Smooth Convergence Away From Singular Sets and Intrinsic Flat Continuity of Ricci Flow

by

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Abstract

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In this thesis we provide a framework for studying the smooth limits of Riemannian metrics away from singular sets. We also provide applications to the non-degenrate neckpinch singularities in Ricci flow. We prove that if a family of metrics, g_i, on a compact Riemannian manifold, M^n, have a uniform lower Ricci curvature bound and converge to g_∞ smoothly away from a singular set, S, with Hausdorff measure, $H^{n-1}(S) = 0$, and if there exists connected precompact exhaustion, W_j, of $M^n \setminus S$ satisfying $\operatorname{diam}_{g_i}(M^n) \le D_0$, $\operatorname{Vol}_{g_i}(\partial W_j) \le A_0$ and $\operatorname{Vol}_{g_i}(M^n \setminus W_j) \le V_j where \lim_{j\to\infty} V_j = 0$ then the Gromov-Hausdorff limit exists and agrees with the metric completion of $(M^n \setminus S, g_\infty)$. This is a strong improvement over prior work of the author with Sormani that had the additional assumption that the singular set had to be a smooth submanifold of codimension two. We have a second main theorem in which the Hausdorff measure condition on S is replaced by diameter estimates on the connected components of the boundary of the exhaustion, ∂W_j. This second theorem allows for singular sets which are open subregions of the manifold. In addition, we show that the uniform lower Ricci curvature bounds in these theorems can be replaced by the existence of a uniform linear contractibility function. If this condition is removed altogether, then we prove that $\lim_{j\to\infty} d_{\mathcal{F}}(M'_j, N') = 0$, in which M'_j and N' are the settled completions of (M, g_j) and $(M_\infty \setminus S, g_\infty)$ respectively and $d_{\mathcal{F}}$ is the Sormani-Wenger Intrinsic Flat distance. We present examples demonstrating the necessity of many of the hypotheses in our theorems.

In the second part of this thesis, we study the Angenent-Caputo-Knopf's Ricci Flow through neckpinch singularities. We will explain how one can see the A-C-K's Ricci flow through a neckpinch singularity as a flow of integral current spaces. We then prove the continuity of this weak flow with respect to the Sormani-Wenger Intrinsic Flat (SWIF) distance.

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Contents

Chapter 1

Introduction

Today, the study of the limit spaces that arise from the smooth convergence of a family of metrics spaces away from a singular set (a set on which we have no knowledge of the behavior the metrics) has broad applications in geometric analysis. It is always useful to know that a coarse limit of a sequence of Riemannian manifolds satisfying certain conditions in fact inherits a rich geometric structure and regularity.

This thesis is devoted to paving the way in the study of these limits and also to demonstrating some fruitful applications in the theory of geometric flows and in particular, Ricci flow. In fact, our motivation for considering this framework was to provide new tools in studying the singular limits arising in real and Kähler Ricci flows.

In the first part of this thesis (Chapters 2, 3 and 4), we will provide criteria that tell us when the smooth limit away from a singular set, S, of a sequence of Riemannian metrics g_i on a compact manifold M coincides with the Gromov-Hausdorff (GH) and Sormani-Wenger Intrinsic Flat (SWIF) limits of this sequence. In the second part of this thesis (Chapters 5 and 6), we will study the applications of our results to the Ricci flow neckpinch in order to view the Ricci flow through neckpinch singularities as a weak flow of integral current spaces.

1.1 Smooth Convergence away from Singular Sets

One definition of smooth convergence away from singularities is as follows:

Definition 1.1.1. *For $k \geq 1$ an integer and $0 < \alpha < 1$, will say that a sequence of Riemannian metrics g_i on a compact manifold M^n converges* ***smoothly away from*** *$S \subset M^n$ to a Riemannian metric g_∞ on $M^n \setminus S$ if for every compact set $K \subset M^n \setminus S$, g_i converge $C^{k,\alpha}$ smoothly to g_∞ as tensors.*

Right away from the definition, it is apparent that the global geometry is not well controlled under such convergence. It is natural to ask under what additional conditions the original sequence of manifolds, $M_i = (M^n, g_i)$ have the expected **Gromov-Hausdorff (GH)** and **Sormani-Wenger Intrinsic Flat (SWIF) limits** [25] [45]. Recall that there are examples of sequences of metrics on spheres which converge smoothly away from a point singularity which have no subsequence converging in the GH or the SWIF sense, so additional conditions are necessary (c.f. [35]).

Many results concerning GH limits of the M_i have appeared in the literature. For example, Anderson in [2] studies the convergence of Einstein metrics to orbifolds. Bando-Kasue-Nakajima in [8] study the singularities of the Einstein ALF manifolds. Eyssidieux-Guedj-Zeriahi in [19] prove similar results for the solutions to the complex Monge-Ampere equation. Also Huang in [28] , Ruan-Zhong in [42] , Sesum in [43], Tian in [46] and Tosatti in [50] study the convergence of Kähler-Einstein metrics and Kähler-Einstein orbifolds. However, even in this setting, the relationship is not completely clear and the limits need not agree (see [7]). In Tian-Viaclovsky [47], compactness results for various classes Riemannian metrics in dimension four were obtained in particular, for anti-self-dual metrics, Kähler metrics with constant scalar curvature, and metrics with harmonic curvature. Also the relation between different notions of convergence for Ricci flow is studied in [31]; Rong-Zhang [41] is concerned with the convergence of Ricci-flat Kähler metrics. The The results proven in this thesis may be applied to all of these settings.

Here we first study SWIF limits of sequences of manifolds which converge away from a singular set and then prove the SWIF and GH limits agree using techniques developed in prior work of the author with Sormani in [LS]. All necessary background on these techniques and on SWIF convergence is reviewed in Section 2.

Theorem 1.1.2. *Let $M_i = (M^n, g_i)$ be a sequence of compact oriented Riemannian manifolds such that there is a subset, S, with $H^{n-1}(S) = 0$ and connected precompact exhaustion, W_j, of $M \setminus S$ satisfying (1.8) with g_i converge smoothly to g_∞ on each W_j,*

$$diam_{M_i}(W_j) \leq D_0 \qquad \forall i \geq j, \tag{1.1}$$

$$\mathrm{Vol}_{g_i}(\partial W_j) \leq A_0, \tag{1.2}$$

and

$$\mathrm{Vol}_{g_i}(M \setminus W_j) \le V_j \text{ where } \lim_{j\to\infty} V_j = 0. \tag{1.3}$$

Then

$$\lim_{i\to\infty} d_{\mathcal{F}}(M_i', N') = 0. \tag{1.4}$$

where M_i' and N' are the settled completion of (M, g_i) and $(M \setminus S, g_\infty)$ respectively.

Here, $\mathrm{diam}_M(W)$ is the **extrinsic diameter** found by

$$\mathrm{diam}_M(W) = \sup\{d_M(x,y) : \, x, y \in W\} \tag{1.5}$$

where d_M is the extrinsic distance measured in M rather than W:

$$d_M(x,y) = \inf\{L(C) : \, C : [0,1] \to M, \; C(0) = x, \; C(1) = y\}. \tag{1.6}$$

We write $M_i = (M, g_i)$. The intrinsic diameter of W is then $\mathrm{diam}_W(W)$. See Remark 4.2.3 for the necessity of the hypotheses in Theorem 1.1.2.

Under the conditions of Theorem 1.1.2, if we assume in addition that the manifolds in the sequence have a uniform lower bound on Ricci curvature, then the SWIF and GH limits agree. So we obtain the following new theorem relating the GH limit to the metric completion of the smooth limit away from the singularity:

Theorem 1.1.3. *Let $M_i = (M, g_i)$ be a sequence of oriented compact Riemannian manifolds with uniform lower Ricci curvature bounds,*

$$\mathrm{Ricci_{g_i}(V,V) \ge (n-1)H\, g_i(V,V) \qquad \forall V \in TM_i} \,, \tag{1.7}$$

which converges smoothly away from a singular set, S, with $H^{n-1}(S) = 0$. If there is a connected precompact exhaustion, W_j, of $M \setminus S$,

$$\bar{W}_j \subset W_{j+1} \text{ with } \bigcup_{j=1}^{\infty} W_j = M \setminus S, \tag{1.8}$$

satisfying

$$diam(M_i) \leq D_0, \tag{1.9}$$

$$\mathrm{Vol}_{g_i}(\partial W_j) \leq A_0, \tag{1.10}$$

and

$$\mathrm{Vol}_{g_i}(M \setminus W_j) \leq V_j \text{ where } \lim_{j\to\infty} V_j = 0, \tag{1.11}$$

then

$$\lim_{j\to\infty} d_{GH}(M_j, N) = 0, \tag{1.12}$$

where N is the metric completion of $(M \setminus S, g_\infty)$.

See Remark 4.2.3 for the necessity of our hypotheses in Theorem 1.1.3. We may replace the Ricci condition by a condition on contractibility (see Theorem 4.2.6). For the necessity of the hypotheses in this theorem see [35, Remark 6.8].

Theorems 1.1.2- 1.1.3 improve upon a prior result of the author and Sormani in [LS] because we no longer require the singular set to be a smooth submanifold of codimension 2 as was required there. In fact, we may even allow the singular set to be an open domain as long as we have sufficiently strong controls on the diameters of the exhaustion's boundaries as seen in the following theorem:

Theorem 1.1.4. *Let $M_i = (M, g_i)$ be a sequence of Riemannian manifolds such that there is a closed subset, S, and a connected precompact exhaustion,*

W_j, of $M \setminus S$ satisfying (1.8) such that g_i converge smoothly to g_∞ on each W_j.

If each connected component of $M \setminus W_j$ has a connected boundary,

$$\limsup_{i\to\infty} \left\{ \sum_\beta diam_{(\Omega_j^\beta, g_i)}(\Omega_j^\beta) : \Omega_j^\beta \text{ connected component of } \partial W_j \right\} \le B_j, \tag{1.13}$$

where $\lim_{j\to\infty} B_j = 0$, and if we have

$$diam_{(W_j,g_i)}(W_j) \le D_{int}, \tag{1.14}$$

$$\mathrm{Vol}_{g_i}(\partial W_j) \le A_0, \tag{1.15}$$

$$\mathrm{Vol}_{g_i}(M \setminus W_j) \le V_j \text{ where } \lim_{j\to\infty} V_j = 0, \tag{1.16}$$

then

$$\lim_{j\to\infty} d_{\mathcal{F}}(M'_j, N') = 0. \tag{1.17}$$

where N' is the settled completion of $(M \setminus S, g_\infty)$.

See Remark 4.3.2 for the necessity of our hypotheses in Theorem 1.1.4.

In presence of a uniform lower Ricci curvature bound, Theorem 1.1.4 can be applied to prove the following theorem:

Theorem 1.1.5. *Let $M_i = (M, g_i)$ be a sequence of oriented Riemannian manifolds with uniform lower Ricci curvature bounds,*

$$\mathrm{Ricci_{g_i}(V,V) \ge (n-1)H\, g_i(V,V) \qquad \forall V \in TM_i} \,, \tag{1.18}$$

which converges smoothly away from a closed singular set, S.

If there is a connected precompact exhaustion, W_j, of $M \setminus S$, satisfying (1.8) such that each connected component of $M \setminus W_j$ has a connected boundary,

$$\limsup_{i\to\infty} \left\{ \sum_\beta diam_{(\Omega_j^\beta, g_i)}(\Omega_j^\beta) : \Omega_j^\beta \text{ connected component of } \partial W_j \right\} \le B_j, \tag{1.19}$$

where $\lim_{j\to\infty} B_j = 0$, *and if we have*

$$diam(M_i) \le D_0, \tag{1.20}$$

$$diam_{(W_j,g_i)}(W_j) \le D_{int}, \tag{1.21}$$

and volume controls:

$$\mathrm{Vol}_{g_i}(\partial W_j) \le A_0, \tag{1.22}$$

$$\mathrm{Vol}_{g_i}(M \setminus W_j) \le V_j \text{ such that } \lim_{j\to\infty} V_j = 0, \tag{1.23}$$

then

$$\lim_{j\to\infty} d_{GH}(M_j, N) = 0. \tag{1.24}$$

where N is the metric completion of $(M \setminus S, g_\infty)$.

In Theorems 1.1.3 and 1.1.5, the diameter hypothesis $\mathrm{diam}(M_i) \le D_0$ is not necessary when the Ricci curvature is nonnegative (see Lemma 3.3.2).

The Ricci curvature condition in Theorems 1.1.3 and 1.1.5 may be replaced by a requirement that the sequence of manifolds have a uniform linear contractibility function (see Theorem 4.3.5 and Theorem 4.2.6). See Definition 2.2.9 for the definition of a contractibility function. Recall that Greene-Petersen have a compactness theorem for sequences of manifolds with uniform contractibility functions and upper bounds on their volume [23].

In Chapter 2, we will review the background needed to study the convergence of metric spaces and in particular, we will provide the rudiments of integral current spaces [25] [45]. Chapter 3 is devoted to reviewing my published pre-dissertation work with Professor Sormani on the smooth convergence away from singular sets [35]. Chapter 4 contains the proofs of Theorems 1.1.4, 1.1.5 and 1.1.3. In Chapters 5 and 6 we turn to the study of Ricci flow. The theorems in these chapters are described in Section 1.2.

1.2 Applications to Ricci Flow Through Singularities

We next apply our work to study the Sormani Wenger Intrinsic Flat continuity of a weak Ricci flow through a neck pinch singularity. In particular, we prove that the flow proposed by Angenent-Caputo-Knopf is continuous with respect to SWIF distance when the Riemannian manifolds owing through the neckpinch singularity and the resulting singular spaces are viewed as integral current spaces.

There are many parallels between Hamilton's Ricci Flow and Mean Curvature Flow. While Ricci Flow with surgery was proposed by Hamilton (see [27]) and modified and developed by Perelman (see [39] and [38]), a *canonical* Ricci Flow through singularities which could be a gateway to a notion of Weak Ricci Flow is only being explored and defined recently (see [3]). There are also other approaches to weak Ricci flow using tools from optimal transport (see [49], [22] and [34]). In contrast, weak MCF was developed by Brakke by applying Geometric Measure Theory and viewing manifolds as varifolds. Recently White proved that Brakke Flow is continuous with respect to the Flat distance, when the varifolds are viewed as integral currents (see [52]). For the Ricci Flow - in contrast with MCF - there is no apriori ambient metric space so one needs to work with intrinsic notions of convergence; So it is natural to study a weak notion of Ricci flow as a flow of integral current spaces as long as it is continuous with respect to the intrinsic flat distance.

Consider the Ricci Flow on the $\mathbb{S}^{n+1}$ starting from a rotationally symmetric metric g_0. Angenent-Knopf in [4] showed that if g_0 is pinched enough, then the flow will develop a neckpinch singularity (see Definition 5.2.1) in finite time T and they computed the precise asymptotics of the profile of the solution near the singular hypersurface and as $t \nearrow T$.

Later in [3], Angenent-Caputo-Knopf proved that one can define a **smooth forward evolution** of Ricci Flow through the neckpinch singularity. They achieved that basically by taking a limit of Ricci Flows with surgery and hence showed that Perelman's conjecture that a *canonical* Ricci Flow with surgery exists is actually true in the case of the sphere neckpinch. Since the smooth forward evolution performs a surgery at the singular time $T = 0$ and on scale 0, therefore at all positive times the flow consists of two disjoint smooth Ricci flows on a pair of manifolds.

