II above. Hanani [11] and [12] has shown that for $\lambda = 1$ and $k = 4$, and for $\lambda = 1$ and $k = 5$, (5) and (6) are sufficient conditions.

The dual of a BIBD with $\lambda^* = 1$ and parameters v^*, b^*, k^*, r^* is a partially balanced incomplete block design with $v = b^*$, $b = v^*$, $k = r^*$, $r = k^*$, and $\lambda i = 0,1$. If we take the v elements as players and the b blocks as the minimal winning coalitions, we have a block design game as shown in [14]. In non-game theoretic terms we can define a blocking coalition as a set of elements that intersects every block but contains no complete block. In order to show that a block design game has no equitable main simple solution, it is sufficient to show that there exists a blocking coalition, B, such that $|B| < k$ [14]. Hoffman and Richardson [14] show that the dual of every non-trivial STS has $|B| = k$. The non-trivial STS are those that have $r^* - k^* > 0$. In the following we seek $|B| < k$ and some information concerning minimum blocking coalitions.

4. SOME PROPERTIES OF STEINER TRIPLE SYSTEMS

Lemma 1. In a non-trivial STS, D^*, any block belongs to a set of at least three blocks, no pair of which have an element in common.

Proof. Form the incidence matrix of D^*, as follows. Call the elements $v_1, v_2, \ldots, v_{v^*}$ and the blocks $b_1, b_2, \ldots, b_{b^*}$. Let row i show which blocks contain v_i by entering 1 in the ith row and jth column if b_j contains v_i and 0 if b_j does not contain v_i. Let the elements of b_1 be labelled v_1, v_2, v_3. Since any element is contained in r^* blocks, let $b_1, b_2, \ldots, b_{r^*}$ contain v_1. Similarly, let $b_1, b_{r^*+1}, b_{r^*+2}, \ldots, b_{2r^*-1}$ contain v_2 and let

$b_1, b_{2r^*}, b_{2r^*+1}, \ldots, b_{3r^*-2}$ contain v_3. These b_j, $j = 1,2,\ldots,3r^* - 2$, are all distinct since $\lambda = 1$. We note now that for Case I,

$$3r^* - 2 = 9t - 2 < 6t^2 + t = b \quad \text{for all} \quad t \geq 2.$$

For Case II,

$$3r^* - 2 = 9t - 1 < (3t + 1)(2t + 1) = b \quad \text{for all} \quad t \geq 1.$$

It follows that for all non-trivial STS there is a block not accounted for above, say b_{3r^*-1}, which contains three new elements, say v_4, v_5, v_6. Since $\lambda = 1$, the 4th row of the incidence matrix will contain 1 in one and only one of columns $2,3,\ldots,r^*$ and 1 in one and only one of columns $r^* + 1$, $r^* + 2,\ldots,2r^* - 1$ and 1 in one and only one of columns $2r^*,\ldots,3r^* - 2$. The same result holds for the 5th row and the 6th row. In other words, in the incidence matrix, each of rows 4,5,6 is incident with four previously noted blocks and $r^* - 4$ others. Hence to accommodate all the blocks that contain v_4 or v_5 or v_6, we add $3(r^* - 4)$ to the $3r^* - 1$ blocks given above for a total of $6r^* - 13$ blocks.

In Case I, $6r^* - 13 = 18t - 13 < 6t^2 + t = b$ for $t \geq 2$. In Case II, $6r^* - 13 = 18t - 7 < 6t^2 + 5t + 1 = b$ for $t \geq 1$. These conditions upon t take care of all STS having $r^* - k^* > 0$. Therefore in any non-trivial STS the number of blocks containing elements of two disjoint blocks is less than the total number of blocks. It follows that D^* contains for any pair of disjoint blocks a third block forming with the pair a set of three mutually disjoint blocks. The condition $r^* - k^* > 0$ ensures the existence of at least one block that is disjoint to any given block.

Lemma 2. In a non-trivial Steiner triple system D^*, q mutually disjoint blocks, $q \geq 3$, have elements in common with at most $3q(r^* - q)$ additional blocks.

Proof. Let p represent the number of blocks of D^* that contain elements of, and are distinct from, the q mutually disjoint blocks. These blocks will be called p-blocks and q-blocks respectively. We form a submatrix of the adjacency matrix for the blocks of D^* as follows. Let any pair of blocks having exactly one element in common be called adjacent. Form a $q \times p$ submatrix by entering 1 in the ith row, jth column if the ith q-block and the jth p-block are adjacent. Enter 0 in the ith row, jth column if the ith q-block and the jth p-block are not adjacent. Let this matrix be designated by A.

If the jth p-block meets u of the q mutually disjoint blocks, $u = 1,2,3$, then the jth column of A will contain u ones. Let p_u designate the number of columns containing u ones and we shall refer to such columns as p_u-columns. Then

(9) $$p = p_1 + p_2 + p_3.$$

We note that the q mutually disjoint blocks contain $3q$ distinct elements each of which is replicated r^* times in D^*, which means $r^* - 1$ times in the set of p-blocks. Therefore, the total number of ones in A is

(10) $$3q(r^* - 1) = 3p_3 + 2p_2 + p_1.$$

By eliminating p_1 in (9) and (10) and solving for p, we obtain

(11) $$p = 3q(r^* - 1) - (2p_3 + p_2).$$

We seek a minimum value for $(2p_3 + p_2)$, given that $p_2, p_3 \geq 0$, and a further condition that we now derive. From A we obtain a relation involving p_2 and p_3 by considering the 2×2 submatrices consisting entirely of ones. Any pair of the q mutually disjoint blocks will generate in A nine 2×1 submatrices consisting entirely of ones, and any (unordered) pair of these nine submatrices will form a 2×2 submatrix of the required form; conversely, it is clear that any 2×2 submatrix consisting entirely of ones occurs in this way. Consequently the number of such matrices is

(12) $$_qC_2 \cdot {}_9C_2 = 18q(q - 1).$$

We now count in two ways the number of incidences of ones with 2×2 submatrices consisting entirely of ones. First of all, the total number of such incidences is four times the number of 2×2 submatrices consisting entirely of ones. Secondly, we note that if a 1 appears in a p-column, it is in eight of the specified 2×2 submatrices; and if a 1 is in a p_3 column it is in sixteen of the specified 2×2 submatrices. By equating these two values for the number of incidences of ones with the specified 2×2 submatrices, we obtain:

(13) $$4 \cdot 18q(q - 1) = (2p_2)8 + (3p_3)16$$

or

(14) $$2p_2 + 3(2p_3) = 9q(q - 1).$$

In (14) we have $2 < 3$, $p_2 \geq 0$, $2p_3 \geq 0$, and $9q(q - 1)$ a positive constant. Therefore, $p_2 + 2p_3$ will be a minimum when $p_2 = 0$. But $p_2 = 0$ implies from (14) that

(15) $$2p_3 = 3q(q - 1).$$

We may now conclude that

(16) $$\min(2p_3 + p_2) = 3q(q - 1).$$

By substituting $3q(q-1)$ for $\min(2p_3 + p_2)$, we obtain, from (11),

(17) $$p \leq 3q(r^* - q),$$

as required.

<u>Lemma 3.</u> Any block of a non-trivial Steiner triple system belongs to a set of t mutually disjoint blocks and any block of a non-trivial Steiner triple system with $r^* = 3t + 1$ belongs to a set of $t + 1$ mutually disjoint blocks.

