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GAUGE FIELDS WITH NONINVARIANT INTERACTIONS

by

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PREFACE

This work is composed of two parts, the first of which has been published in the Journal of Mathematical Physics $\underline{9}$, 73 (1968), under the dual authorship of this author and his thesis advisor, Professor S. Schiminovich. In PART I we consider general theory and possible applications. In PART II one of these applications of the general theory is further developed.

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GAUGE FIELDS WITH NONINVARIANT INTERACTIONS

1. INTRODUCTION

This paper deals with the physical content of theories in which a gauge field, such as the electromagnetic, Yang-Mills or gravitational field, is coupled in a gauge-noninvariant way to a source field. Customarily, such theories are excluded by the requirement of global gauge covariance. This requirement is imposed either to assure the existence of conserved quantities or for heuristic reasons. Such motivations are questionable. Although gauge covariance and the existence of a conserved current are inseparably joined in single-field theories by Noether's procedures, we show that, for interacting fields, conserved currents can follow as a result of the existence of Bianchi identities satisfied by the gauge-field variables.

In proposing a study of the gauge-noninvariant interactions of gauge fields, our point of departure is opposite that taken by Utiyama,[1] who generates interactions using the requirements of global gauge covariance. Such an invariant theoretical approach to interactions has proven

1. R. Utiyama, Phys. Rev. 101, 1597 (1956).

sterile up to the present and is subject to criticism.[2] In Utiyama's procedure, the source field, invariant under its constant-parameter gauge group, is the all-important initial element; in our procedure the gauge field invariant under its coordinate-dependent gauge group, is the basis upon which new theories are constructed.

In order to give meaning to the solutions of a gauge-invariant theory, one has to supplement the theory with coordinate conditions. These can be given at the level of the action principle with the aid of auxiliary Lagrange multiplier fields coupled through gauge-noninvariant terms. The multiplier field can be made a dynamical field by introducing its free Lagrangian into the action. It is hoped that such dynamical multiplier fields may describe properties of gauge-field sources known to exist in nature.

Because the theory is now gauge noninvariant, all the components of the original gauge field have physical meaning. This means that properties of the sources are described not only by the new Lagrange multiplier fields but also by the components of the gauge field. In this way our work will provide, when applied to the electromagnetic field, an extension to Dirac's classical theory of

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2. V. I. Ogievetski and I. V. Polubarinov, Nuovo Cimento 23, 173 (1962); V. Fock, The Theory of Space Time and Gravitation (The Macmillan Company, New York, (1964).

electrons.[3]

The existence of physically meaningful gauge-field components is further related to the discussion and work on the physical meaning of potentials by Aharanov and Bohm. The interpretation of the physical meaning of potentials is bound to the determination of the role played by the phase variables; for, as we shall see, the potentials acquire physical meaning when the phase variables have been completely eliminated from the formulation. Additional points of contact exist with the work of Aharanov and Wisniveski.[5]

There are occasions when the Lagrange multiplier field can be omitted and the original gauge invariance broken merely by the introduction of noninvariant terms into the free gauge-field Lagrangian. In such a case the properties of the source fields are described solely by the gauge field. Thus our study, when applied to the gravitational field, will exhibit a relationship with the work of Wheeler, Rainich, and Misner expressed in a particular coordinate frame.

3. P. A. M. Dirac, Proc. Roy. Soc. (London) 209A, 291 (1951; 212A, 330 (1952); 223A, 438 (1954).
4. Y.Aharanov and D. Bohm, Phys. Rev. 115, 485 (1959); 123, 1511 (1961).
5. Y. Aharanov and D. Wisniveski (to be published).
6. See, for example, Louis Witten, in Gravitation: An Introduction To Current Research, L. Witten, Ed. (John Wiley & Sons, Inc., New York, 1962).

For the present we limit our attention to theories that can be derived from an Utiyama-type theory. It is clear that any such theory can be cast in a frame that is manifestly noninvariant by choosing a particular gauge. This procedure is not trivial as a result of the following:

(a) We require that the particular gauge chosen leaves the theory written explicitly and solely in terms of physically meaningful fields (i.e., in terms of fields not subject to any remaining gauge conditions).

(b) We require that the choice of gauge reduces the number of dynamical variables. The possibility of decreasing the number of variables indicates that the original Utiyama-type theory contains an element of arbitrariness: Gauge covariance is accomplished by the introduction of fictitious fields and by the subsequent redefinition of the observable fields in a manner which compensates for these fictitious fields.

In general it is possible to formally choose the gauge which reduces the number of explicit variables in the Lagrangian. For particular solutions, however, this may require discontinuous or singular gauge transformations, especially when the solutions are defined in non-simple connected demains. The physical content of the new and old theories may then be different.

To begin with we show how we can use the gauge freedom in the Utiyama-type theories to formally reduce the number of explicit dynamical variables and yet guarantee the existence of conserved quantities by reason of the Bianchi identities satisfied by the gauge-field variables. We then consider the example of the electromagnetic field coupled to a Klein-Gordon field and show this formulation provides an extension to Dirac's considerations. Beams of classical charges described by this theory enjoy some of the properties of quantized charges. In particular, static distributions of charge in a Coulomb field are possible with the same multiplicity of solutions and energies as the stationary states of a hydrogen atom, provided self-energies are neglected. We then briefly consider the cases of the gravitational and Yang-Mills fields.

2. GENERAL THEORY

We first consider the properties of theories of the Utiyama type and how in such theories the gauge degrees of freedom can be used to reduce the number of explicit dynamical variables. These theories can be generally characterized by actions of the form

$$\text{Action} = \int_\Omega d^4x \left\{ \mathcal{L}_G(q) + \mathcal{L}_I(q,p) + \mathcal{L}_M(p) \right\} , \tag{2.1}$$

where $\mathcal{L}_G(q)$ and $\mathcal{L}_I(q,p) + \mathcal{L}_M(p)$ are scalar densities under a group of global gauge transformations (i.e., coordinate-dependent transformations whose parameters vary arbitrarily in space-time), $\mathcal{L}_G(q)$ is the free gauge-field Lagrangian, $\mathcal{L}_M(p)$ is the free source-field Lagrangian, and $\mathcal{L}_I(q,p)$ is the coupling term. The gauge invariance properties can be expressed as

$$\delta^* \int_\Omega d^4x \; \mathcal{L}_G(q) = 0 \quad , \tag{2.2}$$

$$\delta^* \int_\Omega d^4x \; \{ \mathcal{L}_I(q,p) + \mathcal{L}_M(p) \} = 0 \quad , \tag{2.3}$$

where δ^* means that the variables q and p are varied by an infinitesimal amount determined by an arbitrary infinitesimal transformation of the global gauge group. The variables q are the gauge-field variables and the variables p are the source-field variables.

In Utiyama's procedure one starts with an action $\mathcal{L}_M(p)$ which is invariant only under the group of gauge transformations depending on n constant parameters. Imposing the requirement that the theory be invariant under the more general group of global gauge transformations, for which the n parameters may be arbitrary functions of space-time, forces the introduction of a set of gauge-field variables q through the term $\mathcal{L}_I(q,p)$. The theory is then completed by adding a free gauge-field Lagrangian which is restricted in form by the requirement of global gauge covariance.

We denote the change induced in the field variable p by an infinitesimal global gauge transformation as $\delta^* p$ and write

$$\delta^* p = O_p \, \delta\xi \quad , \tag{2.4}$$

where O_p is an operator which may depend on the p . We assume that O_p is a differential matrix operator applied on $\delta\xi$ which stands for the set of n infinitesimal parameters $\delta\xi_{i}(x)$ of the global gauge group. Thus $\delta\xi$ is arbitrary. Similarly we have

$$\delta^{*}q = O_q \delta\xi \quad . \tag{2.5}$$

We now take advantage of the arbitrary variability of $\delta\xi$ over space-time to make it vanish over the boundaries of the volume of integration Ω . This restriction simplifies the calculation without impairing the generality of the conclusions provided we keep Ω arbitrary. Assuming locality for Eqs. (2.4) and (2.5), it follows that $\delta^{*}q$ and $\delta^{*}p$ also vanish at the boundaries of Ω .

From (2.2) we have

$$\int_{\Omega} d^4x \, \frac{\delta \mathcal{L}_G(q)}{\delta q} \, \delta^{*}q = \int d^4x \, \frac{\delta \mathcal{L}_G(q)}{\delta q} \, O_q \delta\xi = 0 \quad , \tag{2.6}$$

or

$$\int_{\Omega} d^4x \, O_q^H \left(\frac{\delta \mathcal{L}_G(q)}{\delta q} \right) \delta\xi = 0 \quad , \tag{2.7}$$

where $\frac{\delta \mathcal{L}_G}{\delta q}$ is the variational derivative of $\mathcal{L}_G$ and O_q^H is the Hermitian conjugate of O_q . In the same way, from (2.3) we have

$$\int d^4x \left\{ O_q^H \left(\frac{\delta \mathcal{L}_I}{\delta q} \right) + O_p^H \left(\frac{\delta \mathcal{L}_I}{\delta p} + \frac{\delta \mathcal{L}_M}{\delta p} \right) \right\} \delta\xi = 0 \quad , \tag{2.8}$$

which, because of the arbitrariness of $\delta\xi$, gives

$$O_q^H \, T = - O_p^H \left(\frac{\delta \mathcal{L}_I}{\delta p} + \frac{\delta \mathcal{L}_M}{\delta p} \right) \quad , \tag{2.9}$$

where $T \equiv \delta \mathcal{L}_I / \delta q$. The equations of motion that follow from the action (2.1) are

$$\delta \mathcal{L}_G / \delta q + \delta \mathcal{L}_I / \delta q = 0 \quad , \tag{2.10}$$

and

$$\delta \mathcal{L}_I / \delta p + \delta \mathcal{L}_M / \delta p = 0 \quad . \tag{2.11}$$

Thus when Eq. (2.11) is satisfied, Eq. (2.8) implies

$$O_q^H \, T = 0 \quad , \tag{2.12}$$

which is traditionally called the "conservation law" of the theory. Equation (2.12), being a consequence of the equations of motion (2.11) and the global gauge-invariance property (2.3), is identically satisfied in q .

This property of global gauge covariance is by no means necessary, however, for (2.12) to be true when p fields interact with q fields; for from (2.7) we also have

$$O_q^H \; \delta \mathcal{L}_G / \delta q = 0 \quad , \tag{2.13}$$

identically in q. These are the "Bianchi identities" of the theory and from them and (2.10) we have that, for any solution of the dynamical problem, (2.12) has to be satisfied as a consistency requirement of (2.10). The "conservation laws" follow for any theory in which fields are coupled to a gauge field whose equations of motion satisfy a set of "Bianchi identities," whether or not the total theory is globally gauge covariant. Relations (2.12) are true conservation laws only when $O_q = \partial_\nu$. A sufficient condition for the existence of a Bianchi identity with such an O_q is that there exists a term in the Lagrangian that is a function only of $F_{\mu\nu} \equiv (\partial_\mu q_\nu - \partial_\nu q_\mu)$. This same condition assures the existence of a conserved current.

Coming back to the Utiyama-type theories, we assume that a transformation among the p dynamical variables can be found which splits them into two groups, p^{ph} and p^{in}, such that O_p becomes

$$O_p = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} ,$$

for $$p = \begin{pmatrix} p^{ph} \\ p^{in} \end{pmatrix} , \tag{2.14}$$

with a suitable definition for $\delta\xi$. With such a redefinition the dynamical variables are split into two groups: the p^{in} are invariants of the constant parameter gauge group and the remaining p^{ph} alone are affected by the transformations of the group. We now choose the p^{ph} to be the very descriptors of the gauge group and we shall call them "phase" variables. As the descriptors of the gauge group, the p^{ph} can be reduced to zero by a suitable finite global gauge transformation. This means that, with a suitable choice of gauge, $p^{ph} = 0$ and that this set of dynamical variables drops out of the description of the system.

If one wonders how the information in the equations of motion of the phase variables is to be preserved, it is sufficient to inspect Eq. (2.9) and notice that, by virtue of (2.14), we have

$$O_q^H T = \delta \mathcal{L}_I / \delta p^{ph} + \delta \mathcal{L}_M / \delta p^{ph} . \tag{2.15}$$

This tells us that the "conservation laws" of the theory

are just the equations of motion obtained by variation of the phase variables. In the gauge-noninvariant formulation written-in terms of the variables q' and p^{in}, the physical con
written-in terms of the variables q' and p^{in}, the physical
of the Bianchi identities of the free gauge field.

Even though the reduction in the explicit number of dynamical variables is in principle the result of a particular choice of gauge, one cannot be certain that the physical content of the covariant Utiyama-type theory is the same as that of the corresponding non-covariant theory, if one requires that in both cases the solutions to the equations of motion be single-valued and continuous. This problem arises whenever the transformation of variables from q and p to q' and p^{in} and p^{ph} is singular.

