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Finite Fourier Transform Approximation and Riemann Sum Approximation for Functions that Decay in Time and Frequency

by

Jeffrey Litwin

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Professor Richard Sacksteder

Professor Alphonse Vasquez

Supervisory Committee

THE CITY UNIVERSITY OF NEW YORK

Abstract

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Adviser: Professor Louis Auslander

For functions with finite time and frequency energy moments, we find upper bounds for the error of finite Fourier transform approximation to the Fourier transform. The error can be measured as the maximum error over all of the points of the FFT, or using a discrete $\mathbf{L}^2$ distance, or using a continuous $\mathbf{L}^2$ distance.

Using the machinery developed, we also find an upper bound for the error of approximating the $\mathbf{L}^2$ norm of a function by a Riemann sum. From this result, an upper bound is also derived for the error of approximating the $\mathbf{L}^2$ inner product, as well as the error of approximating the integral of an $\mathbf{L}^1$ function, by a Riemann sum.

As an application of the $\mathbf{L}^2$ norm approximation theorem, we prove an analog of the Landau-Pollak-Slepian approximate dimension theorems for a certain set of functions that is approximately time-and-bandlimited for large duration N and bandwidth M. This set can be approximately parameterized with NM parameters, with the error approaching zero as NM approaches infinity.

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Contents

0 Introduction

One of the most important operations in the signal processing industry is taking the Fourier transform of a function. With digital signal processing, the finite Fourier transform (FFT) is used instead. A question of great practical importance immediately arises: How good an approximation is the FFT?

The Fourier transform at a particular frequency is nothing more than an integral. And, the FFT at a particular frequency is a special case of numerical integration using Riemann sum approximation. It would not be surprising if the theory of Riemann sum approximation allows us to derive an error bound for the FFT. In fact, we will do just the opposite. We will use an error bound for the FFT to derive an error bound for more general Riemann sum approximation.

To obtain any results, it is clear that the class of functions must be restricted somehow. For example, if there are no restrictions on the oscillations, knowing the function at a finite set of points tells us nothing about the function anywhere else. We will consider functions that decay sufficiently fast in time and frequency (in the sense that will be described below). For this class of functions, we will derive an error bound for FFT approximation to the Fourier transform and an error bound for more general Riemann approximation.

0.1 Background

Previous Results

All previous error bounds that I am aware of come out of the observation that we can rearrange the terms of the Poisson summation formula and the formula becomes the desired error bound. It is not clear who first discovered this. See Butzer-Stens [BS] and Briggs-Henson [BH, Chapter 6].

Assume that f satisfies whatever conditions are necessary so that the Poisson summation formula is valid. Consider the case of approximating the Fourier transform, denoted $\hat{f}(\gamma)$, by the FFT at $\gamma = 0$. That is, we will approximate

$$\hat{f}(0) = \int f(t)dt \quad \text{by} \quad \frac{1}{M} \sum_{k=-NM/2}^{NM/2-1} f(k/M).$$

N and M will be assumed throughout this introduction to be even positive numbers. N is the timelength of the approximation and M is the sampling rate.

Note 1: By focusing on what seems to be just the special case of FFT approximation where $\gamma = 0$, we are actually considering the general case of Riemann sum approximation.

Note 2: For the case of $\gamma \neq 0$, we can apply the following method to $fe^{-2\pi it\gamma}$. In fact, by the same method, we can obtain an upper bound for the maximum error over all the points upon which the FFT is defined. (See Section 5.1.)

One form of the Poisson summation formula is

$$\sum_{k\in\mathbf{Z}} \hat{f}(kM) = \frac{1}{M}\sum_{k\in\mathbf{Z}} f(k/M).$$

A rearrangement of the terms yields the following error bound:

$$\left|\hat{f}(0) - \frac{1}{M}\sum_{k=-\frac{1}{2}NM}^{\frac{1}{2}NM-1} f(k/M)\right| \le \left|\frac{1}{M}\Big(\sum_{k<-\frac{1}{2}NM} + \sum_{k>\frac{1}{2}NM-1}\Big) f(k/M)\right| + \left|\sum_{k\neq 0}\hat{f}(kM)\right|.$$

The error bound can be interpreted as follows. Discretization results in two types of error. Truncation error comes from the fact that the FFT only utilizes function values over a finite duration. Aliasing error comes from the fact that the FFT is limited to a finite sampling rate. The first term of the error bound is primarily caused by the truncation error. The second term is primarily caused by the aliasing error.

The form of the error bound provides practical information about how the error can be reduced. Let's say the first term of the error bound is much greater than the second. Then increasing N will be much more effective in reducing the error than increasing M.

At this point, we can impose various conditions on f to obtain many different theorems. One possibility is as follows. Let j be an integer greater than 1. Let's assume that $f^{(j-1)}$ is absolutely continuous (which implies that $f^{(j)}$ exists almost everywhere) and that $f^{(k)} \in \mathbf{L}^1(\mathbf{R})$ for all $k \le j$.

Then, using integration by parts j times,

$$\hat{f}(\gamma) = \int f(t)e^{-2\pi i t\gamma}dt = (\frac{1}{2\pi i\gamma})^j \int f^{(j)}(t)e^{-2\pi i t\gamma}dt.$$

We will show in the next paragraph that the boundary terms from all of the integrations by parts are zero.

Consider the following two typical terms involved in one of the the above j integrations by parts:

$$\lim_{A,B\to\infty}(\frac{1}{2\pi i\gamma})^k \int_{-A}^{B} f^{(k)}(t)e^{-2\pi i t\gamma}dt \quad \text{and} \quad \lim_{A,B\to\infty}(\frac{1}{2\pi i\gamma})^{k+1} \int_{-A}^{B} f^{(k+1)}(t)e^{-2\pi i t\gamma}dt.$$

Because of our L^1 assumption, these limits exist for any fixed γ and all $0 \leq k \leq j-1$. Therefore, the boundary term from each of the integrations by parts must also approach a limit as A, B approaches infinity. The boundary term is

$$\lim_{A,B\to\infty}\left(-(\frac{1}{2\pi i\gamma})^{k+1} f^{(k)}(B)e^{-2\pi i B\gamma} + (\frac{1}{2\pi i\gamma})^{k+1} f^{(k)}(-A)e^{2\pi i A\gamma}\right).$$

Because of the exponential factors, the only way that this limit can exist is if $\lim_{|t|\to\infty} f^{(k)}(t) = 0$. Therefore, the boundary term also equals zero.

From the expression for $\hat{f}$ in terms of $f^{(j)}$, we see that $\hat{f}$ is $O(|\gamma|^{-j})$. Now, let's estimate the second error term in the rearrangement of the Poisson summation formula. The decay of $\hat{f}$ tells us that the second error term is bounded by a constant times $\sum(kM)^{-j}$, which is $O(M^{-j})$.

We can also make the same assumptions for $\hat{f}$ that we did for f. Assume that $\hat{f}^{(j-1)}$ is absolutely continuous and that $\hat{f}^{(k)} \in \mathbf{L}^1(\mathbf{R})$ for all $k \leq j$. By similar reasoning, we see that f is $O(|t|^{-j})$. Therefore, the first error term is bounded by a constant times $\int_{N/2}^{\infty} t^{-j} dt$, which is $O(N^{-(j-1)})$.

$\mathbf{L}^2$ Approach

The prior results show that if the derivatives of f and $\hat{f}$ are absolutely continuous and in $\mathbf{L}^1(\mathbf{R})$, then we can obtain error bounds for both FFT approximation and Riemann sum approximation. What happens if we replace the hypothesis that the derivatives are in $\mathbf{L}^1$ with the hypothesis that the derivatives are in $\mathbf{L}^2$? That is the question that we will address.

Note that we are weakening the hypothesis (on the first $j-1$ derivatives). We saw above that if $f^{(k)}$ is absolutely continuous and $f^{(k)}, f^{(k+1)} \in \mathbf{L}^1$, then $\lim_{|t|\to\infty} f^{(k)}(t) = 0$. This implies that $|f^{(k)}(t)|^2 < |f^{(k)}(t)|$ for sufficiently large t which implies that $f^{(k)} \in \mathbf{L}^2$. Since our hypothesis is weaker, we should expect weaker results.

There is a simpler way to describe the conditions in our hypothesis. The assumption that $f^{(j-1)}$ is absolutely continuous and that $f^{(k)} \in \mathbf{L}^2(\mathbf{R})$ for all $k \leq j$ is equivalent to the assumption that the j^{th} frequency moment is finite. This moment is defined by

$$D_j^2 = \int \gamma^{2j} |\hat{f}|^2 d\gamma.$$

This result is essentially the content of Lemmas 1.3 and 1.4.

The dual of this result is also true. The assumption that $\hat{f}^{(j-1)}$ is absolutely continuous and that $\hat{f}^{(k)} \in \mathbf{L}^2(\mathbf{R})$ for all $k \leq j$ is equivalent to the assumption that the j^{th} time moment is finite. This moment is defined by

$$C_j^2 = \int t^{2j} |f|^2 dt.$$

This equivalence may make our results more practical than the results obtained with the $\mathbf{L}^1$ hypothesis. With the $\mathbf{L}^1$ hypothesis, the error bound depends on the $\mathbf{L}^1$ norms of the j^{th} derivatives. This information is usually difficult to calculate. Our approach uses energy moments, which may be easier to calculate.

0.2 Notation and Definitions

The version of the Fourier transform that we use is

$$\hat{f}(\gamma) = \lim_{A\to\infty} (\text{in } \mathbf{L}^2) \int_{-A}^{A} f(t) e^{-2\pi i t\gamma} dt.$$

The limit refers to convergence in $\mathbf{L}^2(\mathbf{R})$.

$||f||$ denotes the norm of f in $\mathbf{L}^2(\mathbf{R})$. The $\mathbf{L}^2([-N/2, N/2])$ norm is denoted by $||f||_{[-N/2,N/2]}$.

The following norm is described in more detail in Section 2.2. Let $\frac{1}{M}\mathbf{Z}_N$ be the set of points $\{k/M\}_{k=-\frac{1}{2}NM,\ldots,\frac{1}{2}NM-1}$. This notation is supposed to remind the reader of the set of points $\{k/M\}$ modulo N. The $\mathbf{L}^2$ norm for functions defined on $\frac{1}{M}\mathbf{Z}_N$ is defined to be

$$||f||^2_{D(M,N)} = \frac{1}{M}\sum_{k=-\frac{1}{2}NM}^{\frac{1}{2}NM-1} |f(k/M)|^2.$$

Our version of the FFT is an operator from $\mathbf{L}^2(\frac{1}{M}\mathbf{Z}_N)$ onto $\mathbf{L}^2(\frac{1}{N}\mathbf{Z}_M)$ defined by

$$\tilde{f}(n/N) = \frac{1}{M}\sum_{k=-\frac{1}{2}NM}^{\frac{1}{2}NM-1} f(k/M)e^{-2\pi ikn/NM}, \quad n = -\frac{1}{2}NM,\ldots,\frac{1}{2}NM-1.$$

The FFT of f will be denoted by $\tilde{f}$.

Note: The nonstandard scaling in the definitions of the FFT and the discrete norm are used to make the FFT is a unitary operator. With these definitions,

$$||\tilde{f}||_{D(N,M)} = ||f||_{D(M,N)}.$$

(See Section 2.2 for proof.)

The FFT can easily be extended to all of $\mathbf{R}$ so that it is a periodic function with period M. The *extended FFT*, denoted by $f^\sharp$, is

$$f^\sharp(\gamma) = \frac{1}{M}\sum_{k=-\frac{1}{2}NM}^{\frac{1}{2}NM-1} f(k/M)e^{-2\pi ik\gamma/M}.$$

Note: The extended FFT is also a unitary operator. By Parseval's equality for periodic functions,

$$||f^{\sharp}||_{[-M/2,M/2]} = ||f||_{D(M,N)}.$$

The periodization operator P_N is defined by

$$(P_N f)(t) = \sum_{k\in\mathbf{Z}} f(t+kN), \quad f \in \mathbf{L}^2(\mathbf{R}) \text{ and decays sufficiently fast.}$$

See Section 2.2 for a more precise definition.

$\overline{x}$ means the the complex conjugate of x.

$\lfloor x \rfloor$ means the greatest integer less than or equal to x.

0.3 Executive Summary

FFT Approximation

From now on, throughout this paper, all functions are assumed to be in $\mathbf{L}^2(\mathbf{R})$, unless stated otherwise. We also assume that f and $\hat{f}$ are continuous. (In fact, most of our results depend on the existence of time and frequency energy moments, which is a stronger condition than the continuity of f and $\hat{f}$.)

Because of our assumptions, both the Fourier transform and the FFT exist. We would like to measure the error in approximating the Fourier transform by the FFT.

The error in the FFT approximation can be measured in many ways. One way is to measure the maximum difference over all of the points upon which the FFT is defined.

Since $\hat{f}$ is assumed continuous, we can evaluate $\hat{f}$ at $\frac{1}{N}\mathbf{Z}_M$. Therefore, we can also measure the error by calculating the discrete $\mathbf{L}^2$ distance, $||\tilde{f} - \hat{f}||_{D(N,M)}$. We can also measure the error by calculating the continuous $\mathbf{L}^2$ distance, $||f^\sharp - \hat{f}||_{[-M/2,M/2]}$.

With the $\mathbf{L}^1$ hypothesis, a key step in deriving an error bound was to show that f and $\hat{f}$ decay at a sufficiently fast rate. We obtain a similar result easily in Lemma 1.5. If C_j and D_1 are finite, then

$$|f^2(t)| \leq A|t|^{-j},$$

where A is a constant. Unfortunately, this lemma doesn't give us any information about what the constant is.

However, a different approach yields an even stronger result. Define the discrete tail energy to be

$$\tilde{E}^2_{N,M} = \frac{1}{M} \sum_{|k|>\frac{1}{2}NM} |f(k/M)|^2.$$

In Lemma 2.3, we show that if C_j and D_1 are finite, then the discrete tail energy is $O(N^{-j})$, and we also have an upper bound for the constant involved.

Notice that this implies that $f^2(t)$ is $O(|t|^{-j})$. At this point, we could plug this information into the rearrangement of the Poisson summation formula as we did with the $\mathbf{L}^1$ hypothesis to obtain a bound for the maximum error. This approach results in the first FFT approximation theorem (Theorem 5.1).

By proceding somewhat differently, we can obtain bounds for the discrete and continuous $\mathbf{L}^2$ error. The following briefly describes how we derive an upper bound for the discrete $\mathbf{L}^2$ error.

Since the discrete energy in the tail is bounded, one would expect that we can find an upper bound for the discrete $\mathbf{L}^2$ distance between a function and its periodization. In fact, we prove the following periodization comparison lemma (Lemma 2.8). If C_j and D_1 are finite, where $j > 2$, then

$$||P_N f - f||_{D(M,N)} \leq K N^{-j/2},$$

where K depends only on C_j and D_1 (and inversely on N and M).

This lemma allows us to replace $\tilde{f}$ and $\hat{f}$ in the definition of the discrete $\mathbf{L}^2$ distance between the FFT and the Fourier transform with periodizations (using the triangle inequality). Of course, replacing $\tilde{f}$ with $(P_N f)^\sim$ introduces an error of $K_1 N^{-j/2}$ and replacing $\hat{f}$ with $P_M \hat{f}$ introduces an error of $K_2 M^{-j/2}$.