<u>Proof.</u> From Lemma 2, q mutually disjoint blocks have elements in common with at most $3q(r^* - q)$ additional blocks. Therefore, a set of q mutually disjoint blocks can be extended to a set of $q + 1$ mutually disjoint blocks whenever

(18) $$q + 3q(r^* - q) < b^*.$$

We now show that this inequality holds for any $q < t$. In particular, when $r^* = 3t + 1$, the inequality holds for any $q < t$. By substituting $k = 3$ in (7) and (8), we have

(19) $$\text{Case I:} \quad v^* = 6t + 1, \quad b^* = 6t^2 + t, \quad r^* = 3t,$$

and

(20) $$\text{Case II:} \quad v^* = 6t + 3, \quad b^* = 6t^2 + 5t + 1, \quad r^* = 3t + 1.$$

We now seek q such that

(21) $$\text{Case I:} \quad q + 3q(3t - q) < 6t^2 + t$$

and

(22) $$\text{Case II:} \quad q + 3q(3t + 1 - q) < 6t^2 + 5t + 1.$$

The inequality (21) above may be rewritten in the form of a quadratic inequality:

(23) $$3q^2 - q(9t + 1) + (6t^2 + t) > 0, \quad t > 0,$$

which has for its solution set all real numbers q such that

(24) $$q < t \quad \text{or} \quad q > 2t + 1/3.$$

We note that v^*/k^* represents an upper bound for the number of mutually disjoint blocks in a BIBD so that $q > 2t + 1/3$ is impossible.

Similarly, the inequality (22) can be rewritten as

(25) $$3q^2 - q(9t + 4) + (6t^2 + 5t + 1) > 0, \quad t > 0,$$

which has for its solution set all real numbers q such that

(26) $$q < t + 1/3 \quad \text{or} \quad q > 2t + 1.$$

Again $q > 2t + 1$ is impossible.

We have now shown that given a set of q mutually disjoint blocks, $q \geq 3$; we may add a block so as to form a set of $q + 1$ mutually disjoint blocks whenever, for Case I, $q < t$ and, for Case II, $q \leq t$. From Lemma 1, $q = 3$ for all non-trivial STS. Lemma 3 now follows by induction.

5. SOME GAME-THEORETIC RESULTS

The following theorem is fundamental.

<u>Theorem 1.</u> Let D be the dual of a balanced incomplete block design D^* with $\lambda^* = 1$, $r^* - k^* > 0$. If the set of blocks of D^* has a subset, Q, containing q mutually disjoint blocks, $q > 1$, with the property that $\mathcal{C}\{v_q\}$, the complement of $\{v_q\}$, which is the set union of the elements contained in the blocks of Q, can be partitioned into subsets of $k^* - 1$ collinear elements, then the q mutually disjoint blocks together with the blocks determined in the partitioning of $\mathcal{C}\{v_q\}$ form a blocking coalition in D.

<u>Proof.</u> Denote the set of blocks determined in the partitioning of $\{v_q\}$ by C and let B_q be the coalition described in the conclusion of the theorem so that $B_q = Q \cup C$. We note, first of all, that the hypothesis $r^* - k^* > 0$ ensures the existence in D^* of at least two mutually disjoint blocks and the hypothesis concerning the partitioning of $\mathcal{C}\{v_q\}$ requires $k^*q \equiv v^* \pmod{k^* - 1}$.

B_q blocks since any element of D^* is in $\{v_q\}$ or in $\mathcal{C}\{v_q\}$. We now show that B_q does not contain every block on a single element by showing that any element, x, of D^* is contained in at least one block not in B_q. We note that any block of B_q contains either at least $k^* - 1$ elements of $\mathcal{C}\{v_q\}$ or exactly k^* elements of $\{v_q\}$ (but obviously not both since a block contains exactly k^* elements.)

Suppose that $x \in \{v_q\}$; then $x \in b_x$, some block of Q. Let y be an element of D^* such that $y \notin b_x$ and $y \in \{v_q\}$. Such a y exists since by hypothesis $q > 1$. (In fact, $k^*q \equiv v^* \pmod{k^* - 1}$ requires $q > 2$ whenever $k^* \neq 2$.) Then the block determined by x and y is not in Q since the blocks of Q form a partition of $\{v_q\}$, and the block determined by x and y is not in C since it contains at most $k^* - 2$ elements of $\mathcal{C}\{v_q)$. Hence for any $x \in \{v_q\}$ there is a block containing x and not in B_q.

Suppose that $x \in \mathcal{C}\{v_q\}$. Then x is in exactly one of the $k^* - 1$ subsets in the partition of $\mathcal{C}\{v_q\}$. The block of C containing the $(k^* - 1)$-subset that contains x has at most one element of $\{v_q\}$. Moreover, if there is a block of C that contains x and also contains an element of $\{v_q\}$, there is only one such block because of the partitioning of $\mathcal{C}\{v_q\}$. There is then an element, w, of $\{v_q\}$ that is not in the same block of C as x. The elements, w and x then determine a block of D^* that is not in Q and not in C, hence not in B_q.

Since any element of D^* is either in $\{v_q\}$ or in $\mathcal{C}\{v_q\}$, we can conclude that B_q cannot contain every block on a single element of D^* but every element of D^* is in some block of B_q and therefore the blocks of B_q when taken as elements in the dual of D^* form a blocking coalition in D.

Corollary 1. If in the dual D of a BIBD, D^*, with $r^* \equiv 0, 1 \pmod{k^*}$ and $\lambda^* = 1$, there exists a blocking coalition of the type described in the theorem with $q = (k^* - 1)n + 1$, then this blocking coalition has $k - n$ members.

Proof. $|B_q|$ is the sum of the number of blocks in Q and the number of blocks in C, that is,

$$|B_q| = q + (v^* - k^*q)/(k^* - 1). \tag{27}$$

By substituting for v^* the expressions given in (7) and (8), we obtain in both cases $|B_q| = k - n$. $(r^* = k)$.

Corollary 2. Any game that is dual to a BIBD with $r^* \equiv 0, 1 \pmod{k^*}$ and $\lambda^* = 1$ and has a blocking coalition of the type described in the theorem with $q = k^*$ has no equitable main simple solution.

Proof. It is sufficient to prove the existence of a blocking coalition such that $|B| \le k$ [14]. Corollary 2, therefore, follows from Corollary 1.

The preceding theorem has particular application to those games that are dual to Steiner triple systems.

Corollary 3. Any player in a game that is dual to a non-trivial Steiner triple system belongs to a blocking coalition of $k - 1$ members.

Proof. From Lemma 1 we have $q = 3$ and a partition of $\mathcal{C}\{v_q\}$ into pairs is always possible when q is odd. Corollary 3 then follows from Corollary 1.

Corollary 4. Any player in a game that is dual to a non-trivial Steiner triple system D^* belongs to a blocking coalition of $k - n$ members, where n is any positive integer such that for t odd, $n < (t - 1)/2$, and for t even $n \leq t/2$ when $r^* = 3t+1$ and $n \leq (t - 2)/2$ when $r^* = 3t$.

Proof. We take a $(2n + 1)$-subset of the t, or $t + 1$, mutually disjoint blocks which from Lemma 3 include the given block and form the appropriate blocking coalition by partitioning $\mathcal{C}\{v_q\}$ into pairs. Since $q = 2n + 1$, we have $|B| = k - n$ from Corollary 1. Since for Case I, $q < t$ and for Case II, $q \leq t$, n may assume the values stated in the corollary accordingly as t is odd or even.

The last two corollaries yield also the result that any game that is dual to a non-trivial Steiner triple system has no equitable main simple solution. However, we do not state this explicitly here since this result has been proved by Hoffman and Richardson [14], who give a construction for a blocking coalition of k members.

Corollary 5. If a BIBD with $\lambda = 1$ contains a set of v^*/k^* mutually disjoint blocks, the dual D has a minimum blocking coalition of v^*/k^* members.

