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THE DESIGN OF FEEDBACK CONTROL FOR SYSTEMS WITH STATIONARY STOCHASTIC DISTURBANCES

by

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ABSTRACT

The design of feedback control for typical regulator problems in the process industry, but with random disturbances, is treated in this work. The disturbances used have a negative exponential autocorrelation function whose characteristic frequency parameter varies over four decades around the plant's intrinsic frequency.

The performance of the optimal controller for a step disturbance with respect to random disturbances is satisfactory only in the very low characteristic frequency range. This performance deteriorates and becomes worse than that of the uncontrolled plant for medium and high characteristic frequency disturbances. Since disturbances with characteristic frequencies around the plant's intrinsic frequency occur frequently in the process industry, a random controller is justified.

The application of Wiener's optimal design method to the control of a plant corresponding to an ideally stirred tank reactor cascade, leads to an infinite feedback controller gain. When such a plant includes a plug flow section the feedback controller becomes unrealizable. In both cases the feedback controller is unusable in practice.

The introduction of a constraint on the controller effort into the Wiener procedure, using the Lagrangian multiplier tech-

nique, results in practically unacceptable results. For plants with delay, realizability of the feedback controller limits the controller effort to a fraction of what is usually available and the output performance potential of the equivalent open loop controller is not utilized. Better output performance can be achieved with a conventional controller at an acceptable controller effort, particularly for low characteristic frequency disturbances. A similar situation exists for delayless plants where the largest possible controller effort, obtained when the constraint tends to zero, corresponds to the standard deviation of the disturbance. Thus the commonly used constrained formulation of the Wiener optimization problem is not well posed for these feedback regulator problems.

For plants with delay, non optimal random controllers which perform well over the range of disturbances considered, can be constructed from Wiener designs for a two parameter disturbance. Their performance is not too far from the optimal random performance of the equivalent open loop controller, provided they contain a low pass filter to mask them from those high frequency content disturbances for which the optimal performance is practically that of the uncontrolled plant. The step responses of these controllers are satisfactory and their offset acceptable.

The probability structure of a system with a Wiener designed feedback controller and a two state stationary Markov process disturbance is derived. The densities of the systems outputs are of the beta type and the distributions converge to the normal ones for high characteristic frequency disturbances. The differences between these distributions and the normal one are largest for low characteristic frequencies and their significance is due partly to the type of disturbance used.

1. INTRODUCTION

Feedback control is the most widely used control strategy in the Chemical process industry. This reflects the incomplete information that is available about most of the industry's rather complex units, as well as of the nature of the disturbances whose effect has to be regulated. Of primary importance in the process industry is the control of "load" disturbances, that is the regulation problem as opposed to the servo control problem which treats set point, or reference, changes. Set point changes occur rather infrequently upon changing the steady state operating conditions.

Most feedback controller design methods are based on adequate regulation of the response to a single step disturbance or, in an equivalent formulation, on the return of the system to equilibrium, starting from an off equilibrium initial state. One of the most commonly used feedback controllers is the three mode controller whose coefficients are set by optimizing the step response.

It is however, intuitively clear that the fluctuating nature of load disturbances is best described by a random process. For a random disturbance the system variables also become random and so the usual magnitude performance objectives can be defined only in terms of probabilities.

Stocastic, or random, processes were used as process inputs in Chemical Engineering publications but these dealt mainly with open loop properties of processing units and their identification through the measurement of correlation function.(20 - 24)

The difficutly with such an approach to control problems is twofold: 1. Analysis and design methods become more difficult. 2. The probability structure of the disturbance process is not knwon and is quite difficult to measure.

The analysis of control systems in terms of the probability structure can be accomplished with a reasonable effort only for simple situations. There seem to be no probability based methods for controller design. This,however, seems compatible with the effort necessary in acquiring the probability description of the disturbance.

For zero mean stationary variables and linear systems, however, analysis and optimal design methods for the variances of the variables were developed by Wiener (4, 9, 13). In these methods the random processes are characterized by their autocorrelation functions or their Fourier transforms the spectra. The autocorrelation funtion of the disturbance is more accessible to meaurement than is its probability structure, never the less, it is usually not known and its measurement is not trivial. The autocorrelation function is, however, more closely related to the physical background of the disturbance and its character, although not its exact form, can sometimes be inferred from the physical mechanism from which the disturbance arises.It is known (4, 14) that a wide variety of circumstances; applicable to the process industry, give rise to autocorrelation functions of negative exponential shape or can be approximated by several elements of this type (1). The exact parameter values of such autocorrelation function representations, in particular the decay rates, or characteristic frequencies, must be determined from measurements. Further the use of the variance raises the question of its adequacy in estimating the probability of the specified performance objective.

The Wiener optimazation is performed on the equivalent open loop system consisting of the plant and a cascaded compensator from which the feedback controller is then reconstructed. The physical realizability of the cascaded compensator is guaranteed by the procedure but not that of the feedback controller derived from it. Wieners method will make the utmost of the properties of the plant and disturbance properties to deliver a truly optimal design. Some plant model idealizations, which in other applications are of no consequence, might be detrimental in the development of the optimal random controller and make it practically useless. The procedure must therefore be guided into yielding practical results by realistic plants and suitable constraints. These will, of course, depend on the physical context in which the Wiener optimization is used.

The method was developed and used mainly in communication engineering (25) and servo mechanism design applications (4) and, therefore, its applications to the process industry must be done with due consideration to the specific physical background. So for example Newton, Gould and Kaiser in their comprehensive book (4) aimed predominantly towards servo mechanism design do not encounter a situation where the realizability of the feedback controller imposes a contraint on the applicability of the Wiener result. Also, whereas for the regulator problem infinite gain feedback controllers always result from minimum phase plants this is not necessarely the case with the servo problem.

Recently two applications of the Wiener optimization technique to process control were published (11, 12). Lim and Bankoff (12) formulate the procedure for plants with rational transfer functions and a constrained controller output variance. In several examples they show that the feedback controller is rational and in some situations consists of combinations of conventional controller elements. They warn that the feedback controller might be unrealizable but do not investigate this aspect in relationship to the systems performance. Luecke and McGuire (11) apply the Wiener procedure to the design of composite feedforward - feedback controllers. They overlook the fact that their feedback controller is unrealizable and could dispose of the infinite gain difficulty by shifting the emphasis on the feedforward.

portion of their controller. They did, however, investigate the performance of their system with the disturbance for which it was designed thus implying that it is known. They conclude, on the basis of a particular situation, which proves to be an exceptional one, that the design and performance are relatively insensitive to the characteristic frequencies of the negative exponential autocorrelation function they used to represent the disturbance. They therfore proceed to complete their investigation with a single disturbance parameter.

In none of the preceding works attention is given to the fact that the exact description of the disturbance is unknown and, therefore, to the important practical question of the optimal designs sensitivity to disturbances different from those for which it was designed. Also the severe shortcomings of the optimal step controller when the disturbance is random seem not to have been published.

This leads up to the main objective of this work which is the design of a random feedback controller whose performance is satisfactory over the whole spectrum of disturbances that might occur. This would hopefully eliminate the need for the measurement of the autocorrelation function in most cases and point out the situations in which such an investigation would be warranted. The performance of such an overall controller should of course be compared with that of the optimal step controller on the one hand

and with that with the optimal random controller for each specific disturbance on the other. Such an overall controller must also provide adequate protection against an occasional step disturbance due, for example, to an undetected equipment failure. In particular satisfactory offset, or steady state response, to such a disturbance must be assured.

Prior to developing the overall random controller the peculiarities of the Wiener feedback controller design method, when applied to a typical class of single input output plants occuring in the process industry, will be exposed. The rest of the work will be performed with a typical single input - output plant which was used extenisvely to represent plants in the process industry. It corresponds to a reactor consisting of an ideally stirred tank in series with a plug flow section in which a first order irreversible reaction takes place. A negative exponential type autocorrelation function will be used to represent a variety of disturbances occuring in the Chemical industry with its characteristic frequency varying over a wide range. The disturbance will be applied at both the input and output of the plant thus covering a variety of control situations. The output disturbance can be used as representing internally generated randomness such as occur in fluidized beds or the ill defined effect of environmental conditions.

The relationship between the variance and the probability distribution of the variables of a regulator problem will also be further investigated. It has been recommended (4, 12) that the normal distribution be used for this purpose. Although it is known (9) that a normal process remains normal when operated upon by a linear system there is no assurance that the disturbance be normal. The probabilities for the variables of a Wiener control system with a non normal Markov dis-disturbance is analysed and compared to the normal distribution.

2. CHARACTERISATION AND MODELS OF DISTURBANCES

Introduction

The performance objectives in the Chemical Process industry are usually defined in terms of the output deviation from a desired set value. Often it takes the form of specifying the allowable output deviation, or a premium is attached to the maghitude reduction of these deviations. Similarly the controller action usually has a limited range due to its final element, such as a valve, and this leads to a constraint on its magnitude. Treating the disturbance as a random process all other system variables also become random and the above performance specifications and controller limitations can only be expressed as probability distributions. For a random process one can only specify the probability for the deviation to stay within a given range. For a stationary and ergodic process, (10), such a statement is equivalent to specifying the fraction of the time that the deviation will stay inside the specified limits as the total observation time tends to infinity.

The solution of control problems in terms of the probability structure of their variables is mathematically tractable only in some simple situations even for linear systems. Moreover, the disturbance probability structure, which is needed for such a solution, is rarely available and its measurements is a formidable task.

Random control problems are therefore commonly treated in terms of the variances of the systems variables. The analysis and optimization of linear systems in terms of their variances, for zero mean variables, are well known through the work initiated by Wiener (9, 13). The statistic needed for the solution is the autocorrelation function which is more accesible than the probabilities of the disturbance.

The variance gives larger weight to larger deviations and seems intuitively appealing as performance criterion. It is, however, less satisfactory in describing the controller action which has a hard magnitude constraint. The variance and zero mean will, of course, completely describe the probability distribution of a variable if its distribution function can be defined by these two parameters. As in other engineering applications it has been recommended (4, 12) to assume the normal distribution for this purpose. This assumption is correct for the output and controller output of the system if the disturbance is a normal process. It is known (9) that linear operations on a normal process preserve its normal form and affect only its parameters. But even in such a case the normal process assumption for the disturbance remains to be varified.

Based on the variance alone Chebyshev's inequality does, however, afford a lower limit on the probability for the variable, x, to be inside a given range around its mean. It can be expressed as:

$$p[|x - \langle x\rangle| \leq n\sigma] \geq 1 - 1/n^2 \qquad 2.1$$

where $\langle x\rangle$ is the mean of x, σ its standard deviation and n a positive number. Table 2.1 compares these probabilities with the corresponding ones obtained from the normal distribution. It is seen that the inequality offers a rather low limit particularly for small ranges around the mean. On the other hand, whereas a n of 3 for practical purposes includes the complete distribution for a normal process the inequality gives only 88.89%.

In chapter 12 the applicablility of the normal distribution will be investigated further. An example of a system excited by a non-normal disturbance will be solved and its results compared with those of the normal estimate.

The rest of this chapter deals with the statistic of random processes needed for the bulk of this work namely, the autocorrelation function. Its properties in relationship to linear systems will be reviewed. Several examples for disturbnce autocorrelation functions which can arise in the process industry will also be given. The measurement of disturbance autocorrelation functions can be quite laborious. In many situations at least its character can be arrived at by understanding the physical circumstances by which it arises. A control study performed with such an estimate will then indicate whether the determination of a more detailed and accurate autocorrelation function is warranted.

Table 2.1

Comparison of the Normal Distribution and Chebyshev's Inequality

$n = \frac{\lvert x - \langle x \rangle \rvert}{\sigma}$	Q-Ch.	Q-N
1.0	0	68.26
1.5	55.56	86.61
2.0	75.	95.45
2.5	84.	98.77
3.0	88.89	99.73
3.5	91.84	99.96
4.0	93.74	99.994

$$Q = p\left[\,\lvert x - \langle x \rangle \rvert \leqslant n\sigma\,\right]$$

Autocorrelation Functions and Linear Systems

As was mentioned above the performance of control systems with random excitations is most commonly expressed in terms of the signals variances. Usually control problem variables are defined as variations from their means, or steady states, so that their variances are equivalent to the second moments. For the evaluation or design of linear systems in terms of signal variances, the autocorrelation function of the disturbance is the only statistic, necessary. For a stationary signal, x(t), this is defined as :

$$\varphi_{xx}(\tau) = \langle x(t)\, x(t+\tau)\rangle \qquad 2.2$$

where the pointed brackets indicate taking the expected values, i. e., ensemble averages.Because the process, x(t), is stationary, $\varphi_{xx}(\tau)$ depends only on the translation τ of the signal rather than the absolute time,t, and further $\varphi_{xx}(\tau)$ is an even function of τ . Similarly the cross-correlation function for two different stationary processes is defined as:

$$\varphi_{xy}(\tau) = \langle x(t)\, y(t+\tau)\rangle \qquad 2.3$$

and since it also depends on the translation τ only

$$\varphi_{xy}(\tau) = \varphi_{yx}(-\tau) \qquad 2.4$$

From the definition of $\varphi_{xx}(\tau)$ the variance of x(t), for a zero mean process, is:

$$\langle x^2(t)\rangle = \varphi_{xx}(0) \qquad 2.5$$

As is well known (9) the power spectrum, $S_{xx}(s)$, and the auto-correlation function form a Fourier transform pair:

$$S_{xx}(s) = \int_{-\infty}^{+\infty} \varphi_{xx}(\tau) e^{-s\tau} d\tau \qquad 2.6a$$

and

$$\varphi_{xx}(\tau) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} S_{xx}(s) e^{s\tau} ds \qquad 2.6b$$

where $s = i\omega$. The Fourier transform pair is of course, subject to the usual restrictions on $\varphi_{xx}(\tau)$ and $S_{xx}(s)$. In particular

$$\int_{-\infty}^{+\infty} |\varphi_{xx}(\tau)| d\tau < \infty \qquad 2.7$$

which allows only signals whose autocorrelation functions tends to zero as τ approaches infinity. This excludes signals which contain periodic components or possess a non zero mean.

Further from equations 2.6b and 2.5

$$\langle X^2(t) \rangle = \varphi_{xx}(0) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} S_{xx}(s) ds \qquad 2.8$$

so that from the knowledge of the spectrum, or the auto-correlation function, the variance can be calculated from eqation 2.8 or 2.5 respectively. Remembering that $s = i\omega$ the spectrum can be interpreted as the density distribution of the second moment, or power, along the ω axis.

The transmission properties of power spectra through linear systems can be derived from the above and the definition of the impulse response, or weighting function, of the system and its Fourier transform, the transfer function. The response q(t) of a linear system with impulse response w(t) to an input v(t) is given by:

$$q(t) = \int_{-\infty}^{+\infty} w(t_1)\, v(t-t_1)\, dt_1 \qquad 2.9$$

w(t) represents a causal or physically realizable system only if:

$$w(t) = 0 \quad \text{for} \quad t < 0 \qquad 2.10$$

since otherwise the system would have a response before an input was applied to it. For a casual system therefore the lower limit of integration can be replaced by zero.
Also, the upper limit could be replaced by t for inputs which start at t = 0 and vanish for $t < 0$. It will be convenient to retain the symmetrical $\pm\infty$ in what follows.
For visual clarity, however, they will be ommited from the equations.

From equations 2.9 we have for the product q(t) and $q(t+\tau)$:

$$q(t)\,q(t+\tau) = \int w(t_1)\, v(t-t_1)\, dt_1 \int w(t_2)\, v(t+\tau-t_2)\, dt_2$$

$$= \iint w(t_1)\, w(t_2)\, v(t-t_1)\, v(t+\tau-t_2)\, dt_1\, dt_2 \qquad 2.11$$

Taking expected values of both sides of equation 2.11 it becomes:

$$\varphi_{qq}(\tau) = \iint w(t_1)\, w(t_2)\, \varphi_{vv}(\tau+t_1-t_2)\, dt_1\, dt_2 \qquad 2.12$$

The desired power spectrum of the output is obtained from the Fourier transform of equation 2.12. Noticing that the right hand side of equation 2.12 is a double convolution and using the theorem about the transformation of convolutions we get:

$$S_{qq}(s) = w(s)\, w(-s)\, S_{vv}(s) \qquad 2.13$$

or

$$S_{qq}(s) = |w(s)|^2 S_{vv}(s) \qquad 2.14$$

The last line follows from the fact that the impulse response is a real function. The same symbol is used for the weighing function and its Fourier transform with the argument replaced by s. This convention will be followed throughout this work for weighing functions and signals. Equation 2.14 expresses the desired spectrum transmission properties of linear time invariant systems.

Examples of Autocorrelation Functions

Physically the autocorrelation function and its Fourier transform, the spectrum, indicate the frequency content of a random process. A rapidly changing random process will have a spectrum with contributions at high values of ω . Such a process will show correlation between itself and its translation for relatively small τ only and thus the autocorrelation function will rapidly decrease in magnitude for increasing τ . On the other hand a slow moving process will manifest correlation with itself over longer translation times and therfore its autocorrelation function will possess appreciable magnitude for higher τ . The corresponding power spectrum, however, will hardly extend to large values of ω , or frequency .

An extreme and idealized situation is the case of white noise which is completely uncorrelated with its translation. Its autocorrelation function, $\mathcal{P}(\tau)$, is given by:

$$\mathcal{P}(\tau) = \sigma^2 \delta(\tau) \qquad 2.15$$

and its spectrum, $\mathcal{S}(s)$, therefore becomes:

$$\mathcal{S}(s) = \sigma^2 \qquad 2.16$$

The spectrum is seen to have a constant power contribution at all frequencies. Such a spectrum cannot exist practically because it implies an infinite second moment. However, it is a useful model for approximating uniform spectra with a finite cutoff frequency.

If white noise is passed through a first order low pass filter with the transfer function

$$W(s) = \frac{\sqrt{2\nu}}{s+\nu} \qquad 2.17$$

the output spectrum according to equation 2.14 becomes:

$$S_{qq}(s) = \frac{2\nu\sigma^2}{\nu^2 - s^2} = \frac{2\nu\sigma^2}{\nu^2+\omega^2} \qquad 2.18$$

This spectrum corresponds to the autocorrelation function:

$$\varphi_{qq}(\tau) = \sigma^2 e^{-\nu|\tau|} \qquad 2.19$$

The significance of ν in the filtered process is that of an inverse characteristic correlation time or characteristic frequency. Large ν correspond to a relatively fast decay of the autocorrelation function and therfore the process will contain high frequency components. The bell shaped spectrum is broad and drops only for higher frequencies as a function of ω. The opposite situation results from small ν which indicates long correlation times and low frequency content.

This last pair of autocorrelation function and spectrum occurs frequently and additional examples of its occurance in cases of practical interest for this work will be given subsequently. Because it is mathematically convenient it is also used for developing orthogonal series with which other more complex autocorrelation functions are approximated (1).

Distubance Models in the Process Industry

The character of the autocorrelation functions of disturbances occuring in the chemical industry can be frequently discovered by considering the physical circumstances through which they arise. Such an analysis is worthwhile even if some simplifications are necessary to carry it through because of the considerable effort which the actual measurements of the autocorrelation functions requires. The approximate analysis will yield an estimate of predominant characteristics correlation times. The control analysis could then be completed with these estimates and the need for a more accurate autocorrelation function established in the process .

As a relatively simple example of such a deduction consider the variation in concentration of successive catalyst batches fed to a catinuous reactor. The time period during which a particular batch is fed to the reactor is constant and equals $1/\nu$. Assume further that the concentrations, x, in the batches are independent, identically distributed random variables with zero mean and variance σ^2 . A realization of such a process will be a step function of step duration $1/\nu$ with random levels. Also assumed is that the exact location of the switching times is uniformly distributed for different realizations and therfore the process is stationary. For $\tau > 1/\nu$ the autocorrelation function then becomes:

$$\varphi_{xx}(\tau) = \langle x(t)\, x(t+\tau)\rangle = \langle x(t)\rangle \langle x(t+\tau)\rangle = 0 \qquad 2.20$$

since the levels in different batches are independent and have a zero mean. For $\tau < 1/\nu$ two situations exist:

1. $x(t)$ and $x(t+\tau)$ are within the same batch and then:

$$\langle x(t)\, x(t+\tau)\rangle = \sigma^2 \qquad 2.21$$

which has the probability P(1) of occuring:

$$P(1) = \frac{1/\nu - |\tau|}{1/\nu} = 1 - \nu|\tau| \qquad 2.22$$

2. $x(t)$ and $x(t+\tau)$ extend over two successive batches in which case, by the same reasoning as for $\tau > 1/\nu$ the expected value of their product vanishes.

Combining equations 2.21 and 2.22 the autocorrelation function becomes:

$$\varphi_{xx}(\tau) = \begin{cases} \sigma^2(1-\nu|\tau|) & \text{for } |\tau| < 1/\nu \\ 0 & \text{for } |\tau| \geq 1/\nu \end{cases} \qquad 2.23$$

which has a simple triangular shape. Its spectrum, however, is:

$$S_{xx}(s) = \frac{\sigma^2}{\nu}\left(\frac{\sin(\omega/2\nu)}{\omega/2\nu}\right)^2 \qquad 2.24$$

It is transcendental and therefore not as convenient as the rational spectrum for the negative exponential auto-correlation function of the previous section. Comparing the shapes of the two autocorrelation functions it becomes clear that the triangular function can be approximated by several negative exponential terms.

The same autocorrelation function would of course result if the concentration in each batch could assume only two levels, a and b. This would mean that the probability density for the concentration in each batch consists of two delta functions of strength α and β. From the definition of the probability density function, p(x), we obtain:

$$\int_{-\infty}^{+\infty} p(x)dx = \int_{-\infty}^{+\infty} [\delta(a-x)\alpha + \delta(b-x)\beta]dx = \alpha + \beta = 1 \qquad 2.25$$

The requirement that the mean vanish becomes:

$$\langle x(t)\rangle = \int_{-\infty}^{+\infty} [\delta(a-x)a\alpha + \delta(b-x)b\beta]dx = a\alpha + b\beta = 0 \qquad 2.26$$

so that prescribing the two concentration levels a and b determines α and β as:

$$\alpha = b/(b-a), \qquad \beta = a/(a-b) \qquad 2.27$$

Taking the batch switching points to be Poisson distributed in time rather than occur at constant intervals, yields the negative exponential autocorrelation function.

The Poisson distribution arises when the concentration changes are mutually independent events occuring with equal probability at any time, (10). The probability, p(n), for n such events to occur in the time interval 0, t, where the points are distributed according to Poisson with an average rate ν, is:

$$p(n) = (\nu t)^n \frac{e^{-\nu t}}{n!} \qquad 2.28$$

The probability density, p(t), of the times between switches is in this case a negative exponential one:

$$p(t) = \nu e^{-\nu t} \qquad 2.29$$

which has a mean of $1/\nu$. The autocorrelation function will be equal to the product of the concentration variance and the probability Q(t) that both t and t + τ are within the period of one batch. If the two times extend over two batches then their contribution to the autocorrelation function will vanish due to the independance of the batch concentrations and their zero mean, as in the previous example.

The probability Q(t) is the sum over all possible batch periods l of the product of the probability density that t is an interval of length l and the probability that t + τ is in the same interval. The first probability density is obtained from that of the time between switches and is, (14):

$$\frac{l \nu e^{-\nu l}}{\langle l \rangle} = l \nu^2 e^{-\nu l} \qquad 2.30$$

The probability that both t and t + τ are contained in a specified interval l is as before $(l - |\tau|)/l$,for $\tau < l$. Combining these results and integrating over all l from τ to ∞ Q(τ) becomes:

$$Q(\tau) = \int_{|\tau|}^{\infty} l \nu^2 e^{-\nu l} \frac{(l - |\tau|)}{l} \, dl = e^{-\nu |\tau|} \qquad 2.31$$

so that the autocorrelation function becomes:

$$\varphi_{xx}(\tau) = \sigma^2 Q(\tau) = \sigma^2 e^{-\nu |\tau|} \qquad 2.32$$

As in the case of the triangular autocorrelation function the above derivation includes the case where the concentrations

$$p(t) = \nu e^{-\nu t} \qquad 2.29$$

which has a mean of $1/\nu$. The autocorrelation function will be equal to the product of the concentration variance and the probability Q(t) that both t and t + τ are within the period of one batch. If the two times extend over two batches then their contribution to the autocorrelation function will vanish due to the independance of the batch concentrations and their zero mean, as in the previous example.

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$$Q(\tau) = \int_{|\tau|}^{\infty} \ell \nu^2 e^{-\nu \ell} \frac{(\ell - |\tau|)}{\ell} d\ell = e^{-\nu |\tau|} \qquad 2.31$$

so that the autocorrelation function becomes:

$$\varphi_{xx}(\tau) = \sigma^2 Q(\tau) = \sigma^2 e^{-\nu |\tau|} \qquad 2.32$$

As in the case of the triangular autocorrelation function the above derivation includes the case where the concentrations in each batch assume only two fixed values provided that their mean is zero. If the two levels are equal in magnitude and opposite in sign the process is known as a random telegraph signal.

It will be shown in chapter 12 that the autocorrelation function of a two level stationary Markov process with zero mean also has the negative exponential form. If the two level switching intensities are λ_0 and λ_1, then ν becomes their sum.

The negative exponential autocorrelation function is seen to result from several situations which can be expected to occur in the chemical industry. In cases where it does not apply directly one or more exponential elements can be used to capture the salient features of commonly occuring autocorrelation functions. In additon its spectrum is rational and therfore mathematically convenient, particularly for the type of work done in this thesis. For these reasons the negative exponential autocorrelation function will be used to characterize disturbances in this work with the characteristic correlation time $1/\nu$ varying over a four decade range centered around the plant's time constant. It might be noted that the same autocorrelation function can result from a range of processes with drastically different probability distributions. The selection of one autocorrelation function for this work does therefore allow for a wide variety of disturbance sources.

3. PLANT MODELS AND CONTROL SYSTEM CONFIGURATIONS

Introduction

In this chapter the plant models which will be used in this work will be derived. The derivation will be made for the case of a chemical reactor but the resulting dynamic models can clearly be used to represent other plants with similar dynamics. Next two feedback control configurations for these plants will be described which differ in the point where the disturbance is applied to the plant. The disturbance to the plant can be due to fluctuations in its feedstream which leads to an input disturbance. On the other hand, fluctuations can be generated in the plant due to an internally random phenomenon, such as occuring in a fluidized bed. These and fluctuations due to ill defined effects of environmental conditions on the plant can be represented by introducing an equivalent disturbance at the plants output. Both situations exist in the process industry and can be expected to give different results.

The Plant

The representative transfer functions that will be used in this work will now be developed for a typical chemical reactor model with first order irreversible reaction kinetics. Consider a chemical reactor which can be modeled as a plug flow section in series with a perfectly stirred tank. For first order kinetics this model will represent any reactor having the same residence time distribution (15). Such a reactor with its feedback control loop is shown in figure 3.1 where the disturbance enters through the reactors input or feed. The C's, in this figure represent concentrationsof the reactant of interest, C_o is the stationary random disturbance, F is a constant volumetric flow rate, V's are the constant reactor sections volumes, H(t) is the feedback controller wheighing function and M(t) the manipulated variable. Mass balances around the two sections of the reactor, assuming constant molar densities, yield:

$$V_s \frac{dc(t)}{dt} = F C_p(t) - (F + kV_s) c(t) \qquad 3.1$$

$$c_p = e^{-kV_p/F} c_i(t - V_p/F) \qquad 3.2$$

where k stands for the reaction rate constant. Introducing the mormalized time, θ, as:

$$\theta = t(1/\tau_s + k) = t\lambda/\tau_s \qquad 3.3$$

where

$$\tau_s = V_s/F \quad \text{and} \quad \lambda = 1 + k\tau_s$$

and combining equations 3.1 and 3.2 yields:

$$\frac{dc(\theta)}{d\theta} = \frac{1}{\lambda} e^{-k\tau_p} c_i(\theta - \Delta) - c(\theta) \qquad 3.4$$

where

$$\tau_p = V_p/F \quad \text{and} \quad \Delta = \tau_p \lambda / \tau_s$$

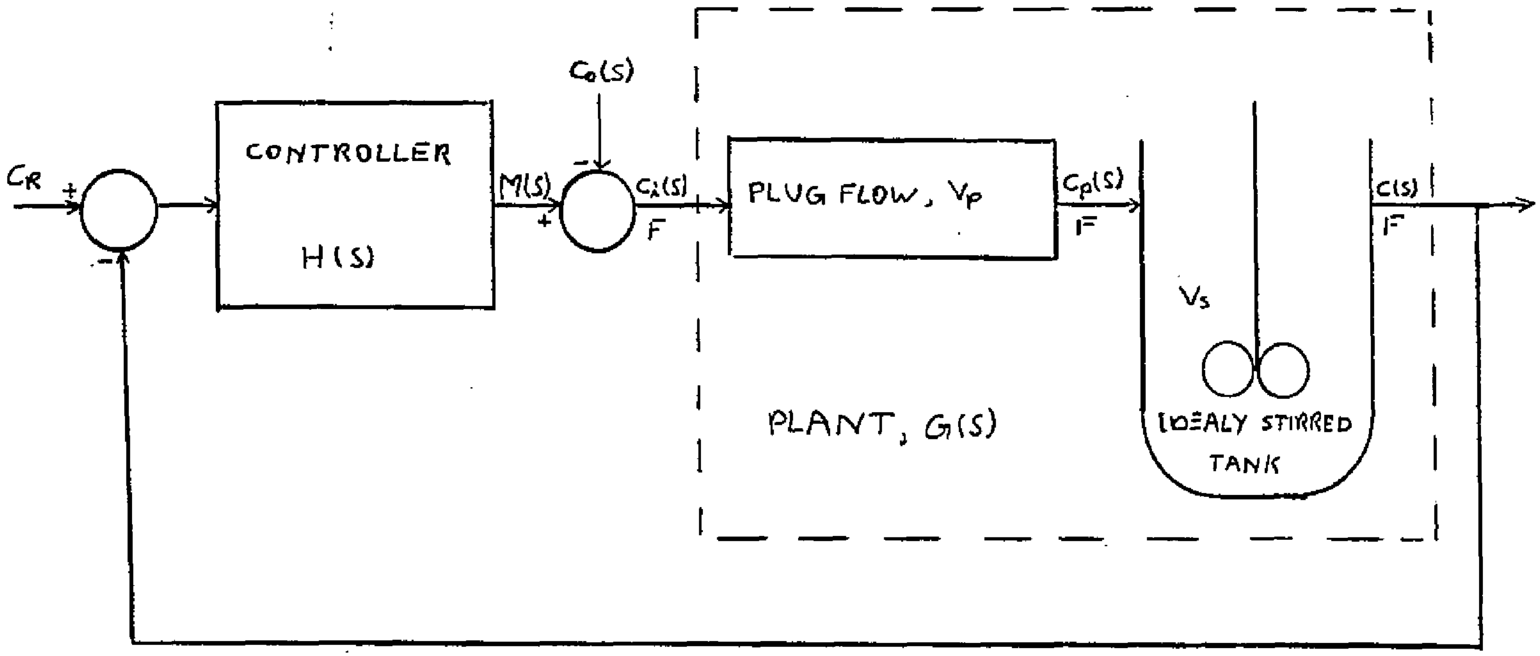

Figure 3.1 FEEDBACK CONTROLLER FOR TYPICAL PLANT WITH INPUT DISTURBANCE

For a constant input the steady state conditon for equation 3.4 becomes:

$$\frac{c}{c_i} = \frac{1}{\lambda} e^{-k\tau_p} = A \qquad 3.5$$

which is the steady state conversion. Taking the expected value of equation 3.4 and imposing stationarity, i. e.:

$$\frac{d\langle c(\theta)\rangle}{d\theta} = 0 \qquad 3.6$$

the conversion for the expected values of the concentration also becomes A, i.e.:

$$\langle c\rangle / \langle c_i\rangle = A \qquad 3.7$$

We can now formulate the plants differential equations in terms of deviations of the expected values. With:

$$\tilde{c}(\theta) = c(\theta) - \langle c\rangle \;, \qquad \tilde{c}_i = c_i(\theta) - \langle c_i\rangle \qquad 3.8,9$$

the differential equation becomes:

$$\frac{d\tilde{c}(\theta)}{d\theta} = A\tilde{c}_i(\theta-\Delta) - \tilde{c}(\theta) \qquad 3.10$$

Input Disturbance and the Control Loop

In figure 3.1 the disturbance enters at the plants input. In this feedback configuration the input to the reactor C_i, becomes:

$$c_i(\theta) = c_o(\theta) + M(\theta) \qquad 3.11$$

with

$$M(\theta) = -\int_{-\infty}^{\theta} H(t)[c(\theta-t) - c_R]\,dt \qquad 3.12$$

where t now serves as the dummy integration variable of the convolution. Since M(θ) is to be stationary its mean must be independant of time. Taking the mean of equation 3.12 it becomes clear that the set point C_R, must be at C which also makes $\langle M \rangle = 0$. We now can formulate the equation for $C_i(\theta)$ in terms or the deviation from its mean. The systems equations than become:

$$\frac{d\tilde{C}(\theta)}{d(\theta)} = A\,C_i(\theta-\Delta) - \tilde{C}(\theta) \qquad 3.14$$

$$\tilde{C}_i(\theta) = \tilde{C}_o(\theta) + M(\theta) \qquad 3.15$$

$$M(\theta) = -\int_{-\infty}^{\theta} H(t)\,\tilde{C}(\theta-t)\,dt \qquad 3.16$$

It is conveniant now to introduce the following normalization for the deviations variables:

$$y = \frac{\tilde{C}_o}{\sigma} \;, \quad Z = \frac{\tilde{C}}{A\sigma} \;, \quad m = \frac{M}{\sigma} \qquad 3.17$$

where σ^2 is the variance of C_o. The systems equations than become:

$$\frac{dZ(\theta)}{d(\theta)} = \frac{\tilde{C}_i}{\sigma}(\theta-\Delta) - Z(\theta) \qquad 3.18$$

$$\frac{C_i(\theta)}{\sigma} = y(\theta) + m(\theta) \qquad 3.19$$

$$m(\theta) = -\int_{-\infty}^{\theta} A\,H(t)\,Z(\theta-t)\,dt \qquad 3.20$$

In this form the system can be studied by varying a single parameter Δ , while the conversion of the mean concentrations becomes a scale factor for the output and a magnitude factor of the feedback controller weighing function $H(t)$. In this way the output fluctuations are compensated for their attenuation due to conversion and taking the input mean as base z shows the output fluctuations relative to their attenuated mean.

Taking the transforms of equations 3.18, 3.19 and 3.20 and substituting for $\frac{\tilde{C}_i(s)}{\sigma}$ in equation 3.18 its value from equation 3.19 the system equations become:

$$z(s) = \frac{e^{-\Delta s}}{1+s}\left[y(s) + m(s)\right] \qquad 3.21$$

$$m(s) = -A H(s) z(s) \qquad 3.22$$

Denoting the transfer function of the reactor by G(s), i.e.:

$$G(s) = \frac{z(s)}{\tilde{C}_i(s)/\sigma} = \frac{e^{-\Delta s}}{1+s} \qquad 3.23$$

and solving equations 3.21 and 3.22 for z(s) and m(s) we get:

$$z(s) = \frac{G(s)}{1 + AHG(s)}\, y(s) \qquad 3.24$$

$$m(s) = \frac{-AHG(s)}{1 + AHG(s)}\, y(s) \qquad 3.25$$

The plant transfer function in equation 3.23 has been used extensively in the process control literature as a typical representative for a wide variety of real processes. Examples range from the classical work of Cohen and Coon for the setting of conventional three mode controller coeficients (2) to a recent publication on the design of direct digital control (Koppel, 3). The justification for this is, that the dynamic response of many real processes can be represented by such a model. It was used as a model for as complex a unit as a fluidized bed reactor (16) with the plug flow and stirred tank sections representing the bubble flow and particulate phase respectively. This model will also be used in this work as representing the typical plant in the process industry.

A special case of this transfer function is the reactor which consists of a stirred tank alone, that is where $\Delta = 0$.
A model consisting of a cascade of n equal stirred tanks will therefore have the following transfer function:

$$G(s) = \left(\frac{1}{1+s}\right)^n \qquad 3.26$$

Output Disturbance Configuration

For the system with output disturbance as shown in figure 3.2 the same plant differential equation holds, i.e.:

$$\frac{d\,\tilde{C}(\theta)}{d(\theta)} = A\tilde{C}_i(\theta-\Delta) - \tilde{C}(\theta) \qquad 3.27$$

with $C_i(\theta)$, however, given by

$$\tilde{C}_i(\theta) = M(\theta) - \int_{-\infty}^{\theta} H(t)\tilde{C}_1(\theta-t)\,dt \qquad 3.28$$

where

$$\tilde{C}_1(\theta) = \tilde{C}(\theta) + \tilde{C}_o(\theta) \qquad 3.29$$

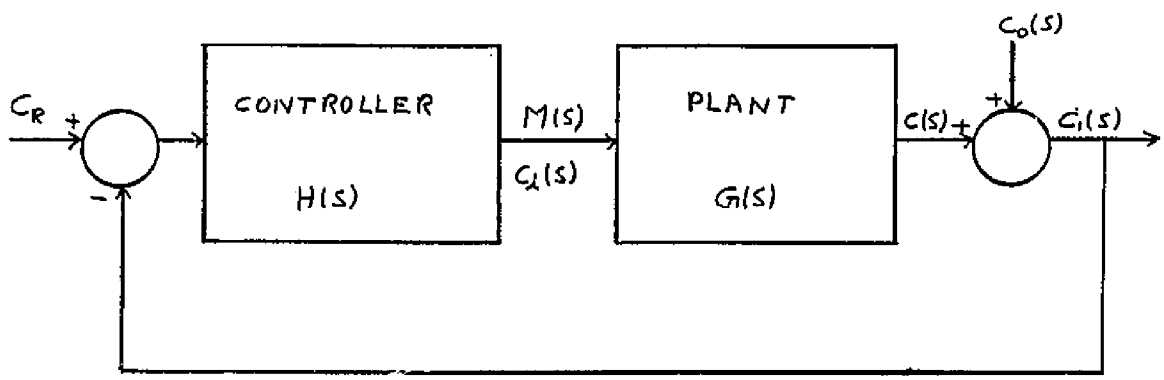

Figure 3.2 FEEDBACK CONTROL OF A PLANT WITH OUTPUT DISTURBANCE

Introducing a somewhat different normalization than the one used for the input disturbance case:

$$y(\theta) = \frac{\tilde{c}_0(\theta)}{\sigma}, \quad z(\theta) = \frac{\tilde{c}_1(\theta)}{\sigma}, \quad w(\theta) = \frac{\tilde{c}(\theta)}{\sigma}, \quad m(\theta) = \frac{\Gamma(\theta)}{\sigma / A} \qquad 3.30$$

the systems equations become:

$$\frac{d w(\theta)}{d\theta} = m(\theta - \Delta) - w(\theta) \qquad 3.31$$

$$z(\theta) = y(\theta) + w(\theta) \qquad 3.32$$

$$m(\theta) = -\int_{-\infty}^{\theta} A H(t) z(\theta - t)\, dt \qquad 3.33$$

where again z, y and m stand for the systems ouput, input and controller action respectively. The normalization achieves the same effect here, as in the input disturbance case. However, since the disturbance enters at the plants output and the disturbance mean is again used as base it is the controller output which is compensated for attenuation, caused by conversion. Taking Fourier transform of equations 3.31, 3.32, and 3.33, eliminating w and retaining the same definition for G, the equations for z and m become:

$$z(s) = y(s) + G\, m(s) \qquad 3.34$$

$$m(s) = -A H z(s) \qquad 3.35$$

Solving for z and m in terms of y one obtains:

$$z(s) = \frac{1}{1 + AGH(s)}\, y(s) \qquad 3.36$$

$$m(s) = \frac{-AH(s)}{1 + AGH(s)} y(s) \qquad 3.37$$

Unified System Description and its Extensions

Both models can be represented with one block diagram shown in figure 3.3 where:

u(s) = Gy(s) for input disturbances

u(s) = y(s) for output disturbances

n(s) = measurement noise

$$G(s) = \frac{e^{-\Delta s}}{1 + s} \qquad 3.38$$

and H incorporates the conversion factor A. Note, however, that the output and controller signals have been normalized differently in both cases.

The above representation still holds if the measurement of the output involves a delay as is often the case in the chemical industry. The normalized delay, Δ , in G then represents the sum of the process and measurement delay.

The role played by measurement error, or noise, can be appreciated from considering the block diagramm in figure 3.3. If n, the measurement error, has a zero mean then:

$$z(s) = \frac{1}{1 + GH(s)} \left[u(s) - GHn(s) \right] \qquad 3.39$$

In other words the input now consists of a sum of two random signals u(s) and GH(s).

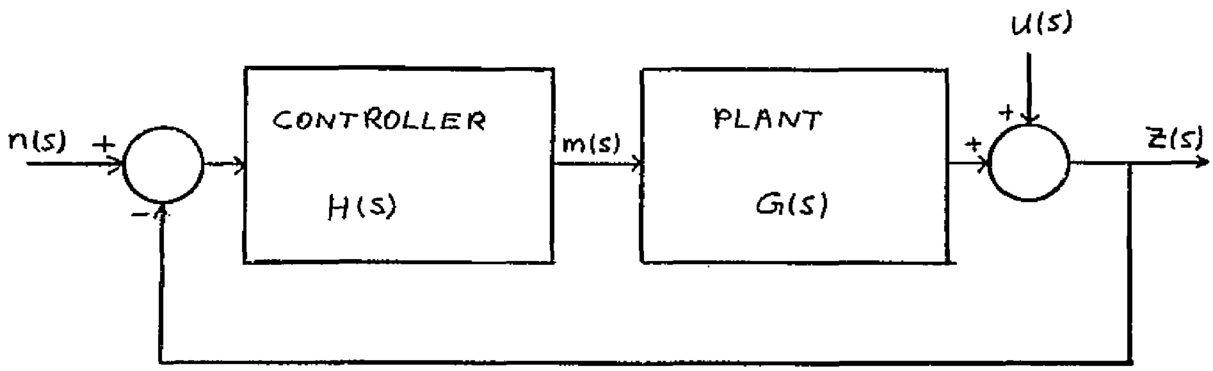

Figure 3.3 UNIFIED BLOCK DIAGRAM FOR INPUT AND OUTPUT DISTURBANCE CONFIGURATION

4. THE CONVENTIONAL CONTROLLER AND ITS PERFORMANCE

The Conventional Controller

Controllers in the process industry are almost exclusively designed with respect to a deterministic input. A step or its equivalent, an initial condition which is off from the equilibrium value, are ususally used to represent the disturbance.

The most widespread feedback controller in use is the three mode controller that is one with proportional together with either or both derivative and integral action on the error signal. Its transfer function, H(s), can be presented as:

$$H(s) = K_g(1 + b/s + d\,s) \qquad 4.1$$

where K_g is the controllers gain, b and d are the reset rate and derivative time respectively. Often the three mode controller is used to approximate more complex controller functions resutling from optimal control theory methods. The determination of the coefficients of the three mode controller was subject to considerable investigation. A stability consideration together with the requirement that the response be fast and with little undershoot, i.e., dead beat response, was used in the pioneering work of Ziegler and Nichols (17). A similar criterion toghether with that of minimizing the area under the response curve was used by Cohen and Coon (2). An integral square error was used by Hazebroek and Van der Waerden (18). The results of all these studies are surprisingly close and a recent comparison of several methods for the plant used in this work is given by Koppel (19). The optimal settings given by Cohen and Coon were developed for this plant and are one of the better known and successful ones.

The three mode controller with the optimal coefficients according to Cohen and Coon will be used here as representing the optimal step controller. Its performance with respect to random disturbances will be determined and compared with that of controllers designed in later chapters for random disturbances.

The Optimal Three Mode Controller

The Cohen and Coon procedure is based on the response to a step disturbance in the input of a plant which corresponds to the reactor with an ideally stirred tank in series with a pure delay that was presented in chapter 3. The transfer function of any given plant is then, according to Cohen and Coon, approximated by adjusting the parameters of the model plant in terms of which the controller constants are defined.

The performance criteria used for the determination of the three mode controllers constants, are: 1. a stability margin corresponding to 75% decay for the amplitude of successive oscillations as determined from the response corresponding to the dominant root pair of the systems characteristic equation. 2. A dead beat response, i.e., practically no undershoot below the equilibrium level. 3. Minimum area under the response curve. With the plants transfer function G(s), given as:

$$G(s) = \frac{e^{-\Delta s}}{1+s} \tag{4.2}$$

the procedure gives the following values for the controller settings, K_g, b and d:

$$\left.\begin{aligned} K_g &= \frac{4}{3\Delta} + \frac{1}{4} \\ b &= \frac{1}{\Delta}\,\frac{13+8\Delta}{32+6\Delta} \\ d &= \frac{4\Delta}{11+2\Delta} \end{aligned}\right\} \tag{4.3}$$

For the proportional-integral controller, performance criteria 1 and 3 only can be satisfied in setting the two constants. The minimum area criterium is, however, somewhat compromised to yield a faster response. The justification for this compromise lies in the nature of the system. The area under the response curve is simply equal to $1/(K_g b)$. This can be seen by applying the final value theorem to the La Place transfer function. For a specified stability margin the relationship between $K_g b$ and the controllers gain shows a flat peak. It is therefore possible to select the two constants somewhat off the peak so that the systems response will be faster than that which would result from the peak values. This is accompanied by only a small increase in the area under the response curve. The recommended setting for this case become:

$$\left.\begin{aligned} K_g &= \frac{9}{10\Delta} + \frac{1}{12} \\ b &= \frac{1}{\Delta}\,\frac{9+20\Delta}{30+3\Delta} \end{aligned}\right\} \qquad 4.4$$

These settings were arrived at from considering the characteristic function of the overall system. They are therefore applicable to both input and output disturbance configuration, which of course have identical characteristic functions.

Performance Computation for Random Disturbances

In the rest of this chapter the performance computing procedure for the conventional controller will be described, in the case of random disturbances. The typical disturbance with negative exponential autocorrelation function, which was described in chapter 2, will be used and the systems output and controller output variances will be calculated for a range of disturbance parameters ν and plant parameter Δ . Two cases will be treated, namely, with the disturbance applied to the output and input of the plant.

a. Output disturbance case.