Assume that the freedom allowed by covariance can be used to remove the explicit appearance of the "phase variables" at least in some approximation. The noncovariant theory which results has such desirable properties as: (a) all variables are physically meaningful; (b) the number of variables is fewer than in the corresponding covariant theory; (c) the gauge field can in certain cases carry all the information previously carried by the source field.

If the noncovariant theory is exactly equivalent to the corresponding covariant theory, a reinterpretation of the covariant theory is in order. If the noncovariant theory is only an approximation to the covariant theory, then the noncovariant theory with its desirable properties can be used as the basis upon which new theories are constructed. In particular, the classical noncovariant theory to be presented in Sec. 3, which exhibits certain "quantum" properties and which does not require the introduction of the charge , can be made the basis for a quantized theory of interactions.

The elimination of phase variables from covariant theories leads to a class of gauge-invariant theories of the type studied in this paper. Still open is the question whether any noncovariant theory can be made covariant by the introduction of enough phase variables with suitable gauge-group transformation properties.

3. MAXWELL-SCHRODINGER THEORY

Instead of systematically reviewing all the possible gauge-noninvariant interactions for the Maxwell field A_μ, we consider only the theory determined by the action,

$$Action = \int d^4x \left\{ -\tfrac{1}{4} F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2} R^2 (A^\mu A_\mu - 1) + \tfrac{1}{2} R^{,\mu} R_{,\mu} \right\} , \qquad (3.1)$$

which is one of the simplest Lorentz-covariant possibilities. We have chosen the Lorentz metric to be $\eta_{00} = 1$, $\eta_{rr} = -1$, and $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. This theory is related to the Utiyama-type theory in which the electromagnetic field is in minimal coupling with a complex scalar Klein-Gordon field. In such a theory the global gauge group of transformations is

$$\psi' = \psi\, e^{i\phi(x)} \quad , \tag{3.2}$$

$$A'_\mu = A_\mu - \partial_\mu \phi \ . \tag{3.3}$$

The transition from variables ψ , ψ^* to the new variables R , S through

$$\psi = R \exp iS \quad , \tag{3.4}$$

is a transition to a phase variable S and a field R invariant under the group of gauge transformations. If we formally set $S = 0$ in the complex Klein-Gordon theory, we recover the form of the action expressed in Eq. (3.1). This can be achieved by a gauge transformation whenever is a continuous and single-valued function of position. The continuity and single valuedness of ψ , however, do not imply the same for R and S . Thus, for the theory described by Eq. (3.1), a physical content different from that of the original Utiyama theory may be expected.

We have written the action (3.1) in a completely nondimensional form. If one introduces a unit of length $1/k$, Eq. (3.1) can be reexpressed in terms of

$$x'_\mu = x_\mu / k \quad , \tag{3.5}$$

as

$$\text{Action} = \int d^4x' k^2 \left\{ -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + \tfrac{1}{2} k^2 R^2 \left(A^\mu A_\mu - 1 \right) + \tfrac{1}{2} R^{,\mu} R_{,\mu} \right\} \ . \tag{3.6}$$

We could further introduce the more customary definitions for the fields

$$A'_\mu = k A_\mu \quad , \qquad R' = k R \quad , \tag{3.7}$$

and set $k = m/e$, the ratio of mass to charge. Then (3.6) becomes

$$\text{Action} = \int d^4x \left\{ -\tfrac{1}{4} F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2} R^2 (A^\mu A_\mu - k^2) + \tfrac{1}{2} R^{,\mu} R_{,\mu} \right. , \tag{3.8}$$

where we have dropped all primes. This is a clear extension of Dirac's work.[3] Here the Lagrange multiplier field R^2 has been given a dynamics of its own by including the $\frac{1}{2} R^{,\mu} R_{,\mu}$ term in the Lagrangian. Thus, our theory describes all the features of classical charges, as does Dirac's theory, in the limit of negligible fluctuations of the R field. On the other hand, the connection with the complex Klein-Gordon theory and, in the nonrelativistic limit, the Schrodinger theory makes it apparent that such fluctuations will incorporate some quantum features into the description of the classical clouds of charge. We present some of these features by studying the static configurations that such clouds may assume in the presence of an external Coulomb field.

The equations of motion derived from (3.1) are

$$F^{\mu\nu}{}_{,\nu} = R^2 A^\mu \quad , \tag{3.9}$$

$$\Box R = R \, (A^\mu A_\mu - 1) \quad . \tag{3.10}$$

According to Eq. (3.9), the electric charge and current densities are

$$J^\mu = R^2 A^\mu \quad . \tag{3.11}$$

They are conserved because the Bianchi identities require

$$(R^2 A^\mu)_{,\mu} = F^{\mu\nu}{}_{,\mu\nu} \equiv 0 \quad . \tag{3.12}$$

In the nonrelativistic limit, the complex Klein-Gordon theory is equivalent to the Schrodinger theory. In such a limit, Eq. (3.10) is the real part and Eq. (3.11) is the imaginary part of the Schrodinger equation written in the gauge $S=0$. This nonrelativistic limit is obtained by setting

$$A_0 = 1+\phi \quad , \quad \phi \ll 1 \quad , \tag{3.13}$$

where ϕ is a first-order field, and by neglecting ϕ^2 and second time derivatives.

To determine the observables in this classical field theory we compute the energy tensor $T_{\mu\nu}$ as well as the current J^μ. This task may prove delicate in our gauge-noninvariant theory because the integrals over infinite surfaces which are usually dropped now may become important. Indeed, constant terms in the potentials are of physical significance because of the lack of gauge invariance. We compute the $T_{\mu\nu}$ tensor by writing (3.1) in a system of curvilinear coordinates with the help of the $g^{\mu\nu}$ metric and then finding

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \left(\mathcal{L} \sqrt{-g} \right) . \tag{3.14}$$

By this procedure we obtain

$$T_{\mu\nu} = -F^{\lambda}{}_{\mu} F_{\lambda\nu} + \tfrac{1}{4} g_{\mu\nu} F^{\sigma\lambda} F_{\sigma\lambda} + R_{,\mu} R_{,\nu} - \tfrac{1}{2} g_{\mu\nu} g^{\sigma\lambda} R_{,\sigma} R_{,\lambda} + \tfrac{1}{2} R^2 \left(2 A_\mu A_\nu - g_{\mu\nu} g^{\sigma\lambda} A_\sigma A_\lambda + g_{\mu\nu} \right) . \tag{3.15}$$

Setting $g^{\mu\nu} = \eta^{\mu\nu}$ (the Lorentz metric), we obtain the energy of the system as the positive-definite quantity

$$E = \int d^3x\, T_{oo} = \int d^3x \left\{ \tfrac{1}{2} (E^2 + H^2 + R_{,0} R_{,0} + R_{,r} R_{,r} + R^2 A_o A_o + R^2 A_r A_r + R^2) \right\} . \qquad (3.16)$$

In order to study the static configurations that the clouds of charge may assume in the presence of external fields, we proceed in the same manner after modifying the original Lagrangian by introducing the static external contravariant current vector $J^{\mu(ext)} = (0, 0, 0, \rho^{ext})$ and the fixed external field $A^{\mu(ext)}$ generated by such a current. For static solutions the energy is

$$E = \int d^3x \left\{ \tfrac{1}{2} A_{o,r} A_{o,r} + R^2 A_o A_o \right\} , \qquad (3.17)$$

where $A_o = A_o^{(s)} + A_o^{(ext)}$, and we have made use of the equations of motion after integrating the R variable by parts.

Truly static situations will arise when the electric currents $\vec{J}$ are zero. We therefore look for solutions in which

$$A_\mu^{(s)} = (0, 0, 0, 1 + C + \phi^{(cl)}) , \qquad (3.18)$$

where we allow for a constant $1 + C$ that may be significant and where $\phi^{(cl)}$ is the potential generated by the cloud of charge:

$$\nabla^2 \phi^{cl} = R^2 A_o , \qquad (3.19)$$

which is one of the equations of motion. The other equation of motion

$$- R_{,rr} = R \, (A_o A_o - 1) , \qquad (3.20)$$

reduces to the nonrelativistic time-independent Schrodinger equation in the limit in which $A_o = 1 + \phi$, $\phi \ll 1$

and squares of ϕ are neglected. Indeed, Eq. (3.20) under these conditions becomes

$$-\tfrac{1}{2} R_{,rr} - \phi^{cl} R - \phi^{ext} R = C R \quad , \tag{3.21}$$

so that C is the eigenvalue $C^{(n)}$ of the Schrodinger equation.

We compute the energy (3.17) corresponding to a solution $R^{(n)}$ for a cloud with total charge equal to -1, so that

$$\int d^3x \, R^2 A_0 = 1 \quad . \tag{3.22}$$

We have, from Eq. (3.17),

$$E^{(n)} = \int d^3x \left\{ \tfrac{1}{2}\phi^{cl}_{,r}\phi^{cl}_{,r} + \phi^{cl}_{,r}\phi^{ext}_{,r} + \tfrac{1}{2}\phi^{ext}_{,r}\phi^{ext}_{,r} + R^2 A_0 \left(1 + C^{(n)} + \phi^{cl} + \phi^{ext}\right) \right\} \quad , \tag{3.23}$$

or, by virtue of Eqs. (3.19) and (3.22),

$$E^{(n)} = 1 + C^{(n)} - \tfrac{1}{2}\int d^3x \, \phi^{cl}_{,r}\phi^{cl}_{,r} + \tfrac{1}{2}\int d^3x \, \phi^{ext}_{,r}\phi^{ext}_{,r} \quad . \tag{3.24}$$

This result, of general validity for static solutions, shows that the energy is equal to a constant term, the rest-mass energy, plus $C^{(n)}$, the usual energy eigenvalue for a stationary state. This eigenvalue reflects the distortion of the cloud due to Coulomb self-interaction. The next term in Eq. (3.24) subtracts the Coulomb self-energy of the cloud.

In the case of $A_0^{ext} = 1/4\pi r$, a complete set of solutions for R in the nonrelativistic case is given by the complete set of real solutions to the hydrogen atom

$$R^{nlm}(r,\theta,\phi) = R^{nl}(r)\, P^{ln}(\theta) \begin{cases} \cos m\phi \\ \sin m\phi \end{cases} \quad . \tag{3.25}$$

Thus our extension of Dirac's theory of classical electrons allows the clouds of charge some quantum features explainable in the framework of a classical and consistent theory of electromagnetism. At the same time, the difficulties encountered in Dirac's theory are removed. For example, charges at rest in a constant magnetic field are described as satisfactorily by this theory as by the Schrodinger theory. In addition, one realizes that a large constant term in the A_0 component of the gauge field is important for the inertial properties of the cloud and that a rotational motion proportional to the external magnetic field intensity is related to the diamagnetic properties of the cloud.

The full extent to which quantum features can be explained in purely classical terms is outside the scope of this paper. We wish to point out, however, that the essential quantum properties seem to be particle number, spin, and statistics rather than the features for which the quantum formalism was originally developed. It is further possible that other ideas,[7] developed in a different context, may prove fruitful when applied to the topological properties of such classical field solutions.

7. D. Finkelstein, J. Math. Phys. 7, 1218 (1966).

4. GRAVITATIONAL FIELD

Einstein's theory of gravitation is a theory of the Utiyama type considered in Sec. 2. It is covariant under the group of general coordinate transformations and has the metric field as the associated gauge field. We consider two cases in which the gravitational field is coupled to a source characterized by a vector field: A^μ (the electromagnetic field) and

$$\sum_i \int \frac{d\xi^{\mu(i)}}{d\tau} \delta^4(x-\xi^{(i)})\, d\tau \quad ,$$

(the field of matter-point singularities). Because all components of these fields are phase variables, their dynamical meaning, short of coordinate conditions, is completely arbitrary. The metric field is capable of absorbing all the information usually carried by such source-field variables.

A. Gravitational-Electromagnetic Field

We assume that the A^μ vector field is timelike in character. This may seem an arbitrary restriction especially if gauge transformations $A'_\mu = A_\mu + \partial_\mu \phi$ are allowed. We have in mind, however, the gauge-noncovariant theory of electromagnetism developed in the last section, for which it can be seen that A^μ is a timelike vector field, at least in the nonrelativistic limit in which $A^\mu = \delta^\mu_0 + \phi^\mu,\ \phi^\mu \ll 1$. All components of the A^μ field are phase variables, since they can be reduced to constant values and eliminated as dynamical variables. We call such a set of constant values a "standard form." The only restriction imposed on such a standard form is that it has the timelike character of A^μ. We choose

$$A^{\mu\,(\text{standard})} = \delta^\mu_0 \quad . \tag{4.1}$$

It should be noted that in general a timelike vector field can be brought to a standard form only in its contravariant components. Indeed, one chooses the field to be the time-coordinate field.