Now we must calculate $||(P_N f)^\sim - P_M \hat{f}||_{D(N,M)}$. For this, we use the generalized Poisson summation formula (Theorem 4.1). The generalized Poisson summation formula says that under the hypotheses of the Poisson summation formula, $(P_N f)^\sim$

is equal to the sampled $P_M\hat{f}$. Therefore, in our case, where the hypotheses are met because of energy moment conditions, the norm of the difference is zero.

We have just derived the desired upper bound for the discrete $\mathbf{L}^2$ error (Theorem 5.2). If C_j and D_j are finite, where $j > 2$, then

$$||\tilde{f} - \hat{f}||_{D(N,M)} \leq K_1 N^{-j/2} + K_2 M^{-j/2}.$$

K_1 depends only on C_j and D_1 (and inversely on N and M). K_2 depends only on D_j and C_1 (and inversely on N and M).

We also obtain an upper bound for the continuous $\mathbf{L}^2$ error (Theorem 5.3, see Section 5.3 for the proof). If C_j and D_j are finite, where $j > 2$, then

$$||f^\sharp - \hat{f}||_{[-M/2,M/2]} \leq K_1 N^{-j/2} + K_2 M^{-j}.$$

K_1 depends only on C_j and D_1 (and inversely on N and M). K_2 depends only on D_j. Notice that the second error term is better than the corresponding term in the previous result.

Riemann Sum Approximation

Since we are focusing on the $\mathbf{L}^2$ norm error, our approach does not automatically provide an error bound for more general Riemann sum approximation. However, from the third FFT approximation theorem, we derive the $\mathbf{L}^2$ norm approximation theorem (Theorem 6.1). Then, we bootstrap our way to derive error bounds for more general Riemann sum approximation.

From the third FFT approximation theorem and the triangle inequality, we have

$$\Big| \, ||f^\sharp||_{[-M/2,M/2]} - ||\hat{f}||_{[-M/2,M/2]} \Big| \leq K_1 N^{-j/2} + K_2 M^{-j},$$

As mentioned above, the first term of this inequality, $||f^\sharp||_{[-M/2,M/2]}$, is equal to $||f||_{D(M,N)}$ by Parseval's equality for periodic functions.

The second term, $||\hat{f}||_{[-M/2,M/2]}$, can be replaced by $||\hat{f}|| = ||f||$ at the expense of increasing K_2. This is a consequence of Chebyshev's inequality.

Making these two substitutions proves the $\mathbf{L}^2$ norm approximation theorem. If C_j and D_j are finite, where $j > 2$, then

$$\Big| \, ||f|| - ||f||_{D(M,N)} \Big| \leq K_1 N^{-j/2} + K_2 M^{-j}.$$

K_1 depends only on C_j and D_1 (and inversely on N and M). K_2 depends only on D_j.

From the $\mathbf{L}^2$ norm approximation theorem, we derive an upper bound for the error of approximating the inner product of two $\mathbf{L}^2$ functions by a Riemann sum approximation (Theorem 7.2). The key observation is that the inner product can be expressed as a linear combination of $\mathbf{L}^2$ norms by the polarization identity:

$$\langle f, g \rangle = \frac{1}{4}(||f+g||^2 - ||f-g||^2 + i||f-ig||^2 - i||f+ig||^2).$$

This yields the inner product approximation theorem. Let $f, g \in \mathbf{L}^2(\mathbf{R})$. If the j^{th} energy moments are finite, where $j > 2$, then the error in discretizing the inner product is bounded by

$$K_1 N^{-j/2} + K_2 M^{-j}.$$

The K_i depend on the energy moments and $||f|| + ||g||$ (and inversely on N and M in the case of K_1).

The next step is to note that any $\mathbf{L}^1$ function can be expressed as the product of two $\mathbf{L}^2$ functions. If the two factors decay sufficiently fast in time and frequency, then we can also derive an upper bound for this general case of Riemann sum approximation (Theorem 7.3).

Approximate Parameterization

Section 8 contains another application of the $\mathbf{L}^2$ norm approximation theorem. We derive an analog of the Landau-Pollak-Slepian approximate dimension theorem for a set of functions with bounded energy moments. This set is approximately time-and-bandlimited for sufficiently large N and M. We show that this set can be approximately parameterized with the number of parameters equal to NM. As NM approaches infinity, the error of the approximate parameterization approaches zero.

1 Energy and Energy Moments

Subsection 1 contains some definitions about energy and energy moments. Various results about energy moments that we will need are then given in Subsections 2 and 3.

1.1 Background

$\int_a^b |f|^2 dt$ is referred to as the *(continuous) energy* of f between a and b. We will present several other definitions that describe how the energy of a function is distributed. N and M represent arbitrary positive numbers, unless stated otherwise.

The *continuous tail energy* of a function f, denoted by $E_N^2(f)$ or E_N^2, is defined as

$$E_N^2 = E_N^2(f) = \int_{|t|>\frac{1}{2}N} |f|^2 dt.$$

Where the function is understood, we will write E_N^2 for $E_N^2(f)$. The square root of E_N^2 will be denoted by E_N.

We also define an analogous concept for the discrete energy in the tail. This definition will depend on two parameters. The *discrete tail energy* of a function f, denoted $\tilde{E}_{N,M}^2$, is defined as

$$\tilde{E}_{N,M}^2 = \tilde{E}_{N,M}^2(f) = \frac{1}{M} \sum_{|k|>\frac{1}{2}NM} |f(k/M)|^2.$$

This expression can be considered a Riemann sum approximation to the integral defining E_N^2. The square root of $\tilde{E}_{N,M}^2$ will be denoted by $\tilde{E}_{N,M}$.

Gabor [G] introduced the idea of energy moments. The k^{th} *time energy moment*, denoted C_k^2, is defined as

$$C_k^2 = C_k^2(f) = \int_{-\infty}^{\infty} t^{2k}|f|^2 dt, \quad k \in \mathbf{Z}^+.$$

(Technically, we should call this the $2k^{th}$ moment, but we opt for the simpler definition that corresponds to the subscript on C_k.) The square root of C_k^2 will be denoted by C_k. One may also think of C_k as the $\mathbf{L}^2(\mathbf{R})$ norm of $t^k f$.

Similarly, the k^{th} *frequency energy moment*, denoted D_k^2, is defined as

$$D_k^2 = D_k^2(f) = \int_{-\infty}^{\infty} \gamma^{2k}|\hat{f}|^2 d\gamma, \quad k \in \mathbf{Z}^+.$$

The square root of D_k^2 will be denoted by D_k.

Note that all of our results except those of Section 1.2 can be extended to the case where k is not an integer.

The following simple lemma will be needed in later sections. Note that for any statement about C_k, there is a dual statement about D_k, and vice versa.

Lemma 1.1 *The finiteness of C_k implies the finiteness of C_j, $j < k$.*

Proof: This can be seen from

$$C_j^2 = \int_{|t|<1} t^{2j}|f|^2 dt + \int_{|t|>1} t^{2j}|f|^2 dt \leq ||f||^2 + C_k^2.$$

□

1.2 Frequency Energy Moments as Time Integrals

This section derives a time integral expression for D_k. Though not explicitly discussed, for any statement about D_k, there is a dual statement about C_k. The case of $k = 1$ is the only case that we will use in later sections.

For functions in the Schwartz class of $\mathbf{C}^\infty$ rapidly decreasing functions, it is easy to see that D_k^2 can be expressed as an integral in time space. Integrating by parts k times, we have

$$\gamma^k \hat{f} = \gamma^k \int f(t) e^{-2\pi i t\gamma} dt = (\frac{1}{2\pi i})^k \int f^{(k)}(t) e^{-2\pi i t\gamma} dt.$$

This gives us the fact that $\gamma^k \hat{f}$ and $(\frac{1}{2\pi i})^k f^{(k)}$ are a Fourier transform pair in $\mathbf{L}^2(\mathbf{R})$.

Now, apply Parseval's equality. Parseval's equality states that for functions in $\mathbf{L}^2(\mathbf{R})$, $\int f\overline{g} = \int \hat{f}\overline{\hat{g}}$. This yields the alternative expression for D_k^2:

$$D_k^2 = \int \gamma^{2k} |\hat{f}|^2 d\gamma = \int (\gamma^k \hat{f})(\gamma^k \overline{\hat{f}}) d\gamma = (\frac{1}{2\pi})^{2k} \int (f^{(k)})(\overline{f^{(k)}}) dt = (\frac{1}{2\pi})^{2k} \int |f^{(k)}|^2 dt.$$

It turns out that we can extend these results to all functions in $\mathbf{L}^2(\mathbf{R})$. Along the way, we prove some lemmas that are of interest in their own right.

Lemma 1.2 *If $D_k(f)$ is finite, then $\gamma^{k-1}\hat{f} \in \mathbf{L}^1(\mathbf{R})$.*

Proof: We will break this into two steps. Let $[-A, A]$ be any interval containing the origin. First, we show that $\gamma^{k-1}\hat{f} \in \mathbf{L}^1([-A, A])$. Since by assumption $\gamma^k \hat{f} \in \mathbf{L}^2(\mathbf{R})$, Lemma 1 tells us that $\gamma^{k-1}\hat{f} \in \mathbf{L}^2(\mathbf{R})$. Then $\gamma^{k-1}\hat{f} \in \mathbf{L}^2([-A, A]) \subset \mathbf{L}^1([-A, A])$.

The second step is to prove that $\gamma^{k-1}\hat{f} \in \mathbf{L}^1(\{|\gamma| > A\})$. Write $\gamma^{k-1}\hat{f}$ as $\frac{1}{\gamma}(\gamma^k \hat{f})$. Since by assumption $\gamma^k \hat{f} \in \mathbf{L}^2(\mathbf{R})$, $\gamma^k \hat{f} \in \mathbf{L}^2(\{|\gamma| > A\})$. Also $\frac{1}{\gamma}$ is in $\mathbf{L}^2(\{|\gamma| > A\})$. Then, $\gamma^{k-1}\hat{f} \in \mathbf{L}^1(\{|\gamma| > A\})$ by the Cauchy-Schwarz inequality.

Since $\gamma^{k-1}\hat{f} \in \mathbf{L}^1([-A, A]) \cup \mathbf{L}^1(\{|\gamma| > A\})$, the lemma is proven. □

The following lemma is essentially due to Titchmarsh [T, page 92].

Lemma 1.3 *The finiteness of D_k is implies that $f^{(k-1)}$ is absolutely continuous. (Since f, as an $\mathbf{L}^2$ function, is technically an equivalence class of functions, this means that one element of the equivalence class has the desired continuity.) Then, by a standard theorem from real analysis [R, page 53], since $f^{(k-1)}$ is absolutely continuous, $f^{(k)}$ exists almost everywhere.*

Proof: Let ϕ be the $\mathbf{L}^2$ inverse Fourier transform of $\gamma^k \hat{f}$. Then

$$\begin{aligned}
\int_0^t \phi(s) ds &= \int_0^t \int_{-\infty}^{\infty} \gamma^k \hat{f} e^{2\pi i s\gamma} d\gamma ds \\
&= \int_{-\infty}^{\infty} \int_0^t \gamma^k \hat{f} e^{2\pi i s\gamma} ds d\gamma \\
&= \int_{-\infty}^{\infty} \gamma^k \hat{f} \frac{e^{2\pi i t\gamma} - 1}{2\pi i\gamma} d\gamma
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \gamma^{k-1} \hat{f} \frac{e^{2\pi i t\gamma} - 1}{2\pi i} d\gamma \\
&= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \gamma^{k-1} \hat{f} e^{2\pi i t\gamma} d\gamma - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \gamma^{k-1} \hat{f} d\gamma.
\end{aligned}
$$

The above changing of the order of integration is valid under Fubini's theorem since the exponential factor (which is continuous) is in $\mathbf{L}^1([0,x], ds)$ for all γ and $\gamma^{k-1}\hat{f}$ is in $\mathbf{L}^1(\mathbf{R}, d\gamma)$ by the prior lemma.

The first integral on the last line of the above equation is the $\mathbf{L}^1$ inverse Fourier transform of $\frac{1}{2\pi i}\gamma^{k-1}\hat{f}$. By the next lemma (even though later we use this lemma to prove the next lemma, this is not circular reasoning since we are only using the next lemma here in the case $k-1$), the $\mathbf{L}^2$ inverse Fourier transform of $\frac{1}{2\pi i}\gamma^{k-1}\hat{f}$ is $(\frac{1}{2\pi i})^k f^{(k-1)}$. Since both the $\mathbf{L}^2$ inverse Fourier transform and the $\mathbf{L}^1$ inverse Fourier transform exist, they are equal. Therefore, from the above equation, we see that

$$
\int_0^t \phi(s) ds = a f^{(k-1)}(t) - b,
$$

where a and b are constants, and thus $f^{(k-1)}$ is absolutely continuous. □

Lemma 1.4 *If $D_k(f)$ is finite, then*

- $f^{(k)} \in \mathbf{L}^2(\mathbf{R})$,
- $f^{(k)}$ *and* $(2\pi i\gamma)^k\hat{f}$ *are a Fourier transform pair in* $\mathbf{L}^2(\mathbf{R})$, *and*

- $D_k^2 = (\frac{1}{2\pi})^{2k} \int_{-\infty}^{\infty} |f^{(k)}|^2 dt.$

Conversely, if $f^{(k-1)}$ is absolutely continuous and $f^{(k)} \in \mathbf{L}^2(\mathbf{R})$, then D_k is finite.

Proof: We show the proof for the case $k = 1$. By the previous lemma, if D_1 exists, then f is absolutely continuous. By results from real analysis [R, pages 53-54], the absolute continuity of f allows the following integration by parts.

$$\gamma \int_{-A}^{A} f(t)e^{-2\pi i t\gamma} dt = -\frac{1}{2\pi i} f(t)e^{-2\pi i t\gamma}\Big]_{-A}^{A} + \frac{1}{2\pi i}\int_{-A}^{A} f'(t)e^{-2\pi i t\gamma} dt.$$

Now let $A \to \infty$. The left hand side converges to $\gamma\hat{f}$ in $\mathbf{L}^2(\mathbf{R})$.

From Lemma 2, we know that $\hat{f} \in \mathbf{L}^1(\mathbf{R})$. Therefore, the $\mathbf{L}^1$ inverse Fourier transform of $\hat{f}$ exists and is the same as the $\mathbf{L}^2$ inverse Fourier transform of $\hat{f}$, which is f. Since f is an $\mathbf{L}^1$ inverse Fourier transform, by the Riemann-Lebesgue lemma, $f(t) \to 0$ as $t \to \infty$. This implies that the first term on the right hand side converges to zero uniformly in γ, and therefore also converges in $\mathbf{L}^2(\mathbf{R})$.

Due to the convergence of the other terms in $\mathbf{L}^2(\mathbf{R})$, the second term on the right hand side must converge to $\gamma\hat{f}$ in $\mathbf{L}^2(\mathbf{R})$. This means that $\gamma\hat{f}$ and $\frac{1}{2\pi i}f'$ are a Fourier transform pair in $\mathbf{L}^2(\mathbf{R})$. This proves the first two items of the lemma. (This argument was taken from [T, page 92].)

The converse is proven in a similar way. Here, the convergence in $\mathbf{L}^2(\mathbf{R})$ of the two terms on the right hand side force the convergence of the term on the left hand side.