In this case the system output z(s) and controller output m(s) become:

$$Z(s) = \frac{y(s)}{1+GH(s)} = \frac{s(s+1)}{s(s+1) + e^{-\Delta s}(K_c s + B + Ds^2)}\, y(s) \tag{4.5}$$

$$m(s) = \frac{-H(s)}{1+GH(s)}\, y(s) = \frac{-s(s+1)(K_c s + B + Ds^2)}{s(s+1) + e^{-\Delta s}(K_c s + B + Ds^2)}\, y(s) \tag{4.6}$$

where H(s) is given in the form:

$$H(s) = K_c + B/s + Ds \tag{4.7}$$

The variances of z and m are obtained from integrating their respective spectra over the whole imaginary axis. These spectra are:

$$S_{zz}(s) = \left|\frac{1}{1+GH(s)}\right|^2 S_{yy}(s) \tag{4.8}$$

$$S_{mm}(s) = \left|\frac{-H(s)}{1+GH(s)}\right|^2 S_{yy}(s) \tag{4.9}$$

where

$$S_{yy}(s) = \frac{2\nu}{\nu^2 - s^2} \tag{4.10}$$

We note, from equation 4.6, 4.9 and 4.10 that the integral of $S_{mm}(s)$ does not converge because the integrand tends to a constant for large absolute s. The exponential is of no consequence in this respect being merely a sine - cosine oscillation. This difficulty comes from the fact that the input spectrum tends to zero only for infinte ω , or frequency. Any real disturbance would have a finite cutoff frequency beyond which it will vanish. For such a disturbance the integral $S_{mm}(s)$ will, of course, converge but the variance will depend on the exact cutoff frequency. If a stray high frequency disturbance, such as instrument noise, is present in a system whose output spectrum does not vanish as ω tends to infinity it will contribute heavily to the variance. In other words small amplitude high frequency noises will be amplified and tend to saturate the controller, which will no longer deliver the required action. Setting the derivative action to zero, D = 0, eliminates this difficulty. This corresponds to the established practice of avoiding derivative action for noisy error signals to prevent controller saturation. In this case where the disturbance is introduced directly at the plants output, the controller is indeed excited by the raw random disturbance.

The derivative action could, of course, be retained if the disturbance spectrum would have a denominator of at least two orders higher in s but such disturbances are not a frequent occurance. Alternatively, a first order low pass filter would be necessary in the feedback line if derivative action is to be retained. Such a filter, however, will cancel the benifit of the derivative action. For the output disturbance configuration, therefore, a proportional-integral controller, set according to Cohen and Coon, will be used as comparative conventional controller.

Upon introducing $s = i\omega$ for the Fourier transform variable in equations 4.8, 4.9 and 4.10, the integration of the spectra is performed over the whole real ω axis. The integrals for the variances then become:

$$\sigma_z^2 = \frac{2\nu}{\pi}\int_0^{\infty} \frac{\omega^2(1+\omega^2)}{(\nu^2+\omega^2)\,Q(\omega)}\,d\omega \qquad 4.11$$

$$\sigma_m^2 = \frac{2\nu}{\pi}\int_0^{\infty} \frac{(1+\omega^2)(\omega^2 K_c^2+B^2)}{(\nu^2+\omega^2)\,Q(\omega)}\,d\omega \qquad 4.12$$

$$Q(\omega) = \omega^4 + \omega^2(1+K_c^2) + 2\omega^2(K_c - B)\cos\Delta\omega$$
$$- 2(\omega^3 K_c + \omega B)\sin\Delta\omega + B^2 \qquad 4.13$$

where the even properties of spectra were used. Because of the trigonometric functions in $Q(\omega)$ these integrals can be evaluated only numerically. The integration has to proceed to large enough values of ω where the contribution to the integral becomes negligable. For large ω the integrands in equation 4.11 and 4.12 can be approximated by $1/\omega^2$ and K_c^2/ω^2 respectively, and the integrals evaluated analytically. The integrals, therefore will converge rather slowly, as $1/\omega$, for large ω . In the numerical integration, the contribution to the integral over $\frac{4\pi}{\Delta}$ increments of ω were compared with those of the approximate integral over the same increment. From the point where the two quantities were sufficiently close to one another the approximate integral was used out to infinity. Simpsons rule was used as a quadrature formula and the increment of ω halved until satisfactory convergence was reached. An example of the results is presented in figure 4.1 for $\Delta = 1.5$, in which case the controller settings are $K_c = 0.683$ and $B = 0.515$. A con-

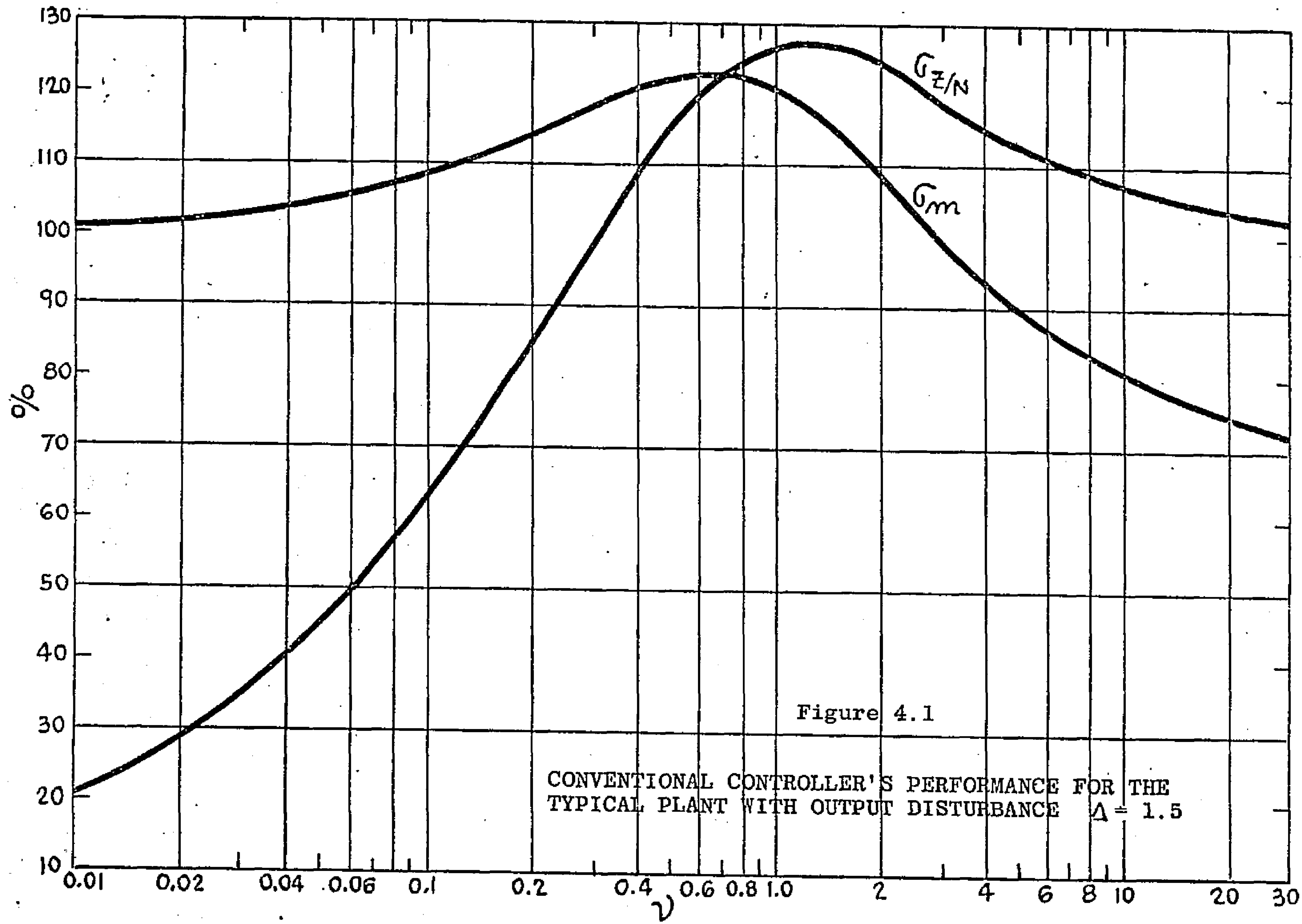

Figure 4.1

CONVENTIONAL CONTROLLER'S PERFORMANCE FOR THE TYPICAL PLANT WITH OUTPUT DISTURBANCE $\Delta = 1.5$

venient yardstick for the output standard deviation is that of the system with no control. The no control output variance, in this case, is obtained from the integral over all ω of the dis-tubance spectrum alone, which, because of the normalization used, becomes 1. The notation $\sigma_{z/N}$ serves to indicate that the output root mean square, rms, is in fact scaled by that of the system with no control. Both standard deviations for the systems output and controller output are shown as a function of the disturbance parameter ν .

From this example it is seen that from a value of $\nu = 0.3$ on a deterministic controller leads to worse output performance than no controller. At the same time the controller output rms is higher than that of the disturbance. The worst behaviour is for ν in the neighborhood of the system time constant, 1, after which the output standard deviation drops to that of no control accompanied by a drop of the controller effort.
Thus a deterministically designed controller ceases to be effective for a disturbance with a characteristic correlation time of about ten times that of systems time constant and becomes unneccessary and wasteful for a disturbance with a characteristic correlation time three times shorter. The relatively good performance for low frequency distrubances was to be expected from the deterministic design of the controller.
b. Input disturbance case.
Proceeding as before in euqations 4.5 to 4.9 with y(s) replaced by Gy(s) and Syy(s) by $|G(s)|^2 S_{yy}(s)$ it is clear that the full three mode controller is applicable. The integrals to be evalu-

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ated after replacing s by $i\omega$ are:

$$\sigma_z^2 = \frac{2\nu}{\pi}\int_0^{\infty} \frac{\omega^2}{(\nu^2+\omega^2)\,Q(\omega)}\,d\omega \qquad 4.14$$

$$\sigma_m^2 = \frac{2\nu}{\pi}\int_0^{\infty} \frac{\omega^4 D^2 + \omega^2(K_c^2 - 2BD) + B^2}{(\nu^2+\omega^2)\,Q(\omega)}\,d\omega \qquad 4.15$$

where:

$$Q(\omega) = \omega^4(D^2+1) + \omega^2(K_c^2 + 1 - 2BD) + B^2 + [\omega^4 D + \omega^2(K_c - B)]2\cos\Delta\omega$$

$$-[\omega^3(K_c - D) + B\omega]2\sin\Delta\omega \qquad 4.16$$

For large ω the integrand of σ_m^2 can be approximated by

$$S_{mm}(s) \cong \frac{2\nu D^2}{(D^2+1+2D\cos\Delta\omega)\,\omega^2} \qquad 4.17$$

and again the integral converges only as $1/\omega$. However, only the lower and upper limits of the tail integrals can be given analytically corresponding to $\cos\Delta\omega = \pm 1$. These limits, to the tail integral from ω to ∞ , σ_{mT}^2 , are given by:

$$\frac{2\nu D^2}{\pi(D+1)^2}\frac{1}{\omega} < \sigma_{mT}^2 < \frac{2\nu D^2}{\pi(D-1)^2}\frac{1}{\omega} \qquad 4.18$$

and similarely for σ_z^2 for which the integral converges as $1/\omega^3$. The integration procedure adopted here, was the same as in the output disturbance case using the avarage of the upper and lower limit for the tail integral as its approximation for large ω .

The output for no control is obtained from:

$$\sigma_N^2 = \frac{1}{\pi i}\int_0^{i\infty} |G(s)|^2 S_{gg}(s)\,ds = \frac{2\nu}{\pi}\int_0^{\infty}\frac{d\omega}{(\nu^2+\omega^2)(1+\omega^2)} = \frac{1}{\nu+1} \qquad 4.19$$

and was used as a scale factor for the output variance. An example of the results is given in figure 4.2 were the value of $\Delta = 1.5$ was again taken and the resulting Cohen and Coon settings are: $K_c = 1.139$, $B = .422$ and $C = .445$.

In this case $\sigma_{z/N}$ crosses the 1.0 mark at $\nu = 1$ compared with 0.3 in the output disturbance case. The curve is shifted by a corresponding amount but reaches a peak of about the same height, 1.25 , from which level it drops only slightly. Again the deterministic design is not very effective from a $\nu = .2$ and becomes worse than no control at $\nu = 1.0$. In this case, however, the control effort decreases from the point where $\sigma_{z/N} = 1.0$ but is still at over 0.5 for $\nu = 100$.

The results for other plants will be presented together with the performance of Wiener type controllers. In both control configurations it is seen that the optimal step controllers performance with respect to the more realistic random disturbance is satisfactory only for disturbances with very low frequency content. As the characteristic frequency is increased the effectiveness of the conventional controller decreases and it becomes worse than no control for characteristic frequencies in the neighborhood of the plants time constant and higher. In effect the conventional controller amplifies the disturbance instead of attenuating it while using up considerable control effort. These facts seem not to be appreciated in the few process controller applications which treat the disturbances as random (11, 12).

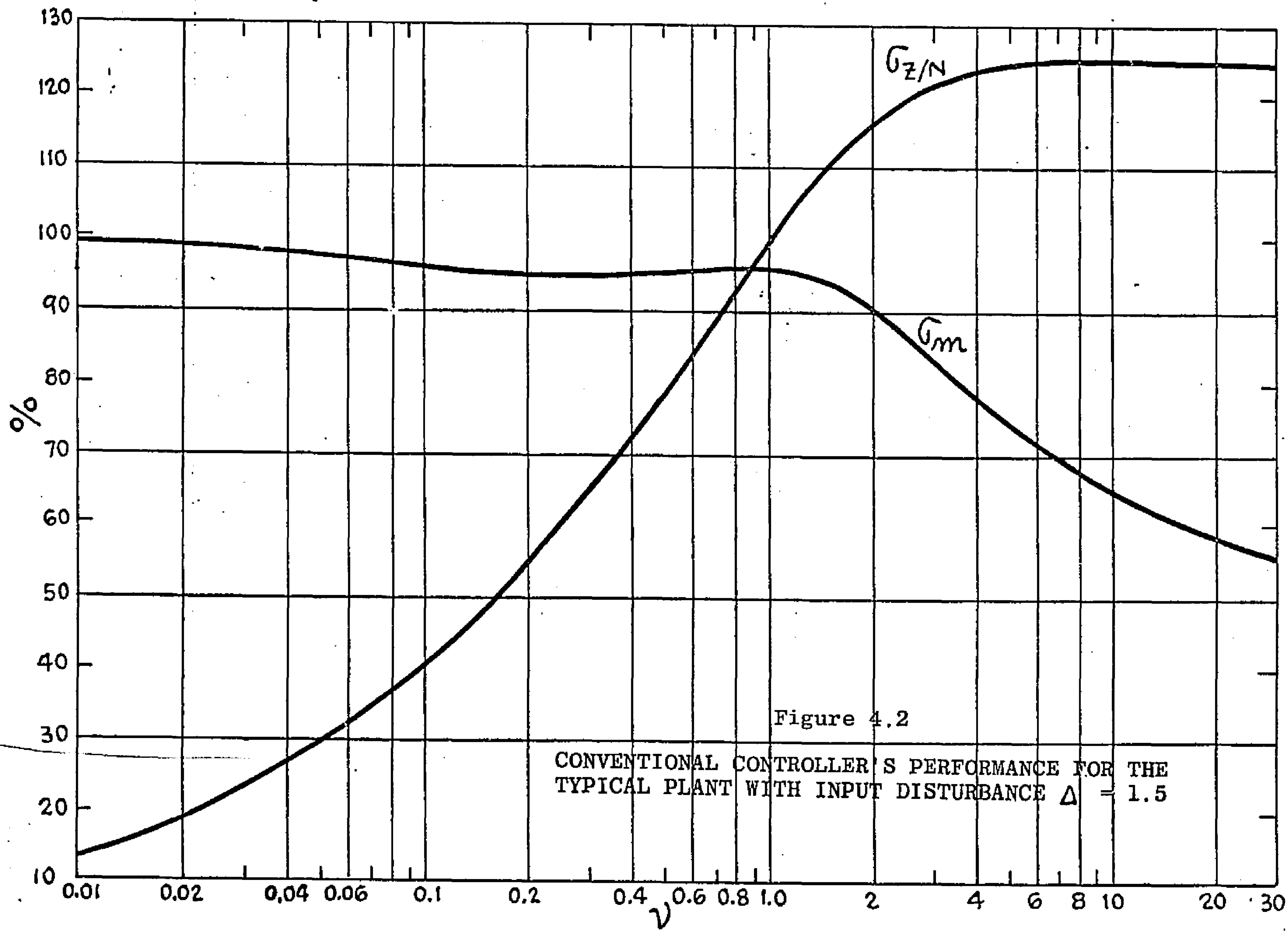


Figure 4.2

CONVENTIONAL CONTROLLER'S PERFORMANCE FOR THE TYPICAL PLANT WITH INPUT DISTURBANCE $\Delta = 1.5$

5. WIENER FILTER - PREDICTOR PROBLEM

Introduction

In this chapter Wiener's basic filter-predictor problem will be formulated and solved. The solution proceeds in two distinct steps: 1. An integral equation is developed using calculus of variations techniques. This equation is known as the Wiener-Hopf equation of the second kind. 2. The integral equation is solved via Wiener's spectrum factorization technique.

The filter predictor problem cannot be applied directly to the regulator problems of this work. However, some simple regulator problems, unconstrained ones, can be recast into the filter predictor problem formulation.

In constrained problems the integral equation must be developed afresh along the same lines as in the filter predictor problem. A somewhat more general integral equation results which can, however, be solved by the same techniques. To accomodate more complex integral equations the solution will be developed and explicit formulas presented in a more general nomenclature than the current problem calls for. The development presented in this chapter follows that of Newton, Gould and Kaiser (4).

The Problem

With reference to figure 5.1, v and i are stationary random processes with zero means which are specified in the terms of the

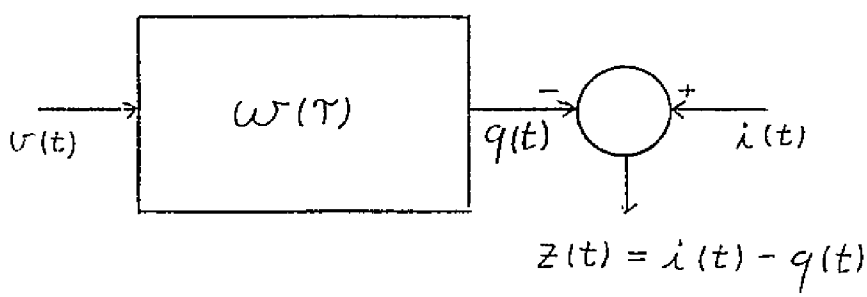


Figure 5.1 THE FILTER-PREDICTOR PROBLEM

autocorrelation function of v, $\varphi_{vv}(\tau)$, and the cross-correlation function between v and i, $\varphi_{vi}(\tau)$, i.e.:

$$\varphi_{vv}(\tau) = \langle v(t)\, v(t+\tau) \rangle \qquad 5.1$$

$$\varphi_{vi}(\tau) = \langle v(t)\, i(t+\tau) \rangle \qquad 5.2$$

The corresponding power spectra, that is, the Fourier transform of the correlation function, are assumed to exist:

$$S_{vv}(s) = \int_{-\infty}^{+\infty} \varphi_{vv}(\tau)\, e^{-s\tau}\, d\tau \qquad 5.3$$

$$S_{vi}(s) = \int_{-\infty}^{+\infty} \varphi_{vi}(\tau)\, e^{-s\tau}\, d\tau \qquad 5.4$$

It is desired to design a linear, physically realizable and time invariant system, with impulse response w(t), which minimizes the mean square deviation of the output q from the ideally desired one i. That is minimize:

$$\langle \varepsilon^2 \rangle = \langle (i - q)^2 \rangle \qquad 5.5$$

where:

$$q(t) = \int w(t_1)\, v(t-t_1)\, dt_1 \qquad 5.6$$

and where w(t) the systems weighting function fulfills the condition of physical realizability:

$$w(t) = 0 \quad \text{for} \quad t < 0 \qquad 5.7$$

Taking v(t) to be for example the sum of a random signal r(t) and a corrupting noise n(t):

$$v(t) = r(t) + n(t) \qquad 5.8$$

and the desired ideal output i(t) to be :

$$i(t) = r(t+\tau) \qquad 5.9$$

the situation becomes that of a filter predictor problem for positive τ .For $\tau \leq 0$ it is merely a filtering problem.

Derivation of the Integral Equation

Using the properties of linear systems $\langle z^2 \rangle$ can be computed as follows:

$$\langle z^2 \rangle = \langle (i-q)^2 \rangle = \langle i^2 \rangle - 2\langle iq \rangle + \langle q^2 \rangle \qquad 5.10$$

where

$$\langle i^2 \rangle = \sigma_i^2 = \varphi_{ii}(0) \qquad 5.11$$

and

$$q^2(t) = \int w(t_1)v(t-t_1)dt_1 \int w(t_2)v(t-t_2)dt_2$$

$$= \iint w(t_1)w(t_2)v(t-t_1)v(t-t_2)dt_1dt_2 \qquad 5.12$$

For convenience the limits of integration were droped as in chapter 2. Taking the expected values of both sides of equation 5.12 and applying the definiton of $\varphi_{vv}(\tau)$ one gets:

$$\langle q^2(t) \rangle = \iint w(t_1)w(t_2)\varphi_{vv}(t_1-t_2)dt_1dt_2 \qquad 5.13$$

Also:

$$i(t)q(t) = i(t)\int w(t_1)v(t-t_1)dt_1 = \int w(t_1)v(t-t_1)i(t)dt_1 \qquad 5.14$$

and from the definiton of $\varphi_{vi}(\tau)$:

$$\langle i(t)q(t) \rangle = \int w(t_1)\varphi_{vi}(t_1)dt_1 \qquad 5.15$$

so that $\langle z^2 \rangle$ becomes:

$$\langle z^2 \rangle = \varphi_{ii}(0) - 2\int w(t_1)\varphi_{vi}(t_1)dt_1 + \iint w(t_1)w(t_2)\varphi_{vv}(t_1-t_2)dt_1dt_2 \qquad 5.16$$

Following standard methods of the calculus of variations w(t) is replaced by:

$$w(t) = w_m(t) + \varepsilon\, w_\varepsilon(t) \qquad 5.17$$

where $w_m(t)$ is the function which minimizes $\langle z\rangle$, $w_\varepsilon(t)$ is an arbitrary but realizable function and ε is a parameter. Inserting this expression for w(t) into equation 5.16 and taking the derivative with respect to ε , remembering that for stationary processes to autocorrelation function is even, one obtains:

$$\frac{\partial\langle z^2\rangle}{\partial\varepsilon} = -2\int w_\varepsilon(t_1)\,\varphi_{vi}(t_1)\,dt_1$$

$$+\,2\iint w_m(t_2)\,w_\varepsilon(t_1)\,\varphi_{vv}(t_1-t_2)\,dt_1 dt_2$$

$$+\,2\varepsilon\iint w_\varepsilon(t_1)\,w_\varepsilon(t_2)\,\varphi_{vv}(t_1-t_2)\,dt_1 dt_2 \qquad 5.18$$

Setting:

$$\frac{\partial\langle z^2\rangle}{\partial\varepsilon} = 0 \qquad at \qquad \varepsilon = 0 \qquad 5.19$$

results in:

$$\int w_\varepsilon(t_1)\,dt_1\left[\int w_m(t_2)\,\varphi_{vv}(t_1-t_2)\,dt_2 - \varphi_{vi}(t_1)\right] = 0 \qquad 5.20$$

Since $w_\varepsilon(t)$ is an arbitrary but physically realizable function we have as a result, for $t_1 \geqslant 0$, the following equation for $w_m(t)$:

$$\int w_m(t_2)\,\varphi_{vv}(t_1-t_2)\,dt_2 - \varphi_{vi}(t_1) = 0 \qquad for \quad t_1 \geqslant 0 \qquad 5.21$$

This is the integral equation for the optimal systems wheighing function. Equations of the type:

$$\int K(t)\,\Delta(\tau - t)\,dt - \Gamma(\tau) = 0 \qquad for\ \tau \geqslant 0 \qquad 5.22$$

are known as Wiener Hopf equations of the second kind and occur in this type of work with $\Delta(t)$ and $\Gamma(t)$ taking on different roles as prescribed by the particular situation. As in the special case of equation 5.22 $\Delta(t)$ is always an even function. This fact was used in the above derivation of equation 5.21 in the step leading from equation 5.17 to 5.18

Solution of the Integral Equation

The solution of the integral equation will be demonstrated in its slightly more general notation of equation 5.22, so that the result can be used in different situations. Without the condition $\tau \geqslant 0$ on equation 5.22 an explicit solution could be obtained by taking the Fourier transform of equation 5.22. The method of solution devised by Wiener, manipulates eqation 5.22 into a form where this restriction is eliminated and the solution can proceed by Fourier transform methods.

Defining:

$$\left.\begin{aligned} \Delta^{+}(\tau) &= 0 \quad \text{for} \quad \tau < 0 \\ \Delta^{-}(\tau) &= 0 \quad \text{for} \quad \tau > 0 \end{aligned}\right\} \qquad 5.23a$$

in such a way that their time domain convolution yields $\Delta(\tau)$, i.e.:

$$\Delta(\tau) = \int \Delta^{-}(t_2)\Delta^{+}(\tau - t_2)dt_2 \qquad 5.23b$$

Further define $\gamma(\tau)$ by:

$$\Gamma(\tau) = \int \Delta^{-}(t_2)\gamma(\tau - t_2)dt_2 \qquad 5.24$$

Equations 5.23 and 5.24 are now used to substitute for $\Delta(\tau-t)$ and $\Gamma(\tau)$ in equation 5.22 to give:

$$\int \Delta^{-}(t_2)\,dt_2 \left[\int K(t_1)\Delta^{+}(\tau-t_1-t_2)\,dt_1 - \gamma(\tau-t_2) \right] = 0 \quad \text{for } \tau \geqslant 0 \qquad 5.25$$

From the definition of $\Delta^{-}(t_2)$ the bracketed quantity in equation 5.25 is identically zero for:

$$\tau - t_2 \geqslant 0 \qquad 5.26$$

Replacing $\tau-t_2$ with t in equation 5.25 leads to :

$$\int K(t_1)\Delta^{+}(t-t_1)\,dt_1 - \gamma(t) = 0 \quad \text{for } t \geqslant 0 \qquad 5.27$$

The first term on the left hand side of equation 5.27 is now identically zero for $t < 0$ by virtue of the definition of $\Delta^{+}(t)$ and the condition that K(t) be physically realizable. Further defining:

$$\gamma(t) = \gamma_{+}(t) + \gamma_{-}(t) \qquad 5.28$$

such that

$$\gamma_{+}(t) = 0 \quad \text{for } t < 0 \qquad 5.29$$

$$\gamma_{-}(t) = 0 \quad \text{for } t > 0 \qquad 5.30$$

it follows that the equation:

$$\int K(t_1)\Delta^{+}(t-t_1)\,dt_1 - \gamma_{+}(t) = 0 \qquad 5.31$$

holds for all t. Applying Fouriers transformation to equation 5.31 yields the explicit solution for K(s):

$$K(s) = \frac{\gamma_{+}(s)}{\Delta^{+}(s)} \qquad 5.32$$

where the same notation is used for the time functions and their Fourier transforms with the argument t replaced by s. Also transforming equations 5.5, 5.6 and 5.8:

$$\Delta(s) = \Delta^{+}(s)\,\Delta^{-}(s) \qquad 5.33$$

$$\Gamma(s) = \Delta^{-}(s)\,\gamma(s) \qquad 5.34$$

$$\gamma(s) = \gamma_{+}(s) + \gamma_{-}(s) \qquad 5.35$$

Note also that in order for K(s) to be physically realizable $\Delta^{+}(s)$ must have no zeros in the right hand s plane. In summery therefore the procedure for solving equation 5.27:

$$\int K(t)\,\Delta(\tau - t)\,dt - \Gamma(\tau) = 0 \quad \text{for } \tau \geqslant 0 \qquad 5.27$$

consists of the following steps:

1. Factor $\Delta(s)$ into a product of two functions $\Delta^{+}(s)$ and $\Delta^{-}(s)$ so that:

   $\Delta^{+}(s)$ has poles and zeros in the left hand s plane

   $\Delta^{-}(s)$ has poles and zeros in the right hand s plane

and:

$$\Delta(s) = \Delta^{+}(s)\,\Delta^{-}(s) \qquad 5.28$$

2. Form:

$$\gamma(s) = \frac{\Gamma(s)}{\Delta^{-}(s)} \qquad 5.29$$

3. Extract from $\gamma(s)$ the portion, $\gamma_{+}(s)$, corresponding to the positive time behaviour of $\gamma(t)$ such that:

$$\gamma(s) = \gamma_{+}(s) + \gamma_{-}(s) \qquad 5.30$$

This can be achieved by a partial fraction expansion of when the latter is a rational function of s. When $\gamma(s)$ containes a transcendental function such as an exponential, recourse must be made to the defining equation of the Fourier transform:

$$\gamma(t) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \gamma(s) e^{st} ds \qquad 5.30$$

$$\gamma_+(s) = \int_0^\infty \gamma(t) e^{-st} dt = \frac{1}{2\pi i}\int_0^\infty e^{-st} dt \int_{-i\infty}^{+i\infty} \gamma(s) e^{st} ds \qquad 5.31$$

4. Form K(s):

$$K(s) = \frac{\gamma_+(s)}{\Delta^+(s)} \qquad 5.32$$

This is then the explicit solution for the Fourier transform of the impulse response, or weighing function, of the desired filter - predictor.

6. WIENER'S METHOD FOR THE UNCONSTRAINED AND CONSTRAINED REGULATOR PROBLEMS

The Unconstrained Regulator Problem

A block diagram of the regulator problems considered here is shown in figure 3.3. In that figure G(s) is the plant to be regulated, H(s) the feedback controller to be designed, m the controller output, and z the systems output. The disturbance y is related to u as follows:

$$U(s) = G y(s) \qquad \text{for input disturbance} \qquad 6.1$$

$$U(s) = y(s) \qquad \text{for output disturbance} \qquad 6.2$$

G(s) is the Fourier transform of the plants impulse response and therfore only stable plants can be treated by Wieners methods. Unstable plants have to be stabilized first, by using, for example, a feedback loop around them, before the following procedure can be applied. In this case the equivalent open loop transfer function for the plant and stabilizing feedback loop take the place of G(s).

From figure 3.3 the output and controller output become:

$$Z(s) = \frac{1}{1 + GH(s)} U(s) \qquad 6.3$$

$$m(s) = \frac{-H(s)}{1 + GH(s)} U(s) \qquad 6.4$$

An equivalent open loop cascaded compensator K(s) is now introduced so that Wieners method can be applied to the feed-back system. K(s) is defined through:

$$\frac{1}{1+GH(s)} = 1 - KG(s) \qquad 6.5$$

which leads to the following relationships between K(s) and H(s):

$$K(s) = \frac{H(s)}{1+GH(s)} \qquad 6.6$$

$$H(s) = \frac{K(s)}{1-KG(s)} \qquad 6.7$$

The output of this system in terms of K(s) becomes:

$$Z(s) = [1 - KG(s)]U(s) = U(s) - K(s)[GU(s)] \qquad 6.8$$

and the controller output then follows from equations 6.4 and 6.6:

$$m(s) = -K\,U(s) \qquad 6.9$$

From equations 6.8 and 6.9 an equivalent open loop block diagram can be drawn for the feedback system of figure 3.3 and is shown in figure 6.1. This block diagram can be thought of as representing a positon servo mechanism with fixed elements G(s) and a cascaded compensator K(s) where the objective is to follow the input as closely as possible. It is clear that designing K(s), so as to minimize the tracking error of this servo system is equivalent to minimizing the deviations from zero of the output of the regulator problem in figure 3.3. Note that the controller output, m(s) is conveniently available at the output of K(s). Alternatively -K(s) can be interpreted

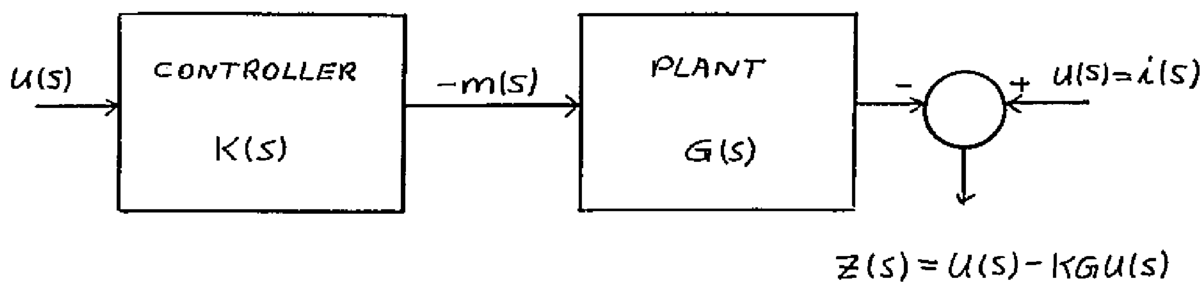


Figure 6.1 EQUIVALENT OPEN LOOP BLOCK DIAGRAMM FOR FEEDBACK REGULATOR PROBLEMS

as a feed- foreward controller acting on the disturbance u(s). Again -Ku(s) will represent the control action.

Equation 6.8 can now be interpreted in terms of the Wiener problem of chapter 5. Identifying u(s) and Gu(s) with i(s) and v(s) respectively the problem of output variance minimization of the feedback system by choice of H(s) is identical with that of designing K(s) via Wieners method of chapter 5. The power spectra necessary for the solution are $S_{vv}(s)$ and $S_{vi}(s)$. The first spectrum can be obtained, according to chapter 2 from:

$$S_{vv}(s) = |G(s)|^2 S_{uu}(s) \tag{6.10}$$

which is clearly even for real systems. This equation expresses the spectrum transmission property of linear systems and was derived in chapter 2. The spectrum $S_{vi}(s)$ is obtained as follows:

$$v(t) = \int G(t_1) u(t-t_1) dt_1 \tag{6.11}$$

Forming the product $v(t) \cdot i(t+\tau)$, and applying the definition of $\varphi_{vi}(\tau)$, i.e.:

$$\varphi_{vi}(\tau) = \langle v(t)\, i(t+\tau) \rangle \tag{2.2}$$

to the result $\varphi_{vi}(\tau)$ becomes:

$$\varphi_{vi}(\tau) = \langle [\int G(t_1) u(t-t_1) dt_1]\, u(t+\tau) \rangle$$

$$= \int G(t_1) \langle u(t-t_1)\, u(t+\tau) \rangle dt_1$$

$$= \int G(t_1) \varphi_{vv}(\tau+t_1) dt_1 \tag{6.12}$$

Taking Fourier transform of this convolution in the time domain , the desired spectrum becomes:

$$S_{vi}(s) = G(-s)\, S_{uu}(s) \qquad 6.13$$

Thus both $S_{vv}(s)$ and $S_{vi}(s)$ are known from the spectrum of the disturbance, $S_{uu}(s)$, and the plants transfer function G(s). Identifying therefore

$$\Delta(s) \quad \text{with} \quad |G(s)|^2 S_{uu}(s) \qquad 6.14$$

and

$$\Gamma(s) \quad \text{with} \quad G(-s)\, S_{uu}(s) \qquad 6.15$$

one can use the explicit solution formula developed in the last chapter for K(s).

The Constrained Regulator Problem

A constraint on the controller effort can be conveniently introduced into the Wiener procedure by imposing a constraint on σ_m^2 the controller output variance. The constraint will be applied by using the familiar Lagrangian multiplier technique. In the constrained problem, therefore, the functional to be minimized, F, is:

$$F = \langle z^2 \rangle + l \langle m^2 \rangle \qquad 6.15$$

where l is the Lagrangian multiplier.

In the equivalent cascaded compensator formulation of our regulator problem, as shown in figure 6.1, the negative manipulated variable, -m(s), is available at the output of K(s).

The procedure of chapter 5 will now be applied to derive the integral equation for the optimum K(s) in this case. The main steps are as follows: From equation 6.9 the manipulated variable is:

$$m(t) = -\int K(t_1)\, u(t-t_1)\, dt_1 \qquad 6.16$$

and therefore its variance becomes:

$$\langle m^2 \rangle = \iint K(t_1) K(t_2)\, \varphi_{uu}(t_1-t_2)\, dt_1 dt_2 \qquad 6.17$$

Similarly from equation 6.8 the output is:

$$z(t) = u(t) - \iint K(t_1) G(t_2)\, u(t-t_1-t_2) dt_2 \qquad 6.18$$

and its variance becomes:

$$\langle z^2 \rangle = \varphi_{uu}(0) - 2\iint K(t_1) G(t_2)\, \varphi_{uu}(t_1+t_2) dt_1 dt_2$$

$$+ \iiiint K(t_1) K(t_3) G(t_2) G(t_4)\, \varphi_{uu}(t_1+t_2-t_3-t_4) dt_1 dt_2 dt_3 dt_4 \qquad 6.19$$

Again we replace K(t) by:

$$K(t) = K_0 + \varepsilon K_\varepsilon(t) \qquad 6.20$$

where $K_0(t)$ is the optimal transfer function, $K_\varepsilon(t)$ an arbitrary but physically realizable function and ε a free parameter. Next the derivative of F with respect to ε is taken and set to zero at $\varepsilon = 0$. Recalling that $\varphi_{uu}(\tau)$ is an even function, the result is:

$$\int K_\varepsilon(t_1)\Big\{ K_0(t_3)\Big[\iint G(t_2) G(t_4)\, \varphi_{uu}(t_1+t_2-t_3-t_4) dt_2 dt_4 + \ell\, \varphi_{uu}(t_1-t_3)\Big] dt_3$$

$$- \int G(t_2)\, \varphi_{uu}(t_1+t_2) dt_2 \Big\} dt_1 = 0 \qquad 6.21$$

Since $K_\varepsilon(t)$ is physically realizable:

$$K_\varepsilon(t) \neq 0 \qquad \text{for } t_1 \geq 0 \qquad 6.22$$

so that for $t_1 \geq 0$ setting the curly bracket in equation 6.21 equal to zero yields the integral equation for $K_0(t_3)$:

$$K_0(t_3)\left[\iint G(t_2)G(t_4)\,\varphi_{uu}(t_1+t_2-t_3-t_4)dt_2dt_4 + \ell\varphi_{uu}(t_1-t_3)\right]dt_3$$

$$-\int G(t_2)\,\varphi_{uu}(t_1+t_2)dt_2 = 0 \qquad \text{for} \quad t_1 \geq 0 \qquad 6.23$$

Equivalent terms can now be identified between the general integral equation of chapter 5, equation 5.22, and equation 6.23. In terms of Fourier transforms we identify:

$$\Delta(s) \quad \text{with}\ [G(s)G(-s)+\ell]\,S_{uu}(s) \qquad 6.24$$

and

$$\Gamma(s) \quad \text{with} \quad G(-s)\,S_{uu}(s) \qquad 6.25$$

The general solutionfor K(s) from the last chapter can then be used:

$$K(s) = \frac{\gamma_+(s)}{\Delta^+(s)} \qquad 6.26$$

Again, given the disturbance spectrum and the plants transfer function, K(s) can be detetmined. However, the solution will contain the Lagrangian multiplier l. For any value of l the variance of m can be determined from its spectrum Smm(s):

$$S_{mm}(s) = |K(s)|^2 S_{uu}(s) \qquad 6.27$$

l, therefore, can be adjusted so that the constraint condition on the variance of m is satisfied.

Physical Realizability of the Feedback Controller

As is clear from the Wiener solution for the integral equation the cascaded compensator K(s) is guarantied to be physically realizable.

The overall system, $1 - KG(s)$, is therefore, also physically realizable, since $G(s)$ is a realizable plant. The controller to be applied to the original feedback system, is $H(s)$ and care must be taken that it be realizable. Nothing in the derivation assures the physical realizability of $H(s)$. From equation 6.7 the criterion for the realizability of $H(s)$ is that $1 - KG(s)$ have no zeros in the right hand s plane. In the case of designs subject to a constrained controller effort it can be intuitively anticipated that physical realizability will depend on the Lagrangian multiplier l. A l of zero, no constraint, might cause unrealizability. An infinite l which corresponds to no control, is always a perfectly realizable feedback controller with zero output.

7. UNCONSTRAINED WIENER DESIGNS

Cascades with no Delay

It was shown in chapter 3 that the transfer function, G(s), for a cascade of n ideally stirred tanks in which a first order irreversible reaction takes place is given by:

$$G(s) = 1/(1+s)^n \tag{7.1}$$

The output z(s) and the manipulated variable m(s) are related to the input y(s) as follows:

$$z(s) = \frac{1}{1+GH(s)}\, u(s) \tag{7.2}$$

$$m(s) = \frac{-H(s)}{1+GH(s)}\, u(s) \tag{7.3}$$

where

$$u(s) = \begin{cases} Gy(s) & \text{for input disturbance} \\ y(s) & \text{for output disturbance} \end{cases} \tag{7.4}$$

H(s) is the feedback controller whose gain is multiplied by the steady state conversion of the reactor. z(s), m(s) y(s) and the time are scaled as shown in chapter 3. The transfer function K(s) of the equivalent cascaded compensator is defined by:

$$\frac{1}{1+GH(s)} = 1 - KG(s) \tag{7.5}$$

where

$$H(s) = \frac{K(s)}{1-KG(s)} \tag{7.6}$$

and

$$m(s) = -Ku(s) \tag{7.7}$$

In chapter 6 it was shown that K(s) can be designed as if it were a filter predictor, via Wieners method, with input Gu(s) and an ideally desired output u(s). To use the explicit solution formula described in chapter 5 the following identifications are made:

$$\Delta(s) \text{ with } |G(s)|^2 S_{uu}(s) \qquad 7.8$$

$$\Gamma(s) \text{ with } G(-s) S_{uu}(s) \qquad 7.9$$

Taking as a representative disturbance one with a negative exponential autocorrelation function and considering the case of input disturbance one gets:

$$S_{yy}(s) = \frac{2\nu}{\nu^2 - s^2} \qquad 7.10$$

$$S_{uu}(s) = \frac{2\nu}{(1-s^2)^n(\nu^2 - s^2)} \qquad 7.11$$

and therefore:

$$\Delta(s) = \frac{2\nu}{(1-s^2)^{2n}(\nu^2 - s^2)} \qquad 7.12$$

$$\Gamma(s) = \frac{2\nu}{(1-s)^n(1-s^2)^{2n}(\nu^2 - s^2)} \qquad 7.13$$

The explicit solution method of chapter 5 can now be applied. Thus factoring $\Delta(s)$ according to poles and zeros:

$$\Delta^+(s) = \frac{1}{(1+s)^{2n}(\nu + s)} \qquad 7.14$$

$$\Delta^-(s) = \frac{2\nu}{(1-s)^{2n}(\nu - s)} \qquad 7.15$$

$\gamma(s)$ becomes:

$$\gamma(s) = \frac{\Gamma(s)}{\Delta^-(s)} = \frac{1}{(1+s)^n(\nu + s)} \qquad 7.16$$

Since $\gamma(s)$ has no poles in the right hand s plane

$$\gamma(s) = \gamma_+(s) \tag{7.17}$$

and

$$K(s) = \frac{\gamma_+(s)}{\Delta^+(s)} = (1+s)^n \tag{7.18}$$

Following the same steps for the output disturbance case results in the same K(s). In both cases the equivalent overall open loop transfer function for the system is identically zero, since:

$$K(s) = 1/G(s) \tag{7.19}$$

and

$$Z(s) = [1 - KG(s)]\, u(s) \tag{7.20}$$

This implies that all disturbances of this type are perfectly regulated. Because of the same reason, however, the feedback controller H(s) will have an infinite gain. These results are true for any regulator problem with a minimum phase plant, that is one whose transfer function has no zeros in the right hand s plane. This can be seen from the explicit solution formula when written directly in terms of the plants transfer function, i.e.:

$$K(s) = \frac{\left[\dfrac{G(-s)\, S_{uu}(s)}{[G(s)G(-s)\, S_{uu}(s)]^-}\right]_+}{[G(s)G(-s)\, S_{uu}(s)]^+} \tag{7.20a}$$

Since G (s) is real and physically realizable and does not possess right hand plane zeros the expression for K(s) becomes:

$$K(s) = \frac{\left[\dfrac{G(-s)\, S_{uu}(s)}{G(-s)\, S_{uu}^-(s)}\right]_+}{G(s)\, S_{uu}^+(s)} = \frac{1}{G(s)}\, \frac{[S_{uu}^+(s)]_+}{S_{uu}^+(s)} = \frac{1}{G(s)} \tag{7.20b}$$

This solution for K(s) is therefore independant of the disturbance spectrum $S_{uu}(s)$.

The equivalent result for the servomechanism problem is given by Newton Gould and Kaiser(4). In this case the desired ideal output for the servomechanism in figure 6.1 is not neccessarely equal to the input, u(s), so that their result for K(s) is:

$$K(s) = \frac{1}{G(s)} \frac{\left[S_{ui}(s)/S_{uu}^{-}(s)\right]_{+}}{S_{uu}^{+}(s)} \qquad 7.20c$$

This solution depends on the spectrum $S_{uu}(s)$ and $S_{ui}(s)$ but reduces to equation 7.20 b for the case where i(s) = u(s).

The result $K(s) = G^{-1}(s)$ can also be arrived at by inspecting the block diagram in figure 6.1, since it clearly yields an output, z, which vanishes identically. A zero output is, of course, the optimal solution and it fulfills the condition of physical realizability of K(s) for a minimum phase plant. If K(s) is interpreted as a feedforward controller this result is also known as the nominal or ideal feedforward controller (11).

In addition to being impractical the infinte gain for the feedback controller raises the question of the overall realizability of the system. It is known that, whereas, an infinite proportional feedback controller is realizable for a cascade of up to two tanks it is unrealizable for cascades for three and more tanks. This can be seen from the transfer function of our control system in either configuration, i.e.:

$$Z(s) = \frac{1}{1+GH(s)} U(s) \qquad 7.20d$$

For a cascade of n tanks and proportional controller K_g equation 7.20 d becomes:

$$Z(s) = \frac{(1+s)^n}{(1+s)^n + K_g} U(s) \qquad 7.20e$$

The zeros, s_o, of the characteristic function are, therefore determined from:

$$(1+s_o)^n = -K_g \qquad 7.20f$$

which leads to the following expression for the zeros:

$$s_o = K_g^{1/n} e^{i(\pi+2m\pi)/n} - 1 \qquad 7.20g$$

where $m = 0,1,.,n-1$. For large K_g the first term of equation 7.20 g dominates and yields n roots. Their phase is π for a single tank, $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ for two tanks. For three tanks, however, there are two roots in the right hand plane with phases $\frac{\pi}{3}$ and $\frac{5\pi}{3}$, and the overall transfer function of equation 7.20 d becomes unrealizable.