If a covariant vector field could be brought to such a form, it would obviously have a vanishing ordinary curl in the new system of coordinates. Because the ordinary curl of a covariant vector is a tensor, it follows that only rotationless covariant vector fields can be brought to standard form.

Condition (4.1) can, equivalently, be imposed as a coordinate condition on the $g_{\mu\nu}$

$$g_{0\mu} = A_\mu \quad . \tag{4.2}$$

We conclude then that the theory determined by

$$A = \int d^4x \sqrt{-g}\,\Big\{ G + \tfrac{1}{2}\, g_{0\lambda,\rho}\, g_{0\mu,\nu}\,(g^{\mu\rho} g^{\nu\lambda} - g^{\nu\rho} g^{\mu\lambda}) + \tfrac{1}{2} R^2 (g_{00} - 1) + \tfrac{1}{2} g^{\mu\nu} R_{,\mu} R_{,\nu} \Big\} \quad , \tag{4.3}$$

describes the gravitational plus electromagnetic properties of the classical clouds of charge found in the last section. The expression (4.3) for the action was obtained from the expression for the action (3.1), made generally covariant through the introduction of the $g_{\mu\nu}$, supplemented with the Einstein free Lagrangian G, and with coordinate condition (4.2) taken into account. The content of the theory of Sec. 2 is a consequence here of the

Bianchi identities satisfied by the Einstein equations.

The coordinates are, of course, no longer arbitrary but are restricted by the theory. Their physical meaning is disclosed by (4.2), which allows us to interpret the physical meaning of the $g_{0\mu}$ components of the metric. The meaning of the coordinate frame can be made clearer by noting that the electric chargecurrent 4-vector is now

$$J^{\mu} \equiv R^2 A^{\mu} = R^2 \delta^{\mu}_{0} \quad . \tag{4.4}$$

Thus we are in the frame of reference in which all electric charges are at rest.

We do not claim to have a theory of gravitation-electromagnetism in the sense of unified theories. The inclusion of electromagnetism is unsatisfactory to the extent to which the addition to the Lagrangian of the gauge noninvariant term

$$g_{0\lambda,\rho}\, g_{0\mu,\nu}\, (g^{\mu\rho} g^{\nu\lambda} - g^{\nu\rho} g^{\mu\lambda}) \quad ,$$

is ad hoc. The situation is no worse, however, than the inclusion of the ordinary Maxwell Lagrangian in the usual theory in terms of the A_{μ} . The advantage of the formulation in terms only of the metric is that the phase variables A_{μ} are eliminated and physical meaning is imparted to the components of the metric field. In the following simple case of chargeless-point singularities, we have an example of how the metric components can carry the information usually shared between the metric field and the phase variables.

B. Gravitational Field Interacting with Matter-Point Singularities

It is conceivable that this theory may arise as an idealized limit of the one considered in the previous section. An analogous coordinate condition results from choosing a frame such that the timelike 4-vector $d\xi^{\mu(i)}/d\tau$ is

$$d\xi^{\mu(i)}/d\tau = \delta_0^{\mu(i)} \quad , \tag{4.5}$$

or

$$d\xi_\mu^{(i)}/d\tau = g_{0\mu} \quad \text{at} \quad x^\mu = \xi^{\mu(i)} \quad , \tag{4.6}$$

namely the particles remain always at rest. The only non-vanishing contravariant component of the energy-momentum tensor is the constant

$$T_{00} = m \quad , \tag{4.7}$$

defined over the world line of the particle. The covariant tensor for the matter-point sources is then

$$T_{\mu\nu} = m\, g_{0\mu}(\xi) g_{0\nu}(\xi) \quad \text{for } x^\mu = \xi^\mu \quad . \tag{4.8}$$

The dynamical variables ξ^μ of the particles are now fixed constants which determine the constant position of the particles. The physical content usually associated with the ξ^μ is now carried by the values of the $g_{0\mu}$ at the position of the particles and the dynamical development is determined solely by the Bianchi identities associated with Einstein's equations.

In this formulation, in which the particle coordinates are no longer dynamical variables, the problem of motion is most easily approached through

$$T^{\mu\nu}{}_{;\nu} = T^{\mu\nu}{}_{,\nu} + \Gamma^{\nu}_{\beta\nu} T^{\mu\beta} + \Gamma^{\mu}_{\beta\lambda} T^{\beta\lambda} = 0 \quad , \tag{4.9}$$

or, remembering that $T^{\mu\nu} = m\delta^{\mu}_{0}\delta^{\nu}_{0}$

$$\Gamma^{r}_{00} = 0 \quad , r \neq 0 \quad , \tag{4.10}$$

$$\Gamma^{0}_{00} + \Gamma^{\mu}_{0\mu} = 0 \quad \text{for } x^{\mu} = \xi^{\mu(i)} \ , \tag{4.11}$$

or

$$\tfrac{1}{2} g^{r\mu} (2g_{0\mu,0} - g_{00,\mu}) = 0 \quad , \tag{4.12}$$

$$\tfrac{1}{2} g^{0\mu} (2g_{0\mu,0} - g_{00,\mu}) + \frac{1}{2g} g_{,0} = 0 \quad , \tag{4.13}$$

for $x^{\mu} = \xi^{\mu(i)}$. Because the $d\xi^{\mu}/d\tau$ are not independent, we can impose the further coordinate condition

$$-g\, g_{00} = 1 \quad , \tag{4.14}$$

which satisfies (4.11) identically by virtue of (4.12) and gives, from (4.12),

$$g_{0r,0} = \tfrac{1}{2} g_{00,r} - \tfrac{1}{2} g_{0r} (\ln g_{00})_{,0} \quad , \tag{4.15}$$

for $x^{\mu} = \xi^{\mu(i)}$, as the equations of motion for the point particles. This equation can be roughly interpreted as giving the acceleration on the left-hand side equal to a force derivable from a potential and a velocity-dependent force. In the nonrelativistic limit it gives Newton's law of gravitation when g_{00} is evaluated from the Einstein equations of the same order. This formulation may be derived from the noninvariant action,

$$\text{Action} = \int d^4x \sqrt{-g}\, G + \int d\tau\, m^{(i)} \sqrt{g_{00}} \quad , \tag{4.16}$$

which results from the usual action when the coordinate conditions (4.5) and (4.14) are used.

The above formulation of the motion of gravitating singularities makes it clear that the motion of a point particle cannot be an absolute attribute. One can speak only of locations in space where there is or is not such a particle. If we label positions by means of the point particles, what is usually called their equation of motion becomes the condition (4.15), which is satisfied by the $g_{0\mu}$ along the world lines of the particles. This is in accord with the identification of (4.6), which gives the physical meaning of such $g_{0\mu}$.

A noncovariant theory, which could not be made generally covariant by the introduction of new dynamical variables, might prove even more interesting than the case considered. Whether such theories exist and whether the existence of a limit of the form of (4.7) for the vanishing of some length parameter assures all the verifiable content of a theory of space, time, and gravitation, remain open questions.

5. YANG-MILLS FIELD

In the gauge noninvariant theory of Sec. 3, the electromagnetic field is coupled to a real scalar field R ; yet, the theory describes charged particles of both signs. It might be thought that the introduction of more complicated internal symmetries or, equivalently, the description of higher multiplets will necessarily require that the R field be generalized to a multi-component field. To show that this is not the case, we briefly consider the Yang-Mills field $\mathbf{B}_\mu$.

The action,

$$\text{Action} = \int d^4x \left\{ -\tfrac{1}{4}\, \mathbb{F}_{\mu\nu}\cdot \mathbb{F}_{\mu\nu} + \tfrac{1}{2} R^2 (\mathbb{B}_\mu \cdot \mathbb{B}_\mu - 1) + \tfrac{1}{2} R_{,\mu} R_{,\mu} \right\} \quad , \tag{5.1}$$

where

$$\mathbb{F}_{\mu\nu} = \partial_\mu \mathbb{B}_\nu - \partial_\nu \mathbb{B}_\mu - \tfrac{1}{2} (\mathbb{B}_\mu \mathbb{B}_\nu - \mathbb{B}_\nu \mathbb{B}_\mu) \quad ,$$

and R is a scalar real field, describes the charge-independent interaction of a Yang-Mills field with an isospinor scalar meson. This action results from the ordinary expression when the prescription of Sec. 2 is used. We have written the usual isospinor scalar field ψ as

$$\psi = \begin{pmatrix} a+ib \\ c+id \end{pmatrix} = e^{\tau\cdot s} \begin{pmatrix} R \\ 0 \end{pmatrix} \quad ,$$

where τ are the three isospinor matrices and R^2 is the isospinor invariant

$$R^2 = (\psi, \psi) = a^2 + b^2 + c^2 + d^2 \quad .$$

Substitution into the usual action with $s = 0$, which can always be accomplished by a gauge transformation if s is a single-valued function of position, leads to expression (5.1).

Although the action (5.1) is not invariant under isospinor transformations, the symmetry implied by charge independence is assured by the symmetric way in which $\mathbb{B}_\mu$ appears. Conservation of the isospin current $\mathbb{J}^\mu = \mathbb{J}^\mu_{(B)} + R^2 \mathbb{B}^\mu$ where $\mathbb{J}^\mu_{(B)}$ is the isospin current contributed solely by the Yang-Mills field, is a consequence of the Bianchi identities. The contribution of the meson field to the isospin current is $\mathbb{J}^{\kappa(\text{meson})} = \mathbb{R}^2 \mathbb{B}^{\kappa}$.

The action (5.1) is the simplest and most obvious generalization of the electromagnetic theory of Sec. 3. it shows how internal symmetries of the particles and multiplet structure can be described solely by an enrichment of the properties of the gauge field. In addition, the breaking of the symmetry and the appearance of an electromagnetic axis in the theory can be accomplished for both source and gauge fields by modifying only the gauge-field action in a way previously proposed.[8]

8. D. Finkelstein, J. Jauch, S. Schiminovich, and D. Speiser, J. Math. Phys. 4, 788 (1963)

CLASSICAL GAUGE NONCOVARIANT ELECTROMAGNETISM

1. INTRODUCTION

In Part I of this work we discussed gauge noninvariant interactions for gauge fields and showed, among other things, how Dirac's classical theory of the electron is related to the theory of the Schrodinger field in interaction with a Maxwell field. In particular the theory allowed for the existence of static configurations of charge density in the presence of an external centrally symmetric Coulomb field with classical energies closely paralleling the energy levels of the quantized hydrogen atom.

The question arises naturally of how far this classical picture can be pursued without giving up classical concepts, by which we mean, without introducing the quantum concept of the state of a system and the probabilistic interpretation.

One of the obstacles we met with was the self energy problem. Even though self energies are finite, they shift the energy levels from the correct values and the self fields distort the configurations of the clouds, as noted in our first paper. It is not clear to our minds whether this problem is solved consistently or is just ignored in the conventional quantum theory. It is clear, however, that we cannot ignore it for a consistent classical formulation. This problem was the more vexing in our gauge noninvariant formulation in terms of real electrom fields because there the self interaction assures the continuity equation and the equivalence with the Schrodinger theory. We were not allowed to switch on and off the interaction.

In Part II of this work we review the status of the gauge noncovariant classical theory of distributed charge. We then show how the unphysical self interaction corrections can be removed by employing a formalism in which a particle field interacts only with those gauge fields with sources in the other particle fields. Then by indicating a modification for this classical theory which enables it to account for such phenomena as the Zeeman effect we clarify the relation which exists among the existence of states with integral nonzero values of L_z, the single-valuedness requirement for quantum mechanical wave functions, and Dirac's quantization condition for magnetic monopoles. Whereas in the quantum formalism the quantization of magnetic monopoles is taken to be a result of the quantum properties of matter, in the classical formalism with which we are concerned the "quantum" property of matter exhibited by the Zeeman effect is accounted for by allowing for such magnetic fields as are produced by Dirac magnetic monopoles. We next apply the newly modified theory to obtain a complete solution for the static configurations of the classical hydrogen-like atom in the presence of a uniform external magnetic field.

In our considerations we have left out spin, statistics, and particle number for the matter field which properties, we feel, may be the essential quantum features.