In order to define a weak Ricci flow, we must view the pair of manifolds M_1 and M_2 as a single **integral current space**. Recall that an integral current space (X, d, T) defined in [45] is a metric space (X, d) endowed with an integral current structure T using the Ambrosio-Kirchheim notion of an integral current [1] so that X is the set of positive density of T. Following a suggestion of Knopf, we endow $M = M_1 \sqcup M_2$ with a metric restricted from a metric space obtained by gluing the manifolds at either end of a thread of length $L(t)$. The resulting integral current space does not include the thread (nor the point of singularity at time $t = T$) because every point in an integral current space has positive density. We will consider this approach and prove the following continuity result:

Theorem 1.2.1. *Let* $(X(t), D(t), T(t))$ *be a smooth rotationally and reflection symmetric Ricci flow on* $\mathbb{S}^{n+1}$ *for* $t \in (-\epsilon, 0)$ *developing a neckpinch singularity at* $T = 0$ *and continuing for* $t \in (0, \epsilon)$ *as a disjoint pair of manifolds joined by a thread of length* $L(t) > 0$ *with* $L(0) = 0$ *undergoing Ricci flow as in [3]. Then, this is continuous in time with respect to the SWIF distance.*

Notice that the assumption $T = 0$ in Theorem 1.2.1 is only for the sake of simplicity . The reflection symmetry in Theorem 1.2.1 is there to guarantee the finite diameter at the singular time. In general, we get the following corollary:

Corollary 1.2.2. *Let* $(X(t), D(t), T(t))$ *be a smooth rotationally symmetric Ricci flow on* $\mathbb{S}^{n+1}$ *for ,* $t \in (-\epsilon, 0)$ *developing a neckpinch singularity at* $T = 0$ *with finite diameter and continuing for* $t \in (0, \epsilon)$ *as a disjoint pair of manifolds joined by a thread of length* $L(t) > 0$ *with* $L(0) = 0$ *undergoing Ricci flow as in [3]. Then,* $X(t)$ *is continuous in time with respect to the SWIF distance.*

In order to prove Theorem 1.2.1, in Theorem 6.1.6, we adapt a result from the pre-dissertation work of the author with Sormani [35] to estimate the SWIF distance between our spaces. In Lemmas 6.2.1, 6.4.2 and 6.5.1, we prove the continuity of the flow prior, at and post the singular time respectively.

We will review Ricci flow and the work of Angenent-Caputo-Knopf [3] in Chapter 5 and Chapter 6 will be devoted to the proving Theorem 1.2.1 and Corollary 1.2.2. In Chapter 7, we will mention a few directions for possible future research.

Chapter 2

Background

2.1 Metric Spaces

For any metric space X, one can construct a complete metric space $\bar{X}$, which contains X as a dense subspace. It has the following universal property: if Y is any complete metric space and f is any uniformly continuous function from X to Y, then there exists a unique uniformly continuous function $\bar{f}$ from $\bar{X}$ to Y, which extends f. The space $\bar{X}$ is determined up to isometry by this property, and is called the completion of X.

Definition 2.1.1 (Metric Completion)**.** *The* ***completion*** *of* (X, d_X) *can be constructed as a set of equivalence classes of Cauchy sequences in* X*. For any two Cauchy sequences* $\{x_n\}$ *and* $\{x_n\}$ *in* X*, we may define their distance as*

$$d(x, y) = \lim_{n \to \infty} d\left(x_n, y_n\right) \tag{2.1}$$

and two Cauchy sequences are considered equivalent if their distance is zero. The original space is embedded in this space via the identification of an ele-

ment x of X with the equivalence class of constant sequence $\{x\}$ This defines an isometry onto a dense subspace.

Remark 2.1.2. *It is worth noting that any Lipschitz function $f : X \to Y$ extends to a Lipschitz function $f : \bar{X} \to Y$ via $f(\{x_n\}) = \lim_{n\to\infty} f(x_n)$ provided that Y is a complete metric space.*

Definition 2.1.3 (Isometric Embedding)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. A map $\phi : X \to Y$ is called an* ***isometric embedding*** *or* ***distance preserving*** *if for any $a, b \in X$ one has*

$$d_Y\left(\phi(a), \phi(b)\right) = d_X(a, b). \tag{2.2}$$

A metric d on a space X induces a length structure on X:

Definition 2.1.4 (The Length Induced by a Metric)**.** *Let (X, d) be a metric space and γ be a path in X (i.e. a continuous map $\gamma : [a, b] \to X$). The length of γ induced by d is given by*

$$L_d(\gamma) = \sup_P \sum_{i=1}^{N} d\left(\gamma(p_{i-1}), \gamma(p_i)\right) \tag{2.3}$$

where the supremum is taken over all partitions $P = \{a = p_0 \leq p_1 \leq p_2 \leq \cdots \leq p_N = b\}$ of the interval $[a, b]$. Whenever $L_d(\gamma) < \infty$, γ is called a rectifiable path.

A **component of accessibility by rectifiable paths**, $C \subset X$ is a maximal subset of the metric space X in which any two points can be joined by a rectifiable path. The length L_d when restricted to the components that are accessible by rectifiable paths gives rise to an intrinsic metric as below:

Definition 2.1.5 (Induced (Intrinsic) Metric)**.** *Let $Z \subset X$ be a subset of the metric space (X, d_X) and suppose any two points $x, y \in Z$ can be joined by a rectifiable path inside Z. The **intrinsic metric** on Z induced by L_d is given by:*

$$d_Z(x, y) := \inf \{L_d(\gamma) | \gamma : [a, b] \to Z \textit{ and } \gamma(a) = x \textit{ and } \gamma(b) = y\} \tag{2.4}$$

Remark 2.1.6. *Let $\mathcal{M}_{rect}$ denote the category of all metric spaces in which any two points can be joined by a rectifiable path. The induction of intrinsic metrics can be thought of as a functor* Int : $\mathcal{M}_{rect} \to \mathcal{M}_{rect}$ *given by*

$$(X, d) \xrightarrow{\text{Int}} (X, d_X) \tag{2.5}$$

For a subset $Z \subset X$ of a metric space (X, d_X), one can also associate a subspace metric which is given as

Definition 2.1.7 (Extrinsic Metric)**.** *Let $Z \subset X$ be a subset of a metric space (X, d_X) then the restriction of the metric d to the subset Z which we also denote by d_X is called the extrinsic metric on $Z \subset X$.*

Remark 2.1.8. *It is very crucial to notice that in general, for a component of accessibility by rectifiable paths $Z \subset X$, the induced (intrinsic) metric d_Z and the extrinsic (restricted) metric d_X are different metrics.*

Extrinsic and induced intrinsic metrics give rise to the notions of extrinsic and intrinsic diameter of a subset as follows:

Definition 2.1.9. *Let $Z \subset X$ be a component of accessibility by rectifiable paths of the metric space (X, d_X). Then the **extrinsic diameter** of Z is defined by*

$$diam_X(Z) = \sup \{d_X(x, y) | x, y \in Z\} \tag{2.6}$$

while the ***intrinsic diameter*** *of Z is given by*

$$diam_Z(Z) = \sup\{d_Z(x,y)|x,y \in Z\} \tag{2.7}$$

2.2 The Gromov-Hausdorff Convergence

2.2.1 Gromov Hausdorff Distance

Definition 2.2.1 (Hausdorff Distance)**.** *For any two subsets $X, Y \subset M$ of a metric space (M,d), the* ***Hausdorff distance*** *between X, Y is defined by*

$$d_H^M(X,Y) = \inf\{\epsilon | X \subset T_\epsilon(Y) \; and \; Y \subset T_\epsilon(X)\}, \tag{2.8}$$

*where $T_\epsilon(X)$ (called the $\epsilon-$**fattening**,* ***tubular neighborhood*** *or* ***generalized ball*** *around X) is defined given by*

$$T_\epsilon(X) = \{z \in M | d(z,X) \leq \epsilon\} \tag{2.9}$$

The Gromov-Hausdorff (GH) distance was defined by Gromov in order to turn the Hausdorff distance to an intrinsic distance (independent of the ambient metric space).

Definition 2.2.2. *The* ***Gromov-Hausdorff distance*** *between two metric spaces (X_1, d_1) and (X_2, d_2) is given by*

$$d_{GH}(X_1,X_2) = \inf\left\{d_H^Z(\varphi_1(X_1),\varphi_2(X_2)) : \; \varphi_i : X_i \to Z\right\} \tag{2.10}$$

where the infimum is taken over all common metric spaces, Z, and all isometric embeddings *(distance preserving), $\varphi_i : X_i \to Z$.*

Remark 2.2.3. *In [25], Gromov observed that d_{GH} is an honest* ***distance*** *when restricted to compact metric spaces; this means that for any two compact metric spaces X_1 and X_2, we have*

$$d_{GH}(X_1, X_2) = 0 \quad \textbf{iff} \quad X_1 \textit{ and } X_2 \textit{ are isometric} \tag{2.11}$$

Because of this, one needs to take the metric completions of precompact spaces when they study the Gromov-Hausdorff distance of spaces.

2.2.2 Gromov's Compactness Theorem

The Gromov's Compactness Theorem gives us a criteria for when a family of metric spaces have a subsequential Gromov-Hausdorff limit. We denote the class of all compact metric spaces by $\mathcal{M}$.

Definition 2.2.4. *Let (X, d) be a compact metric space and $\epsilon > 0$; cap(X, ϵ) and cov(X, ϵ) are defined as follows:*

$$cap(X, \epsilon) = \max\left\{n | X \textit{ contains } n \textit{ disjoint } \frac{\epsilon}{2} - \textit{balls}\right\}, \tag{2.12}$$

and,

$$cov(X, \epsilon) = \min\left\{n | X \textit{ is covered by } n \ \epsilon - \textit{balls}\right\}, \tag{2.13}$$

Theorem 2.2.5 (The Gromov's Compactness Theorem)**.** *Let $\mathcal{C} \subset \mathcal{M}$ be a class of compact metric spaces. Then the following statements are equivalent*

(i) *$\mathcal{C}$ is precompact i.e. any sequence in $\mathcal{C}$ contains a subsequence that Gromov-hausdorff converges in $\mathcal{M}$.*

(ii) *There is a function* $N : (0, \alpha) \to (0, \infty)$ *such that for all* $X \in \mathcal{C}$ *we have cap*$(X, \epsilon) \leq N(\epsilon)$*,*

(iii) *There is a function* $N : (0, \alpha) \to (0, \infty)$ *such that for all* $X \in \mathcal{C}$ *we have cov*$(X, \epsilon) \leq N(\epsilon)$*,*

It is also useful to know that any convergence sequence of compact metric spaces and their limit space can be embedded in a common compact metric space. This fact is known as the Gromov's embedding theorem and was proven by Gromov in [24, p. 65].

Theorem 2.2.6 (Gromov's Embedding Theorem)**.** *For a sequence of compact metric spaces* X_i *with* $X_j \xrightarrow{Gromov-Hausdorff} X_\infty$*, one can find a compact metric space* Z *and distance-preserving maps* $\phi_i : X_i \to Z$ *and* $\phi_\infty : X_\infty \to Z$ *such that*

$$\phi_i(X_i) \xrightarrow{Hausdorff} \phi_\infty(X_\infty) \tag{2.14}$$

2.2.3 GH Convergence and Ricci Curvature

A very popular frame-work in geometry is the study the rigidity (of some sort) of geometric structures in the presence of a lower bound on the curvature. One well-known result in this direction is the Bishop-Gromov volume comparison result which gives an upper bound on the volume growth of balls. In a nutshell, the volume of balls in an Alexandrov space with curvature bounded below grow no faster than the volume of balls in the comparison Alexandrov space of the same dimension. Below, we will give the more general version of this comparison theorem which can easily be specified to manifolds with lower Ricci curvature $\geq (n-1)\kappa$.

Theorem 2.2.7 (Bishop-Gromov Comparison [10]). *Let X be a locally compact Alexandrov space of curvature $\geq \kappa$ and n be a positive integer, then for every $p \in X$ the ratio*

$$\frac{\mu_n\left(B_r(p)\right)}{\operatorname{Vol}\left(V_r^\kappa\right)} \tag{2.15}$$

is nonincreasing in r, where $\mu_n(B_r(p))$ is the $n-$ dimensional Hausdorff measure of a ball of radius r and V_r^κ is the volume of the $r-$ball in the space form, M_κ^n of curvature κ and dimension n.

In fact, the Theorem 2.2.7 states that if $R \geq r > 0$, then

$$\frac{\mu_n\left(B_R(p)\right)}{V_R^\kappa} \leq \frac{\mu_n\left(B_r(p)\right)}{V_r^\kappa} \tag{2.16}$$

Using the Bishop-Gromov comparison result and the control on the volume growth of balls imposed by a lower curvature bound, One can modify the Gromov's Compactness Theorem to prove the following theorem (Also known as Gromov's Compactness Theorem).

Theorem 2.2.8 (Gromov's Compactness Theorem II [25]). *A sequence of compact Riemannian manifolds, (M_j, g_j), such that diam$(M_j) \leq D$ and Ricci$_{M_j} \geq -H$, has a subsequence converging in the Gromov-Hausdorff sense to a metric space (X, d).*

These limit spaces have been extensively studied in the work of Cheeger and Colding [11] [12] [13].

2.2.4 GH Convergence and Contractibility

A different way to approach compactness theorems is to assume the existence of contractibility functions instead of a lower curvature bound.

Definition 2.2.9 (Contractibility Function). *A function* $\rho : [0, r_0] \to [0, \infty)$ *is a* ***contractibility function*** *for a manifold* M *with metric* g *if every ball* $B_p(r)$ *is contractible within* $B_p(\rho(r))$*.*

Theorem 2.2.10 (Greene-Petersen [23]). *A sequence of compact Riemannian manifolds,* (M_j, g_j)*, such that* $\mathrm{Vol}(M_j) \leq V$ *and such that there is a uniform contractibility function,* $\rho : [0, r_0] \to [0, \infty)$*, for all the* M_j*, has a subsequence converging in the Gromov-Hausdorff sense to a metric space* (X, d)*.*

Remark 2.2.11. *Notice that in Theorem 2.2.10, a volume upper bound is assumed together with a contractibility assumption as opposed to the diameter and lower curvature bound assumptions in Theorem 2.2.8. Ferry-Okun [20] showed that* M_n *can converge to an infinite dimensional space if one removes the upper volume bound from the assumptions.*

2.3 Sormani-Wenger Intrinsic Flat Convergence

Though the Gromov-Hausdorff notion of convergence has provided a very important frame-work in the study of limit spaces, it fails to provide a limit in cases which the assumptions of the Gromov's compactness theorem fails (for example see the example of many splines [33, Example 3.11]). One place in which these pathological sequences of spaces come up is taking a minimizer sequence for the Plateau's problem. To deal with this problem, Federer and Fleming invented the notion of **integral currents** and **flat convergence** of integral currents. Their work deal with the integral currents in the Euclidean space and hence applicable to smooth compact manifolds.

To wit, a $k-$dimensional current $\mathbf{T}$ is a linear functional on the space of smooth $k-$forms. For example any smooth $k-$dimensional compact oriented manifold M (with boundary) has a current structure $\mathbf{T}$ defined by

$$T(\omega) := \int_M \omega. \tag{2.17}$$

A set X is called $\mathcal{H}^k$ countably rectifiable if it can be covered by the images of a countable collection of Lipschitz maps $\phi_i : E_i \to X$ where, E_i's are Borel subsets of the $k-$dimensional Euclidean space. A $k-$dimensional **integral current T** with multiplicity function θ on an $\mathcal{H}^k$ countably rectifiable canonical set X is defined by

$$\mathbf{T}(\omega) := \int_X \theta\, \omega = \sum_{i=1}^{\infty} \int_{E_i} (\theta \circ \phi_i)\, \phi_i^* \omega, \tag{2.18}$$

The multiplicity function θ is required to be an integer valued Borel function on the canonical set X. An integral current and its boundary $\partial\mathbf{T}$ given by $\partial\mathbf{T}(\omega) := \mathbf{T}(d\omega)$ are also required to have finite mass i.e. one needs to have:

$$\mathbf{M}(\mathbf{T}) := \int_X \theta \, \mathrm{d}\mathcal{H}^k < \infty \tag{2.19}$$

and

$$\mathbf{M}(\partial\mathbf{T}) < \infty. \tag{2.20}$$

Definition 2.3.1 (Flat Distance)**.** *The flat distance between two* $k-$*dimensional currents* $\mathbf{T_1}$ *and* $\mathbf{T_2}$ *is given by*

$$\begin{aligned}&d_F\left(\mathbf{T}_1, \mathbf{T}_2\right) := \\ &\inf\left\{\mathbf{M}(\mathbf{A}) + \mathbf{M}(\mathbf{B}) \mid \mathbf{A} \textit{ and } \mathbf{B} \textit{ are integral currents and } \mathbf{T_1} - \mathbf{T_2} = \mathbf{A} + \partial\mathbf{B}\right\}\end{aligned}$$

Federer and Fleming proved a compactness theorem for the flat distance which guarantees a rectifiable subsequential limit with finite mass for a sequence of integral currents with a uniform upper bound on their masses and the mass of their boundaries. This compactness result was a key step in proving the Plateau problem in Euclidean space.