<u>Proof.</u> The proof of the theorem follows for $\mathcal{C}\{v_q\} = \emptyset$. $|B| = v^*/k^*$ is obviously a minimum.

The duals of most of the block designs with $r^* - k^* > 0$, $\lambda^* = 1$, $r^* \equiv 1 \pmod{k^*}$ that have been given by direct construction have v^*/k^* mutually disjoint blocks and therefore have minimum blocking coalitions of v^*/k^* elements. Some of these are indicated in a later section of this paper. We make explicit the case of the finite Euclidean geometries in the following corollary.

<u>Corollary 6.</u> If D^* is the system of lines in the finite Euclidean space $EG(m,p^n)$ of m dimensions over the Galois field $GF(p^n)$ for $m \geq 2$ and $p^n \geq 2$, the dual D has a minimum blocking coalition of $p^{n(m-1)}$ members and has no equitable main simple solution.

<u>Proof.</u> A set of $p^{n(m-1)}$ mutually parallel lines in D^* provides the required minimum blocking coalition in D and $p^{n(m-1)} < r^* = k$; hence D has no equitable main simple solution.

We note that the duals of the finite projective spaces are not so readily handled since only the odd-dimensional cases satisfy $r^* \equiv 1 \pmod{k^*}$. We can, however, dispose of the problem of whether the dual of the system of lines in a finite projective space has an equitable main simple solution by enlarging the construction devised by Hoffman and Richardson [14] to obtain a blocking coalition of k members in the dual of a Steiner triple system.

<u>Theorem 2.</u> If D^* is the system of lines in the finite projective space $PG(m,p^n)$ of m dimensions over the Galois field $GF(p^n)$ with $m \geq 3$ and $p^n \geq 2$, then the dual D has a blocking coalition of

$1 + p^n + p^{2n} + \ldots + p^{(m-1)n}$ members and hence has no equitable main simple solutions.

Proof. Let O be any point in $PG(m,p^n)$. Consider the pencil of lines on O. Since $\lambda^* = 1$, every point of $PG(m,p^n)$ is on some line of the pencil. Choose $p^n + 1$ coplanar lines on O. Call the plane containing these lines P. Let A be a point in P such that $A \neq O$. Then on A there is in P a pencil of lines containing every point of P. In the pencil of lines of D^* on O replace the lines that are in P by the previously noted plane pencil on A. The resulting set of lines forms a blocking coalition of $r^* = 1 + p^n + {}^2p^n + \ldots + p^{(m-1)n}$ in the dual. Since we have $|B| = r^* = k$, D has no equitable main simple solution.

The method of construction of the blocking coalition in Theorem 1 enables us to solve the problem of determining the minimum number of elements in a blocking coalition in the dual of a Steiner triple system, at least to the extent of expressing a minimum blocking coalition in terms of the maximum number of mutually disjoint blocks in the system. For any given STS a minimum blocking coalition in the dual can easily be found by application of the following theorem.

Theorem 3. Let D^* denote a non-trivial Steiner triple system. In the set of blocks $\{b_i\}$ in D^* let M_q be the largest subset having the property that the members of M_q are mutually disjoint blocks of D^* and M_q contains q members where q is odd. Let $\{v_q\}$ be the subset of elements of D^* contained in the blocks of M_q. In the dual D let B_q be the blocking coalition consisting of the members of M_q and the blocks of D^* determined by a partition of $\mathcal{C}\{v_q\}$ into pairs. Then B_q is a minimum blocking coalition in D.

Proof. That B_q is a blocking coalition in D follows from Theorem 1. We now show that $|B_q|$ is a minimum. Suppose the contrary. Then there exists a blocking coalition B_x such that $|B_x| < |B_q|$. Let $\{b_1, b_2 \ldots, b_x\}$ be the largest mutually disjoint subset of B_x and let $|B_x| = z$. Then

(28) $$B_x = \{b_1, b_2, \ldots, b_x, b_{x+1}, \ldots . b_z\}.$$

B_x blocks, therefore,

(29) $$\{v_{v*}\} = \{b_1 \cup b_2 \cup \ldots \cup b_x \cup b_{x+1}, \ldots, \cup b_z\},$$

where $\{v_{v*}\}$ is the set of all the elements of D^*.

Since, in the expression for $\{v_{v*}\}$ above, the first x blocks are mutually disjoint and since there are not $x + 1$ mutually disjoint blocks in B_x, it follows that

(30) $$|(b_1 \cup b_2 \cup \ldots \cup b_j) \cap b_{j+1}| \geq 1 \quad \text{for} \quad j \geq x.$$

In computing $|B_x|$, therefore, any $b_j, j > x$, adds at most two new elements to the set union. Suppose, now, that x is odd. Then z is at least so large as to satisfy

(31) $$v^* = 3x + 2(z - x)$$

and (31) implies that $z \geq (v^* - x)/2$. But by assumption $z < |B_q|$; hence

(32) $$(v^* - x)/2 < q + (v^* - 3q)/2 = (v^* - q)/2.$$

By (32), $x \geq q$. This is impossible since q is the largest odd-numbered mutually disjoint subset of the set of blocks in D^*.

Suppose x is even. Then at least one block, say b_j, for some $j > x$, adds only one new element to the set union so that z is at least large enough to satisfy

(33) $$v^* = 3x + 2(z - 1) - x) + 1,$$

which implies that $z \geq (v^* - (x - 1))/2$. But $z < |B_q|$; hence

(34) $$(v^* - (x - 1))/2 < (v^* - q)/2,$$

which implies that $x - 1 > q$. However, any subset of a set of mutually disjoint blocks is a set of mutually disjoint blocks. Since x is even, any $(x - 1)$-subset of the x mutually disjoint blocks of B_x will be an odd-numbered set of mutually disjoint blocks in D^*. The inequality $x - 1 > q$, therefore, contradicts the assumption that M_q is the largest odd-numbered set of mutually disjoint blocks in D^*. Hence $|B_q|$ is a minimum.

6. EXAMPLES

We conclude this discussion of block design games by presenting a number of examples illustrating the preceding results.

Example 1. The dual of a Kirkman triple system of order $v^* = 6t + 3$ has a minimum blocking coalition of $2t + 1$ members. By definition, a Kirkman triple system of order $6t + 3$ is a Steiner triple system with the additional stipulation that the set of $b^* = (2t + 1)(3t + 1)$ triples be partitioned into $3t + 1$ components, each of which is a $(2t + 1)$-subset of triples with each element of the triple system appearing exactly once in a component [22. p. 101]. Each component is thus a set of $2t + 1$ mutually disjoint blocks and Corollary 5 applies.

Example 2. The "method of symmetrically repeated differences," due to Bose [4], for the construction of a BIBD yields, when applied to designs with $v^* = (k^* - 1)k^*t + k^*$ and $\lambda^* = 1$, $k^*t + 1$ transitive constituents (with respect to blocks) of $(k^* - 1)t + 1$ blocks each, and has an automorphism of order $(k^* - 1)t + 1$; and one of the transitive constituents consists of a set of $(k^* - 1)t + 1$ mutually disjoint blocks that constitute a minimum blocking coalition in the dual. For example, a Steiner triple system on 15 letters can be constructed by applying the transformation

$$T = (1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10)(11\ 12\ 13\ 14\ 15)$$

to the basis blocks:

$$B_1 = (1,4,10), \quad B_6 = (2,3,10), \quad B_{11} = (6,9,15),$$
$$B_{16} = (7,8,15), \quad B_{21} = (11,14,5), \quad B_{26} = (12,13,5),$$
$$B_{31} = (5,10,15).$$

The last five blocks; (5,10,15), (1,6,11), (2,7,12), (3,8,13), and (4,9,14), which are based on B_{31}, are a set of five mutually disjoint blocks [9,p. 117].