2. THE GAUGE COVARIANT AND GAUGE NONCOVARIANT THEORIES

To illustrate the relation between them the gauge noncovariant Lagrangian is derived from the gauge covariant Klein-Gordon-Maxwell Lagrangian. The Lagrangian density for a Klein-Gordon field ψ interacting with its Maxwell field A_μ can be written as[1]

$$\mathcal{L} = (\partial^\mu + i A^\mu)\psi^+ (\partial_\mu - i A_\mu)\psi - \lambda^2 \psi^+\psi - \frac{1}{\alpha} F^{\mu\nu}F_{\mu\nu} \ , \tag{2.1}$$

where $\lambda = \frac{mc}{\hbar}$ is a reciprocal length, α is the fine structure constant, and $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. Writing the wave function as

$$\psi = R \exp iS \ , \tag{2.2}$$

where R and S are respectively the real amplitude and phase of ψ, (2.1) becomes

$$\mathcal{L} = \partial^\mu R \partial_\mu R + R^2 (A^\mu - \partial^\mu S)(A_\mu - \partial_\mu S) - \lambda^2 R^2 - \frac{1}{\alpha} F^{\mu\nu}F_{\mu\nu} \ . \tag{2.3}$$

Defining the new variable

$$A'_\mu = A_\mu - \partial_\mu S \ , \tag{2.4}$$

we can rewrite (2.3) in the form

$$\mathcal{L} = \partial^\mu R \partial_\mu R + R^2 (A^{\mu\prime} A'_\mu - \lambda^2) - \frac{1}{\alpha} F^{\mu\nu\prime} F'_{\mu\nu} \ . \tag{2.5}$$

where $F'_{\mu\nu} = A'_{\nu,\mu} - A'_{\mu,\nu}$ and we have assumed that $\partial_\mu S$ is integrable. Hereafter we drop the primes in referring to (2.5) which is the desired generalization of the gauge noncovariant classical Lagrangian for distributed charge which Dirac[2] gives as

$$\mathcal{L} = R^2 (A^\mu A_\mu - \lambda^2) - F^{\mu\nu} F_{\mu\nu} \ . \tag{2.6}$$

1 The metric signature (−−−+) is used throughout this paper. Greek (Latin) indices refer to the space-time (spatial) coordinates and components.
2 P. A. M. Dirac, Proc. Roy. Soc. (London) 209A, 291 (1951)

Equation (2.5) is a generalization of (2.6) in as much as it contains the dynamics of the source as well as the dynamics of the Maxwell field. Although (2.5) is gauge noncovariant, it is Lorentz covariant.

The Lagrangian (2.5) is equivalent to that of (2.1) only under the circumstance that the gradient of the phase of the wave function is integrable. It is because this condition is not satisfied in general that a more detailed examination of the theory (2.5) and of its relation to the quantum theory becomes necessary. We begin with a comparison of the equations of motion obtained in the two theories.

The Euler-Lagrange Equations

Independent variation of the six variables ψ , ψ^+ , A_μ in the Klein-Gordon-Maxwell Lagrangian (1.1) leads to the six Euler-Lagrange equations

$$\frac{\delta \mathcal{L}}{\delta \psi^+} = (\partial^\mu - i A^\mu)(\partial_\mu - i A_\mu)\psi + \lambda^2 \psi = 0 \quad , \tag{2.7}$$

$$\frac{\delta \mathcal{L}}{\delta \psi} = (\partial^\mu + i A^\mu)(\partial_\mu + i A_\mu)\psi^+ + \lambda^2 \psi^+ = 0 \quad , \tag{2.8}$$

$$\frac{\delta \mathcal{L}}{\delta A_\mu} = i \psi^+ \overleftrightarrow{\partial^\mu} \psi + 2 A^\mu \psi^+ \psi - \frac{4}{\alpha} F^{\mu\nu}{}_{,\nu} \quad , \tag{2.9}$$

where δ is the variational derivative. From the Maxwell equations (2.9) and the Bianchi identity

$$F^{\mu\nu}{}_{,\mu\nu} = 0 \quad , \tag{2.10}$$

or by the combination of the Lagrange equations (2.7) and (2.8)

$$\psi^+ \frac{\delta \mathcal{L}}{\delta \psi^+} - \psi \frac{\delta \mathcal{L}}{\delta \psi} = 0 \quad , \tag{2.11}$$

we obtain the continuity equation

$$\partial_\mu \left(i \psi^+ \overleftrightarrow{\partial^\mu} \psi + 2 A^\mu \psi^+ \psi \right) = 0 \quad . \tag{2.12}$$

Independent variation of the five variables R , A_μ in the generalized Dirac theory (2.5) leads to the five Euler-Lagrange equations

$$\frac{\delta\mathcal{L}}{\delta R} = -2\,\Box R + 2R\,(A^\kappa A_\kappa - \lambda^2) = 0 \quad , \tag{2.13}$$

$$\frac{\delta\mathcal{L}}{\delta A_\mu} = 2R^2 A_\mu - \frac{4}{\alpha}F^{\mu\nu}{}_{;\nu} = 0 \quad . \tag{2.14}$$

From the Maxwell equations (2.14) and the Bianchi identity (2.10) we obtain the continuity equation

$$\partial_\mu\,(R^2 A^\mu) = 0 \quad . \tag{2.15}$$

Although the gauge covariant quantum theory (2.1) depends on six variables ψ , ψ^+ , A_μ and leads to six Euler equations, whereas the five variables R , A_μ of the gauge noncovariant classical theory (2.5) yield but five Euler equations, this does not of itself require that the physical content of the two theories be essentially different. The covariance property of the quantum theory indicates that not all six of the equations of motion are independent. That the number of independent equations of motion is less than six is seen from the fact that the Lagrange equation obtained in the quantum theory (2.3) by variation of the phase is just the continuity equation

$$\frac{\delta\mathcal{L}}{\delta S} = 2\,\partial_\mu\left\{R^2\,(A^\mu - \partial^\mu S)\right\} = 0 \ , \tag{2.16}$$

which is a consequence of the Bianchi identity and the Maxwell equations according to

$$2\,\partial_\mu\{R^2(A^\mu - \partial^\mu S)\} = \frac{4}{\alpha}F^{\mu\nu}{}_{;\mu\nu} \equiv 0 \quad . \tag{2.17}$$

Nonrelativistic Theory

The nonrelativistic limit of the generalized Dirac theory (2.5) can be obtained either by applying the transformations (2.2) and (2.4) to the Schrodinger-Maxwell Lagrangian given by

$$\mathcal{L} = \pm 2\lambda A_0 \psi^\dagger\psi \pm i\lambda \psi^\dagger \overleftrightarrow{\partial_0} \psi + (\partial^r + iA^r)\psi^\dagger(\partial_r - iA_r)\psi - \frac{1}{\alpha} F^{\kappa\nu} F_{\mu\nu} \quad , \quad (2.18)$$

where the signs $\pm$ are determined by the sign of the particle's charge, or directly from the relativistic classical Lagrangian (2.5) as follows.

Setting

$$A_0 = A_0^* \pm \lambda \quad , \quad (2.19)$$

in (2.5), where λ is the positive constant previously defined, we obtain

$$\mathcal{L} = \partial^r R \partial_r R \pm 2\lambda A_0^* R^2 + A^r A_r R^2 - \frac{1}{\alpha} F^{\kappa\nu} F_{\mu\nu} + R^2 A_0^{*2} + \partial^0 R \partial_0 R . \quad (2.20)$$

The nonrelativistic approximation consists of taking

$$|A_0^*| << \lambda \quad , \quad (2.21)$$

$$\partial^0 R \partial_0 R = \frac{1}{c^2} R_{,t} R_{,t} << |\partial^r R \partial_r R| \quad . \quad (2.22)$$

The first of these conditions requires that changes in time be relatively slow and the second indicates that the largest term in the energy is the mass term. The nonrelativistic Lagrangian for the generalized Dirac theory then becomes

$$\mathcal{L} = \partial^r R \partial_r R \pm 2\lambda A_0 R^2 + A^r A_r R^2 - \frac{F^{\kappa\nu} F_{\mu\nu}}{\alpha} \quad , \quad (2.23)$$

where we have dropped the star on the A_0 variable.

The relativistic action (2.5) describes a self interacting single-particle distribution for either sign of charge and is invariant under the transformation

$$A_\mu \Rightarrow - A_\mu \quad . \quad (2.24)$$

Having removed the constant $\pm\lambda$ from A_0 with the intention of imposing the condition (2.22), we find that the theory

(2.23) is invariant under the simultaneous replacement of A_μ by $-A_\mu$ and of positive (negative) charge by negative (positive) charge.

The Euler equations which follow from (2.23) are

$$-\partial_r\partial^r R \pm 2\lambda A_0 R + A^r A_r R = 0 \quad , \tag{2.25}$$

$$\frac{2}{\alpha} F^{0r}{}_{,r} = \pm \lambda R^2 \quad , \tag{2.26}$$

$$\frac{2}{\alpha} F^{r\mu}{}_{,\mu} = R^2 A^\mu \quad . \tag{2.27}$$

The continuity equation now reads

$$\pm \lambda \partial_0 R^2 + \partial_r (R^2 A^r) = 0 \quad . \tag{2.28}$$

The Maxwell equation (2.26) obtained by variation of the potential A_0 contains no time derivatives of A_0 and can therefore be considered as a constraint equation on this variable. Similarly (2.25) is a constraint equation for R . The dynamical development of the R field is determined by the continuity equation (2.28).

3. INITIAL VALUES AND CONTINUITY CONDITIONS

Specification of a theory is not complete with the determination of the Lagrangian. Among the other necessary information are the conditions to be imposed on the field variables. Although in the Klein-Gordon theory one customarily requires for problems that do not involve singular model-potentials that the wave function ψ be analytic in each of the cartesian coordinates, this restriction cannot be applied to R and S if one hopes to obtain all the solutions that are found in the quantum theory. In particular, the analytic quantum mechanical eigenstates of L_z, which are proportional to $e^{im\phi}$ where

ϕ is the azimuthal angle, have a real phase $S = m\phi$ which is not analytic along the z axis when m is different from zero. On the z axis neither S nor its derivatives are defined.

What we can say is that

$$R^2 = \psi^+ \psi \qquad , \qquad (3.1)$$

and

$$2iR^2 \partial_\mu S = \psi^+ \overleftrightarrow{\partial_\mu} \psi \qquad , \qquad (3.2)$$

are analytic where ψ is analytic. Furthermore, in any region throughout which R is nonzero, $\partial_\mu S$ is analytic and we can define an analytic R field. The R field cannot vanish over any finite spatial volume; for, if it were to do so, then by analytic continuation we could show that it vanished everywhere. Because R can vanish only over spatial surfaces, lines, and at points we see that space is divided into regions inside each of which R and $\partial_\mu S$ are analytic. The extent to which these fields can be analytically continued across the boundaries separating the regions of analyticity is discussed in a later section.

The Cauchy Problem

In the Klein-Gordon-Maxwell theory (2.1) it is not possible to independently specify the values of all the field variables and their first time derivatives over a space-like hypersurface. This will be clear from the Euler equation (3.5) below, which is first order in time. If one restricts the variables by a gauge condition on $A_r^{long.}$, then it is possible to give as the initial data the values of

ψ , A_r^{tr} , $\partial_0\psi$, $\partial_0 A_r^{tr}$ over all space at the time t, where A_r^{long} and A_r^{tr} refer respectively to the longitudinal and transverse components of A_r . With this initial data and with the aid of the Lagrange equations we show that one can determine the values of the independently specified quantities on the hypersurface infinitesimally displaced in time from the initial surface.

From the Lagrange equations, which we write as

$$\partial_\mu\partial^\mu\psi - 2iA^\mu\partial_\mu\psi - A^\mu A_\mu\psi - i\psi\partial_\mu A^\mu + \lambda^2\psi = 0 \quad , \quad (3.3)$$

$$\partial_\mu\partial^\mu\psi^+ + 2iA^\mu\partial_\mu\psi^+ - A^\mu A_\mu\psi^+ + i\psi^+\partial_\mu A^\mu + \lambda^2\psi^+ = 0 \quad , \quad (3.4)$$

$$-\frac{4}{\alpha}(A^{r,0} - A^{0,r})_{,r} + i\psi^+\overleftrightarrow{\partial^0}\psi + 2A^0\psi^+\psi = 0 \quad , \quad (3.5)$$

$$\frac{4}{\alpha}(A^{r,0} - A^{0,r})_{,0} - \frac{4}{\alpha}(A^{s,r} - A^{r,s})_{,s} + i\psi^+\overleftrightarrow{\partial^r}\psi + 2A^r\psi^+\psi = 0 \quad , \quad (3.6)$$

and from the values of ψ , A_r^{tr} , $\partial_0\psi$, $\partial_0 A_r^{tr}$ over all space at the time t we require the values of ψ , A_r^{tr} , etc. over all space at the time $t+\delta t$. We assume that the initial values have been given for the variables in the Coulomb gauge

$$\partial_r A^r = 0 \quad , \quad (3.7)$$

This gauge condition requires that the longitudinal component of A_r be identically zero so that

$$A_r = A_r^{tr} \quad , \quad (3.8)$$

$$\partial_0 A_r = \partial_0 A_r^{tr} \quad . \quad (3.9)$$

Equation (3.5) is a constraint equation for A_0 in that the values of the independent variables over the space-like hypersurface determine A_0 on that surface. Thus, with $A^r{}_{,r} = 0$ in (3.5) we find $A_0|^t$ from the initial data. From (3.6) we obtain $(A^{r,0} - A^{0,r})_{,0}|^t$ and hence $(A^{r,0} - A^{0,r})|^{t+\delta t}$. Taking the divergence of this last and noting that $A^r{}_{,r} = 0$ we have $A^{0,r}{}_r|^{t+\delta t}$ from which we can determine $A_0|^{t+\delta t}$. We can now find $A_{0,0}|^t$ and use it in (3.3) to obtain $\partial_0^2 \psi|^t$ and in (3.6) to obtain $\partial_0^2 A_r|^t$. From these second time derivatives we can find $\partial_0 \psi|^{t+\delta t}$ and $\partial_0 A_r|^{t+\delta t}$ thereby completing the determination of the independent variables on the time displaced hypersurface.