To derive the desired expression for D_1, apply Parseval's equality:

$$D_1^2 = \int \gamma^2 |\hat{f}|^2 d\gamma = \int (\gamma \hat{f})(\gamma \overline{\hat{f}}) d\gamma = (\frac{1}{2\pi})^2 \int (f')(\overline{f'}) dt = (\frac{1}{2\pi})^2 \int |f'|^2 dt.$$

By similar methods, which involve integrating by parts k times and applying Parseval's equality, we get

$$D_k^2 = (\frac{1}{2\pi})^{2k} \int_{-\infty}^{\infty} |f^{(k)}|^2 dt.$$

□

Remark: The converse without the continuity condition is *not* true. Even though $\int |f^{(k)}|^2$ is finite, D_k may be infinite. An example is the characteristic function of $[-1/2, 1/2]$. Its Fourier transform is $\frac{\sin \pi\gamma}{\pi\gamma}$. D_1 is infinite, even though f' is zero almost everywhere and thus in $\mathbf{L}^2$. (The problem with f not being continuous is that the integration by parts in the above proof is not valid.) □

1.3 Convergence to Zero at Infinity

For the generalized Poisson summation fromula, we will need the hypothesis that f is $O(\frac{1}{|t|^{1+\epsilon}})$. The following lemma shows that we can use moment conditions to guarantee sufficiently fast convergence to zero at infinity.

Lemma 1.5 *Let f be a function with C_k and D_1 finite. Then, f is $o(\frac{1}{|t|^{k/2}})$.*

Proof: Let $g(t) = t^k f^2(t)$. From Lemma 1.3, we know that g is absolutely continuous on any finite interval. Therefore, from a standard result in real analysis ([R, page 53]), $g(t) = \int_0^t g'(s)ds$. Assume for the moment that $g' \in \mathbf{L}^1(\mathbf{R})$. Then,

$$\lim_{t\to\infty} \int_0^t g'(s)ds = \lim_{t\to\infty} g(t) = \lim_{t\to\infty} t^k f^2$$

exists.

Since by assumption $t^k f \in \mathbf{L}^2(\mathbf{R})$, $t^{k/2} f \in \mathbf{L}^2(\mathbf{R})$. (For k even, this statement is just a special case of Lemma 1.1. For k odd, the proof of Lemma 1.1 proves this statement just as easily.) Therefore, the only possible value for $\lim_{t\to\infty} t^k f^2$ is zero. A similar argument shows that $\lim_{t\to-\infty} t^k f^2 = 0$. These limits being equal to zero is equivalent to the conclusion of the lemma that f is $o(\frac{1}{|t|^{k/2}})$.

The only fact left to prove is that $g' \in \mathbf{L}^1(\mathbf{R})$. This can be seen from

$$(t^k f^2)' = kt^{k-1} f^2 + 2t^k f f'.$$

Since $t^{k-1} f$ and f are both in $\mathbf{L}^2$, the first term on the right hand side is in $\mathbf{L}^1$ by the Cauchy-Schwarz inequality. Similarly, the second term is in $\mathbf{L}^1$ because $t^k f$ and f' are both in $\mathbf{L}^2$. $\square$

2 Discrete Tail Energy and Periodization Comparison

This section discusses several key inequalities that will be used in later secions.

Subsection 1 derives upper bounds for the decay of the continuous and discrete tail energy.

Subsection 2 derives upper bounds for the continuous and discrete $\mathbf{L}^2$ distances between a function and its periodization.

2.1 Bounds on the Discrete Tail Energy

Recall that E_N^2 and $\tilde{E}_{N,M}^2$ are the continuous and discrete tail energy, respectively. The following lemma provides a bound for $\tilde{E}_{N,M}^2$. Unfortunately, this bound does not decay sufficiently fast as $N, M \to \infty$. However, at the cost of additional assumptions, this is remedied in Lemma 3.

Lemma 2.1 *Let N and M be positive integers, with N or M even. Assume f is absolutely continuous. (By Lemma 1.3, this condition will be met if D_1 is finite.) Then,*

$$\tilde{E}_{N,M}^2 \leq E_N^2 + 2\pi E_N D_1/M.$$

Proof: The idea behind this proof is to use Euler's summation formula to change the discrete sum to integrals. A similar method was used by Landau-Pollak to prove a different result in [LP].

First, consider the case $M = 1$. In this case, N is even.

Euler's summation formula [I, Volume II, page 1109] states that

$$\sum_{k=\alpha+1}^{\beta-1} g(k) + \frac{1}{2}(g(\alpha) + g(\beta)) = \int_\alpha^\beta g(t)dt + \int_\alpha^\beta (t - \lfloor t \rfloor - \frac{1}{2})g'(t)dt,$$

where g is absolutely continuous and α and β integers. The proof of this formula consists of applying integration by parts to the second integral. (Note that the absolute continuity of g implies that g' exists almost everywhere and that integration by parts is valid [R, pages 53-54].)

Assume that g is a positive function. Then letting $\beta \to \infty$, we have the inequality

$$\sum_{k=\alpha+1}^{\infty} g(k) \le \int_\alpha^\infty g(t)dt + \int_\alpha^\infty |(t - \lfloor t \rfloor - \frac{1}{2})g'(t)|dt. \tag{1}$$

Now, apply (1) to $g(t) = |f(t)|^2$ and $g(t) = |f(-t)|^2$ with $\alpha = N/2$, and add the results. In this case, the inequality becomes

$$\begin{aligned}\sum_{|k|>\frac{1}{2}N} |f(k)|^2 &\le \int_{|t|>\frac{1}{2}N} |f(t)|^2 dt + \int_{|t|>\frac{1}{2}N} \left|(t - \lfloor t \rfloor - \frac{1}{2})\,(f(t)\overline{f(t)})'\right| dt \\ &= \int_{|t|>\frac{1}{2}N} |f(t)|^2 dt + \int_{|t|>\frac{1}{2}N} \left|(t - \lfloor t \rfloor - \frac{1}{2})\; 2\ \mathrm{Real}(f(t)\overline{f'(t)})\right| dt.\end{aligned}$$

The first integral on the right hand side of the inequality is E_N^2. To find an upper bound for the second integral, first note that $|2(t - \lfloor t \rfloor - \frac{1}{2})| \leq 1$. Applying the Cauchy-Schwarz inequality, we have the following upper bound for the second integral:

$$(\int_{|t|>\frac{1}{2}N} |f|^2 dt)^{1/2} (\int_{|t|>\frac{1}{2}N} |f'|^2 dt)^{1/2} \leq E_N 2\pi D_1,$$

where we have used the fact that $\int_{-\infty}^{\infty} |f'|^2 dt = D_1^2$.

This proves the lemma for $M = 1$.

For $M > 1$, let $h(x) = f(x/M)$ and apply the lemma to $h(x)$ with $M' = 1$ and $N' = NM$. Then

$$\begin{aligned}
\sum_{|k|>\frac{1}{2}NM} |f(k/M)|^2 &\leq E_{NM}^2(h) + 2\pi E_{NM}(h) D_1(h) \\
&= \int_{|s|>\frac{1}{2}NM} |f(s/M)|^2 ds + (\int_{|s|>\frac{1}{2}NM} |f(s/M)|^2 ds)^{1/2} (\int_{-\infty}^{\infty} |f'(s/M)|^2 ds)^{1/2} \\
&= \int_{|t|>\frac{1}{2}N} |f(t)|^2 M dt + (\int_{|t|>\frac{1}{2}N} |f(t)|^2 M dt)^{1/2} (\int_{-\infty}^{\infty} |f'(t) \frac{1}{M}|^2 M dt)^{1/2} \\
&= M E_N^2(f) + (M^{1/2} E_N(f))(2\pi M^{-1/2} D_1(f)) \\
&= M E_N^2(f) + 2\pi E_N(f) D_1(f).
\end{aligned}$$

Dividing both sides by M proves the lemma. □

Remark: One may ask whether a similar bound would hold if we use different sampled values in the tail with the same sampling rate. We will show that a similar bound holds for the following modified discrete tail energy:

$$\frac{1}{M}\sum_{|k|>\frac{1}{2}NM}|f(\frac{k+\lambda}{M})|^2, \quad |\lambda|\le 1.$$

To see this, apply the inequality (1) to $g(t)=|f(t+\lambda)|^2$ and $g(t)=|f(-t+\lambda)|^2$ with $\alpha=N/2$, and add the results.

As in the proof of the previous lemma, we obtain

$$\sum_{|k|>\frac{1}{2}N}|f(k+\lambda)|^2 \le \int_{|t|>\frac{1}{2}N}|f(t+\lambda)|^2dt+\int_{|t|>\frac{1}{2}N}\left|(t+\lambda-\lfloor t+\lambda\rfloor-\frac{1}{2})\ (f(t+\lambda)\overline{f(t+\lambda)})'\right|dt.$$

The right hand side is bounded by

$$\int_{|t|>\frac{1}{2}(N-1)}|f(t)|^2dt+\int_{|t|>\frac{1}{2}(N-1)}\left|(t-\lfloor t\rfloor-\frac{1}{2})\ (f(t)\overline{f(t)})'\right|dt.$$

By proceding in the same way as the previous lemma, we see that the modified discrete tail energy is bounded by

$$E_{N-1}^2+2\pi E_{N-1}D_1/M.$$

□

The next lemma gives us a bound for the continuous tail energy.

Lemma 2.2 *Let N be a positive number and let k be a positive integer. Then,*

$$E_N^2\le\frac{C_k^2}{(\frac{1}{2}N)^{2k}}.$$

Proof: The lemma is just Chebyshev's inequality. The one-line proof is shown for the sake of completeness.

$$\int_{|t|>\frac{1}{2}N}|f(t)|^2dt \leq \int_{|t|>\frac{1}{2}N}\left(\frac{t}{(\frac{1}{2}N)}\right)^{2k}|f(t)|^2dt \leq \frac{\int_{-\infty}^{\infty}t^{2k}|f(t)|^2dt}{(\frac{1}{2}N)^{2k}}.$$

□

The following lemma follows immediately from Lemmas 1 and 2.

Lemma 2.3 *Assume f is absolutely continuous. Let N and M be positive integers, with N or M even. Let k be a positive integer. Then,*

$$\tilde{E}^2_{N,M} = \frac{1}{M}\sum_{|m|>\frac{1}{2}NM}|f(m/M)|^2 \leq \frac{C_k^2}{(\frac{1}{2}N)^{2k}} + 2\pi\frac{C_kD_1}{(\frac{1}{2}N)^kM}.$$

If D_1 and C_k are finite, then $\tilde{E}^2_{N,M} = \frac{1}{M}\sum_{|k|>\frac{1}{2}NM}|f(k/M)|^2$ is bounded by $\frac{A_1}{N^{2k}} + \frac{A_2}{N^kM}$. A_1 varies proportionately with C_k^2 and A_2 varies proportionately with C_k and D_1. A simpler but less precise bound is $\frac{A}{N^k}$. A varies directly with C_k and D_1, and inversely with N and M.

Since the Fourier transform is an isometry of $\mathbf{L}^2(\mathbf{R})$ onto $\mathbf{L}^2(\mathbf{R})$, we have analogous propostions for $\hat{f}$. For example —

Lemma 2.4 *Assume $\hat{f}$ is absolutely continuous. Let N and M be positive integers, with N or M even. Then,*

$$\frac{1}{N}\sum_{|n|>\frac{1}{2}NM}|\hat{f}(n/N)|^2 \leq \frac{D_k^2}{(\frac{1}{2}M)^{2k}} + 2\pi\frac{D_kC_1}{(\frac{1}{2}M)^kN}.$$

If C_1 and D_k are finite, then the discrete tail energy of $\hat{f}$, $\frac{1}{N}\sum_{|k|>\frac{1}{2}NM}|\hat{f}(k/N)|^2$, is bounded by $\frac{A_1}{M^{2k}} + \frac{A_2}{M^k N}$. A_1 varies proportionately with D_k^2 and A_2 varies proportionately with D_k and C_1. A simpler but less precise bound is $\frac{A}{M^k}$. A varies directly with D_k and C_1, and inversely with N and M.

2.2 Distance Between a Function and Its Periodization

We derive the continuous and discrete periodization comparison lemmas, which will be used to prove subsequent theorems. Lemmas 5 and 6 are essentially due to Frank Geshwind (unpublished). They provide an upper bound for the $\mathbf{L}^2([-\frac{1}{2}N, \frac{1}{2}N])$ distance between a truncated function and a periodized function. In Lemmas 7 and 8, Geshwind's methodology is used to obtain the analogous results in the discrete case.

The following definitions will be used. Let $\frac{1}{M}\mathbf{Z}_N$ be the set of points m/M, where $m = -\frac{1}{2}NM, \ldots, \frac{1}{2}NM - 1$. (The notation should remind the reader of the set of fractions k/M mod N.) We will be mostly be concerned with the case where N and M are even integers, but the definitions in this section also make sense when N and M are arbitrary positive numbers. $\mathbf{L}^2(\frac{1}{M}\mathbf{Z}_N)$ is the space of functions defined on $\frac{1}{M}\mathbf{Z}_N$.

The *(scaled) discrete norm* on $\mathbf{L}^2(\frac{1}{M}\mathbf{Z}_N)$, which will be denoted $||\cdot||_{D(M,N)}$, is

$$||f||^2_{D(M,N)} = \frac{1}{M} \sum_{m=-\frac{1}{2}NM}^{\frac{1}{2}NM-1} |f(m/M)|^2.$$

As usual, the *discrete* $\mathbf{L}^2(\frac{1}{M}\mathbf{Z}_N)$ *distance* between two functions, f and g, is defined as the norm of their difference, $||f-g||_{D(M,N)}$.

Remark 1: Though it is sometimes convenient to have the points of $\frac{1}{M}\mathbf{Z}_N$ arranged symmetrically around the origin, at other times it will be convenient to use a nonsymmetric definition. This alternative definition of $\frac{1}{M}\mathbf{Z}_N$ is the set of points m/M, where $m = 0, \ldots, NM-1$. We will note when we are using the nonsymmetric definition.

In general, changing the definition of $\frac{1}{M}\mathbf{Z}_N$ changes the values of $\tilde{f}$ (see Section 0.2 for the definition of the FFT). However, if f is a periodic function with period N, then this alternative definition does not change $\tilde{f}$. Similarly, since $\tilde{f}$ is a periodic function with period M, applying the alternative definition of the inverse FFT yields the original function.