<u>Example 3.</u> There is another construction due to Bose [4] for BIBD with $(k^* - 1)k^*t + k^*$ elements and $\lambda^* = 1$, where the k^*t nonzero elements of a finite field with $k^*t + 1$ elements are used $k^* - 1$ times to obtain $(k^* - 1)t$ basis blocks to which is added a single basis block containing an element ∞ and $k^* - 1$ replications of the zero element of the field in the form

$$s(v^* - 1)/(k^* - 1),$$

where $s = 1,2,\ldots,k^* - 1$, so that when the transformation $T = (1\ 2\ldots m)(m + 1\ldots 2m)(2m + 1\ldots 3m)\ \ldots\ ((k^* - 2)m\ldots v^* - 1)$, where $m = (v^* - 1)/(k^* - 1)$, is applied to the $(k^* - 1)t + 1$ basis blocks, each element appears once with every other element and each element occurs $k^*t + 1$ times in all. Here the basis blocks are mutually disjoint and form in the dual a minimum blocking coalition as noted in Theorem 1, Corollary 5.

For example, when $v^* = 40$, $b^* = 130$, $r^* = 13$, $k^* = 4$, $\lambda^* = 1$, the ten basis blocks are:

$B_1 = (1,12,18,21)$, $B_{14} = (4,9,20,19)$,

$B_{27} = (3,10,15,24)$, $B_{40} = (14,25,31,34)$,

$B_{53} = (17,22,33,32)$, $B_{66} = (16,23,28,37)$,

$B_{79} = (27,38,5,8)$, $B_{92} = (30,35,7,6)$,

$B_{105} = (29,36,2,11)$, $B_{118} = (\infty,13,26,39)$,

and the remaining blocks are obtained by applying

$$T = (1\ 2\ 3 \ldots 13)(14\ 15 \ldots 26)(27\ 28 \ldots 39)$$

[7, p. 89]. The ten basis blocks, when taken as players in the dual, form a minimum blocking coalition.

Example 4. It is well known that a non-symmetric BIBD with $r^* \equiv 1 (\mathrm{mod}\ k^*)$ and $\lambda^* = 1$ can be constructed whenever an appropriate difference set can be found. For example, take $v^* = 21$, $b^* = 3(21) + 7$, $k^* = 3$, $r^* = 10$, $\lambda^* = 1$. An appropriate difference set is given by

$$S = (0,3,9 | 0,1,5 | 0,2,10 | 0,7,14),$$

so that the design is constructed by taking as blocks:

$(x, x + 3, x + 9)$, $(x, x + 1, x + 5)$, $(x, x + 2, x + 10)$, $(x, x + 7, x + 14)$,

where $x = 0,1,2,\ldots,20$ and sums are taken (mod 21). The last basis block provides a set of seven mutually disjoint blocks which form a minimum blocking coalition in the dual.

A number of non-isomorphic Steiner triple systems on 15 letters may be constructed by this method. We give two examples:

STS_1: $(x, x + 1, x + 4)$, $(x, x + 2, x + 8)$, $(x, x + 5, x + 10)$;

STS_2: $(x, x + 1, x + 4)$, $(x, x + 6, x + 13)$, $(x, x + 5, x + 10)$;

We note that STS_1 above is PG(3,2) [24,p.203]. The dual of any BIBD with $\lambda^* = 1$ and constructed by this method has a minimum blocking coalition of v^*/k^* members and hence no equitable main simple solution since

$$v^*/k^* < r^* = k.$$

Example 5. We present, finally, an example of a block design that has less than v^*/k^* mutually disjoint blocks. Let $v^* = 15, b^* = 35$, $k^* = 3$, $r^* = 7$, $\lambda^* = 1$, and let the first 15 letters of the alphabet represent the elements. Then the blocks of a BIBD with these parameters may be represented by

B_1 = ABC,

B_2 = ADE, B_8 = BDF, B_{14} = CDG,

B_3 = AFG, B_9 = BEG, B_{15} = CHK,

B_4 = AHI, B_{10} = BHJ, B_{16} = CIJ, B_{20} = DHL, B_{24} = EFH,

B_5 = AJK, B_{11} = BIK, B_{17} = CLO B_{21} = DIN, B_{25} = EJM,

B_6 = ALM, B_{12} = BLN, B_{18} = CNE, B_{22} = DKM, B_{26} = EKO,

B_7 = ANO, B_{13} = BMO B_{19} = CFM, B_{23} = DJO, B_{27} = EIL,

B_{28} = FOI, B_{30} = FKN, B_{32} = GHO, B_{34} = GJN,

B_{29} = FJL, B_{31} = HMN, B_{33} = GIM, B_{35} = GKL.

This design does not contain a set of five mutually disjoint blocks. However, by Lemma 1 any block belongs to a set of three mutually disjoint blocks. Consider B_1. The blocks B_1, B_{20}, B_{28} form a set of three mutually disjoint blocks containing elements A, B, C, D, H, L, F, O, I. By partitioning the remaining elements into pairs, we obtain EG, JK, MN, which determine B_9, B_5, B_{31} respectively. A blocking coalition of $k - 1$ elements in the dual is given by $B_1, B_{20}, B_{28}, B_9, B_5, B_{31}$. This is a minimum blocking coalition since the largest odd-numbered set of mutually disjoint blocks in D^* contains three blocks.

The design used in this example was constructed by trial and error. We note, however, that all the 80 non-isomorphic triple systems on 15 letters have been enumerated by Cole, White, and Cummings [5] and independently by Hall and Swift [10]. No attempt has been made to classify either the preceding design or those of Examples 2 and 4 in the context of these two references.

CHAPTER II. BLOCK DESIGNS AND GRAPH THEORY*

1. INTRODUCTION

The purpose of the following discussion is to demonstrate the relation of balanced incomplete block designs to certain concepts of graph theory. The set of blocks of a balanced incomplete block design with $\lambda = 1$ is shown to be related to a maximum internally stable set of vertices of a suitably defined graph. The development yields also an upper bound for the internal stability number of a large subclass of a class of graphs which we call "graphs on binomial coefficients." In a different but related context every balanced incomplete block design with $\lambda = 1$ is shown to be a solution of a suitably defined irreflexive relation. Some examples of relativizations and extensions of solutions of irreflexive relations (as developed by Richardson [18-20]) are generated as a result of the concepts derived.

Following a suggestion implied by Berge [2] we introduce the

Definition. A graph on the binomial coefficient $\binom{v}{k}$ with edge parameter λ, written $G\binom{v}{k}_{\lambda}$, is a graph whose vertices are the $\binom{v}{k}$ possible k-tuples which can be formed from v elements and having as adjacent vertices those pairs of vertices which have more

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than λ and less than k elements in common.

Obviously the definition requires $k - \lambda > 1$ and $v > k$.

An internally stable set of vertices of any graph G is a set of vertices no pair of which are adjacent. A maximum internally stable set of vertices is an internally stable set of vertices the number of which is greater than or equal to the number of vertices in any other internally stable set. A graph may have several maximum internally stable sets of vertices. The number of vertices in a maximum internally stable set is the internal stability number of G and is symbolized by $\alpha(G)$ (cf. Berge [2, p.472])

Berge [2] indicates that any maximim internally stable set of vertices of the particular graph $G\binom{15}{3}_1$ is a BIBD with $(v, b, r, k, \lambda) = (15,35,7,3,1)$. We note, however, that there are some binomial coefficients for which maximum internally stable sets of vertices are not BIBD at all or, if they are BIBD, the parameter v of the graph is not the same as the v of the block design. The observation we have just made can be seen more clearly from the following remarks.