Because it is not gauge covariant the generalized Dirac theory (2.5) avoids the complication of dealing with a gauge condition. From the Lagrange equations

$$\partial_\mu \partial^\mu R + R(\lambda^2 - A^\mu A_\mu) = 0 \quad , \tag{3.10}$$

$$\frac{2}{\alpha}(A^{r,0} - A^{0,r})_{,0} - \frac{2}{\alpha}(A^{s,r} - A^{r,s})_{,s} + R^2 A^r = 0 \quad , \tag{3.11}$$

$$-\frac{2}{\alpha}(A^{r,0} - A^{0,r})_{,r} + R^2 A^0 = 0 \quad , \tag{3.12}$$

and from the values of the independent variables R, A_r, $\partial_0 R$, $\partial_0 A_r$ over all space at the time t we require the values of R, A_r, etc. at the time $t+\delta t$. Equation (3.12) is the constraint equation for A_0 determining its value on the space-like surface in terms of the values of the independent variables on that surface. Thus, from (3.12) we have $A_0|^t$. Using this in (3.10) we obtain $\partial_0^2 R|^t$. From R and $\partial_0 R$, $\partial_0 R$ and $\partial_0^2 R$, A_r and $\partial_0 A_r$ at t we determine respectively R, $\partial_0 R$ and A_r at $t+\delta t$. From (3.11) we find $(A^{r,0} - A^{0,r})_{,0}|^t$ which in turn gives $(A^{r,0} - A^{0,r})|^{t+\delta t}$. Together with $R|^{t+\delta t}$ this yields $A_0|^{t+\delta t}$ from (3.12) Knowing $A_0|^t$ and $A_0|^{t+\delta t}$ we can determine $\partial_0 A_0|^t$ and use it in (3.11) to calculate $\partial_0^2 A_r|^t$. Finally, from $\partial_0^2 A_r|^t$ and $\partial_0 A_r|^t$ we obtain $\partial_0 A_r|^{t+\delta t}$.

Each of the independent variables ψ, A_r^{tr} chosen for the quantum theory is composed of two components. These four independent components and their first time derivatives are eight functions which when specified over an initial space-like plane form a complete set of initial data in some gauge. In the generalized Dirac theory we choose as initial data the four variables R , A_r and their first time derivatives $\partial_0 R$, $\partial_0 A_r$ or again a total of eight independent functions on the initial surface.

In the gauge covariant quantum theory one speaks of the values of the variables with reference to a gauge. If a gauge is not chosen, then only the values of the gauge invariants $\psi^+\psi$, $\psi^+(\partial_\mu - i A_\mu)\psi$, etc. are meaningful. Although in the classical gauge noncovariant theory all the variables are physically meaningful in the sense that all have definite values, yet they are not all independent; for one variable A_0 is still related to the others not by a dynamical equation but through an equation of constraint (3.12).

The Nonrelativistic Cauchy Problem

In passing from the relativistic case (2.13) to the nonrelativistic limit (2.25) the Lagrange equation for R goes from second order to zeroth order in the time derivative. As a result the number of independent functions which form a complete set of initial data over a space-like hypersurface goes from eight to six.

Thus for example one might choose as the initial data the values of A_r , $\partial_0 A_r$ over all space at time t. The values of these independent functions over the infinitesimally time displaced hypersurface can be calculated from the Lagrange equations as follows. Equation (2.26) can be solved for A_0 as a function of R and the result used in (2.25) to find $R|^t$. Equation (2.25) then gives $A_0|^t$. From the continuity equation (2.28) we can now determine $R|^{t+\delta t}$. Again employing (2.25) we find $A_0|^{t+\delta t}$ from which we obtain $\partial_0 A_0|^t$. This in (2.27) gives $\partial_0^2 A_r|^t$ which in turn yields $\partial_0 A_r|^{t+\delta t}$ completing the program.

Another possible choice of initial data is A_r^{tr}, A_r^{long}, $\partial_0 A_r^{tr}$, R over all space at time t. This set of data differs from that just discussed in that the single independent function $\partial_0 A_r^{long}|^t$ has been replaced by $R|^t$. From this new set of data we can recover $\partial_0 A_r^{long}|^t$ in the following manner. From (2.25) one can determine $A_0|^t$ with which (2.26) gives $\partial_0 \partial_r A^{r\,long}|^t$ or $\partial_r A^{r\,long}|^{t+\delta t}$. From this last we can find $A_r^{long}|^{t+\delta t}$ and hence $\partial_0 A_r^{long}|^t$.

4. SIMPLE APPLICATIONS

To illustrate the utility and the limitations of the classical gauge noncovariant theory thus far developed we apply the Euler equations to several simple problems.

Hydrogen-like Atom

We consider a single distributed classical charge in the presence of an external point source Coulomb field. This is the analog of the hydrogen atom problem in the usual quantum theory. We solve only for the static configurations.

With the external source

$$j^{\mu(x,t)} = (0, 0, 0, \delta(\vec{r})) \quad , \tag{4.1}$$

the nonrelativistic Euler equations (2.25), (2.26), (2.27) become respectively

$$-\partial_r \partial^r R + A^r A_r R \pm 2\lambda A_0 R = 0 \quad , \tag{4.2}$$

$$-\frac{2}{\alpha}(A^{0,r} - A^{r,0})_{,r} = \pm \lambda R^2 + \frac{2}{\alpha}\delta(\vec{r}) \; , \tag{4.3}$$

$$\frac{1}{\alpha} F^{r\mu}{}_{,\mu} = R^2 A^r \quad . \tag{4.4}$$

The continuity equation (2.28) is

$$\pm \lambda \partial_0 R^2 + \partial_r (R^2 A^r) = 0 \quad . \tag{4.5}$$

For the static solutions which we seek one expects the spatial currents to vanish

$$j^r = \frac{\alpha}{2} R^2 A^r = 0 \quad . \tag{4.6}$$

This condition on the current can be satisfied by taking

$$A^r = 0 \quad , \tag{4.7}$$

which then requires by reason of the continuity equation that

$$\partial_0 R = 0 \quad . \tag{4.8}$$

The Euler-Lagrange equations therefore become

$$-\partial_r \partial^r R \pm 2\lambda A_0 R = 0 \quad , \tag{4.9}$$

$$-\frac{2}{\alpha}\partial_r \partial^r A_0 = \pm \lambda R^2 + \frac{2}{\alpha}\delta(\vec{r}) \; . \tag{4.10}$$

Solving (4.10) we obtain

$$A_0(\vec{r}) = -\frac{1}{4\pi r} \mp \frac{\lambda\alpha}{8\pi}\int d^3r' \, \frac{R^2(r')}{|r - r'|} - E \quad , \tag{4.11}$$

where E is a constant. Upon entering this solution for A_0 into the Schrodinger equation for R (4.9) we have

$$\nabla^2 R \mp \frac{\lambda}{2\pi}\frac{R}{r} - \frac{\lambda^2\alpha}{4\pi}\left\langle R(r') \middle| \frac{1}{|r - r'|} \middle| R(r') \right\rangle R = \pm 2\lambda E R \; , \tag{4.12}$$

in which the constant E appears as the eigenvalue. This equation is nonlinear in R and therefore depends upon the normalization of the R field. This is determined by requiring that the atom be electrically neutral; that is, we take

$$\int d^3r \, (j^0 + j^{0(ext)}) = 0 \quad . \tag{4.13}$$

Integrating the right hand side of (4.3) we obtain

$$\int d^3r \left\{ \pm \lambda R^2 + \frac{2}{\alpha} \delta(\vec{r}) \right\} = 0 \quad , \tag{4.14}$$

or

$$\mp \int d^3r \, R^2(\vec{r}) = \frac{2}{\alpha\lambda} \quad . \tag{4.15}$$

That cloud and external source be charges of opposite sign requires then that the lower sign in (4.15) and elsewhere be chosen in our equations. Changing the normalization of R from (4.15) to

$$\int d^3r \, R^2(\vec{r}) = +1 \quad , \tag{4.16}$$

(4.12) becomes

$$\nabla^2 R + \frac{\lambda}{2\pi} \frac{R}{r} - \frac{\lambda}{2\pi} \langle R(r') | \frac{1}{|r-r'|} | R(r') \rangle R = -2\lambda E R \, . \tag{4.17}$$

If we neglect the nonlinear self interaction term in (4.17) we find that R has the complete set of real time independent solutions

$$R^{nlm}(r,\theta,\phi) = \mathcal{R}^{nl}(r) \, P^{lm}(\theta) \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix} \quad , \tag{4.18}$$

where $\mathcal{R}^{nl}$ are the hydrogen atom radial functions and P^{lm} are the associated Legendre polynomials with m integral. Still neglecting the self interaction term we find that the energy eigenvalue ratios $E_n / E_{n'}$ for the solutions (4.18) agree with those of the quantum mechanical hydrogen atom solutions.

With the set of solutions (4.18), however, the self interaction term is not negligible. Equation (4.17) is the

Schrodinger equation for a real density of charge R^2 distributed about a point charge at the origin. On the average the potential energy arising from the self interaction among the parts of the cloud is of the same order as that arising from the interaction with the nucleon; that is,

$$\langle R | \tfrac{1}{r} | R \rangle \approx \langle R(r) | R(r') | \tfrac{1}{|r-r'|} | R(r') | R(r) \rangle \quad . \tag{4.19}$$

For example, with the normalization (4.16) and for the ground state

$$R^{100} = \frac{1}{\sqrt{\pi}} \left(\frac{\lambda}{4\pi}\right)^{3/2} e^{-\frac{\lambda r}{4\pi}} \quad , \tag{4.20}$$

we have that

$$\langle R^{100} | \tfrac{1}{r} | R^{100} \rangle = \frac{8}{5} \langle R^{100}(r) | R^{100}(r') | \tfrac{1}{|r-r'|} | R^{100}(r') | R^{100}(r) \rangle . \tag{4.21}$$

The solutions obtained $R^{n\ell m}$ are just the real and imaginary components of the quantum mechanical wave functions $\psi^{n\ell m}$.

To show that the Schrodinger eigenvalues E_n in (4.17) are indeed the nonrelativistic energies we rewrite the action (2.5) in a system of curvilinear coordinates with the help of the $g_{\mu\nu}$ metric and calculate

$$T_{00} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{00}} \left(\mathcal{L} \sqrt{-g} \right) \quad . \tag{4.22}$$

Having done this we restrict the metric to the Lorentz form and use the Euler equations to obtain

$$T_{00} = \partial_0 R \partial_0 R - R \partial_0^2 R + 2 R^2 A_0 A_0 - \tfrac{2}{\alpha} F_0{}^r F_{0r} + \tfrac{1}{\alpha} F^{rs} F_{rs} - \partial_r (R \partial^r R) . \tag{4.23}$$

For a static localized system this becomes

$$T_{00} = 2 R^2 A_0 A_0 - \tfrac{2}{\alpha} F_0{}^r F_{0r} \quad . \tag{4.24}$$

Assuming the presence of an external source we write the Maxwell potential as

$$A_0 = \phi^{el} + \phi^{ext} - E \quad , \tag{4.25}$$

where ϕ^{el} and ϕ^{ext} are respectively the spatially
where ϕ^{el} and ϕ^{ext} are respectively the spatially
dependent components of the potential with sources in the distributed charge and in the external source. The constant term E is the relativistic energy which is computed from the Klein-Gordon equation for R . Thus we have

$$T_{00} = 2R^2 A_0 (\phi^{el} + \phi^{ext} - E) + \frac{2}{\alpha}\left(\phi^{el}_{,r}\phi^{el}_{,r} + 2\phi^{el}_{,r}\phi^{ext}_{,r} + \phi^{ext}_{,r}\phi^{ext}_{,r}\right) . \tag{4.26}$$

With the Maxwell equations

$$\frac{4}{\alpha}\nabla^2\phi^{el} = 2R^2 A_0 \equiv j^0 \quad , \tag{4.27}$$

$$\frac{4}{\alpha}\nabla^2\phi^{ext} = j^{0(ext)} \quad , \tag{4.28}$$

where j^0 and $j^{0(ext)}$ are the internal and external current sources respectively, (4.26) becomes after partial integrations

$$T_{00} = -j_0 E - \frac{2}{\alpha}\phi^{el}_{,r}\phi^{el}_{,r} + \frac{2}{\alpha}\phi^{ext}_{,r}\phi^{ext}_{,r} \quad . \tag{4.29}$$

Normalizing the internal charge to -1 we obtain after integrating over space

$$\int d^3r\, T_{00} = E - \frac{2}{\alpha}\int d^3r\, \phi^{el}_{,r}\phi^{el}_{,r} + \frac{2}{\alpha}\int \phi^{ext}_{,r}\phi^{ext}_{,r} \quad . \tag{4.30}$$

The second term on the right hand side subtracts the self energy of the distributed charge while the third term is the self energy of the external source. In the nonrelativistic limit and neglecting self effects the energy reduces to the mass term plus the Schrodinger eigenvalue E_n .