It was shown, however, that Wieners procedure leads to realizable open loop cascaded compensators and therefore to realizable overall transfer functions. The resolution of this apparent contradiction lies in the structure of the feedback controller for infinite gains. This can be discovered by going to the limit of an infinite gain controller more carefully and will be deferred to a later section.

The infinite feedback controller gain was encountered by Luecke and McGuire (11) and in chapter 9 of Newton Gould and Kaiser (4). The first authors shifted emphasis to the feedforward portion of the composite controller treated by them on encountering this situation. The later authors unveiled the structure of the feedback controller through the use of an artifice. In this work the structure of the feedback controller will follow naturally from a Wiener design for a plant that in the limit becomes the minimum phase cascade.

The fact that perfect control is possible for this system intuitively suggests that the plant model used is an over-idealized one for this type of analysis. A cascade of ideally stirred tanks will have an instantaneous response to an input, which for large n might be very small, but is never the less finite. Real plants, of course, do not have an instantaneous response, even when they are fairly well mixed tanks. Furthermore, even if one had an ideally stirred tank, a real sensing devise would show no response below its sensitivity limit. As a result any real plant has a finite delay before the controller can act correctively upon the output error. The infinity gain controller obtained in the analysis will, of course, pick up the small instantaneous response of the model and turn it into a significant corrective action. One is therefore tempted to circumvent the impracticality of the Wiener solution by applying the method to a more realistic model of the plant.

Cascades with Delay and Output Disturbance

A natural improvement to the stirred tank cascade model would be the application of a pure time delay in series with it. That is :

$$G(s) = e^{-\Delta s}/(1+s)^n \qquad 7.21$$

It was pointed out in chapter 3 that such a delay can represent a transport delay in the plant as well as a measurement delay with Δ being their sum. For the case of output disturbance $u(s) = y(s)$ and the effective input spectrum $S_{uu}(s)$ equals that of the disturbance $S_{yy}(s)$. Therefore:

$$\Delta(s) = |G(s)|^2 S_{yy}(s) = \frac{2\nu}{(1-s^2)^n(\nu^2-s^2)} \qquad 7.22$$

and

$$\Gamma(s) = G(-s)S_{yy}(s) = \frac{2\nu\, e^{\Delta s}}{(1-s)^n(\nu^2-s^2)} \qquad 7.23$$

Factoring $\Delta(s)$ according to zeros and poles:

$$\Delta^+(s) = \frac{1}{(1+s)^n(\nu+s)} \qquad 7.24$$

$$\Delta^-(s) = \frac{2\nu}{(1-s)^n(\nu-s)} \qquad 7.25$$

$\gamma(s)$ becomes:

$$\gamma(s) = \frac{e^{\Delta s}}{\nu+s} \qquad 7.26$$

$\gamma_+(s)$ will be that additive portion of $\gamma(s)$ corresponding to the positive time part of $\gamma(t)$. Since $\gamma(s)$ is transcendental it will be inverted and the positive time portion of $\gamma(t)$ transformed again:

$$\gamma(t) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \gamma(s)e^{st}ds = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{e^{s(\Delta+t)}}{(s+\nu)}ds \qquad 7.27$$

To evaluate this integral the integration path is complemented with a large semi circle around the finite left hand s plane, centered at the origin. For $\Delta+t>0$ integration in the counter clock wise direction along this contour will equal the integral along the imaginary axis, as there will be no contribution to the integral along the semi circle by Jordan's lemma. The residue theorem can, therefore, be applied to the integral with the result:

$$\gamma(t) = e^{-\nu(\Delta+t)} \qquad \text{for } \Delta+t>0 \tag{7.28}$$

This result includes the positive time portion, $\gamma_+(t)$, since $\Delta > 0$. The other part of the inversion is obtained for from integration along the semi circle around the right hand s plane. Since there are no poles in the right hand s plane $\gamma(t)$ vanishes for $\Delta+t<0$. The complete inversion of $\gamma(s)$ is therefore:

$$\gamma(t) = \begin{cases} e^{-\nu(\Delta+t)} & \text{for } t > -\Delta \\ 0 & \text{for } t < -\Delta \end{cases} \tag{7.29}$$

$\gamma_+(s)$ is then obtained by inverting $\gamma(t)$ along the positive time axis:

$$\gamma_+(s) = \int_0^\infty \gamma(t)\, e^{-st} dt = \frac{e^{-\Delta\nu}}{s+\nu} \tag{7.30}$$

Hence K(s) becomes:

$$K(s) = \gamma_+(s)/\Delta^+(s) = (1+s)^n e^{-\Delta\nu} \tag{7.31}$$

The overall transfer function of the system becomes:

$$1 - KG(s) = 1 - e^{-\Delta(\nu+s)} \tag{7.32}$$

and the output power spectrum $S_{ZZ}(s)$ is therefore:

$$S_{zz}(s) = |1 - KG(s)|^2 S_{\eta\eta}(s) = |1 - e^{-\Delta(\nu+s)}|^2 \frac{2\nu}{\nu^2 - s^2} \qquad 7.33$$

The output variance is obtained from integrating the output power spectrum along the imaginary axis, i.e.:

$$\sigma_z^2 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} S_{zz}(s)\,ds = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\left[1 + e^{-2\Delta\nu} - 2\,e^{\Delta(\nu+s)}\right]\frac{2\nu}{\nu^2 - s^2}\,ds = 1 - e^{-2\Delta\nu} \qquad 7.34$$

(The integration is performed by using the residue theorem after completing the integration path with a semicircle around the finite right hand s plane. Because of the exponential term in the integrand the contribution to the integral vanishes along the right side semi circle only for $\Delta > 0$.)

Note that the output performance of the system improves for small delays or low ν disturbances, and reaches perfect control for $\Delta = 0$, which concurs with our previous result. For large delays or high frequency content disturbances the output rms approaches one which is the same as that with no controller present.

From the definition of K(s) the feedback controller H(s) is:

$$H(s) = \frac{K(s)}{1 - KG(s)} = \frac{(1+s)^r\, e^{-\Delta\nu}}{1 - e^{-\Delta(\nu+s)}} \qquad 7.35$$

which does no longer have an infinite gain. As was mentioned in chapter 6 Wieners method guarantees a physically realizable cascaded compensator K(s). Therefore the overall system 1 - KG(s), is physically realizable since G(s) represents a physically realizable plant. However, there is no assurance that the feedback controller H(s) will be physically realizable and this must be assertained for each solution.

In order for H(t) to be physically realizable it must be identically zero for negative t. Applying the residue theorem to the Fourier transform inversion integral the condition for the transform can be established for which its time function will vanish for negative time. For a transform which does not grow faster than a polynomial for large s, the absence of poles in the right hand s plane insures that the inverted time function will vanish for negative time.

Applied to H(s) this means that its denominator, 1 - KG(s), should not vanish in the right hand s plane, i.e.:

$$e^{-\Delta(s+\nu)} \neq 1 \qquad \text{in the right hand s plane} \qquad 7.36$$

substituting for $s = \alpha + i\omega$, the right hand side of equation 7.36 is written in polar form as:

$$e^{-\Delta(\nu+\alpha)} e^{i\Delta\omega} = r e^{i\theta} \qquad 7.37$$

For s to be in the right hand plane α must be positive. It follows that for $\Delta > 0, r < 1$, so that $r e^{i\theta}$ never becomes 1 and H(s) is always physically realizable.

Any real controller has an upper limit on the magnitude of its action corresponding, for example, to the maximum opening of a control valve. An optimal controller will, of course, not yield the designed optimal performance if it calls for larger control action than is available. The manipulated variable, or controller action, m, is a stationary random process whose probability distribution is not available. As with the output z the variance of m will be computed instead and serve as a measure for the magnitude of its fluctuations. The applicability of such a procedure is discussed in chapters 2 and 12.

The variance of the controller action m is calculated by integrating its power spectrum over the whole imaginary axis:

$$\sigma_m^2 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} S_{mm}(s)\,ds = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} |K(s)|^2 S_{uu}(s)\,ds$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{(1-s^2)^n e^{-2\Delta\nu}}{(\nu^2 - s^2)}\, 2\nu\, ds \qquad 7.38$$

Even for one tank, n = 1, the integral does not converge and the variance is therefore infinite. For a disturbance with a finite frequency cutoff the variance will be finite but sensitive to the exact cutoff frequency. A high cutoff frequency is usually taken to give possible stray high frequency noises adequate representation. These noises will be amplified by the present system and lead to large variances and therefore to controller saturation. This would have an adverse effect on the control quality and is unacceptable in practice. This means that large control signals might be necessary to implement the optimum controller, and again the result is not of practical use.

These results suggest that in addition to using a realistic model of the plant a constraint on the controllers action must be built into the development of a usable Wiener type controller.

Cascades with Delay and Input Disturbance

The derivation of the Wiener controller for the cascade with delay and the input disturbance configuration can be made along similar lines. In this case, however, the solution can not be written down in general for an n tank cascade. For illustration purposes the development of the result for a single tank with delay will be presented. Similar results are obtained for a larger number of tanks but the algebra gets more cumbersome.

The plants transfer function G(s) is as in the last section with n = 1:

$$G(s) = e^{-\Delta s}/(1+s) \qquad 7.39$$

From chapter 3:

$$Z(s) = \frac{1}{1+GH(s)}\, Gy(s) \qquad 7.40$$

so that with $u(s) = Gy(s)$

$$S_{uu}(s) = \frac{2\nu}{(1-s^2)(\nu^2-s^2)} \qquad 7.41$$

and the Wiener procedure of the last section is repeated with u(s) replacing y(s). In this case therefore:

$$\Delta(s) \text{ becomes: } |G(s)|^2 S_{uu}(s) = \frac{2\nu}{(1-s^2)^2(\nu^2-s^2)} \qquad 7.42$$

$$\Gamma(s) \text{ becomes: } G(-s)S_{uu}(s) = \frac{2\nu\, e^{\Delta s}}{(1-s)(1-s^2)(\nu^2-s^2)} \qquad 7.43$$

Following Wieners explicit solution formula the result for K(s) is:

$$K(s) = (1+s)(g_0+g_1 s) \qquad 7.44$$

where g_o and g_1 are defined as:

$$g_0 = \frac{1}{1-\nu}(e^{-\Delta\nu} - \nu e^{-\Delta}); \quad g_1 = \frac{1}{1-\nu}(e^{-\Delta\nu} - e^{-\Delta}) \quad \text{for } \nu \neq 1 \qquad 7.45$$

$$g_0 = (1+\Delta)e^{-\Delta}; \quad g_1 = \Delta e^{-\Delta} \quad \text{for } \nu = 1 \qquad 7.46$$

The feedback controller in this case becomes:

$$H(s) = \frac{K(s)}{1-KG(s)} = \frac{(1+s)(g_0+g_1 s)}{1-e^{-\Delta s}(g_0+g_1 s)} \qquad 7.47$$

For physical realizability the following equation must not be satisfied in the right hand s plane.

$$e^{\Delta s} = g_0 + g_1 s \qquad 7.48$$

Again introducing $s=\alpha+i\omega$, equation 7.48 becomes:

$$e^{\alpha\Delta} e^{i\Delta\omega} = g_0 + g_1\alpha + i g_1\omega \qquad 7.49$$

For a constant α the left hand side of equation 7.49 describes a circle, as a function of ω , of radius $e^{\alpha\Delta}$ centered at the origin. The right hand side represents a vertical line, as a function of ω , which intercepts the real axis at $g_0+g_1\alpha$. For positive α the radius of the left hand side circle increases exponentially. Noting that g_o and g_1 are always positive the vertical line's intercept, with the real axis, increases linearely with increasing positive α . It follows that there will always be some positive α at which the circle and line intersect. This means that equation 7.48 always has a root with positive real part of s, i. e., in the right hand s plane, and consequently H(s) is always unrealizable.

The possibility that a feedback controller, which is reconstructed from a Wiener designed cascaded compensator, will be unrealizable, is not mentioned in Newton, Gould and Kaiser (4). Possibly this is due to the rarety with which such situations occur in the classical control field, that is servomechanism design, to which the book is mainly addressed. Luecke and McGuire calculate the feedback controller for the same situation as in this section, equation 23, page 176 of (11). Their result is, however, in error and is correct only for $\tau_c = 0$, in which case it is equivalent to equation 7.47. The wrong controller has the term $e^{\tau_c s}$ in the numerator and they state that the controller becomes realizable for $\tau_c = 0$, i.e., the controller of equation 7.47 . Lim and Bankoff in their treatment of rational non minimum phase plants recognize that unrealizable feedback controllers might arise occasionally. They suggest to approximate them by realizable controllers but do not elaborate on this matter. It should be pointed out that an unrealizable feedback controller is obtained only in the input disturbance configuration and not for the output disturbance case treated in the last section.

In addition the spectrum $S_{mm}(s)$, of m is again not integrable. This spectrum is:

$$S_{mm}(s) = |K(s)|^2 S_{uu}(s) = \frac{g_0^2 - g_1^2 s^2}{\nu^2 - s^2} 2\nu \qquad 7.50$$

The Feedback Controller for Delayless Cascades

In a previous section it was seen that the Wiener feedback controller for minimum phase plants had an infinite gain. This apparantly contradicts the realizability of the overall system for a cascade with three tanks or more. It is recalled that the Wiener procedure guarantees this realizability and that therefore the resolution of this difficulty must lie in the structure of the feedback controller which was washed out in the above derivation. A natural way of obtaining this structure is by designing a Wiener controller for the minimum phase plant with asuperimposed delay and computing its limiting form as the delay tends to zero. This will be done for the stirred tank cascades in both input and output disturbance configurations.

For the cascade with output disturbance the feedback controller was:

$$H(s) = (1+s)^n / (e^{-\Delta\nu} - e^{-\Delta s}) \qquad 7.51$$

Expanding the exponentials in power series in Δ and retaining first order terms only, i.e.:

$$\left.\begin{aligned} e^{\Delta\nu} &\cong 1+\Delta\nu \\ e^{-\Delta s} &\cong 1-\Delta s \end{aligned}\right\} \quad \text{for } \Delta \downarrow 0 \tag{7.52}$$

results in

$$H(s) = \frac{(1+s)^n}{\Delta(\nu+s)} \quad \text{for } \Delta \downarrow 0 \tag{7.53}$$

For a single tank this is equivalent to

$$H(s) = \frac{1}{\Delta}\left(1 + \frac{1-\nu}{s+\nu}\right) \tag{7.54}$$

which is a proportional controller together with a first order low pass filter where the controller gain tends to infinity. The filter parameters are tuned to that of the disturbance spectrum and the sign of its gain changes from negative to positive as ν becomes larger than one. The feedback controller for two, $H_2(s)$, and three tanks, $H_3(s)$, cascades, become:

$$H_2(s) = \frac{1}{\Delta}\left[2-\nu+s+\frac{(1-\nu)^2}{\nu+s}\right] \tag{7.54a}$$

$$H_3(s) = \frac{1}{\Delta}\left[3(1-\nu)+\nu^2+(3-\nu)s+s^2+\frac{(1-\nu)^3}{\nu+s}\right] \tag{7.54b}$$

It is therefore seen that the Wiener procedure uses higher order derivatives to make the overall system realizable for very large gains. From a practical point of view these derivatives exclude the use of very large gains as an approximation to the infinite gain feedback controller.

To obtain the structure of the Wiener feedback controller for stirred tank cascades with input disturbance the Wiener design for cascades with delay were derived. The derivation follows the one shown in the last section for the one tank with delay plant. The results for the open loop compensator $K_n(s)$ can be expressed as:

$$K_n(s) = (1+s)^n \sum_{i=0}^{n} g_i s^i \qquad 7.54c$$

The corresponding feedback controllers $H_n(s)$ are given by:

$$H_n(s) = \frac{(1+s)^n \sum_{i=0}^{n} g_i s^i}{1 - e^{-\Delta s} \sum_{i=0}^{n} g_i s^i} \qquad 7.54d$$

The values of the coefficients in these equations for two and three tank cascades are given in the appendix.

Again going to the limit as the delay tends to zero the feedback controller becomes:

$$H_n(s) = (1+s)^n/(\Delta s) \qquad 7.54e$$

The controllers for one, two and three tanks are:

$$H_1(s) = \frac{1}{\Delta}\left(1 + \frac{1}{s}\right) \qquad 7.54f$$

$$H_2(s) = \frac{1}{\Delta}\left(2 + \frac{1}{s} + s\right) \qquad 7.54g$$

$$H_3(s) = \frac{1}{\Delta}\left(3 + \frac{1}{s} + 3s + s^2\right) \qquad 7.54h$$

The feedback controller for a single tank is a proportional-integral controller with a very large gain. For two tanks derivative action is added and for three an additional second order derivative is used. These controllers are different from those obtained for the output disturbance configuration in two respects. a. They have regular integral action in place of the first order filter action. b. The coefficients of the controllers are independant of the disturbance parameter ν . These differences seem plausible since in the output disturbance case the raw disturbance enters the controller. A filtering action tuned to the disturbance is therefore to be expected. In the input disturbance case, on the other hand, the disturbance is attenuated by the plant before it is detected by the controller.

Again realizability of the infinite gain feedback controller is achieved through the use of a first and second derivative action which make large gain approximations unappealing for practical use.

8. CONCLUSIONS AND DEFINITION OF THE MAIN PROBLEM

Conclusions from the Unconstrained Designs

The discussion of the last chapter indicates some of the pitfalls of an indiscriminate application of Wieners design method to feedback regulator problems. The method guarantees the best linear, physically realizable cascaded compensator, with the output variance of the overall system as performance criterion. But while the overall, i. e. input to output, cascaded system, which made the analysis possible, seems flawless the reconstruction of the original feedback controller exposes the difficulties.

Since the method truly delivers the best possible overall system, for a given plant and disturbance, it will make the maximum use of the models properties to this end. In the process some of the models idealizations which are of little consequence for other applications may dominate the nature of the Wiener solution and make it useless in practice. This seems to be the reason for the failure of the procedure when applied to a cascade of ideally stirred tank reactors which resulted in infinite feedback controller gain. Introducing a delay element into the cascade and thus removing some of the overidealization in the plants model did yield finite feedback controller gains.

However, it pointed out to other spectres. In the case of the input disturbance applied to a single stirred tank with delay plant, the feedback controller is physically unrealizable and thereby even theoretically impossible to implement. In both input and output disturbance configurations an infinite controller action might be anticipated. This would mean that if the derived controller was to be implemented it would frequently saturate, i. e. reach its physical limit, and not deliver the action prescribed by the optimal controller. As a result the performance will not be optimal. The large control actions obtained do suggest that the Wiener design should be performed with some constraint on the control action incorporated into the procedure.

This can be done conveniently by limiting the variance of the controller output. The controller variance pertaines to its amplitude in the same way as does the systems output variance and the comments made previously apply here also. Further consideration of this aspect will be given in chapters 12 of this work.

The constraint on the controllers output variance will be introduced via the Lagrangian multiplier technique, so that the functional, F, which the Wiener method will be asked to minimize, becomes:

$$F = \langle z^2 \rangle + l \langle m^2 \rangle$$

where l is the Lagrangian multiplier. The basic procedure for the design was developed in chapter 6. Note that for l = 0 there is no constraint and this limit should yield the results of the previous chapter. For l = ∞ there is complete supression of control action, in other words, no controller is present.

It is intuitively clear that the physical realizability will also be determined by the Lagrangian multiplier. The feedback controller for a stirred tank with delay and input disturbance, obtained in the last chapter with no constraint on m, was unrealizable. On the other hand $l = \infty$, or no controller, is always a realizable solution. Thus one anticipates that the realizability analysis will yield a limiting value of l below which the feedback controller will become unrealizable.

Definition of the Main Problem

As was mentioned in chapter 2 the measurement of autocorrelation functions, or the equivalent power spectra, of disturbances is a laborious and expensive task in reality. On the other hand understanding the origin of the disturbance can often lead to the derivation of the autocorrelation function as was shown. In the worst case some typical correlation times of the disturbance can be observed. This brings up two practical questions which have to be answered. 1. How bad can the performance of a plant become if the random character of the disturbance is overlooked and the feedback controller is designed conventionally for a determinstic input, such as a step. For what random disturbances would such a procedure be most harmful and what can be achieved by an optimal design for random distrubances. 2. Having designed an optimal Wiener type controller for the expected type of random disturbance, what would its performance be for disturbances of different frequency content, which might occur only occasionally, or have escaped recognition in a first analysis.

In other words, how far is the performance of the best design for a random disturbance from the performance of a conventional design for a deterministic disturbance, when excited with a random input. Is the benefit of the best random design limited to a narrow range of input disturbances and inferior to the conventional design in other ranges which might be present. The ultimate goal, of course, would be a robust overall controller design. That is, a controller which will perform well over the whole disturbance range, which might be encountered, and whose performance will be close enough to the optimal one, so that a more detailed analysis will be warranted only in special situations. It follows from this that the performance of the optimal design for a specific random disturbance must be evaluated over a whole range of possible disturbances and compared with the performance of a conventional deterministically designed controller over the same range of disturbances.

To answer these questions and in view of the results obtained in the last chapter constrained feedback controllers will be designed for typical plant models and disturbances. Their performance will be evaluated over a wide range of input disturbances and compared with that of a conventional controller excited by the same disturbances.

The plant model corresponding to a reactor consisting of an ideally stirred tank in series with a plug flow section, in which a first order irreversible reaction takes place, will be used. This model was described in detail in chapter 3 where its merits as a typical model for plants in the process industry were pointed out. The application of the disturbance at the input and output of this plant will cover a wide range of situations occuring in the chemical industry. The output disturbance will serve as a simulation for random processes generated in the plant itself.

The negative exponential type autocorrelation function will be used to represent the disturbance. It was shown in chapter 2 that it occurs frequently, is convenient and can be used as an element in representing more complex autocorrelation functions. It seems therefore appropriate to use it as a typical disturbance.

The conventional optimal controller to be used for comparison is the three mode controller with coefficients set by the Cohen and Coon procedure. This controller and the procedure used to calculate its performance for random disturbances was discussed in chapter 4.

The Wiener designs and the performance evaluation of both the Wiener controller and the conventional one, will be made with the disturbance parameter ν varying between 0.01 and 100. This constitues a four decades range around the plants time constant which, with the time scaling adopted in chapter 3, is unity.

Disturbances with a ν value larger than 100 are of little interest in the process industry because most processes have typical low pass filtering characteristics. Disturbances with smaller than 0.01 can be considered as deterministic and would be accounted for in the conventional manner.

9. CONSTRAINED WIENER DESIGN FOR OUTPUT DISTURBANCES

In the next two chapters Wiener controllers will be designed and their performance evaluated for the standard plant and disturbance models of this work. Once a controller is designed, for say a disturbance of characteristic frequency ν_0 , and its physical realizability assertained, its performance will be evaluated for distrubances of the whole range of interest. This will be done for the output disturbance case in this chapter and for the input disturbance configuration in the next chapter.

Derivation of the Wiener Controller

In chapter 6 the Wiener method was extended to feedback regulator problems, with constraint on the manipulated variable, as shown in figure 6.1. It was shown that the explicit Wiener solution of chapter 5 could be used in this situation if the following identifications are made:

$$\Delta(s) \equiv [G(s)G(-s)+\ell]\, S'_{uu}(s) \qquad 9.1$$

$$\Gamma(s) \equiv G(-s)\, S'_{uu}(s) \qquad 9.2$$

The plant transfer function G(s) and the effective disturbance spectrum $S_{uu}(s)$ in this case are:

$$G(s) = \frac{e^{-\Delta s}}{1+s} \qquad 9.3$$

$$S_{uu}(s) = S_{\gamma\gamma}(s) = \frac{2\nu_o}{\nu_o^2 - s^2} \qquad 9.4$$

After some manipulations designed to put $\Delta(s)$ into factored form, it becomes:

$$\Delta(s) = \frac{2\nu_o \ell (R^2 - s^2)}{(\nu_o^2 - s^2)(1 - s^2)} \; ; \quad \text{with} \quad R^2 = 1 + \frac{1}{\ell} \qquad 9.5$$

and $\Gamma(s)$ is:

$$\Gamma(s) = \frac{2\nu_o e^{\Delta s}}{(1 - s)(\nu_o^2 - s^2)} \qquad 9.6$$

The factoring of $\Delta(s)$ is simple and leads to a $\gamma(s)$:

$$\gamma(s) = \frac{\Gamma(s)}{\Delta^-(s)} = \frac{e^{\Delta s}}{(\nu_o + s)(R - s)} \qquad 9.7$$

Because of the exponential in the numerator of $\gamma(s)$, extraction of that portion of $\gamma(s)$ corresponding to the positive time behaviour, $\gamma_+(s)$, will be done by inverting and retransforming $\gamma(s)$. This was done in detail in chapter 7 for the design of the unconstrained controller. Following the same procedure:

$$\gamma(t) = \frac{e^{-\nu_o(\Delta + t)}}{R + \nu_o} \qquad \text{for} \quad \Delta + t > 0 \qquad 9.8$$

which for $\Delta > 0$ is all that is necessary. Transforming equation 9.8 along the positive time axis $\gamma_+(s)$ becomes:

$$\gamma_+(s) = \frac{e^{-\Delta\nu_o}}{(R + \nu_o)(s + \nu_o)} \qquad 9.9$$

and K(s) is therefore:

$$K(s) = g\,\frac{1+s}{R+s} \qquad 9.10$$

with $$g = \frac{e^{-\Delta \nu_0}}{\ell(R+\nu_0)} \qquad 9.11$$

The Feedback Controller and its Realizability

The cascaded compensator of equation 9.10 leads to the following feedback controller, H(s):

$$H(s) = \frac{K(s)}{1-KG(s)} = \frac{g(1+s)}{R+s-ge^{-\Delta s}} \qquad 9.12$$

Note that for the no delay case, $\Delta = 0$, H(s) becomes, after partial fraction expansion:

$$H(s) = g\left\{1+\frac{1+g-R}{s+R-g}\right\} = \frac{1}{\ell(R+\nu_0)}\left\{1+\frac{(1-\nu_0)(R-1)}{s(R+\nu_0)+1+R\nu_0}\right\} \qquad 9.13$$

This feedback controller has proportional action in parallel with a first order low pass filter. This result corresponds to that obtained for the unconstrained case where the controller had the same structure but an infinite gain.

For $\Delta \neq 0$ the physical realizability of H(s) is determined from the location of its poles. H(s) will be physically realizable if the denominator in equation 9.12 has no zeros in the right hand s plane. Thus with $s = \alpha + i\omega$ the following equation must be satisfied for $\alpha > 0$:

$$g\,e^{-\Delta\alpha}\,e^{-i\Delta\omega} = R + \alpha + i\omega \qquad 9.14$$

Plotting both sides of the equation in the complex plane it is seen that the condition for no roots of equation 9.14 is $g e^{-\Delta\alpha} < R+\alpha$ which is satisfied if $g < R$ for all $\alpha > 0$. This is equivalent to the condition $g/R < 1$ since $R > +1$.

From the definiton of g:

$$\frac{g}{R} = \frac{e^{-\Delta\nu_0}}{R\,l(R+\nu_0)} = \frac{e^{-\Delta\nu_0}}{l+1+\nu_0\sqrt{l(l-1)}} \qquad 9.15$$

The denominator of equation 11.15 represents a straight line as a function of ν_0 with a positive slope. Since ν_0 is always positive the smallest value of the line is its intercept $l+1$, which is always larger than 1. The numerator is always smaller than 1, since $\Delta > 0$, and it follows that:

$$g/R < 1 \qquad 9.16$$

So that the feedback controller is always physically realizable. This was to be expected since the unconstrained controller, 1 = 0, derived in chapter 7, was always realizable.

The Performance of the Wiener Controller

What remains is the calculation of the system output variance and that for the manipulated variable m. This will be done for the same disturbance type used in the design but for a range of parameters ν . The variances are calculated by integrating the respective spectra, i.e.:

$$\sigma_z^2 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} |1 - KG(s)|^2 S_{yy}(s)\,ds \qquad 9.17$$

$$\sigma_m^2 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} |K(s)|^2 S_{yy}(s)\,ds \qquad 9.18$$

Substituting for $S_{yy}(s)$:

$$S_{yy}(s) = \frac{2\nu}{\nu^2 - s^2} \qquad 9.19$$

and also for G(s) and K(s), equations 9.17 and 9.18 become:

$$\sigma_z^2 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{2\nu}{(\nu^2 - s^2)(R^2 - s^2)}\left[R^2 + g^2 - s^2 - 2g(R+s)e^{\Delta s}\right] ds \qquad 9.20$$

$$\sigma_m^2 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{2\nu g^2(1 - s^2)}{(\nu^2 - s^2)(R^2 - s^2)}\, ds \qquad 9.21$$

Both equations are integrated by the method of residues. For the integration of equation 9.21 the residues of the poles in either the right or left hand s plane can be used. The integral along the semicircle surrounding any half plane vanishes since the order of the denominator is larger by two than that of the numerator. The integration of equation 9.20 requires more caution because of the exponential term. This integral can be evaluated from the residues of the poles in the left hand s plane only. By Jordans lema the integral along the infinite semicircle vanishes for $\Delta > 0$ along the left hand semicircle. The integration results are:

$$\sigma_z^2 = 1 + \frac{g^2}{R(R+\nu)} - 2g\,\frac{e^{-\Delta\nu}}{R+\nu} \qquad 9.22$$

$$\sigma_m^2 = g^2\,\frac{\rho\nu+1}{R(R+\nu)} \qquad 9.23$$

It is of interest to note that the calculation of variances for feedback systems with Wiener designed controllers is possible analytically, even for plants with delay. This is unlike the situation encountered for the same plant but with a con - ventional three mode feedback controller where numerical integration must be applied. The reason for this benefit is that for the Wiener analysis the system is designed in its equivalent open loop form, 1 - KG(s). Whereas K(s) is rational, G(s) introduces the exponential in the numerator and such a situation can be handled by residues.

For each plant with normalized delay Δ a controller can be designed by specifying a design characteristic frequency ν_0 and Lagrangian multiplier l. With the controller fixed the system output variance and controllers effort can be evaluated for a range of disturbances specified by their characteristic frequency ν. The Lagrangian multiplier l is, however, only an auxiliary variable and as such of no intrinsic value. What is of interest is the controller effort which l determines for a certain set of conditions.

To gain an overall view of the systems performance over the range of disturbances considered the results will be presented in the following way. For a fixed plant, Δ , and design characteristic frequency, ν_o , the output standard deviation, σ_z , will be plotted as a function of the manipulated variables standard deviation σ_m , with the exciting disturbance characteristic frequency, ν , as parameter. This will be repeated for different values of 1. Lines of constant 1 will show the behaviour of a fixed controller over the range of disturbances. Lines connecting performance points of different controllers in response to the same disturbance will indicate the trend in σ_z resulting from the variation of the controller effort. In this way one plot per design characteristic frequency is obtained for a particular plant.

Examples of such a plot are given in figures 9.1 and 9.2, where as in chapter 4 the output standard deviation is scaled by that of the system with no controller. These examples are for $\Delta = 0$, that is, the plant is an ideally stirred tank reactor, and ν_o is 0.01 and 1.0. In both cases the control systems output, is always better than that of an uncontrolled one ($\sigma_{z,N}$=100%) although for high frequency disturbances, as seen from the ν =100 line, the controller achieves little despite conciderable controller effort. For each input disturbance the output performance improves for an increased controller effort. Ultimately as the controller effort is further increased the output performance approaches a constant value for each disturbance. This

Figure 9.1 - THE PERFORMANCE OF THE WIENER DESIGN FOR OUTPUT DISTURBANCE $\Delta = 0$, $\nu_0 = 0.01$

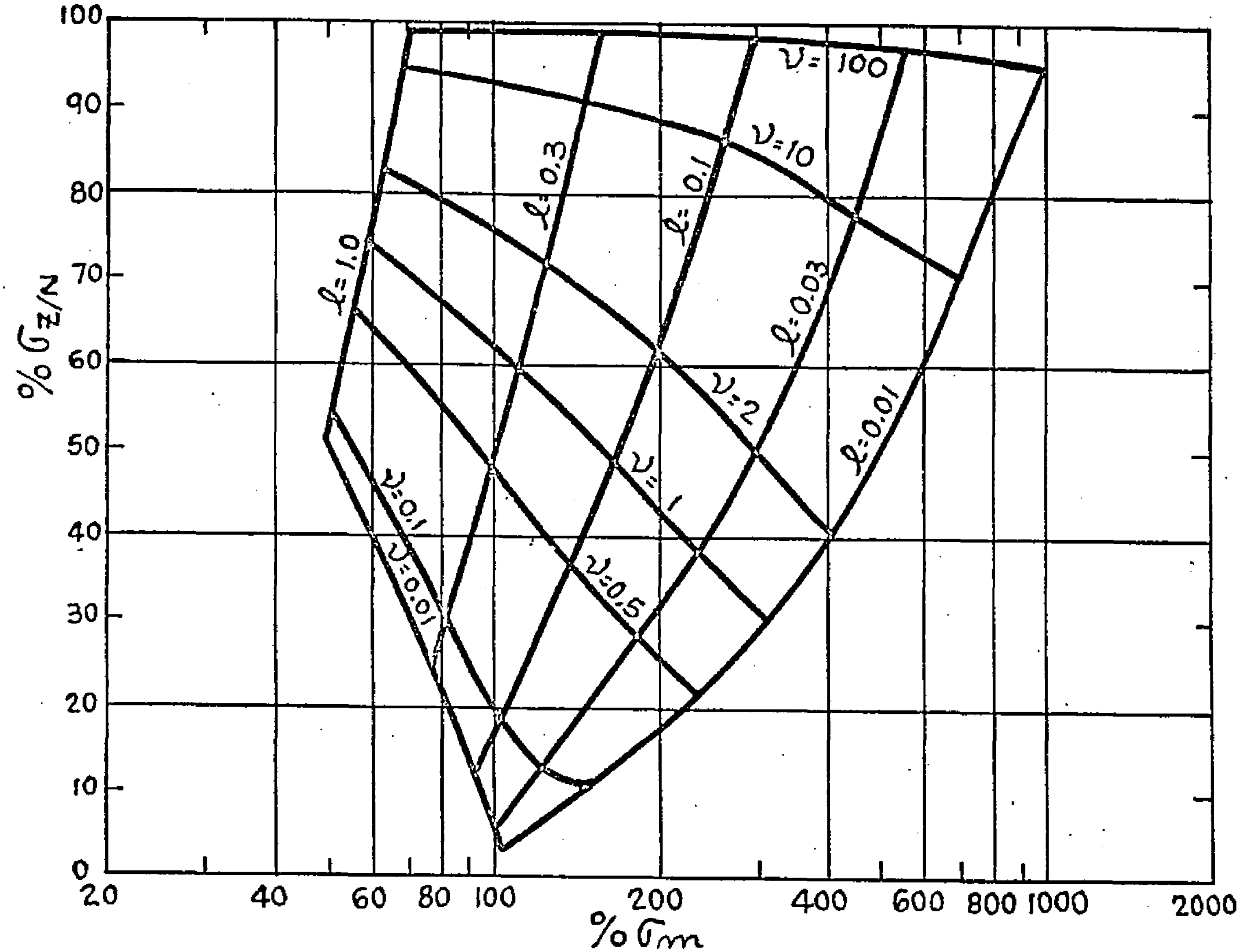
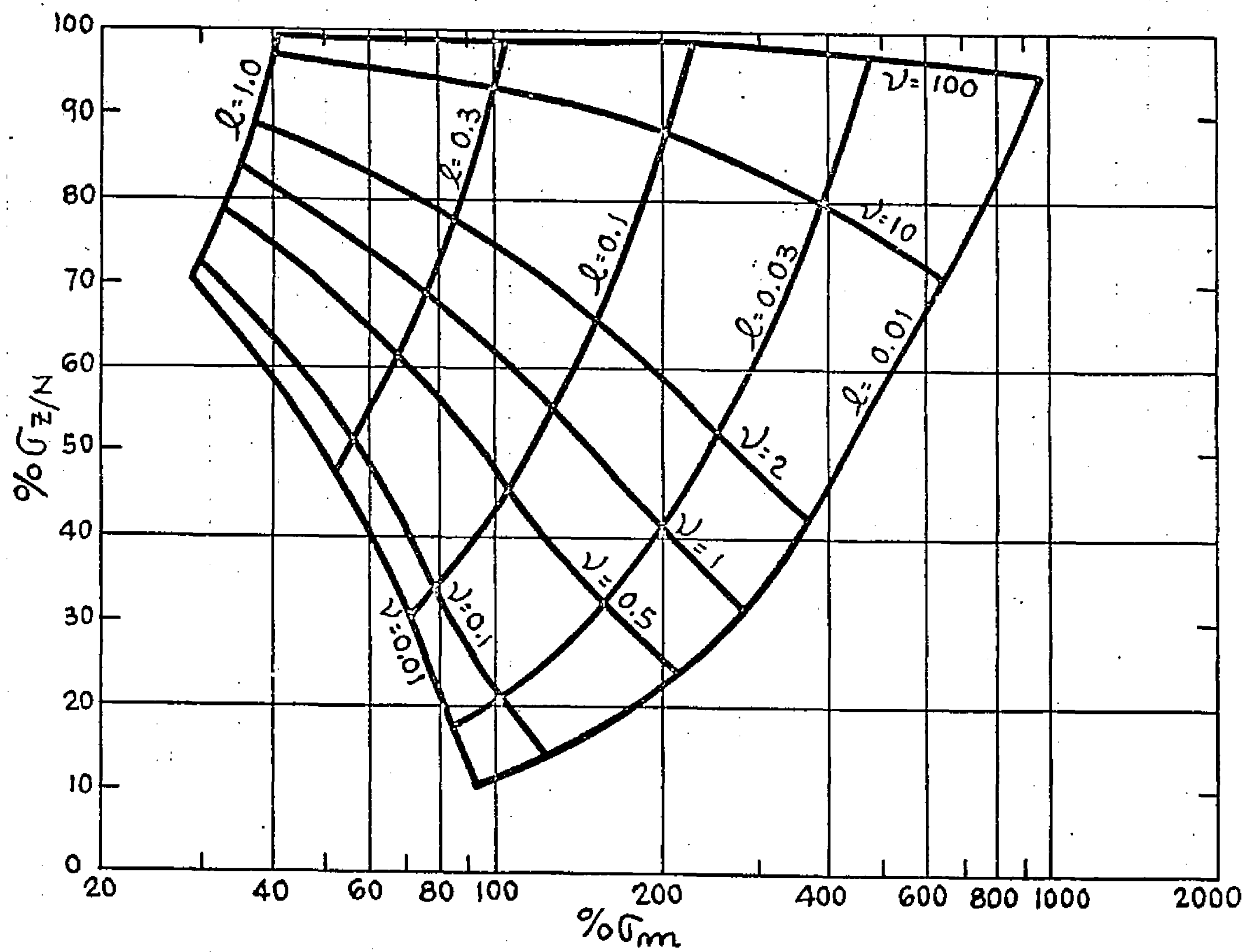


Figure 9.2 THE PERFORMANCE OF THE WIENER DESIGN FOR OUTPUT DISTURBANCE $\Delta = 0$, $\nu_0 = 1.0$



corresponds to the limit of 1 = 0 of equations 9.22 and 9.23 at which the controller output variance becomes infinite but that of the output tends to a finite limit. These surfaces all converge to the no control point, i.e. $\sigma_{z/N} = 100\%$, $\sigma_m = 0$ which corresponds to an infinite 1. Note that the constant ν lines are practically identical regardless of the design characteristic frequency. Clearly the performance of the system with respect to the disturbance for which it was designed must be better than the performance of the controller which was designed for a different disturbance. It is seen, however, from the two figures, that the difference between the $\nu = 1$ lines, for example, for the designs of $\nu_0 = 0.01$ and $\nu_0 = 1.0$, is small. It should be noted that the same performance for two different designs over a range of input disturbances cannot be achieved. This is seen from the constant 1 lines which indicate the performance of a specific controller for different exciting disturbances. The shape of these lines do show some variation for differnet design characteristic frequency.

If the main source of disturbance is in the range of $\nu = 0.01$ to $\nu = 0.1$, for which an output standard deviation of 10 to 20% is tolerable the design of ν_0 =0.01 and 1 = 0.1 is the better choice. Occasional, or unforeseen, higher frequency content disturbances can still be handled with this controller without an excessive controller effort, σ_m, because of the steep slope of the 1 = 0.1 line. If the predominant disturbance has a ν value of 1 and higher, slightly better output performance in this range can be obtained with the $\nu_0 = 1$, $1 = .01$ design at the expense of a larger, up to $\sigma_m = 10$, controller effort.

The results, for a plant with $\Delta = .5$ designed for the characteristic frequencies $\nu_0 = 0.01$ and $\nu_0 = 1$, are shown in figures 9.3 and 9.4 respectively. Designs for higher ν_0 values tend to squeeze the surface into the left hand corner of $\sigma_{z/N} = 100\%$ and $\sigma_m = 0$. (Note the change of $\sigma_{z/N}$ scale in figure 9.4) . At $\nu_0 = 10$ the surface practically degenerates to that point. In figure 9.3 the behaviour of a conventional controller (C.C) is shown for the same range of disturbances, i.e.: $\nu = 0.01$ to 100. This controller has proportional integral action and was set according to Cohen and Coon as described in chapter 4. Its performance is not drastically inferior to that of the compatible Wiener design of $\nu_0 = 0.01$, $1 = .1$ up to a ν of about .1. Both controllers, however, do become worse than no control for higher ν disturbances with the conventional one's output rms peaking at $\nu \approx 1$ after which it drops accompanied by a drop in controller effort. The Wiener controller continues to show better output performance than the conventional one in this region but at the expense of considerably larger controller effort.

The relatively good behaviour of the Cohen and Coon controller for low ν disturbances can be accounted for heuristically.