To begin, we can define $G\binom{v}{k}_\lambda$ for any integers v, k, λ provided $k - \lambda > 1$ and $v > k$. However, for a BIBD it is necessary that the parameters satisfy

(1) $$vr = bk,$$

(2) $$r(k - 1) = \lambda(v - 1)$$

Therefore, while $G\binom{v}{k}_\lambda$ exists for any v,k,λ with $v > k$, $k-\lambda > 1$ and although such graphs have maximum internally stable sets of vertices, these will certainly not be BIBD having the v of $G\binom{v}{k}_\lambda$ as a parameter for those values of v,k,λ for which we cannot find integers b and r so as to satisfy Eqs,(1) and (2). Moreover, there are quintuples of parameters v,b,r,k,λ which satisfy Eqs.(1) and (2) but for which a BIBD does not exist. A notable example is the case

$$(v,b,r,k,\lambda) = (36,42,7,6,1).$$

A BIBD with these parameters, if it existed, would be a finite affine plane with 36 points and 42 lines. Such a plane has long been known not to exist. We seek, therefore, to establish a characterization of those maximum internally stable sets of vertices which are balanced incomplete block designs.

2. THE INTERNAL STABILITY NUMBER FOR $G\binom{v}{k}_{\lambda=1}$

Our first theorem introduces an upper bound for the internal stability number of a graph on a binomial coefficient and having edge parameter $\lambda = 1$.

Theorem 4. The internal stability number of a graph G on the binomial coefficient $\binom{v}{k}$ with edge parameter $\lambda = 1$ is less than or equal to $\lambda v(v - 1)/k(k - 1)$. (In symbols, $\alpha(G\binom{v}{k}_{\lambda=1}) \leqq \lambda v(v - 1)/k(k-1)$.)

<u>Proof</u>. Let S be an internally stable set of vertices of $G\binom{v}{k}_{\lambda=1}$ so that

(3) $$S = \{s_1, s_2, \ldots, s_n\}.$$

Let x be an element appearing in some vertex s_j. x can appear only once in a given vertex since by definition a k-tuple contains k distinct elements. Any k-tuple in which x appears contains $k - 1$ distinct elements which are also distinct from x. Moreover, the element x cannot appear with the same element in more than one k-tuple, for suppose the vertices s_i, and s_j, $i \neq j$, each contain the pair of elements x,y with $x \neq y$. Then s_i and s_j are adjacent since they have more than $\lambda = 1$ elements in common, thus violating the assumption that S is internally stable. Each of the v elements, then, can appear in at most $\lambda(v - 1)/(k - 1)$, for $\lambda = 1$, blocks in an internally stable set of vertices. The product of the number of elements by the number of times each element may appear in S without violating the internal stability of S, that is, $\lambda v(v - 1)/(k - 1)$, counts each possible appearance of the same element k times. Therefore, the total number of k-tuples which can accommodate a maximum number of appearances of each element and still form an internally stable set of vertices is $\lambda v(v - 1)/k(k - 1)$, where $\lambda = 1$.

<u>Theorem 5</u>. If $\alpha(G\binom{v}{k}_{\lambda=1}) = \lambda v(v - 1)/k(k - 1)$ then a maximum internally stable set of vertices is a balanced incomplete block design with the given v,k,λ as parameters.

Proof. From the proof of Theorem 1 it follows that, if

$$\alpha(G\binom{v}{k}_{\lambda=1}) = \lambda v(v-1)/k(k-1),$$

then in a maximum internally stable set of vertices each of the v elements appears exactly $(v-1)/(k-1)$ times. This means that each of the v elements appears once with every other element so that a maximum internally stable set with $\lambda v(v-1)/k(k-1)$ members, $\lambda = 1$, forms a BIBD with parameters

(4) $$(v,b,r,k,\lambda) = (v,\alpha(G), (v-1)/(k-1), k, 1).$$

Theorem 6. A balanced incomplete block design with $\lambda = 1$ and parameters v,k is a maximum internally stable set of vertices of $G\binom{v}{k}_{\lambda=1}$.

Proof. If D is a BIBD with v,k given and $\lambda=1$, we can assume $v > k$ and $k - \lambda > 1$; otherwise the design is trivial. Then from Eqs. (1) and (2)

(5) $$r = \lambda(v-1)/(k-1) \quad \text{and} \quad b = \lambda v(v-1)/k(k-1), \ \lambda = 1.$$

Moreover, each pair of blocks have at most one element in common since, from the definition of BIBD when $\lambda = 1$, each pair of distinct elements determines a unique block. Therefore, the set of blocks is an internally stable set of vertices of $G\binom{v}{k}_{\lambda=1}$. This set is a maximum since the number of blocks is equal to the upper bound established in Theorem 4 for the internal stability number of a graph on a binomial coefficient and having $\lambda = 1$.

3. INEQUALITIES AMONG INTERNAL STABILITY NUMBERS

We consider now families of graphs on binomial coefficients. Two graphs which have the same k and the same λ will be considered to be members of the same family. We establish some inequalities for the internal stability numbers of distinct graphs of the same family of graphs on binomial coefficients.

Theorem 7. If $v_1 > v_2$ then for a fixed k and a fixed λ

$$\alpha(G\binom{v_1}{k}_\lambda) \geqq \alpha(G\binom{v_2}{k}_\lambda).$$

Proof. The result follows immediately from the observation that, if λ and k are fixed and $v_1 > v_2$, the k-tuples which form a maximum internally stable set of vertices of $G\binom{v_2}{k}_\lambda$ form an internally stable set of vertices of $G\binom{v_1}{k}_\lambda$.

Theorem 8. If $v_1 - v_2 \geqq n(k - 1)$ for some integer $n > 0$ then

$$\alpha(G\binom{v_1}{k}_{\lambda=1}) \geqq \alpha(G\binom{v_2}{k}_{\lambda=1}) + n.$$

Proof. Let M_a represent a maximum internally stable set of vertices of $G\binom{v_2}{k}_{\lambda=1}$. Then M_a is an internally stable set of vertices of $G\binom{v_1}{k}_{\lambda=1}$ which uses at most only $v_1 - n(k - 1)$ of the v_1 elements from which the k-tuples forming the vertices of $G\binom{v_1}{k}_{\lambda=1}$ are chosen. To the set M_a we add n additional k-tuples in the following manner: Partition $n(k - 1)$ elements which are in the set of v_1 elements and not in the set of v_2 elements into n mutually disjoint subsets of $k - 1$ elements each. To each of these $(k - 1)$-tuples add an element which appears in M_a to yield n k-tuples which, when combined with the k-tuples in M_a, form an internally stable set of vertices of $G\binom{v_1}{k}_{\lambda=1}$. Either this set which contains $\alpha(G\binom{v_2}{k}_{\lambda=1}) + n$ vertices or some larger set is a maximum.

The preceding theorems have some interesting applications.

Example 1. Since there is no BIBD for $v = 36$, $k = 6$, $\lambda = 1$, we know from Theorems 4 and 5 that $\alpha(G\binom{36}{6}_1) < 42$. From the existence of a projective plane of order 5 we have from Theorem 6 that $\alpha(G\binom{31}{6}_1) = 31$. Hence from Theorem 8, since $36 - 31 = 1(6 - 1)$, it follows that $\alpha(G\binom{36}{6}_1) \geqq 32$.

Example 2. Since the existence of a finite affine plane of order 10 is an unsolved problem, $\alpha(G\binom{100}{10}_1)$ is unknown. However, from the existence of a finite projective plane of order 9, we know from Theorem 6 that $\alpha(G\binom{91}{10}_1) = 91$. We can conclude from Theorem 8 that $\alpha(G\binom{100}{10}_1) \geqq 92$. We know also from Theorem 4 that $\alpha(G\binom{100}{10}_1) \leqq 110$.