Later on, in order to study the Plateau problem in general metric spaces, Ambrosio and Kirchheim [1] generalized integral currents machinery to arbitrary metric spaces. In doing so, they employed De Giorgi's ideas for defining $k-$differential forms on a metric space that is considering $(k+1)-$tuples $(f, \omega_1, \omega_2, \ldots, \omega_k)$ of Lipschitz functions to resemble the $k-$form $f \ \mathrm{d}\omega_1 \wedge \mathrm{d}\omega_2 \wedge \cdots \wedge \mathrm{d}\omega_k$ satisfying some axioms. Using this new machinery, they were able to generalize the Plateau's problem to Banach spaces [1]. Wenger [51] generalized the Plateau's problem to a larger class of metric spaces and defined a flat distance between integral currents in these spaces.

Inspired by the Gromov-Hausdorff distance, Sormani-Wenger [45] introduced the notion of intrinsic flat distance between integral current spaces $(X_1, d_1, \mathbf{T}_1)$ and $(X_1, d_1, \mathbf{T}_1)$ as the infimum of $d_{\mathcal{F}}^{Z}\left((\phi_1)_\#\mathbf{T}_1, (\phi_2)_\#\mathbf{T}_2\right)$ over all common metric spaces Z and isometric embeddingd $\phi_i : X_i \to Z$.

2.3.1 Ambrosio-Kirccheim Integral Currents on Metric Spaces

In this section, we will briefly and rather informally review the required material for the later sections. The reader is advised to consult the original papers by Ambrosio-Kirchheim [1]. The reader should be alert that we may suppress the term "a.e." (almost everywhere) in some statements.

Definition 2.3.2 ($\mathcal{H}^k-$ Rectifiability)**.** *A metric space X is called $\mathcal{H}^k-$* ***countably rectifiable*** *if there is a countable collection of Lipschitz maps $\phi_i : E_i \to X$ from Borel subsets $E_i \subset \mathbb{R}^k$ such that*

$$\mathcal{H}^k\left(X \setminus \left(\cup_{i=1}^{\infty} \phi_i(E_i)\right)\right) = 0. \tag{2.21}$$

Based on the work of Kirchheim [30], one can assume that the maps ϕ_i (**chart maps**) are in fact **bi-Lipschitz** hence we can talk about their **transition maps** which turn out to be also bi-Lipschitz. so Similar to the manifold setting, a collection of bi-Lipschitz charts for X is called an **atlas**.

Resembling the manifold theory, an **oriented atlas** is a a positive atlas i.e. all the resulting transition maps have positive Jacobians (This can easily be made rigorous using the Rademachers Theorem). The **orientation** induced by an oriented atlas $\{\phi_i\}$ which is denoted by $[\{\phi_i\}]$ is the equivalence class of all oriented atlases that are positively related to $\{\phi_i\}$.

Another important notion in Geometric Measure Theory is the notion of the density of a Borel measure μ.

Definition 2.3.3. *(Density) For a Borel measure μ on a metric space X*

and a point $p \in X$, the lower $m-$dimensional density of μ at p is defined as

$$\Theta_*^m(p) := \liminf_{r \to 0} \frac{\mu\left(B_r(p)\right)}{\omega_m r^m}, \tag{2.22}$$

where, ω_m is the volume of the unit ball in $\mathbb{R}^m$.

Roughly speaking, a current works like integration of forms and only the portions of space with enough density would contribute in the integration.

Definition 2.3.4 (Completely Settled Space)**.** *A weighted oriented $\mathcal{H}^k-$countably rectifiable metric space $(X, d, [\{\phi_i\}], \theta)$is called* ***completely settled*** **iff**

$$X = \left\{p \in Closure(X) \mid \Theta_*^k(p) > 0\right\}. \tag{2.23}$$

Now we are ready to give the Ambrosio-Kirchheim's definition of Currents on arbitrary metric spaces. Following the ideas of De Giorgi [16], Ambrosio-Kirchheim [1] replaced the notion of $k-$differential forms by the set $\mathcal{D}^k(X)$ which by definition is the set of all $(k+1)-$tuples $(f, \pi_1, \pi_2, \ldots, \pi_k)$ of real-valued Lipschitz functions in X with $f \in \mathrm{Lip}_b(X)$ (f is a bounded Lipschitz function). In the case of $k = 0$, $\mathcal{D}^0(X) = \mathrm{Lip}_b(X)$.

Definition 2.3.5 (Currents)**.** *Let $k \geq 0$ be an integer and X a complete metric space. The vector space of $\mathbf{M}_k(X)$ of currents in X is the set of all real-valued multilinear functionals $\mathbf{T}$ on $\mathcal{D}^k(X)$ which satisfy the following properties.*

(i) Continuity: *If $\pi_j^i \to \pi_j$ pointwise in X and $\mathrm{Lip}(\pi_j^i) \leq Const.$ then, $\lim_{i\to\infty} \mathbf{T}\left(f, \pi_1^i, \pi_2^i, \ldots, \pi_k^i\right) = \mathbf{T}\left(f, \pi_1, \pi_2, \ldots, \pi_k\right)$.*

(ii) Locality: *If for some $i \in \{1, 2, \ldots, k\}$, π_i is constant on a neighborhood of $\{f \neq 0\}$ then,* $\mathbf{T}(f, \pi_1, \pi_2, \ldots, \pi_k) = 0$.

(i) Finite Mass: *There exists a finite Borel measure μ on X such that*

$$|\mathbf{T}(f, \pi_1, \pi_2, \ldots, \pi_k)| \leq \prod_{i=1}^{k} \mathrm{Lip}(\pi_i) \int_X |f| \, \mathrm{d}\mu \tag{2.24}$$

for all $(f, \pi_1, \pi_2, \ldots, \pi_k) \in \mathcal{D}^k(X)$. The minimal measure μ satisfying 2.24 is called the mass of $\mathbf{T}$ and is denoted by $\|\mathbf{T}\|$.

Similar to differential forms, one can define an **exterior differential** operator on $\mathcal{D}^k(X)$ as follows:

Definition 2.3.6 (Exterior Differential)**.** *The exterior differential* $\mathrm{d} : \mathcal{D}^k(X) \to \mathcal{D}^{k+1}(X)$ *is given by:*

$$\mathrm{d}\omega = \mathrm{d}(f, \pi_1, \pi_2, \ldots, \pi_k) := (1, f, \pi_1, \pi_2, \ldots, \pi_k). \tag{2.25}$$

Also for a map $\phi \in \mathrm{Lip}(X, Y)$, one can define a **pullback** operator as follows:

Definition 2.3.7 (Pullback)**.** *For complete metric spaces X, Y and for* $\phi \in \mathrm{Lip}(X, Y)$ *the pullback operator* $\phi^{\#} : \mathcal{D}^k(Y) \to \mathcal{D}^k(X)$ *is given by:*

$$\phi^{\#}\omega = \phi^{\#}(f, \pi_1, \pi_2, \ldots, \pi_k) := (f \circ \phi, \pi_1 \circ \phi, \pi_2 \circ \phi, \ldots, \pi_k \circ \phi). \tag{2.26}$$

By dualizing these two operators, one gets the **boundary** and **pushforward** operators on currents as is given in below:

Definition 2.3.8 (Boundary Operator). *For $k \geq 1$ and $\mathbf{T} \in \mathbf{M}_k(X)$, the boundary of $\mathbf{T}$, $\partial\mathbf{T}$ is defined as follows:*

$$\partial\mathbf{T}(\omega) := \mathbf{T}(\partial\omega). \tag{2.27}$$

for all $\omega \in \mathcal{D}^{k-1}(X)$.

It is straightforward to check that using locality property, we have $\mathrm{d}^2 = 0$ and hence $\partial^2 = 0$. One should notice that $\partial\mathbf{T}$ is not necessarily a current.

Definition 2.3.9 (Normal Currents). *A current $\mathbf{T} \in \mathbf{M}_k(X)$ ($k \geq 1$) is called a* ***normal current*** *if also $\partial\mathbf{T}$ is a current (i.e. $\partial\mathbf{T} \in \mathbf{M}_{k-1}(X)$). The class of all normal currents in X is denoted by $\mathbf{N}_k(X)$ which is a Banach space equipped with the norm*

$$\mathbf{N}(\mathbf{T}) := \|\mathbf{T}\|(X) + \|\partial\mathbf{T}\|(X). \tag{2.28}$$

Definition 2.3.10 (Pushforward). *for the Lipschitz map $\phi_X \to Y$ and $\mathbf{T} \in \mathbf{M}_k(X)$, the pushforward $\phi_\#\mathbf{T} \in \mathbf{M}_k(Y)$ is given by*

$$\phi_\#\mathbf{T}(\omega) := \mathbf{T}(\phi^\#\omega), \tag{2.29}$$

For any $\omega \in \mathbf{M}_k(Y)$.

By construction, the boundary and pushforward operators commute with each other i.e.

$$\phi_\#(\partial\mathbf{T}) = \partial(\phi_\#\mathbf{T}). \tag{2.30}$$

Another important operation is the **restriction** of currents to forms (similar to contraction in differential geometry) which is defined as:

Definition 2.3.11 (Restriction)**.** *For* $\mathbf{T} \in \mathbf{M}_k(X)$ *and* $\omega = (g, \alpha_1, \alpha_2, \ldots, \alpha_l) \in \mathcal{D}^l(X)$ *(*$l \leq g$*), the restriction* $\mathbf{T} \llcorner \omega \in \mathbf{M}_{k-l}(X)$ *is given by:*

$$\mathbf{T}\left(f, \pi_1, \pi_2, \ldots, \pi_{k-l}\right) := \mathbf{T}\left(fg, \alpha_1, \alpha_2, \ldots, \alpha_l, \pi_1, \pi_2, \ldots, \pi_{k-l}\right). \tag{2.31}$$

Here, we give a key example of current which later on, will be used to define an **integer rectifiable current**.

Example 2.3.12. *Any* L^1-*function* $g : A \subset \mathbb{R}^k \to X$ *gives rise to a* $k-$*current* $[[g]]$ *given by*

$$\begin{aligned} [[g]]\left(f, \pi, \ldots, \pi_k\right) &:= \int_{A \subset \mathbb{R}^k} gf \det(\nabla \pi) \, \mathrm{d}\mathcal{L}^k & (2.32)\\ &= \int_{A \subset \mathbb{R}^k} gf \; \mathrm{d}\pi_1 \wedge \mathrm{d}\pi_2 \wedge \cdots \wedge \mathrm{d}\pi_k \, \mathrm{d}\mathcal{L}^k. & (2.33) \end{aligned}$$

2.3.2 Ambrosio-Kirchheim Compactness Theorem

In this section we briefly recall the compactness theorem proved by Ambrosio-Kirchheim [1]. To do so, we need to define the notion of **weak convergence** as has appeared in [1, Definition 3.6].

Definition 2.3.13 (Weak Convergence of Currents)**.** *A sequence* $\mathbf{T}_i \in \mathbf{M}_k(X)$ *is said to weakly converge to* $\mathbf{T} \in \mathbf{M}_k(X)$ *if* $\mathbf{T}_i$ *converge to* $\mathbf{T}$ *pointwise (as functionals on* $\mathcal{D}^k(X)$*) i.e. if we have*

$$\lim_{i \to \infty} \mathbf{T}_i\left(f, \pi_1, \ldots, \pi_k\right) = \mathbf{T}\left(f, \pi_1, \ldots, \pi_k\right), \tag{2.34}$$

for all $(f, \pi_1, \ldots, \pi_k) \in \mathcal{D}^k(X)$*.*

The Ambrosio-Kirchheim's compactness theorem states that:

Theorem 2.3.14 (Ambrosio-Kirchheim's Compactness Theorem)**.** *For a given complete metric space X, let $\mathbf{T}_i \in \mathbf{N}_k(X)$ be a sequence of normal currents with*

$$\mathbf{N}(\mathbf{T_i}) = \|\mathbf{T}_i\|(X) + \|\partial \mathbf{T}_i\|(X) \leq Const; \tag{2.35}$$

If for any integer $m \geq 1$, there exists a compact set $K_m \subset X$ such that

$$\|\mathbf{T}_i\|(X \setminus K_m) + \|\partial \mathbf{T}_i\|(X \setminus K_m) < \frac{1}{m}, \tag{2.36}$$

then, there exits a subsequence $\{\mathbf{T}_{i(n)}\}$ that converges to a current $\mathbf{T} \in \mathbf{N}_k(X)$ satisfying

$$\|\mathbf{T}\|\left(X \setminus \bigcup_{m=1}^{\infty} K_m\right) + \|\partial \mathbf{T}\|\left(X \setminus \bigcup_{m=1}^{\infty} K_m\right) = 0. \tag{2.37}$$

2.3.3 Integral Current Spaces

In this section we will review the **integer rectifiable currents** defined by Ambrosio-Kirchheim [1] and **integral current spaces** defined by Sormani-Wenger [45].

We first need to know what a **rectifiable current** is;

Definition 2.3.15 (Rectifiable Current)**.** *A current $\mathbf{T} \in \mathbf{M}_k(X)$ ($k \geq 1$) is called a rectifiable current if*

(i) *The mass measure $\|\mathbf{T}\|$ is **concentrated** on a countably $\mathcal{H}^k-$rectifiable set;*

(ii) *The mass measure $\|\mathbf{T}\|$ vanishes on Borel sets N with $\mathcal{H}^k(N) = 0$ ($\mathcal{H}^k-$negligible sets).*

Definition 2.3.16 (Integer Rectifiable Current)**.** *A rectifiable current* $\mathbf{T}$ *is called* ***integer rectifiable*** *if for any* $\phi \in \mathrm{Lip}(X, \mathbb{R}^k)$ *and any open set* $A \subset X$*, one has* $\phi_\#(\mathbf{T} \llcorner A) = [[\theta]]$ *for some* $\theta \in L^1(X)$*.*

Notation. The set of rectifiable currents in X is denoted by $\mathcal{R}_k(X)$ and the set of integer rectifiable currents on X is denoted by $\mathcal{I}_k(X)$.

Definition 2.3.17 (Integer Rectifiable Current Structure)**.** *A* k*-dimensional integer rectifibale current structure on a metric space* (X, d) *is an integer rectifiable current* $\mathbf{T} \in \mathcal{I}_k(\bar{X})$ *on the completion,* $\bar{X}$*, of* X *such that* $\mathrm{set}(\mathbf{T}) = X$*. Such a space is called an* ***integer rectifiable current space*** *and is denoted by* $(X, d, \mathbf{T})$*.*

Definition 2.3.18 (Integral Current [1])**.** *An* ***integral current*** *is an integer rectifiable curent which is also a normal current i.e.* $\partial\mathbf{T}$ *is also a current.*

Finally, we define the notion of **integral current space**;

Definition 2.3.19 (Integral Current Space [45])**.** *A* $k-$*dimensional integral current space is an integer rectifiable current space,* $(X, d, \mathbf{T})$*, whose current structure,* $\mathbf{T}$ *is an integral current (that* $\partial\mathbf{T}$ *is an integer rectifiable current in* $\bar{X}$*). The boundary of* $(X, d, \mathbf{T})$ *is then the integral current space:*

$$\partial(X, d_X, \mathbf{T}) := (\mathrm{set}(\partial\mathbf{T}), d_{\bar{X}}, \partial\mathbf{T}) \tag{2.38}$$

Remark 2.3.20. *By the definition, integral current spaces are metric measure spaces with bi-Lipschitz charts and integer valued Borel weight functions. Integral currrent spaces are also completely settled.*

Remark 2.3.21. *Though* $\mathrm{set}(\partial\mathbf{T})$ *might not be a subset of* $\mathrm{set}(\mathbf{T}) = X$*, we always have* $\mathrm{set}(\partial\mathbf{T}) \subset \bar{X}$*.*

Here we give two basic examples of integral current spaces. Notice that Example 2.3.23 demonstrates that while an integral current space is completely settled, it is not necessarily compact.