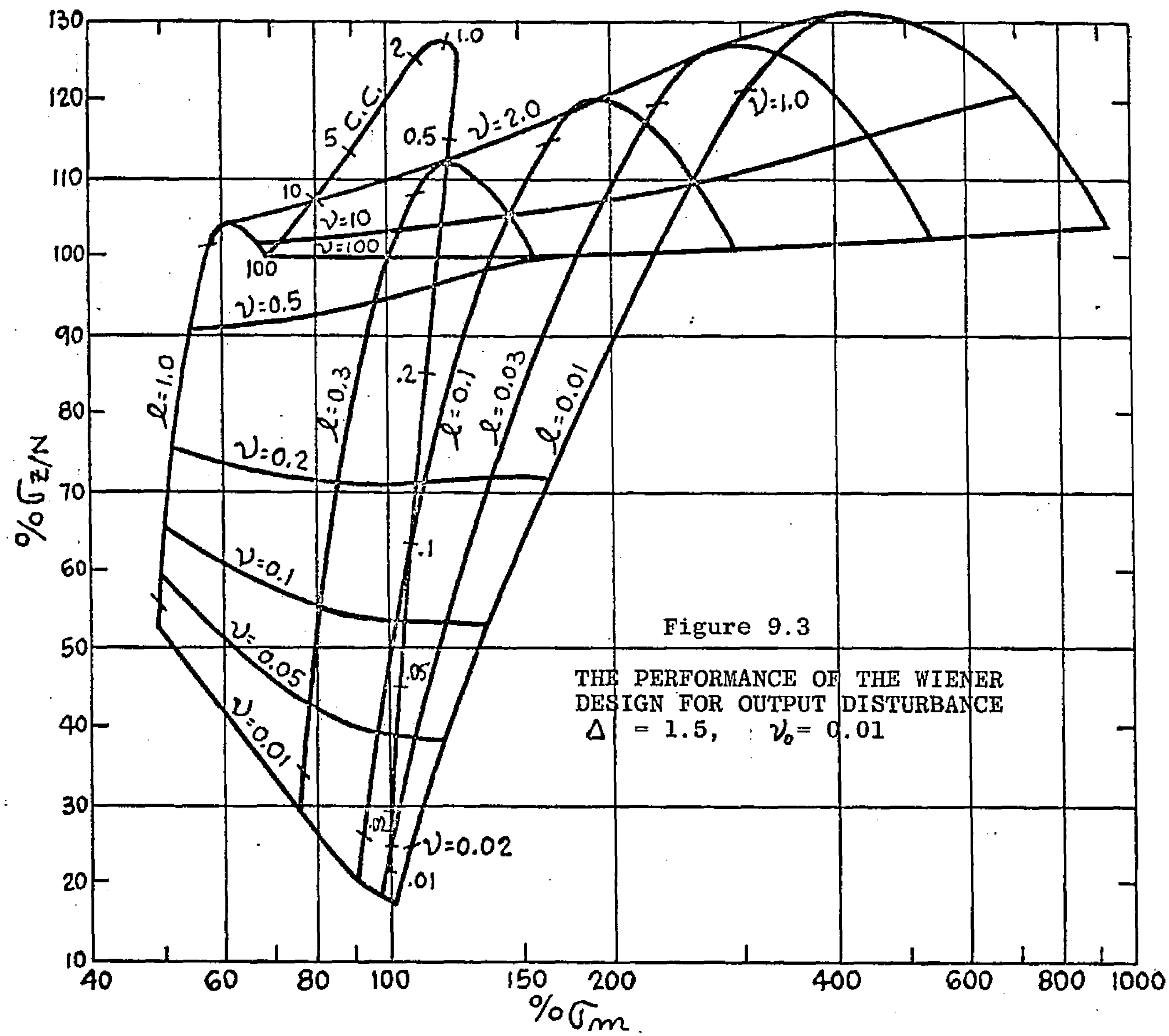
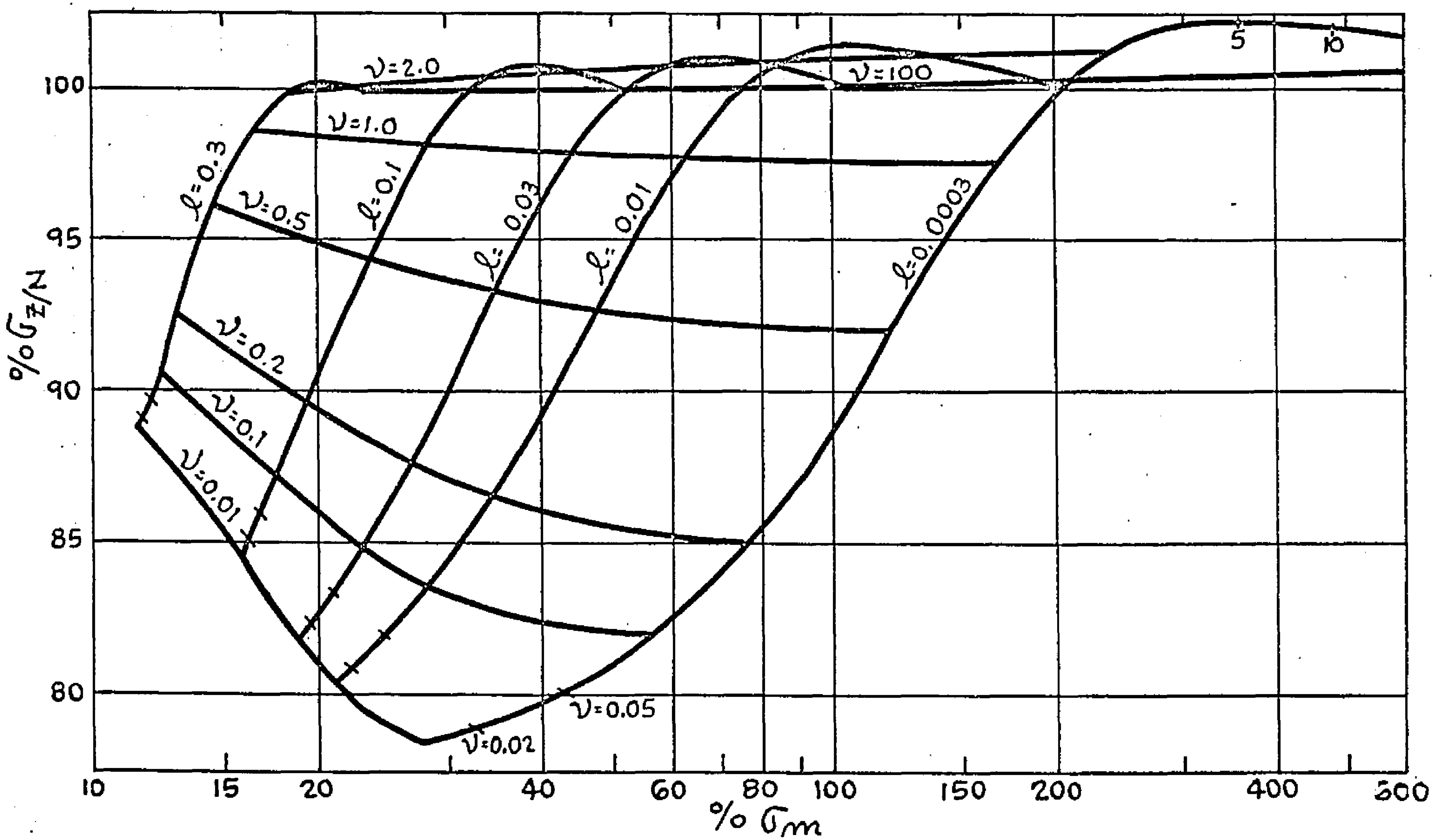


Figure 9.3

THE PERFORMANCE OF THE WIENER DESIGN FOR OUTPUT DISTURBANCE $\Delta = 1.5$, $\nu_0 = 0.01$

Figure 9.4 THE PERFORMANCE OF THE WIENER DESIGN FOR OUTPUT DISTURBANCE Δ = 1.5, ν_0 = 1.0



A low ν disturbance can be thought of as originating from independant random levels which are constant between Poisson distributed switching times. The time between switches is long compared with the plants time constant, ν being the inverse average of these times. (See chapter 2 and 12). Thus the system most probably always reaches a steady state condition before the next level change occurs and therefore the response to a level change is the same as to a single step. The Cohen and Coon controller was in fact designed with respect to a single step disturbance minimizing the area under the response curb in addition to producing a fast and stable response. Further, a closer look at the spectrum transfer property, that is the absolute value squared of the overall transfer function for the system with conventional controller, reveals that the integral action plays a decisive role. Because of the integral action the absolute value squared, decays to zero for low frequencies, ω , at which most of the disturbance spectrum for low ν values is concentrated. The contribution to the integral of the output spectrum, therfore, remains small over all frequencies.

In both Wiener designs shown the output performance for high ν disturbances is over 100%, that is worse, than no control. This does not invalidate the assertion that the Wiener controller is optimal. It is guaranteed optimal for its design characteristic frequency, but not necessarely for others. On the contrary for higher ν disturbances, i.e.: $\nu \geqslant 0.5$ for the $\nu_o = 0.01$ design and $\nu \geqslant 2$ for the $\nu_o = 1.0$ design, a larger control effort increases the output rms. The Wiener answer in fact for the disturbances of $\nu \geqslant 10$ is no control.

For these reasons a satisfactory single Wiener controller for the whole disturbance range of $\nu = 0.01$ to 100 is not possible. In the absence of disturbances with $\nu \geqslant 0.5$ the $\nu_o = 0.01$ design is quite satisfactory over this range. The best choice would be a controller corresponding to $l = .1$. Further decrease of l and thereby increase of controller effort will not improve output performance any more. Note that in the disturbance characteristic frequency range of $\nu = 0.01 - .1$ this controller offers a relatively small advantage, when compared with the conventional one, which might not warrant the use of a more complex controller. Such a controller will bring the output standard deviation to 120% for $\nu = 2$ and will be worse than no control from ν of .5.

For a disturbance whose predominant frequency content corresponds to ν of .5 - 1.0 the $\nu_o = 1$ design commends itself with a l of .01 to .03. Such a controller will prevent the output standard deviation for high ν disturbances from exceeding the 100% line excessively but at the price of very poor control for low ν disturbances.

Form this one concludes that the low ν_0 design, i.e. .01, is capable of handling a wider range of disturbances than can the $\nu_0 = 1$ design. Because of the relatively small reduction of output variance possible for disturbances of $\nu = 1$ and higher the best overall controller seems to be one designed for $\nu_0 = 0.01$ whose input is filtered with a low pass filter. The filter will make the controller inactive with respect to higher disturbances at which no control performances will therefore result. It should be noted that this corresponds to the wide spread practice of filtering noisy controller inputs to conventional controllers. This controller offers better output performance than the conventional one in the low frequency content disturbance range. For disturbances with a characteristic frequency larger than .5 no amplification of the disturbance variance will occur as it does for the conventional controller. In fact in the higher ν range this controller will be equal to the optimal controller designed for that range that is no controller. It seems therefore, that if detailed information about the disturbance spectrum is lacking, a quite satisfactory controller can be designed which can be improved upon by a specific optimal design only marginally in the 0.1 - 1.0 characteristic frequency range. A more accurate determination of the disturbance would therefore be warranted only in special cases.

The performance of the Wiener controllers for a plant with $\Delta = 0.5$ designed for $\nu_0 = 0.01$ and $\nu_0 = 1.0$ is shown in figures 9.5 and 9.6 respectively. In both figures the performance of the Cohen and Coon controller is also shown. Note that although the output rms of the Cohen and Coon controller for a disturbance with a ν of up to 1.0 is lower than that for a plant with the longer delay, the output rms peaks to a considerably higher value of 144% and the entire curve lies at a higher range of controller effort, even for the high ν disturbance range. Also in this case the output rms exceeds the no control mark of 100% only for $\nu > 0.5$ as opposed to $\nu > 0.2$ for the Δ =1.5 plant.

The performance surfaces for the Wiener controllers are similar in shape to those obtained for the Δ = 1.5 plant. Again the ν_0 = 0.01, say at 1 = .1, produces a better output performance than the Cohen and Coon controller over the whole ν range. The output rms exceeds the nocontrol line only at about ν =2. and peaks to 122%. The performance at the medium disturbance range is quite satisfactory offering a considerable advantage when compared to the performance of the conventional controller. The control effort levels that the Wiener controller uses in this case are rather large. The controller corresponding to 1 = 0.3 offers reasonable control effort at the expense, however, of output rms for low ν disturbances. This design's output rms will overshoot the 100% line for higher ν disturbances to a smaller extend. This, however, is of lesser importance, since, if such disturbances are to be expected,

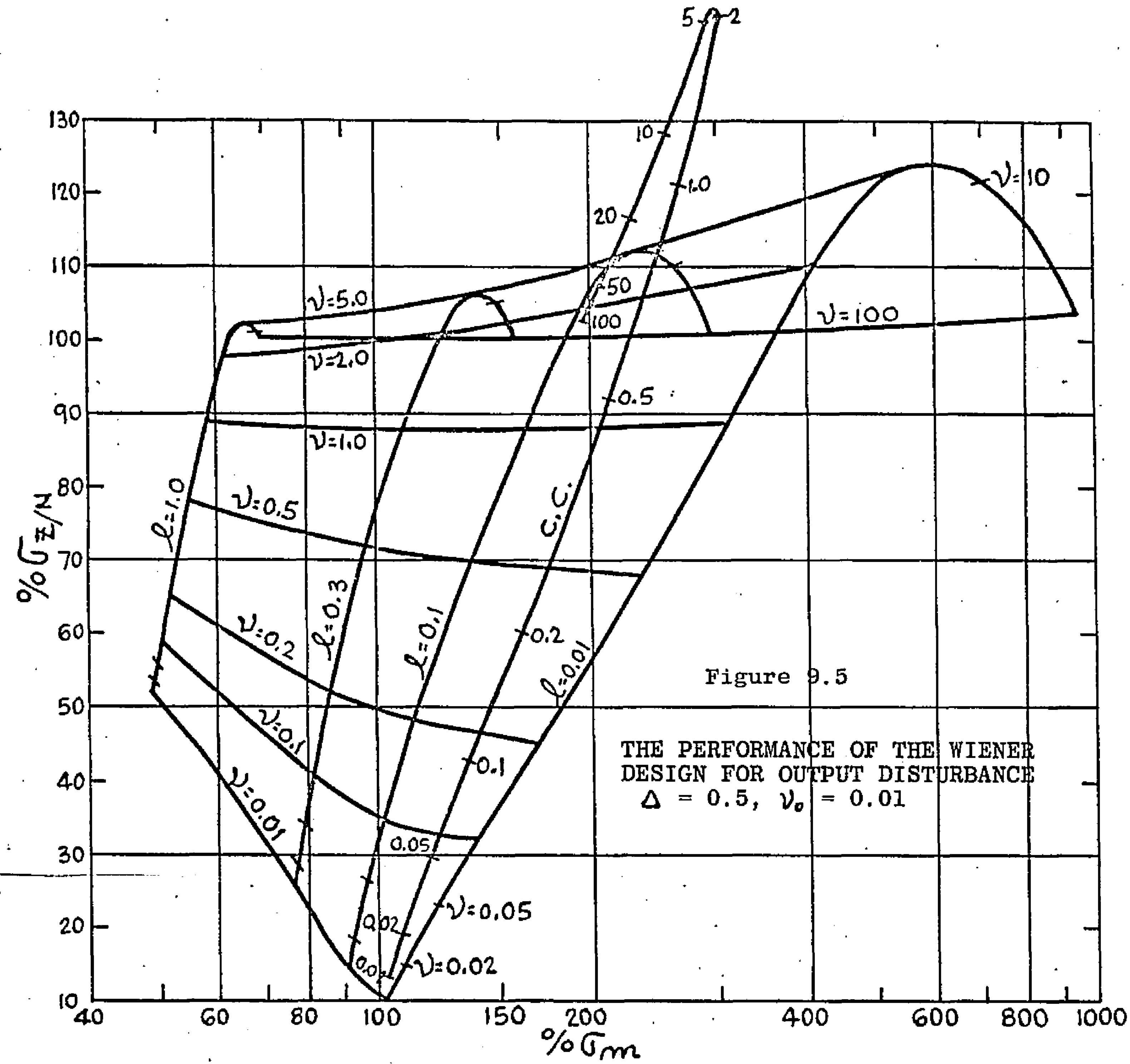


Figure 9.5

THE PERFORMANCE OF THE WIENER DESIGN FOR OUTPUT DISTURBANCE
$\Delta = 0.5, \ \nu_o = 0.01$

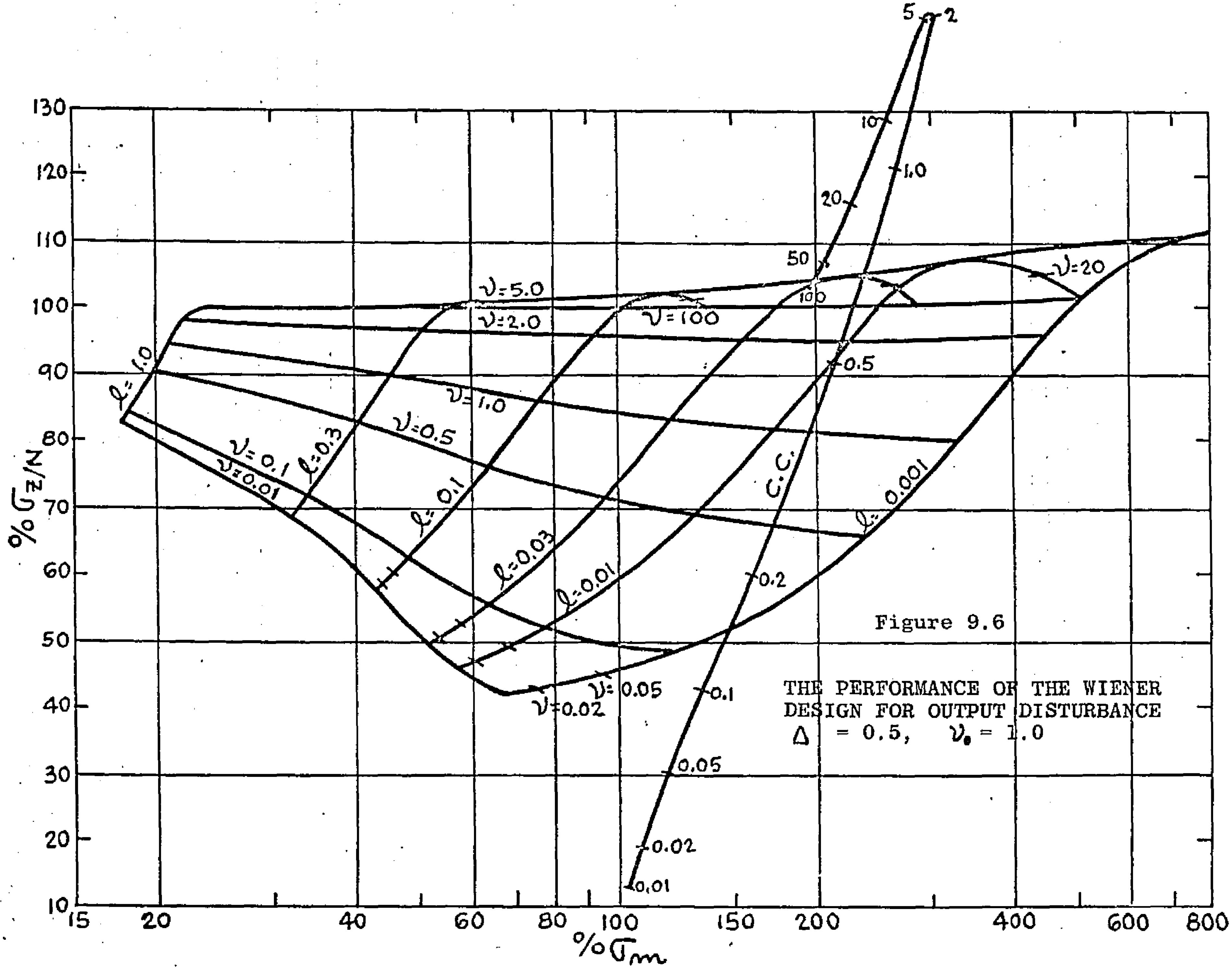


Figure 9.6

THE PERFORMANCE OF THE WIENER DESIGN FOR OUTPUT DISTURBANCE $\Delta = 0.5$, $\nu_0 = 1.0$

the controller should be used in series with a low pass filter which will produce no control performances for these disturbances. No control performance in this case is, of course, the same as in the previous one, namely, $\sigma_{z/N}$ = 100%, $\sigma_m = 0$.

A Wiener design for $\nu_0 = 1.0$ allows the achievement of lower output rms for disturbances in a narrow range around $\nu = 1$. This range extends down to a disturbance of $\nu = 0.5$ only. The output rms exceeds the 100% mark only for a disturbance with ν of about 5 and to a smaller extend in this design. This fact is of little practical significance if a low pass filter is to be used in series with the controller. If on the other hand the high ν disturbance portion of the input is relatively small one could eliminate the filter when using this design.

The recommendations,therefore, for both an overall controller and a specific one for the plant with $\Delta = 0.5$ are essentially the same as for the Δ =1.5 plant. Note, however, that larger controller efforts are required in this case, particularely if one fully utilizes the output performance reduction potential of the Wiener controller. The slope of the performance lines for a specific controller is smaller than in the previous case. This might force the selection of a higher 1 controller, with worse output performance, in order that the larger controller effort for higher ν disturbances might still be accomodated by the available controller effort.

The results of this chapter can now be summarized as follows. The Wiener feedback controller for the output disturbance control configuration is always realizable. The conventional controllers performance is satisfactory only in the low end of the disturbance characteristic frequency range and becomes worse than no control in the upper half of the range considered. In the high range the optimal design recommendation is no control. In each specific disturbance characteristic frequency range the Wiener design indicates the lowest possible output rms regardless of additional controller effort beyond a necessary minimum. Increase of controller effort for a Wiener design can also cause an increase in output rms for a disturbance away from that for which the design was made, while still improving the output performance for the design disturbance. An overall Wiener design could be recommended which offered better performance than the conventional controller particularely in the region of $\nu > 0.2$. When coupled with a low pass filter this controller will not exceed no control output performance for the high ν range. For short delay plants such a controller calls for rather large controller effort. The advantages of the Wiener controller designed for a specific distrubance range are not large compared with the suggested overall controller.

10. CONSTRAINED WIENER DESIGNS FOR INPUT DISTURBANCE

Derivation of the Wiener Controller

For this case the Wiener solution developed in chapter 6 will again be used. The plant model, G(s), and disturbance spectrum, $S_{yy}(s)$, are as in the last chapter, i.e.:

$$G(s) = \frac{e^{-\Delta s}}{1+s} \qquad 10.1$$

$$S_{yy}(s) = \frac{2\nu_0}{\nu_0^2 - s^2} \qquad 10.2$$

However, $S_{uu}(s)$ expresses the fact that we are dealing with an input disturbance situation:

$$S_{uu}(s) = |G(s)|^2 S_{yy}(s) = \frac{2\nu_0}{(1-s^2)(\nu_0^2 - s^2)} \qquad 10.3$$

Δ(s) and Γ(s) in this case become after some manipulation:

$$\Delta(s) = \frac{2\nu_0 \ell (R^2 - s^2)}{(1-s^2)^2(\nu_0^2 - s^2)} \qquad 10.4$$

$$\Gamma(s) = \frac{2\nu_0 e^{\Delta s}}{(1-s)(\nu_0^2 - s^2)(1-s^2)} \qquad 10.5$$

where as in the output disturbance case:

$$R^2 = 1 + \frac{1}{\ell} \qquad 10.6$$

Preceeding as in the last chapter γ(s) becomes:

$$\gamma(s) = \frac{e^{\Delta s}}{\ell(\nu_o+s)(1+s)(R-s)} \qquad 10.7$$

from which the portion corresponding to the positive time behaviour $\gamma_+(s)$ is again obtained by inverting $\gamma(s)$ and retransforming the result. Thus:

$$\gamma_+(s) = \frac{g_o + g_1 s}{(\nu_o+s)(1+s)} \qquad 10.8$$

where g_o and g_1 are constants whose definiton depends on whether ν_o equals 1 or not. For $\nu_o \neq 1$ they become:

$$g_o = \frac{\nu_o(R+\nu_o)e^{-\Delta} - (R+1)e^{\Delta\nu_o}}{\ell(\nu_o-1)(R+\nu_o)(R+1)} \qquad 10.9$$

$$g_1 = \frac{(R+\nu_o)e^{-\Delta} - (R+1)e^{-\Delta\nu_o}}{\ell(\nu_o-1)(R+\nu_o)(R+1)} \qquad 10.10$$

For $\nu_o = 1$ they are:

$$g_o = \frac{e^{-\Delta}}{\ell(R+1)^2}\left[1+(R+1)(\Delta+1)\right] \qquad 10.11$$

$$g_1 = \frac{e^{-\Delta}}{\ell(R+1)^2}\left[1+\Delta(R+1)\right] \qquad 10.12$$

The difference between the two cases comes from the fact that for $\nu_o = 1$ the simple pole at s = -1 of $\gamma(s)$ becomes a double pole. This effects the inversion of $\gamma(s)$ which for $\Delta+t>0$ uses the residue theorem in the left hand s plane.

The equivalent open loop cascaded controller, K(s), the becomes:

$$K(s) = \frac{(1+s)(g_0+g_1 s)}{R+s} \qquad 10.13$$

and the feedback controller H(s), is:

$$H(s) = \frac{K(s)}{1-KG(s)} = \frac{(1+s)(g_0+g_1 s)}{(R+s)-(g_0+g_1 s)e^{-hs}} \qquad 10.14$$

For the case, where there is no delay in the plant, the feedback controller becomes:

$$H(s) = \frac{(1+s)(g_0+g_1 s)}{(1-g_1)[s+(R-g_0)/(1-g_1)]} \qquad 10.15$$

Since, however, $(R-g_0)/(1-g_1)$ is identically one, H(s) simply becomes a conventional proportional - derivative controller:

$$H(s) = \frac{g_0}{1-g_1} + \frac{g_1}{1-g_1}s = K_c + Ds \qquad 10.16$$

The controller coeficients are:

$$\left.\begin{aligned} K_c &= \frac{g_0}{1-g_1} = \frac{R+\nu_0+1}{\ell(R+\nu_0)(R+1)-1} \\ D &= \frac{g_1}{1-g_1} = \frac{1}{\ell(R+\nu_0)(R+1)-1} \end{aligned}\right\} \qquad 10.16a$$

This is in contrast to the no delay output disturbance case where the feedback controller obtained consisted of a proportional action plus a first order filter. Both K_c and D are always positive so that the controller has true negative feedback and the overall system is of course realizable as guaranteed by the Wiener solution method. Furthermore as 1 tends to 0, i. e., no constraint, K_c and D tend to infinity, a result which was obtained for the unconstrained designs of chapter 7.

The Realizability of the Feedback Controller

It was shown in chapter 7 that the unconstrained Wiener controller when $\Delta \neq 0$ is unrealizable. Also, it is clear that the completely constrained controller output, $1 = \infty$, which is the same as an uncontrolled system, is realizable. There will be therefore a limiting value of 1 below which the feedback controller will become unrealizable.

H(s) will be unrealizable if it possesses poles in the right hand s plane and poles of H(s) correspond to zeros of 1 - KG(s). The realizability limit therefore, is obtained by determining that value of 1 which causes any root of 1 - KG(s) to cross over to the right hand s plane. This is conveniently done via the Nyquist criterion for KG(s) with respect to the point +1+i0.

Along the imaginary axis KG(s) can be represented in polar form as : (with $s = i\omega$)

$$KG(\omega) = \rho e^{i\phi} \qquad 10.17$$

where:

$$\rho = |KG(\omega)| = \left[\frac{g_0^2 + g_1^2\omega^2}{R^2 + \omega^2}\right]^{1/2} \qquad 10.18$$

$$\varnothing = -\Delta\omega + tg^{-1}\frac{g_1}{g_0}\omega - tg^{-1}\frac{\omega}{R} \qquad 10.19$$

Along the semi circle surrounding the right hand s plane, $s = re^{i\lambda}$ where r tends to infinity. KG(s) then become a function of λ which varies from $-\pi/2$ to $+\pi/2$.

$$KG(\lambda) = e^{-\Delta re^{i\lambda}}\frac{g_0 + g_1 re^{i\lambda}}{R + re^{i\lambda}} = g_1 e^{-\Delta re^{i\lambda}} \qquad \text{for } r\uparrow\infty \qquad 10.20$$

The usual application of the Nyquist mapping procedure becomes somewhat vague in this case because of the vanishing of KG(s) along the semicircle only. For this reason the integration used in developing the Nyquist procedure will be applied directly to the present case.

From the residue theorem it follows that the number of zeros of KG(s) in the right hand s plane, ennumerating each zero with its multiplicity and assuming the absence of poles, is:

$$z_r = \frac{1}{2\pi i}\int_{D}\frac{[KG(s)]'}{KG(s)}ds \qquad 10.21$$

$[KG(s)]'$ is the derivative and the integration path is in the positive direction along the imaginary axis and the right hand semicircle. Breaking the integration up into the sum of integrants along these pathes z_r becomes:

$$z_r = \frac{1}{2\pi i}\int_{-\pi/2}^{+\pi/2}\frac{[KG(\lambda)]'}{KG(\lambda)}rie^{i\lambda}d\lambda + \frac{1}{2\pi}\int_{+\infty}^{-\infty}\frac{[KG(\omega)]}{KG(\omega)}d\omega \qquad 10.22$$

Along the semicircle the derivative of $KG(\lambda)$ with respect to λ becomes:

$$[KG(\lambda)]' = g_1(-\Delta r i)e^{i\lambda} e^{-\Delta r e^{i\lambda}} \qquad 10.23$$

Substituting this value into the first integral of equation 10.22 and performing the integration:

$$\frac{1}{2\pi i}\int_{-\pi/2}^{+\pi/2} \frac{[KG(\lambda)]'}{KG(\lambda)} r i e^{i\lambda} d\lambda = \frac{1}{2\pi i}\int_{-\pi/2}^{+\pi/2} \Delta r^2 e^{i2\lambda} d\lambda = 0 \qquad 10.24$$

The expression for z_r then becomes:

$$z_r = \frac{-1}{2\pi i}\int_{-\infty}^{+\infty} d\,\ln[KG(\omega)] = \frac{-1}{2\pi i}[\ln \rho + \emptyset]_{-\infty}^{+\infty} \qquad 10.25$$

The magnitudes ρ at the two extremes are equal, the $KG(\omega)$ being conjugates of each other, so that the criterion becomes:

$$z_r = \frac{1}{2\pi}[\emptyset(-\infty) - \emptyset(+\infty)] \qquad 10.26$$

This is the usual criterion where the net change in phase of KG(s) is considered, with s, however, traversing the imaginary axis only. In other words the contribution to the phase along the semicircle is taken as zero. The number of zeros of 1 - KG(s) will therefore equal the net positive encirclements around the point +1+i0 of KG(s) where s traverses the imaginary axis from $+\infty$ to $-\infty$.

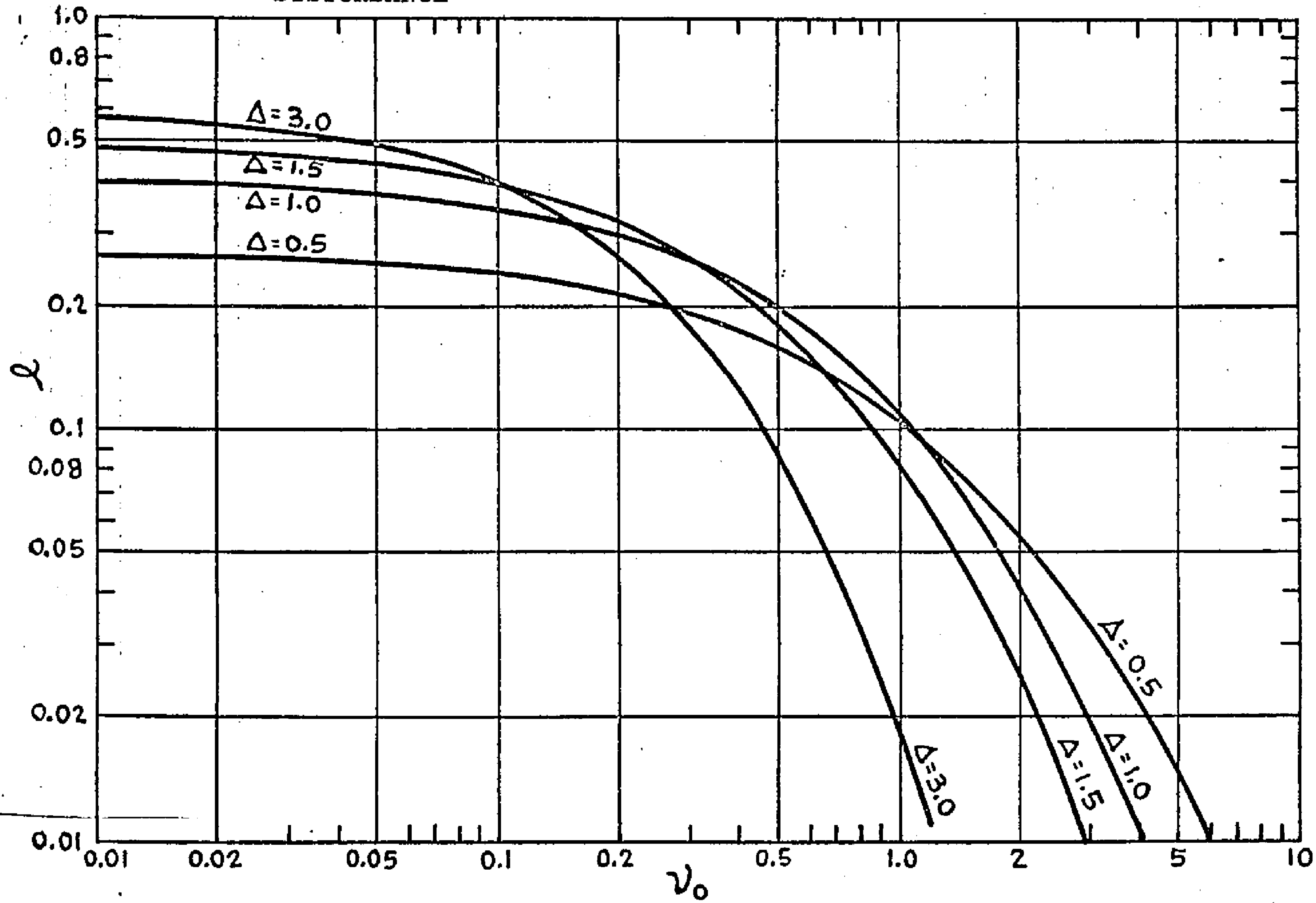
At $\omega = 0$ $KG(\omega)$ becomes g_0/R as seen from equation 10.18 and 10.19. As ω tends toward $-\infty$ or recedes from $+\infty$ $KG(\omega)$ repeatedly describes a circle, centered at the origin, of radius

g_1 in the positive direction. It follows therefore from the Nyquist criterion that if $g_1 > g_o/R$ the necessaty and sufficient condition for realizability is $g_1 < 1$. If $g_o/R > g_1$, the condition $g_1 < 1$ is necessary but not sufficient and a detailed mapping is required. In this case the additional condition that , $g_o/R < 1$ is certainly sufficient but not necessary.

These observations were applied to H(s) in the ν_o range of 0.01 to 10 and Δ range of .5 to 3. At each set of ν_o and Δ values l was varied from 0.001 to 10.0 and g_1 and g_o/R calculated. In this range of parameters the point where the g_1 versus l curve crosses 1, g_o/R was less than 1 so that the $g_1 = 1$ point fulfills the necessary and sufficient condition for the limit of realizability. A numerical search was then made for the exact value of l where g_1 becomes 1. It is seen, therefore, that the realizability on l for the Wiener feedback controller can be implicitly expressed in terms of the coefficients of the equivalent open loop compensator. That is, an actual Nyquist plot, is not necessary.

The results are presented in figure 10.1 where the limiting l is plotted vs. ν_o with Δ as a parameter. The exact relative position of these lines is not directly significant since the same l value at identical ν_o but different Δ corresponds to different performance.

Figure 10.1 THE REALIZABILITY LIMIT ON l OF THE WIENER DESIGN FOR INPUT DISTURBANCE



The Performance of the Wiener Controller

The output variance, σ_z^2 and the controller output variance, σ_m^2, can now be computed by integrating the respective power spectra over the imaginary s axix, i.e. :

$$\sigma_z^2 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} |1-KG(s)|^2 S_{uu}(s)\,ds \qquad 10.27$$

$$\sigma_m^2 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} |K(s)|^2 S_{uu}(s)\,ds \qquad 10.28$$

The input spectrum $S_{uu}(s)$ in this case is:

$$S_{uu}(s) = \frac{2\nu}{(1-s^2)(\nu^2-s^2)} \qquad 10.29$$

Substituting in equation 10.28 for K(s), G(s) and $S_{uu}(s)$ and evaluating the absolute values indicated in the integrands σ_z^2 and σ_m^2 becomes:

$$\sigma_z^2 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \left\{1 + \frac{g_0^2 - g_1^2 s^2}{R^2 - s^2} - 2\,\frac{(g_0 + g_1 s)e^{-\Delta s}}{R+s}\right\}\frac{2\nu}{(\nu^2-s^2)(1-s^2)}\,ds \qquad 10.30$$

$$\sigma_m^2 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{(g_0^2 - g_1^2 s^2)2\nu}{(R^2-s^2)(\nu^2-s^2)}\,ds \qquad 10.31$$

Again these integrals are determined via the residue theorem. The residues of the poles in the right hand s plane must be used for the term, in the σ_z^2 integral, containing the exponential. Either half plane residues can be used for the other terms of both integrals provided the contour integration is performed in the posititve sense. The results are:

$$\sigma_m^2 = \frac{\nu(g_0^2 - g_1^2 R^2)}{R(\nu^2 - R^2)} + \frac{g_0^2 - g_1^2 \nu^2}{R^2 - \nu^2} \qquad 10.32$$

$$\sigma_z^2 = \frac{1}{\nu+1} + \frac{\nu(g_0^2 - g_1^2 R^2)}{R(\nu^2 - R^2)(1-R^2)} + \frac{g_0^2 - g_1^2\nu^2}{(R^2-\nu^2)(1-\nu^2)} + \frac{\nu(g_0^2 - g_1^2)}{(R^2-1)(\nu^2-1)}$$

$$+ \frac{2(g_0 + g_1\nu)e^{-\Delta\nu}}{(1-\nu^2)(R+\nu)} + \frac{2\nu(g_0+g_1)e^{-\Delta}}{(R+1)(\nu^2-1)} \quad \text{for } \nu \neq 1 \qquad 10.33$$

These results, of course, hold for the no delay case as well.

The performance of the Wiener controllers for plants with no delay, designed for ν_0 of 0.01, 1.0, and 10 are shown in figures 10.2, 10.3 and 10.4 respectively. Again the output standard deviation is scaled by that of the plant with no control, that is:

$$\sigma_{z/N}^2 = \sigma_z^2 (\nu + 1) \qquad 10.33b$$

The surfaces in these figures are of crescent shape which converges at its ends to the points $\sigma_{z/N} = 1.$, $\sigma_m = 0$, that is no control, and $\sigma_{z/N} = 0$, $\sigma_m = 1.$, which corresponds to the no constraint case. In the latter case the controller has an infinite gain but the rms of the controller output stays equal to that of the input disturbance. This immediately raises a suspicion against the variance as an adequate measure of controllers action magnitude. This aspect is elaborated on in chapter

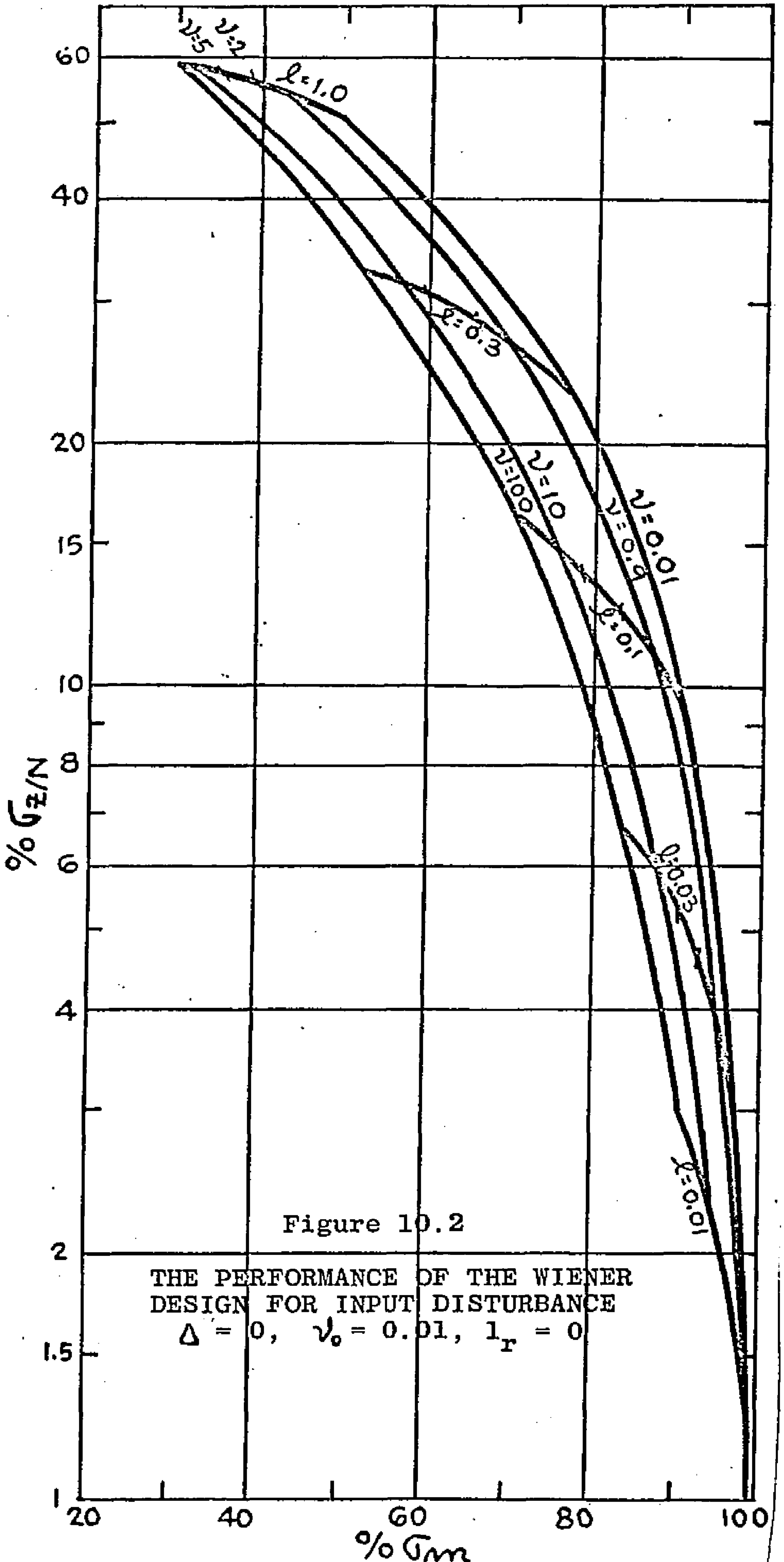


Figure 10.2

THE PERFORMANCE OF THE WIENER DESIGN FOR INPUT DISTURBANCE

$\Delta = 0, \quad \nu_0 = 0.01, \quad 1_r = 0$

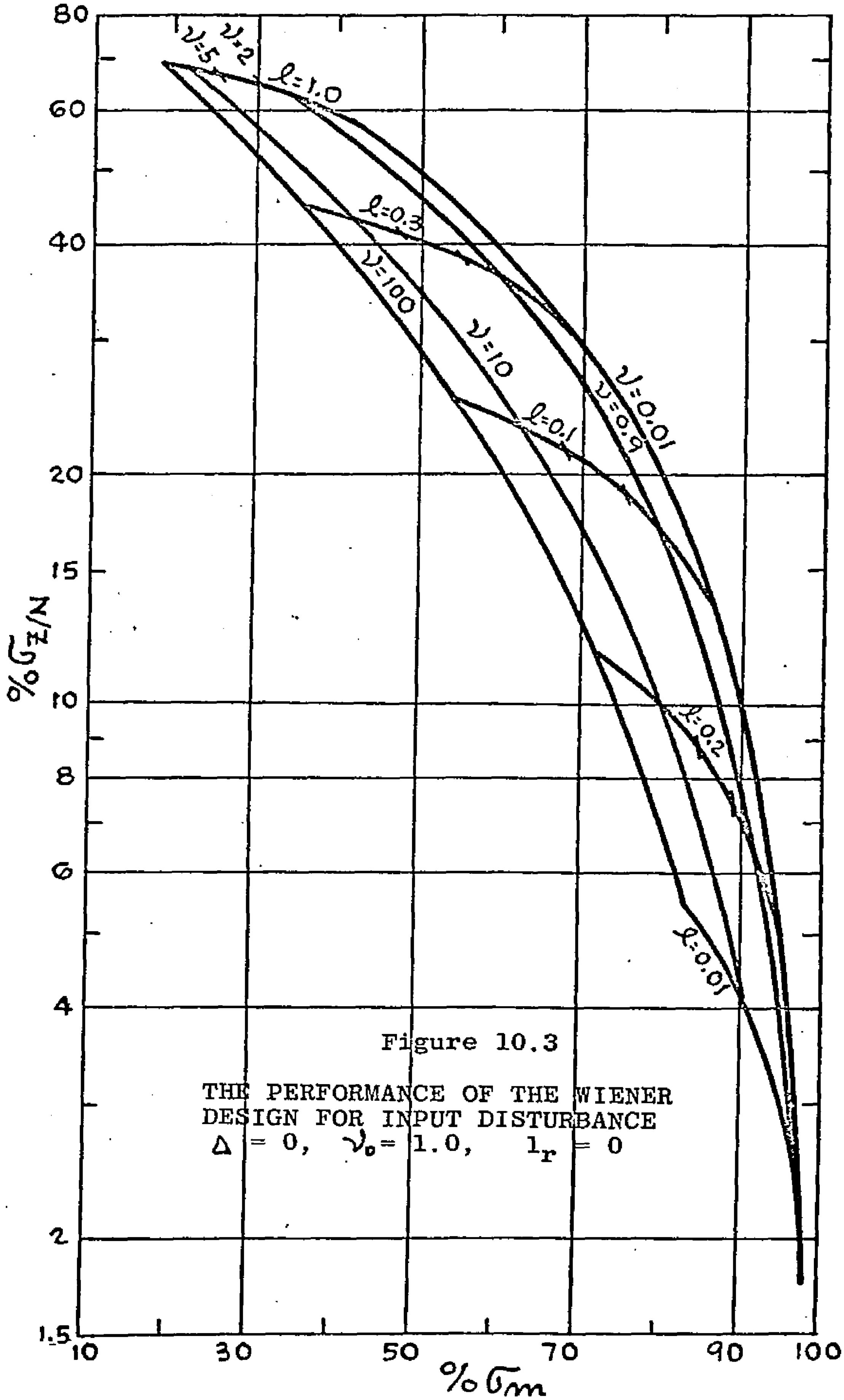


Figure 10.3

THE PERFORMANCE OF THE WIENER DESIGN FOR INPUT DISTURBANCE
$\Delta = 0, \quad \nu_0 = 1.0, \quad l_r = 0$

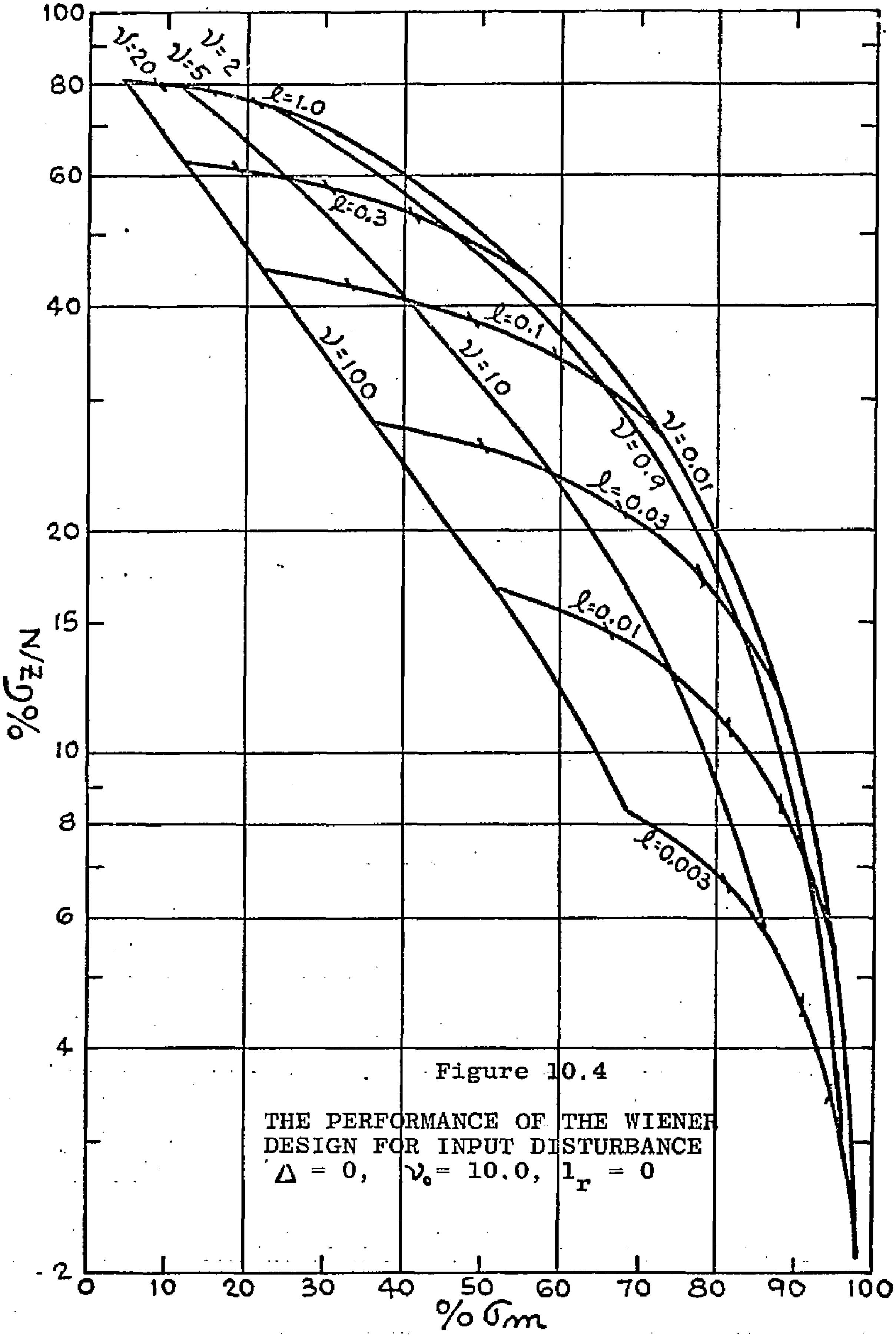


Figure 10.4

THE PERFORMANCE OF THE WIENER DESIGN FOR INPUT DISTURBANCE

$\Delta = 0$, $\nu_0 = 10.0$, $l_r = 0$

2 and 12 of this work.