4. BLOCK DESIGNS AS INTERNALLY STABLE SETS

From the set of k-tuples which form the vertices of $G\binom{v}{k}_{\lambda=1}$ we can select an internally stable set which uses only u elements for some $u \leq v$. We inquire which such internally stable sets are balanced incomplete block designs. We present two theorems that are generalizations of the previous development.

We establish by definition a set of parameters for internally stable sets of vertices of graphs on binomial coefficients and having $\lambda = 1$. Let S be an internally stable set of vertices of $G\binom{v}{k}_{\lambda=1}$ and let b_s represent the number of vertices in S. Let v_i be the number of elements which appear r_i times in S. If the set union of the vertices belonging to S contains u distinct elements then, obviously, $\Sigma\, v_i = u$. Since the u elements in S are a subset of the v elements in the vertices of $G\binom{v}{k}_{\lambda=1}$, then S internally stable in G implies that S is an internally stable set of vertices of $G\binom{u}{k}_{\lambda=1}$; b_s, then, has an upper bound established in Theorem 1. Represent this upper bound by b so that

(6) $$b_s \leq b = \lambda u(u-1)/k(k-1),\ \lambda = 1.$$

We call u, k, b_s, r_i, $\lambda = 1$, the parameters of the internally stable set of vertices and we establish the following relations.

<u>Theorem 9</u>. If S is an internally stable set of vertices of $G\binom{v}{k}_{\lambda=1}$ then the parameters of S satisfy $r_i(k-1) \leq \lambda(u-1)$, $\lambda = 1$, and $\Sigma\, v_i r_i = b_s k \leq bk$.

<u>Proof.</u> Since $\lambda = 1$ the maximum number of times which any element can appear in S is $(u - 1)/(k - 1)$, otherwise two vertices of S would have more than $\lambda = 1$ elements in common, which contradicts the assumption that S is internally stable. This means that

(7) $$r_i \leq \lambda(u - 1)/(k - 1), \ \lambda = 1.$$

The first inequality stated in the theorem now follows from (7).

To prove the second inequality we note that the product bk gives the maximum number of possible element appearences since it counts the number of places to be filled in forming an internally stable set of vertices in which each element appears a maximum number of times and where there are u elements and $\lambda = 1$. $\Sigma v_i r_i$ counts the total number of actual element appearances in S so that

(8) $$\Sigma v_i r_i = b_s k \leq bk.$$

<u>Theorem 10.</u> If S, an internally stable set of vertices of $G\binom{v}{k}_{\lambda=1}$, has $u(u - 1)/k(k - 1)$ vertices and contains u distinct elements, then S is a balanced incomplete block design with parameters

$$(v,b,r,k,\lambda) = (u,u(u - 1)/k(k - 1), (u - 1)/(k - 1),k,1).$$

<u>Proof.</u> As noted above, S is internally stable in $G\binom{u}{k}_{\lambda=1}$ and, therefore, Theorem 10 follows from Theorem 5.

5. BLOCK DESIGNS AS SOLUTIONS OF IRREFLEXIVE RELATIONS

We have sought in the foregoing development to characterize from among internally stable sets of vertices of graphs on binomial coefficients those sets which are balanced incomplete block designs. The concepts involved and even the possibility of such a characterization is related to a study made in a different terminology and in a vastly more general context by Richardson [18-20] in an investigation of solutions of irreflexive relations. (See also Harary and Richardson [13]). We proceed now to show that every balanced incomplete block design with $\lambda = 1$ is a solution of the irreflexive relation established by each of several suitably defined graphs from the same family of graphs.

We note, first of all, that the relation of adjacency of vertices is a binary relation on the domain of vertices of $G\binom{v}{k}_{\lambda}$. We symbolize this relation by $\succ$ and $x \succ y$ means "x and y are adjacent." If x and y are vertices of some $G\binom{v}{k}_{\lambda}$, then $x \succ y$ if and only if x and y have more than λ and less than k elements in common. The relation $\succ$ is irreflexive since $x \succ x$ would imply that the k-tuple denoted by x has less than k elements in common with itself, which is impossible. The most general irreflexive relations make no assumption concerning either symmetry or transitivity. However, the relation we have called "adjacency" implies symmetry and is indifferent to transitivity. The implication of symmetry may be arrived at in two ways. We may consider the graph to be unoriented. In this case, since no orientation has been assigned, $x \succ y$ implies $y \succ x$. We may, on the other hand, consider the graph to be a symmetric directed graph, which means that if y is adjacent from x then x is adjacent from y and, again, $x \succ y$ implies $y \succ x$. The fact that the irreflexive relation we are considering here is symmetric is not catastrophic. A great deal of work has been done with irreflexive relations in which asymmetry is implied as, for example, in the so-called ancestor relations. Analogously the presence of symmetry does not preclude an investigation of the nature of solutions.

A solution of an irreflexive relation is defined as follows: Let P be the set of all vertices for which the relation $\succ$ has been defined. A subset S of P is a solution of the irreflexive relation if the following two conditions hold:

(1) For any two vertices $a, b \in S$ it is not true that $a \succ b$.

(2) For any $a \in P - S$, there exists $b \in S$ such that $b \succ a$.

(This definition is the same as that used in Richardson [18] except that we have called the members of the set P "vertices" rather than "elements" since we have reserved the word "elements" for the v entities from which the k-tuples are chosen to form the vertices of the graph.) We are now in a position to explore the relation of balanced incomplete block designs to solutions of certain irreflexive relations. We do this for designs having $\lambda = 1$.

<u>Theorem 11.</u> Every balanced incomplete block design with parameters $(v,b,r,k,\lambda = 1)$ is a solution of an irreflexive relation defined by a graph on the binomial coefficient $\binom{v}{k}$ and having edge parameter $\lambda = 1$.

<u>Proof.</u> Let S be the set of k-tuples which are the blocks of the given BIBD. Then S is a subset of the set of vertices defined by $G\binom{v}{k}_{\lambda=1}$. Let $\succ$ be the irreflexive relation defined by G, that is, for any two vertices $x,y, x \succ y$ if and only if x and y have

more than λ and less than k elements in common. If $a \in S$ and $b \in S$ then $a \not\succ b$ since no two blocks of a BIBD having $\lambda = 1$ have more than one element in common.

Let P be the set of vertices of $G\binom{v}{k}_{\lambda=1}$. If $a \in P - S$ consider any pair of elements in a, say v_1, v_2. Since $\lambda = 1$ the pair v_1, v_2 determines a unique block in the BIBD and this block will have two elements in common with a. This means that for any $a \in P - S$ there is $b \in S$ such that $b \succ a$. The BIBD is, therefore, a solution, since it satisfies both relations required in the definition of solution.

The preceding theorem can be generalized to show that a BIBD with $\lambda = 1$ is a solution to each of the irreflexive relations defined by several graphs from the same family of graphs.

<u>Theorem 12.</u> If S is the set of blocks of a BIBD with parameters $(v,b,r,k,\lambda = 1)$ then S is a solution of each of the irreflexive relations defined respectively by $G\binom{v+q}{k}_{\lambda=1}$ for each q such that $0 \leqq q \leqq k - 2$ and S is not a solution for $q \geqq k - 1$.

<u>Proof.</u> The proof is the same as that for Theorem 11. Clearly if $a \neq b$ and $a \in S$ and $b \in S$ then $a \not\succ b$, since no two distinct blocks of S have more than one element in common, and if $a = b$ then $a \not\succ b$ as noted previously. Also, since $q \leqq k - 2$, any k-tuple in P, the set of all vertices of $G\binom{v+q}{k}_{\lambda=1}$, contains a pair of elements, say v_1, v_2, which belong to the set of v elements contained in the blocks of S considered as k-sets. It follows that if

$a \in P - S$ there exists $b \in S$, where b is determined by v_1, v_2, such that $b \succ a$.