Example 2.3.22 (Cone)**.** *There are metrics g_j on the sphere M^3 such that (M^3, g_j) converge smoothly away from a point singularity $S = \{p_0\}$ and the metrics g_j form a conical singularity at p_0. The Gromov-Hausdorff and intrinsic flat limits agree with the metric completion of $(M \setminus S, g_\infty)$ which is the sphere including the conical tip.*

Proof. More precisely the metrics g_j are defined by

$$g_j = dr^2 + f_j^2(r) g_{S^2} \text{ for } r \in [0, \pi] \tag{2.39}$$

where $f_j(r) = (1/j)\sin(r) + (1 - 1/j)\, f(r)$ in which, $f(r)$ is a smooth function such that:

$$f(r) = \sin(r) \text{ for } r \in [0, \pi/2], \tag{2.40}$$

and,

$$f(r) = -\frac{2}{\pi}\,(r - \pi) \text{ for } r \in [3\pi/4, \pi]. \tag{2.41}$$

For any $\delta > 0$, f_j converge to f smoothly on $[0, \pi - \delta]$. Thus g_j converge smoothly on compact subsets of $M \setminus S$ to

$$g_\infty = dr^2 + f^2(r) g_{S^2}. \tag{2.42}$$

The metric completion of $(M \setminus S, g_\infty)$ then adds in a single point p_0 at $r = \pi$. Since

$$\liminf_{r \to 0} \mu(B_{p_0}(r))/r^3 = \frac{4}{3\pi^2} vol(S^2) = \frac{16}{3\pi} > 0, \tag{2.43}$$

the point, p_0, is also included in the settled completion of $(M \setminus S, g_\infty)$. To complete the proof of the claim we could apply Theorem 1.1.2. $\square$

Example 2.3.23 (Cusp)**.** *There are metrics g_j on the sphere M^3 such that (M^3, g_j) converge smoothly away from a point singularity $S = \{p_0\}$ and the metrics g_j form a cusp singularity at p_0. The Gromov-Hausdorff agree with the metric completion of $(M \setminus S, g_\infty)$ which is the sphere including the cusped tip. However the intrinsic flat limit of $(M \setminus S, g_\infty)$ does not include the cusped tip because it has 0 density. So the intrinsic flat limit is the settled completion of $(M \setminus S, g_\infty)$ which in this case is $(M \setminus S, g_\infty)$*

Proof. More precisely the metrics g_j are defined by

$$g_j = dr^2 + f_j^2(r) g_{S^2} \text{ for } r \in [0, \pi] \tag{2.44}$$

where $f_j(r) = (1/j)\sin(r) + (1 - 1/j)\, f(r)$ in which, $f(r)$ is a smooth function such that:

$$f(r) = \sin(r) \text{ for } r \in [0, \pi/2], \tag{2.45}$$

and,

$$f(r) = \frac{4}{\pi^2}\left(r - \pi\right)^2 \text{ for } r \in [3\pi/4, \pi]. \tag{2.46}$$

For any $\delta > 0$, f_j converge to f smoothly on $[0, \pi - \delta]$. Thus g_j converge smoothly on compact subsets of $M \setminus S$ to

$$g_\infty = dr^2 + f^2(r) g_{S^2}. \tag{2.47}$$

The metric completion of $(M \setminus S, g_\infty)$ then adds in a single point p_0 at $r = \pi$. Since

$$\liminf_{r \to 0} \mu(B_{p_0}(r))/r^3 = \liminf_{r \to 0} \frac{4}{5\pi^2} r^2 vol(S^2) = 0, \tag{2.48}$$

the point, p_0, is not included in the settled completion of $(M \setminus S, g_\infty)$.

This Gromov-Hausdorff and Intrinsic Flat limits in this example were proven to be as claimed in the Appendix of [45]. One may also apply Theorem 3.1.1 to reprove this. $\square$

As we will see in chapter 6, the Ricci flow neckpinch is a cusp and hence when we think of Ricci flow (and the weak solutions of Ricci flow past the singular time) as integral current spaces, at the singular time, the cusp point will not be included (since the density is zero) and therefore the space splits into two disjoint parts.

2.3.4 Sormani-Wenger Intrinsic Flat Distance

Definition 2.3.24 (Sormani-Wenger's Intrinsic Flat Distance [45])**.** *The* ***intrinsic flat distance*** *between two integral current spaces* $(X_1, d_1, \mathbf{T}_1)$ *and* $(X_2, d_2, \mathbf{T}_2)$ *is defined as*

$$d_{\mathcal{F}}\left((X_1, d_1, \mathbf{T}_1), (X_2, d_2, \mathbf{T}_2)\right) := \inf\left\{ d_F^Z\left((\phi_1)_\# \mathbf{T}_1, (\phi_2)_\# \mathbf{T}_2\right) \mid \phi_i : X_i \to Z \right\}, \tag{2.49}$$

where the infimum is taken over all common complete metric spaces, Z*, and all isometric embeddings* $\phi_i : X_i \to Z$*.*

The Sormani-Wenger's intrinsic flat distance (SWIF) enjoys many nice properties for example as is proven in Sormani-Wenger [45], for two **precompact** integral current spaces $(X_1, d_1, \mathbf{T}_1)$ and $(X_1, d_1, \mathbf{T}_1)$, we have

$$d_{\mathcal{F}}\left((X_1, d_1, \mathbf{T}_1), (X_2, d_2, \mathbf{T}_2)\right) = 0 \tag{2.50}$$

iff there is a **current-preserving** isometry between X_1 and X_2.

Riemannian Setting. As we explained in the beginning of this chapter, one can think of Riemannian manifolds as integral current spaces in which the current structure is given by integration of top dimensional differential forms.

The SWIF distance, $d_{\mathcal{F}}(M_1, M_2)$, is estimated by explicitly constructing a filling manifold, B^{m+1}, between the two given manifolds, finding the excess boundary manifold A^m satisfying

$$\int_{\varphi_1(M_1)} \omega - \int_{\varphi_2(M_2)} \omega = \int_B d\omega + \int_A \omega, \tag{2.51}$$

and summing their volumes

$$d_{\mathcal{F}}(M_1^m, M_2^m) \leq \mathrm{Vol}_m(A^m) + \mathrm{Vol}_{m+1}(B^{m+1}). \tag{2.52}$$

2.3.5 Intrinsic Flat Convergence

Now that we have gotten familiar with the rudiments of the theory of current structures and integral current spaces, it is time to mention a few compactness theorem for sequences of integral current spaces.

Theorem 2.3.25 (Sormani-Wenger's Compactness Theorem [45]). *Let* $(X_i, d_i, \mathbf{T}_i)$ *be a sequence of* $k-$*diemsnional integgral current spaces wuch that the underlying spaces,* X_i *are equicompact and equibounded fartheremore, assume that* $\mathbf{N}(\mathbf{T}_i)$ *are uniformly bounded above, then there exists a subsequence* $\big(X_{i(n)}, d_{i(n)}, \mathbf{T}_{i(n)}\big)$ *such that*

$$\big(X_{i(n)}, d_{i(n)}\big) \xrightarrow{\textit{Gromov-Hausdorff}} (Y, d_Y) \tag{2.53}$$

and,

$$\big(X_{i(n)}, d_{i(n)}, \mathbf{T}_{i(n)}\big) \xrightarrow{\textit{SWIF}} (X, d, \mathbf{T}) \tag{2.54}$$

where either $(X, d, \mathbf{T})$ *is a* $k-$*dimensional integral current space with* $X \subset Y$ *or it is the* $\mathbf{0}$ *current space.*

Here, we also mention two resulting compactness theorems which deal with sequences of Riemannian manifolds.

Theorem 2.3.26 (Sormani-Wenger [45])**.** *If a sequence of oriented compact Riemannian manifolds,* (M_j, g_j)*, with a uniform linear contractibility function,* $\rho : [0, \infty) \to [0, \infty)$ *and a uniform upper bound on volume,* $\mathrm{Vol}(M_j) \leq V$*, converges in the Gromov-Hausdorff sense to* (X, d)*, then it converges in the intrinsic flat Sense to* (X, d, T) *(see Theorem 4.14 of [45]).*

Theorem 2.3.27 (Sormani-Wenger)**.** *If a sequence of oriented compact Riemannian manifolds,* (M_j, g_j)*, such that diam*$(M_j) \leq D$ *and Ricci*$_{M_j} \geq 0$ *and vol*$(M_j) \geq V_0$ *converges in the Gromov-Hausdorff sense to* (X, d)*, then it converges in the intrinsic flat Sense to* (X, d, T) *(see Theorem 4.16 of [45]).*

Chapter 3

Review of Smooth Convergence Away from Singular Sets

This chapter is devoted to a brief review of my pre-dissertation work with my doctoral advisor Professor Christina Sormani. For more details, the reader is gested to consult Lakzian-Sormani [35].

3.1 Estimating the GH and SWIF Distances

We can estimate both of these distances by applying the following theorem which was proven in prior work of the author with Sormani [35] by constructing an explicit space Z and isometric embeddings φ_i. Here we have cut and pasted the exact theorem statement along with the corresponding figure from that paper:

Theorem 3.1.1. *Suppose $M_1 = (M, g_1)$ and $M_2 = (M, g_2)$ are oriented precompact Riemannian manifolds with diffeomorphic subregions $U_i \subset M_i$*

and diffeomorphisms $\psi_i : U \to U_i$ *such that*

$$\psi_1^* g_1(V,V) < (1+\epsilon)^2 \psi_2^* g_2(V,V) \qquad \forall V \in TU, \tag{3.1}$$

and

$$\psi_2^* g_2(V,V) < (1+\epsilon)^2 \psi_1^* g_1(V,V) \qquad \forall V \in TU. \tag{3.2}$$

Taking the extrinsic diameters,

$$D_{U_i} = \sup\{ diam_{M_i}(W) : W \text{ is a connected component of } U_i\} \le diam(M_i). \tag{3.3}$$

we define a hemispherical width,

$$a > \frac{\arccos(1+\epsilon)^{-1}}{\pi} \max\{D_{U_1}, D_{U_2}\}. \tag{3.4}$$

Taking the difference in distances with respect to the outside manifolds,

$$\lambda = \sup_{x,y \in U} \left| d_{M_1}(\psi_1(x), \psi_1(y)) - d_{M_2}(\psi_2(x), \psi_2(y)) \right|, \tag{3.5}$$

we define heights,

$$h = \sqrt{\lambda(\max\{D_{U_1}, D_{U_2}\} + \lambda/4)}, \tag{3.6}$$

and

$$\bar{h} = \max\{h, \sqrt{\epsilon^2 + 2\epsilon}\, D_{U_1}, \sqrt{\epsilon^2 + 2\epsilon}\, D_{U_2}\}. \tag{3.7}$$

Then the Gromov-Hausdorff distance between the metric completions *is bounded,*

$$d_{GH}(\bar{M}_1, \bar{M}_2) \le a + 2\bar{h} + \max\left\{ d_H^{M_1}(U_1, M_1), d_H^{M_2}(U_2, M_2) \right\}, \tag{3.8}$$

and the intrinsic flat distance between the settled completions *is bounded,*

$$\begin{aligned} d_{\mathcal{F}}(M_1', M_2') \;\; &\le \;\; \left(2\bar{h} + a\right) \Big(\mathrm{Vol}_m(U_1) + \mathrm{Vol}_m(U_2) + \mathrm{Vol}_{m-1}(\partial U_1) + \mathrm{Vol}_{m-1}(\partial U_2) \Big) \\ &\quad + \mathrm{Vol}_m(M_1 \setminus U_1) + \mathrm{Vol}_m(M_2 \setminus U_2), \end{aligned}$$

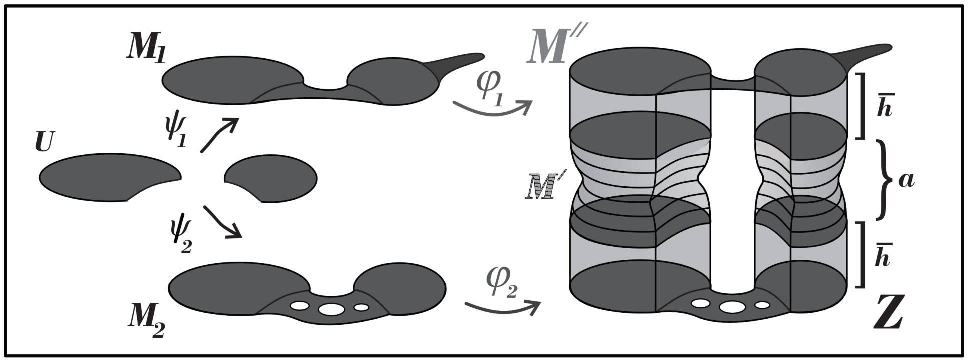

Figure 3.1: Creating Z for Theorem 3.1.1.

Note that permission to reprint this figure along with the statement of Theorem 3.1.1 has been granted by the author and Christina Sormani who own the copyright to this figure that first appeared in [35].

3.2 Uniform Well Embeddedness

Definition 3.2.1. *Given a sequence of Riemannian manifolds $M_i = (M, g_i)$ and an open subset, $U \subset M$, a connected precompact exhaustion, W_j, of U satisfying (1.8) is* ***uniformly well embedded*** *if there exist a λ_0 such that*

$$\limsup_{j\to\infty} \limsup_{k\to\infty} \limsup_{i\to\infty} \lambda_{i,j,k} \le \lambda_0, \tag{3.9}$$

and

$$\limsup_{k\to\infty} \lambda_{i,j,k} = \lambda_{i,j} \text{ where } \limsup_{i\to\infty} \lambda_{i,j} = \lambda_j \text{ and } \lim_{j\to\infty} \lambda_j = 0. \tag{3.10}$$

where,

$$\lambda_{i,j,k} = \sup_{x,y\in W_j} \left| d_{(W_k,g_i)}(x,y) - d_{(M,g_i)}(x,y) \right| \tag{3.11}$$

The author and Sormani in [35] have proven:

Theorem 3.2.2. *Let $M_i = (M, g_i)$ be a sequence of Riemannian manifolds such that there is a closed subset, S, and a uniformly well embedded connected precompact exhaustion, W_j, of $M \setminus S$ satisfying (1.8) such that g_i converge smoothly to g_∞ on each W_j with*

$$diam_{M_i}(W_j) \le D_0 \qquad \forall i \ge j, \tag{3.12}$$

$$\mathrm{Vol}_{g_i}(\partial W_j) \le A_0, \tag{3.13}$$

and

$$\mathrm{Vol}_{g_i}(M \setminus W_j) \le V_j \text{ where } \lim_{j\to\infty} V_j = 0, \tag{3.14}$$

Then

$$\lim_{j\to\infty} d_{\mathcal{F}}(M'_j, N') = 0. \tag{3.15}$$

where N' is the settled completion of $(M \setminus S, g_\infty)$.