If both N and M are odd (this case is mentioned once in passing at the beginning of Section 4), we can use another alternative definition of $\frac{1}{M}\mathbf{Z}_N$ that is even more symmetric. We can define $\frac{1}{M}\mathbf{Z}_N$ to be the set of points m/M, where $m = -\lfloor NM/2 \rfloor, \ldots, \lfloor NM/2 \rfloor$. This definition changes the points at which f evaluated from m/M to $m/M + 1/(2M)$. Since the points used are no longer the same, this definition changes the values of $\tilde{f}$. □

Remark 2: With our nonstandard scaling in the definitions of the discrete norm and the FFT (see Section 0.2 for the definition of the FFT), the FFT is a unitary operator. To see this, consider the function $f_p \in \mathbf{L}^2(\frac{1}{M}\mathbf{Z}_N)$ defined by

$$f_p = \begin{cases} \sqrt{M} \text{ at the point } p \in \frac{1}{M}\mathbf{Z}_N \\ 0 \text{ elsewhere.} \end{cases}$$

$\tilde{f}_p$ is a function with constant absolute value equal to $\frac{1}{\sqrt{M}}$. Easy calculations show that $||\tilde{f}||_{D(N,M)} = ||f||_{D(M,N)} = 1$. Since the set of all f_p is an orthonormal basis of $\mathbf{L}^2(\frac{1}{M}\mathbf{Z}_N)$, we have shown that $||\tilde{f}||_{D(N,M)} = ||f||_{D(M,N)}$. □

Remark 3: Similarly, because of our nonstandard scaling in the definitions of the discrete norm and the extended FFT (see Section 0.2 for the definition of the extended FFT), the extended FFT is a unitary operator. By Parseval's equality for periodic functions,

$$||f^{\sharp}||_{[-M/2,M/2]} = ||f||_{D(M,N)}.$$

□

The *periodization operator* $P_N : \{f \in \mathbf{L}^2(\mathbf{R}) : f(t) \text{ is } O(\frac{1}{|t|^{1+\epsilon}}), \epsilon > 0\} \longrightarrow \mathbf{L}^2([-N/2, N/2])$ is defined by:

$$(P_N f)(t) = \sum_{k \in \mathbf{Z}} f(t + kN) \text{ for all } |t| \leq N/2.$$

Remark: We can also define $P_N f$ if C_1 is finite. In this case, by Fubini's theorem, we have

$$\int_{-\infty}^{\infty} t^2|f|^2 dt = \sum_{k\in\mathbf{Z}} \int_0^N (t+kN)^2|f(t+kN)|^2 dt = \int_0^N \sum_{k\in\mathbf{Z}} (t+kN)^2|f(t+kN)|^2 dt,$$

and the sum converges for almost all $t \in [0, N]$.

View $(t+kN)f(t+kN)$ as a function of k. The convergence of the sum for almost all t is equivalent to $(t+kN)f(t+kN) \in \mathbf{L}^2(\mathbf{Z})$ for almost all t. Also, $1/(t+kN) \in \mathbf{L}^2(\mathbf{Z})$ for t not equal to a multiple of N. Apply the Cauchy-Schwarz inequality to see that $f(t+kN) \in \mathbf{L}^1(\mathbf{Z})$ for almost all t. Thus, the definition of $P_N f$ makes sense for almost all t. $\square$

The following is our first result about the $\mathbf{L}^2([-N/2, N/2])$ distance between a function and its periodization.

Lemma 2.5 *Let N be a positive number. Assume*

$$E_N^2 = \int_{|t|>\frac{1}{2}N} |f|^2 dt \le \frac{A}{N^j},$$

where $A \in \mathbf{R}^+$ and j is an integer greater than 2. Also assume f is $O(\frac{1}{|t|^{1+\epsilon}})$ (so that f is in the domain of P_N). Then,

$$||P_N f - f||^2_{[-N/2,N/2]} \le \frac{AB}{N^j},$$

where

$$B = \left(\sum_{k=1}^{\infty} \frac{1}{(2k-1)^{j/2}}\right)^2.$$

Proof:

$$\begin{aligned}
\|P_N f - f\|_{[-N/2,N/2]} &= \left(\int_{|t|<\frac{1}{2}N}\left|\left(\sum_{k\in\mathbb{Z}} f(t+kN)\right) - f(t)\right|^2\right)^{1/2} \\
&= \left(\int_{|t|<\frac{1}{2}N}|\sum_{k\neq 0} f(t+kN)|^2\right)^{1/2}.
\end{aligned}$$

View $f(t+kN)$ as k different functions of t and apply the triangle inequality. This yields

$$\begin{aligned}
\|P_N f - f\|_{[-N/2,N/2]} &\leq \sum_{k\neq 0}\left(\int_{|t|<\frac{1}{2}N}|f(t+kN)|^2\right)^{1/2} \\
&= \sum_{k=1}^{\infty}\left(\int_{\frac{1}{2}(2k-1)N<|t|<\frac{1}{2}(2k+1)N}|f(t)|^2\right)^{1/2} \\
&\leq \sum_{k=1}^{\infty}\left(\int_{|t|>\frac{1}{2}(2k-1)N}|f(t)|^2\right)^{1/2} \\
&= \sum_{k=1}^{\infty} E_{(2k-1)N}.
\end{aligned}$$

Therefore, from our assumed bound on the tail energy,

$$\|P_N f - f\|_{[-N/2,N/2]} \leq \sum_{k=1}^{\infty}\left(\frac{A}{((2k-1)N)^j}\right)^{1/2} = (\frac{A}{N^j})^{1/2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^{j/2}}.$$

Squaring both sides proves the lemma. □

Remark: For values of j divisible by 4, there is an explicit formula for B. Let

$$\alpha(n) = \sum_{k=1}^{\infty}\frac{1}{(2k-1)^n},$$

so that $B = \alpha(j/2)^2$. When n is an even integer, the following explicit formula is known for $\alpha(n)$ ([I, Volume IV, page 1759]):

$$\alpha(2n) = \frac{(2^{2n}-1)\pi^{2n}}{2(2n)!}B_{2n},$$

where B_{2n} is the $2n^{th}$ Bernoulli number.

For example, in the case $j = 4$, using the fact that $B_2 = 1/6$,

$$B = (\alpha(2))^2 = (\frac{(2^2-1)\pi^2}{2(2)!}B_2)^2 = \frac{\pi^4}{64}.$$

□

References to the continuous periodization comparison lemma in later sections apply to the following lemma.

Lemma 2.6 (Continuous Periodization Comparison) *Let N and M be even positive integers. Let j be an integer greater than 2. If C_j is finite, then*

$$||P_N f - f||_{[-N/2,N/2]} \leq \frac{2^j C_j B^{1/2}}{N^j},$$

where

$$B = \left(\sum_{k=1}^{\infty} \frac{1}{(2k-1)^{j/2}}\right)^2.$$

Proof: It is easily seen that the required hypotheses of Lemma 5 are satisfied. We know from Lemma 1.5 that f is $o(\frac{1}{|t|^{3/2}})$. And from Lemma 2,

$$E_N^2 \leq \frac{C_j^2}{(\frac{1}{2}N)^{2j}}.$$

Therefore, from Lemma 5,

$$||P_N f - f||^2_{[-N/2,N/2]} \le \frac{2^{2j} C_j^2 B}{N^{2j}}.$$

Taking the square root proves the lemma. □

The following lemma is the discrete analog of Lemma 5.

Lemma 2.7 *Let $N > 1$ and $M \ge 2$. Assume*

$$\tilde{E}^2_{N,M} = \frac{1}{M} \sum_{|k| > \frac{1}{2}NM} |f(k/M)|^2 \le \frac{A}{N^j},$$

where $A \in \mathbf{R}^+$ and j is an integer greater than 2. Also assume f is $O(\frac{1}{|t|^{1+\epsilon}})$, $\epsilon > 0$, and continuous (so that f is in the domain of P_N and P_N). Then

$$||P_N f - f||^2_{D(M,N)} \le \frac{AB}{(N-1)^j},$$

where

$$B = \left(\sum_{k=1}^{\infty} \frac{1}{(2k-1)^{j/2}} \right)^2.$$

Proof:

$$\begin{aligned} ||P_N f - f||_{D(M,N)} &= \left(\frac{1}{M} \sum_{a=-\frac{1}{2}NM}^{\frac{1}{2}NM-1} \left| \left(\sum_{k \in \mathbf{Z}} f(a/M + kN) \right) - f(a/M) \right|^2 \right)^{1/2} \\ &= \left(\frac{1}{M} \sum_{a=-\frac{1}{2}NM}^{\frac{1}{2}NM-1} |\sum_{k \neq 0} f(a/M + kN)|^2 \right)^{1/2} \\ &= ||\sum_{k \neq 0} f(a/M + kN)||_{D(M,N)}. \end{aligned}$$

View $f(a/M+kN)$ as k different functions of a and apply the triangle inequality. This yields

$$\begin{aligned}
||P_N f - f||_{D(M,N)} &= ||\sum_{k\neq 0} f(a/M+kN)||_{D(M,N)} \\
&\leq \sum_{k\neq 0} ||f(a/M+kN)||_{D(M,N)} \\
&= \sum_{k\neq 0} \left(\frac{1}{M} \sum_{a=-\frac{1}{2}NM}^{\frac{1}{2}NM-1} |f(a/M+kN)|^2 \right)^{1/2} \\
&\leq \sum_{k=1}^{\infty} \left(\frac{1}{M} \sum_{\frac{1}{2}(2k-1)NM \leq |a| \leq \frac{1}{2}(2k+1)NM} |f(a/M)|^2 \right)^{1/2} \\
&\leq \sum_{k=1}^{\infty} \left(\frac{1}{M} \sum_{|a| > \frac{1}{2}(2k-1)NM-1} |f(a/M)|^2 \right)^{1/2} \\
&\leq \sum_{k=1}^{\infty} \left(\frac{1}{M} \sum_{|a| > \frac{1}{2}(2k-1)(N-1)M} |f(a/M)|^2 \right)^{1/2} \\
&= \sum_{k=1}^{\infty} \tilde{E}_{(2k-1)(N-1),M}.
\end{aligned}$$

Therefore, from our assumed bound on the discrete tail energy,

$$\begin{aligned}
||P_N f - f||_{D(M,N)} &\leq \sum_{k=1}^{\infty} \left(\tilde{E}^2_{(2k-1)(N-1),M} \right)^{1/2} \\
&\leq \sum_{k=1}^{\infty} \left(\frac{A}{((2k-1)(N-1))^j} \right)^{1/2} \\
&= (\frac{A}{(N-1)^j})^{1/2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{j/2}}.
\end{aligned}$$

Squaring both sides proves the lemma. □

References to the discrete periodization comparison lemma in later sections apply to the following lemma.

Lemma 2.8 (Discrete Periodization Comparison) *Let N and M be even positive integers. Let j be an integer greater than 2. If C_j and D_1 are finite, then*

$$||P_N f - f||_{D(M,N)} \leq \sqrt{\frac{2^j C_j^2}{N^j} + \frac{2\pi C_j D_1}{M}} \frac{2^{j/2} B^{1/2}}{(N-1)^{j/2}}.$$

where

$$B = \left(\sum_{k=1}^{\infty} \frac{1}{(2k-1)^{j/2}} \right)^2.$$

Proof: It is easily seen that the required hypotheses of Lemma 7 are satisfied. From Lemma 1.3 we know that f is absolutely continuous, and from Lemma 1.5 we know that f is $o(\frac{1}{|t|^{3/2}})$. From Lemma 3,

$$\tilde{E}_{N,M}^2 \leq \left(\frac{2^j C_j^2}{N^j} + \frac{2\pi C_j D_1}{M} \right) \frac{2^j}{N^j}.$$

Therefore, from Lemma 7,

$$||P_N f - f||_{D(M,N)}^2 \leq \left(\frac{2^j C_j^2}{N^j} + \frac{2\pi C_j D_1}{M} \right) \frac{2^j B}{(N-1)^j}.$$

Taking the square root proves the lemma. □

Remark: The conclusion of the lemma is that $||P_N f - f||_{D(M,N)}$ is $O(\frac{1}{N^{j/2}})$. However, depending on how $N, M \to \infty$, the bounds given in the lemma can be improved.

If $N, M \to \infty$, with N^j approaching infinity faster than M, the first term approaches zero faster than the second. Then, asymptotically, $||P_N f - f||_{D(M,N)}$ is $O(\frac{1}{N^{j/2} M^{1/2}})$ — faster than claimed!

If $N, M \to \infty$, with M approaching infinity faster than N^j, the second term of Lemma 3 approaches zero faster than the first. Then, asymptotically, $||P_N f - f||_{D(M,N)}$ is $O(\frac{1}{N^j})$ — also faster than claimed! □

3 Weil Space

The Weil transform is an important tool in the study of properties of functions that depend on time and frequency. The Weil transform will be used later to prove the generalized Poisson summation formula and the $\mathbf{L}^2$ norm approximation theorem. We give alternative proofs of these results using classical methods, but the Weil transform adds geometric insight for a better understanding of these theorems.

Another application of Weil space is as follows. All of our approximation theorems require the finiteness of certain energy moiments. A sufficient condition that f have j^{th} time and frequency energy moments is that Θf be $\mathbf{C}^j$ in Weil space. (See part 4 of Theorem 2.)

Before discussing the properties of the Weil transform, we give a hueristic explaination of how the Weil transform might arise. We start by creating a function of two variables out of a function of one variable. Let $f \in \mathbf{L}^2(\mathbf{R})$. Define $g(x,k) = f(t)$, where $x = t - \lfloor t \rfloor$ and $k = \lfloor t \rfloor$. From Fubini's theorem, we know that

$$\int_{\mathbf{R}} |f|^2 = \int_0^1 \sum_{k\in\mathbf{Z}} |f(t+k)|^2 = \int_0^1 \sum_{k\in\mathbf{Z}} |g(x,k)|^2,$$

and $g(x,k)$ is an $\mathbf{L}^2$ function of $k \in \mathbf{Z}$ for almost every $x \in [0,1)$. To change the discrete variable to a continuous variable, define

$$h(x,y) = \sum_k g(x,k)e^{2\pi iky} = \sum_k f(x+k)e^{2\pi iky}.$$

This defines h for almost every x. For almost every fixed x, h is the periodic function corresponding to the Fourier coefficients, $g(x,k)$. h is known as the Zak transform.

h can also be extended so that it is defined for $x, y \in \mathbf{R}$. However, h will not have desirable continuity properties. Even if f and $\hat{f}$ are $\mathbf{C}^\infty$, h will not be continuous in x. The Weil transform, described below, is a modification of the Zak transform that, under prescribed conditions, preserves the smoothness of f and $\hat{f}$.

The Weil transform is also important for its relation to the Dirac representation of the Heisenberg group. This is not discussed here since we do not utilize these properties in this paper.

Subsection 1 discusses the theory of the original Weil transform. Subsection 2 discusses a discrete analog.

3.1 Continuous Theory

This section is a summary of basic facts about the Weil transform and Weil space. It describes the definitions and notation that we use. For additional details, see [A] and [Ge]. For a survey article on the related Zak transform, see [J]. For the connections with nilmanifold theory, see [AT]. For the relation to wavelets, see [HW].

Let $\mathcal{N}$ denote the real three dimensional Heisenberg group with coordinates (x, y, z), $x, y, z \in \mathbf{R}$ and multiplication defined by

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2).$$

Let Γ be its subgroup of integers. Let $\Gamma\backslash\mathcal{N}$ be the left coset space. The cosets are the equivalence classes of $\mathcal{N}$ determined by the orbits under left multiplication by elements of Γ. $\Gamma\backslash\mathcal{N}$ is the unit cube $0 \leq x, y, z \leq 1$ with certain identifications on the boundary.