Over the range of interest the performance curves of different controllers to a specific disturbance do show lower output rms for an increase in controller effort. For a particular controller an increase in frequency content of the disturbance causes an increase in the output rms but a decrease in controller effort. The closeness of the performance points for disturbances with a ν of 0.01 to .1 for all fixed controller designs indicates that as ν is further decreased there is no change in the output rms and this corresponds to the offset of a porportional-derivative action controller in response to a step. The output rms for low ν disturbances can be reduced by decreasing l and thereby increasing the controller effort. The limitation, however, is a large increase in feedback controller gain.

For a disturbance with a ν value of up to 10 the performance for the three designs is practically identical. Moreover, the constant l lines, which show the performance of a specific controller over the range of disturbances, are practically parallel to each other. This means that the same performance can be achieved with controllers, designed for different ν_0 in response to disturbances with a ν of up to 10. For different ν_0 designs the same performance will correspond to a different value of l but this is immaterial since l is merely an auxiliary variable used to adjust the performance at a desired controller effort level. In other words , therefore, the output rms in this particular case depends on the controller effort and is insensitive to the particular disturbance characteristic frequency used for the design. This behaviour is different from the one observed for the output disturbance configuration where the constant ν lines coincided for different ν_o designs but the 1 lines were not parallel, so that the choice of a specific controller depended on ν_o .

Notice also that the performance curves for ν between .01 and 5 for all designs form a rather narrow band. This means that the performance in this case is not only insensitive to the design characteristic frequency but also to the disturbance frequency content itself. This fact was observed by Lueck and McGuire who were concerened with the design of feedback feedforeward composite controllers via Wieners methods,(11) for the same type of system and disturbance used in this chapter. Their overall Wiener design is equivalent to the present one and differs only in its implementation as a composite controller . The performance is therefore equivalent, but they investigate their system only with respect to the disturbance for which it was designed. Testing the effect on performance of the disturbance characteristic frequency with a delayless plant they observe the fact just described, namely the relative small effect of the disturbance frequency content on performance. On this basis they proceed with the rest of their investigation with a single characteristic frequency in the neighborhood of the plants time constant for design and performance evaluation. It was seen, however, that the system with output disturbance did not behave similarly even for a plant with no delay and it will be shown that for the system with input disturbance where $\Delta \neq 0$ the performance does depend on ν_0 and ν .

Performance of a Plant with Delay

The performance of the Wiener controller for a plant with a delay of 1.5 is shown in figures 10.5, 10.6 for $\nu_0 = 0.01$ and 1.0 respectively. For comparison the performance of a Cohen and Coon controller is also shown. The lines of constant disturbance parameter ν all converge to the point $\sigma_{z/N} = 1.,\ \sigma_m = 0$, that is no control. As the controller rms is increased the output rms first decreases and then increases for high ν disturbnces or stays constant for low ν disturbances. Clearly controllers which operate in points to the right of the minimum are undesirable since the additional controller effort results in inferior output performance.

As the design characteristic frequency, ν_0 , is increased to 1 the whole surface is squeezed horizontally from high and low output rms towards the line $\sigma_{z/N} = 1.0$. For $\nu_0 = 10$ the whole surface is below this line with the $\nu = 10$ and 100 lines ranging between 96% and 100% output rms for all controller

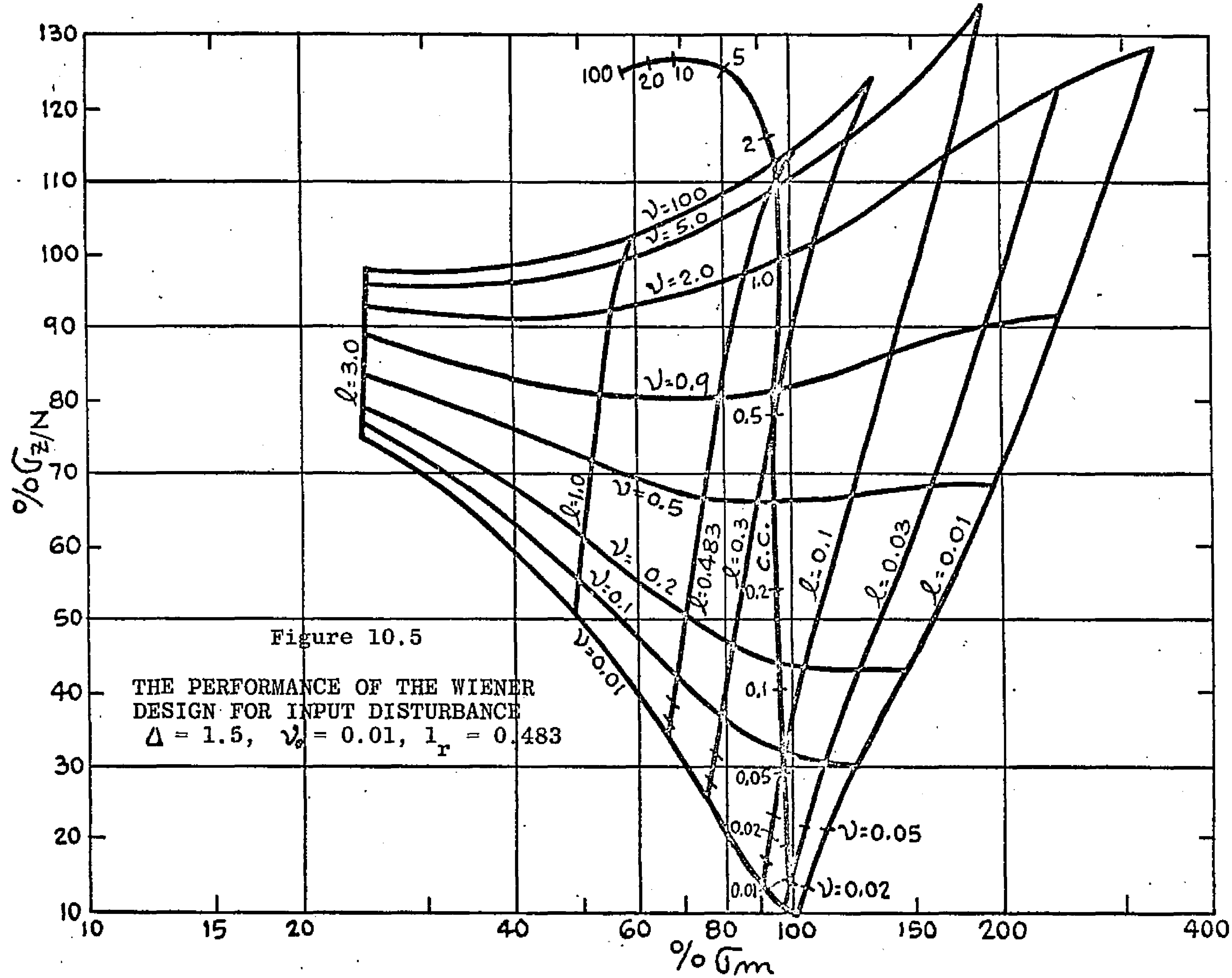


Figure 10.5

THE PERFORMANCE OF THE WIENER DESIGN FOR INPUT DISTURBANCE

$\Delta = 1.5$, $\nu_0 = 0.01$, $l_r = 0.483$

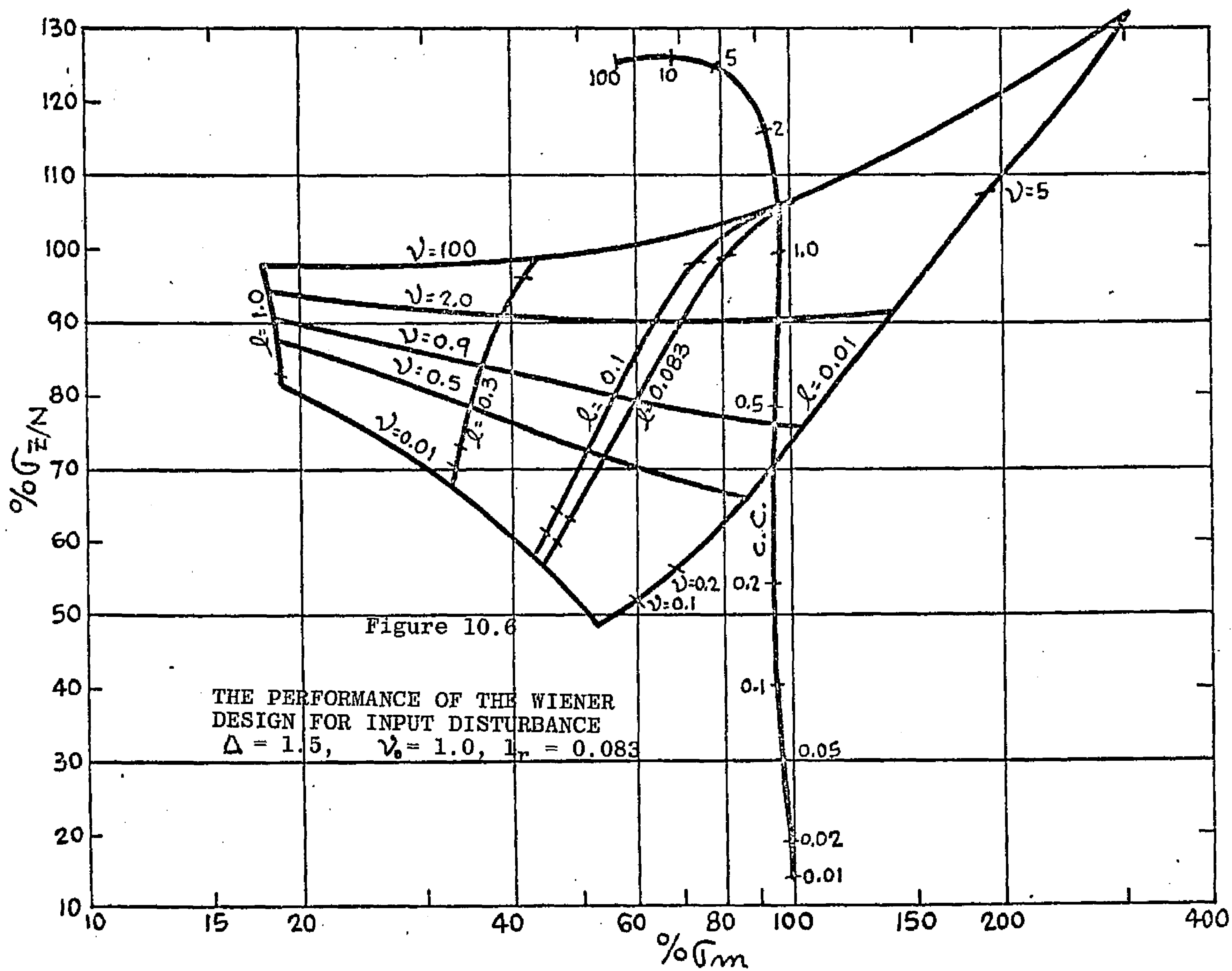


Figure 10.6

THE PERFORMANCE OF THE WIENER DESIGN FOR INPUT DISTURBANCE
$\Delta = 1.5$, $\nu_0 = 1.0$, $l_r = 0.083$

efforts. The line for $\nu = 0.01$ lies between $\sigma_{z/N}$ of 74 - 80% and σ_m of 20 - 37%. This behaviour indicates, that for a $\Delta = 1.5$ plant and disturbances of $\nu \geqslant 10$ the Wiener controller does not lead to an appreciably lower output rms than no controller, regardless of controller effort. Since the Wiener controller is the best possible one the search for an improved controller for the high range disturbance can be abandoned and no controller, i.e., a Wiener controller with $l = \infty$, becomes the practical recommendation for these disturbances. It should be noted that the Cohen and Coon controller crosses the $\sigma_{z/N} = 1.$ line for $\nu = 1.0$ and its output rms becomes larger than 125% for higher ν disturbances.

The performance of Wieners controller for low ν disturbances is limited by the realizability of the feedback controller. The performance of the controller at the realizability limit on l, l_r, is shown in these figures. All controllers with a $l < l_r$ are unrealizable and cannot be used. The best low ν performance is shown by the Wiener controllers designed for $\nu_0 = 0.01$ at the realizability limit l_r. Still its output performance is higher than that of the Cohen and Coon controller up to a disturbance of $\nu = 0.2$. From then on the Wiener design is better than that of Cohen and Coon. It does, however, also cross the $\sigma_{z/N} = 1$ line for $\nu = 0.2$ but overshoots it to a smaller extend and at higher ν disturbances. The Wiener controller designed for $\nu_0 = 1$ and l_r is superior to the $\nu_0 = 0.01$ design for disturbances with $\nu > 0.5$ and its output rms overshoots the 100% mark even less. The advantage of this design as compared to the $\nu_0 = 0.01$ one seems smaller than the loss of output performance with respect to low ν disturbances.

The observed fact that better output performance can be achieved with the Cohen and Coon controller than with a Wiener design of $\nu_0 = 0.01$ in the low 0.01 - 0.1 range does not contradict the optimality of the Wiener design. Comparison must be made at comparable control efforts and the Wiener design at the realizability limit uses considerably less controller effort than does the conventional design. The controller effort could be increased beyond the limit if the open loop compensator K(s) would be employed as a feedforeward controller KG(s), acting on y(s) which is always realizable. Thus the constraint on the controller effort which was introduced to limit the controller action and to produce a realizable feedback controller results in a design which is realizable but only up to a lower controller effort than is usually available, i.e., about 100%. The realizability limitation on the Wiener controller designed and excited in the range of 0.2 = 0.5 is less severe since the 1_r line crosses the constant ν lines of this range very close to their minimum.

It is therefore not possible to recommend a compromise Wiener design for the whole disturbance range as was done for the output disturbance configuration. The recommendation for a controller effective over a narrow range of disturbances, where controller efforts of up to 100% are available, would be: Cohen and Coon for disturbances with $\nu = 0.01$ to 0.5 Wiener design of $\nu_0 = 1$, at 1_r, for the disturbance range of

ν = 0.5 - 2. No controller for the high ν disturbance range.

The situation for a plant with Δ = 0.5 and ν_0 = 0.01 and 1.0 is shown in figures 10.7 and 10.8 respectively. Comparing the two figures shows that the surfaces are very similar to one another in the realizable range, where the small differences lie mainly in the slopes of the constant 1 lines. The performance of these controllers with respect to disturbances in the range ν = 10 to 100 can be improved only slightly if the controller is designed specifically for these disturbances. If this is done reduction by two to three % in the design disturbance region is achieved while there is a gain of about 10% in the low disturbance range when compared to the two designs shown. The lowest possible output rms for disturbances of ν = 10 and 100 is 72% and 78% respectively. Unlike the Δ = 1.5 case this is significantly lower than the no control output rms, i. e. 100% . The conventional Cohen and Coon controller has an output rms of over 100% from about a $\nu=10$ disturbance but it exceeds this value by only 8% at ν = 100. Again at low disturbances the Cohen and Coon controller yields a lower output rms than is possible with the realizable Wiener controller.

As in the case of the Δ = 1.5 plant an overall Wiener design cannot be recommended. In the low region of up to ν = 0.5 the optimal step controller delivers the best performance at a controller effort of about 100%. For the rest of the

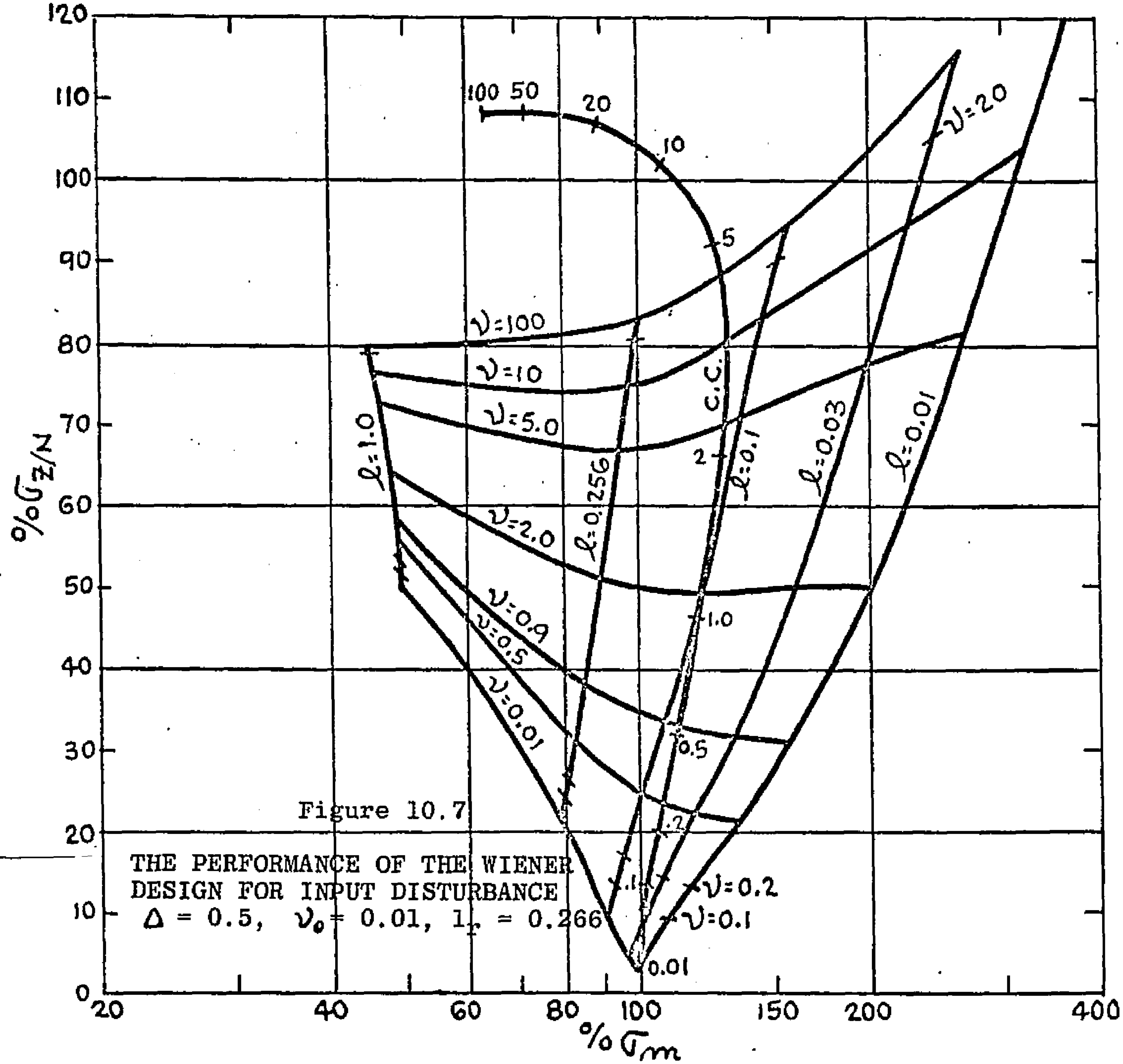


Figure 10.7

THE PERFORMANCE OF THE WIENER DESIGN FOR INPUT DISTURBANCE

$\Delta = 0.5, \quad \nu_0 = 0.01, \quad l_1 = 0.266$

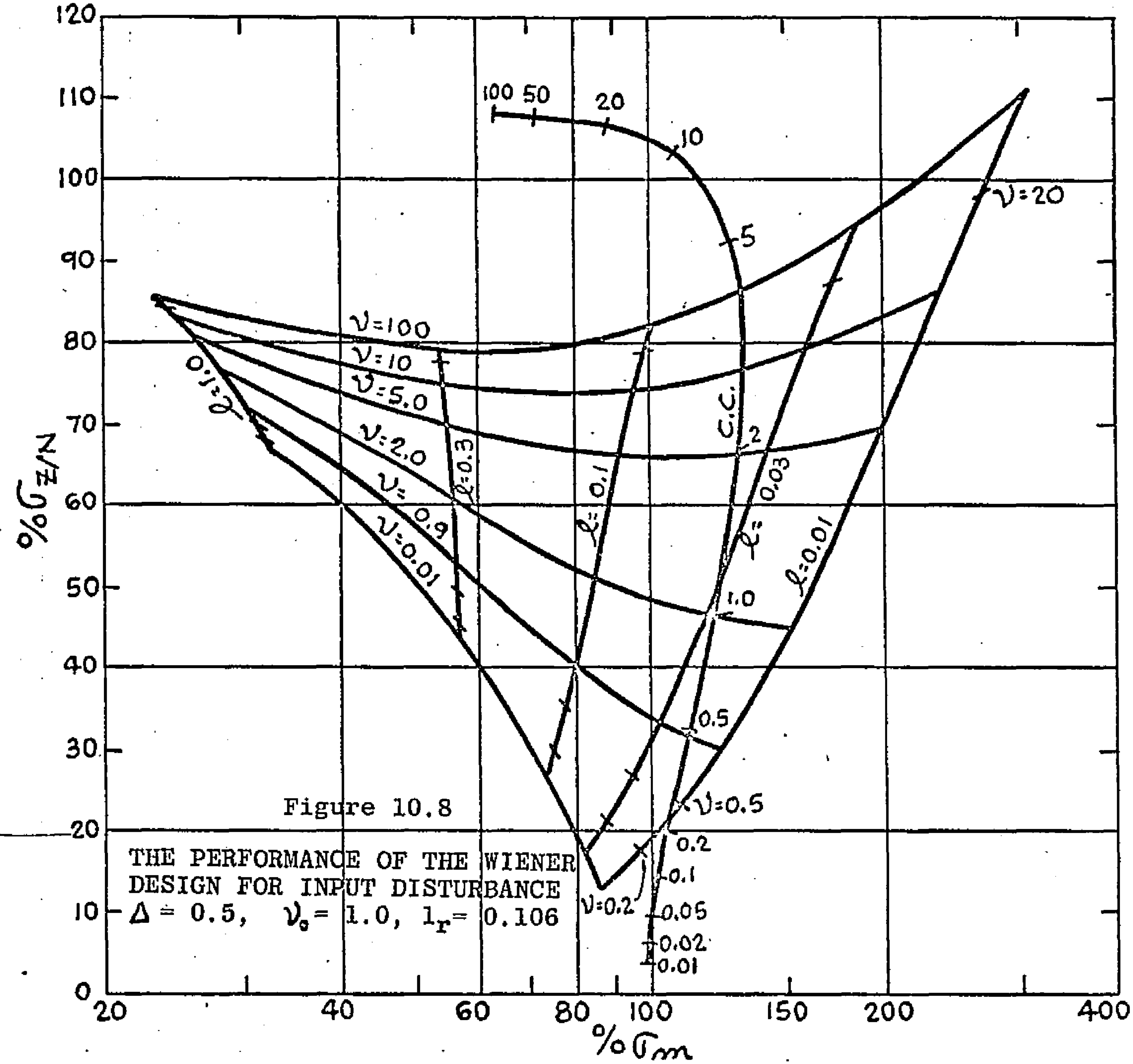


Figure 10.8

THE PERFORMANCE OF THE WIENER DESIGN FOR INPUT DISTURBANCE $\Delta = 0.5$, $\nu_o = 1.0$, $l_r = 0.106$

disturbance range there is little difference between the $\nu_0 = 0.01$ and $\nu_0 = 1.0$ designs at their respective l_r's . This behaviour is markedly different from the of a $\Delta = 1.5$ plant but is to be expected for smaller values of Δ . It was observed above that in the case of the delayless plant the performance of the Wiener controller is, for practical purposes, independant of ν_0 .

No filter should be used for the input signal of the controller in this case since its output performance with respect to high ν disturbances is significantly better than that of the uncontrolled plant.

Conlusion

In conclusion it is noted that in the input configuration the introduction of constraint on the controller action produces realizable Wiener feedback controllers. However, realizability limits the controller effort, and, therefore, the output performance, considerably for low ν disturbances where the potential of the equivalent open loop Wiener design is not utilized. In this disturbance region the optimal step controller produces a lower output rms at an acceptable controller effort. In the medium and high ν disturbance ranges Wiener designs can be specified which are superior to the performance of the conventional one. In particular, the worse than no control performance of the step controller, can be avoided in this region.

11. CONSTRAINED TWO PARAMETERS DISTURBANCE DESIGNS

A. Input Disturbance Configuration

Motivation of the two disturbance design

It was seen in the last chapter that the Wiener designed feedback controller for input disturbances is limited by its realizability. The designs developed in this chapter are intended to produce a feedback controller which will be suitable for the whole disturbance range and not suffer from realizability limitations.

The general shape of the performace surfaces, particularely if realizability limits are overlooked for a moment, suggest a possible way of achieving this goal. Consider a low ν_0 Wiener design in the 1 range were increasing controller effort still decreases the output rms appreciably for a disturbance with $\nu \approx \nu_0$. In this region the point is reached where the output rms for high ν disturbances increases sharply with an increase of controller effort. Comparing this behaviour with that of a system designed for $\nu_0 = 1$ or higher, shows that this adverse effect, for disturbances with $\nu > \nu_0$, is considerably reduced. If good overall performace is to be achieved, again while disregarding realizability limitation, one is temped to bring into one design the virtues of the low and high ν_0 designs by designing the Wiener controller for a sum of uncorrelated disturbances, one with low ν_0 and the other with a high ν_0.

That is, the design disturbance will have two characteristic frequency parameters. One would hope that the proper relative intensity of the two design disturbances would produce good performance for ν disturbances while supressing the bad performance for high ν disturbances which accompany the low ν_0 single disturbance design. The realizability limit on l, l_r, for this two disturbance design should, of course, be such that use can be made of the improved performance surface.

The Two Parameter Disturbance Design

As suggested in the preceeding section the Wiener controller will be designed for a disturbance consisting of the sum of two uncorrelated disturbances. The two disturbances will each be of the familiar type, but with different characteristic frequencies ν. The spectrum for such a two parameter disturbance $S_{yy}(s)$ becomes:

$$S_{yy}(s) = \frac{2\nu_0}{\nu_0^2 - s^2} + x\frac{2\nu_1}{\nu_1^2 - s^2} \qquad 11.1$$

where $\nu_0 < \nu_1$ and the spectrum is normalized with σ_0^2 so that $x = \sigma_1^2/\sigma_0^2$. The development of the Wiener controller can now be made following the same steps as in the proceeding chapter. Thus $S_{uu}(s)$ becomes:

$$S_{uu}(s) = |G_1(s)|^2 S_{yy}(s) = \frac{2(R_1^2 - s^2)(\nu_0 + x\nu_1)}{(\nu_0^2 - s^2)(\nu_1^2 - s^2)(1 - s^2)} \qquad 11.2$$

where

$$R_1 = \nu_0\nu_1\frac{\nu_1 + x\nu_0}{\nu_0 + x\nu_1} \qquad 11.3$$

Also the functions $\Delta(s)$ and $\Gamma(s)$ can be put in factored form and become :

$$\Delta(s) = \frac{2\ell(R^2-s^2)(R_1^2-s^2)(\nu_0+x\nu_1)}{(1-s^2)^2(\nu_1^2-s^2)(\nu_0^2-s^2)} \qquad 11.4$$

$$\Gamma(s) = \frac{2(R_1^2-s^2)(\nu_0+\nu_1 x)\,e^{\Delta s}}{(\nu_0^2-s^2)(\nu_1^2-s^2)(1-s^2)(1-s)} \qquad 11.5$$

From these $\gamma(s)$ becomes:

$$\gamma(s) = \frac{(R_1+s)\,e^{\Delta s}}{(\nu_0+s)(\nu_1+s)(1+s)(R-s)\,\ell} \qquad 11.6$$

which can be broken up into the sum $\gamma_+(s)+\gamma_-(s)$, corresponding to the positive and negative time parts of $\gamma(s)$, by inverting $\gamma(s)$ and retransforming $\gamma_+(t)$ as was done before. $\gamma_+(s)$ only is of interest and becomes:

$$\gamma_+(s) = \frac{g_2 s^2 + g_1 s + g_0}{(s+\nu_0)(s+\nu_1)(s+1)} \qquad 11.7$$

The constants g_o, g_1 and g_2 are given as a function of Δ, ν_0, ν_1, and 1 in the appendix. They are given for the case of $\nu_0 = 1, \nu_1 \neq 1$ and $\nu_0 \neq 1$, $\nu_1 \neq 1$. Again the different values in these two cases arise from the inversion of $\gamma(s)$ using the residue theorem in the left hand s plane. For $\nu_0 = 1$ the pole at s = -1 becomes a second order pole.

The equivalent cascaded compensator K(s) therefore becomes:

$$K(s) = \frac{(1+s)(g_2 s^2 + g_1 s + g_0)}{(R+s)(R_1+s)} \qquad 11.8$$

which is of one order higher in s in both nominator and denominator than was the single parameter disturbance design.

As before the feedback controller, H(s), is obtained from:

$$H(s) = \frac{K(s)}{1 - KG(s)} \qquad \text{II.9}$$

The realizability limit on l is again conveniently determined from a Nyquist analysis of KG(s).

$$KG_1(s) = \frac{g_2 s^2 + g_1 s + g_0}{(R+s)(R_1+s)} e^{-\Delta s} \qquad \text{II.10}$$

As was the case for the single parameter disturbance design KG(s) vanishes along the semicircle surrounding the right hand s plane. By the same reasoning used in the last chapter it can be shown that in this case the Nyquist encirclement criterion can be applied when s is varied along the imaginary axis only, i.e., for $s = i\omega$. KG(s) then become:

$$KG(\omega) = \frac{-g_2\omega^2 + i g_1 \omega + g_0}{(R+i\omega)(R_1+i\omega)} \qquad \text{II.11}$$

At $\omega = 0$ this becomes:

$$KG(0) = g_0/RR_1 \qquad \text{II.12}$$

As ω tends to $\pm\infty$ KG(ω) becomes:

$$KG(\omega) = g_2 e^{-i\Delta\omega} \qquad \text{II.13}$$

Thus as ω recedes from $+\infty$ and approaches $-\infty$, KG(ω) repeatedly describes a circle centered at the origin, of radius g_2 in the positive sense. Neither K(s) nor G(s) have poles in the right hand s plane. The first because it is designed via the Wiener procedure, the second because it is a realizable plant. It follows that their product also does not posses poles in the right hand s plane and so the number of positive encirclements of the point + 1 + i0 by KG(s) as s traverses the whole imaginary axis equals the number of zeros of 1 - KG(s) in the right hand s plane. From the behaviour of KG(ω) one concludes, that if

$$g_2 > g_0 RR_1 \qquad 11.14$$

the necessary and sufficient conditons for H(s) to be realizable is $g_2 < 1$. If on the other hand

$$g_2 < g_0 / RR_1 \qquad 11.5$$

the condition $g_2 < 1$ is necessary for realizability but is not sufficient. The additional condition $g_0/RR < 1$ is certainly sufficient for realizability but not necessary and realizability must in this case be determined from a detailed mapping of KG(ω). In all cases of interest for our purposes g_2 became 1 while g_0/RR_1 was less than 1. The value of 1, 1_r, corresponding to the limit of realizability was then that at which $g_2 = 1$ and this was determined by a search procedure on 1 for each set of values Δ, ν_0, ν_1 and x of interest. Some of the results are presented in tables 11.1 and 11.2 for Δ = 0.5 and 1.5 respectively for values of ν_0, ν_1 and x which prove to be of interest in the performance studies.

The Performance of the Two Parameter Disturbance Design

The performance of the two parameter disturbance Wiener design will now be evaluated for the regular distubance used before.

Table 11.1

Realizability Limits, l_r, on the Two Parameter Disturbance Wiener Feedback Controller.

$\Delta = 0.5$

ν_1 / ν_0	X	0.1	1.0	10
0.01	0.1	0.257	0.142	0.010
	1.0	0.245	0.115	0.005
	10.	0.240	0.108	0.004
0.1	0.1		0.188	0.122
	1.0		0.130	0.009
	10.		0.110	0.005
1.0	0.1	0.111		0.074
	1.0	0.130		0.022
	10.	0.188		0.005

Table 11.2

Realizability Limit, l_r, for the Two Parameter Disturbance Wiener Feedback Controller.

$\Delta = 1.5$

ν_0 \ ν_1	χ	0.1	1.0	10
0.01	0.1	0.455	0.170	0.010
	1.0	0.418	0.106	0.002
	10.	0.402	0.089	0.001
0.1	0.1		0.277	0.188
	1.0		0.140	0.007
	10.		0.0093	0.002
1.0	0.1	0.093		0.050
	1.0	0.140		0.009
	10.	0.277		0.001

Equations 10.27, 10.28 and 10.29 are again applicable with $|1 - KG(s)|^2$ and $|K(s)|^2$ replaced by their current expression. The integration then becomes:

$$\sigma_z^2 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\left[1 + \frac{(g_2 s^2 + g_0)^2 - g_1^2 s^2}{(R^2 - s^2)(R_1^2 - s^2)} - 2\frac{(g_2 s^2 - g_1 s + g_0)e^{\Delta s}}{(R - s)(R_1 - s)}\right]\frac{2\nu\, ds}{(1 - s^2)(\nu^2 - s^2)} \qquad 11.16$$

$$\sigma_m^2 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{(g_2 s^2 + g_0)^2 - g_1^2 s^2}{(\nu^2 - s^2)(R^2 - s^2)(R_1^2 - s^2)}\, 2\nu\, ds \qquad 11.17$$

Again both of these integrals are evaluated by using the residue theorem with the results:

$$\sigma_z^2 = \frac{1}{\nu + 1} + A\frac{\nu}{\nu^2 - R^2} + B\frac{\nu}{\nu^2 - R_1^2} + C\frac{\nu}{\nu^2 - 1} + \frac{(\nu^2 g_2 + g_0)^2 - \nu^2 g_1^2}{(R^2 - \nu^2)(R_1^2 - \nu^2)(1 - \nu^2)}$$

$$- 2\frac{(\nu^2 g_2 + \nu g_1 + g_0)e^{-\Delta\nu}}{(R + \nu)(R_1 + \nu)(1 - \nu^2)} \qquad 11.18$$

$$\sigma_m^2 = A\frac{\nu(1 - R^2)}{(\nu^2 - R^2)} + B\frac{\nu(1 - R_1^2)}{(\nu^2 - R_1^2)} + \frac{(\nu^2 g_2 + g_0)^2 - \nu^2 g_1^2}{(R^2 - \nu^2)(R_1^2 - \nu^2)} \qquad 11.19$$

where the constants A, B, and C depend only on the design parameters Δ, ν_0, ν_1, x and l are specified in the appendix. A variety of two disturbance designs were tried. For a plant with Δ = 1,5 the objective was to improve the performance for the low disturbance range, over that of the single parameter disturbance design for ν_0 = 0.01 at l_r. A successful design is shown in figure 11.1 where ν_0 =0.01, ν_1 =10, x = 0.1.

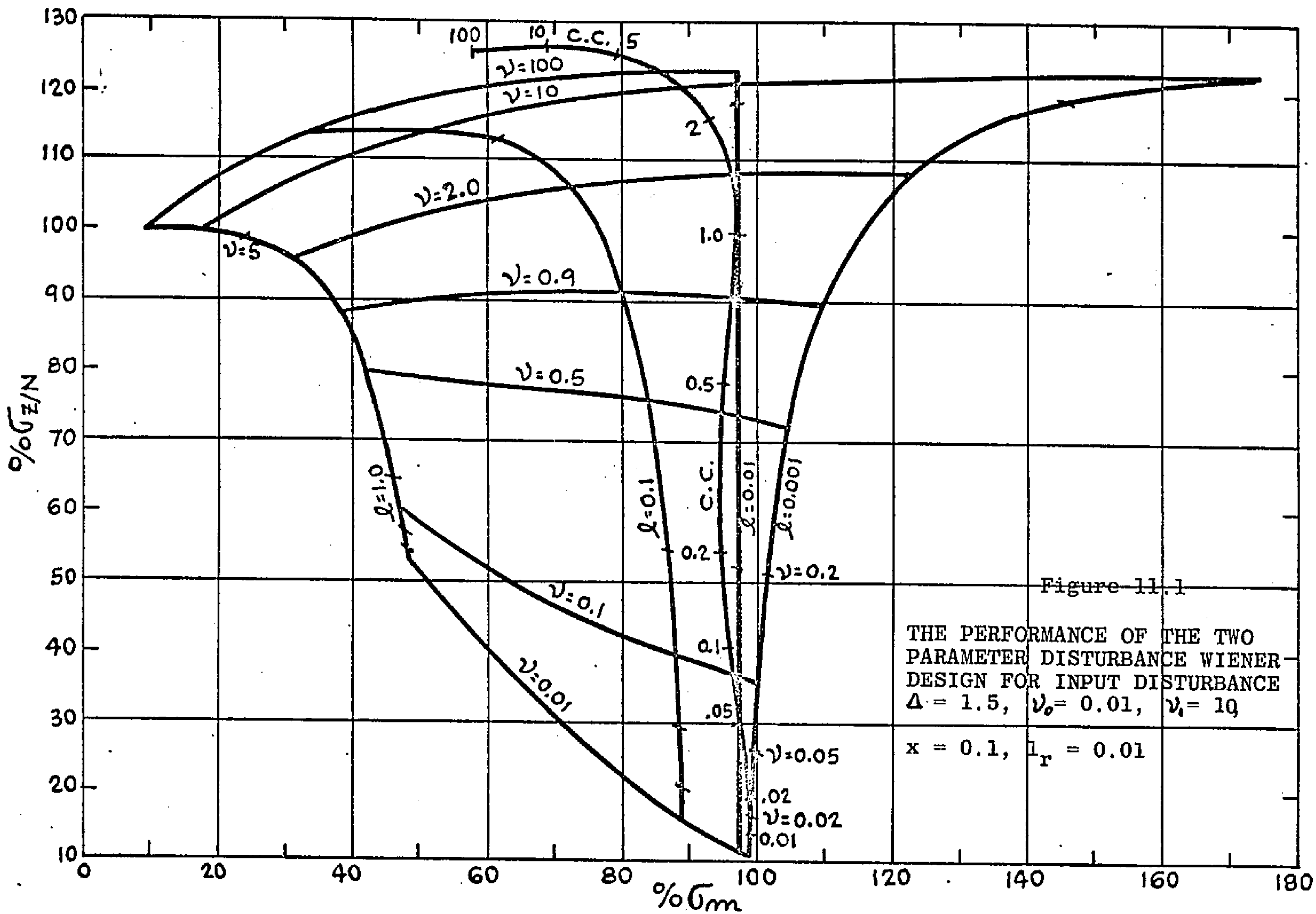


Figure 11.1

THE PERFORMANCE OF THE TWO PARAMETER DISTURBANCE WIENER DESIGN FOR INPUT DISTURBANCE $\Delta = 1.5$, $\nu_0 = 0.01$, $\nu_1 = 10$

$x = 0.1$, $l_r = 0.01$

When compared with the single parameter disturbance design for $\nu_0 = 0.01$ it is seen that adding the higher frequency content disturbance to the design results in an increase of the allowable controller effort for low ν distrubances. The output rms for low ν disturbances of this design is hardly hindered by realizability from reaching its potential which is not substentially larger than that of the single parameter disturbance design of $\nu_0 = 0.01$. The Wiener controller at 1_r has a lower output rms than that of the optimal step controller over the comlete disturbance range. In their useful region, that is where their output rms is lower than that of the uncontrolled plant, the two controllers use comparable control effort. However, the improvement of the output rms, by the Wiener controller in the lower ν disturbance region occurred at the expense of a deterioration in the $\nu = 1.0$ neighborhood as compared to the $\nu_0 = 0.01$ 1_r design for a single parameter disturbance. This makes the difference in performance between the two parameter disturbance Wiener design and the step controller too small to justify the more complex Wiener controller.

The two parameter disturbance design of $\nu_0 = 0.01$, $\nu_1 = 1.0$, $\chi = 0.1$ shown in figure 11.2 retains the output performance of the single parameter disturbance design of $\nu_0 = 0.01, 1_r$, in the middle ν region, while maintaining an output rms of 21% at $\nu = 0.01$ for the realizability limit controller. Although the output rms of this controller is somewhat higher than that of the conventional one up to a ν of 0.1 its output perfor-

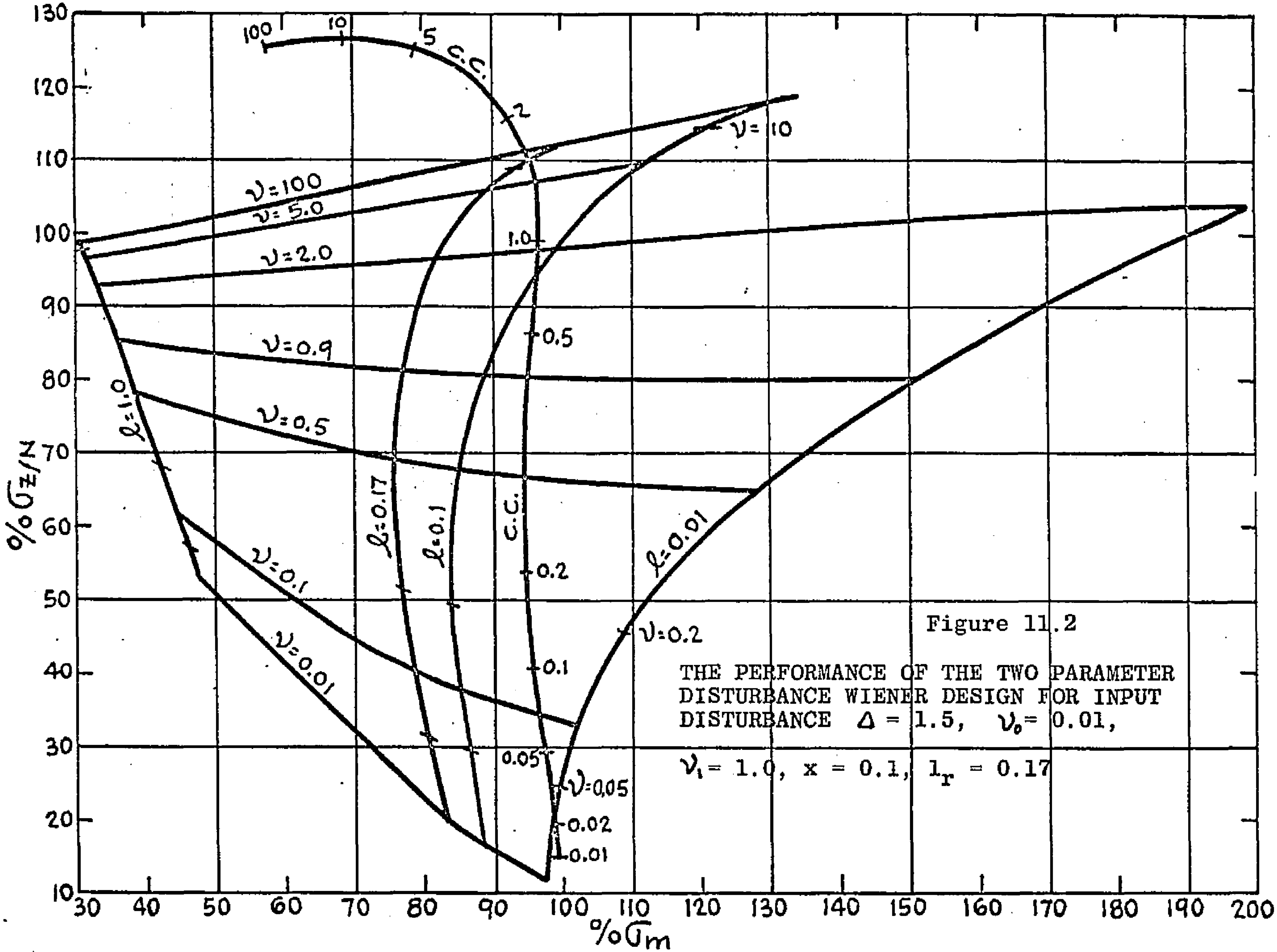


Figure 11.2

THE PERFORMANCE OF THE TWO PARAMETER DISTURBANCE WIENER DESIGN FOR INPUT DISTURBANCE $\Delta = 1.5$, $\nu_0 = 0.01$,

$\nu_1 = 1.0$, $x = 0.1$, $l_r = 0.17$

mance is significantly better than that of the optimal step controller in the $\nu = 1.0$ disturbance region. Since it also uses a 20% smaller controller effort to achieve this it is an attractive overall controller when used with a low pass filter on its input to produce no control performance with respect to high ν disturbances. It is therefore seen that the two parameter disturbance design succeed in producing a significantly better overall controller than the optimal step controller.