Remark 3.2.3. *Example 4.1.6 demonstrates the necessity of well-embeddedness condition in Theorem 3.2.2.*

3.3 Lemmas on Volume and Diameter

Lemma 3.3.1. *Let $M_i = (M, g_i)$ be a sequence of Riemannian manifolds such that there is a closed subset, S, and a connected precompact exhaustion, W_j, of $M \setminus S$ satisfying (1.8) such that g_i converge smoothly to g_∞ on each W_j. If $\mathrm{Vol}_{g_\infty}(M \setminus S) < \infty$ and*

$$\mathrm{Vol}_{g_i}(M \setminus W_j) \le V_j \text{ where } \lim_{j\to\infty} V_j = 0, \tag{3.16}$$

then there exists a uniform $V_0 > 0$ such that

$$\mathrm{Vol}_{g_i}(M) < V_0. \tag{3.17}$$

Lemma 3.3.2. *Suppose we have a sequence of manifolds, $M_j = (M, g_j)$ with nonnegative Ricci curvature and*

$$\operatorname{Vol}(M_j) \leq V_0, \tag{3.18}$$

converging smoothly away from a singular set to $(M \setminus S, g_\infty)$ then

$$diam_{M_i}(W_j) \leq diam(M_i) \leq D_0 \qquad \forall i \geq j. \tag{3.19}$$

3.4 Ricci Curvature

Proposition 3.4.1. *Suppose we have a sequence of manifolds, $M_j = (M, g_j)$ with a uniform lower bound on Ricci curvature and*

$$\operatorname{Vol}(M_j) \leq V_0, \tag{3.20}$$

converging smoothly away from a singular set to $(M \setminus S, g_\infty)$. Suppose also that (M, g_j) converge in the intrinsic flat sense to N' where N' is the settled completion of $(M \setminus S, g_\infty)$. Then

$$d_{GH}(\bar{M}_j, \bar{N}) \to 0, \tag{3.21}$$

and $\bar{N} = N'$.

3.5 Contractibility

Theorem 3.5.1. *Let $M_i = (M, g_i)$ be a sequence of compact oriented Riemannian manifolds with a uniform linear contractibility function, ρ, which converges smoothly away from a singular set, S. If there is a uniformly well* *embedded connected precompact exhaustion of $M \setminus S$ as in (1.8) satisfying the volume conditions (4.140) and (4.141) then*

$$\lim_{j\to\infty} d_{GH}(M_j, N) = 0, \tag{3.22}$$

where N is the settled and metric completion of $(M \setminus S, g_\infty)$.

Proof. By Lemma 3.3.1, we have

$$\mathrm{Vol}(M_i) \leq V_0. \tag{3.23}$$

This combined with the uniform contractibility function allows us to apply the Greene-Petersen Compactness Theorem. In particular, we have a uniform upper bound on diameter:

$$\mathrm{diam}(M_i) \leq D_0, \tag{3.24}$$

We may now apply Theorem 3.2.2 to obtain

$$\lim_{j\to\infty} d_{\mathcal{F}}(M_j, N') = 0 \tag{3.25}$$

We then apply Theorem 2.3.26 to see that the flat limit and Gromov-Hausdorff limits agree due to the existence of the uniform linear contractibility function and the fact that the volume is bounded below uniformly by the smooth limit. In particular, the metric completion and the settled completion agree.

□

Chapter 4

Diameter Controls and Smooth Convergence Away From Singular Sets

4.1 Examples

In this section we present some examples which helps in understanding the notions we have mentioned so far. Some examples will prove the necessity of some conditions in Theorem 1.1.5.

4.1.1 Unbounded Limits

The following examples show why some sort of bounded geometry is necessary in this context.

Example 4.1.1. *There are metrics* g_j *on the sphere* M^3 *with a uniform* *upper bound on volume such that* (M^3, g_j) *converge smoothly away from a point singularity* $S = \{p_0\}$ *to a complete noncompact manifold. There is no Gromov-Hausdorff limit in this case. The intrinsic flat limit is* $(M \setminus S, g_\infty)$.

Proof. Let

$$g_0 = h^2(r)dr^2 + f^2(r)g_{S^2}, \tag{4.1}$$

be defined on $M^3 \setminus S$ as a complete metric such that

$$\int_0^\pi h(r)dr = \infty, \tag{4.2}$$

and

$$\int_0^\pi \omega_2 h(r) f^2(r) dr < \infty, \tag{4.3}$$

so that $\mathrm{diam}(M \setminus S, g_0) = \infty$ and $\mathrm{Vol}(M \setminus S, g_0) < \infty$.

We set

$$g_j = h_j^2(r)dr^2 + f_j^2(r)g_{S^2}, \tag{4.4}$$

such that

$$\begin{aligned} h_j(r) &= h(r) \qquad r \in [0, \pi - 1/j], & (4.5)\\ f_j(r) &= f(r) \qquad r \in [0, \pi - 1/j], & (4.6)\end{aligned}$$

and extend smoothly so that g_j is a metric on S^3.

Metrics g_j converge smoothly to g_0 away from $S = \{p_0\} = r^{-1}(\pi)$ and, since $(M \setminus S, g_0)$ is noncompact, (M, g_j) has no Gromov-Hausdorff limit. The intrinsic flat limit of (M, g_j) is the settled completion of $(M \setminus S, g_0)$ by Theorem 1.1.2, taking $W_j = r^{-1}[0, \pi - 1/j)$ since

$$\int_{\pi - 1/k}^\pi \omega_2 h(r) f^2(r) dr = 0, \tag{4.7}$$

by the finiteness of (4.3) and we also have $\mathrm{Vol}_{g_i}(\partial(W_j)) \le f^2(r)$. In this case the settled completion is just $(M \setminus S, g_0)$ because it is already a complete metric space with positive density. $\square$

Example 4.1.2. *There are metrics* g_j *on* $M^3 = S^3$ *converging smoothly away from a singular set* $S = \{p_0\}$ *to a complete noncompact manifold of infinite volume.* (M, g_j) *have no intrinsic flat or Gromov-Hausdorff limit since, if such a limit existed it would have to contain the smooth limit and the smooth limit has infinite diameter and volume.*

Proof. We define a metric g_0 on $M \setminus S$ exactly as in Example 4.1.1 except that we replace (4.3) with

$$\int_0^{\pi} \omega_2 h(r) f^2(r) dr = \infty, \tag{4.8}$$

so that $\mathrm{diam}(M \setminus S, g_0) = \infty$ and $\mathrm{Vol}(M \setminus S, g_0) = \infty$.

Selecting g_j also as in that example, we have (M, g_j) converge smoothly away from S to $(M \setminus S, g_0)$. However there is no Gromov-Hausdorff limit because the diameter diverges to infinity [25] and there is no intrinsic flat limit because the volume diverges to infinity [45]. $\square$

One may define pointed Gromov-Hausdorff and pointed intrinsic flat limits to deal with unboundedness. However even assuming boundedness, we see in [35, Example 3.11] that the Gromov-Hausdorff limit need not exist.

4.1.2 Ricci Example

This example shows that the mere uniform lower bound for Ricci curvature does not imply the existence of the Gromov-Hausdorff limit.

Example 4.1.3. *There are metrics g_j on $M^3 = S^3$ with negative uniform lower bound on Ricci curvature, converging smoothly away from a singular set $S = \{p_0\}$ to a complete noncompact manifold of finite volume.*

Proof. Consider the metric g_0 on $S^3 \setminus \{p_0\} = \mathbb{R} \times S^2$ given by

$$\bar{g}(t) = dt^2 + (\bar{f}(t))^2 g_{S^2}, \tag{4.9}$$

where, $\bar{f}$ is a nonzero smooth function such that

$$\bar{f}(t) = \sin(t) \text{ for } t \in [0, \pi/2], \tag{4.10}$$

$$\bar{f}(t) = \exp -t \text{ for } t \in [\pi/2 + 1, \infty), \tag{4.11}$$

with $\bar{f}''(t) < \bar{f}(t)$ elsewhere. hence, $\bar{g}$ has Ricci curvature bounded below by

$$-\Lambda = -2 \max \frac{\bar{f}''}{\bar{f}} > -\infty. \tag{4.12}$$

We can extract warped metrics $\bar{g}_j$ on $[0, j+1] \times S^2$

$$\bar{g}(t) = dt^2 + \bar{f}_j(t)^2 g_{S^2}, \tag{4.13}$$

where $\bar{f}_j$ is a nonzero smooth function satisfying

$$\bar{f}_j(t) = \bar{f}(t) \text{ for } t \in [0, j-1], \tag{4.14}$$

$$\bar{f}_j(t) = \exp -j \sin(\exp j(\pi + t - j - 1)) \text{ for } t \in [j, j+1], \tag{4.15}$$

and

$$-2 \max \frac{\bar{f}_j''}{\bar{f}_j} \geq -\Lambda. \tag{4.16}$$

Note that in fact we are cutting off a part of $\bar{f}$ and replacing it with a less concave function which closes up like a sin function hence obtaining a metric on S^3, with lower bound on Ricci curvature.

It is clear that

$$\mathrm{Vol}(M, g_0) \leq \infty. \tag{4.17}$$

Let $\phi : [0, \pi] \to [0, \infty)$ be a smooth increasing function such that

$$\phi(r) = r \text{ for } r \in [0, \pi/2], \tag{4.18}$$

with

$$\lim_{r \to \pi} \phi(r) = \infty. \tag{4.19}$$

For $j > 2$, let $\phi_j(r) : [0, \pi] \to [0, L_j = j + \pi/2 + 1]$ be a smooth increasing function such that

$$\phi_j(r) = \phi(r) \text{ for } r \in [0, \phi^{-1}(j + \pi/2)], \tag{4.20}$$

and

$$\phi_j(r) = j + r - \pi/2 + 1 \text{ for } r \text{ near } \pi. \tag{4.21}$$

we construct metrics

$$g_j(r) = \phi_j^* \left(\bar{g}_j\right), \tag{4.22}$$

with Ricci bounded below by $-\Lambda$ converging smoothly away from $\{p_0\}$ to $\phi^*(\bar{g})$. Taking $W_j = r^{-1}([0, \pi - 1/j))$ we observe that W_j satisfies all the hypotheses in Theorem 1.1.5 except that $\mathrm{diam}_{M_i}(W_j)$ is not bounded. The Gromov-Hausdorff limit does not exist because $(M \setminus S, g_0)$ is complete noncompact. Both intrinsic flat limit and the metric completion coincide with the complete noncompact manifold $(M \setminus S, g_0)$. □

Remark 4.1.4. *From [53], we know that any complete noncompact manifold with nonnegative Ricci curvature has infinite volume, so Example 4.1.3 can not be constructed with the sequence having nonnegative Ricci curvature.*

4.1.3 Pinching a torus

Example 4.1.5. *There are* (M^2, g_j) *all diffeomorphic to the torus,* $S^1 \times S^1$ *which converge smoothly away from a singular set,* $S = \{0\} \times S^1$*, to*

$$(M \setminus S, g_\infty) = \left((0, 2\pi) \times S^1, dt^2 + \sin^2(\frac{t}{2})\, ds^2\right). \tag{4.23}$$

So the metric completion and the settled completions are both homeomorphic to

$$M_\infty = [0, 2\pi] \times S^1 / \sim, \tag{4.24}$$

where

$$(0, s_1) \sim (0, s_2) \;\; \textit{and} \;\; (2\pi, s_1) \sim (2\pi, s_2) \qquad \forall s_1, s_2 \in S^1. \tag{4.25}$$

However the Gromov-Hausdorff and Intrinsic Flat limits identify these two end points.

Proof. Let g_j on M be defined by

$$g_j = dt^2 + f_j^2(t) ds^2, \tag{4.26}$$

where $f_j : S^1 \to (0, 1]$ are smooth with $\left|f_j'(t)\right| \leq 1$ that decrease uniformly to $\sin(\frac{t}{2})$ and $f_j(t) = \sin(\frac{t}{2})$ for $t \in [1/j, 2\pi - 1/j]$. □

4.1.4 Examples of Slit Tori

Example 4.1.6. *Let* (M^2, g) *be the standard flat 2 torus* $S^1 \times S^1$ *and* $S \subset M^2$ *a vertical geodesic segment of length* $\leq \pi$*, then if* g_j *are a constant sequence of the standard flat metric, we see that* (M^2, g_j) *converges smoothly to itself and thus the intrinsic flat and Gromov-Hausdorff limits are both the flat torus.*

However, the metric completion of $(M \setminus S, g_\infty)$ *has two copies of the slit,* S *(with end points identified hence the limit has fundamental group* $= \mathbb{Z}$*) one found taking limits of Cauchy sequences from the right and the other found taking limits of Cauchy sequences from the left. This example shows necessity of uniform well embeddedness condition in our Theorems.*

Proof. Let $M^2 = S^1 \times S^1 = [0, 2\pi] \times [0, 2\pi]/\sim$ such that $(x, 0) \sim (x, 2\pi)$ and $(0, y) \sim (2\pi, y)$. Without loss of generality, we can assume $S = \{(\pi, y) : y \in [\pi/2, 3\pi/2]\}$. Then the metric completion of $M^2 \setminus S$ is

$$M_\infty = \frac{S^1 \times S^1 \times \{0\} \sqcup S^1 \times S^1 \times \{1\}}{\sim}, \tag{4.27}$$

where,

$$(x, y, 0) \sim (x, y.1) \ \text{ for } \ (x, y) \notin S, \tag{4.28}$$

with the distance d_∞ given by

$$d_\infty\left([x, y, l], [x', y', l']\right) = \lim_{\delta \to 0^+} d_{M^2 \setminus S}\left(\left(x + (-1)^l \delta \ , \ y\right), \left(x' + (-1)^{l'} \delta, y'\right)\right) \tag{4.29}$$

for $l, l' = 0, 1$. In particular,

$$d_\infty\left([\pi, \pi, 0], [\pi, \pi, 1]\right) = \pi. \tag{4.30}$$

Notice that M_∞ is not a manifold (not even Hausdorff as $B_r([\pi, \pi, 0]) \cap B_r([\pi, \pi, 1]) \neq \emptyset$ for all r, r'.) Taking the connected precompact exhaustion

$$W_j = M^2 \setminus \left([\pi/2 - 1/j, 3\pi/2 + 1/j] \times [\pi - 1/j, \pi + 1/j]\right), \tag{4.31}$$

we observe that

$$\begin{gathered} \operatorname{diam}_{M_i}(W_j) \leq \operatorname{diam}(M^2) \\ \operatorname{Vol}(M_i) = \operatorname{Vol}(M^2) \\ \operatorname{Vol}_{g_i}(\partial W_j) \leq 2\pi + 4, \end{gathered} \tag{4.32}$$

are uniformly bounded, also

$$\lim_{j\to\infty} \mathrm{Vol}_{g_i}(N \setminus W_j) = \lim_{j\to\infty}(2/j)(\pi + 2/j) = 0, \tag{4.33}$$

but,

$$\begin{aligned}
\lambda_{i,j,k} &= \sup_{x,y\in W_j} \left| d_{(W_k,g_i)}(x,y) - d_{(M,g_i)}(x,y) \right| \qquad (4.34)\\
&\geq \left| d_{(W_k,g_i)}\left((\pi - 1/j, \pi), (\pi + 1/j, \pi)\right) - d_{(M,g_i)}\left((\pi - 1/j, \pi), (\pi + 1/j, \pi)\right) \right|\\
&\geq \pi + 2/j + 2/k,
\end{aligned}$$

Therefore,

$$\lim_{j\to\infty} \limsup_{i\to\infty} \limsup_{k\to\infty} \lambda_{i,j,k} \geq \lim_{j\to\infty} \limsup_{i\to\infty} \limsup_{k\to\infty} (\pi + 2/j + 2/k) = \pi, \tag{4.35}$$

□

Example 4.1.7. *Let (M^2, g_0) be the standard flat torus with S as in Example 4.1.6. Let $W_j = T_{1/j}(S)$ with respect to the flat norm. Let g_j be the flat metric on $M^2 \setminus W_j$. There exists smooth metrics g_j on M^2 which agree with g_0 on $M^2 \setminus W_j$ such that the Gromov-Hausdorff and Intrinsic Flat limits are the metric space created by taking the flat torus and identifying all points in S with each other. Then, g_j converges smoothly away from S to $g_\infty = g_0$. The metric completion of $(M \setminus S, g_\infty)$ is the slit torus as described in example 4.1.6. These metrics demonstrate that the diameter condition may not be replaced by an extrinsic diameter condition in Theorem 1.1.4 and in Theorem 4.3.5 but not the Ricci theorem since they have negative curvature.*