Zeeman Effect

To show the limitations of the theory (2.5) we consider a hydrogen-like atom in the presence of an external constant magnetic field whose vector potential is given by

$$\vec{A} = \frac{1}{2}\vec{H}\times\vec{r} \tag{4.31}$$

The static Klein-Gordon equation for R becomes

$$\nabla^2 R + R\,(A_0 A_0 - \frac{1}{4} H^2 r^2 \sin^2\theta - \lambda^2) = 0 \tag{4.32}$$

Again writing

$$A_0 = \phi^{el.} + \phi^{ext} - E \tag{4.33}$$

we find that (4.32) is aside from self effects the usual Klein-Gordon equation less the linear Zeeman term. The energies calculated from (4.32) will therefore not show the linear Zeeman splittings.

Minimum Wave Packet

In another example of the application of the gauge noncovariant classical theory we consider the nonrelativistic time development of the minimum wave packet in the absence of external fields.

The Euler equations are given by (2.25), (2.26), and (2.27) and the conservation law is (2.28). As in the hydrogen atom a finite self interaction arises because we have allowed the R field to act as a source term in the Maxwell equations. Disregarding this source term in order to eliminate the self interactions the Lagrange equations become

$$-\partial_r \partial^r R \pm 2\lambda\phi R + A_r A^r R = 0 \tag{4.34}$$

$$F^{\mu\nu}{}_{,\nu} = 0 \tag{4.35}$$

The conservation law we keep as

$$\pm \lambda \partial_0 R^2 + \partial_r (R^2 A^r) = 0 \tag{4.36}$$

We choose the initial data to be

$$R(\vec{r},0) = (2\pi)^{-3/4}\, \delta^{-3/2} \exp\{-r^2/4\delta^2\} \quad , \tag{4.37}$$

$$\vec{A}^{tr} = 0 \quad , \tag{4.38}$$

$$\partial_0 \vec{A}^{tr} = 0 \quad , \tag{4.39}$$

$$\vec{A}^{long} = 0 \quad , \tag{4.40}$$

where $\vec{A}^{tr}$ and $\vec{A}^{long}$ are the transverse and longitudinal components of $\vec{A}$ respectively.

The Maxwell equations (4.35) are in three dimensional notation

$$\nabla^2 \phi + \nabla \cdot \partial_0 \vec{A}^{long} = 0 \quad , \tag{4.41}$$

$$\nabla \times \nabla \times \vec{A}^{tr} + \nabla \partial_0 \phi + \partial_0^2 \vec{A}^{tr} + \partial_0^2 \vec{A}^{long} = 0 \quad . \tag{4.42}$$

Equation (4.41) requires that

$$\partial_0 \vec{A}^{long} + \nabla \phi = 0 \quad , \tag{4.43}$$

which with (4.42) leads to

$$\nabla \times \nabla \times \vec{A}^{tr} + \partial_0^2 \vec{A}^{tr} = 0 \quad . \tag{4.44}$$

By repeated differentiation with respect to time of (4.44) and by the use of the initial conditions for $\vec{A}^{tr}$ we find that all the time derivatives of $\vec{A}^{tr}$ vanish and hence

$$\vec{A}^{tr}(\vec{r},t) = \vec{A}^{tr}(\vec{r},t_0) = 0 \quad . \tag{4.45}$$

The Maxwell equation (4.43) determines the dynamical development of $\vec{A}^{long}$; the continuity equation (4.36) determines the dynamics of the R field; and the Schrodinger equation for R (4.34) is a constraint equation which determines A_0 .

These equations can be solved in a Taylor expansion in time as follows. From (4.34) and the initial data we obtain $\phi|^{t_0}$. This and (4.43) yield $\vec{A}^{long}|^{t_0+\delta t}$. Using (4.36) and the initial data we obtain $R|^{t_0+\delta t}$. In this way we determine the necessary independent variables at the displaced time $t_o + \delta t$.

The complete solution, which can be checked by substitution in the dynamical equations, is given by

$$R = (2\pi)^{-3/4}\left(\delta^2 + \frac{x_0^2}{4\lambda^2\delta^2}\right)^{-3/2} \exp\left\{-\frac{r^2}{4\left(\delta^2 + \frac{x_0^2}{4\lambda^2\delta^2}\right)}\right\}, \tag{4.46}$$

$$\phi = \pm\frac{3}{4\lambda}\left(\delta^2 + \frac{x_0^2}{4\lambda^2\delta^2}\right)^{-1} \mp \frac{r^2}{8\delta^2\lambda}\left(\delta^2 - \frac{x_0^2}{4\lambda^2\delta^2}\right)\left(\delta^2 + \frac{x_0^2}{4\lambda^2\delta^2}\right)^{-2}, \tag{4.47}$$

$$A_r = \pm\frac{r x_0}{4\lambda\delta^2}\left(\delta^2 + \frac{x_0^2}{4\lambda^2\delta^2}\right)^{-1}, \tag{4.48}$$

where x_0 refers to the time coordinate ct . The usual solution obtained from the Schrodinger wave theory is identical with the above solutions under the transformations (2.2) and (2.4).

5. THE SELF ENERGY PROBLEM

Both the usual quantum theory (2.1) and the classical (2.5) suffer from an unphysically large self interaction. When applying the quantum theory one alternative is to neglect the self interaction completely by dropping that part, $F^{\mu\nu}F_{\mu\nu}$, of the Lagrangian which depends solely upon the Maxwell field and, having thus eliminated the Maxwell equations which indicate what are the sources of the Maxwell field, to reinterpret the remaining variables, A_μ , as external fields. This procedure is not possible in the gauge noncovariant theory, because the continuity equation which is a dynamical equation for the R field follows from the Bianchi identity and the Maxwell equations. If one neglects the Maxwell equations, one loses the continuity equation and the conservation of charge. The classical theory differs in this respect from the quantum theory for which charge conservation is still guaranteed by reason of the covariance under the constant parameter gauge group even if the free Maxwell Lagrangian is dropped. The other alternative for the quantum theory lies in the procedures of quantum electrodynamics which in spite of its practical successes is not without its own difficulties.

In accord with our attempt to treat the fields classically and to interpret the current density as a real charge density rather than as a probability density we are led next to a formalism in which a particle field interacts only with the gauge fields whose sources lie in other particles. Such a program eliminates the self interactions except in so far as a particle influences the sources of the gauge fields with which it interacts. The situation is quite similar to the manner in which one introduces the Coulomb interaction between point charges while neglecting the infinite self interaction of a point charge with the singular field it produces at its own position.

We assume that Coulomb's law is applicable between the continuous concentrations of distributed charges with the restriction that a particle's charge density at any point does not interact with the Maxwell potentials produced by the other parts of that particle's distribution. For a system of interacting distributed charges we then have

$$\frac{dW}{dt} = \sum_{\substack{i,j \\ (i \neq j)}} \int d^3r \, \vec{J}_i \cdot \vec{E}_j \quad , \tag{5.1}$$

where $\frac{dW}{dt}$ is the rate at which work is done, $\vec{J}_i$ is the current of the i^{th} charge distribution, $\vec{E}_j$ is the electric field produced by the j^{th} charge distribution, and the integral extends over all space. Assuming further that Maxwell's equations relate the fields produced by the j^{th} charge distribution to the four current of that distribution we have

$$\begin{aligned} \nabla \cdot \vec{H}_j &= 0 \quad , \\ \nabla \times \vec{E}_j + \frac{1}{c} \frac{\partial \vec{H}_j}{\partial t} &= 0 \quad , \\ \nabla \cdot \vec{E}_j &= 4\pi \rho_j \quad , \\ \nabla \times \vec{H}_j - \frac{1}{c} \frac{\partial \vec{E}_j}{\partial t} &= \frac{4\pi}{c} \vec{J}_j \quad . \end{aligned} \tag{5.2}$$

Using the Maxwell equations to eliminate the sources from (5.1) we can write

$$\frac{dW}{dt} = -\frac{1}{2} \sum_{\substack{i,j \\ (i \neq j)}} \frac{1}{4\pi} \int d^3r \left\{ c \nabla \cdot (\vec{E}_i \times \vec{H}_j + \vec{E}_j \times \vec{H}_i) + \frac{\partial}{\partial t} (\vec{E}_i \cdot \vec{E}_j + \vec{H}_i \cdot \vec{H}_j) \right\} . \tag{5.3}$$

This suggests defining the Poynting vector and the energy density respectively as

$$\begin{aligned} \vec{S} &= \frac{c}{4\pi} \sum_{\substack{i,j \\ (i \neq j)}} (\vec{E}_i \times \vec{H}_j + \vec{E}_j \times \vec{H}_i) \quad , \\ u &= \frac{1}{8\pi} \sum_{\substack{i,j \\ (i \neq j)}} (\vec{E}_i \cdot \vec{E}_j + \vec{H}_i \cdot \vec{H}_j) \quad . \end{aligned} \tag{5.4}$$

For point charges the Lorentz force on particle i due to the fields produced by the other charges is given by

$$\vec{F}_i = q_i \sum_{\substack{j \\ (j\neq i)}} \left(\vec{E}_j + \frac{\vec{v}_i}{c} \times \vec{H}_j \right) . \qquad (5.5)$$

Generalizing we obtain the rate of change of mechanical momentum for an interacting system of distributed charges as

$$\frac{d\vec{P}_{mech}}{dt} = \sum_{\substack{i,j \\ (i\neq j)}} \int d^3r \left\{ \rho_i \vec{E}_j + \frac{1}{c} \vec{J}_i \times \vec{H}_j \right\} . \qquad (5.6)$$

Using the Maxwell equations to eliminate the sources from (5.6) we can write

$$\begin{aligned} \frac{d\vec{P}_{mech}}{dt} = \frac{1}{8\pi} \sum_{\substack{i,j \\ (i\neq j)}} \int d^3r \Big[\nabla \cdot \Big\{ & \vec{H}_i\vec{H}_j + \vec{H}_j\vec{H}_i + \vec{E}_i\vec{E}_j + \vec{E}_j\vec{E}_i \\ & - \overleftrightarrow{I} (\vec{H}_i\cdot\vec{H}_j + \vec{E}_i\cdot\vec{E}_j) \Big\} - \frac{1}{c}\frac{\partial}{\partial t} \left\{ \vec{E}_i \times \vec{H}_j + \vec{E}_j \times \vec{H}_i \right\} \Big] , \end{aligned} \qquad (5.7)$$

where $\overleftrightarrow{I}$ is the unit dyadic. This suggests defining the stress tensor as

$$\overleftrightarrow{T} = \frac{1}{8\pi} \sum_{\substack{i,j \\ (i\neq j)}} \left\{ \vec{H}_i\vec{H}_j + \vec{H}_j\vec{H}_i + \vec{E}_i\vec{E}_j + \vec{E}_j\vec{E}_i - \overleftrightarrow{I} (\vec{H}_i\cdot\vec{H}_j + \vec{E}_i\cdot\vec{E}_j) \right\} . \qquad (5.8)$$

Combining the results of (5.3) and (5.8) we obtain the energy momentum tensor

$$T_{\mu\nu} = \frac{1}{8\pi} \sum_{\substack{i,j \\ (i\neq j)}} \left\{ F_\mu^{\ \lambda(i)} F_{\nu\lambda}^{\ \ (j)} - \frac{1}{4} g_{\kappa\nu} F^{\kappa\lambda(i)} F_{\kappa\lambda}^{\ \ (j)} \right\} , \qquad (5.9)$$

which can be obtained from the Lagrangian

$$\mathcal{L} = \int d^4x \sum_{\substack{i,j \\ (i\neq j)}} F^{\kappa\nu(i)} F_{\mu\nu}^{\ \ (j)} , \qquad (5.10)$$

where the extra indices on the gauge fields refer them to their sources. Equations (5.9) and (5.10) are of course the same equations that are usually obtained less those terms which result from self interactions.

The distributed sources can be coupled to the gauge fields through the currents by taking the interaction term in the Lagrangian to be

$$\mathcal{L}_{interaction} = \sum_{\substack{i,j \\ (i \neq j)}} \int d^4x \; J^{\mu(i)} A_\mu^{(j)} \; . \tag{5.11}$$

Recalling the continuity equation (2.15) we expect the currents to be of the form

$$J^{\mu(i)} = R^{2(i)} \sum_{\substack{j \\ (j \neq i)}} A^{\mu(j)} \; , \tag{5.12}$$

where self interaction has been avoided by not allowing the current $J^{\mu(i)}$ to depend on $A^{\mu(i)}$.