If $q \geqq k - 1$ then $P - S$ contains a k-tuple consisting of $k - 1$ elements not in S and some other element. This k-tuple is not adjacent to any k-tuple in S. Therefore, S is not a solution when $q \geqq k - 1$.

We pause now to consider the significance of Theorems 11 and 12. It is well known that every symmetric irreflexive relation has a solution. This is merely a restatement from graph theory that every symmetric graph without loops has a kernel, since "kernel" is synonymous with "solution" in the sense that "kernel" is used to designate from the set of vertices of a graph a subset that satisfies the same conditions as those needed for a solution when adjacency of vertices is taken as the irreflexive relation (cf. Berge [1,p.47]). A graph on a binomial coefficient, as we have defined it above, is a symmetric graph without loops; hence it has a kernel and the irreflexive relation which it defines has a solution. The significance of Theorems 11 and 12 is that they provide a particular solution which, if it exists, is a maximum internally stable set of vertices.

We consider now the number of vertices in a solution of the irreflexive relation determined by a graph on the binomial coefficient $\binom{v}{k}$ and having $\lambda = 1$. We show by example that the number is not constant for a given relation. Consider $G\binom{15}{3}_1$, which has for its vertices all possible triples that can be formed from 15 elements. Any one of the 80 Steiner triples systems on 15 letters (elements) (see [5] and [7]), is a solution containing 35 vertices. The number 35 is a maximum since $\alpha(G\binom{15}{3}_1 \leq 35$ by Theorem 1 and any solution is internally stable. We can find a kernel of a symmetric graph without loops by maximizing any internally stable set of vertices. Any kernel of $G\binom{13}{3}_1$ can be maximized to form a kernel of $G\binom{15}{3}_1$. This is shown in

Example 3. There are exactly two BIBD having $(v,b,r,k,\lambda) = (13,26,6,3,1)$ (cf. Hall [7]). One such design is given by taking the following triples as blocks:

1, 2, 3				
1, 4, 5	2, 4, 6	4, 3, 8	7, 8, 13	3, 5, 12
1, 6, 7	2, 5, 7	4, 7, 9	7, 10, 12	3, 6, 10
1, 8, 9	2, 8, 10	3, 10, 13	8, 5, 11	3, 9, 13
1, 10, 11	2, 9, 12	4, 11, 12	8, 6, 12	5, 6, 13
1, 12, 13	2, 11, 13	7, 3, 11	6, 9, 11	5, 9, 10.

To this set of 26 triples, we add an additional triple in the manner of Theorem 8, say the triple (1, 14,15). These 27 triples form a solution of the irreflexive relation defined by $G\binom{15}{3}_1$. This is easily verified in the following way. The block design is internally stable and the triple (1, 14, 15) is not adjacent to any of the 26 triples in the block design. Call the set of 27 triples S. From the preceding remarks we see that S is internally stable. Any vertex of G contains either two elements of the first 13 elements or only one element of the first 13 elements. If the latter, the vertex contains the pair 14, 15. In either case any vertex not in S is adjacent to a vertex in S. It follows that S is a solution for $G\binom{15}{3}_1$.

We have, then, for the same graph on a given binomial coefficient a solution containing 35 vertices and a solution containing 27 vertices. The larger solution is a BIBD and the smaller is not a BIBD. It seems logical to ask whether there can exist a graph on a binomial coefficient and having $\lambda = 1$ which has two solutions of differing cardinal numbers but such that both solutions are BIBD. We now show that this is impossible.

We have been considering those balanced incomplete block designs which have $\lambda = 1$ and we have noted that the parameters of a BIBD must satisfy Eqs. (1) and (2). If we consider all quintuples of parameters that satisfy Eqs. (1) and (2) for a fixed k and for $\lambda = 1$, then these equations provide a system of equations for which a solution in integers v, b, r is required. The following lemma gives a general necessary relation between r and k.

<u>Lemma.</u> A balanced incomplete block design with $\lambda = 1$, $k = p^m q$ where q is a positive integer and p is a prime such that p does not divide q, has $r \equiv 0,1 \pmod{p^m}$.

<u>Proof.</u> We have from Eqs. (1) and (2) that $vr = bk$ and $r(k - 1) = \lambda(v - 1)$. Let $k = p^m q$, and $\lambda = 1$ and $r \equiv x \pmod{p^m}$ so that $r = p^m t + x$ for some non-negative integers t and x. By substituting in the basic relations (1) and (2) and eliminating v, we obtain

$$(9) \qquad (p^m t + x)^2 (p^m q - 1) + (p^m t + x) = bp^m q.$$

This implies that

(10) $(p^m t + x) \mid bp^m q.$

We now consider the possibilities for relation (10). There are three cases which we consider separately:

Case 1: $(p^m, p^m t + x) = p^m$. Then $r \equiv 0 \pmod{p^m}$ as required.

Case 2: $(p^m, p^m t + x) = 1$. Then Eq. (10) implies

(11) $(p^m t + x) \mid bq.$

We now divide each member of Eq. (9) by $p^m t + x$ and obtain upon simplification

(12) $p^m q(p^m t + x) - p^m t - (x - 1) = (bq/(p^m t + x))p^m,$

where $bq/(p^m t + x)$ is an integer. Eq. (12) implies that $p^m \mid (x - 1)$. This, in turn, implies $r \equiv 1 \pmod{p^m}$.

Case 3: $(p^m, p^m t + x) \neq 1, p^m$. Then since p is a prime

(13) $(p^m, p^m t + x) = p^{m-n}$, for some $n < m$,

so that $x = ap^{m-n}$ for some integer a. When Eq. (13) is applied in evaluating the right member of Eq. (12), that term assumes the form $(bq/(p^n t + x))p^n$, where $bq/(p^n t + x)$ is an integer. Eq. (12) now implies $p^n \mid (x - 1)$. It follows that $x - 1 = cp^n$ for some integer c. If $c = 0$ we have the required result that $x = 1$. If $c \neq 0$ then

(14) $x = ap^{m-n} = cp^n + 1,$

which is impossible. This completes the proof of the lemma.

From Eq. (2) and the lemma it is obvious that we can generate all possible values of v which belong to quintuples of parameters v, b, r, k, λ having a given k and having $\lambda = 1$ and satisfying Eqs. (1) and (2) by taking all values of r such that $r \equiv 0, 1 \pmod{p^m}$ for all p^m such that $k = p^m q$ and $(p^m, q) = 1$. Moreover, distinct values of r yield distinct values of v and from Eq. (2) it follows that, for a fixed k, v is a monotonically increasing function of r. In particular, if $r_1 > r_2$ then $r_1 - r_2 \geqq 1$ and, therefore, from Eq. (2) with $\lambda = 1$,

(15) $$v_1 - v_2 \geqq k - 1, \quad \text{for} \quad v_1 > v_2.$$

Eq. (15) implies, in view of Theorem 12, that for a fixed k and $\lambda = 1$, a BIBD with v_2 elements cannot be a solution of the irreflexive relation defined by $G\binom{v_1}{k}_{\lambda=1}$ for any v_1, v_2 such that $v_1 > v_2$ and v_1, v_2 are possible parameters of a BIBD with the given k and with $\lambda = 1$. We have now proved

Theorem 13. All balanced incomplete block designs which are solutions of an irreflexive relation defined by a graph on a given binomial coefficient $\binom{v}{k}$ and having $\lambda = 1$ have the same parameters.