Proof. Let $g_j = dt^2 + f_j(s,t)^2 ds^2$ where $f_j(s,t) = 1$ on W_j and $f_j(s,t) = 1/j$ on S, and smooth with values in $[1/j, 1]$ everywhere. Let $\sim$ be defined as follows:

$$x \sim y \;\; \textbf{iff} \;\; x, y \in S \tag{4.36}$$

To estimate the GH and SWIF distance between (M^2, g_j) and $(\frac{M^2}{\sim}, d_0)$ we use the Theorem 3.1.1. First we need to find an estimate on the distortion λ_j , which is defined by

$$\lambda_j = \sup_{x,y\in W_j} \left| d_{M_j}(x,y) - d_{\frac{M}{\sim}}(x,y) \right| . \tag{4.37}$$

Now let $P : M^2 \to \frac{M^2}{\sim}$ be the quotient map and Suppose $x_j, y_j \in \bar{W}_j$ achieve the maximum in the definition of λ_j . Since $\frac{M^2}{\sim}$ is flat outside $\frac{S}{\sim}$, any shortest path, $\bar{C}_{x_j,y_j}$, joining x_j, y_j has to be a straight line. As a result, $P^{-1}(\bar{C}_{x_j,y_j})$ is either the straight line , C_{x_j,y_j} , in M^2 joining x_j, y_j or the same straight line union the singular set S. And since the metric in (M^2, g_j) is smaller than the flat metric on outside W_j and coincide with the flat metric in W_j, we get

$$\begin{aligned} \lambda_j &\leq L(\bar{C}_{x_j,y_j}) - L(C_{x_j,y_j}) && (4.38)\\ &\leq \operatorname{diam}_{(M^2,g_j)}(M^2 \setminus W_j) + \operatorname{diam}_{\left(\frac{M^2}{\sim},d\right)}\left(\frac{M^2 \setminus W_j}{\sim}\right). && (4.39) \end{aligned}$$

Any two points in $M^2 \setminus W_j$ can be joined by a few horizontal segments, whose lengths add up to at most $2/j$ and a segment in S with length less than π/j and vertical segments, whose lengths add up to $2/j$ therefore,

$$\operatorname{diam}_{(M^2,g_j)}(M^2 \setminus W_j) \leq \frac{\pi + 4}{j}, \tag{4.40}$$

and projecting these segments by P we get

$$\operatorname{diam}_{\left(\frac{M^2}{\sim},d\right)}\left(\frac{M^2 \setminus W_j}{\sim}\right) \leq 4/j, \tag{4.41}$$

hence,

$$\lambda_j \leq \frac{\pi + 8}{j} \to 0 \text{ as } j \to \infty. \tag{4.42}$$

Now letting $\epsilon = 0$ in Theorem 3.1.1, we have a = 0 and

$$\bar{h}_j = h_j = \sqrt{\lambda_j \left(\max\left\{ \mathrm{diam}(W_j), \mathrm{diam}\left(\frac{W_j}{\sim}\right) \right\} + \lambda_j/4 \right)} \to 0 \ \text{ as } \ j \to \infty. \tag{4.43}$$

So we conclude that

$$\begin{aligned} d_{GH}\left((M^2, g_j), \left(\frac{M^2}{\sim}, d \right) \right) \leq a \ &+ \ 2\bar{h} && (4.44) \\ &+ \ \max\left\{ d_H^{M^2}(W_j, M^2), d_H^{\frac{M^2}{\sim}}\left(\frac{W_j}{\sim}, \frac{M^2}{\sim} \right) \right\} \\ & \ \to 0, \end{aligned}$$

as $j \to \infty$ and also, it is easy to see that

$$\begin{aligned} & d_{\mathcal{F}}\left((M^2, g_j), \left(\frac{M^2}{\sim}, d \right) \right) && (4.45) \\ \leq \ & \left(\bar{h} + a\right) \left(\mathrm{Vol}_2(W_j) + \mathrm{Vol}_2\left(\frac{W_j}{\sim}\right) + \mathrm{Vol}_1(\partial W_j) + \mathrm{Vol}_1\left(\frac{\partial W_j}{\sim}\right) \right) \\ & + \mathrm{Vol}_2(M^2 \setminus W_j) + \mathrm{Vol}_2\left(\frac{M^2}{\sim} \setminus \frac{W_j}{\sim} \right) \to 0 \ \text{ as } \ j \to \infty. && (4.46) \end{aligned}$$

As we observed,

$$\lim_{j \to \infty} \mathrm{diam}_{(M^2, g_j)}(\partial W_j) \leq \lim_{j \to \infty} \frac{\pi + 4}{j} = 0, \tag{4.47}$$

but,

$$\lim_{j \to \infty} \mathrm{diam}_{(W_j, g_j)}(\partial W_j) \geq \pi. \tag{4.48}$$

□

4.1.5 Splines with Positive Scalar Curvature.

In this section, we will present two examples that demonstrate that in our Theorems, the uniform lower Ricci curvature bound condition can not be replaced by a uniform scalar curvature bound. In the first example we construct a sequence of metrics on the 3 - sphere which converge to the canonical sphere away from a singular point and also in the intrinsic flat sense but converges to a sphere with an interval attached to it in the Gromov-hausdorff sense. In the second example of this section, we will construct a sequence of metrics with positive scalar curvature which converge to the 3 - sphere away from a singular point and also in the intrinsic flat sense while having no Gromov-hausdorff limit. The second example was in fact presented by Tom Ilmannen in a talk in 2004 at Columbia without details. Both examples play an important role in [45] however the fact that they have positive scalar curvature was never presented in detail in that paper.

Lemma 4.1.8. *For any* $L > 0$ *and* $0 < \delta < 1$*, there exists a smooth Riemannian metric on the* 3*-sphere with positive scalar curvature which is obtained by properly gluing a spline of length* $L + O(\delta^{\frac{1}{2}})$ *and width* $\leq \delta$ *to the unit* 3 *- sphere.*

Example 4.1.9. *There are metrics* g_j *on the sphere* M^3 *with positive scalar curvature such that* $M_j = (S^3, g_j)$ *converge smoothly away from a point singularity* $S = \{p_0\}$ *to the sphere,* S^3*, with* $diam(M_j) \leq \pi + L + 2$ *and such that*

$$d_{\mathcal{F}}\left(M_j, S^3\right)\ ,\ d_{s\mathcal{F}}\left(M_j, S^3\right) \to 0, \tag{4.49}$$

and,

$$d_{GH}\left(M_j, M_0\right) \to 0, \tag{4.50}$$

where $M_0 = S^3 \sqcup [0, L]$ *(the round sphere with an interval of length* L *attached to it).*

Remark 4.1.10. *Example 4.1.9 demonstrates that the uniform lower Ricci curvature bound condition in Theorem 1.1.3 and 1.1.5 can not be replaced by a uniform lower bound on the scalar curvature.*

Example 4.1.11. *There are metrics g_j on the sphere M^3 with positive scalar curvature such that $M_j = (S^3, g_j)$ converge smoothly away from a point singularity $S = \{p_0\}$ to the sphere, S^3, with diam$(M_j) \leq \pi + L + 2$ and such that*

$$d_{\mathcal{F}}\left(M_j, S^3\right)\ ,\ d_{s\mathcal{F}}\left(M_j, S^3\right) \to 0, \tag{4.51}$$

and there is no Gromov-Hausdorff limit.

Proof. of Lemma 4.1.8. The goal here is to attach a spline of finite length and arbitrary small width to a sphere whith positive scalar curvature. For this, we need to employ some ideas related to the Mass of rotationally symmetric manifolds. (c.f. [36]). The construction goes as follows; we first find an admissible Hawking mass function (c.f. [36]) which will provide us with a three manifold embedded in $\mathbb{E}^4$ which is a hemisphere to which spline of finite length and small width is attached; we then, attach a hemisphere along its boundary.

Let $\delta < 1$ (this later will become the width of the spline). and let $r_{min} = 0$. Now we take an admissible Hawking mass function, $m_H(r)$ (which has to be smooth and increasing) that satisfies (ϵ to be determined later)

$$m_H(r) = r(1-\epsilon^2)/2 \text{ for } r \in [0, \delta^3], \tag{4.52}$$

and,

$$m_H(r) = r^3/2 \text{ for } r \in [\delta, 1]. \tag{4.53}$$

As in [36], define the function $z(r)$ via

$$z(\bar{r}) = \int_{r_{min}}^{\bar{r}} \sqrt{\frac{2m_H(r)}{r - 2m_H(r)}}\, \mathrm{d}r. \tag{4.54}$$

Note that z depend on δ.

$z(r)$ is unique up to a constant and gives our desired three manifold as a graph over $\mathbb{E}^3$. By our choice of $m_H(r)$ we get,

$$z'(r) = \sqrt{\frac{1-\epsilon^2}{\epsilon^2}} \text{ for } r \in [0, \delta^3], \tag{4.55}$$

and,

$$z'(r) = \sqrt{\frac{r^2}{1-r^2}} \text{ for } r \in [\delta, 1], \tag{4.56}$$

and, since

$$\frac{\delta^3}{2}(1-\epsilon^2) \le m_H(r) \le \delta^3/2 \text{ for } r \in [\delta^3, \delta], \tag{4.57}$$

one obtains

$$\sqrt{\frac{\delta^3(1-\epsilon^2)}{r - \delta^3(1-\epsilon^2)}} \le z'(r) \le \sqrt{\frac{\delta^3}{r-\delta^3}} \text{ for } r \in [\delta^3, \delta]. \tag{4.58}$$

Now, choose the ϵ that solves

$$\delta^3 \sqrt{\frac{1-\epsilon^2}{\epsilon^2}} = L, \tag{4.59}$$

For some fixed L. From (4.55), we have,

$$\bar{L}(\delta) = z(\delta) - z(\delta^3) \le \int_{\delta}^{\delta^3} \sqrt{\frac{\delta^3}{r-\delta^3}} = 2\delta^{3/2}\left(\delta - \delta^3\right)^{1/2} < 2, \tag{4.60}$$

which goes to 0 as δ goes to 0.

We also get

$$z(\delta^3) - z(0) = \delta^3 \sqrt{\frac{1-\epsilon^2}{\epsilon^2}} = L. \tag{4.61}$$

The metric in terms of the distance from the pole, can be written as

$$\bar{g}_\delta = ds^2 + f^2(s)g_{S^2} = (1 + [z'(r)]^2)dr^2 + r^2 g_{S^2}. \tag{4.62}$$

In the virtue of the Theorem 5.4 in [36], we know that when $r \in [\delta, 1]$, we are on a unit sphere, and since

$$\lim_{r\to 1^-} z'(r) = \infty, \tag{4.63}$$

we have

$$\lim_{r\to 1^-} f'(s) = \lim_{r\to 1^-} r'(s) = \lim_{r\to 1^-} \frac{1}{\sqrt{1 + [z'(r)]^2}} = 0. \tag{4.64}$$

Therefore, the boundary $r = 1$ is in fact a great 2-sphere along which we can smoothly attach a 3 - hemisphere. as follows

So far we have got the metric

$$\bar{g}_\delta = (1 + [z'(r)]^2)dr^2 + r^2 g_{S^2} \text{ for } r \in [0, 1]. \tag{4.65}$$

Letting $r = \sin(\rho)$, one sees that

$$\bar{g}_\delta = (1 + [z'_\delta(\sin(\rho))]^2)\cos^2(\rho)d\rho^2 + \sin^2(\rho)g_{S^2} \text{ for } \rho \in [0, \pi/2]. \tag{4.66}$$

Therefore, on the sphere we define g_δ to be

$$(1 + [z'_\delta(\sin(\rho))]^2)\cos^2(\rho)d\rho^2 + \sin^2(\rho)g_{S^2} \text{ for } \rho \in [0, \pi/2], \tag{4.67}$$

and,

$$d\rho^2 + \sin^2(\rho)g_{S^2} \text{ for } \rho \in [\pi/2, \pi], \tag{4.68}$$

which has positive scalar curvature when $\rho \leq \pi/2$ because it is isometric to $\bar{g}_\delta$ and has positive scalar curvature when $\rho \geq \pi/2$ because it is isometric to a round hemisphere. g_δ is smooth at $\rho = \pi/2$ because by 4.56 near $\rho = \pi/2$,

$$z'_\delta(\sin(\rho)) = \sqrt{\frac{\sin^2(\rho)}{1 - \sin^2(\rho)}} = \tan(\rho). \tag{4.69}$$

So,

$$\begin{aligned}(1+[z_\delta'(\sin(\rho))]^2)\cos^2(\rho)d\rho^2+\sin^2(\rho)g_{S^2} &= (1+\tan^2(\rho))\cos^2(\rho)+\sin^2(\rho)g_{S^2}\\ &= d\rho^2+\sin^2(\rho)g_{S^2}. \qquad (4.70)\end{aligned}$$

The key idea is that, using this method, one can attach symmetric spline of length $L+\bar{L}(\delta)$ and arbitrary small width $\delta<1$ to a sphere while keeping the scalar curvature positive and $\mathrm{diam}(M_j)\leq \pi+L+2$. And the metric found can actually be written as a warped metric.

□

Proof. of Example 4.1.9. Now let $\delta_j\to 0$ and take the sequence $M_j=(S^3,g_{\delta_j})$, where g_{δ_j} is given by the above construction for δ_j. we are going to prove that M_j converges to M_0 in Gromov-Hausdorff sense where M_0 is the unit three sphere to which an interval of length L is attached; and M_j converges to S^3 in intrinsic flat sense.

First notice that M_j contains a subdomain U_j which is isometric to $U_j'=S^3\setminus B_p(\arcsin(\delta_j))$ also letting $V_j=M_j\setminus U_j$ and $V_j'=S^3\setminus U_j'$ one observes that since

$$\begin{aligned}\mathrm{Vol}(V_j) &\leq \int_0^\delta (4\pi r^2)(1+[z'(r)]^2)^{1/2}\,\mathrm{d}r\\ &\leq \int_0^\delta (4\pi r^2)(1+|z'(r)|)\mathrm{d}r \qquad (4.71)\\ &\leq (4\pi\delta^2)\left(\delta+L+\bar{L}(\delta)\right),\end{aligned}$$

one gets $\mathrm{Vol}(V_j)\to 0$ as $\delta_j\to 0$. Also it is obvious that $\mathrm{Vol}(V_j')\to 0$ as $\delta_j\to 0$.

Now, to be able to use Theorem 3.1.1, we need an estimate on

$$\lambda_j=\sup_{x,y\in U_j}\left|d_{M_j}(x,y)-d_{S^3}(x,y)\right|. \qquad (4.72)$$

Let $x, y \in U_j$ and let γ and $c_{x,y}$ be the minimizing geodesic connecting x and y in (M_j, g_j) and S^3 (resp.). If γ lies completely in U_j, then, so does $c_{x,y}$ and $\gamma = c_{x,y}$ hence, $d_{M_j}(x, y) = d_{S^3}(x, y)$. If $\gamma \not\subset U_j$, therefore $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ where $x \in \gamma_1$, $y \in \gamma_3$ and $\gamma_1, \gamma_3 \subset U_j$ and $\gamma_2 \subset V_j$. We are in either of the following cases

Case I: $c_{x,y} \subset U_j$

Obviously $L(\gamma_2) \leq 2\pi\delta_j$, also we have

$$|d_{S^3}(x, p) - L(\gamma_1)| \leq \arcsin \delta_j, \tag{4.73}$$

and

$$|d_{S^3}(y, p) - L(\gamma_3)| \leq \arcsin \delta_j. \tag{4.74}$$

Since $\delta_j \to 0$, for j large enough,

$$L(\gamma) \approx d_{S^3}(x, p) + d_{S^3}(y, p) > L(c_{x,y}), \tag{4.75}$$

which is a contradiction.

Case II: $c_{x,y} \not\subset U_j$.