For a plant with $\Delta = 0.5$ the objective is again the improvement of the performance for low ν disturbances. In this case, however, better than no control performance can be achieved for high ν disturbances with a single parameter distrubance design for that range. Therefore, the performance over the complete disturbance range must be considered when choosing a controller and masking of high frequency content disturbances from the controller might not be acceptable. Figure 11.3 shows the design for $\nu_0 = 0.01$ $\nu_1 = 10$ and $x = 0.1$ which at $l = l_r$ has the same output rms as the Cohen and Coon controller in the $\nu = 0.01 - 0.1$ disturbance range together with a smaller controller effort. From a disturbance with $\nu = 0.1$ and up, the output performance is superior to that the Cohen and Coon controller with considerable smaller controller effort up to a disturbance of $\nu = 10$. At the high ν end of the disturbance range the output rms is somewhat higher than that achieved with the single parameter disturbance

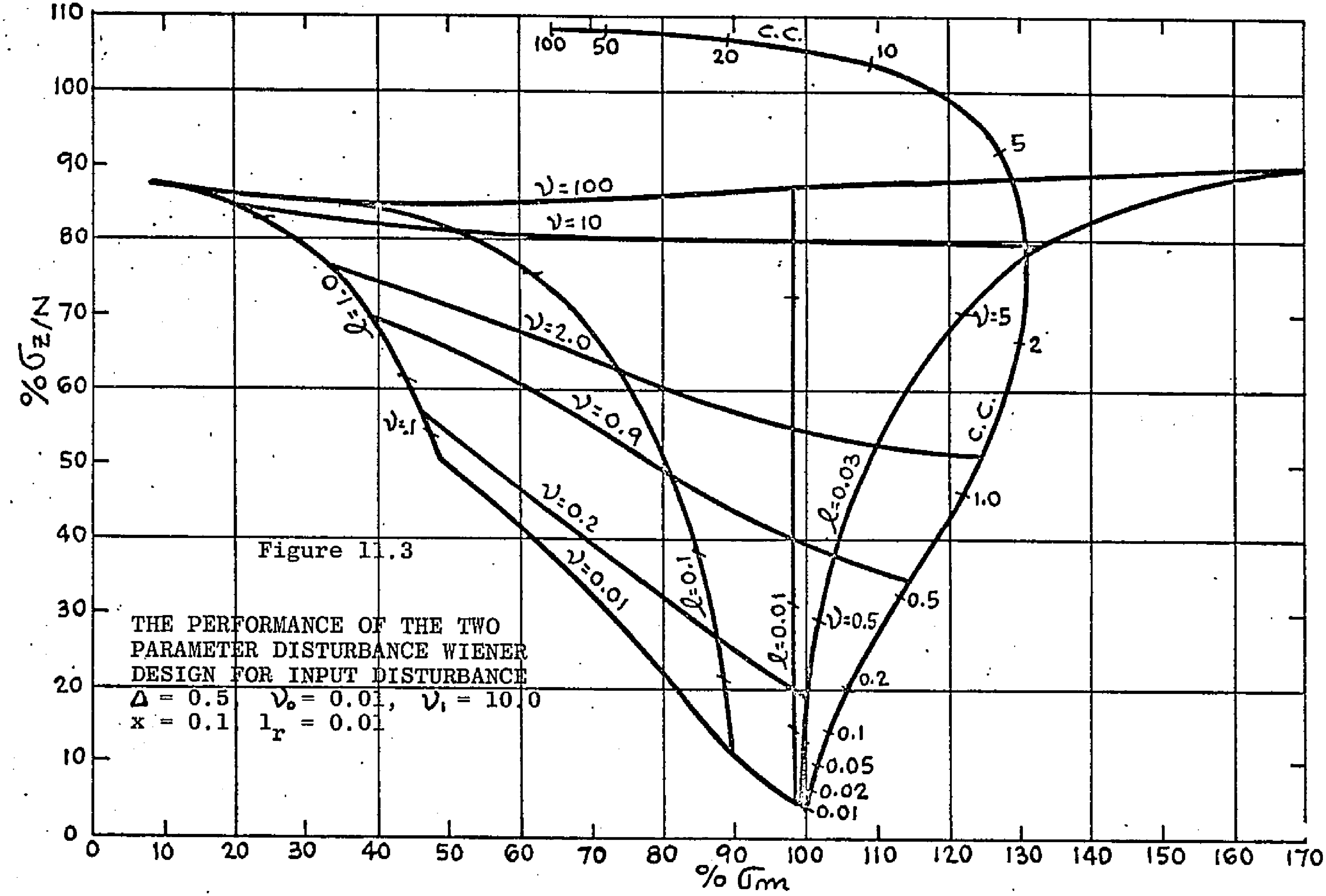


Figure 11.3

THE PERFORMANCE OF THE TWO PARAMETER DISTURBANCE WIENER DESIGN FOR INPUT DISTURBANCE

$\Delta = 0.5$, $\nu_0 = 0.01$, $\nu_1 = 10.0$

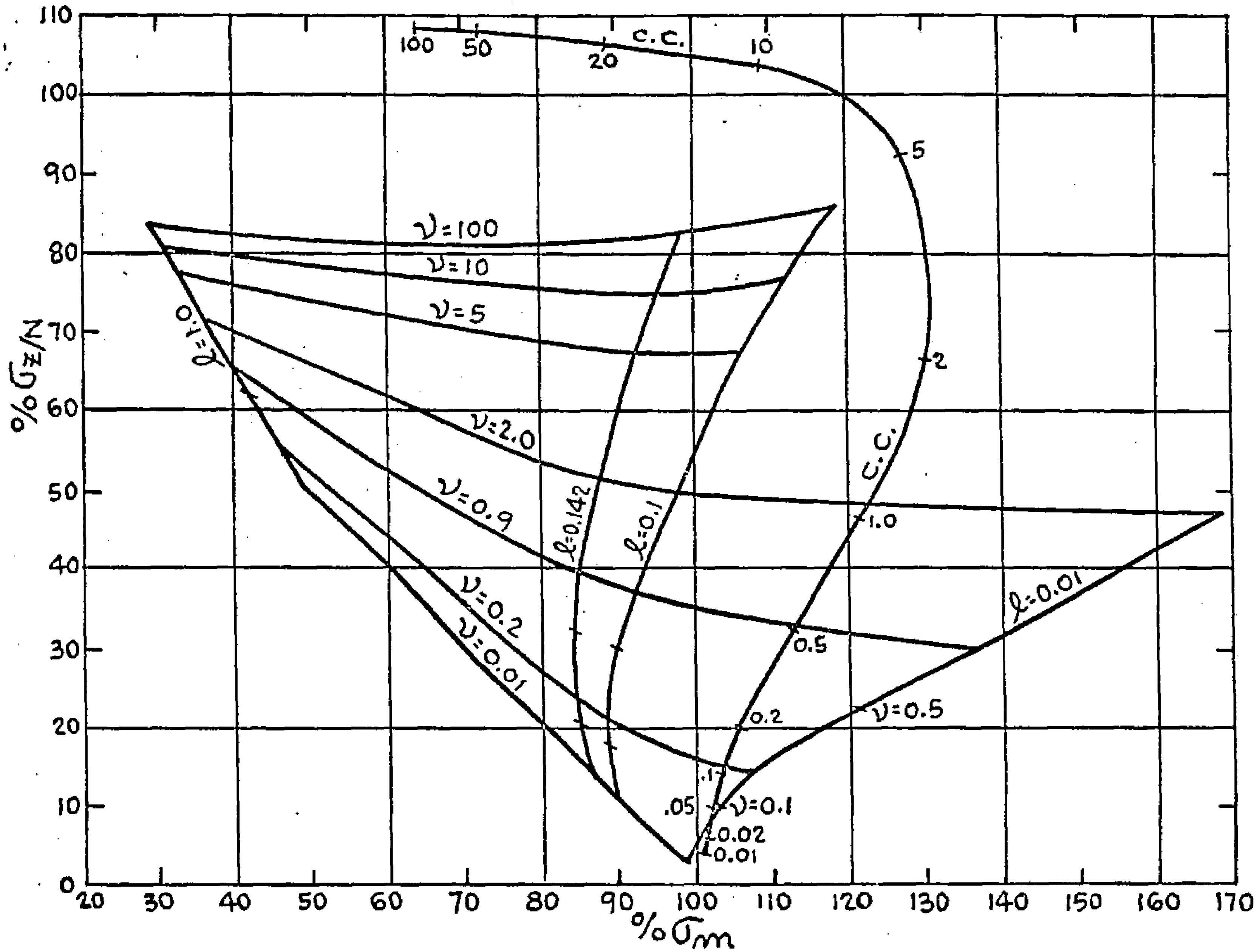
$x = 0.1$, $l_r = 0.01$

Wiener controller at its realizability limit. A considerably more attractive overall controller is shown in figure 11.4 for the design parameters $\nu_0 = 0.01$, $\nu_1 = 1.0$, x = 0.1. Although its low ν output rms is somewhat higher than that of the optimal step controller its output perfomance in the rest of the range is considerably better than the design of figure 11.3. In the middle to high characteristic frequency disturbance range its output performance is comparable to the single parameter disturbance design for $\nu_0 = 1.0$ at l_f which was hardly limited by its realizability conditions. In addition, the controller effort of this two parameter disturbance design is considerably smaller than that of the conventional one and thus makes it an attractive overall design.

From these examples it is seen that the good low ν disturbance performance of the Cohen and Coon controller can be matched by a Wiener design for a two parameter disturbance. This is done by adding an uncorrelated high frequency content disturbance to a low ν design disturbance. It allows the use of higher controller effort for low ν disturbances without the feedback controllerbecoming unrealizable and the full potential of the controller in this region is usable.

This is achieved, however, with the penalty of worse output rms than the single disturbance design in the middle ν region and brings it close to the conventional controllers output performance. A compromise in output rms for the low region

Figure 11.4 THE PERFORMANCE OF THE TWO PARAMETER WIENER DESIGN FOR INPUT DISTURBANCE $\Delta = 0.5$, $\nu_i = 0.01$, $\nu_1 = 1.0$, $x = 0.1$, $l_r = 0.142$



allows the two parameter disturbance design to be a very attractive overall design with good performance in the middle and high disturbance characteristic frequency regions. For a plant with the longer delay of $\Delta = 1.5$ a low pass filter is necessary in conjuction with this design.

B. Output Disturbance Configuration

The two parameter disturbance design

Even though the motivation for a two parameter disturbance design came from the performance of a single parameter disturbance Wiener design for a plant with input disturbance, the two parameter disturbance design will be applied to the output disturbance case for completeness. The same development as in the input disturbance case will be followed with $S_{yy}(s)$ the actual disturbance spectrum taken for $S_{uu}(s)$, the effective input spectrum for the derivation, i.e.:

$$S_{uu}(s) = S_{yy}(s) = \frac{2(R_1^2 - s^2)(\nu_0 + x\nu_1)}{(\nu_0^2 - s^2)(\nu_1^2 - s^2)} \qquad 11.20$$

R_1, R and x have the same meaning in the equation 11.20 as they do in the last section. Again identifying $\Delta(s)$ and $\Gamma(s)$ for this case as:

$$\Delta(s) = 2\ell \frac{(\nu_0 + x\nu_1)(R_1^2 - s^2)(R^2 - s^2)}{(1 - s^2)(\nu_0 - s^2)(\nu_1^2 - s^2)} \qquad 11.21$$

$$\Gamma(s) = \frac{(R_1^2 - s^2)(\nu_0 + x\nu_1)2e^{\Delta s}}{(\nu_0^2 - s^2)(\nu_1^2 - s^2)(1 - s)} \qquad 11.22$$

$\gamma(s)$ becomes:

$$\gamma(s) = \frac{(R_1+s)e^{\Delta s}}{\ell(\nu_0+s)(\nu_1+s)(R-s)} \qquad 11.23$$

inverting it for $t+\Delta > 0$ and transforming the result over the positive time axis yields $\gamma_+(s)$:

$$\gamma_+(s) = \frac{g_1 s + g_0}{(\nu_0+s)(\nu_1+s)} \qquad 11.24$$

where in this case g_1 and g_0 are defined as

$$g_1 = f_0 + f_1 \,, \qquad g_0 = f_0\nu_1 + f_1\nu_0 \qquad 11.25$$

$$f_0 = \frac{(R_1-\nu_0)e^{-\Delta\nu_0}}{\ell(R+\nu_0)(\nu_1-\nu_0)} \,, \qquad f_1 = \frac{(R_1-\nu_1)e^{-\Delta\nu_1}}{\ell(R+\nu_1)(\nu_0-\nu_1)} \qquad 11.26$$

The resulting equivalent cascaded compensator K(s) becomes:

$$K(s) = \frac{(s+1)(g_1 s + g_0)}{(R+s)(R_1+s)} \qquad 11.27$$

For the feedback controller H(s) to be physically realizable 1 - KG(s) should have no zeros in the right hand s plane. For each set of design parameters, Δ, ν_0, ν_1 and x the realizability of the unconstrained design, i.e. l = 0, will be investigated. It is then assumed that the realizability of the unconstrained controller assures that of the constrained design. This assumption, which seems sound intuitively, was borne out in the realizability analysis for the single parameter disturbance design for the output disturbance configuration. Also, in both single and two parameter disturbance designs for the input disturbance case, realizability imposes a lower limit on l. The analysis is simplified by considering the unconstrained design and in all cases of interest here the feedback controller was realizable for this design. In this way the search for a limiting value of l_r was unneccessary.

The limit of K(s) for l = 0 can be arrived at conveniently by mulitplying and dividing K(s) by $\sqrt{\ell}$ and taking the limit of the result as l tends to zero. Remembering that:

$$R = (1 + 1/\ell)^{1/2} \tag{11.28}$$

it follows that

$$\lim_{\ell \to 0} \sqrt{\ell} R = 1 \tag{11.29}$$

so that the limit of the denomiator of K(s) becomes, after multiplication by $\sqrt{\ell}$:

$$\lim_{\ell \to 0} \sqrt{\ell}(R+S)(R_1+S) = R_1 + S \tag{11.30}$$

Similarly it can be seen from the definition of g_o and g_1 that the following limits:

$$\lim_{\ell \to 0} \sqrt{\ell}\, g_0 = g_0' , \qquad \lim_{\ell \to 0} \sqrt{\ell}\, g_1 = g_1' \tag{11.31}$$

exist and can be written as

$$g_0' = f_0' \nu_1 + f_1' \nu_0 , \qquad g_1' = f_0' + f_1' \tag{11.32a}$$

where f_o' and f_1' become:

$$f_0' = \frac{(R_1 - \nu_1)}{(\nu_1 - \nu_0)} e^{-\Delta \nu_0} , \qquad f_1' = \frac{(R_1 - \nu_1)}{(\nu_0 - \nu_1)} \tag{11.32b}$$

K(s) for 1 = o then becomes

$$\lim_{\ell \to 0} K(s) = \lim_{\ell \to 0} \frac{\sqrt{\ell}\,(g_1 s + g_0)(1+s)}{\sqrt{\ell}\,(R+s)(R_1+s)} = \frac{(g_1' s + g_0')(1+s)}{(R_1+s)} \qquad 11.33$$

The realizability condition then prohibits right hand s plane roots of the equation KG(s) = 1, i.e.:

$$\frac{(g_1' s + g_0')}{R_1 + s}\, e^{-\Delta s} = 1 \qquad 11.34$$

The equation which determines realizability for the single disturbance design in the input disturbance configuration was of the same form, namely

$$\frac{(g_0 + g_1 s)}{(R+s)}\, e^{-\Delta s} = 1 \qquad 11.35$$

Naturally the criteria developed there hold for the present case. It is recalled that a sufficient condition in that case for no roots of equation 11.35 to exist in the right hand s plane was

$$g_0/R < 1 \quad \text{and} \quad g_1 < 1 \qquad 11.36$$

This condition becomes for the present case:

$$g_0'/R_1 < 1 \quad \text{and} \quad g_1' < 1 \qquad 11.37$$

In all cases of interest which were studied this sufficient condition was fulfilled and therefore the feedback controller was realizable.

The Performance of the Two Parameter Disturbance Design

The variances of the output and controller output are again evaluated by integrating their respective spectra over the imaginary s axis:

$$\sigma_z^2 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} |1 - KG(s)|^2 S_{\gamma\gamma}(s)\,ds$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \left\{ 1 + \frac{g_0^2 - g_1^2 s^2}{(R^2 - s^2)(R_1^2 - s^2)} - \frac{2(g_0 + g_1 s)e^{-\Delta s}}{(R+s)(R_1+s)} \right\} \frac{2\nu}{(\nu^2 - s^2)}\,ds \qquad 11.38$$

$$\sigma_m^2 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} |K(s)|^2 S_{\gamma\gamma}(s)\,ds = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{(g_0^2 - g_1^2 s^2)(1 - s^2)\,2\nu\,ds}{(R^2 - s^2)(R_1^2 - s^2)(\nu^2 - s^2)} \qquad 11.39$$

As in the previous cases these integrals were evaluated by using the residue theorem with the results:

$$\sigma_z^2 = 1 + A\frac{\nu}{\nu^2 - R^2} + B\frac{\nu}{\nu^2 - R_1^2} + \frac{g_0^2 - g_1^2\nu^2}{(R^2 - \nu^2)(R_1^2 - \nu^2)} - \frac{2(g_1\nu + g_0)e^{-\Delta\nu}}{(R+\nu)(R_1+\nu)} \qquad 11.40$$

$$\sigma_m^2 = A\frac{\nu(1 - R^2)}{\nu^2 - R^2} + B\frac{\nu(1 - R_1^2)}{\nu^2 - R_1^2} + \frac{(g_0^2 - g_1^2\nu^2)(1 - \nu^2)}{(R^2 - \nu^2)(R_1^2 - \nu^2)} \qquad 11.41$$

where A and B are constants which depend on the design parameters Δ, ν_0, ν_1, X and 1:

$$A = \frac{g_0^2 - g_1^2 R^2}{R(R_1^2 - R^2)} \quad ; \quad B = \frac{g_0^2 - g_1^2 R_1^2}{R_1(R^2 - R_1^2)} \qquad 11.42$$

The performance surfaces over a wide range of parameters were calculated for this case and compared with those of the single parameter disturbance Wiener design. In most combinations of parameters the two parameters disturbance design did not seem to offer any advantages over the single parameter disturbance design. The two parameter disturbance design with $\nu_0 = 0.01$, $\nu_1 = 1.0$, x = 0.1 for example does, however, have some attractive features for both $\Delta = 0.5$ and $\Delta = 1.5$ plants. The corresponding surfaces are shown in figures 11.5 and 11.6. It was seen in the discussion of the single parameter disturbance Wiener controller for a $\Delta = 0.5$ plant that a controller which has low output rms for low ν disturbances, i.e. the l = 0.1 controller in figure 9.5, would involve quite large control efforts in the middle ν disturbance range. Limiting the controller effort at that range, although not harming the output rms at this range, causes an appreciable increase in the output performance in the lower ν disturbance range. The reason for this is the comparatively shallow slope of the constant l lines. With the two parameter disturbance design of figure 11.5 steeper slopes are achieved up to a disturbance characteristic frequency of about 2.0. For higher ν disturbances the curves have a lower peak than that of the single parameter disturbance design. The controller of l = 0.1 achieves practically the same output rms as the single disturbance design $\nu_0 = 0.01$ l = 0.1, of figure 9.5 but at an appreciablby smaller controller effort. The most seizable difference in the output performance of the two controllers is in the low end of the characteristic frequency range. It does seem to offer a better overall controller, in conjuction with a low pass filter than does the single parameter disturbance design.

A similar advantage can be realized with the $\nu_0 = 0.01, \nu_1 = 1.0$,

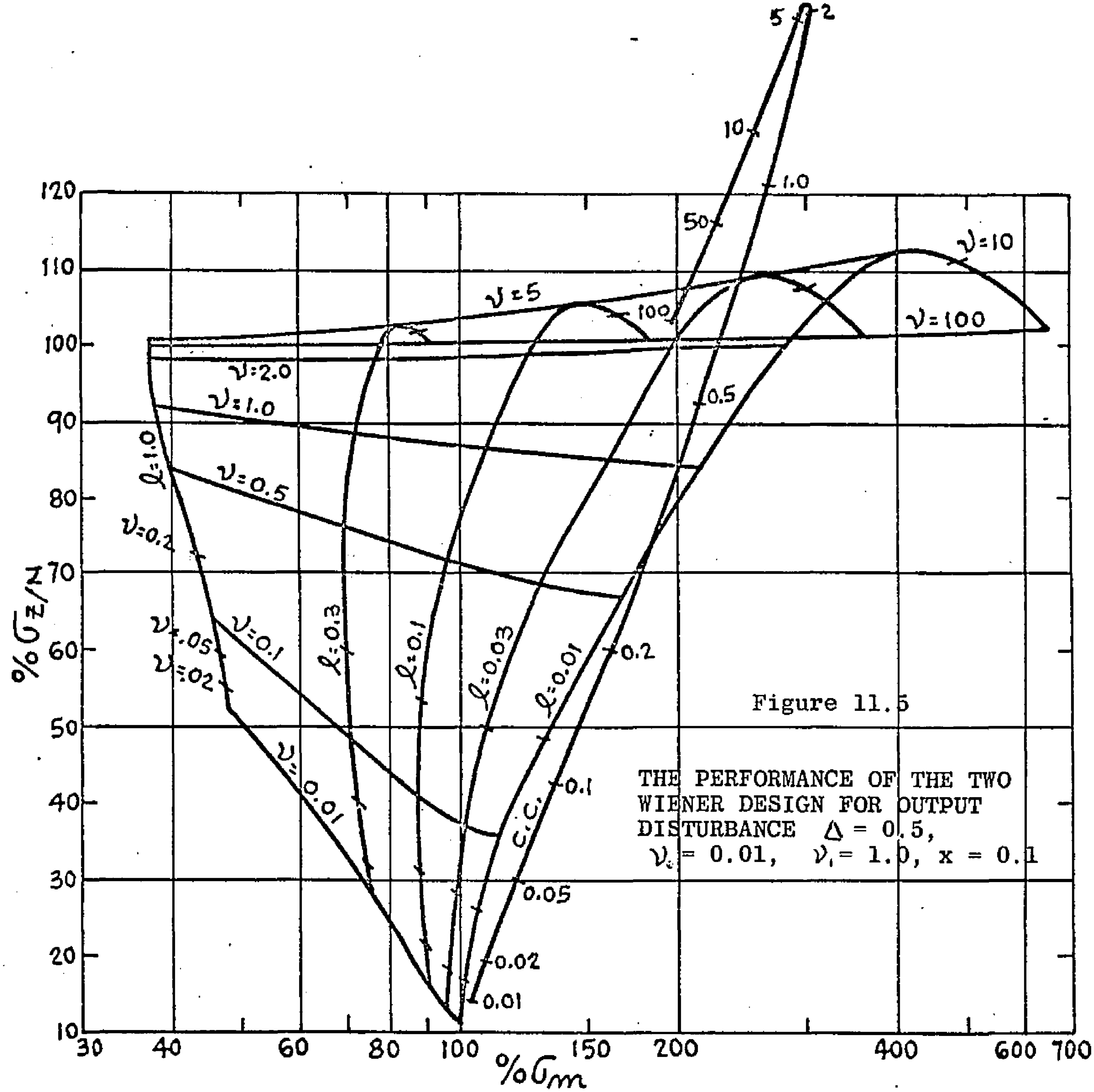


Figure 11.5

THE PERFORMANCE OF THE TWO WIENER DESIGN FOR OUTPUT DISTURBANCE $\Delta = 0.5$, $\nu_c = 0.01$, $\nu_1 = 1.0$, $x = 0.1$

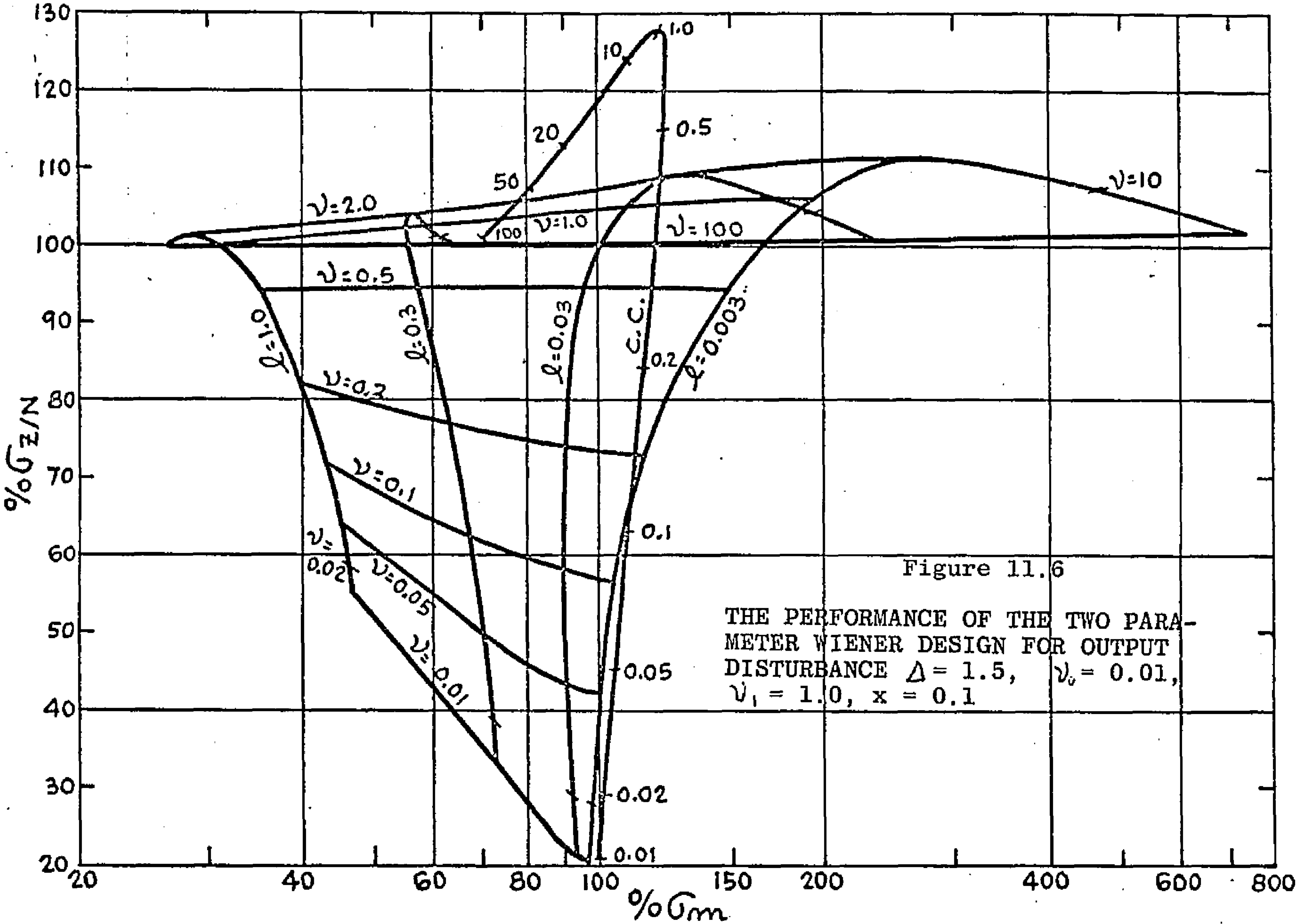


Figure 11.6

THE PERFORMANCE OF THE TWO PARAMETER WIENER DESIGN FOR OUTPUT DISTURBANCE $\Delta = 1.5$, $\nu_0 = 0.01$, $\nu_1 = 1.0$, $x = 0.1$

$x = 0.1$ design for a $\Lambda = 1.5$ plant, the effect, however, is less pronounced. Note that although the recommended design parameters for an overall Wiener controller are the same as for the input disturbance configuration, the controller is quite different.

C. Step Response Characteristic of the Wiener Designs

In this section the main characteristics of the step response of the Wiener designed systems will be described. Whereas, ordinary disturbances are best described as random process occasional step type disturbances might occur in the process industry. These might result from equipment failures which are not detected immediately and against which a controller must provide adequate protection . In particular the offset of the step response, that is the steady state response, must be sufficiently small. In the conventional controller the integral action provides for a zero offset response. The largest controller action to be expected as a result of a step disturbance is also important for controllers of limited magnitude.

The general relationship between the input y(s) and output z(s) and controller output m(s) are shown in chapter 3 to be :

$$z(s) = [1 - KG(s)]\,u(s) \qquad 11.43$$

and

$$m(s) = -K\,u(s) \qquad 11.44$$

where

$$\left.\begin{aligned} u(s) &= Gy(s) && \text{for input disturbances} \\ u(s) &= y(s) && \text{for output disturbances} \end{aligned}\right\} \quad 11.45$$

Both transfer functions K(s) and G(s) have no poles in the right hand s plane. The first because it was designed by the Wiener method and the second because it represents a realizable plant. Hence, these functions also represent the Laplace transforms of their respective weighing functions. The Laplace transfer functions are more convenient to deal with when evaluating the step response and in particular use can be made of the initial and final value theorems.

In what follows, the response to a unit step of the Wiener designed systems of chapters 9, 10 and 11 will be described with particular attention to the offset of their output and controller action magnitude.

a. Single parameter disturbance design for input disturbances. In this case the transfer function of the responses to a unit step, 1/s, become:

$$Z(s) = \frac{e^{-\Delta s}}{s(1+s)} - \frac{(g_0 + g_1 s)\,e^{-2\Delta s}}{s(1+s)(R+s)} \qquad 11.46$$

From this the time response becomes:

$$z(t) = [1 - e^{-(t-\Delta)}]u(t-\Delta) - \left\{ \frac{g_0 - g_1}{R-1}[1 - e^{-(t-2\Delta)}] \right.$$

$$\left. + \frac{g_0 - g_1 R}{R(1-R)}[1 - e^{-R(t-2\Delta)}] \right\} u(t-2\Delta) \qquad 11.47$$

where u(t) is the unit step function. Since $g_0 > g_1$ and $R > 1$ both coefficients in the curly brackets of equation 11.47 are positive. It follows that from the time, $t = 2\Delta$, that the effect of corrective action on z(t) can be observed, it consists of a sum of two fast decaying exponential. The exponents of these decays are 1 and R which is always larger than 1. The decay rate used in the Cohen and Coon controller corresponds to an exponent of 1.38. The magnitude of the output deviation up to that point depend on the plant's delay only. The offset can be determined from this response as t tends to infinity, or directly from the transfer function by the use of the final value theorem. It is seen to be $1 - g_0/R$.

The controller action response to a step input in this case is:

$$m(s) = -(g_0 + g_1 s)e^{-\Delta s}/[s(R+s)] \qquad 11.48$$

The corresponding time response therefore becomes:

$$m(t) = -[g_1 - (g_1 - g_0/R)(1 - e^{-R(t-\Delta)})]u(t-\Delta) \qquad 11.49$$

This represents a decaying exponential starting with the value of g_1 at $t = \Delta$ and a steady state value of g_0/R. The highest value is the initial response g_1 and this could have been evaluated from the transform directly by use of the initial value theorem.

These two responses are quite acceptable. Their two important features in this context, i. e. the offset and the largest controller action can be readily evaluated from the transfer functions directly. The offset $1 - g_0/R$ and the largest controller action g_1 are the parameters which also determine the realizability of the feedback controller. In chapter 9 the realizability conditons on 1 was the point where $g_1 = 1$ at which g_0/R was always less than unity. A few values of these two qualities are given in table 11.3 for the designs whose performance was presented in chapter 9 and 1 values in the range corresponding to a realizable feedback controller. It is seen that for a realizable controller the largest controller action is always smaller than the step size and decreases with decreasing controller effort, i. e. an increase of 1. With the increase of 1 an increase of offset is observed. The smallest offset in table 11.1, i.e. for $\nu_0 = 0.01$ at $1 = 0.4$ for a plant with $\Delta = 0.5$, is usually unacceptable in practice.

The lower offset obtained from the design for a lower frequency content disturbance is to be expected. As described in chapter

Table 11.3

Parameters of the Single Disturbance Wiener Design for Input Disturbances

	$\Delta=0.5$, $\nu_o=0.01$		$\Delta=0.5$, $\nu_o=1.0$		$\Delta=1.5$, $\nu_o=0.01$		$\Delta=1.5$, $\nu_o=1.0$	
ℓ	g_1	g_o/R	g_1	g_o/R	g_1	g_o/R	g_1	g_o/R
0.1							0.895	0.426
0.2			0.694	0.642			0.579	0.368
0.4	0.802	0.711	0.448	0.522			0.359	0.296
0.6	0.632	0.622	0.338	0.422	0.867	0.617	0.266	0.249
0.8	0.526	0.553	0.273	0.384	0.711	0.548	0.212	0.216
1.0	0.452	0.497	0.230	0.340	0.605	0.493	0.177	0.191

9 a low characteristic frequency disturbance might originate from constant random levels switching their values at Poisson distributed times with very low avarage switching frequency. Thus the system usually reaches equilibrium before the next disturbance change occurs. Under these circumstances the offset would contribute significantly to the variance and a Wiener design would attempt to reduce it.

b. Two parameter disturbance design for input disturbances. The transformed step responses of the system in this case are given by:

$$Z(s) = \frac{e^{-\Delta s}}{s(1+s)} - \frac{(g_2 s^2 + g_1 s + g_0)\, e^{-2\Delta s}}{s(1+s)(R+s)(R_1+s)} \qquad 11.50$$

and

$$m(s) = \frac{-(g_2 s^2 + g_1 s + g_0)}{s(R+s)(R_1+s)}\, e^{-\Delta s} \qquad 11.51$$

Both these responses again represent sums of delayed decaying exponentials in the time domaine. Using the final value theorem the offset is obtained from equation 11.50:

$$\lim_{t\to\infty} z(t) = \lim_{s\to 0} s Z(s) = 1 - g_0/RR_1 \qquad 11.52$$

The initial controller action is obtained from equation 11.51 by applying the initial value theorem to its rational portion. This will give the response at t = Δ :

$$m(\Delta) = g_2 \qquad 11.53$$

As in the single parameter disturbance case g_2 and g_o/RR_1, are the parameters determining the realizability of the feedback controller. In a previous section it was seen that $g_2 = 1$ determines the limiting value of l for realizability and that g_o/RR_1 is smaller than unity. A few values of these parameters in the l range corresponding to a realizable controller are given in table 11.4 for the cases presented earlier in this chapter. When l is increased the same trend observed with the single parameter disturbance design is seen in this case. Note, however, that much lower offset results in this case for designs that are close to their realizability limits. It is therefore seen that the improved output performance of these designs with respect to low characteristic frequency disturbances carries over to their step response offset. The designs with $\nu_1 = 1$ have a larger offset then those with $\nu_1 = 10$ reflecting their somewhat higher output rms for low frequency output disturbances. The offset for the recommended design, i. e., that of $\nu_1 = 1.0$ is acceptable in many situations.

c. Single parameter disturbance design for output disturbances. In the output disturbance configuration the Laplace transform for the system become:

$$Z(s) = \frac{1}{s} - g\frac{e^{-\Delta s}}{s(R+s)} \qquad 11.54$$

and

$$m(s) = -g\frac{1+s}{s(R+s)} \qquad 11.55$$

Table 11.4

<u>Parameters for a Two Parameter Disturbance Wiener Design for Input Disturbances</u>

	$\Delta=0.5$, $x=0.1$ $\nu_0=0.01$ $\nu_1=10.$		$\Delta=0.5$, $x=0.1$ $\nu_0=0.01$ $\nu_1=1.0$		$\Delta=1.5$, $x=0.1$ $\nu_0=0.01$, $\nu_1=10.$		$\Delta=1.5$, $x=0.1$ $\nu_0=0.01$, $\nu_1=1.0$	
ℓ	g_2	g_o/RR_1	g_2	g_o/RR_1	g_2	g_o/RR_1	g_2	g_o/RR_1
0.01	0.985	0.985			0.977	0.975		
0.02	0.693	0.975			0.687	0.965		
0.06	0.392	0.937			0.389	0.928		
0.2	0.202	0.827	0.830	0.826	0.200	0.818	0.908	0.814
0.4	0.132	0.708	0.553	0.707	0.131	0.701	0.585	0.696
0.8	0.082	0.550	0.347	0.549	0.081	0.544	0.359	0.541
1.0	0.070	0.495	0.295	0.494	0.069	0.490	0.303	0.486

The corresponding time functions are:

$$z(t) = u(t) - g/R[1 - e^{-R(t-\Delta)}]u(t-\Delta) \qquad 11.56$$

$$m(t) = - g/R[1 + (R-1)e^{-Rt}] \qquad 11.57$$

These equations reflect the fact that the step is applied at the plants output. Therefore there is an immediate output for both z(s) and m(t). These responses are again quite acceptable since they have the character of fast decaying exponentials. The offset and the largest control action magnitude are in this case 1 - g/R and g respectively. In the realizability discussion of chapter 9 it is shown that g/R is always less than unity. Since it is also positive the offset is always smaller than unity. These parameters, g and g/R are given in table 11.5 for the cases presented in chapter 9.

Again it is seen that the ν_o = 0.01 design leads to much lower offset for compatible magnitudes of the controller action. Even for very small l values and the associated large controller action magnitudes the offset of the ν_o = 1.0 design remains rather high. This design therefore in both input and output disturbance configurations cannot be used to accomodate an occasional step disturbance. It was seen in the predeeding two chapters that this design is useful for random disturbances only in the narrow range of characteristic

Table 11.5

Parameters of the Single Parameter Disturbance Wiener Design for Output Disturbances

ℓ	$\Delta=0.5$, $\nu_0=0.01$		$\Delta=0.5$, $\nu_0=1.0$		$\Delta=1.5$, $\nu_0=0.01$		$\Delta=1.5$, $\nu_0=1.0$	
	g	g/R	g	g/R	g	g/R	g	g/R
0.0003			34.42	0.596			12.66	0.219
0.001			18.58	0.587			6.84	0.216
0.01	9.89	0.984	5.49	0.546	9.79	0.974	2.02	0.201
0.03	5.65	0.965	2.95	0.503	5.60	0.955	1.08	0.185
0.1	2.99	0.902	1.41	0.424	2.96	0.893	0.52	0.156
0.3	1.59	0.762	0.66	0.315	1.57	0.754	0.24	0.116
1.0	0.7	0.494	0.25	0.178	0.79	0.489	0.09	0.065

frequencies around the design value. One can therefore hardly expect such a design to yield good step response characteristics which would be reflected in good performance of the controller when excited by low frequency disturbances.

d. Two parameter disturbance design for output disturbances. Finally the step response charactereistics of the two parameter disturbance Wiener design for the output disturbance configuration will be described. The Laplace transforms of the step responses in this case are:

$$z(s) = \frac{1}{s} - \frac{(g_1 s + g_0)\, e^{-\Delta s}}{s(R+s)(R_1+s)} \qquad 11.58$$

$$m(s) = -\frac{(g_1 s + g_0)(1+s)}{s(R+s)(R_1+s)} \qquad 11.59$$

The response consists of constants and decaying exponentials properly delayed in the case of z(t). From the final value theorem the offset is seen to be $1 - g_0/RR_1$. The initial magnitude of the controller action, g_1, follows from equation 11.59 by using the initial value theorem. The values of these parameters for the cases presented earlier in this chapter are shown in table 11.6. Comparing these values with those of table 11.5 for $\nu_o = 0.01$ shows that for the same controller action magnitudes lower offset can be achieved with the two disturbance design. The recommended overall controllers for both plants, i.e. 1 = 0.1 and 1 = 0.03 for the plants with Δ = 0.5 and Δ = 1.5 respectively, both possess accep-

Table 11.6

Parameters of the Two Parameter Disturbance Wiener Design for Output Disturbances.

$\nu_o = 0.01$ $\nu_i = 1.0$ $x = 0.1$

	$\Delta = 0.5$		$\Delta = 1.5$	
1	g_1	g_o/RR_1	g_1	g_o/RR_1
0.003			5.02	0.907
0.01	5.85	0.941		
0.03	3.16	0.918	1.453	0.878
0.1	1.53	0.853		
0.3	0.73	0.718	0.351	0.692
1.0	0.29	0.458	0.142	0.451

table offset. These are slightly higher than those of the recommended overall single disturbance designs, i.e. $\nu_0 = 0.01$ with $1 = 0.1$ for both plants, but have a considerably lower control action magnitude.

In conclusion it might be noted that the general form of the step responses of systems with Wiener design controllers is satisfactory. All these responses have an offset whose magnitude becomes smaller as the system output performance improves for low characteristic frequency disturbances. Thus the single disturbance design at $\nu_0 = 1.0$ for both configuration show quite large offsets in response to a step. The single parameter disturbance designs for the input configuration at $\nu_0 = 0.01$ also shows unacceptable high offset. The two parameter disturbance designs produce an offset which would be acceptable in may instances. In the output disturbance configuration low offset can be obtained with the single parameter disturbance design at $\nu_0 = 0.01$. It does, however, require considerable controller action magnitude. Good offset at an acceptable controller action can again be achieved with the previously recommended two parameter disturbance design.

Occasionally the offset of the recommended Wiener designs might not be satisfactory and a smaller or zero offset mith be required. This could be achieved by adding a weak integral action, whose input is filtered with a very low pass

filter, is parallel to the Wiener controller. The integral action will produce zero offset whereas the filter will prevent the auxiliary controller from action on any but very low frequency content random disturbances and steps.
In the low characteristic frequency region where some overlap might occur between the auxiliary controllers action and that of the Wiener controller, its effect would not damage the random performance. In this region the optimal random performance is only slightly better than that of the conventional controller and for the recommended overall two parameter disturbance designs the output rms is somewhat higher than that of the conventional controller. The good performance of the conventional controller, in this disturbance range can be at least partly attributed to the integral action as noted in chapter 9. It seems therefore, that in the combined controller the respective virtues of the two portions will complement each other.

12. PROBABILITY DISTRIBUTIONS IN THE REGULATOR PROBLEM

The Problem

In the process industry the specification of controller performance objectives is usually in terms of the magnitude of the deviation from the desired output. For random outputs such a specification is expressed in terms of the deviation's probability distribution. The solution of control problems in terms of the probabilities is mathematically tractable only in a few simple situations. Furthermore, the probability distribution of the disturbance is rarely known and its determination requires considerable effort.

In the previous chapters , therefore, the output variance was used as a criterion for the controller performance and the controller output variance as an indication for the magnitude of its action. This was done mainly so that tractable mathemathics result and can be strictly justified only if the distributions of these random processes are determind by their variance and mean alone, and their type is known. It was, however shown in chapter 2 that the variance yields the lower limit for the deviations distribution through the use of Chebyshev's inequality.

It has been common practice in many engineering applications to assume a normal distribution in situations where the actual distribution is not known. This is then justified by reference to the central limit theorem of probability theory (10). Occasionally a convincing argument of this type can be made for disturbances in the process industry. It is known, see for example Parzen (9), that linear operations on a normal process preserve its normal form changing its parameters only. In the case of normal disturbances, therefore, to linear systems the outputs will be normal and their variance and mean will fully specify their probability distribution. In many cases, however, the assumption of normal distributions is not justified and the probability estimation for a variable to remain within specified limits on the basis of the normal distribution can be quite misleading. Although the use of the normal distribution is the recommended and used procedure (4, 11), no consideration was given to its validity.

It should be pointed out, however, that one would expect the output distribution of a linear control system to approach the normal one for disturbances with high frequency content as compared with the characteristic frequency of the system. This can be seen by considering the convolution of the systems input and weighing function as a sum by breaking the continuous integration variable into increments. The higher the frequency content of the disturbance, more segments will be needed over the stretch of time during which the weighing function is significantly different from zero. Also the contributions from these segments approach statistical independance with the increase in frequency content of the disturbance. Thus the output approaches the sum of independant variables whose number of terms becomes very large and these are the conditions satisfying the central limit theorem in its general form(10).

In what follows a simple example will be described in which the probability structure of the system can be solved for a concievable non normal disturbance. This will demonstrate that output and controller output distributions of drastically different character than the normal distribution can occur and that, in this case, they degenerate into the normal distribution for much too high frequency content disturbances for such an approximation to be of practical use.

The disturbance to be used will be a stationary Markov process. It will be applied to the input of a plant, corresponding to a single stirred tank reactor with no delay, around which a feedback control loop is closed. The feedback controller is designed according to Wiener with a constraint on the controller action as described in chapter 10. Under these circumstances the system output and disturbance jointly form a Markov process whose distribution can be solved for. From this joint distribution those of the output and manipulated variable can be obtained.

The Disturbance Process

The disturbance to be used for the following development is a stationary two state Markov process. A Markov process is a random process in which the probabilities of the process's future states are determined from its present alone,and are independant of past states. It is convenient in this context to deal with the transition probabilities from one state to another which for a homogenuous, or time independant, process depend only on the transition interval, but not on the absolute time of the transition. The definition of the Markov process leads directly to the Chapman-Kolmogorov equation which for a homogenuous discrete valued process becomes:

$$p_{ij}(s+t) = \sum_{k} p_{ik}(s)\, p_{kj}(t) \qquad 12.1$$

where $p_{ij}(t)$ is the transition probability from state i to j in the time interval t. This equation expresses the fact that the transition from an intermediate state k to the final stage j is independant from the transition from i to k and therefore the product of the corresponding transition porbabilities yields the probability of the overall transition. Equation 12.1 can be used to develop a differential equation for the transsition probability once the very short time behaviour of the transition probabilities is given. For a two state process with states 0, 1 whose levels are a_o and a_1 we define the following short time behaviour:

$$p_{01}(t) = \lambda_0 t \ , \quad \text{for } t \downarrow 0 \qquad 12.2a$$

and therefore

$$p_{00}(t) = 1-\lambda_0 t \,, \quad for \; t\downarrow 0 \qquad 12.2b$$

Similarely

$$p_{10} = \lambda_1 t \,; \quad p_{11}(t) = 1-\lambda_1 t \,; \quad for \; t\downarrow 0 \qquad 12.3$$

where λ_0 and λ_1 are constants and have the significance of state transition intensities or rates. Note that if $\lambda_0 = \lambda_1 = \lambda$ and the transitions themselves are considered the events of a random process, the above specifications are equivalent to those of a Poison process with parameter λ . Hence, this process is closely related to the process in chapter 2 which was formed by independantly distributed levels switching their values at Poisson distributed times.

Writing the Chapman-Kolmogorov equation for $p_{01}(t+s)$ and letting s the increment for the second transition tend to zero we get:

$$p_{01}(t+s) = p_{01}(t)\,p_{11}(s) + p_{00}(t)\,p_{01}(s)$$

$$= p_{01}(t)(1-\lambda_1 s) + p_{00}(t)\lambda_0 s \qquad for \; s\downarrow 0 \qquad 12.4$$

From the definition of the derivative equation 12.4 becomes:

$$\frac{d\,p_{01}(t)}{dt} = -\,p_{01}(t)\lambda_1 + p_{00}(t)\lambda_0 \qquad 12.5$$

In a similar way the differential equations for $p_{00}(t)$, $p_{10}(t)$ and $p_{11}(t)$ can be derived. Defining the matrices:

$$\underline{\underline{P}}(t) = \begin{bmatrix} p_{00}(t) & p_{01}(t) \\ p_{10}(t) & p_{11}(t) \end{bmatrix} \quad 12.6; \quad \underline{\underline{\Lambda}} = \begin{bmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{bmatrix} \qquad 12.7$$

The four equations can be put in matrix form:

$$\frac{d\,\underline{\underline{P}}(t)}{dt} = \underline{\underline{P}}(t)\,\underline{\underline{\Lambda}} \tag{12.8}$$

with the initial condition

$$\underline{\underline{P}}(0) = \underline{\underline{I}} \tag{12.9}$$

where $\underline{I}$ is the identity matrix.