We have introduced the formalism of attached fields for the purpose of eliminating the unphysical self interactions from the static configurations for which we solved earlier in this paper. The effect on the Lagrangian has been to remove the self effects from both the current and the gauge field terms. In particular, although they are usually considered to be significant as energy-momentum carriers in radiative processes, we have eliminated the squared quantities E_i^2 and H_i^2 from the Lagrangian and thus from the energy.

In the ordinary treatment of radiation from an isolated emitter the squared fields appear to carry off energy in the expanding wave. The subsequent collapse of this wave and the appearance of the entire quantum of energy at some localized absorber then lead to the interpretation of the quantity $E_i^2 + H_i^2$ as a photon probability density. In the attached field formalism the phenomenon of radiation and absorbtion between two systems would appear to be interpretable in terms of the transmission of an energy density $\vec{E}_1 \cdot \vec{E}_2 + \vec{H}_1 \cdot \vec{H}_2$ from one system to the other, where the subscripts 1, 2 on the fields refer them to their respective sources in the radiating system and in the absorbing system. The formalism is in this way related to the other absorber theories of radiation. Thus, although the attached fields have been introduced merely as an aid in the attempt at a consistent classical nonprobabilistic interpretation of the current densities, we find that they suggest a similar classical interpretation for the Maxwell radiation fields. The extent to which the formalism of attached fields is validly applicable to radiative phenomena is outside the scope of this work.

Euler Equations and Energy

In order to show that the formalism of attached fields accomplishes the task for which it was introduced we determine the equations of motion and the energy for the static configurations of a distributed charge in the presence of an external point charge. We take the Lagrangian density to be

$$\mathcal{L} = \partial^\kappa R \partial_\kappa R + R^2 (\bar{A}^\kappa \bar{A}_\kappa - \lambda^2) - \frac{1}{\alpha} F^{\kappa\nu} \bar{F}_{\kappa\nu} + \bar{R}^2 (A^\kappa A_\kappa - \bar{\lambda}^2) \quad , (5.13)$$

where the quantities under the bars refer to the external source and its attached Maxwell potential. The internal and external sources appear symmetrically in the Lagrangian but the fluctuations of the external source have been neglected. The Euler equations which follow from this Lagrangian are given by

$$\Box R = R(\bar{A}^{\mu}\bar{A}_{\mu}-\lambda^2) \quad , \tag{5.14}$$

$$P(A^{\mu}A_{\mu}-\bar{\lambda}^2)=0 \quad , \tag{5.15}$$

$$\frac{2}{\alpha}F^{\mu\nu}{}_{|\nu} = 2R^2\bar{A}^{\mu} \equiv j^{\mu} \quad , \tag{5.16}$$

$$\frac{2}{\alpha}\bar{F}^{\mu\nu}{}_{|\nu} = 2P^2A^{\mu} \equiv \bar{j}^{\mu} \quad . \tag{5.17}$$

The continuity equations are

$$\partial_{\mu}(2R^2\bar{A}^{\mu})=0 \quad , \tag{5.18}$$

$$\partial_{\mu}(2P^2A^{\mu})=0 \quad . \tag{5.19}$$

After writing (5.13) in curvilinear coordinates by introducing the metric we compute T_{00} from

$$T_{00} \equiv \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{00}}(\mathcal{L}\sqrt{-g}) \quad , \tag{5.20}$$

where δ is the variational derivative operator. In this way we obtain after restricting the metric to the Lorentz form

$$\begin{aligned} T_{00} = {} & \partial_0 R\,\partial_0 R - R\partial_0^2 R - \partial_r(R\partial^r R) + 2R^2\bar{A}_0\bar{A}_0 \\ & - \frac{4}{\alpha}F_0{}^r\bar{F}_{0r} + \frac{1}{\alpha}F^{\mu\nu}\bar{F}_{\mu\nu} + 2P^2A_0A_0 \quad . \end{aligned} \tag{5.21}$$

For static configurations in which only the time components of the Maxwell potentials are nonzero this becomes

$$T_{00}^{\,static} = j_0\bar{A}_0 + \bar{j}_0A_0 - \frac{2}{\alpha}F_0{}^r\bar{F}_{0r} \quad . \tag{5.22}$$

We write the external Maxwell potential $\bar{A}_0$ as

$$\bar{A}_0 = \bar{\Phi} - E \quad , \tag{5.23}$$

where $\bar{\Phi}$ contains only the spatial dependence of $\bar{A}_0$ and E is a constant. In this way we obtain after integrating the term $F_0{}^r\bar{F}_{0r}$ by parts

$$T_{00}^{\,static} = j_0\bar{A}_0 + \bar{j}_0A_0 + \frac{2}{\alpha}(\bar{\Phi}F_0{}^r)_{|r} - j_0\bar{\Phi} \quad . \tag{5.24}$$

When integrated over all space the divergence term vanishes and the remaining terms can be written

$$\int d^3x\, T_{00}^{\,static} = -\int d^3x\, E j_0 + \int d^3x\, A_0 \bar{j}_0 \quad . \tag{5.25}$$

With the external current given by

$$\bar{j}_0 = \delta(\vec{r}) \quad , \tag{5.26}$$

the Euler equation (5.15) gives

$$A_0(\vec{r}=0) = \bar{\lambda} \quad . \tag{5.27}$$

Requiring that the atom be electrically neutral we obtain the condition

$$\int d^3x\, j_0 = -1 \quad . \tag{5.28}$$

Thus, we obtain the energy as

$$\int d^3x\, T_{00}^{\,static} = E + \bar{\lambda} \quad , \tag{5.29}$$

where E is the relativistic energy eigenvalue, as can be seen from (5.14), and $\bar{\lambda}$ is the mass energy of the external particle.

The Cauchy Problem

To complete the introduction of the attached fields we treat the Cauchy problem as it appears in this formalism for a distributed charge in the presence of a localized external charge whose fluctions we neglect. The Euler equations can be written as

$$-\partial_0^2 R - \partial_r \partial^r R + R(\bar{A}^\kappa \bar{A}_\kappa - \lambda^2) = 0 \quad , \tag{5.30}$$

$$\frac{2}{\alpha}(A^{r,0}_{long} - A^{0,r})_{,r} = 2R^2 \bar{A}_0 \quad , \tag{5.31}$$

$$\frac{2}{\alpha}(A^{s,r}_{tr} - A^{r,s}_{tr})_{,s} + \frac{2}{\alpha}(A^{0,r} - A^{r,0})_{,0} = 2R^2 \bar{A}^r \quad , \tag{5.32}$$

$$\bar{R}(A^\kappa A_\kappa - \bar{\lambda}^2) = 0 \quad , \tag{5.33}$$

$$\frac{2}{\alpha}(\bar{A}^{r,0} - \bar{A}^{0,r})_{,r} = 2\bar{R}^2 A^0 \quad , \tag{5.34}$$

$$\frac{2}{\alpha}(\bar{A}^{s,r} - \bar{A}^{r,s})_{,s} + \frac{2}{\alpha}(\bar{A}^{0,r} - \bar{A}^{r,0})_{,0} = 2\bar{R}^2 A^r \quad , \tag{5.35}$$

where the quantities A_r^{tr} and A_r^{long} refer respectively to the transverse and longitudinal components of A_r . This division of A_r into components is necessitated by the fact that wherever the $\overline{R}$ field vanishes the theory is covariant under the group of gauge transformations

$$A_\mu \Rightarrow A_\mu + \partial_\mu \xi(x^\nu) \quad , \tag{5.36}$$

where $\xi(x^\nu)$ is a single valued and continuous function of the space time coordinates. In addition to the Euler equations we employ the continuity equations

$$\overline{A}_o \partial_o R^2 + R^2 \partial_o \overline{A}_o + (R^2 \overline{A}^s)_{,s} = 0 \quad , \tag{5.37}$$

$$\partial_o (\overline{R}^2 A^o) + (\overline{R}^2 A^s)_{,s} = 0 \quad , \tag{5.38}$$

which follow from the Bianchi identities satisfied by the attached gauge fields. As the initial data we assume that at time t the values of R , $\partial_o R$, A_r^{tr} , $\partial_o A_r^{tr}$, $\overline{R}$, $\overline{A}_r$, $\partial_o \overline{A}_r$ are given over all space while the values of A_r^{long} and $\partial_o A_r^{long}$ are given where $\overline{R}^2$ is nonzero. From the initial data we obtain at $t+\delta t$ the values of R , A_r^{tr} , $\overline{A}_r$ over all space and the values of A_r^{long} where $\overline{R}^2$ is nonzero. From (5.33) we determine $A_o|^t$ and $A_o|^{t+\delta t}$ where $\overline{R}^2$ is nonzero. This gives us $\overline{R}^2 A_o|^t$ which with (5.38) yields $\overline{R}^2 A_o|^{t+\delta t}$. Knowing $A_o|^{t+\delta t}$ where $\overline{R}^2$ is nonzero we obtain $\overline{R}|^{t+\delta t}$. Next we compute $\overline{A}_o|^t$ from (5.34) and use it in (5.30) to determine $\partial_o^2 R|^t$ which then gives $\partial_o R|^{t+\delta t}$. From (5.37) we now obtain $\partial_o \overline{A}_o|^t$ and use it to find $\overline{A}_o|^{t+\delta t}$. Next we compute $(A_{o,r} - A_{r,o})_{,o}|^t$ from (5.32). From (5.31) we then calculate $(A_{r,o}^{long} - A_{o,r})|^t$ and $(A_{r,o}^{long} - A_{o,r})|^{t+\delta t}$. Using the latter we can determine $\partial_o A_r^{long}|^{t+\delta t}$ where

$\overline{R}^2$ is nonzero. From what we have thus far we can extract $(A_{r,0}^{long}-A_{0,r})_{,0}|^t$ and then obtain $\partial_0^2 A_r^{tr}|^t$. From this last we compute $\partial_0 A_r^{tr}|^{t+\delta t}$. Finally we determine $(\overline{A}_{0,r}-\overline{A}_{r,0})_{,0}|^t$ from (5.35) and from it find $\partial_0^2 \overline{A}_r|^t$ and in turn from this compute $\partial_0 \overline{A}_r|^{t+\delta t}$. This then completes the determination at the time $t+\delta t$ of the quantities given initially at the time t.

6. MAGNETIC MONOPOLES AND L_z EIGENSTATES

As we saw in Sec. 4 the hydrogen-like atom in the classical gauge noncovariant theory suffered from two significant defects. The first of these, the appearance of an unphysically large self energy perturbation in the Schrodinger equation is removed in the attached-field formalism presented in Sec. 5. The second difficulty, the absence of the linear Zeeman effect, is considered here.

In Sec. 2 we derived the form of the classical theory (2.5) from that of the quantum theory (2.1). The quantum Lagrangian (2.3) written in terms of the phase and amplitude of the wave function can be divided into three gauge invariant parts:

the free amplitude terms

$$\partial^\mu R \partial_\mu R - \lambda^2 R^2 \quad , \tag{6.1}$$

the interaction term

$$R^2 (A^\mu - \partial^\mu S)(A_\mu - \partial_\mu S) \quad , \tag{6.2}$$

and the free Maxwell term

$$F^{\mu\nu} F_{\mu\nu} \quad . \tag{6.3}$$

The first two of these, (6.1) and (6.2), are invariant under the group of gauge transformations

$$A_\mu \Rightarrow A_\mu + \partial_\mu \xi(x^\nu) \quad ,$$
$$S \Rightarrow S + \xi(x^\nu) \quad , \qquad (6.4)$$

where $\xi(x^\nu)$ is a completely arbitrary function of the space time coordinates. The last term (6.3) is invariant only under the restricted gauge group of transformations for which $\partial_\mu \xi(x^\nu)$ is an integrable function of the coordinates. A transformation with nonintegrable $\partial_\mu \xi$ would introduce into $F_{\mu\nu}$ a physical electromagnetic field which was not present beforehand. The Maxwell equations, which read

$$\frac{1}{\alpha} F^{\mu\nu}{}_{,\nu} = R^2 (A^\mu - \partial^\mu S) \quad , \qquad (6.5)$$

would then have the same source but different $\vec{E}$ and $\vec{H}$ fields before and after the gauge transformation with nonintegrable $\partial_\mu \xi$.

The eigenfunctions of the operator $H_z L_z$, which determines the linear Zeeman energy, are proportional to $e^{im\phi}$ where ϕ is the azimuthal angle. The phase of such an eigenfunction is not defined along the z axis and the gradient of the phase is nonintegrable on the z axis. The gauge transformation which reduces to zero the phase of an eigenfunction of L_z is therefore not a member of the covariance group of (6.3). The absence of the linear Zeeman effect is then a particular case of the more general problem of explaining the handling of the gauge transformations with nonintegrable functions $\partial_\mu \xi$. Because we are not interested in just any gauge transformation with a nonintegrable $\partial_\mu \xi$, but only in those which transform to zero the nonintegrable components of $\partial_\mu S$, we must determine the possible nonintegrabilities of $\partial_\mu S$.