Assume $f(t)$ is $O(\frac{1}{|t|^{1+\epsilon}})$ and $\hat{f}(\gamma)$ is $O(\frac{1}{|\gamma|^{1+\epsilon}})$, with $\epsilon > 0$. The *Weil transform*, Θ, maps f into the set of continuous functions in $\mathbf{L}^2(\Gamma\backslash\mathcal{N})$, as follows:

$$(\Theta f)(x, y, z) = e^{2\pi i z} \sum_{a \in \mathbf{Z}} f(x + a) e^{2\pi i a y}.$$

Θ extends to a unitary isomorphism between $\mathbf{L}^2(\mathbf{R})$ and a closed subspace $\mathbf{H}$ of $\mathbf{L}^2(\Gamma\backslash\mathcal{N})$. We refer to $\Theta f \in \mathbf{H}$ as the function f in *Weil space.*

A function on $\mathcal{N}$ is Γ-*invariant* if the function is constant on each left coset equivalence class. (The equivalence classes are the orbits under left multiplication by elements of Γ.) Note that any function on $\mathcal{N}$ that is Γ-invariant can be considered a function on $\Gamma\backslash\mathcal{N}$. Conversely, any function defined on $\Gamma\backslash\mathcal{N}$ can be extended to $\mathcal{N}$ by assigning the function value of each equivalence class to all elements of the equivalence class.

Any function in $\mathbf{H}$ satisfies the following relations:

1. $F(x,y,z) = F(x,y+1,z)$ (due to Γ-invariance)

2. $F(x,y,z) = F(x+1,y,z+y)$ (due to Γ-invariance)

3. $F(x,y,z+u) = e^{2\pi iu}F(x,y,z)$ (an additional requirement)

The $\mathrm{L}^2(\Gamma\backslash\mathcal{N})$ *inner product* is

$$\int_0^1\int_0^1\int_0^1 F(x,y,z)\overline{G}(x,y,z)dxdydz, \quad F,G \in \mathbf{L}^2(\Gamma\backslash\mathcal{N}).$$

When $F,G \in \mathbf{H}$, from the third relation above, we see that $F\overline{G}$ does not depend on z. Therefore, the $\mathbf{L}^2$ inner product in $\mathbf{H}$ is

$$\begin{aligned}\int_0^1\int_0^1\int_0^1 F(x,y,z)\overline{G}(x,y,z)dxdydz &= \int_0^1\int_0^1\int_0^1 F(x,y,0)\overline{G}(x,y,0)dxdydz \\ &= \int_0^1\int_0^1 F(x,y,0)\overline{G}(x,y,0)dxdy.\end{aligned}$$

From this, we see that when only $\mathbf{L}^2$ properties are involved, we can set $z = 0$ and view Θf as a function of two variables in the unit square. This special case of the Weil transform when $z = 0$ is the Zak transform discussed above.

The $\mathbf{L}^2$ *norm for* $\mathbf{H}$ is

$$||F||_W^2 = \int_0^1 \int_0^1 |F(x, y, 0)|^2 dx dy.$$

The *inverse Weil transform* is

$$(\Theta^{-1}F)(t) = \int_0^1 F(t, y, 0) dy.$$

The following theorem is intimately connected with the Poisson summation formula. It can be proven using the Poisson summation formula and will be used later to prove the generalized Poisson summation formula.

Theorem 3.1 *Let* $f \in \mathbf{L}^2(\mathbf{R})$. *Let* $F = \Theta f$ *and* $\hat{F} = \Theta \hat{f}$. *Then*

$$\hat{F}(x, y, z) = F(y, -x, z - xy) = e^{-2\pi i x y} F(y, -x, z).$$

In words, except for a phase factor, $\hat{F}$ *is* F *with a* 90° *rotation of the* x–y *coordinates.*

From part 4 of the following theorem, we see that smoothness properties of the Weil transform imply finiteness conditions on the energy moments. The theorem is proven in [Ge] by Frank Geshwind.

Theorem 3.2 *Let $\mathcal{S}$ be the Schwartz class of $\mathbf{C}^\infty(\mathbf{R})$ rapidly decreasing functions. Let $\mathcal{S}'$ be the distributions on $\mathcal{S}$ (i.e. the tempered distributions). Let $\mathcal{W} = \mathbf{H} \cap \mathbf{C}^\infty(\Gamma\backslash\mathcal{N})$. (Note that only the function itself and not its derivatives are required to be Γ-invariant.) Let $\mathcal{W}'$ be the distributions on $\Gamma\backslash\mathcal{N}$.*

1. *We can extend the Weil transform to be an isometry from $\mathcal{S}'$ to $\mathcal{W}'$.*

2. *Let $m, n \in \mathbf{Z}^+$. Let $T \in \mathcal{S}'$. Then,*

$$\Theta(t^n(\frac{\partial}{\partial t})^m T) = (x + \frac{1}{2\pi i}\frac{\partial}{\partial y})^n(\frac{\partial}{\partial x})^m(\Theta T).$$

3. *If the first j time and energy moments are finite, then $\Theta(t^j f)$ and $\Theta(\gamma^j \hat{f})$ are in $\mathbf{L}^2(\Gamma\backslash\mathcal{N})$.*

4. *If $\Theta f \in \mathbf{C}^j(\Gamma\backslash\mathcal{N})$, then the first j time and energy moments are finite.*

3.2 Discrete Theory

In this section, we discuss a discrete analog of the continuous Weil transform. Since continuity considerations do not apply, the third variable of the continuous Weil transform is ignored.

Let $(\mathbf{Z}_M, \mathbf{Z}_N)$ be the points $\{(\frac{m}{M}, \frac{n}{N}) : 0 \leq n < N,\ 0 \leq m < M\})$, where N and M are positive integers. (We are forced to use this nonsymmetric definition of $\mathbf{Z}_M$

and $\mathbf{Z}_N$ for purposes of the Weil transform. This is because the Weil transform is not periodic in the x variable, as discussed below.)

Define a *Weil space sampling operator* $S_{M,N} : \mathbf{H} \cap \mathbf{C}(\Gamma \backslash \mathcal{N}) \longrightarrow \mathbf{L}^2(\mathbf{Z}_M, \mathbf{Z}_N)$ by:

$$(S_{M,N}F)(\frac{m}{M}, \frac{n}{N}) = F(\frac{m}{M}, \frac{n}{N}, 0), \quad 0 \leq n < N, \; 0 \leq m < M.$$

Assume $f(t)$ is $O(\frac{1}{|t|^{1+\epsilon}})$ and $\hat{f}(\gamma)$ is $O(\frac{1}{|\gamma|^{1+\epsilon}})$. Then $F = \Theta f$ is continuous. Expressing $S_{M,N}F$ in terms of f, we have

$$(S_{M,N}F)(\frac{m}{M}, \frac{n}{N}) = \sum_{a \in \mathbf{Z}} f(m/M + a) e^{2\pi i a n/N}.$$

Comparing this equation with the definition of Θ, we note that $S_{M,N}F$ is a discrete analog of the Weil transform (with $z = 0$). Thus, it would appear that $S_{M,N}F$ is the desired finite Weil transform of f.

However, let's simplify the expression for $S_{M,N}F$ as follows:

$$\begin{aligned}
(S_{M,N}F)(\frac{m}{M}, \frac{n}{N}) &= \sum_{a \in \mathbf{Z}} f(m/M + a) e^{2\pi i a n/N} \\
&= \sum_{a'=0}^{N-1} (\sum_{k \in \mathbf{Z}} f(m/M + kN + a')) e^{2\pi i a' n/N} \\
&= \sum_{a'=0}^{N-1} (P_N f)(m/M + a') e^{2\pi i a' n/N}.
\end{aligned}$$

The second line in the equation above is obtained by writing $a = kN + a'$, where $k \in \mathbf{Z}$ and $0 \leq a' \leq N - 1$, and reordering the sum. The reordering is valid since f is $O(\frac{1}{|t|^{1+\epsilon}})$, and thus the sum is absolutely convergent.

We see from the last line of the equation above that the finite Weil transform of f only depends on the sampled values of $P_N f$. Therefore, we define the *finite (M,N) Weil transform* as the operator from $\mathbf{L}^2(\frac{1}{M}Z_N) \longrightarrow \mathbf{L}^2(\mathbf{Z}_M, \mathbf{Z}_N)$ that takes $P_N f$ to $S_{M,N}F$.

The last line of the equation tells us even more about the finite Weil transform. Let $(P_N f)(m/M + \cdot)$ denote $P_N f$ as a function in $\mathbf{L}^2(\mathbf{Z}_N)$ with m fixed. Let $(S_{M,N}F)(m/M, \cdot)$ denote $S_{M,N}F$ as a function in $\mathbf{L}^2(\frac{1}{N}\mathbf{Z}_1)$ with m fixed. Then, from the last line of the above equation, we see that $(S_{M,N}F)(m/M, \cdot)$ is the inverse FFT of $(P_N f)(m/M + \cdot)$. (The fact that the index runs from 0 to $N-1$ instead of from $-N/2$ to $N/2-1$ makes no difference because of the periodicity of $P_N f$.)

In fact, $S_{M,N}F$ can be extended by the equation above so that it defined on all of the points of $\{(\frac{m}{M}, \frac{n}{N}) : m, n \in \mathbf{Z}\}$. $S_{M,N}F$ is periodic in n/N with period 1 and also satisfies

$$S_{M,N}F(x+1, y) = e^{-2\pi i y} S_{M,N}F(x,y), \quad (x,y) \in \{(\frac{m}{M}, \frac{n}{N}) : m, n \in \mathbf{Z}\}.$$

For $F \in \mathbf{L}^2(\mathbf{Z}_M, \mathbf{Z}_N)$, define a *Weil space discrete norm* by

$$||F||^2_{W(M,N)} = \frac{1}{MN} \sum_{m \in \mathbf{Z}_M} \sum_{m \in \mathbf{Z}_N} |F(\frac{m}{M}, \frac{n}{N})|^2.$$

Note that this norm is well-defined for any function in $\mathbf{H} \cap \mathbf{C}(\Gamma \backslash \mathcal{N})$.

Using the fact that discrete norm of a function and its FFT are equal and the definitions of the discrete norms, we see below that the finite Weil transform of

$P_N f$ (as a function in $\mathbf{L}^2(\frac{1}{M}\mathbf{Z}_N)$) is unitary.

$$\begin{aligned}
\frac{1}{MN}\sum_{m,n}|F(\frac{m}{M},\frac{n}{N})|^2 &= \frac{1}{M}\sum_m \frac{1}{N}\sum_n |F(\frac{m}{M},\frac{n}{N})|^2 \\
&= \frac{1}{M}\sum_m ||F(m/M,\cdot)||^2_{D(N,1)} \\
&= \frac{1}{M}\sum_m ||(P_N f)(m/M+\cdot)||^2_{D(1,N)} \\
&= ||P_N f||^2_{D(M,N)}.
\end{aligned}$$

4 Generalized Poisson Summation Formula

In this section, we state and prove the generalized Poisson summation formula. This generalization will be used to prove the second FFT approximation theorem.

The Poisson summation formula

$$\sum_{n\in\mathbf{Z}} f(n) = \sum_{n\in\mathbf{Z}} \hat{f}(n)$$

is known to hold when $f(t)$ is $O(\frac{1}{|t|^{1+\epsilon}})$ and $\hat{f}(\gamma)$ is $O(\frac{1}{|\gamma|^{1+\epsilon}})$, with $\epsilon > 0$. (See for example [SW, page 252]. They prove the Poisson summation formula under the decay condition that $f(t)$ be $O(\frac{1}{(1+|t|)^{1+\epsilon}})$, which is equivalent to the decay condition being used here when f and $\hat{f}$ are bounded. $\hat{f}$ (and similarly f) is bounded since our decay condition ensures that $f \in \mathbf{L}^1$ and the $\mathbf{L}^\infty$ norm of $\hat{f}$ is bounded by the $\mathbf{L}^1$ norm of f [SW, page 2]. The additional condition mentioned there that the Fourier inversion formula hold pointwise for f and $\hat{f}$ is satisfied when, as in our case, f and $\hat{f}$ are in $\mathbf{L}^1$ [SW, page 11].)

A second form of the Poisson summation formula can be derived as follows. Apply the Poisson summation formula to $f(tN)$. This yields

$$\sum_{n\in\mathbf{Z}} f(nN) = \frac{1}{N}\sum_{n\in\mathbf{Z}} \hat{f}(n/N).$$

A third form of the Poisson summation formula can also be derived. Apply the second form of Poisson summation formula to $f(t+t')$ as a function of t. Since the Fourier transform of $f(t+t')$ is $\hat{f}(\gamma)e^{2\pi i\gamma t'}$, we have

$$(P_N f)(t') = \sum_{n\in\mathbf{Z}} f(t'+nN) = \frac{1}{N}\sum_{n\in\mathbf{Z}} \hat{f}(n/N)e^{2\pi i n t'/N}.$$

This form of the Poisson summation formula states that the Fourier coeffients of $P_N f$ are $\frac{1}{N}$ times the sampled values of $\hat{f}$. Let's apply periodization and sampling in both the time and frequency domains. One may then expect that there would be some Fourier-type relation between the sampled $(P_N f)^{\sim}$ and the sampled $\hat{f}$. This is in fact true in the following sense.

Theorem 4.1 (Generalized Poisson Summation Formula) *Assume $f(t)$ is $O(\frac{1}{|t|^{1+\epsilon}})$ and $\hat{f}(\gamma)$ is $O(\frac{1}{|\gamma|^{1+\epsilon}})$, with $\epsilon > 0$. Let N and M be positive integers. Then*

$$(P_N f)^{\sim}(n/N) = P_M \hat{f}(n/N), \quad n = -\frac{1}{2}NM, \ldots, \frac{1}{2}NM - 1.$$

Note that we get the first form of the Poisson Summation Formula if $M = N = 1$

(using the alternative definition of the FFT for N, M odd in Section 2.2, Remark 1).

The first mention of this result, without discussing specific conditions that f must satisfy or the Poisson summation formula, was apparently in the engineering literature in an article by Cooley-Lewis-Welch [CLW]. Briggs-Henson derive the generalized Poisson summation formula, with an elementary proof based on the classical Poisson summation formula, in a book that was published this year [BH, page 196]. This proof is shown in Subsection 2. Louis Auslander found a proof using ideas based on nilmanifolds and the Weil transform (unpublished). A shortened version of this proof is shown in Subsection 1.

4.1 Proof Using Weil Space

This proof is based on the Weil space interpretation of the Cooley–Tukey algorithm given in [AGT]. A summary of properties of the Weil transform is contained in Section 3.

$P_N f$ exists and is continuous because of our assumption about the decay of f and $\hat{f}$. Now, apply the Cooley–Tukey algorithm to $P_N f$:

$$\begin{aligned}(P_N f)^{\sim}(\frac{n'}{N}+m') &= \frac{1}{M}\sum_m\sum_n (P_N f)(\frac{m}{M}+n)e^{-2\pi i(m+nM)(n'+m'N)/NM} \\ &= \frac{1}{M}\sum_m\left(\left[\sum_n (P_N f)(\frac{m}{M}+n)e^{-2\pi inn'/N}\right]e^{-2\pi imn'/NM}\right)e^{-2\pi imm'/M},\end{aligned}$$

where $0 \le n, n' < N$ and $0 \le m, m' < M$. (Here, we use the nonsymmetric version of $\frac{1}{M}\mathbf{Z}_N$, which is equivalent to the symmetric definition in this case because of the periodicity of $P_N f$.)