Equation 12.8 which holds for any number of states is known as Kolmogorovs foreward differential equation and contains the complete probability structure of the process. It will be used to develop equations for the probabilities of the process states, i.e., for the single time probabilities, and for the joint two time probabilities. The latter are necessary for the calculation of the autocorrelation function of the process.

The probability of the process states can be obtained from the initial probabilities and the transition probabilities by using the Markov property. The probability of being in state i at time t, $p_i(t)$, is the sum of products of the initial probabilities $p_j(0)$ and the transition probabilities form state j to i in the increment t, summed over all initial states j. Thus:

$$p_i(t) = \sum_j p_j(0)\, p_{ji}(t) \tag{12.10}$$

In matrix notation this becomes:

$$\underline{p}^T(t) = \underline{p}^T(0)\,\underline{\underline{P}}(t) \tag{12.10a}$$

where $\underline{P}^T(t)$ is the transpose of the vector $\underline{P}(t)$. Pre-multiplying both sides of equation 12.8 by the constant vector $\underline{P}^T(0)$ results in:

$$\frac{d\,\underline{P}^T(t)}{dt} = \underline{P}^T(t)\,\underline{\underline{\Lambda}} \qquad 12.11$$

which is the vector equation for the desired porbabilities. The equation can be solved after the initial probabilities $\underline{P}(0)$ are specified. The stationary probabilities, $\overline{p}_i$, are obtained from equation 12.11 by setting the derivative equal to zero. These become for the two level process

$$\left.\begin{aligned} -\overline{p}_0\lambda_0 + \overline{p}_1\lambda_1 = 0 \\ \overline{p}_0\lambda_0 - \overline{p}_1\lambda_1 = 0 \end{aligned}\right\} \qquad 12.12$$

One of this pair of equations together with the condition that

$$\overline{p}_0 + \overline{p}_1 = 1$$

leads to:

$$\overline{p}_0 = \frac{\lambda_1}{\lambda_0 + \lambda_1}, \qquad \overline{p}_1 = \frac{\lambda_0}{\lambda_0 + \lambda_1} \qquad 12.14$$

The stationary probabilities therefore depend only on the ratio $x = \lambda_0/\lambda_1$. The mean, μ , and the variance σ^2 , therefore, become:

$$\mu = a_0\overline{p}_0 + a_1\overline{p}_1 = \frac{a_0\lambda_1 + a_1\lambda_0}{\lambda_0 + \lambda_1} \qquad 12.15$$

$$\sigma^2 = \frac{\lambda_0\lambda_1}{(\lambda_0+\lambda_1)^2}(a_0-a_1)^2 \qquad 12.16$$

For a zero mean process the variance becomes:

$$\sigma^2 = a_0^2\,\lambda_1/\lambda_0 = a_1^2\,\lambda_0/\lambda_1 \qquad 12.17$$

The additional condition for a unity variance leads to:

$$a_0 = -\sqrt{\lambda_0/\lambda_1} = -\sqrt{x}\,; \qquad a_1 = \sqrt{\lambda_1/\lambda_0} = \sqrt{1/x} \qquad 12.18$$

In this case, therefore both the stationary probabilities and the process levels are determined from the ratio of the transition intensities x.

To determine the autocorrelation function of the process its joint two time probabilities are necessary. By definition the autocorrelation function for a discreet state process is given by:

$$\varphi_{xx}(\tau) = \langle x_i(t)\,x_j(t+\tau)\rangle = \sum_i \sum_j x_i(t)\,x_j(t+\tau)\,w_{ij}(t,\tau) \qquad 2.19$$

where $w_{ij}(t,\tau)$ is the probability for the process to be in state i in time t and state j at time $t+\tau$. From the property of Markov processes this probability can be obtained from the product of the single time probability at time t, $p_i(t)$, and the transition probability from i to j in the interval τ i.e.:

$$w_{ij}(t,\tau) = p_i(t)\,p_{ij}(\tau) \qquad 12.20$$

Multiplying every row of $\underline{\underline{P}}$ in equation 12.8 by the corresponding probability $p_i(t)$ results in the differential equation for the two times probabilities:

$$\frac{\partial \underline{\underline{W}}(t,\tau)}{\partial \tau} = \underline{\underline{W}}(t,\tau)\,\underline{\underline{\Lambda}} \qquad 12.21$$

The initial conditions for $\tau = 0$ are clearly given by:

$$w_{ij}(t,0) = p_i(t)\,\delta_{ij} \qquad 12.22$$

where δ_{ij} is Kronecker's delta. In this case the random process is assumed to have reached stationarity and the initial probabilities are the stationary ones of equation 12.14. Omitting the argument t the four equations for the process considered become:

$$\frac{d w_{00}(\tau)}{d\tau} = -\lambda_0 w_{00}(\tau) + \lambda_1 w_{01}(\tau) \qquad 12.23$$

$$\frac{d w_{01}(\tau)}{d\tau} = \lambda_0 w_{00}(\tau) - \lambda_1 w_{01}(\tau) \qquad 12.24$$

$$\frac{d w_{10}(\tau)}{d\tau} = -\lambda_0 w_{10}(\tau) + \lambda_1 w_{11}(\tau) \qquad 12.25$$

$$\frac{d w_{11}(\tau)}{d\tau} = \lambda_0 w_{10}(\tau) - \lambda_1 w_{11}(\tau) \qquad 12.26$$

with the initial conditions:

$$\left.\begin{array}{ll} \omega_{00}(0) = \bar{p}_0 \;; & \omega_{01}(0) = 0 \\ \omega_{01}(0) = 0 \;; & \omega_{11}(0) = \bar{p}_1 \end{array}\right\} \qquad 12.27$$

These equations form two independant pairs and their solution is:

$$\omega_{00}(\tau) = p_0[\bar{p}_0 + \bar{p}_1 e^{-(\lambda_0+\lambda_1)\tau}] \qquad 12.28$$

$$\omega_{01}(\tau) = p_0[\bar{p}_1 - \bar{p}_1 e^{-(\lambda_0+\lambda_1)\tau}] \qquad 12.29$$

$$\omega_{10}(\tau) = \bar{p}_1[\bar{p}_0 - \bar{p}_0 e^{-(\lambda_0+\lambda_1)\tau}] \qquad 12.30$$

$$\omega_{11}(\tau) = \bar{p}_1[\bar{p}_1 + \bar{p}_0 e^{-(\lambda_0+\lambda_1)\tau}] \qquad 12.31$$

Forming the autocorrelation function according to equation 12.19 it becomes:

$$\varphi_{xx}(\tau) = \bar{p}_0\bar{p}_1(a_0 - a_1)^2 e^{-(\lambda_0+\lambda_1)\tau} + (a_0\bar{p}_0 + a_1\bar{p}_1)^2 \qquad 12.32$$

The coefficient of the exponential is the variance of the process and the second term on the right hand side of equation 12.32 is the square of its mean. For the case of a zero mean process and recalling that the autocorrelation function is even in τ it can be written as:

$$\varphi_{xx}(\tau) = \sigma^2 e^{-(\lambda_0+\lambda_1)|\tau|} \qquad 12.33$$

This is the familiar negative exponential autocorrelation function used throughout this work where the characteristic frequency ν equals the sum of the transition intensities $\lambda_0+\lambda_1$. From equation 12.18 and 12.33 it follows that for the zero mean unity variance two level process the ratio of the transition intensities x determines the process levels and its stationary probabilities whereas their sum specifies the autocorrelation function .

The Systems Output Probability Density

The two levels stationary Markov process of the last section will now be applied to the input of the typical plant of this work which is regulated with a Wiener feedback controller. Since Markov processes hold the prospect of a mathematically amenable access to their probability structure it is desirable to define the problem variables as Markovian.

From equations 3.18 - 3.20 it is clear that if the plant has a delay the current and future problem variables depend on their values during the proceeding delay period. It follows therefore that such variables cannot form a Markov process. The no delay situation will therefore be treated where the Wiener feedback controller, designed for a single parameter disturbance, reduces to a proportional - derivative action controller. The systems equations 3.18 - 3.20 then become:

$$\frac{dz(\theta)}{d\theta} = y(\theta) + m(\theta) - z(\theta) \qquad 12.34$$

$$m(\theta) = -\left[K_c z(\theta) + D\frac{dz(\theta)}{d\theta}\right] \qquad 12.35$$

These equations can be combined to give:

$$\frac{dz(\theta)}{d\theta} = g_0 y(\theta) - g_1 z(\theta) \qquad 12.36$$

where

$$g_0 = 1/(1+D), \qquad g_1 = (1+K_c)/(1+D) \qquad 12.37$$

Since the disturbance used is a Markov process, knowledge of the current state of $y(\theta)$ and $z(\theta)$ determines their future development. That is the joint process, $y(\theta)$, $z(\theta)$ is a Markov process. The differential equations governing the probabilities of a joint Markov process with a mixed continuous discrete state space, are derived in the appendix. Since only the stationary probabilities of the process are of interest here and the state space consists of one continuous variable and two discrete states, a somewhat simpler notation than that of the general case, described in the appendix, will be adopted. Thus:

$p_0(z) \equiv$ The probability density for z with the disturbance in state 0.

$p_1(z) \equiv$ The probability density for z with the disturbance in state 1.

Note that these are joint probability densities which must fulfill the condition:

$$\int_z [p_0(z) + p_1(z)]dz = 1 \qquad 12.37a$$

The bracketed sum in the integral is the marginal probability density for z. Similarly the marginal probabilities for the disturbance states emerge from integrating the respective joint densities over the whole z range, i.e.:

$$\bar{p}_\alpha = \int_z p_\alpha(z)\,dz \qquad 12.37b$$

Further the two levels of the zero mean and unity variance disturbance are denoted by l_o and l_1 which according to equation 12.18 become:

$$\ell_0 = -\sqrt{x}\,, \qquad \ell_1 = \sqrt{1/x} \qquad 12.38$$

Equations A.21 of the appendix then become for this case:

$$\frac{d}{dz}\left[(g_0\ell_0 - g_1 z(\theta))p_0(z)\right] = -\lambda_0 p_0(z) + \lambda_1 p_1(z) \qquad 12.39$$

$$\frac{d}{dz}\left[(g_0\ell_1 - g_1 z(\theta))p_1(z)\right] = \lambda_0 p_0(z) - \lambda_1 p_1(z) \qquad 12.40$$

This is a set of two ordinary differential equations which describe the stationary probability densities $p_o(z)$ and $p_1(z)$.

Note that if the feedback controller included integral action an additional continuous variable would be introduced into the system when described in the general form of equation A.1 of the appendix. This would mean that the equations for the stationary densities would become partial differential equations with two independant variables, one for each continuous state variable. Adding equations 12.34 and 12.40 results in:

$$\frac{d}{dz}\left\{(g_0 \ell_0 - g_1 z(\theta))\, p_0(z) + (g_0 \ell_1 - g_1 z(\theta))\, p_1(z)\right\} = 0 \qquad 12.41$$

It follows that:

$$(g_0 \ell_0 - g_1 z(\theta))\, p_0(z) + (g_0 \ell_1 - g_1 z(\theta))\, p_1(z) = A \qquad 12.42$$

where A is a constant. Integrating this equation over the range of z and re-arranging yields:

$$g_0 \int_z [\ell_0 p_0(z) + \ell_1 p_1(z)]\, dz - g_1 \int_z z(\theta)[p_0(z) + p_1(z)]\, dz = \int_z A\, dz \qquad 12.43$$

The first and second integrals on the left hand side of equation 12.43 represent the disturbance mean and the output mean respectively. Since both are identically zero it follows that A also vanishes. Equation 12.42 can now be used to express, say, $p_1(z)$ in terms of $p_o(z)$, i.e.:

$$p_1(z) = \frac{g_1 z(\theta) - g_0 \ell_0}{g_0 \ell_1 - g_1 z(\theta)}\, p_0(z) \qquad 12.44$$

Substituting $p_1(z)$ from this equation into equation 12.39 and integrating the resulting equation, $p_o(z)$ becomes:

$$p_o(z) = E(g_1 z - g_o \ell_o)^{(\lambda_o/g_1)-1} (g_o \ell_1 - g_1 z)^{\lambda_1/g_1} \qquad 12.45$$

where E is the integration constant. The result for $p_1(z)$ then follows from equation 12.44 and 12.45 and becomes:

$$p_1(z) = E(g_1 z - g_o \ell_o)^{\lambda_o/g_1} (g_o \ell_1 - g_1 z)^{(\lambda_1/g_1)-1} \qquad 12.46$$

It is convenient now to normalize z to the range 0-1. From the differential equation for z, equation 12.36 it is seen that it cannot exceed the limits $g_o l_o/g_1$ and $g_o l_1/g_1$. These limits would be reached if the system were allowed to reach steady state when the disturbance is fixed at one of its two levels. The normalized variable z*, defined as:

$$z^* = \frac{z - g_o \ell_o/g_1}{(\ell_1 - \ell_o)\, g_o/g_1} \qquad 12.47$$

varies over the desired range. Introducing z* into equation 12.45 and 12.46 they become:

$$p_o(z^*) = E' z^{*(\lambda_o/g_1)-1} (1 - z^*)^{\lambda_1/g_1} \qquad 12.48$$

$$p_1(z^*) = E' z^{*\lambda_o/g_1} (1 - z^*)^{(\lambda_1/g_1)-1} \qquad 12.49$$

where E' is again a constant. The value of the constant E' is determined from the normalization condition on the probability densities, i.e:

$$\int_0^1 [p_o(z^*) + p_1(z^*)]dz^* = 1 \qquad 12.50$$

Substituting for $p_o(z^*)$ and $p_1(z^*)$ in this equation yields

$$\int_0^1 [p_o(z^*) + p_1(z^*)]dz^* = \int_0^1 E^1 z^{*(\lambda_o/g_1)-1}(1-z^*)^{(\lambda_1/g_1)-1}dz^*$$

$$= E^1 B(\lambda_o/g_1, \lambda_1/g_1) = 1 \qquad 12.51$$

where B(a,b) is the beta function which in terms of the gamma function is:

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \qquad 1252$$

Hence the marginal probability density for z*

$$p(z^*) = p_o(z^*) + p_1(z^*) = \frac{z^{*(\lambda_o/g_1)-1}(1-z^*)^{(\lambda_1/g_1-1)}}{B(\lambda_o/g_1, \lambda_1/g_1)} \qquad 12.53$$

is the beta probability density. The two joint distributions $p_o(z^*)$ and $p_1(z^*)$ can also be written in terms of beta densities as:

$$p_o(z^*) = \frac{\lambda_1}{\lambda_o+\lambda_1} D[z^*, \lambda_o/g_1, (\lambda_1/g_1)+1] \qquad 12.54$$

$$p_1(z^*) = \frac{\lambda_o}{\lambda_o+\lambda_1} D[z^*, (\lambda_o/g_1)+1, \lambda_1/g_1] \qquad 12.55$$

where D(x, a, b) is the beta probability density defined as

$$D(x, a, b) = \frac{x^{a-1}(1-x)^{b-1}}{B(a, b)} \qquad 12.56$$

where:

$$0 \leq x \leq 1 \quad , \quad a > 0 \quad , \quad b > 0 \qquad 12.57$$

Thus it is seen that the probability density for the output of the system is of the beta type which depending on a and b can have drastically different behaviour than that of the normal density. A sketch of its possible shapes is shown in figures 12.1a and 12.1b. From its definition it is seen that the density tends to infinity at $x = 0$ if $a < 1$ and at $x = 1$ if $b < 1$. On the other hand if $a > 1$ the density vanishes at $x = 0$ and if $b > 1$ it will vanish at $x = 1$. Thus the density can be of the concave or convex shape as shown in figure 1a if both a and b are larger or smaller than 1 respectively. A mixture of the two types is also possible as shown in figure 1b. The density becomes symetrical for $a = b$ only. Examples for the densities occuring in the control situations treated will be given later.

The Controller Output Probability

As was mentioned in chapter 2, the probability distribution of the controller output is of critical importance for the applicability and effectiveness of a controller design for random disturbances. In this section the probability density and distribution for the controller output $m(\theta)$ will be deduced from that of the systems output. From equation 12.35 and 12.36

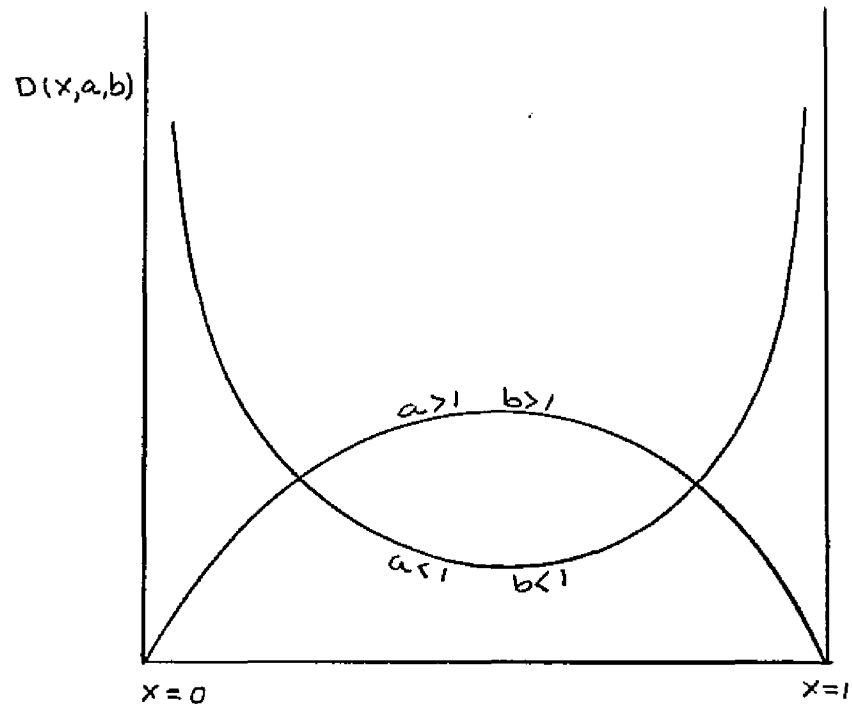


Figure 12.1a

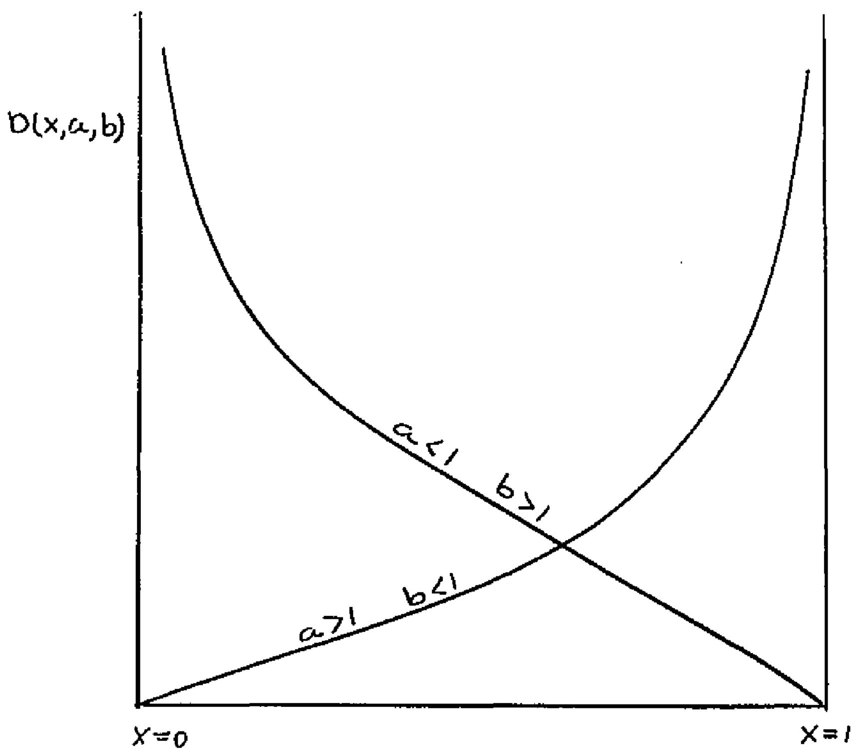


Figure 12.1b

SKETCH OF THE BETA DENSITIES

the controller output is:

$$m(\theta) = g_0[-D\ell_\alpha + (D - K_c)\, z(\theta)] \qquad 12.58$$

Introducing the normalized output variable z* into this equation results in:

$$m(\theta) = h\, z^*(\theta) + k_\alpha \qquad 12.59$$

where the slope h and the intercept k_α are:

$$h = \frac{g_0^2}{g_1}(D-K_c)(\ell_1 - \ell_0) \qquad 12.60$$

$$k_\alpha = \frac{g_0^2}{g_1}(D-K_c)\ell_0 - Dg_0\ell_\alpha \qquad 12.61$$

Note that the intercept of the linear relationship between z* and m can assume two values, depending on the state of the disturbance as indicated by the subscript α of k_α .

To find the probability density of m from that of z* the usual rules for the probability density of a function of a random variable can be applied noting that the density of m, as is that of z*, is joint with the probability of the diturbance states. Defining:

$$z_\alpha^*(m) = \frac{1}{h}(m - k_\alpha)$$

the marginal probability density for m becomes:

$$p(m) = p_0(m) + p_1(m) = \frac{1}{|h|}\left\{p_0[z_0^*(m)] + p_1[z_1^*(m)]\right\} \qquad 12.63$$

Substituting for the joint densities from the preceding section:

$$p(m) = \frac{1}{|h|}\left\{\frac{\lambda_1}{\lambda_0+\lambda_1} D[z^*_0(m), \frac{\lambda_0}{g_1}, \frac{\lambda_1}{g_1}+1] + \frac{\lambda_0}{\lambda_0+\lambda_1} D[z^*_1(m), \frac{\lambda_0}{g_1}+1, \frac{\lambda_1}{g_1}]\right\} \quad 12.64$$

where the definitions of the last section hold. Note that unlike the marginal density for z*, that for m consists of two seperate beta densities. These cannot be combined because $z^*_o(m)$ and $z^*_1(m)$ for the same value of m are different. It follows that the marginal density for m will show a behaviour corresponding to the superposition of two distinct beta densities. Further in the calculation of $z^*_\alpha(m)$ from m according to equation 12.62 values of z* (m) outside the range 0-1 will produce vanishing densities. If both $z^*_o(m)$ and $z^*_1(m)$ are outside these limits for a specific m the marginal density at that point will of course vanish. Needless to say that such behaviour is far of that of the normal distribution.

The probability distribution for both z* and m can be obtained from those of the corresponding beta densities. Care must, however, be exercised in the case of the distribution of m. If h $<$ 0 then short reflection will show that the distribution of m, F(m), defined in the usual manner:

$$F(m) = \int_{-\infty}^{m} p(m)\,dm \qquad 12.65$$

is obtained from the complement of the value obtained by replacing the beta densities in the right hand side of equation 12.64 by their respective distributions.

Examples

The densities and distributions for both z* and m were computed for several cases and the results for a specific controller designed by the Wiener method will be presented. The Wiener controller picked for illustration is the one designed for a characteristic disturbance frequency ν_0 of 1 at the Lagrangian multiplier l of 0.1. This results in the following controller coefficients:

$$K_c = 6.16 \quad , \quad D = 1.16 \qquad 12.66$$

Further, the ratio of the transiton intensities, x was chosen as 0.1. The stationary probabilities for the disturbance states and their levels are completely defined by x according to equations 12.14 and 12.18, i.e.:

$$\bar{p}_0 = 1/(1+x) = 90.91\% \, , \quad \bar{p}_1 = 1/(1+1/x) = 9.09\% \qquad 12.67$$

$$\ell_0 = -\sqrt{x} = -0.3162 \quad , \quad \ell_1 = \sqrt{1/x} = 3.162 \qquad 12.68$$

It is seen that such a disturbance is for 90.91% of the time not too far from its zero mean and only during 9.09% of the time relatively large disturbances are involved. The system becomes completely defined after the specification of this disturbance's characteristic frequency ν. This parameter was varied over the range used in the evaluation of controllers in the previous chapters, i.e. from 0.01 t 100. The transition intensities can be expressed in terms of the parameters x and ν, and become:

$$\lambda_0 = \nu x/(1+x)\,, \qquad \lambda_1 = \nu/(1+x) \qquad 12.69$$

For fixed controller settings and x the transition intensities determine the character of the probability density which will therefore be strongly dependant on the characteristic frequency of the disturbance. Further for low ν values λ_0 and λ_1, becomes smaller than 1.

The necessary beta distributions were computed by using the subroutine supplied in the scientific SubroutinePackage (7) of the IBM 360/50 computer at the City College computation center. The procedure used in this subroutine is valid only when the parameters a and b defined in equation 12.56 are larger than 0.5. It is clear from equation 12.69 that these exponents can become smaller than 0.5 in this example. In these cases the following recursion formular were used (8):

$$I_x(a,b) = I_x(a+1,b) + x^a(1-x)^b \frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)} \qquad 12.70$$

$$I_x(a,b) = I_x(a,b+1) - x^a(1-x)^b \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b+1)} \qquad 12.71$$

where

$$I_x(a,b) = \int_0^x b(x,a,b)dx = \int_0^x x^{a-1}(1-x)^{b-1}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}dx \qquad 12.72$$

Since one exponent in the joint densities of z* is always larger than 1 equation 12.70 can be used when a $<$ 1 and equation 12.71 for the case b $<$ 1.

The system output marginal density over the range 0.1 - 100 of disturbance parameters ν , is shown in figure 12.2. For ν of 0.01 the density is similar to that for of 0.1 shifted down by a decade. The density is plotted for values of z* whose mean for all cases shown is:

$$\langle Z^* \rangle = \frac{\langle Z \rangle - g_0 \ell_0 / g_1}{(\ell_1 - \ell_0) g_0 / g_1} = \frac{-\ell_0}{\ell_1 - \ell_0} = \frac{x}{1+x} = 0.0909 \qquad 12.73$$

As ν increases from 0.1 to 100 the density is seen to cover the whole spectrum of possible forms for the beta density. Note, however, that for a disturbance of ν = 1 the density is still convex and at ν = 10 is of mixed character vanishing at z* = 1 and tending to infinity at z* = 0. Only for the relatively high disturbance characteristic frequency ν = 100 the density becomes concave and centers around its mean. The normal densities corresponding to ν = 10 and 100 are also shown in figure 12.2. It is also seen that at ν = 100 the actual density approaches the normal one even though the former is asymmetric.

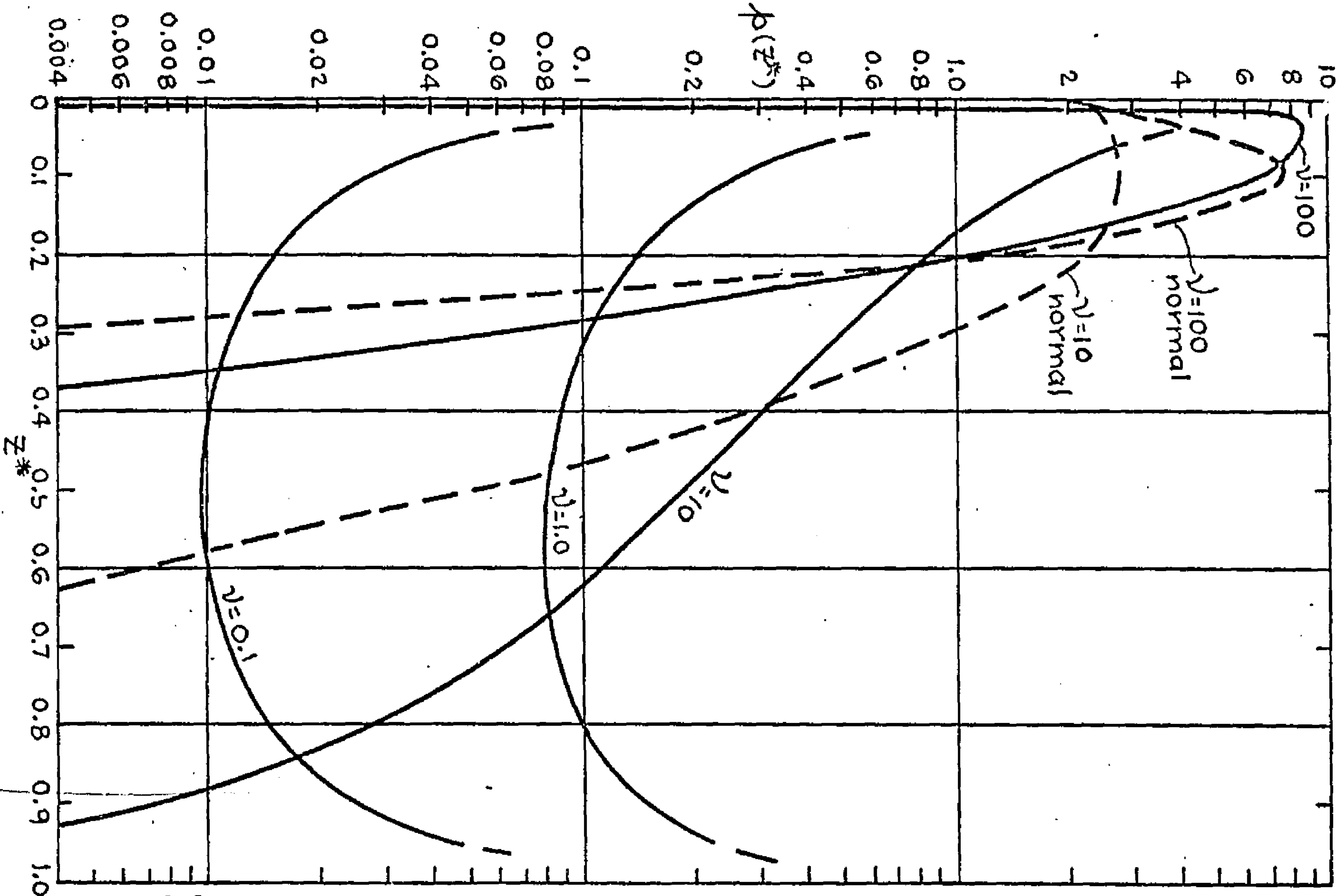


Figure 12.2 SYSTEMS OUTPUT PROBABILITY DESNSITY OF THE WIENER DESIGN FOR INPUT DISTURBANCE $\Delta = 0$, $\nu_0 = 1.0$, $1 = 0.1$, $x = 0.1$

The density for a disturbance with $\nu = 10$ is not too far from the corresponding normal one if one disregards the spike at $z^* = 0$. In other words the normal approximation to the probability desnity becomes applicable only at the upper region of the characteristic frequency range of the disturbance.

The probability densities for the controller output m are shown in figure 12.3 for disturbances with ν in the range 0.1 to 100. The densities are shown for m values expressed in fractions of its total range. In this case the mean of m is at 0.9091. The slope h of the relationship between m and z* is negative and the intercept is such that in a central region of m both z^*_o and z^*_1 are outside the 0-1 range and therefore the density for m vanishes there. On each side of this region a seperate beta density is then observed. These, again, become concave only for $\nu \geqslant 10$. The middle region of vanishing density is one in which the controller output would be found with a 0 probability density and thus it will be located at either of the two side regions. This phenomenon seems less surprising in view of the fact that the disturbance density is localized at its two levels. This localization is already considerably broadened in the density of figure 12.3. It should be noted that as the controller gain decreases the seperation of the two densities for m is reduced and finally disappears. For smaller control gains the controller output will follow the dis-

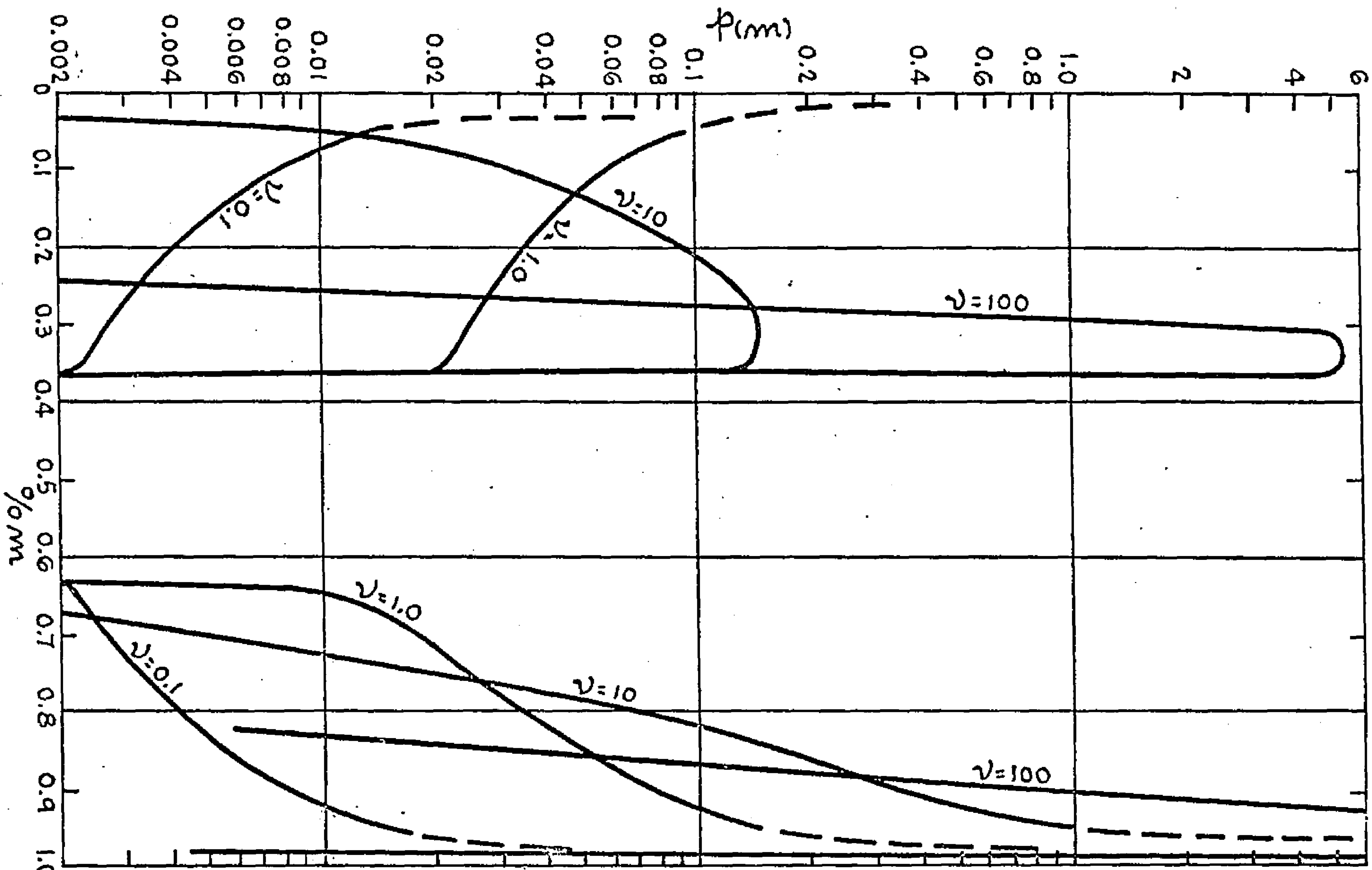


Figure 12.3 CONTROLLER OUTPUT PROBABILITY DENSITY OF THE WIENER DESIGN FOR INPUT DISTURBANCE Δ = 0, ν_0 = 1.0, 1 = 0.1, x = 0.1

turbance changes less forcefully thus allowing its densities to relax and become less localized.Needless to say that such a state of affairs is far from what would be expected from a normally ditributed variable.

The densities shown in this example give a clear picture of the probability structure of the system. Of practical importance in this context, however, are the distributions of z* and m. What is of great interest is the probability of the control system outputs to stay within a certain permissible range. The values obtained from the actual distribution will be compared with the equivalent probability to be expected if the variable would have followed a normal probability distribution.

The results for this case are presented in figures 12.4 and 12.5 where the probability, Q, is shown, for m and z* to be inside 80% and40% of their accessible range. These regions were taken by excluding 20% and 60% of the range on the large excursion side from the mean of these variables. Also shown are the equivalent probabilities for the normally distributed variables all as a function of the disturbance characteristic frequency, ν. In figure 12.4 it is seen that in the range of .01 to 1.0 the probability of being in the extreme 20% of the allowable range of z* and m is about 6 - 9% as compared to 0.5 - 1.4% for a normal distribution. Recalling that the probability for the disturbance to be in its high

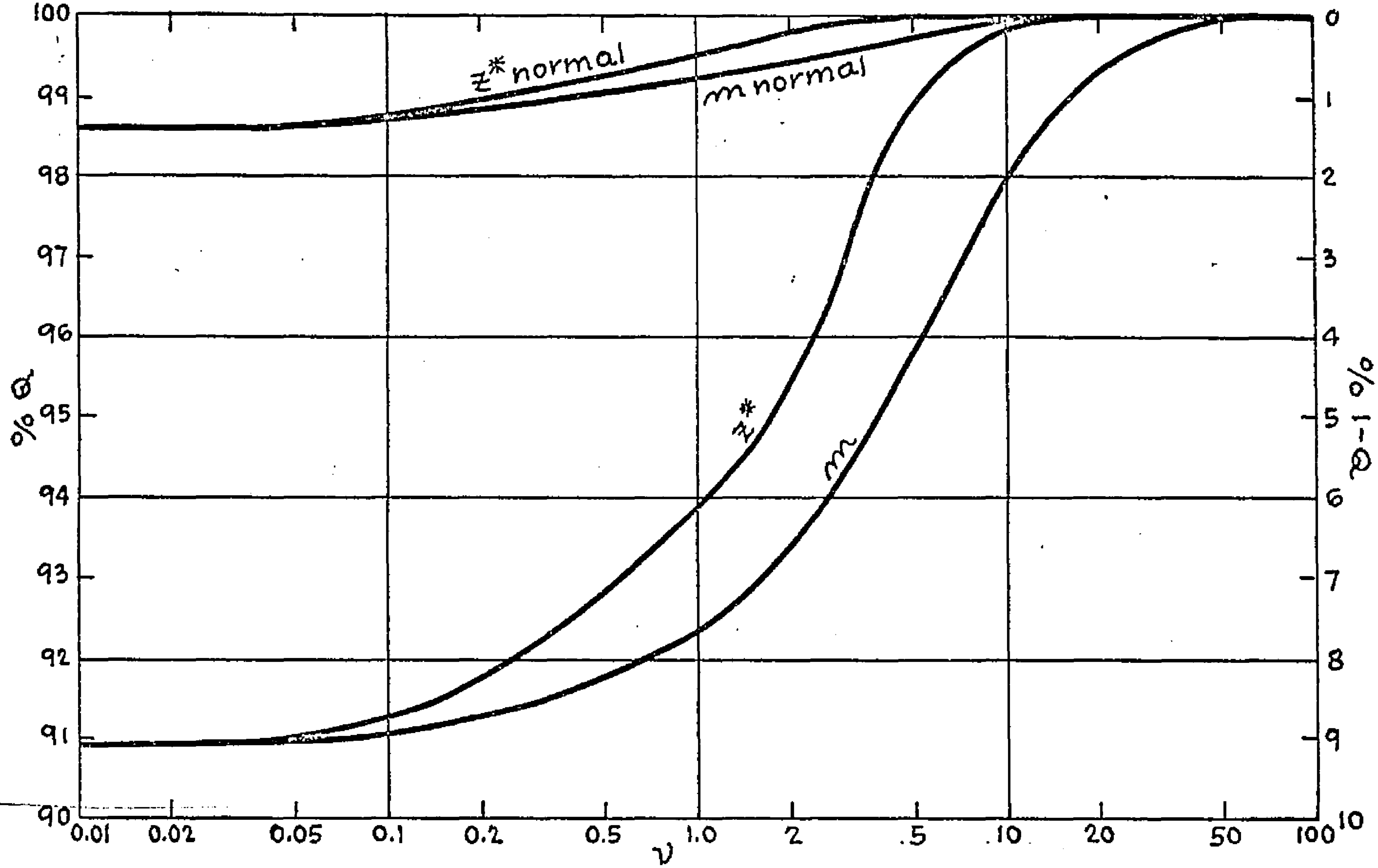


Figure 12.4 80% OF RANGE PROBABILITY FOR OUTPUTS OF THE WIENER DESIGN FOR INPUT DISTURBANCE $\Delta = 0$, $\nu_0 = 1.0$, $1 = 0.1$, $x = 0.1$

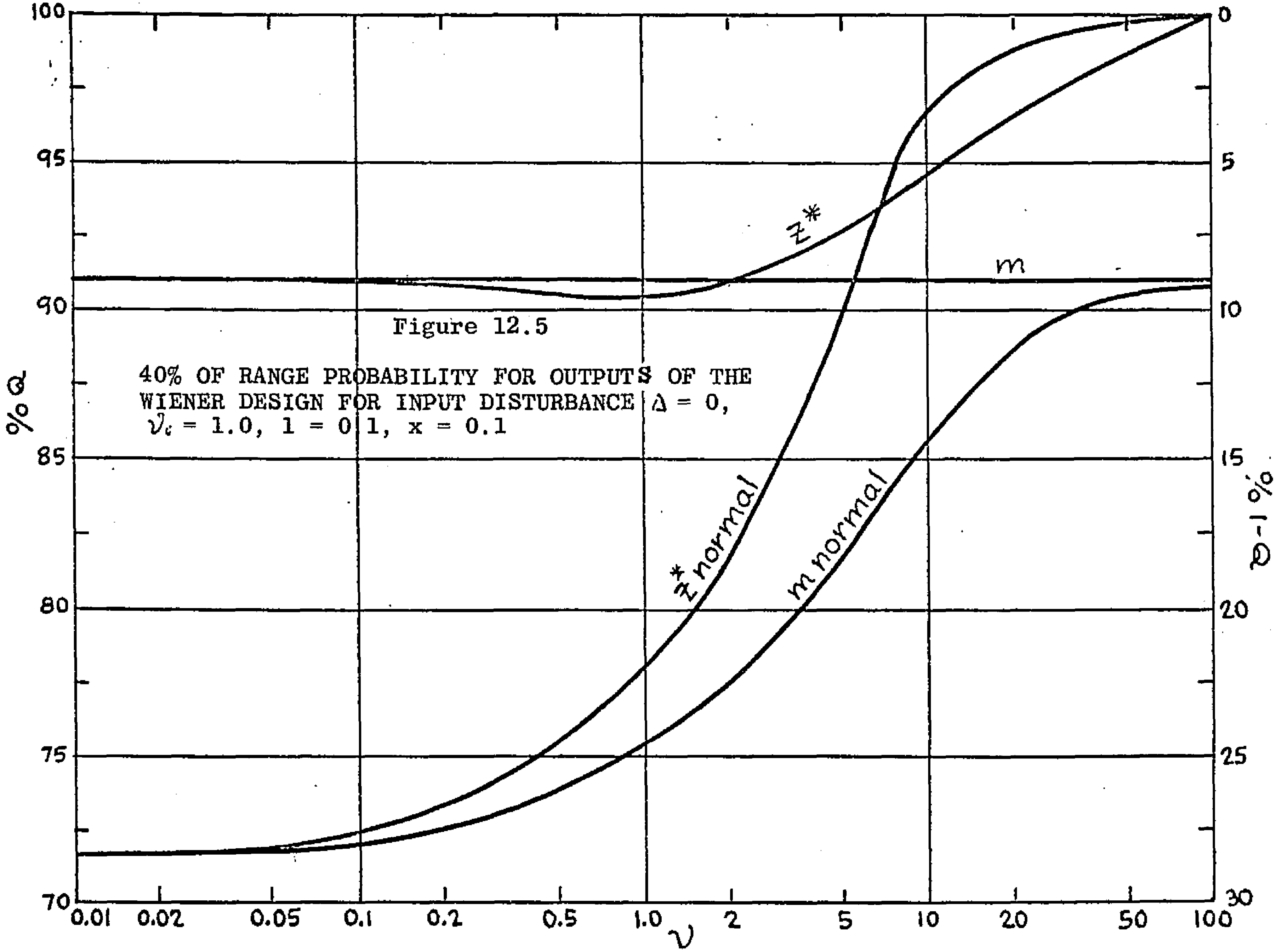


Figure 12.5

40% OF RANGE PROBABILITY FOR OUTPUTS OF THE WIENER DESIGN FOR INPUT DISTURBANCE $\Delta = 0$, $\nu_i = 1.0$, 1 = 0.1, x = 0.1

level $l_{||}$ is 9.09% it is seen that the probability for m and z* to be in the extreme 20% of their range is almost the same. The six fold difference between the normal probability estimate and the actual one for this range is therefore significant in view of the special disturbance used. As the characteristic frequency of the disturbance is increased above .1 the probability for the 80% range begins to rise toward that of the normal one. At $\nu = 1$ the probabilities are still 92.3% and 93.8% for m and z* respectively. The corresponding normal probabilities are 99.2% and 99.5%, so that there still is an appreciable difference. Both the normal and actual curve converge to 100% probability for a further increase of ν. For z* this occurs at $\nu \cong 10$, whereas for m only at $\nu \cong 35$.

For systems with delay it was seen that in this ν range very little regulation can be achieved. For the low and medium ν range where the approach of this work is more fruitful the discrepancy between a normal probability estimate and the actual one is the largest.

Figure 12.5 shows the results for 40% of the accessible range of m and z*. In this case, however, the normal estimation is considerable more conservative than that of the actual distribution. The probability for m is close to 91% for all ν disturbances. The reason for the constant value is that at 40% of the range the density vanishes and the distribution includes the integral over the complete range of one of the beta densities and no contribution from the second. From equations 12.64, 12.65 and the fact that h is negative, it follows that the distribution becomes $1/(1+x)$ and is independant of ν . The normal estimation rises from 72% and reaches the 91% level at $\nu = 100$.

The situation with the probability for the 40% range of z* is somewhat different in that the probability rises from 91% to 100% as ν is increased, whereas the normal estimate rises from 72% to 100% which it reaches at $\nu \cong 100$. The two curves, that is the actual and the normal estimate, do however, cross each other for $\nu \cong 65$ from where on the normal distribution is less conservative than the actual one.