Let $c_{x,y} = c_1 + c_2 + c_3$ where $x \in c_1$, $y \in c_3$ and $c_1, c_3 \subset U_j$ and $c_2 \subset V_j'$, Then, $L(c_2) \leq 2 \arcsin(\delta_j)$ and also

$$|L(\gamma_i) - L(c_i)| \leq 2 \arcsin(\delta_j). \tag{4.76}$$

Therefore,

$$\left| d_{M_j}(x, y) - d_{S^3}(x, y) \right| = |L(\gamma) - L(c_{x,y})| \leq 4 \arcsin(\delta_j) + 2\pi\delta_j. \tag{4.77}$$

This argument shows that $\lambda_j \to 0$ as $j \to \infty$. Since the intrinsic diameter $D_{U_j} \leq \pi$ (both in M_j and S^3), h_j in Theorem 3.1.1 goes to 0 as $j \to \infty$.

Letting $\epsilon = 0$ in Theorem 3.1.1 we get $a = 0$ and $\bar{h}_j = h_j$ therefore,

$$d_{\mathcal{F}}(M_j, S^3) \leq \bar{h_j}\left(2\,\mathrm{Vol}(U_j) + 2\,\mathrm{Vol}(\partial U_j)\right) + \mathrm{Vol}(V_j) + \mathrm{Vol}(V_j'). \tag{4.78}$$

which gives

$$d_{\mathcal{F}}(M_j, S^3) \to 0 \text{ as } j \to \infty. \tag{4.79}$$

To prove that M_j converges to M_0 in Gromov-hausdorff sense, we will estimate $d_{GH}(M_j, M_0)$ using the fact that

$$d_{GH}(M_j, M_0) = \frac{1}{2}\inf_{\mathfrak{R}}\left(\operatorname{dis}\mathfrak{R}\right), \tag{4.80}$$

where, the infimum is over all correspondences $\mathfrak{R}$ between M_j and M_0 and $\operatorname{dis}\mathfrak{R}$ is the distortion of $\mathfrak{R}$ given by

$$\operatorname{dis}\mathfrak{R} = \sup\left\{\left|d_{M_j}(x,x')\right| - \left|d_{M_0}(y,y')\right| : (x,y),(x',y') \in \mathfrak{R}\right\}. \tag{4.81}$$

For details see [10, p. 257].

We need to find correspondences $\mathfrak{R}_j$ between M_j and M_0 such that $\operatorname{dis}\mathfrak{R}_j \to 0$ as $j \to \infty$. Consider $W_j \subset \mathbb{E}^4$ given by

$$W_j = M_j \cup M_0 = M_0 \cup V_j = M_j \cup \left(M_0 \setminus U_j\right). \tag{4.82}$$

In fact, we can picture W_j as the union of sphere, an interval of length L and a spline of length $L + \bar{L}(\delta_j)$ around the spline. Then we have, $M_j \subset W_j$ and $M_0 \subset W_j$ define the following surjective maps $f : W_j \to M_j$ and $g : W_j \to M_0$

$$f|_{M_j} = \mathrm{id}. \tag{4.83}$$

$$f(w) = \left(r\left(z(w)\right), 0, 0, z(y)\right) \in V_j \text{ for } w \in \left(M_0 \setminus U_j\right). \tag{4.84}$$

Note that when $w \in (M_0 \setminus U_j)$, $f(w)$ is the point in $V_j \cap xz-$plane closest to w .

Similarly let

$$g|_{M_0} = \mathrm{id}\,. \tag{4.85}$$

$$g(w) = \text{the point in } (M_0 \setminus U_j) \text{ closest to } w \text{ for } w \in V_j. \tag{4.86}$$

Let $\mathfrak{R}_j$ be the following correspondence between M_j and M_0,

$$\mathfrak{R}_j = \{(f(w), g(w)) : w \in W_j\}. \tag{4.87}$$

Claim: $\operatorname{dis}\mathfrak{R}_j \to 0$ as $j \to \infty$.

Pick $w_1, w_2 \in W_j$, and suppose $\gamma + \lambda$ is the minimal geodesic in M_j connecting $f(w_1)$ and $f(w_2)$ where $\gamma \subset U_j$ and $\lambda \subset V_j$. and let $\gamma' + \lambda'$ be the (possibly) broken minimal geodesic connecting $g(w_1)$ and $g(w_2)$ where, $\gamma' \subset U_j$ and $\lambda' \subset (M_0 \setminus U_j)$. Without loss of generality we assume that $z(w_1) \le z(w_2)$. Next we need estimates on the lengths of $\gamma, \lambda, \gamma', \lambda'$. Let q and q' be starting points on λ and λ' respectively, then

$$\int_q^{f(w_2)} \mathrm{d}z \le \int_q^{f(w_2)} s'(z)\,\mathrm{d}z \le L(\lambda) \le \int_q^{f(w_2)} s'(z)\,\mathrm{d}z + 2\pi\delta_j, \tag{4.88}$$

where, the term $2\pi\delta_j$ is the maximum perimeter of the well and note that any two point on the we can be joined by a radial geodesic followed by a curve of length less than $2\pi\delta_j$.

Since $ds^2 = (1 + [r'(z)]^2)\, dz^2$ we get

$$\begin{aligned} \int_q^{f(w_2)} s'(z)\,\mathrm{d}z \;&\le\; \int_q^{f(w_2)} \mathrm{d}z + \int_q^{f(w_2)} |r'(z)|\,\mathrm{d}z \qquad (4.89) \\ &\le\; \int_q^{f(w_2)} \mathrm{d}z + \delta_j \left(L + \bar{L}(\delta_j)\right). \end{aligned}$$

We also have

$$\int_{q'}^{g(w_2)} \mathrm{d}z \leq L(\lambda') \leq \int_{q'}^{g(w_2)} \mathrm{d}z + 2\arcsin(\delta_j). \tag{4.90}$$

The last inequality comes from the fact that any two points in $M_0 \setminus U_j$ can be joined by a (broken) geodesic which is a straight line followed by a curve of length at most $\operatorname{diam}(V_j') = 2\arcsin(\delta_j)$.

On the other hand by our construction

$$\left| \int_{q}^{q'} \mathrm{d}z \right| \leq \bar{L}(\delta_j), \tag{4.91}$$

and,

$$\left| \int_{f(w_2)}^{g(w_2)} \mathrm{d}z \right| \leq \bar{L}(\delta_j). \tag{4.92}$$

Therefore,

$$\left| \int_{q}^{f(w_2)} \mathrm{d}z - \int_{q'}^{g(w_2)} \mathrm{d}z \right| \leq 2\bar{L}(\delta_j). \tag{4.93}$$

From 4.88 - 4.93, we get

$$|L(\lambda) - L(\lambda')| \leq \delta_j \left(L + \bar{L}(\delta_j) + 2\pi \right) + 2\bar{L}(\delta_j) + 2\arcsin(\delta_j). \tag{4.94}$$

Also one observes that when $w_1 \in U_j$, then γ and γ' are geodesics on the sphere starting from the same point and ending up in V_j' which means that

$$|L(\gamma) - L(\gamma')| \leq \operatorname{diam}(V_j') = 2\arcsin(\delta_j). \tag{4.95}$$

From (4.94) and (4.95),

$$\begin{aligned} \left| d_{M_j}\left(f(w_1), f(w_2)\right) - d_{M_0}\left(g(w_1), g(w_2)\right) \right| &\leq |L(\gamma) - L(\gamma')| + |L(\lambda) - L(\lambda')| \\ &\leq \delta_j \left(L + \bar{L}(\delta_j) + 2\pi \right) \qquad (4.96) \\ &\quad +2\bar{L}(\delta_j) + 4\arcsin(\delta_j). \end{aligned}$$

Therefore,

$$\operatorname{dis}\mathfrak{R}_j \leq \delta_j \left(L + \bar{L}(\delta_j) + 2\pi\right) + 2\bar{L}(\delta_j) + 4\arcsin(\delta_j), \tag{4.97}$$

which shows that $\operatorname{dis}\mathfrak{R}_j \to 0$ as $j \to \infty$. This completes the proof of the claim.

To prove that the convergence off the singular set $S = \{p_0\}$, which is the bottom of the well, let $\rho_0 > 0$, then

$$g_{\delta_j} \to g_{S^3} \text{ on } \rho^{-1}\left([\rho_0, \pi]\right), \tag{4.98}$$

because for j sufficiently large, $\delta_j < \rho_0$, which by our construction means that $g_{\delta_j} = g_{S^3}$ on $\rho^{-1}([\rho_0, \pi])$. □

Proof. of Example 4.1.11. Let $g(p_0, s)$ denote a symmetric spline of length L centered at the point p_0 with width s (as constructed in the previous examples). Also fix a great circle and a point 0 in S^2, so now, when we say a point given by the angle θ, it means a point on this great circle given by the angle θ. On the round sphere, $r = \pi - \frac{1}{2^j}$ is a 2-sphere with radius $\sin(\pi - \frac{1}{2^j})$, Therefore, balls with radius $s_j = \frac{1}{j}\sin(\pi - \frac{1}{2^j})\sqrt{2 - 2\cos(\frac{2\pi}{2^j})}$ centered at points p_k given by the angle $\theta = \frac{2k\pi}{2^j}$ are disjoint. Now for each j, we can glue metrics $g(p_k, s_j)$ which agree with the metric on the spline given in Example 4.1.9. Outside of each $B_{p_k}(s_j)$, we set $g_j = g_0$. It is easy to see that by our construction, g_j agrees with the round metric for $r < \pi - \frac{1}{2^j} - s_j$ and has 2^j splines of length L and width s_j and also the volume of the non spherical part is going to 0 as $j \to \infty$. So by taking $U_j = r^{-1}([0, \pi - \frac{1}{2^j} - s_j])$, we see that again all conditions in Theorem 3.2.2 are satisfied therefore we have the flat convergence to the settled completion. □

4.2 Hausdorff Measure Estimates $\Longrightarrow$ Well-Embeddedness.

We now prove Theorems 1.1.2, 1.1.3 and its counterpart (with Ricci condition replaced by contractibility condition) stated in the introduction. First we must prove the following two lemmas:

Lemma 4.2.1. *Let M^n be compact Riemannian manifold, S a subset of M with $H^{n-1}(S) = 0$, and let $\gamma : [0, L] \to M$ be a shortest geodesic parametrized by arclength with endpoints $x, y \in M \setminus S$. Then, for any small enough $\epsilon > 0$, there exists a path γ_ϵ joining x, y such that $\gamma_\epsilon \cap S = \emptyset$ and*

$$L(\gamma_\epsilon) \leq L(\gamma) + \epsilon. \tag{4.99}$$

Proof. Let $\Gamma : [-\sigma, \sigma]^{n-1} \times [0, L] \subset \mathbb{R}^n \to M$ be the $(n-1)-$th variation of γ given by

$$\Gamma(t_1, t_2, ...t_{n-1}, s) \quad = \quad \exp_{\gamma(s)} \mathbf{F}(t_1, t_2, ...t_{n-1}, s) \tag{4.100}$$

where

$$\mathbf{F}(t_1, t_2, ...t_{n-1}, s) = \sin\left(\frac{\pi s}{L}\right)\Big(\sum_i t_i e_i(s)\Big), \tag{4.101}$$

in which $\{e_i(s)\}$ is a parallel orthonormal frame along γ and $e_0(s)$ is the unit tangent to γ.

For any $\bar{t} = (t_1, ..t_{n-1})$, the curve $\gamma_{\bar{t}}(s) := \Gamma(t_1, t_2, ...t_{n-1}, s)$ is a curve from x to y. If we choose σ sufficiently small then,

$$L(\gamma) \leq L(\gamma_{\bar{t}}) \leq L(\gamma) + \epsilon, \tag{4.102}$$

therefore, to prove the lemma, we need to find a $\bar{t}$ such that

$$\gamma_{\bar{t}} \cap S = \emptyset. \tag{4.103}$$

Claim: There exists some $\sigma > 0$ such that after restricting the domain of Γ accordingly, for any small $\delta > 0$, Γ is bi-Lipschitz on

$$\Lambda_\delta = [-\sigma, \sigma]^{n-1} \times [\delta, L - \delta]. \tag{4.104}$$

To see this, we need to compute the derivative of Γ. Let

$$x(u, s, t_1, \ldots, t_{n-1}) = \exp_{\gamma(s)} \Big(u\mathbf{F}(s, t_1, \ldots, t_{n-1}) \Big), \tag{4.105}$$

for fixed $s, t_1, \ldots, t_{n-1}$, as u ranges from 0 to 1, the curve $x(u, s, t_1, \ldots, t_{n-1})$ is a geodesic segment from $\gamma(s)$ to $\Gamma(s, t_1, \ldots, t_{n-1})$. As s varies, x is a variation through geodesics therefore,

$$D(\Gamma)\left(\frac{\partial}{\partial s}\right)(s, t_1, \ldots, t_{n-1}) = \mathbf{J}(1), \tag{4.106}$$

where, $\mathbf{J}$ is the Jacobi field along this geodesic segment, with the initial conditions $\mathbf{J}(0) = e_0(s)$, $\nabla_{\frac{\partial x}{\partial u}} \mathbf{J}(0) = \frac{\pi}{L} \cos\left(\frac{\pi s}{L}\right) \sum_i t_i e_i(s)$ since,

$$\begin{aligned} \nabla_{\frac{\partial x}{\partial u}} \mathbf{J}(0) &= \nabla_{\frac{\partial x}{\partial u}} \frac{\partial x}{\partial s}(0, s, t_1, \ldots, t_{n-1}) \\ &= \nabla_{\frac{\partial x}{\partial s}} \frac{\partial x}{\partial u}(0, s, t_1, \ldots, t_{n-1}) \\ &= \nabla_{\frac{\partial x}{\partial s}} \mathbf{F}(0, s, t_1, \ldots, t_{n-1}) \\ &= \frac{\pi}{L} \cos\left(\frac{\pi s}{L}\right) \sum_i t_i e_i(s). \end{aligned} \tag{4.107}$$

Also for any $1 \leq i \leq n-1$ we have:

$$D(\Gamma)\left(\frac{\partial}{\partial t_i}\right)(s, t_1, \ldots, t_{n-1}) = \left(\exp_{\gamma(s)}\right)_* \big|_{\mathbf{F}(s, t_1, \ldots, t_{n-1})} \sin\left(\frac{\pi s}{L}\right) e_i(s) \tag{4.108}$$

For $t_1 = \ldots, t_{n-1} = 0$, and $V = \alpha_0 \frac{\partial}{\partial s} + \sum_i \alpha_i \frac{\partial}{\partial t_i}$ with $\|V\| = 1$ we compute:

$$\begin{aligned} \|D(\Gamma)V\| &= \left\| D(\Gamma)\Big(\alpha_0 \frac{\partial}{\partial s} + \sum_i \alpha_i \frac{\partial}{\partial t_i}\Big) \right\| \\ &= \left\| \alpha_0 e_0(s) + \sin\Big(\frac{\pi s}{L}\Big) \sum_i \alpha_i e_i(s) \right\|, \end{aligned} \tag{4.109}$$

Therefore, for any $\delta > 0$, there exist $c(\delta) > 0$ such that on γ,

$$0 < c(\delta) \leq \|D(\Gamma)V\| \leq 1. \tag{4.110}$$

By continuity, for a small enough σ, we will have

$$0 < \frac{c(\delta)}{2} \leq \|D(\Gamma)V\| \leq 3/2, \tag{4.111}$$

on $[-\sigma, \sigma]^{n-1} \times [\delta, L - \delta]$.

Also by making σ smaller, we can assume that

$$\sigma < \frac{r_{focal}}{n}, \tag{4.112}$$

in which, r_{focal} is the focal radius of the geodesic γ. This guarantees that Γ is injective on $[-\sigma, \sigma]^{n-1} \times [\delta, L-\delta]$ which is compact, therefore Γ is a homeomorphism with the derivative bounded away from zero on its domain. By applying the inverse function theorem, we deduce that Γ is a diffeomorphism on $[-\sigma, \sigma]^{n-1} \times [\delta, L - \delta]$ onto its image. As a result, Γ is bi-Lipschitz on Λ_δ.