6. RELATIVIZATIONS AND EXTENSIONS OF SOLUTIONS

As an application of the preceding results we note that we can now generate an infinite set of non-trivial examples of "relativizations" and "extensions" of solutions of irreflexive relations. In this section we shall use interchangeably the terms "solution of a graph" and "solution of an irreflexive relation" An extension of a solution of a graph has been defined by Richardson [20] as follows:

Definition. By a subsystem $(\mathcal{S}_o, \succ)$ of the system $(\mathcal{S}, \succ)$ is meant a system where $\mathcal{S}_o \subset \mathcal{S}$ and the relation $\succ$ for the subsystem is merely the restriction of the relation $\succ$ for the supersystem $(\mathcal{S}, \succ)$. Let G_o be the graph of the subsystem $(\mathcal{S}_o, \succ)$ and let

V_o be a solution of G_o. A solution V of G is termed an extension of V_o if $V \cap \mathcal{D}_o = V_o$; in this case V_o is said to be relativized from V.

We can generate an infinite class of extensions of solutions of irreflexive relations by adding an appropriate k-tuple to any block design. In Example 2, the 27 member solution of $G\binom{15}{3}_1$ is an extension of the 26 member solution of $G\binom{13}{3}_1$. As a generalization of this example we can consider those values of k for which BIBD exist for all values of v satisfying Eqs. (1) and (2) with $\lambda = 1$. If k is a power of a prime we are limited by the lemma, proved above, to choosing either $r = kt$ or $r = kt + 1$. This means that, if k is a power of a prime, then there are exactly two cases for the parameters for $\lambda = 1$:

(16) Case I: $v = k(k - 1)t + 1,\ b = k(k - 1)t^2 + t,\ r = kt$

Case II: $v = k(k - 1)t + k,\ b = (kt + 1)((k - 1)t + 1, r = kt+1$.

Due to the work of Reiss [17] and Moore [16] it is possible to construct a BIBD with $k = 3$, $\lambda = 1$, for every $t \geqq 1$ in Cases I and II above. Hanani [11,12] has shown that designs can be constructed for every $t = 1$ when $k = 4,5$ and $\lambda = 1$. Results for larger values of k are incomplete.

Let k assume some value for which, when $\lambda = 1$, a BIBD exists for each value of t. Let G be a graph $G\binom{k(k-1)t+k}{k}_{\lambda=1}$. Let the block design, V_o, be a solution of $G_o\binom{k(k-1)t+1}{k}_{\lambda=1}$. To V_o add a single k-tuple consisting of an element in V_o and the $k - 1$ elements in $G - G_o$. The $|V_o| + 1$ k-tuples thus formed comprise a solution V of $G\binom{k(k-1)t+k}{k}_{\lambda=1}$ which is an extension of V_o. As in Example 2, V is an extension of a block design but is not a block design. G has, however, another solution which is a block design.

The following question arises from the preceding discussion. Is it possible for both V and V_o to be block designs? In other words, is it possible for a balanced incomplete block design to be an extension of another balanced incomplete block design? The answer is yes. Examples are to be found in the higher dimensional projective geometries. The set of lines of any $PG(m_1,p^n)$ form a block design that is an extension of the block design formed by the set of lines of $PG(m_2,p^n)$ whenever $m_2 < m_1$.

Example 4. Consider $PG(3,2)$. This is a BIBD with

$$(v,b,r,k,\lambda) = (15,35,7,3,1).$$

If we take as points the 15 numerals $0,1,\ldots,14$ then the blocks are obtained by letting x assume each of the values $0,1,\ldots,14$ in the following 3 triples [24,p. 203]: $(x,x+1, x+4)$; $(x,x+2, x+8)$; $(x,x+5, x+10)$, the sums being taken modulo 15. Let V represent the 35 blocks obtained in this way. Let V_o be the following set of 7 blocks:

$$(0,\ 1,\ 4) \qquad (2,\ 4,\ 10) \qquad (4,\ 5,\ 8)$$

$$(0,\ 2,\ 8) \qquad (1,\ 8,\ 10) \qquad (1,\ 2,\ 5)$$

$$(0,\ 5,\ 10)$$

V_o is a BIBD using the numerals $0,1,2,4,5,8,10$ as elements. In fact, V_o is obtained from $PG(3,2)$ by considering the plane generated by the lines (0,1,4) and (0,2,8). V_o is isomorphic to PG(2,2) and is a solution of $G_o\binom{7}{3}_1$. Moreover, V_o is relativized from a solution of $G\binom{35}{3}_1$.

With reference to Example 4, we note that, for block designs, the problem of finding extensions which are also block designs is synonymous with the problem of embedding.

7. CONCLUDING REMARKS

The foregoing discussion may be considered a study in equivalence of concepts. It has been shown that balanced incomplete block designs with $\lambda = 1$ are internally stable sets of vertices of suitably defined graphs and are solutions of certain suitably defined irreflexive relations. These results depend heavily on the proposition that, in a BIBD with $\lambda = 1$, any pair of distinct blocks have at most λ elements in common. This property is characteristic of all symmetric BIBD and of all BIBD with $\lambda = 1$ but is not generally characteristic of BIBD with $\lambda > 1$. A BIBD in which a pair of blocks have more than λ elements in common has been constructed by Bhattacharya [3]. In particular, Bhattacharya's example shows that the property of containing a pair of blocks with more than λ elements in common is a property of the structure rather than of the parameters. His example has parameters $(v,b,r,k,\lambda) = (16,24,9,6,3)$, which are also the parameters of a known BIBD without this property. (cf. Hall [7,p.100]).

With the evidence provided by Bhattacharya's design it is obvious that the relation of block designs to graph theory as developed in this paper cannot be extended to higher values of λ. However, there is some hope of repeating our results for $\lambda = 2$. This would, of course, depend on being able to prove that any pair of distinct blocks of a BIBD with $\lambda = 2$ have at most λ elements in common. As a result of the work of Connor and Hall [6], it can be stated that any pair of distinct blocks of a BIBD having parameters of the form

$$v' = v - k; \quad b' = v - 1; \quad r' = k; \quad k' = k - \lambda; \quad \lambda' = \lambda = 2$$

where the $v, k, \lambda = 2$ are parameters of an existing symmetric design, have at most λ elements in common.

In spite of the limited generality of the concept of balanced incomplete block designs as internally stable sets of vertices of graphs on binomial coefficients having edge parameter λ, it is nevertheless the writer's opinion that the investigation of the special case of $\lambda = 1$ has provided considerable insight into the nature of block designs. Combinatorial mathematics cannot afford to regard block designs as isolated combinatorial phenomena and whatever concepts can revitalize our view of their basic nature should receive careful exposition.

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AUTOBIOGRAPHICAL STATEMENT

Jane Walsh Di Paola was born September 17, 1917 in Brooklyn, New York. She received her early education in Long Beach, New York and was graduated with a Bachelor of Arts degree from St.Joseph's College for Women in Brooklyn, New York, in 1939. She matriculated for a Master's degree in mathematics at Brooklyn College in 1940 and taught mathematics at Long Beach High School during 1941 and 1942. In 1942 she was married to Peter Di Paola, a mathematics teacher, and moved to Buffalo, New York where, until 1945, she was employed in the Research Laboratory of the Curtiss-Wright Corporation. She returned to Long Beach, New York with her husband in September 1945. Their three children are Peter, born August 22, 1946, Jane, born October 1, 1948 and Joseph, born July 24, 1951.

Mrs. Di Paola was readmitted to the Master's degree program at Brooklyn College in September 1958 and completed this degree in June 1962. During this time she taught mathematics in various secondary schools in the New York area. In 1963 she taught at New York City Community College. In January 1964, Mrs. Di Paola was matriculated in the doctoral program in mathematics at the City University of New York. She now resides in New Rochelle, New York where her husband is Assistant Superintendent of Schools of the City of New Rochelle.