For both variables the 40% of their range excludes most of the region they occupy due to the large distrubance level l_1. It does therefore represent the response due to the l_o disturbance level which has a 90.91% probability of occuring. This can be expected to lead to a narrower distribution than due to the normal one. Again it is seen that for high disturbances the normal extimate converges with the actual distribution.

In conclusion it is noted that the densities for the variables of the simple control system treated are drastically different from the normal ones. They are of the beta type whose character depends strongly on the disturbance's characteristic frequency. The controller output density is a composite of two beta densities which in the example presented are seperated by a central region in which the density vanishes . The distribution of the output and controller output are consequently also different than the normal distributions. The probability of being outside 80% of the accessible range almost equals that of being at the corresponding disturbance level and is six times larger than indicated by an equivalent normal distribution. The probability of being outside 40% of this range is larger for the normal distribution than with the one derived. In both cases the distributions converge to the normal ones for disturbances with ν in the 10 - 100 range. The largest divergence from the normal distribution is observed for disturbances with ν in the range 0.01 - 0.1.

These somewhat extreme results reflect the nature of the disturbance used whose probability density is localized at two levels. A more realistic simulation of the disturbance would result from allowing it to assume additional levels. It might be expected that with a disturbance density distributed over more levels less drastic densities for the system outputs would result, so that the normal approximation would fare better in comparison.

Approximate Method for System with Delay

The development of the probability distributions for the output and controller output in the previous situation indicated the inadequacy of the normal approximation unless the characteristic frequency of the disturbance is high. For the high frequency content disturbances at which the normal distribution becomes appropriate the regulator problem solution for plants with delay points toward the application of no control and there is surely no interest in the controller output distribution. For low frequency content disturbances, however, there is considerable incentive for investigating the distributions and it is in this region that the discrepancy with the normal estimation was largest in the examples shown. Unfortunately the Markov description used is not applicable for plants with delay as was discussed. It is, however, possible to get an estimate of the distributions if the characteristic frequency of the distrubance is low and this estimate can, in view of the previous example, serve as a bound on the actual distribution. This approximate approach will be described in what follows.

It was seen in the previous section that the characteristic frequency ν of the two state Markov disturbance is the sum of the transition intensities, or rates. For low ν disturbances these rates are small so that the disturbance on the average stays at one level for a long duration compared with the systems time constant . The systm can therefore be assumed to have approached steady state before the next disturbance transiton takes place. The response to the disturbance could therefore be approximated by that of the system to a sequence of deterministic steps appropriately spaced.

The intervals between successive transitions of a two state Markov process have probability densities p(t) of negative exponential shape whose parameter, or rate constant, is the transiton intensity from the state considered, i.e.:

$$p_\alpha(t) = \lambda_\alpha e^{-\lambda_\alpha t} \qquad 12.74$$

This is so because the transition from a state of a Markov process fulfills the conditions of a first event of a Poisson process. The expected duration of a state is from equation 12.74 $1/\lambda_\alpha$ and this will be used as a duration of a disturbance level in the deterministic imitation of a low ν disturbance. For this disturbance to have a zero mean its levels y_o and y_1 must fulfil the condition:

$$y_0/\lambda_0 + y_1/\lambda_1 = 0 \qquad 12.75$$

Further if the average of the squares is normalized to unity corresponding to a unity variance, the following results:

$$\left(\frac{y_0^2}{\lambda_0} + \frac{y_1^2}{\lambda_1}\right) \Big/ \left(\frac{1}{\lambda_0} + \frac{1}{\lambda_1}\right) = 1 \qquad 12.76$$

From equations 12.75 and 12.76 the two levels are defined by:

$$y_0 = -\sqrt{x} \quad , \qquad y_1 = \sqrt{1/x} \tag{12.77}$$

with x being as before the ratio of the transition intensities: λ_0/λ_1 . This is the same relationship that holds for the random Markov distrubance process. As is the case with the random distrubance ν and x specify it completely. Thus equation 12.69 can be used to determine the average duration of a state from ν and x. So, for example, for $\nu = 1$ and $x = 0.1$ the two levels are -0.316 and 3.162 with average durations 11 and 1.1 respectively. An interval of 11 plant time constants is sufficient time for the system to practically reach steady state and the response to a superimposed box function of high $y_1 - y_0 = 3.478$ and duration 1.1 can therefore be used as an approximation to the random response.

The response of the output and controller output for systems with Wiener designed feedback controllers to deterministic inputs is relatively simple. The corresponding transfer functions 1 - KG(s) and K(s) are of simple rational form with at most a superimposed exponential delay factor. Since G(s) represents a realizable plant and K(s) was designed by Wiener's procedure, both have no poles in the right hand s plane and therefore the systems transfer functions are identical to the Laplace transforms of the corresponding weighing functions. The Laplace transforms are more convenient to use for the evaluation of the response to deterministic inputs and in particular use can be made of the initial and final theorem of Laplace transforms. Thus the response of the system to the two level deterministic disturbance which imitates the corresponding Markov process can be conveniently given as the superposition of the steady state response to y_o/s and the response to:

$$(y_1 - y_0)(1 - e^{-s/\lambda_1})/s \qquad 12.78$$

which represent a box function of duration $1/\lambda_1$.

To illustrate these procedures consider the typical plant with a Wiener feedback controller, designed for a single disturbance, where the disturbance enters at the plants input. In the transformed domaine the controller output was shown in chapter 10 to be:

$$m(s) = -KGy(s) = -\frac{(g_0 + g_1 s)}{(R+s)}e^{-\Delta s}y(s) \qquad 12.79$$

The steady state response to y_o is then:

$$\lim_{\theta \to \infty} m(\theta) = \lim_{s \to 0} sm(s) = \lim_{s \to 0} s e^{-\Delta s}\frac{(g_0 + g_1 s)}{(R+s)}\frac{y_0}{s} = \frac{g_0}{R} \qquad 12.80$$

the transformed response to the box function becomes:

$$m(s) = -e^{-\Delta s}\frac{(g_0 + g_1 s)}{(R+s)}\frac{(y_1 - y_0)}{s}(1 - e^{-s/\lambda_1}) \qquad 12.81$$

from which the time response is:

$$-m(\theta) = [g_1 + (g_1 - g_0/R)(1 - e^{-R(\theta-\Delta)})](y_1 - y_0)u(\theta-\Delta)$$

$$-[g_1 + (g_1 - g_0/R)(1 - e^{-R(\theta-\Delta-1/\lambda_1)})](y_1 - y_0)u(\theta-\Delta-1/\lambda_1) \qquad 12.82$$

The largest value of $m(\theta)$ is seen from this equation to be g_1. From this response, the time $m(\theta)$ spend above a specified level compared with the total time of a complete two disturbances cycle yields an estimate for the probability of exceeding that value.

This is illustrated for a plant of Δ = 0.5 whose Wiener controller was designed for ν_0 = 0.1 and 1 = 0.3. The results for the disturbance discussed before, i.e., one with ν = 1.0, x = 0.1, are shown in figure 12.6 together with the corresponding normal distribution approximation. In this figure Q represents the estimate to the probability that the magnitude of the controller output is whithin the fraction of its accessible range indicated on the abcissa. These results are similar in character to those obtained from the actual calculation of the disturbance for the delayless plant in the proceding section.

The constant probability estimate for the controller output to be within from 24 to 84% of its possible magnitude again reflects the disturbance model used. It originates from the discontinuity in the time function $m(\theta)$ at $\theta = \Delta + 1/\lambda_1$, that is

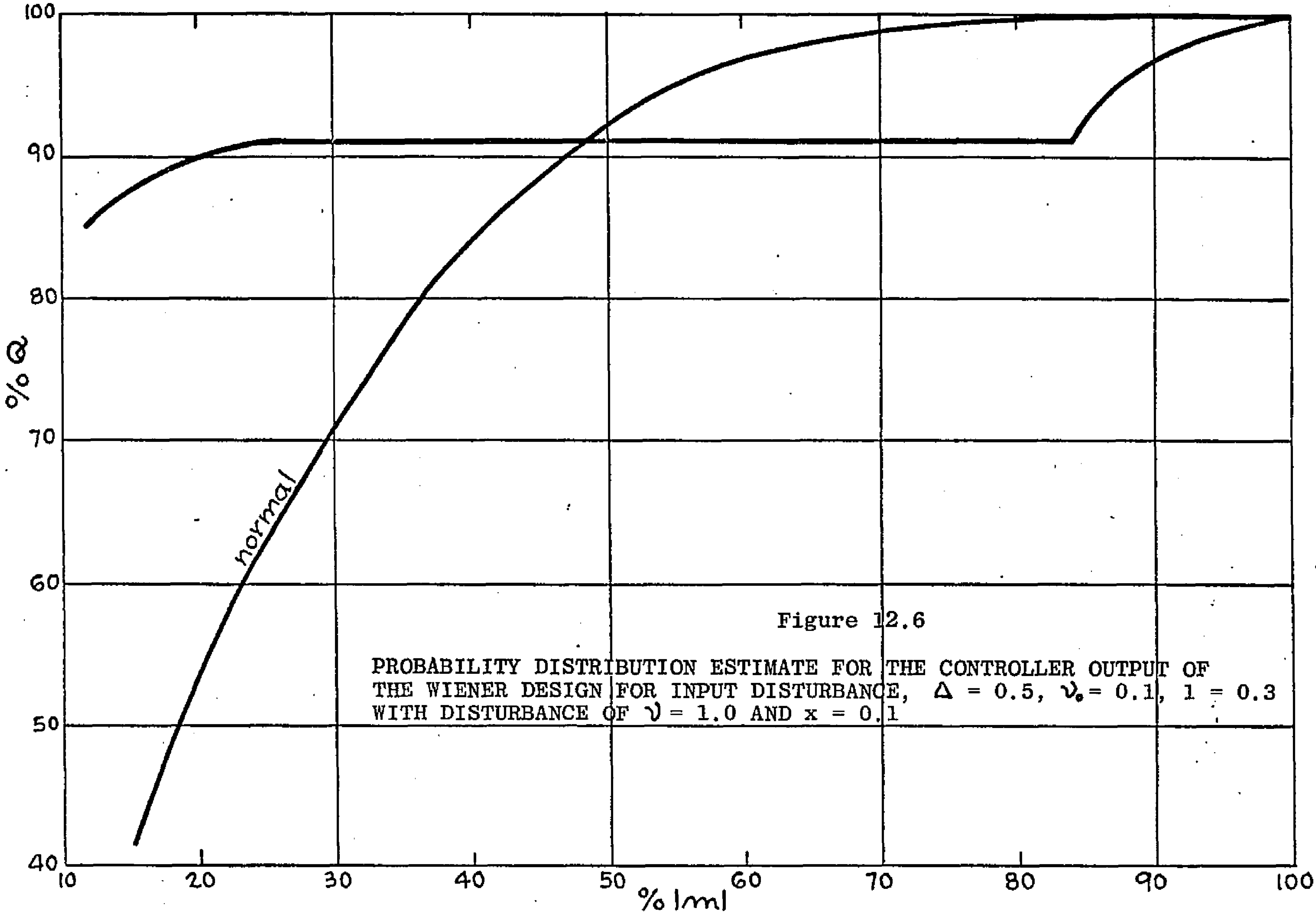


Figure 12.6

PROBABILITY DISTRIBUTION ESTIMATE FOR THE CONTROLLER OUTPUT OF THE WIENER DESIGN FOR INPUT DISTURBANCE, $\Delta = 0.5$, $\nu_o = 0.1$, $1 = 0.3$ WITH DISTURBANCE OF $\nu = 1.0$ AND $x = 0.1$

the point where the disturbance transition, back to y_o, occurs. Again the same remarks made in the last section about the significance of the difference between the normal distribution and the calculated one hold here. This significance derives to a large extent from the special type of disturbances used. Therefore the estimate obtained in this way should be regarded merely as a guide in indicating the magnitudes of controller actions to be expected.

13. SUMMARY AND CONCLUSIONS

Most disturbances occuring in the Chemical process industry are of fluctuating nature and would be best described as random processes. Conventional feedback controller design and optimization procedures almost exclusively treat an isolated step as representing the disturbances. Rarely, if ever, is the probability structure of the disturbance known and the analysis of controllers in terms of the probabilities is possible, at an acceptable effort, only in a few simple cases. The optimal design method originated by Wiener which is based on the variances of control system variables, affords a feasible approach to the problem. This, however, raises the question of the relationship between the performance objective, usually specified in terms of probabilities of the variables magnitude and the variance.

The prerequisite for the execution of Wiener's procedure is the autocorrelation function of the disturbance. This is in general not known and even though it is more accessible to measurement than is the probability structrure of the disturbance its experimental determination is far from trivial. The autocorrelation function, or its Fourier transform, the spectrum, does however bear a closer relationship to the physical origins of the disturbance than does the probability description of a random process. From the physical background it is therfore occasionally possible to establish the main features of the disturbance autocorrelation function. It was shown that such a consideration, applied to several typical situations leads to an autocorrelation function of negative exponential type, or one which could be represented by several elements of this type.

The general purpose of this study was to investigate the performance of the optimal step controller with respect to random disturbances, and to design and evaluate an optimal random controller considering the fact that the exact description of the disturbance is not available. This means that the optimal random controller must perform well over a spectrum of possible disturbances, including an occasional step input, and that consideration must be given to probability distributions of the system variables.

This study treats single input - output linear feedback control systems. The disturbances are applied to the input or output of a typical plant corresponding to a reactor which consists of one or more ideally stirred tanks in series with a plug flow section in which an irreversible first order reaction takes place. The negative exponential autocorrelation function is used to represent the disturbance with its characteristic frequency parameter, ν , varying from 0.01 to 100, when normalized by the plants time constant. Specific results were presented for the plant with the normalized delay parameter, Δ , in the range 0 - 1.5.

The performance of the three mode optimal step controller, after Cohen and Coon, was investigated with respect to random disturbances for a single tank with delay type plant. In the output disturbance configuration only the proportional-integral modes were used, since with a derivative mode the controller output variance does not converge and thus is sensitive; and prone to saturation , by stray high frequency noises. In both configurations this controller proved to produce satisfactory performance for very low frequency content disturbances. However, their output rms considerably exceeds that of an uncontrolled plant for disturbances with higher characteristic frequencies. This occurs at a disturbance characteristic frequency which is somewhat lower or higher than the plants intrinsic frequency for the output and input disturbance cases respectively. This fact was not reported for the process industry and provides a strong motivation for this work.

The straight forward application of Wiener's method to the present regulator problems raises some difficulty partly because a feedback controller is desired whereas the procedure yields an equivalent open loop compensator. The Wiener design for cascades of stirred tanks, or any minimum phase plant, yields perfect regulation by an infinite feedback controller gain. It is shown, however, that an infinite proportional feedback controller for such cascades yield unrealizable overall systems for cascades with three tanks and more. The Wiener procedure on the other hand gurantees a realizable equivalent open loop compensator and therefore also a realizable overall system. This apparent contradiction as well as the pratically unusable infintie gains are overcome by applying the Wiener procedure to a more realistic plant representation, that is by superimposing a pure delay on the cascade model. The Wiener design then leads to finite feedback controller gain and a non zero output variance. Taking the limit of this feedback controller as the delay, Δ, tends to zero reveals the structure of the infinite gain controller for the delayless plants. The feedback controller transfer function for a cascade of n tanks then becomes $(1+s)^n/\Delta s$ and $(1+s)^n/[\Delta(s+\nu)]$ in the input and output disturbance configurations respectively. The Wiener controller thus uses second and higher order derivative action to keep the system with three or more tank cascades realizable. The impracticality of high gains is thus compounded by the higher order derivatives in the controller.

For the typical plant with input disturbance the feedback controller, however, is shown to be unrealizable, a fact which was overlooked in previous work (11). For both input and output disturbance configurations the controller output variance does not converge and thus is sensitive to high frequency disturbances and prone to saturation by stray high frequency noises.

These results indicate that the Wiener optimization technique must be guided into yielding practically usesful results by appropriate constraints. A constraint on the controller output variance was therefore introduced by the use the Lagrangian multiplier technique. Two parameters therefore specify the Wiener controller for a fixed plant: The characteristic freqency of the disturbance used in the design and the Lagrangian multiplier which determines the controller output variance when the system is excited by a specific disturbance. The Wiener design then yields the optimal output rms for any controller effort, i.e., controller output rms, when the system is excited by a distrubance with the same characteristic frequency used for its design but will not be necessarily be optimal for other disturbances.

With the exception of the case of input disturbance to a delayless plant the optimal output performance vs. controller effort curve decreases and then levels off to a constant output rms which is the lowest possible one for the particular disturbance regardless of the controllers effort used. In the very low disturbance characteristic frequency region this optimal performance point is only slightly better than that achieved by the optimal step controller. For high frequency content disturbances the optimal output rms curve approaches the uncontrolled level at any controller effort, so that the practical Wiener controller is no controller. The characteristic frequency at which this occurs depends on the plants delay and systems configuration and in the case of input disturbance to a low delay plant the optimal performance point does reach significantly below the uncontrolled level. In this range the conventional controller shows its worse output performance namely, it is appreciably above the uncontrolled level, and thus it amplifies the disturbance variance rather than attenuated it. In the medium characteristic frequency range the optimal Wiener controller shows considerably better performance than the optimal step controller and the output rms is well below the no control value.

Non optimal performance curves which arise from exciting disturbances different from those used in the controllers design might show an increase of output rms with controller effort. This happens in particular when the design characteristic frequency is lower than that of the exciting disturbance and does lead to higher output rms than that of the uncontrolled plant. The set of such curves obtained when exciting a fixed characteristic frequency design with a whole range of disturbances together with the performance lines of a fixed controller, that is fixed l, form a surface which allows the evaluation of a compromise overall Wiener controller.

For the typical plant with output disturbance a feedback controller which will have good output performance over all disturbance characteristic frequencies can be constructed by using the controller design for $\nu_0 = 0.01$ whose input is filtered with a low pass filter. The filter will render no control performance in the high frequency content disturbance range, whereas in the medium range the output rms is somewhat higher than the optimal but still considerably lower than that of the conventional one. Designing the controller for the middle range disturbances leads to greater loss of performance in the lower range. In order for the suggested controller to produce compatible, or lower, output rms than the conventional controller in the low characteristic frequency ranges rather high controller efforts result for disturbances in the medium range in particular for the shorter delay plants. The step response of such a controller is satisfactory and its offset acceptable, however, the controller action magnitude will be rather large.

Whereas the feedback controller for the output configuration is always realizable that for the input disturbance has a lower limit on the Lagrangian multiplier below which the controller in no longer realizable. The optimal output rms of the feedback controller at its realizability limit is considerably higher than the minimum of the optimal performance curve of the equivalent open loop system, particularly for disturbances of low characteristic frequency. The compromise design for overall performance is again that for a characteristic frequency of 0.01 at a Lagrangian multipier corresponding to the realizability limit. Its output rms is higher than that of the optimal step controller in the low characteristic frequency region and although the form of its step response is satisfactory it shows a large offset. For plants with a short delay the optimal performance for high freuqency content disturbances is lower than that of the uncontrolled plant and no filter is needed for the input of the feedback controller in the compromise design.

The performance surfaces obtained for a system with input disturbance for the typical but delayless plant are rather insensitive to the characteristic frequency used for the design. Thus for practical purposes the same performance over the disturbance range can be achieved by controllers designed for different disturbance characteristic frequenies. Any one controller also shows a performance which varies only little for low and medium frequency content disturbances. These results include the observation that optimal performance curves are close to one another over the low and medium frequency content disturbance region. This was used in previous work (11) as the justification for treating other cases with a single disturbance, in the medium range, for design and evaluation. This observation, however, is valid only in this particular situation.

To overcome the realizability limitation and obtain a better overall Wiener controller for the input disturbance configuration the controller was designed for a sum of two uncorrelated disturbances with low and high characteristic frequencies. Such a controller, designed for the characteristic frequencies 0.01 and 10 with the relative variance ratio of 0.9 at its realizability limit produces compatible performance with that of the conventional controller for low frequency content disturbances. Better output performance in the middle characteristic frequency range at a small loss of output performance for the very low frequency content disturbances can be achieved with the design for the characteristic frequency 0.01, 1.0 with the variance ratio of 0.9, also at its realizability limit. Its controller effort is also lower than the first two parameter disturbance design and its performance is not too far from the optimal one in the middle to lower region. For longer delay plants this recommended feedback controller must be used with a low pass filter on its input to avoid worse than uncontrolled performance for high frequency content disturbances. The offset for the step response for this controller is acceptable in most situations.

The two parameter disturbance design is also advantageous in the output disturbance configuration. The design for the characteristic frequencies 0.0 1 and 1.0 with relative variance ratio of 0.9 allows to achieve an output performance compatible with that of the conventional controller in the very low frequency content disturbance region while practically maintaining the good performance in the middle to low region of the single parameter disturbance design at a considerably lower controller effort than is called for by a previous design in this region. Also the acceptable offset for the step response is maintained but at substantially reduced controller action magnitude and no unrealizable feedback controllers were encountered.

In both configurations therefore the recommended overall feedback controller is one designed for a two parameter disturbance. To the compromise disturbance of the single parameter disturbance design an uncorrelated second disturbance with a characteristic frequency of 1 and a 0.1 relative variance intensity is added. Its output performance is compatible with that of the conventional controller in the very low characteristic frequency region and not too far from the optimal performance in the middle to low region. Whereever necessary a low pass filter at the input of this controller must be used to provide the optimal's controller performance for the high characteristic frequency range. The offset to a step response in both cases will be acceptable in most situations and controller actions are equally reasonable. Should, however, lower or zero offset be requeired a parallel weak integral action controller is suggested whose input is filtered by a very low pass filter. This will insure zero offset without interfereing with the control of random disturbances.

It is thus demonstrated that a feedback controller for random disturbances can be designed using Wieners method which is insensitive to the exact characterisitc frequency of the disturbance and whose performance is not too far from the optimal one at any particular characteristic frequency. This performance is considerably better than that of the conventional controller in any but the very low characteristic frequency range.

Although the controller parameters for the recommended overall design are the same for both input and output disturbance configurations the controllers are quite different. Also, throughout this work the two control configurations show different behaviour for a similar disturbance. The controller design does, therefore depend on the correct recognition of the point at which the disturbance enters the plant.

The probabilities for the output and controller action of a delayless plant with input disturbance and a Wiener controller designed for a single disturbance were developed. The disturbance was a stationary two state Markov process and since the plant has no delay the system output and the disturbance can be described as a joint Markov process which makes the analysis possible. The disturbance autocorrelation function is shown to be of negative exponential type with the sum of the transition intensities of the disturbance process playing the role of the characteristic frequency.

The system output density is shown to be of the beta type and that of the controller action a composite of two beta densities corresponding to the two states of the disturbance.

These densities show all the possible shapes for the beta density, as the characteristic frequencies are varied over the range of 0.01 to 100. Only at the high end of the range the densities become concave and resemble the normal density in shape. The two densities for the controller action seperate, for reasonably sized controller gains, into two distinct densities. The distributions for the two outputs for large and medium excursions from their mean are smaller and larger respectively than the corresponding normal distributions. The differences are not very large but when compared to the probabilities of the disturbance statescorresponding to these variable levels they are quite significant. This significance together with the seperation of the controller output densities seem to reflect the special character of the disturbance namely its confinement to only two states.For high characteristic frequencies these distributions converge to the normal one. In the low range the distributions differ most from the normal estimate. An approximate method is suggested for determining the distributions for low characteristic frequency disturbances by simulating the two level disturbance deterministically.

APPENDIX

I. Coefficients for Wiener Controllers

Unconstrained Design for Cascades with Delay and Input Disturbance.

$$K_n(s) = (1+s)^n \sum_{i=0}^{n} g_i s^i$$

a. $n = 2 \quad \nu \neq 1$

$$g_2 = f_1 + f_2 \; ; \quad g_1 = 2f_1 + f_2(1+\nu) + f_3 \; ; \quad g_0 = f_1 + \nu f_2 + \nu f_3$$

$$f_1 = \frac{e^{-\Delta\nu}}{(1-\nu)^2} \; ; \quad f_2 = \frac{[\Delta(\nu-1)-1]e^{-\Delta}}{(\nu-1)^2} \; ; \quad f_3 = \frac{e^{-\Delta}}{\nu-1}$$

b. $n = 2 \quad \nu = 1$

$$g_2 = f_1 \; ; \quad g_1 = 2f_1 + f_2 \; ; \quad g_0 = f_1 + f_2 + 2f_3$$

$$f_1 = \frac{1}{2}\Delta^2 e^{-\Delta} \; ; \quad f_2 = \Delta e^{-\Delta} \; ; \quad f_3 = \frac{1}{2} e^{-\Delta}$$

c. $n = 3 \quad \nu \neq 1$

$$g_3 = f_1 + f_2 \; ; \quad g_2 = 3f_1 + (\nu+2)f_2 + f_3 \; ; \quad g_1 = 3f_1 + (2\nu+1)f_2 + (\nu+1)f_3 + 2f_4$$

$$g_0 = f_1 + \nu(f_2 + f_3 + 2f_4)$$

$$f_1 = \frac{e^{-\Delta\nu}}{(1-\nu)^3}\ ; \qquad f_2 = [\Delta^2(\nu-1)^2 - 2\Delta(\nu-1) + 2]\frac{e^{-\Delta}}{2(\nu-1)^3}$$

$$f_3 = \frac{e^{-\Delta}}{(\nu-1)^2}[\Delta(\nu-1)-1]\ ; \qquad f_4 = \frac{e^{-\Delta}}{2(\nu-1)}$$

d. $n = 3 \quad \nu = 1$

$$g_0 = \frac{1}{6}\, e^{-\Delta}(\Delta^3 + 3\Delta^2 + 6\Delta + 6)\ ; \qquad g_1 = \frac{1}{2}\, e^{-\Delta}(\Delta^3 + 2\Delta^2 + 2\Delta)$$

$$g_2 = \frac{1}{2}\, e^{-\Delta}\Delta^2(\Delta+1)\ ; \qquad g_3 = \frac{1}{6}\, e^{-\Delta}\Delta^3$$

Constrained Two Parameter Disturbance Design for Typical Plant and Input Disturbance

a. $\nu_0 \neq 1$, $\nu_1 \neq 1$

$$g_2 = f_1 + f_2 + f_3 \; ; \qquad g_1 = f_1(\nu_1 + 1) + f_2(\nu_0 + 1) + f_3(\nu_0 + \nu_1)$$

$$g_0 = f_1 \nu_1 + f_2 \nu_0 + f_3 \nu_0 \nu_1$$

$$f_1 = \frac{(R_1 - \nu_0)\, e^{-\Delta \nu_0}}{\ell(1-\nu_0)(\nu_1 - \nu_0)(R + \nu_0)} \; ; \qquad f_2 = \frac{(R_1 - \nu_1)\, e^{-\Delta \nu_1}}{\ell(1-\nu_1)(\nu_0 - \nu_1)(R + \nu_1)}$$

$$f_3 = \frac{(R_1 - 1)\, e^{-\Delta}}{\ell(\nu_0 - 1)(\nu_1 - 1)(R+1)}$$

b. $\nu_0 = 1$, $\nu_1 \neq 1$

$$g_2 = f_1 + f_2 \; ; \qquad g_1 = 2f_1 + f_2(\nu + 1) + f_3 \; ; \qquad g_0 = f_1 + \nu_1(f_2 + f_3)$$

$$f_1 = \frac{(R_1 - \nu_1)\, e^{-\Delta \nu_1}}{\ell(1-\nu_1)^2 (R + \nu_1)} \; ; \qquad f_3 = \frac{(R_1 - 1)\, e^{-\Delta}}{\ell(\nu_1 - 1)\;(R+1)}$$

$$f_2 = \frac{e^{-\Delta}}{\ell(\nu_1 - 1)^2 (R+1)^2} \left[R_1(\nu_1 - R - 2) + R\nu_1 + 1 + \Delta(\nu_1 - 1)(R+1)(R_1 - 1) \right]$$

c. For any ν_0 and ν_1

$$A = \frac{(R^2 g_2 + g_0)^2 - R^2 g_1^2}{R(R_1^2 - R^2)(1 - R^2)} \; ; \qquad B = \frac{(R_1^2 g_2 + g_0)^2 - R_1^2 g_1^2}{R_1(R^2 - R_1^2)(1 - R_1^2)}$$

$$C = \frac{(g_2 + g_0)^2 - g_1^2}{(R^2 - 1)(R_1^2 - 1)} - \frac{2(g_2 + g_1 + g_0)\, e^{-\Delta}}{(R + 1)(R_1 + 1)}$$

II. Derivation of the Forward Kolmogorov Equation for a Markov Process with Mixed Discrete - Continuous State Space

In this appendix the probability equations will be derived for a homogenuous, or time independant, Markov process which arises form a continuous dynamic system excited by a discrete state Markov process. First the backward Kolmogorov differential equation for the transition probabilities will be derived. From this the forward Kolmogorov, or Fokker Planck, differential equation follows. The one time probability equation for the process can then be obtained from the forward equation. This develpment is due to Katz (5) and a similar development was also presented by Krambeck, Shinnar and Katz (6).

Consider a set of differential equations, in time, for the variables describing the system, $\underline{X}$. The time derivatives of these variables depend on the state of a time independant discrete state Markov process in addition to their dependance on $\underline{X}$. Denoting the states of the Markov process by α the set of differential equations can be written as:

$$\frac{d\underline{X}(t)}{dt} = \underline{f}(\underline{X},\alpha) \qquad A.1$$

Clearly $\underline{X}$ becomes a random process which, however, is not in it self Markovian. The joint process $\underline{X},\alpha$ is, never the less, a Markov process since from its current state its future develpment is determined. The transition probability for the joint process, i.e., the probability of moving from state α, $\underline{X}$ to state β, $\underline{Y}$ in the interval t, $p(\alpha,\underline{X};\beta,\underline{Y};t)$, is a probability for the states β and a probability density for $\underline{Y}$. It therefore fulfils the condition:

$$\sum_{\beta} \int_{\underline{Y}} p(\alpha,\underline{X};\beta,\underline{Y};t)\,d\underline{Y} = 1 \qquad \text{A.2}$$

where the integration is over the complete continuous variable space and the summation is over all discrete states. The Chapman - Kolmogorov equation, which follows from the Markov property takes the form:

$$p(\alpha,\underline{X};\beta,\underline{Y};s+t) = \sum_{\gamma} \int_{\underline{Z}} p(\alpha,\underline{X};\gamma,\underline{Z};s)\,p(\gamma,\underline{Z};\beta,\underline{Y};t)\,d\underline{Z} \qquad \text{A.3}$$

In order to derive the backward equation the short time transition probabilities are necessary, i.e. $p(\alpha,\underline{X};\beta,\underline{Y};t)$ for $t \downarrow 0$. These can be obtained from the transition intensities of the discrete process and the differential equation of $\underline{X}$. Two situations exist:

1. $\beta \neq \alpha$, that is the discrete process changes states in the short time t. In this case the transition probability, neglecting higher than first order terms in t becomes:

$$p(\alpha,\underline{X};\beta,\underline{Y};t) \approx \lambda_{\alpha\beta}\, t\, \delta(\underline{Y}-\underline{X}); \quad \text{for } t \downarrow 0 \qquad \text{A.4}$$

where $\lambda_{\alpha\beta}$ are the constant transition intensities from state α to β of the underlying discrete state process. Equation A.4 indicates no change of the continuous variable vector $\underline{X}$ in the short increment t. A simultanuous change of $\underline{X}$ would involve higher order terms in t since such a change would be goverened by the differential equation A.1.

2. $\beta = \alpha$, that is there is no transition of the discrete

state process in the short time increment t. This leads to the following equation for the transition probability:

$$p(\alpha,\underline{X};\beta,\underline{Y};t) \approx \Big(1 - t\sum_{\beta\neq\alpha}\lambda_{\alpha\beta}\Big)\delta(\underline{Y}-\underline{X}-t\underline{f}(\alpha,\underline{X})) \quad \text{for } t\downarrow 0 \qquad A.5$$

Defining

$$\lambda_{\alpha\alpha} = -\sum_{\beta\neq\alpha}\lambda_{\alpha\beta} \qquad A.6$$

equation A.5 becomes:

$$p(\alpha,\underline{X};\beta,\underline{Y};t) \approx (1+\lambda_{\alpha\alpha}t)\delta(\underline{Y}-\underline{X}-t\underline{f}(\alpha,\underline{X})) \quad \text{for } t\downarrow 0 \qquad A.7$$

In this case a first order approximation for a change in $\underline{X}$ can be accomodated with first order terms in t.

Thetransition probabilities of equations A.5 and A.7 are now introduced into the Chapman - Komogorov equation, equation A.3, for the case where s tends to zero. After the indicated integration is performed, using the properties of delta functions, the result is:

$$p(\alpha,\underline{X};\beta,\underline{Y};s+t) = (1+\lambda_{\alpha\alpha}s)p(\alpha,\underline{X}+s\underline{f}(\alpha,\underline{X});\beta,\underline{Y};t)$$

$$+\sum_{\gamma\neq\alpha}\lambda_{\alpha\gamma}s\,p(\gamma,\underline{X};\beta,\underline{Y};t) \quad \text{for } s\downarrow 0 \qquad A.9$$

Expanding $p(\alpha,\underline{X}+s\underline{f}(\alpha,\underline{X});\beta,\underline{Y};t)$ around $\underline{X}$, retaining only first order terms in s, it becomes:

$$p(\alpha,\underline{X}+s\underline{f}(\alpha,\underline{X});\beta,\underline{Y};t) \cong p(\alpha,\underline{X};\beta,\underline{Y};t)$$

$$+ s\sum_{i} f_i(\alpha,\underline{X})\frac{\partial}{\partial x_i}p(\alpha,\underline{X};\beta,\underline{Y};t) \quad \text{for } s\downarrow 0 \qquad A.9$$

Where x_i and $f_i(\alpha, \underline{X})$ are the i'th component of the variable vector $\underline{X}$ and its derivative vector $\underline{f}(\alpha, \underline{X})$ respectively. Introducing this expression into equation A.8 rearranging and going to the limit of $s \downarrow 0$, as was done in the derivation of equation 12.5, results in:

$$\frac{\partial}{\partial t} = p(\alpha, \underline{X}; \beta, \underline{Y}; t) = \sum_i f_i(\alpha, \underline{X}) \frac{\partial}{\partial x_i} p(\alpha, \underline{X}; \beta, \underline{Y}; t)$$

$$+ \sum_{\gamma} \lambda_{\alpha\gamma} p(\gamma, \underline{X}; \beta, \underline{Y}; t)$$

This is the backward equation and derives its name from the fact that it was obtained by letting the first time interval, s, tend to zero with the result that the operations on the right hand side of equation A.10 are on the first set of arguments of $p(\alpha, \underline{X}; \beta, \underline{Y}; t)$.

It was seen in chapter 12 that the single time probability of a Markov process can be obtained from the initial single time probability and the transition probability using the Markov property. The analog expression for the single time probability, $p(\beta, \underline{Y}; t)$, in this case is:

$$p(\beta, \underline{Y}; t) = \sum_{\alpha} \int_{\underline{X}} p(\alpha, \underline{X}; 0) \, p(\alpha, \underline{X}; \beta, \underline{Y}; t) \qquad A.11$$

where the operations performed on the right hand side are on the first pair of arguments of the transition probability. As in chapter 12 the differential equation for the single time probabilities can be obtained by applying equation A.11 to the forward equation for the transition probability.

The forward Komogorov differential equation for the transiton probabilities can be conveniently obtained by transforming the backward equation whose derivation is easier. The backward equation can be written as:

$$\frac{\partial}{\partial t} p(\alpha, \underline{x}; \beta, \underline{y}; t) = G_{\alpha \underline{x}}\, p(\alpha, \underline{x}; \beta, \underline{y}; t) \qquad \text{A.12}$$

where $G_{\alpha \underline{x}}$ is the linear operator of the right hand side of equation A. 10 and where the subscripts are a reminder for the arguments of p which are involved. Considering again the Chapman - Kolmogorov relationship for this process, equation A.3, the partial derivative with respect to the first time interval can be taken on both sides yielding:

$$\frac{\partial}{\partial t} p(\alpha, \underline{x}; \beta, \underline{y}; t+s) = \sum_{\gamma} \int_{\underline{z}} \frac{\partial}{\partial t}[p(\alpha, \underline{x}; \gamma, \underline{z}; t)]\, p(\gamma, \underline{z}; \beta, \underline{y}; s)\, d\underline{z} \qquad \text{A.13}$$

Taking the partial derivative of equation A.3 with respect to the second time interval, s, gives:

$$\frac{\partial}{\partial s} p(\alpha, \underline{x}; \beta, \underline{y}; s+t) = \sum_{\gamma} \int_{\underline{z}} p(\alpha, \underline{x}; \gamma, \underline{z}; t) \frac{\partial}{\partial s} p(\gamma, \underline{z}; \beta, \underline{y}; s)\, d\underline{z}$$

$$= \sum_{\gamma} \int_{\underline{z}} p(\alpha, x; \gamma, \underline{z}; t)\, G_{\gamma \underline{z}}[p(\gamma, z; \beta, \underline{y}; s]\, d\underline{z} \qquad \text{A.14}$$

Defining the adjoint operator G* as satifying the identity

$$\sum_{\gamma}\int_{\underline{z}} p(\alpha,\underline{x};\gamma,\underline{z};t)\,G_{\gamma,\underline{z}}[p(\gamma,\underline{z};\beta,\underline{y};s)]\,d\underline{z}$$

$$= \sum_{\gamma}\int_{\underline{z}} G^{*}_{\gamma\underline{z}}[p(\alpha,\underline{x};\gamma,\underline{z};t)]\,p(\gamma,\underline{z};\beta,\underline{y};s)\,d\underline{z} \qquad A.15$$

equation A.14 becomes:

$$\frac{\partial}{\partial s}p(\alpha,\underline{x};\beta,\underline{y};t+s) = \sum_{\gamma}\int_{\underline{z}} G^{*}_{\gamma\underline{z}}[p(\alpha,\underline{x};\gamma,\underline{z};t)]\,p(\gamma,\underline{z};\beta,\underline{y};s)\,d\underline{z} \qquad A.16$$

The two time derivatives in equation A.13 and A.16 are identical. Taking the difference of these two equations, gives:

$$\sum_{\gamma}\int_{\underline{z}} p(\gamma,\underline{z};\beta,\underline{y};s)\left\{\frac{\partial}{\partial t}p(\alpha,\underline{x};\gamma,\underline{z};t) - G^{*}_{\gamma\underline{z}}[p(\alpha,\underline{x};\gamma,\underline{z};t)]\right\} = 0 \qquad A.17$$

Taking the limit as s tends to zero and noting that:

$$p(\gamma,\underline{z};\beta,\underline{y};s) = \delta_{\alpha\beta}\,\delta(\underline{z}-\underline{y}) \quad \text{for} \quad s\downarrow 0 \qquad A.18$$

equation A.17 becomes:

$$\frac{\partial}{\partial t}p(\alpha,\underline{x};\beta,\underline{y};t) = G^{*}_{\beta,\underline{y}}[p(\alpha,\underline{x};\beta,\underline{y};t)] \qquad A.19$$

This is the forward equation since $G^{*}_{\beta\underline{y}}$ operates on the forward pair of the transition probability arguments. From the definition of the adjoint operator in equation A.15 the forward equation becomes:

$$\frac{\partial}{\partial t}p(\alpha,\underline{x};\beta,\underline{y};t) = -\sum_{i}\frac{\partial}{\partial y_i}[f_i(\beta,\underline{y})\,p(\alpha,\underline{x};\beta,\underline{y};t)]$$

$$+ \sum_{\gamma}\lambda_{\gamma\beta}\,p(\alpha,\underline{x};\gamma,\underline{y};t) \qquad A.20$$

where the transition probabilities at $\underline{Y}=\infty$ were assumed to vanish. Both sides of equation A.20 can now be multiplied by the initial single time probability $p(\alpha, \underline{x}, 0)$ and the result summed over all α's and integrated over the $\underline{X}$ space. These operations are interchangeable with both the time derivative and the operations of $G^*_{\beta\underline{Y}}$ in equation A.20 since they involve the backward argument while the latter are made with respect to the forward arguments. Using equation A.11 the result is the differential equation for the single time probabilities, i.e.:

$$\frac{\partial}{\partial t}p(\beta,\underline{Y};t) = -\sum_i \frac{\partial}{\partial y_i}\left[f_i(\beta,\underline{Y})p(\beta,\underline{Y};t)\right] + \sum_\gamma \lambda_{\gamma\beta}\, p(\gamma,\underline{Y};t) \qquad A.21$$

This equation governs the probability behaviour of the joint process and can be solved with the adequate initial and boundry, or normalization, conditions. The initial conditions are clearly the initial probabilities of the process.
Note that equation A.21 is a short hand notation for a set of equations whose number equals the number of discrete states β. The dimension of $\underline{Y}$ determines the number of independant varialbes in addition to time in euqation A.21. The stationary probabilities are given by equationA.21 with the time derivative set equal to zero.

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NOMENCLATURE

Note: The same symbols are used for time functions and their Fourier and Laplace transforms with the time argument replaced by s.

A Reactor's conversion parameter, equation 3.5

b Controllers's reset rate

B Controllers's reset rate x gain

C, C_o, C_1, C_i Reactor Concentrations

d Controllers derivative time

D Controllers derivative time x gain

$f_1 \dots f_3$ Coefficients

F Volumetric flow rate

$\underline{f}(\alpha, \underline{X})$ Derivative vector of $\underline{X}$

g, $g_o \dots g_3$ Coefficients

$G(s)$ Plants transfer function

$G_{\alpha \underline{x}}$ Operator of Kolmogorovs backward equations: A.12

$G^*_{\alpha \underline{x}}$ Adjoint operator of $G_{\alpha x}$, see equation A.15

$H(s)$ Feedback controller's transfer function

h See equation 12.60

$i(t)$ Ideal output of filter-predictor

i $\sqrt{-1}$

k Reaction rate constant

k_α See equation 12.61

$K(s)$ Cascaded compensator's transfer function

K_g, K_c Controller gains

l Lagrangian multiplier

l_r Realizability limit for Lagrangian multiplier

l_o, l_1 Normalized random disturbance states

$M(t)$ Controller's output

$m(t)$ Normalized controllers output, see equations 3.17 or 3.30

$n(t)$ Noise signal

n Integer

$p(x)$ Probability density of x

$\underline{P}(t)$ Single time probability vector, discrete space

$p_i(t)$ Element of single time probability vector, discrete space

$\underline{\underline{P}}(t)$ Transition probability matrix, discrete space

$p_{ij}(t)$ Element of transition probability matrix, discrete space

$\bar{p}_o$, $\bar{p}_1$ Stationary probability, discrete space

$p(\alpha, \underline{X}, t)$ Joint single time probability density for $\underline{X}$ and α

$p(\alpha, \underline{X}, \beta, \underline{Y}, t)$ Joint transition probability density for $\underline{Y}$ and β

$p_\alpha(x)$ Shorthand for $p(\alpha, x, t)$ for the stationary case

$q(t)$ Output of linear system

Q Probability distribution

R $\sqrt{1. + 1/1}$

R_1 Coefficient, see equation 11.3

$r(t)$ Random signal

s The complex Fourier and Laplace transform variable

$S_{xx}(s)$ Power spectrum of x

t Time

$u(t)$ Effective disturbance

V_s, V_p Reactor volumes

$v(t)$ Input of linear system

$w(s)$ Transfer function of a linear system

$w(s)$	Normalized reactor output in chapter 3, equation 3.30
$\underline{W}(t)$	Two time probability matrix
$w_{ij}(t)$	Element of the two time probability matrix
x	Variance ratio for the two parameter disturbance, chapter 11.
	Transition intensities ratio, chapter 12
$\underline{X}$	Probability space state vector
$y(t)$	Normalized disturbance, Equations 3.17 or 3.30
$\underline{Y}$	Probability space state vector
y_o, y_1	Disturbance levels
z_r	Zeros in the right hand s plane
z^*	Reduced control system output
$z^*_\alpha(m)$	See equation 12. 62
$\underline{Z}$	Probability space state vector
$z(t)$	Normalized systems output, see equations 3.17 or 3.30

Greek Letters

α	Probability space state index
β	Probability space state index
γ	Probability space state index
$\gamma(s)$	See equation 5.24
$\Gamma(s)$	See equation 5.22
$\Delta(s)$	See equation 5.22
Δ	Normalized plant delay parameter
$\delta(x)$	Dirac delta function
δ_{ij}	Kronecker's delta

ε	Parameter
θ	Normalized time, see equation 3.3
$\underline{\underline{\Delta}}$	Transition intensities matrix
$\lambda_{\alpha\beta}$	Element of transition intensities matrix
λ_0, λ_1	Transition intensities for two state process
λ	Reactor parameter
ν	Disturbance characteristic frequency
ν_0, ν_1	Design disturbance characteristic frequency
ρ	Magnitude of complex function
$\sigma, \sigma_m, \sigma_z$	Standard derivations
$\sigma_{z/N}$	Systems output standard derivation scaled by that of the uncontrolled plant.
τ	Translation time
τ_s, τ_p	Reactor average residence times
ϕ	Phase of complex function
$\varphi_{xx}(\tau)$	Autocorrelation function of x
ω	Frequency in radians

Special Symbols (as applied to x)

$\langle x \rangle$	Expected value of x
$\tilde{x}(t)$	Derivation of x around its mean
$x(s)^+$	Factor of x(s) containing all zeros and poles of left hand s plane
$x(s)^-$	Factor of x(s) containing all zeros and poles of right hand s plane
$x(t)_+$	Positive time portion of x(t)
$x(t)_-$	Negative time portion of x(t)

VITA

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After completing elementary and highschool education in Petach-Tiqwa, Israel, he started studies at the department of Chemistry at the Swiss Institute of Technology (E.T.H), Zurich, Switzerland, in 1954. These studies were concluded, after an interruption for military service (1956 - 1958), with the degree of Dipl. Ing. Chem., in Jannuary, 1961. The author pursued his studies in this country in February 1961, and received his M.S. degree in Chemical Engineering from the Massachusetts Institute of Technology (M.I.T), Cambridge, Mass. in June 1962. The following three years he was employed as research Engineer by Allied Chemical Co., in Morristown, N.J.. Since July of 1965 he was working towards his Ph.D. degree in Chemical Engineering at the City College of New York.