Let's break this calculation down into steps. The first step is to do the N-point FFT in brackets M times, once for each m. By our remarks in Section 3.2, the inverse FFT of $P_N f(\frac{m}{M} + \cdot)$ is $S_{M,N}F(\frac{m}{M}, \cdot)$, where $F = \Theta f$. Therefore, the forward FFT is the inversion, $S_{M,N}F(\frac{m}{M}, -\cdot)$. Doing the FFT for each m yields the function of two variables, $S_{M,N}F(\frac{m}{M}, -\frac{n'}{N})$.

The second step is to multiply by the phase factor. By Theorem 3.1, we have

$$e^{-2\pi i m n'/NM} S_{M,N}F(\frac{m}{M}, -\frac{n'}{N}) = S_{N,M}\hat{F}(\frac{n'}{N}, \frac{m}{M}).$$

The third step is to take the M-point FFT of $S_{N,M}\hat{F}(\frac{n'}{N}, \cdot)$ N times, once for each n'. We know from section 3.2 that the FFT of $S_{N,M}\hat{F}(\frac{n'}{N}, \cdot)$ is equal to the sample values of $P_M \hat{f}(\frac{n'}{N} + \cdot)$. Doing the FFT for each n yields $P_M \hat{f}(\frac{n'}{N} + m')$.

This proves that $(P_N f)^{\sim}$ is equal to the sampled values of $P_M \hat{f}$.

4.2 Proof Using Classical Techniques

From the third form of Poisson summation formula, we have

$$(P_N f)(t') = \sum_{n \in \mathbf{Z}} f(t' + nN) = \frac{1}{N} \sum_{n \in \mathbf{Z}} \hat{f}(n/N) e^{2\pi i n t'/N}.$$

Evaluating $P_N f$ at $t' = m/M,\ -\frac{1}{2}NM \leq m \leq \frac{1}{2}NM - 1$, we have

$$\begin{aligned}(P_N f)(m/M) &= \frac{1}{N}\sum_{n\in\mathbf{Z}} \hat{f}(n/N)e^{2\pi inm/NM} \\ &= \frac{1}{N}\sum_{n'=-\frac{1}{2}NM}^{\frac{1}{2}NM-1}\left(\sum_{k\in\mathbf{Z}} \hat{f}(n'/N + kM)\right) e^{2\pi in'm/NM}.\end{aligned}$$

The last expression is obtained by writing $n = kNM + n'$, where $k \in \mathbf{Z}$ and $-NM/2 \leq n' \leq NM/2 - 1$, and reordering the sum. The reordering is valid since $\hat{f}(\gamma)$ is $O(\frac{1}{|\gamma|^{1+\epsilon}})$, and thus the sum is absolutely convergent.

The equation is saying that the FFT of $P_N f$ is $\sum_{k\in\mathbf{Z}} \hat{f}(n'/N + kM)$ which is $P_M \hat{f}(\frac{n'}{N})$.

5 FFT Approximation

The error in the FFT approximation to the Fourier transform can be measured in many ways. In this section, we obtain upper bounds for several different ways of defining the error.

In Subsection 1, we derive an error bound for the maximum difference between the extended FFT and the Fourier transform at any frequency where the extended FFT is defined. This bound is derived in the same way that we derived a bound for the case of the $\mathbf{L}^1$ hypothesis in Section 0.1.

In the next two subsections, we utilize methods that do not apply to the case of the $\mathbf{L}^1$ hypothesis. In Subsection 2, we derive an error bound for the FFT approximation to the Fourier transform using the discrete $\mathbf{L}^2$ norm. In Subsection 3, we derive an error bound for the FFT approximation to the Fourier transform using the continuous $\mathbf{L}^2$ norm.

The following remark applies to all of the FFT approximation theorems.

Remark: Note that our values of M and N can be reduced if, instead of truncating around zero, we truncate around the average values of $|f|^2$ and $|\hat{f}|^2$. Doing this would replace the energy moments around zero with the lower energy moments around the means. □

5.1 Maximum FFT Approximation Error

We derive an upper bound for the maximum difference between the extended FFT and the Fourier transform at any frequency where the extended FFT is defined.

Let f be a function with C_j and D_1 finite, where j is an integer greater than 2. Let N be an even positive integer. Then, from Lemma 2.3 (and the remark following Lemma 2.1), we know that for any real λ with $|\lambda| \leq 1$.

$$\sum_{|m|>\frac{1}{2}N} |f(m+\lambda)|^2 \leq \frac{A}{(\frac{1}{2}N)^j},$$

where A depends only on C_j and D_1 (and inversely on N and M).

Apply this result to the special case $\lambda = t - \lfloor t \rfloor$ and $N = 2(t - 1 - \lambda)$, where $t \geq 2$. Then,

$$|f(t)|^2 \leq \sum_{|m|>t-1-\lambda} |f(m+\lambda)|^2 \leq \frac{A}{(t-1-\lambda)^j} \leq \frac{A'}{t^j},$$

where A' depends only on C_j and D_1 (and inversely on N and M).

A similar argument for negative values of t shows that $|f|^2$ is $O(\frac{1}{|t|^j})$, or equivalently, that f is $O(\frac{1}{|t|^{j/2}})$.

Now, we can proceed in the same way that we did for the case of the $\mathbf{L}^1$ hypothesis in Section 0.1.

Theorem 5.1 (FFT Approximation I) *Let f be a function with C_j and D_k finite, where j, k are integers greater than 2. Let N and M be even positive integers. Then, for all $\gamma \in [-M/2, M/2]$,*

$$|f^\sharp(\gamma) - \hat{f}(\gamma)| \leq \frac{K_1}{N^{j/2-1}} + \frac{K_2}{M^{k/2}}.$$

The K_i depend only on the following parameters. K_1 varies directly with C_j and D_1 and varies inversely with N and M. K_2 varies directly with D_k and C_1 and varies inversely with N and M.

Proof: Lemma 1.5 ensures that the f is $o(\frac{1}{|t|^{3/2}})$ and $\hat{f}$ is $o(\frac{1}{|\gamma|^{3/2}})$. Therefore, the hypotheses of the Poisson summation formula are met.

A rearrangement of the terms of the Poisson summation formula (see Section 0.1) yields

$$\left|\hat{f}(0)-\frac{1}{M}\sum_{m=-\frac{1}{2}NM}^{\frac{1}{2}NM-1} f(m/M)\right| \leq \left|\frac{1}{M}\Big(\sum_{m<-\frac{1}{2}NM}+\sum_{m>\frac{1}{2}NM-1}\Big)f(m/M)\right|+\left|\sum_{m\neq 0}\hat{f}(mM)\right|.$$

Let's apply this inequality to the modulated function, $f(t)e^{-2\pi i t\gamma}$, which corresponds to a shift in frequency space, instead of f. This yields

$$\left|\hat{f}(\gamma)-\frac{1}{M}\sum_{m=-\frac{1}{2}NM}^{\frac{1}{2}NM-1} f(m/M)e^{-2\pi i m\gamma/M}\right| \leq \frac{1}{M}\Big(\sum_{m<-\frac{1}{2}NM}+\sum_{m>\frac{1}{2}NM-1}\Big)|f(m/M)|+\left|\sum_{m\neq 0}\hat{f}(mM+\gamma\right|$$

From the discussion prior to the statement of the theorem, we see that f is $O(\frac{1}{|t|^{j/2}})$. Therefore, the first error term is bounded by a constant times $\int_{N/2}^{\infty} t^{-j/2}dt$, which is $O(\frac{1}{N^{j/2-1}})$.

Similarly, since D_k and C_1 are finite, $\hat{f}$ is $O(\frac{1}{|\gamma|^{k/2}})$. Therefore, the second error term is bounded by a constant times

$$\sum_{m\neq 0}|mM+\gamma|^{-k/2} = M^{-k/2}\sum_{m\neq 0}|m+\gamma/M|^{-k/2}.$$

Since $|\gamma/M| \leq \frac{1}{2}$, the series is well-defined. Since $k>2$, the series converges. So as claimed, the second error term is bounded by a constant times $M^{-k/2}$, where the constant is

$$\max_{\alpha\in[-1/2,1/2]}\sum_{m\neq 0}|m+\alpha|^{-k/2}.$$

The maximum is obtained since the sum is a uniformly convergent sum of continuous functions of α, and therefore the sum is a continuous function of α. □

Remark: Depending on how $N, M \to \infty$, each term of the bound given in this theorem can be improved, but improving the rate of convergence of one term can decrease the rate of convergence the other term.

We saw before the statement of the theorem that $|f(t)|$ is bounded by the square root of the discrete tail energy (with $N = 2(t-1-\lambda)$). In Lemma 2.3, the discrete tail energy is bounded by a term that is $O(\frac{1}{N^{2j}})$ plus a term that is $O(\frac{1}{N^j M})$.

If $N, M \to \infty$, with N^j approaching infinity faster than M, the first term approaches zero faster than the second. Then, asymptotically, the square root of the discrete tail energy is $O(\frac{1}{N^{j/2}M^{1/2}})$. The $\frac{K_1}{N^{j/2-1}}$ term is a sum of these $O(\frac{1}{N^{j/2}M^{1/2}})$ terms. So, in this case, the $\frac{K_1}{N^{j/2-1}}$ term actually approaches zero like $O(\frac{1}{N^{j/2-1}M^{1/2}})$ — faster than claimed!

If $N, M \to \infty$, with M approaching infinity faster than N^j, the second term of Lemma 2.3 approaches zero faster than the first. Then, asymptotically, the square root of the discrete tail energy is $O(\frac{1}{N^j})$. The $\frac{K_1}{N^{j/2-1}}$ term is a sum of these $O(\frac{1}{N^j})$ terms. So, in this case, the $\frac{K_1}{N^{j/2-1}}$ term actually approaches zero like $O(\frac{1}{N^{j-1}})$ — also faster than claimed!

Similar comments apply to K_2. If $N, M \to \infty$, with M^j approaching infinity faster than N, the $\frac{K_2}{M^{j/2}}$ term approaches zero like $O(\frac{1}{M^{j/2}N^{1/2}})$. If $N, M \to \infty$, with N approaching infinity faster than M^j, the $\frac{K_2}{M^{j/2}}$ term approaches zero like $O(\frac{1}{M^j})$.

We just saw that the $\frac{K_1}{N^{j/2-1}}$ term is $O(\frac{1}{N^{j-1}})$ if M approaches infinity faster than N^j. And, the $\frac{K_2}{M^{k/2}}$ term is $O(\frac{1}{M^k})$ if N approaches infinity faster than M^k. Unfortunately, these are not independent conditions. M approaching infinity faster than N^j and N approaching infinity faster than M^k are mutually exclusive.

If N^j approaches infinity faster M and M^k approaches infinity faster N, then the $\frac{K_1}{N^{j/2-1}}$ term is $O(\frac{1}{N^{j/2-1}M^{1/2}})$ and the $\frac{K_2}{M^{k/2}}$ term is $O(\frac{1}{M^{k/2}N^{1/2}})$. □

5.2 FFT Approximation in $\mathbf{L}^2$ Discrete Norm

We derive an upper bound for the $\mathbf{L}^2(\frac{1}{N}\mathbf{Z}_M)$ distance between the FFT and the Fourier transform.

The idea behind the following FFT approximation theorem is to replace periodization in the generalized Poisson summation formula with truncatation. For functions that decay in time and frequency sufficiently fast, if N and M are large enough, the error will be small.

Theorem 5.2 (FFT Approximation II) *Let f be a function with C_j and D_k finite, where j, k are integers greater than 2. Let N and M be even positive integers.*

Then

$$||\tilde{f} - \hat{f}||_{D(N,M)} \leq \frac{K_1}{N^{j/2}} + \frac{K_2}{M^{k/2}}.$$

The K_i depends only on the following parameters. K_1 varies directly with C_j and D_1 and varies inversely with N and M. K_2 varies directly with D_k and C_1 and varies inversely with N and M.

Proof: By the triangle inequality, the left hand side of the above inequality is less than or equal to

$$||\tilde{f} - (P_N f)^{\sim}||_{D(N,M)} + ||(P_N f)^{\sim} - P_M \hat{f}||_{D(N,M)} + ||P_M \hat{f} - \hat{f}||_{D(N,M)}.$$

Lemma 1.5 ensures that the f is $o(\frac{1}{|t|^{3/2}})$ and $\hat{f}$ is $o(\frac{1}{|\gamma|^{3/2}})$. Therefore, the hypotheses of the generalized Poisson summation formula are met, and the second term is zero.

The first term is equal to $||f - P_N f||_{D(M,N)}$ since the FFT is linear and unitary. Now, apply the discrete periodization comparison lemma and see that the first term is $O(\frac{1}{N^{j/2}})$.

To obtain an upper bound for the third term, apply the dual of the discrete periodization comparison lemma and see that the third term is $O(\frac{1}{M^{k/2}})$. □

Remark: Since both error terms come from the discrete periodization comparison lemma, the remarks after that lemma apply here. Depending on how $N, M \to \infty$, each term of the bound given in this theorem can be improved. However, improving the rate of convergence of one term can decrease the rate of convergence the other term.

The $\frac{K_1}{N^{j/2}}$ term is $O(\frac{1}{N^j})$ if M approaches infinity faster than N^j. The $\frac{K_2}{M^{k/2}}$ term is $O(\frac{1}{M^k})$ if N approaches infinity faster than M^k. Unfortunately, these are not independent conditions. M approaching infinity faster than N^j and N approaching infinity faster than M^k are mutually exclusive.

If N^j approaches infinity faster M and M^k approaches infinity faster N, then the $\frac{K_1}{N^{j/2}}$ term is $O(\frac{1}{N^{j/2}M^{1/2}})$ and the $\frac{K_2}{M^{k/2}}$ term is $O(\frac{1}{M^{k/2}N^{1/2}})$. □

5.3 FFT Approximation in $\mathbf{L}^2$ Continuous Norm

We derive an upper bound for the $\mathbf{L}^2([-M/2, M/2])$ distance between the FFT and the Fourier transform.

Notice that the second error term differs from the corresponding term in the second FFT approximation theorem.

Theorem 5.3 (FFT Approximation III) *Let f be a function with C_j and D_k finite, where j, k are integers greater than 2. Let N and M be even positive integers. Then*

$$||f^{\sharp} - \hat{f}||_{[-M/2,M/2]} \leq \frac{K_1}{N^{j/2}} + \frac{K_2}{M^k}.$$

The K_i depend only on the following parameters. K_1 varies directly with C_j and D_1 and varies inversely with N and M. K_2 varies directly with D_k.