If we assume that ψ and ψ^+ are analytic in each cartesian space time coordinate, then as we saw in Sec. 3

$$R^2 = \psi^+\psi \quad , \tag{6.6}$$

and

$$2i R^2 \partial_\mu S = \psi^+ \overleftrightarrow{\partial}_\mu \psi \quad , \tag{6.7}$$

are analytic everywhere and $\partial_\mu S$ is analytic where R^2 is nonzero. Wherever $\partial_\mu S$ is analytic all of its derivatives exist and it is integrable.

Because $\partial_\mu S$ can be nonintegrable only where ψ vanishes and S is undefined, we conclude that a nonintegrability in $\partial_\mu S$ is physically meaningful only in terms of its nonlocal effects. Thus the statement that over some region where the wave function vanishes

$$\partial_\nu \partial_\mu S - \partial_\mu \partial_\nu S \neq 0 \quad , \tag{6.8}$$

is sensible only in so far as

$$\oint_c \partial_\mu S \, dx^\mu = \int_\Sigma (\partial_\nu \partial_\mu S - \partial_\mu \partial_\nu S) \, d\Sigma^{\mu\nu} \quad , \tag{6.9}$$

is uniquely determined, where Σ is a two dimensional surface in space time at some points of which $\partial_\mu S$ is nonintegrable and c is a curve bounding the surface.

The value of (6.9) will be uniquely determined only if $\partial_\mu S$ is well defined over the full length of the curve c ; and, for this to be so, c must lie completely within a region of nonvanishing ψ . Because $\partial_\mu S$ is integrable wherever ψ does not vanish, we conclude that for (6.9) to be different from zero so that $\partial_\mu S$ will be nonintegrable at some points on the surface Σ , the region of nonzero ψ in which C lies must be multiply connected.

Therefore, a necessary condition for the existence of a physically significant $\partial_\mu S$ nonintegrable in a region of vanishing ψ is that the domain over which ψ vanishes must be encompassed by a multiply connected domain of nonvanishing ψ .

For the static configurations in which we are particularly interested the only analytic wave functions which satisfy this condition are those which vanish over a closed or infinite line in space. For these wave functions the nonintegrable phase gradient ∇S adds to the field tensor $F_{\mu\nu}$ a discontinuous magnetic field on and tangent to the line where ψ vanishes.

Such singular lines of magnetic field appear in Dirac's theory of magnetic monopoles.[2] In fact Dirac arrives at his quantization condition for magnetic monopoles

$$eg = \frac{n\hbar c}{2} \quad , \tag{6.10}$$

where n is an integer and g is the magnetic pole strength by considering the possible nonintegrabilities of the wave function's phase gradient.

Dirac began with the Schrodinger theory and its single valued wave functions and deduced from the "quantum"

2. P. A. M. Dirac, Proc. Roy. Soc. (London) 133A, 60 (1931); Phys. Rev. 74, 817 (1948)

properties of matter that magnetic monopoles, if they exist, must be quantized in terms of electric charge. Our procedure has been to remove the phase from the complex wave function, to add its gradient $\partial_\mu S$ to the A_μ field, and to interpret the resulting quantity $A_\mu - \partial_\mu S$ as a Maxwell potential. This leads to a gauge noncovariant theory which is physically equivalent to the Schrodinger theory only if $\partial_\mu S$ is everywhere integrable. For the time independent solutions of the quantum theory ∇S will be integrable except perhaps along a line in space. The singular magnetic field produced by such a nonintegrable phase gradient is just the magnetic filament which appears in Dirac's theory of magnetic monopoles. Taking the posture opposite to Dirac's we allow for the existence of the type of singular fields produced by quantized magnetic monopoles and find that the time independent "quantum" properties of matter can be explained in a consistent classical gauge noncovariant field theory.

In the quantum theory it is the density $\psi^+\psi$ and not the wave function itself that has a physical interpretation. The imposition of the single valuedness condition on ψ itself is in this regard an additional postulate to the quantum theory. In the same way the allowance for singular quantized magnetic strings is an added condition on the classical field theory.

7. APPLICATIONS OF THE EXTENDED THEORY

In this section we apply to the hydrogen-like atom in the presence of a constant external magnetic field the gauge noncovariant classical theory modified in accordance with the formalism of attached fields of Sec. 5 and extended to allow for magnetic-pole-like strings. We then present a nonrelativistic static solution for the minimum wave packet in the presence of a particular external source.

Zeeman Effect for the Hydrogen-like Atom

In order to maintain the Maxwell equations in their usual form and yet consider the magnetic filaments produced by point magnetic poles we consider the stationary magnetic sources to be located sufficiently far from the localized system of interest that the radial field from the pole is negligible and we delete from the spatial coordinate system the line along which the filament lies. In the region of interest then the magnetic field $\vec{H}^D$ of the pole is zero and the Maxwell equations for the field attached to the pole are

$$\nabla \cdot \vec{H}^D = 0 \quad ,$$
$$\nabla \times \vec{H}^D = 0 \quad . \tag{7.1}$$

The vector potential A_μ^D by which we describe the effects of the pole in the region of interest is, however, not zero. We choose its covariant components to be

$$A_\mu^D = (A_r^D, A_\theta^D, A_\varphi^D, A_0^D) = (0, 0, \pm m, 0) \quad , \tag{7.2}$$

where m is an integer. This potential corresponds to a singular magnetic field along the z axis and passing through the origin. Thus we have

$$\oint_C A_i^D \, dx^i = 2\pi m \quad , \tag{7.3}$$

where C is a closed curve in space surrounding the magnetic string. Equations (7.1) and (7.2) are compatible because we have deleted from the coordinate system the line where the string lies thereby making space multiply connected.

The covariant components of the potential for the constant external magnetic field, which we choose to be in the direction of the z axis, are given by

$$A_\mu^H = (A_r^H, A_\theta^H, A_\varphi^H, A_0^H) = (0, 0, \tfrac{\alpha H}{e\,2} r^2 \sin^2\theta, 0) \quad . \tag{7.4}$$

The covariant components of the potential from the point charge located at the origin are

$$A_\mu^c = (A_r^c, A_\theta^c, A_\varphi^c, A_0^c) = (0, 0, 0, -\tfrac{\alpha}{8\pi r} - E) \,, \tag{7.5}$$

where E is a constant.

Because we are not interested in the interactions among the various external fields, we consider an effective external potential which is the sum of (7.2), (7.4) and (7.5) and obtain

$$\overline{A}_\mu = (\overline{A}_r, \overline{A}_\theta, \overline{A}_\varphi, \overline{A}_0) = (0, 0, \tfrac{\alpha H}{e\,2} r^2 \sin^2\theta \pm m, -\tfrac{\alpha}{8\pi r} - E) \,. \tag{7.6}$$

The Lagrangian is now

$$\mathcal{L} = \partial^\mu R \partial_\mu R + R^2 (\overline{A}^\mu \overline{A}_\mu - \lambda^2) - \tfrac{1}{\alpha} F^{\mu\nu} \overline{F}_{\mu\nu} + \overline{J}^\mu A_\mu \quad , \tag{7.7}$$

where $\overline{J}^\mu$ contains the sources of $\overline{A}^\mu$. With $\overline{J}^\mu$ nondynamical the Euler equations are

$$-\Box R + R\,(\overline{A}_\mu \overline{A}_\mu - \lambda^2) = 0 \quad , \tag{7.8}$$

$$\tfrac{2}{\alpha} \overline{F}^{\mu\nu}{}_{,\nu} = \overline{J}^\mu \quad , \tag{7.9}$$

$$\frac{2}{\alpha} F^{\mu\nu}{}_{,\nu} = 2R^2 \bar{A}^\mu \quad . \tag{7.10}$$

Substituting the value of the external potential into (7.8) we obtain

$$-\Box R + R\left\{ \left(\frac{\alpha H}{2e} r^2 \sin^2\theta \pm m\right)^2 g^{\phi\phi} + \left(\frac{\alpha}{8\pi r} + E\right)^2 - \lambda^2 \right\} = 0 \ , \tag{7.11}$$

or

$$-\Box R + R\left\{ \left(\frac{\alpha H}{2e}\right)^2 r^2 \sin^2\theta \pm \frac{\alpha m H}{e} + \frac{m^2}{r^2 \sin^2\theta} + \left(\frac{\alpha}{8\pi r} + E\right) - \lambda^2 \right\} = 0 \ , \tag{7.12}$$

which is the Klein-Gordon equation for the amplitude of the wave function for a hydrogen atom in a constant magnetic field. The relativistic energy is E . The linear Zeeman term is $\pm \frac{\alpha m H}{e}$.

We compute the energy from T_{00} which for the Lagrangian (7.7) is

$$T_{00} = 2\partial_0 R \partial_0 R + 2R^2 \bar{A}_0 \bar{A}_0 - \frac{4}{\alpha} F_0{}^r \bar{F}_{0r} + 2 \bar{j}^0 A_0 - \mathcal{L} \quad . \tag{7.13}$$

With the aid of the Euler equations this can be expressed for stationary configurations as

$$T_{00} = 2R^2 \bar{A}_0 \bar{A}_0 + \bar{j}^0 A_0 - \bar{j}^r A_r + \frac{1}{\alpha} F^{rs} \bar{F}_{rs} - \frac{2}{\alpha} F_0{}^r \bar{F}_{0r} \quad . \tag{7.14}$$

Writing

$$\bar{A}_0 = \bar{\Phi} - E \quad , \tag{7.15}$$

this becomes after partial integrating and using the Euler equations

$$T_{00} = -E j_0 + \frac{2}{\alpha} (\bar{\Phi} F_0{}^r)_{,r} + \bar{j}_0 A_0 - \frac{2}{\alpha} (A_r \bar{F}^{rs})_{,s} \quad . \tag{7.16}$$

Because $\bar{\Phi} F_0{}^r$ goes to zero faster than $1/r^2$ as r goes to infinity the second term on the right hand side vanishes. Because we have treated the external sources nondynamically the A_0 potential is arbitrary up to a constant which does not depend on the solution for the

R field. We choose this constant to be zero. The last term on the right hand side vanishes because of the symmetry.

Normalizing the current j_0 to -1 we obtain

$$\int d^3r\, T_{00} = E \quad , \tag{7.17}$$

where E is the relativistic energy computed from the Klein-Gordon equation for R (7.12).

Static Minimum Wave Packet

It is of some interest to note that there exists an exact time-independent solution for the minimum wave packet in the presence of a particular external field.

We take the nonrelativistic Lagrangian to be

$$\mathcal{L} = \partial^r R \partial_r R + R^2 (\bar{A}^r \bar{A}_r \pm 2\lambda \bar{A}_0) - \frac{1}{\alpha} F^{\mu\nu} \bar{F}_{\mu\nu} + \frac{2}{\alpha} \bar{j}^\mu A_\mu \ , \tag{7.18}$$

with the nondynamical external source given as

$$\bar{j}^\mu = (0, 0, 0, \xi) \ , \tag{7.19}$$

where ξ is a constant. This external source is to be interpreted as a uniform density of charge distributed spherically symmetrically about the origin. The Euler equations are then

$$-\partial_r \partial^r R + R \bar{A}^r \bar{A}_r \pm 2\lambda \bar{A}_0 R = 0 \quad , \tag{7.20}$$

$$F_0{}^r{}_{,r} = \pm \alpha \lambda R^2 \quad , \tag{7.21}$$

$$F^{r\mu}{}_{,\mu} = \alpha R^2 \bar{A}^r \quad , \tag{7.22}$$

$$\bar{F}^{0r}{}_{,r} = \bar{j}^0 \quad , \tag{7.23}$$

$$\bar{F}^{r\mu}{}_{,\mu} = 0 \quad . \tag{7.24}$$

For the static wave packet

$$R(\vec{r}, t) = (\delta \sqrt{2\pi})^{-3/2} \exp\left\{-\frac{r^2}{4\delta^2}\right\} \ , \tag{7.25}$$

which is distributed symmetrically about the origin, the

Lagrange equations have the solutions

$$A_r = \bar{A}_r = 0 \quad , \tag{7.26}$$

$$A_0 = \mp \frac{\alpha\lambda}{4\pi} \int d^3r' \frac{R^2(r')}{|r-r'|} \quad , \tag{7.27}$$

$$\bar{A}_0 = \mp \frac{r^2}{8\lambda\delta^4} \pm \frac{3}{4\lambda\delta^2} \quad , \tag{7.28}$$

$$\bar{j}_0 = \mp \frac{3}{4\lambda\delta^4} = \xi \quad , \tag{7.29}$$

where the external source has been evaluated in terms of the parameters of the packet.