Proof: By the triangle inequality, the left hand side of the above inequality is less than or equal to

$$||f^\sharp - P_M\hat{f}||_{[-M/2,M/2]} + ||P_M\hat{f} - \hat{f}||_{[-M/2,M/2]}.$$

First, we calculate an upper bound for the first term. Lemma 1.5 ensures that the f is $o(\frac{1}{|t|^{3/2}})$ and $\hat{f}$ is $o(\frac{1}{|\gamma|^{3/2}})$. Therefore, the hypotheses of the Poisson summation formula are met. By the dual of the third form of the Poisson summation formula (beginning of Section 4),

$$(P_N\hat{f})(\gamma) = \sum_{m\in\mathbf{Z}} \hat{f}(\gamma + mM) = \frac{1}{M}\sum_{m\in\mathbf{Z}} f(m/M)e^{-2\pi i m\gamma/M}.$$

Subtracting $f^\sharp$ from $P_M\hat{f}$ yields

$$\frac{1}{M}\Big(\sum_{m<-\frac{1}{2}NM} + \sum_{m>\frac{1}{2}NM-1}\Big)f(m/M)e^{-2\pi i m\gamma/M}.$$

Therefore, by Parseval's equality for periodic functions,

$$\begin{aligned} ||f^\sharp - P_M\hat{f}||^2_{[-M/2,M/2]} &= \frac{1}{M}\Big(\sum_{m<-\frac{1}{2}NM} + \sum_{m>\frac{1}{2}NM-1}\Big)|f(m/M)|^2 \\ &\leq \frac{1}{M}\sum_{|m|>\frac{1}{2}NM-1}|f(m/M)|^2 \\ &\leq \frac{1}{M}\sum_{|m|>\frac{1}{2}(N-1)M}|f(m/M)|^2 \\ &= \tilde{E}^2_{N-1,M}. \end{aligned}$$

Since, by assumption, C_j and D_1 are finite, we know from Lemma 2.3 that the discrete tail energy is $O(\frac{1}{N^j})$. Taking the square root, we see that $||f^\sharp - P_M\hat{f}||_{[-M/2,M/2]}$ is $O(\frac{1}{N^{j/2}})$.

Next, we calculate an upper bound for the second term. From the dual of the continuous periodization comparison lemma, we see that $||P_M\hat{f} - \hat{f}||_{[-M/2,M/2]}$ is $O(\frac{1}{M^k})$. □

Remark: If we follow the trail of constants with a little more care, we can determine some more information about the K_i. K_2 is proportional to D_k since K_2 comes from an application of the dual of the continuous periodization comparison lemma.

Depending on how $N, M \to \infty$, the first term of the error bound given in this theorem can be improved. $\frac{K_1}{N^{j/2}}$ is bounded by the square root of the discrete tail energy. In Lemma 2.3, the discrete tail energy is bounded by a term that is $O(\frac{1}{N^{2j}})$ plus a term that is $O(\frac{1}{N^j M})$. If $N, M \to \infty$, with N^j approaching infinity faster than M, the first term approaches zero faster than the second. Then, asymptotically, the discrete tail energy is bounded by $\frac{A}{N^j}$, where A approaches a number that is proportional to $C_j D_1/M$. This implies that the square root of the discrete tail energy is bounded by $\frac{\sqrt{A}}{N^{j/2}}$. So, in this case, the $\frac{K_1}{N^{j/2}}$ term approaches zero like $O(\frac{1}{N^{j/2}M^{1/2}})$ — faster than claimed!

If $N, M \to \infty$, with M approaching infinity faster than N^j, the second term of Lemma 2.3 approaches zero faster than the first. Then, asymptotically, the discrete tail energy is bounded by $\frac{A}{N^{2j}}$, where A approaches a number that is proportional to C_j^2. This implies that the square root of the discrete tail energy is bounded by $\frac{\sqrt{A}}{N^j}$. So, in this case, the $\frac{K_1}{N^{j/2}}$ term approaches zero like $O(\frac{1}{N^j})$ — also faster than claimed! □

6 $\mathbf{L}^2$ Norm Approximation

The machinery developed allows us to prove several related theorems. The following theorem provides an upper bound for the error in estimating the $\mathbf{L}^2(\mathbf{R})$ norm by the discrete $\mathbf{L}^2$ norm.

Theorem 6.1 ($\mathbf{L}^2$ Norm Approximation) *Let f be a function with C_j and D_k finite, where j, k are integers greater than 2. Let N and M be even positive integers. Then,*

$$\Big| \, ||f|| - ||f||_{D(M,N)} \Big| \leq \frac{K_1}{N^{j/2}} + \frac{K_2}{M^k}.$$

The K_i depend only on the following parameters. K_1 varies directly with C_j and D_1 and varies inversely with N and M. K_2 varies directly with D_k.

Subsection 1 proves a weak version of the $\mathbf{L}^2$ norm approximation theorem from the generalized Poisson summation formula using Weil space. We prove that if $j = k$, then the error is bounded by the weaker bound, $\frac{K_1}{N^{j/2}} + \frac{K_2}{M^{k/2}}$. This proof gives us a geometric understanding of the theorem.

Subsection 2 proves the stated version of the theorem from the third FFT approximation theorem without using Weil space.

6.1 Proof Using Weil Space

This proof of the weak version of the $\mathbf{L}^2$ norm approximation theorem uses the Weil space theory summarized in Section 3.

Lemma 6.2 *Assume $F \in \mathbf{H}$ is continuous. Let N and M be positive integers. As $M, N \to \infty$, the discrete $\mathbf{L}^2$ norm $||F||_{W(M,N)}$ approaches the continuous $\mathbf{L}^2$ norm $||F||_W$.*

Proof: Just note that $||F||^2_{W(M,N)}$ is a Riemann sum approximation to the integral, $||F||^2_W = \int_0^1 \int_0^1 |F(x,y,0)|^2 dx dy$. □

Let f be $O(\frac{1}{|t|^{1+\epsilon}})$ and $\hat{f}$ be $O(\frac{1}{|\gamma|^{1+\epsilon}})$. Let $F = \Theta f$. As noted in Section 3, $||F||_W = ||f||$ and $||F||_{W(M,N)} = ||P_N f||_{D(M,N)}$. Therefore, the previous lemma tells us that $||P_N f||_{D(M,N)}$ converges to $||f||$, as N, M approaches infinity. The following lemma tells us how fast $||P_N f||_{D(M,N)}$ converges, as N approaches infinity.

Lemma 6.3 *Assume $C_j(f)$ and $D_1(f)$ are finite. Let N, N' be even positive integers, with $N' \geq N$. Then*

$$\Big| \, ||P_N f||_{D(M,N)} - ||P_{N'} f||_{D(M,N')} \Big| \leq \frac{K_1}{N^{j/2}},$$

where K_1 only depends on C_j and D_1.

Proof: By the triangle inequality, the left hand side of the above equation is less than or equal to

$$\begin{aligned} \Big| \, ||P_N f||_{D(M,N)} - ||f||_{D(M,N)} \Big| &+ \Big| \, ||f||_{D(M,N)} - ||T_{N'} f||_{D(M,N')} \Big| \\ &+ \Big| \, ||T_{N'} f||_{D(M,N')} - ||P_{N'} f||_{D(M,N')} \Big| \\ \leq ||P_N f - f||_{D(M,N)} &+ \Big| \, ||f||_{D(M,N)} - ||T_{N'} f||_{D(M,N')} \Big| \\ &+ ||T_{N'} f - P_{N'} f|||_{D(M,N')}. \end{aligned}$$

By the discrete periodization comparison lemma, the first and third terms are bounded by $\frac{K'}{N^{j/2}}$, where K' only depends on C_j and D_1.

The second term is bounded by the discrete tail energy. Therefore, by Lemma 2.3, the term is bounded by $\frac{K''}{N^j}$, where K'' only depends on C_j and D_1. □

Lemma 6.4 *Assume f has finite j^{th} energy moments. Let N, M be even positive integers. Then,*

$$\Big| \, ||f|| - ||P_N f||_{D(M,N)} \Big| \leq \frac{K_1}{N^{j/2}} + \frac{K_2}{M^{j/2}},$$

where the K_i only depend on C_1, C_j, D_1 and D_j.

Proof: Let N' and M' be even integers greater than N and M, respectively.

$$\begin{aligned}
&\Big| \, ||P_N f||_{D(M,N)} - ||P_{N'} f||_{D(M',N')} \Big| \\
&\quad \leq \Big| \, ||P_N f||_{D(M,N)} - ||P_{N'} f||_{D(M,N')} \Big| + \Big| \, ||P_{N'} f||_{D(M,N')} - ||P_{N'} f||_{D(M',N')} \Big| \\
&\quad = \Big| \, ||P_N f||_{D(M,N)} - ||P_{N'} f||_{D(M,N')} \Big| + \Big| \, ||P_M \hat{f}||_{D(N',M)} - ||P_{M'} \hat{f}||_{D(N',M')} \Big| .
\end{aligned}$$

The last line of the above inequality uses the generalized Poisson summation formula.

The first term on the last line of the inequality is bounded by $\frac{K_1}{N^{j/2}}$ by Lemma 3. Similarly, the second term is bounded by $\frac{K_2}{M^{j/2}}$. So we have

$$\Big| \, ||P_N f||_{D(M,N)} - ||P_{N'} f||_{D(M',N')} \Big| \leq \frac{K_1}{N^{j/2}} + \frac{K_2}{M^{j/2}}.$$

Let $F = \Theta f$. From Lemma 1.5, we know that f is $O(\frac{1}{|t|^{1+\epsilon}})$ and $\hat{f}$ is $O(\frac{1}{|\gamma|^{1+\epsilon}})$. Recall from the discussion in Section 3 that under these decay conditions, $||P_{N'} f||_{D(M',N')} = ||F||_{W(M',N')}$. Now, let $N', M' \to \infty$, and apply Lemma 2. This yields

$$\Big| \, ||P_N f||_{D(M,N)} - ||F||_W \Big| \leq \frac{K_1}{N^{j/2}} + \frac{K_2}{M^{j/2}}.$$

Since $||F||_W = ||f||$, the lemma is proven. □

Lemma 4 is almost the $\mathbf{L}^2$ norm approximation theorem. Since f has finite j^{th} energy moments, we know from the discrete periodization comparison lemma that

$||P_N f - f||_{D(M,N)}$ is $O(\frac{1}{N^{j/2}})$. Thus, we can replace $||P_N f||_{D(M,N)}$ with $||f||_{D(M,N)}$ (using the triangle inequality) at the expense of increasing K_1. This gives us the weak version of the $\mathbf{L}^2$ norm approximation theorem.

6.2 Proof from FFT Approximation Theorem

The third FFT approximation theorem yields an error bound for $\mathbf{L}^2$ norm approximation quite easily.

From the third FFT approximation theorem and the triangle inequality, we have

$$\left| \, ||f^\sharp||_{[-M/2,M/2]} - ||\hat{f}||_{[-M/2,M/2]} \right| \leq \frac{K_1}{N^{j/2}} + \frac{K_2}{M^k}.$$

We can replace the first term of this inequality, $||f^\sharp||_{[-M/2,M/2]}$ with $||f||_{D(M,N)}$ since both term are equal by Parseval's equality for periodic functions.

We would like to replace the second term, $||\hat{f}||_{[-M/2,M/2]}$, with $||\hat{f}|| = ||f||$. From the dual of Lemma 2.2, we see that

$$||\hat{f}||^2 - ||\hat{f}||^2_{[-M/2,M/2]} = \int_{|\gamma|>\frac{1}{2}M} |\hat{f}(\gamma)|^2 \leq \frac{D_k^2}{(\frac{1}{2}M)^{2k}}.$$

Now, if $x^2 - y^2 \leq \delta^2$, where $x \geq y \geq 0$ and $\delta > 0$, then

$$x^2 \leq y^2 + \delta^2 \leq y^2 + 2y\delta + \delta^2 = (y+\delta)^2.$$

Taking the square roots of the extreme left and the extreme right yields $x \leq y + \delta$. Therefore, we see that $0 \leq x - y \leq \delta$.

Applying this to our case, we see that $||\hat{f}|| - ||\hat{f}||_{[-M/2,M/2]}$ is $O(\frac{1}{M^k})$. Thus, we can make the desired replacement at the expense of increasing K_2.

Making both substitutions proves the $\mathbf{L}^2$ norm approximation theorem.

Remark: If we follow the trail of constants with a little more care, we can determine some more information about the K_i. K_1 is the same as the K_1 in the third FFT approximation theorem. So, as remarked after that theorem, depending on how $N, M \to \infty$, the first error term of this theorem can be improved.

K_2 is the sum of the constant K_2 in the third FFT approximation theorem plus another constant from the application of the dual of Lemma 2.2. Since both of these constants are proportional to D_k, so is the K_2 of this theorem. □

7 General Riemann Sum Approximation

Both the FFT and $\mathbf{L}^2$ norm approximation theorems provide error bounds for special cases of Riemann sum approximation. This section addresses more general cases of Riemann sum approximation.

In Subsection 1, we use the approximation theorem for the $\mathbf{L}^2$ norm to establish an approximation theorem for the $\mathbf{L}^2$ inner product.

In Subsection 2, we use the inner product approximation theorem to establish an approximation theorem for $\mathbf{L}^1$ Riemann sum approximation.

7.1 $\mathbf{L}^2$ Inner Product Approximation

The *inner product* on $\mathbf{L}^2(\mathbf{R})$ is

$$\langle f,g\rangle = \int f(t)\overline{g(t)}.$$

Similarly, define the *discrete inner product* on $\mathbf{L}^2(\frac{1}{M}\mathbf{Z}_N)$ by

$$\langle f,g\rangle_{D(M,N)} = \frac{1}{M}\sum_{m=NM/2}^{NM/2-1} f(m/M)\overline{g(k/M)}.$$

The polarization identity [W, page 9] allows us to express a inner product in terms of norms. The continuous version is:

$$\langle f,g\rangle = \frac{1}{4}(||f+g||^2 - ||f-g||^2 + i||f-ig||^2 - i||f+ig||^2).$$

The discrete version is entirely analogous.

Subtract the discrete polarization identity from the continuous polarization identity. From the result, we see that the error of discretizing the continuous inner product is bounded by the average of the errors of discretizing the four continuous squares of norms in the polarization identity.

To estimate the error of discretizing $||f \pm g||^2$ and $||f \pm ig||^2$, we use the following immediate consequence of the triangle inequality, stated without proof.

Lemma 7.1 (Corollary of the Triangle Inequality) *For any measure $d\mu$,*

$$(\int |f \pm g|^2 d\mu)^{1/2} \leq (\int |f|^2 d\mu)^{1/2} + (\int |g|^2 d\mu)^{1/2},$$

and

$$(\int |f \pm ig|^2 d\mu)^{1/2} \leq (\int |f|^2 d\mu)^{1/2} + (\int |g|^2 d\mu)^{1/2}.$$

In particular, setting $d\mu = t^{2k}dt$, the square root of any energy moment of $f \pm g$ or $f \pm ig$ is bounded by the sum of the corresponding square roots of the energy moments of f and g.

Now, we can easily prove the following theorem.

Theorem 7.2 (Inner Product Approximation) *Let f, g be functions with C_j and D_k finite, where j, k are integers greater than 2. Let $C_i = \max(C_i(f), C_i(g))$ and $D_i = \max(D_i(f), D_i(g))$. Let N and M be even positive integers. Then,*

$$\left| \langle f, g \rangle - \langle f, g \rangle_{D(M,N)} \right| \leq 4 \left(\frac{K_1}{N^{j/2}} + \frac{K_2}{M^k} \right) \left(\left(\frac{K_1}{N^{j/2}} + \frac{K_2}{M^k} \right) + L \right),$$

where the K_i are the same as the K_i in the $\mathbf{L}^2$ norm approximation theorem (but depending on the C_i and D_i above), and

$$L = \max(||f+g||, ||f-g||, ||f+ig||, ||f-ig||) \leq ||f|| + ||g||.$$

Proof: As discussed before the statement of the theorem, the inner product discretization error is bounded by the average of the errors in discretizing the four continuous squares of norms in the polarization identity.

By the $\mathbf{L}^2$ norm approximation theorem, the error in discretizing $||f||$ or $||g||$ is bounded by $(\frac{K_1}{N^{j/2}} + \frac{K_2}{M^{j/2}})$. Now, if $|x-y| \leq K$, where x and y are positive numbers, then

$$|x^2 - y^2| = |x-y|\,(x+y) \leq |x-y|\,(|x-y| + 2x) \leq K(K+2x).$$

Applying this to our case yields

$$\left|\,||f||^2 - ||f||^2_{D(M,N)}\right| \leq \left(\frac{K_1}{N^{j/2}} + \frac{K_2}{M^k}\right)\left(\left(\frac{K_1}{N^{j/2}} + \frac{K_2}{M^k}\right) + 2||f||\right).$$

The same inequality holds for g taking the place of f.

By the corollary of the triangle inequality, we have a similar error bound for $||f \pm g||^2$, but in this case the K_i are higher. By the remarks following the $\mathbf{L}^2$ norm approximation theorem and the third FFT approximation theorem, we see that K_1 is equal to a term proportional to $\sqrt{C_j D_1}$ plus a term proportional to C_j. Thus, doubling the the C's and D's will double K_1. A similar result holds for K_2. Therefore, the discretization of $||f \pm g||^2$ results in an error that is bounded by

$$\left(2\frac{K_1}{N^{j/2}} + 2\frac{K_2}{M^k}\right)\left(\left(2\frac{K_1}{N^{j/2}} + 2\frac{K_2}{M^k}\right) + 2||f \pm g||\right).$$

A similar bound holds for $||f \pm ig||^2$.

Since each term in the average is bounded by

$$4\left(\frac{K_1}{N^{j/2}} + \frac{K_2}{M^k}\right)\left(\left(\frac{K_1}{N^{j/2}} + \frac{K_2}{M^k}\right) + L\right),$$

so is the average. □

Remark 1: In order to utilize this error bound, an estimate for L is required. Considering L as a constant, the error bound is $O(\frac{1}{N^{j/2}}) + O(\frac{1}{M^k})$. □

Remark 2: The remark about the K_i in the $\mathbf{L}^2$ norm approximation theorem at the end of Section 6.2 also applies to this theorem. □

7.2 $\mathbf{L}^1$ Riemann Sum Approximation

The following theorem provides an error bound for the general case of using a Riemann sum to approximate $\int_{-\infty}^{\infty} f dt$, where $f \in \mathbf{L}^1(\mathbf{R})$.

Theorem 7.3 ($\mathbf{L}^1$ Riemann Sum Approximation) *Let f be a function in $\mathbf{L}^1(\mathbf{R})$. Assume that $C_j(f^{1/2})$ and $D_k(f^{1/2})$ are finite, where j, k are integers greater than 2. Let N and M be even positive integers. Then,*

$$\left| \int_{-\infty}^{\infty} f dt - \frac{1}{M} \sum_{k=NM/2}^{NM/2-1} f(k/M) \right| \leq 4 \left(\frac{K_1}{N^{j/2}} + \frac{K_2}{M^k} \right) \left(\left(\frac{K_1}{N^{j/2}} + \frac{K_2}{M^k} \right) + 2||f||_1^{1/2} \right),$$

where the K_i are the same as the K_i in the $\mathbf{L}^2$ norm approximation theorem (but depending on $C_i(f^{1/2})$ and $D_i(f^{1/2})$), and $||f||_1$ is the $\mathbf{L}^1(\mathbf{R})$ norm of f.

Proof: Apply the inner product approximation theorem to estimating $\int f = \int f_1 f_2$, where $f_1 = |f|^{1/2}$ and $f_2 = f/|f|^{1/2}$. Note that $|f_1| = |f_2| = |f|^{1/2}$. This fact together with our assumptions on $C_j(f^{1/2})$ implies that

$$C_j(f_1) = C_j(f_2) = C_j(f^{1/2}) < \infty.$$

The same result holds for D_k. Therefore, the assumptions of the $\mathbf{L}^2$ norm approximation theorem are met.

The error bound in the statement of the $\mathbf{L}^1$ Riemann sum approximation theorem now follows as a special case of the error bound for inner product approximation. In particular, the L of the inner product approximation theorem is bounded by

$$(\int(|f_1| + |f_2|)^2)dt)^{1/2} = (\int(2|f|^{1/2})^2)dt)^{1/2} = (4||f||_1)^{1/2} = 2||f||_1^{1/2}.$$

□

Remark: The remark about the K_i in the $\mathbf{L}^2$ norm approximation theorem at the end of Section 6.2 also applies to this theorem. □

8 Approximate Dimension and Parameterization

Generally, for a function f used as a communication signal, there exists N and M such that the support of f is contained "mostly" in $[-N/2, N/2]$ and the support of

$\hat{f}$ is contained "mostly" in $[-M/2, M/2]$. Engineers had noticed that the number of "approximately" time-and-bandlimited functions that are linearly independent is "approximately" equal to the time-band product NM.

Landau and Pollak formalized these ideas in their seminal paper [LP]. In [S], these ideas were further expanded by Slepian (who also played a major role in the development of the ideas that led up to Landau-Pollak's theorems). They define a set of functions as having approximate dimension if any function from the set can be approximated by a function from a finite-dimensional subspace of $\mathbf{L}^2(\mathbf{R})$. Then they prove that the set of approximately time-and-bandlimited functions has approximate dimension $NM + o(NM)$, as NM approaches infinity.

In Subsection 1, we outline some of their results in a more rigorous manner. We then show that a set of functions with bounded energy moments is approximately time-and-bandlimited. Thus, there is an approximate dimension theorem for this set of functions.

We propose another approach in Subsection 2. Instead of approximating a set of function with a finite dimensional subspace, we approximate the set of functions using a finite number of parameters. Then, using the $\mathbf{L}^2$ norm approximation theorem, we prove that a set of functions with bounded energy moments can be approximately parameterized with the number of parameters equal to the time-band product.

8.1 Approximate Dimension

We start with a few definitions. An *M-bandlimited* function f is a function with $\int_{|\gamma|>M/2} |\hat{f}|^2 d\gamma = 0$. An (N, ϵ)*-approximately timelimited* function is a function with $\int_{|t|>N/2} |f|^2 dt \leq \epsilon^2$.

Note: Our definition of an M-bandlimited function differs from the usual definition, which is a function with $\int_{|\gamma|>M} |\hat{f}|^2 d\gamma = 0$.

Say that a set of functions, S, has ϵ*-approximate dimension* d if there exists a d-dimensional vector space of functions, V (not necessarily in S), such that the following holds for any $f \in S$:

$$\inf_{v \in V} ||f - v|| \leq \epsilon.$$

One of Landau-Pollak's approximate dimension theorems is as follows.

Theorem 8.1 (Landau-Pollak) *Let N, M, and ϵ_0 be positive numbers. Let $S_0 = S_0(M, N, \epsilon_0)$ be the set of M-bandlimited functions that are (N, ϵ_0)-approximately timelimited and have* $\mathbf{L}^2(\mathbf{R})$ *norm equal to 1. For all $\epsilon > \epsilon_0$, the ϵ-approximate dimension of S_0 is $NM + o(NM)$ as $NM \to \infty$.*

Remark: The theorem can be extended to the set $S_0' = S_0'(M, N, \epsilon_0)$ which will denote the set of M-bandlimited functions that are (N, ϵ_0)-approximately timelimited, but do not necessarily have $\mathbf{L}^2(\mathbf{R})$ norm equal to 1.

The proof is as follows. Assume that $f \in S_0'(M, N, \epsilon_0)$ and $f \neq 0$. Then $f/||f|| \in S_0(M, N, \epsilon_0/||f||)$. By the theorem,

$$\forall f \in S_0',\ \forall \epsilon > \frac{\epsilon_0}{||f||} : \quad \inf_{v \in V} ||\frac{f}{||f||} - v|| \leq \epsilon.$$

This is equivalent to

$$\forall f \in S_0',\ \forall \epsilon > \frac{\epsilon_0}{||f||} : \quad \inf_{v \in V} ||\ f - ||f||v\ || \leq ||f||\epsilon,$$

or

$$\forall f \in S_0',\ \forall \epsilon > \frac{\epsilon_0}{||f||} : \quad \inf_{v \in V} ||f - v|| \leq ||f||\epsilon,$$

or

$$\forall f \in S_0',\ \forall \epsilon > \epsilon_0 : \quad \inf_{v \in V} ||f - v|| \leq \epsilon.$$

□

This theorem as stated does not apply to functions that are only approximately bandlimited. (An (M, ϵ)-*approximately bandlimited* function f is a function with $\int_{|\gamma|>M/2} |\hat{f}|^2 d\gamma \leq \epsilon^2$.)

However, at the end of their paper, they extend some of their other theorems to include functions that are only approximately bandlimited. Slepian explicitly extends a version of the dimension theorem to functions that are only approximately bandlimited.

We will say that a set of functions, S, has *ϵ-approximate dimension d in* $\mathbf{L}^2([-N/2, N/2])$ if there exists a d-dimensional vector space of functions, V, (not necessarily in S) such that the following holds for any $f \in S$:

$$\inf_{v \in V} \int_{-N/2}^{N/2} |f - v|^2 \leq \epsilon^2.$$

(Consistent with this definition, we can consider the prior definition of approximate dimension as approximate dimension in $\mathbf{L}^2(\mathbf{R})$.)

Theorem 8.2 (Slepian) *Let N, M, and ϵ_0 be positive numbers. Let $S_1 = S_1(M, N, \epsilon_0)$ be the set of functions that are (N, ϵ_0)-approximately timelimited and (M, ϵ_0)-approximately bandlimited (not necessarily with $\mathbf{L}^2(\mathbf{R})$ norm equal to 1). For all $\epsilon > \epsilon_0$, the ϵ-approximate dimension of S_1 in $\mathbf{L}^2([-N/2, N/2])$ is $NM + o(NM)$ as $NM \to \infty$.*

By Lemma 2.2, a function with bounded first energy moments is (N, ϵ_T)-approximately timelimited and (M, ϵ_B)-approximately bandlimited, where

$$\epsilon_T = \frac{C_1}{\frac{1}{2}N} \quad \text{and} \quad \epsilon_B = \frac{D_1}{\frac{1}{2}M}.$$

So the following is an immediate corollary of Slepian's version of the dimension theorem.

Corollary 8.3 *Let N and M be positive numbers. Let*

$$S_2 = \{f \in \mathbf{L}^2(\mathbf{R}) : C_1 \leq c, D_1 \leq d\}.$$

Let

$$\epsilon_0 = \max\left(\frac{c}{\frac{1}{2}N}, \frac{d}{\frac{1}{2}M}\right).$$

For all $\epsilon > \epsilon_0$, the ϵ-approximate dimension of S_2 in $\mathbf{L}^2([-N/2, N/2])$ is $NM + o(NM)$ as $NM \to \infty$.

8.2 Approximate Parameterization

If a set of functions has approximate dimension d, then any function of the set can be described (within a certain error) by d parameters. To see this, let $\{\phi_i\}$ be an orthonormal basis for the d-dimensional subspace. Then, any function of the set can be described (within a certain error) by the d Fourier coefficients with respect to the $\{\phi_i\}$.

In this section, we introduce another type of approximate parameterization. Then, we show that a set of functions with bounded energy moments can be approximately parameterized.

We will say that a set of functions, S, has *d ϵ-approximate sampling parameters* if there exist d real numbers $\{x_i\}$ such that the following holds:

$$f, g \in S \text{ and } f(x_i) = g(x_i),\ 1 \leq i \leq d \Longrightarrow ||f - g|| \leq \epsilon.$$

Theorem 8.4 (Approximate Parameterization) *Let j, k be integers greater than 2. Let $S_3 = S_3(d_1, c_j, d_k)$ be the following set:*

$$S_3 = \{f \in \mathbf{L}^2(\mathbf{R}) : D_1 \leq d_1, C_j \leq c_j, D_k \leq d_k\}.$$

Let N and M be even positive integers. Assume $f, g \in S_3$ and

$$f(m/M) = g(m/M), \quad -\frac{1}{2}NM \leq m \leq \frac{1}{2}NM - 1.$$

Then,

$$||f - g|| \leq \frac{K_1}{N^{j/2}} + \frac{K_2}{M^k}.$$

The K_i depend only on the following parameters. K_1 varies directly with c_j and d_1 and varies inversely with N and M. K_2 varies directly with d_k.

In other words, S_3 has NM ϵ-approximate sampling parameters, where $\epsilon = \frac{K_1}{N^{j/2}} + \frac{K_2}{M^k}$.

Proof: Let $h = f - g$. We will prove the theorem by finding an upper bound for $||h||$, using the $\mathbf{L}^2$ norm approximation theorem and the corollary of the triangle inequality, Lemma 7.1.

By this corollary, the square root of a first or j^{th} or k^{th} energy moment of h is bounded by two times the corresponding c_i or d_i of $S_3(d_1, c_j, d_k)$. Thus, we can apply the $\mathbf{L}^2$ norm approximation theorem to the set $S_3(2d_1, 2c_j, 2d_k)$, and

$$\Big| \, ||h|| - ||h||_{D(M,N)} \Big| \leq \frac{K_1}{N^{j/2}} + \frac{K_2}{M^k}.$$

where the K_i only depend on the c_i and d_i.

Since $h = 0$ at all of the points of $\frac{1}{M}\mathbf{Z}_N$, the left hand side of this inequality is $||h||$, which gives us the conclusion of the theorem. □

Remark: The remark about the K_i after the $\mathbf{L}^2$ norm approximation theorem also applies to this theorem. □

This theorem raises additional questions. Note that if NM numbers are arbitrarily chosen, it is possible that there is no $f \in S_3$ such that $f(m), m \in \frac{1}{M}\mathbf{Z}_N$ is equal to those numbers. How can we tell whether there is such an f? If such a function exists, how can we construct it from the sampled values?

In some cases, one might answer the first question negatively by approximating the $\mathbf{L}^2$ norm of $t^k f$ (which is C_k) and of $\gamma^k \hat{f}$ (which is D_k) by Riemann sums. The error can be bounded by the $\mathbf{L}^2$ norm approximation theorem, assuming that $t^k f$ and $\gamma^k \hat{f}$ decay in time and frequency sufficiently fast. If one of the Riemann sum approximations of the C_k or D_k exceed the corresponding c_k or d_k plus the error bound, then no such function can exist.

Both of the questions above could be answered if we could find an appropiate generalization of the Heisenberg inequality. The Heisenberg inequality states that

$$C_1(f)D_1(f) \geq \frac{||f||^4}{16\pi^2},$$

and that equality holds only for $f = e^{-at^2}$, with $a > 0$ (See [DM, page 116].) Thus, we can always find a function such that $C_1(f) \leq c_1$ and $D_1(f) \leq d_1$. Our case is much more difficult since there are 3 functionals (C_j, D_1, D_k) to be minimized (among which tradeoffs of one or two functionals decreasing at the expense of one or two functionals increasing can occur) and there are NM constraints.

In practice though, the following procedure for contructing an interpolating function may suffice. Set $g = e^{-at^2}$, with $a > 0$. Divide the given sampled values by $g(k/M)$, $-\frac{1}{2}NM \leq k \leq \frac{1}{2}NM - 1$. Interpolate the result using polynomial interpolation or trigonometric polynomial interpolation. Then multiply the interpolated function by $g(k/M)$. The result will be a function with the given sampled values and with exponential decay in time and frequency.

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