It is rather straightforward to see that for a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$, any subset $A \subset \mathbb{R}^n$ and $0 \leq s < \infty$, we have

$$H^s\left(f(A)\right) \leq [\operatorname{Lip} f]^s H^s(A) \tag{4.113}$$

(see [37, Theorem 3.1.2]). Therefore, bi-Lipschitz preimages of sets of 0 Hausdorff measure, have 0 Hausdorff measure. Since Γ is bi-Lipschitz on Λ_δ, we get:

$$H^{n-1}\left(\Gamma^{-1}(S)\cap\Lambda_\delta\right)=0. \tag{4.114}$$

Now we can compute:

$$\begin{aligned} H^{n-1}\left(\Gamma^{-1}(S)\right) &\leq H^{n-1}\left(\bigcup_i\left(\Gamma^{-1}(S)\cap\Lambda_{1/i}\right)\right) \\ &\quad +H^{n-1}\left(\Gamma^{-1}(S)\cap[-\sigma,\sigma]^{n-1}\times\{0\}\right) \\ &\quad +H^{n-1}\left(\Gamma^{-1}(S)\cap[-\sigma,\sigma]^{n-1}\times\{L\}\right) \\ &= 0. \end{aligned} \tag{4.115}$$

Since any orthogonal projection $\mathrm{Pr}:\mathbb{R}^n\to\mathbb{R}^k$ is distance dicreasing, we have $\mathrm{Lip}\,(\mathrm{Pr})\leq 1$. By (4.113) (see [37, Theorem 3.1.2]), for any $A\subset\mathbb{R}^n$ and any $0\leq s<\infty$, we get

$$H^s\left(\mathrm{Pr}(A)\right)\leq H^s(A) \tag{4.116}$$

Thus for any orthogonal projection Pr onto an $(n-1)-$dimensional face of $[-\sigma,\sigma]^{n-1}\times[0,L]$ we have

$$H^{n-1}\left(\mathrm{Pr}\left(\Gamma^{-1}(S)\right)\right)=0. \tag{4.117}$$

Let Pr_1 and Pr_2 be the projections onto the faces $[-\sigma,\sigma]^{n-1}\times\{0\}$ and $[-\sigma,\sigma]^{n-1}\times\{L\}$ respectively. Setting $E_1=\mathrm{Pr}_1\left(\Gamma^{-1}(S)\right)$ and $E_2=\mathrm{Pr}_2\left(\Gamma^{-1}(S)\right)$, then,

$$H^{n-1}(E_i)=0\ \ for\ \ i=1,2 \tag{4.118}$$

and so,

$$H^{n-1}([-\sigma,\sigma]^{n-1}\setminus E_i)=(2\sigma)^{n-1}\ \ for\ \ i=1,2 \tag{4.119}$$

Any countable union of null sets is a null set (see [21, p. 26]) hence, $E_1 \cup E_2$ is a null set. This means that

$$H^{n-1}\left([-\sigma,\sigma]^{n-1} \setminus E_1 \cup [-\sigma,\sigma]^{n-1} \setminus E_2\right) = (2\sigma)^{n-1}. \tag{4.120}$$

Let $\bar{t} = (t_1, t_2, ..., t_{n-1}) \in (([-\sigma,\sigma]^{n-1} \setminus E_0) \cap ([-\sigma,\sigma]^{n-1} \setminus E_L)$, then the path

$$\gamma_{\bar{t}}(s) = \Gamma(t_1, t_2, t_3, ..., s), \tag{4.121}$$

is a path joining x, y and $\gamma_{\bar{t}} \cap S = \emptyset$ which also satisfies

$$|L(\gamma_\epsilon) - L(\gamma)| \leq \epsilon. \tag{4.122}$$

□

Lemma 4.2.2. *Let M^n be a compact Riemannian manifold, S a set with $H^{n-1}(S) = 0$ and $diam_{g_\infty}(M \setminus S) < \infty$ then, any connected precompact exhaustion, W_j, of $M^n \setminus S$ is uniformly well embedded.*

Proof. Suppose not.

Let $x_{i,j,k}, y_{i,j,k} \subset \bar{W}_j$ achieve to supremum in the definition of $\lambda_{i,j}$.

Since $\bar{W}_j$ is compact, a subsequence as $k \to \infty$ converges to $x_{i,j}, y_{i,j} \subset \bar{W}_j$. Let $\gamma_{i,j}$ be a minimizing geodesic between these points in M with respect to g_i. Since S is a set of codimension strictly larger than than 1, by applying Lemma 4.2.1, we can find a curve $C_{i,j} : [0,1] \to M \setminus S$ between these points such that

$$L_{g_i}(C_{i,j}) \leq d_{M,g_i}(x_{i,j}, y_{i,j}) + \lambda_{i,j}/5, \tag{4.123}$$

Let k be chosen from the subsequence sufficiently large that

$$\begin{aligned} C_{i,j}([0,1]) &\subset W_k, \\ d_{(\bar{W}_j,g_i)}(x_{i,j,k}, x_{i,j}) &< \lambda_{i,j}/10, \\ d_{(\bar{W}_j,g_i)}(y_{i,j,k}, y_{i,j}) &< \lambda_{i,j}/10, \end{aligned} \tag{4.124}$$

Thus

$$\begin{aligned} d_{(\bar{W}_k,g_i)}(x_{i,j,k}, y_{i,j,k}) &\leq d_{(\bar{W}_k,g_i)}(x_{i,j,k}, x_{i,j}) + d_{(\bar{W}_k,g_i)}(x_{i,j}, y_{i,j}) + d_{(\bar{W}_k,g_i)}(y_{i,j}, y_{i,j,k}) \\ &\leq d_{(\bar{W}_j,g_i)}(x_{i,j,k}, x_{i,j}) + L(C_{i,j}) + d_{(\bar{W}_j,g_i)}(y_{i,j}, y_{i,j,k}) \\ &\leq \lambda_{i,j}/10 + d_{M,g_i}(x_{i,j}, y_{i,j}) + \lambda_{i,j}/5 + \lambda_{i,j}/10 \qquad (4.125) \\ &\leq 2\lambda_{i,j}/5 + d_{M,g_i}(x_{i,j}, x_{i,j,k}) + d_{M,g_i}(x_{i,j,k}, y_{i,j,k}) + d_{M,g_i}(y_{i,j,k}, y_{i,j}) \\ &\leq 2\lambda_{i,j}/5 + d_{W_j,g_i}(x_{i,j}, x_{i,j,k}) + d_{M,g_i}(x_{i,j,k}, y_{i,j,k}) + d_{W_j,g_i}(y_{i,j,k}, y_{i,j}) \\ &\leq 3\lambda_{i,j}/5 + d_{M,g_i}(x_{i,j,k}, y_{i,j,k}) \\ &\leq 3\lambda_{i,j}/5 + d_{W_k,g_i}(x_{i,j,k}, y_{i,j,k}) - \lambda_{i,j,k}, \end{aligned}$$

by the choice of $x_{i,j,k}$ and $y_{i,j,k}$. This is a contradiction.

Next we must show

$$\limsup_{j\to\infty} \limsup_{k\to\infty} \limsup_{i\to\infty} \lambda_{i,j,k} \leq \lambda_0. \tag{4.126}$$

Observe that

$$\lambda_{i,j,k} \leq \bar{\lambda}_{i,j,k} = \mathrm{diam}_{(W_k,g_i)}(W_j). \tag{4.127}$$

Since $g_i \to g_\infty$ on W_k we know

$$\limsup_{i\to\infty} \lambda_{i,j,k} \leq \mathrm{diam}_{(W_k,g_\infty)}(W_j). \tag{4.128}$$

Claim:

$$\limsup_{k\to\infty} \mathrm{diam}_{(W_k,g_\infty)}(W_j) \leq \mathrm{diam}_{(M\setminus S,g_\infty)}(W_j). \tag{4.129}$$

Suppose not; then, there exists $s > 0$ and a subsequence $k \to \infty$ such that

$$\limsup_{k\to\infty} \operatorname{diam}_{(W_k, g_\infty)}(W_j) = L > \operatorname{diam}_{(M\setminus S, g_\infty)}(W_j) + 5\delta. \tag{4.130}$$

Pick $x_k, y_k \in W_j$ so that

$$\operatorname{diam}_{(W_k, g_\infty)}(W_j) \leq d_{(W_k, g_\infty)}(x_k, y_k) + \delta. \tag{4.131}$$

$\bar{W}_j$ is compact therefore, after passing to a subsequence,

$$x_k \to x \in \bar{W}_j \tag{4.132}$$

$$y_k \to y \in \bar{W}_j. \tag{4.133}$$

Therefore, there exists a curve $c : [0,1] \to M \setminus S$ such that,

$$\begin{aligned} L_{g_\infty}(c) &< d_{(M\setminus S, g_\infty)}(x, y) + \delta \\ &< \operatorname{diam}_{(M\setminus S, g_\infty)}(\bar{W}_j) + \delta \\ &\leq \operatorname{diam}_{(M\setminus S, g_\infty)}(W_j) + \delta \\ &< L - 4\delta. \end{aligned} \tag{4.134}$$

For k sufficiently large, we have

$$c([0,1]) \subset W_k, \tag{4.135}$$

so

$$\begin{aligned} L - 4\delta > L_{g_\infty}(c) &> d_{(W_k, g_\infty)}(x, y) \\ &> d_{(W_k, g_\infty)}(x_k, y_k) - 2\delta \\ &\geq \operatorname{diam}_{(W_k, g_\infty)}(W_j) - 3\delta. \end{aligned} \tag{4.136}$$

Taking the limit as $k \to \infty$, we get:

$$L - 4\delta \geq L. \tag{4.137}$$

which is a contradiction hence, the claim is proved and we have

$$\limsup_{k\to\infty} \limsup_{i\to\infty} \lambda_{i,j,k} \leq \operatorname{diam}_{(M\setminus S, g_\infty)}(W_j), \tag{4.138}$$

and so

$$\limsup_{j\to\infty} \limsup_{k\to\infty} \limsup_{i\to\infty} \lambda_{i,j,k} \leq \operatorname{diam}_{g_\infty}(M \setminus S). \tag{4.139}$$

□

Proof of Theorem 1.1.2:

Proof. The lemmas 4.2.1 and 4.2.2 prove the well-embeddedness and then applying the Theorem 3.2.2 completes the proof of Theorem 1.1.2. □

Remark 4.2.3. *Example 4.1.5 demonstrates that the connectivity of the exhaustion in Theorems 1.1.2 hence, in Theorems 1.1.3 and 4.2.6 is a necessary condition. The excess volume bound in (1.23) is shown to be necessary in [35, Example 3.7]. All these examples satisfy the uniform embeddedness hypothesis of Theorem 3.2.2 and demonstrate the necessity of these conditions in that theorem as well. By Lemma 3.3.2, the diameter hypothesis is not necessary when the Ricci curvature is nonnegative although the volume condition is still necessary as seen in [35, Example 3.8]. Otherwise we see this is a necessary condition in Example 4.1.3. We were unable to find an example proving the necessity of the uniform bound on the boundary volumes, (1.22), and suggest this as an open question in [35, Remark 3.15]. The Hausdorff* *measure condition $H^{n-1}(S) = 0$ of Theorem 1.1.2 and the uniform embeddedness hypothesis of Theorem 3.2.2 are seen to be necessary for their respective theorems in 4.1.6.*

Proof of Theorem 1.1.3:

Proof. The assumption $H^{n-1}(S) = 0$ along with the hypotheses (3.19), (1.22) and (1.23), allows us to apply Theorem 1.1.2. Therefore, (M_i, g_i) has an intrinsic flat limit and this limit coincides with the settled completion of $(M \setminus S, g_\infty)$. Now by proposition 3.4.1, the Gromov-Hausdorff and Intrinsic Flat limits agree. □

Remark 4.2.4. *Example 4.1.3 proves the necessity of the the condition (3.19) in Theorem 1.1.3.*

Remark 4.2.5. *From [12], Ricci bounded below and $d_{GH}(M_i, M) \to 0$ imply $\mathrm{Vol}(M_i) \to \mathrm{Vol}(M)$ (conjectured by Anderson-Cheeger) and M_i are homeomorphic to M for i sufficiently large (later proved diffeomorphic in [40]). This means that (1.23) in Theorem 1.1.3 is a necessary condition.*

Theorem 4.2.6. *Let $M_i = (M, g_i)$ be a sequence of oriented compact Riemannian manifolds with a uniform linear contractibility function, ρ, which converges smoothly away from a closed singular subset, S, with $H^{n-1}(S) = 0$. If there is a connected precompact exhaustion of $M \setminus S$ as in (1.8) satisfying the volume conditions*

$$\mathrm{Vol}_{g_i}(\partial W_j) \le A_0 \tag{4.140}$$

and

$$\mathrm{Vol}_{g_i}(M \setminus W_j) \le V_j \text{ where } \lim_{j\to\infty} V_j = 0, \tag{4.141}$$

then

$$\lim_{j\to\infty} d_{GH}(M_j, N) = 0, \tag{4.142}$$

where N is the settled and metric completion of $(M \setminus S, g_\infty)$.

Proof. By the proof of Theorem 3.5.1, we see that

$$\mathrm{diam}(M_i) \le D_0, \tag{4.143}$$

This along with $H^{n-1}(S) = 0$, (4.140) and (4.141), allows us to apply the Lemma 4.2.2 to get the well-embeddedness of the exhaustion $\{W_j\}$. Then, we can fully apply Theorem 3.5.1 and that finishes the proof. □

4.3 Diameter Controls $\Longrightarrow$ Well-embeddedness.

In this section we prove Theorems 1.1.4, 1.1.5 and its counterpart (with Ricci condition replaced with contractibility condition). In the theorems of this section, there is no co-dimension condition on the singular set S. We first need to prove the following lemma:

Lemma 4.3.1. *Suppose W_j is a connected precompact exhaustion of $M \setminus S$ with boundaries ∂W_j such that any connected component of $M \setminus W_j$ has a connected boundary. If the intrinsic diameters satisfy*

$$diam_{(W_j, g_i)}(W_j) \le D_{int}, \tag{4.144}$$

and

$$\limsup_{i\to\infty} \left\{ \sum_\beta diam_{(\Omega^\beta_j, g_i)}(\Omega^\beta_j) : \Omega^\beta_j \text{ connected component of } \partial W_j \right\} \le B_j \tag{4.145}$$

satisfies $\lim_{j\to\infty} B_j = 0$ then W_j is uniformly well embedded.

Proof. Recall from Definition 3.2.1 that we have

$$\lambda_{i,j,k} = \sup_{x,y\in W_j} \left| d_{(W_k,g_i)}(x,y) - d_{(M,g_i)}(x,y) \right| \leq \operatorname{diam}_{(W_k,g_i)}(W_j). \tag{4.146}$$

Since

$$\operatorname{diam}_{W_k,g_i}(W_j) \leq \operatorname{diam}_{W_k,g_i}(W_k), \tag{4.147}$$

and g_i converges smoothly on W_k we have,

$$\lim_{i\to\infty} \operatorname{diam}_{(W_k,g_i)}(W_k) = \operatorname{diam}_{(W_k,g_\infty)}(W_k) \leq D_{int}. \tag{4.148}$$

Therefore,

$$\limsup_{j\to\infty} \limsup_{k\to\infty} \limsup_{i\to\infty} \lambda_{i,j,k} \leq \limsup_{j\to\infty} \limsup_{k\to\infty} \lim_{i\to\infty} \operatorname{diam}_{(W_k,g_i)}(W_k) \leq D_{int}. \tag{4.149}$$

Now suppose $x_{ijk}, y_{ijk} \in \partial W_j$ give the supremum in the definition of λ_{ijk}, Let γ_{ijk} and C_{ijk} be shortest paths between x_{ijk} and y_{ijk} in $\bar{W}_k$ and M respectively. Letting $k \to \infty$ and passing to a subsequence if necessary, $x_{ijk} \to x_{ij} \in \bar{W}_j$ and $y_{ijk} \to y_{ij} \in \bar{W}_j$. Passing to a subsequence again if necessary, C_{ijk} converges to C_{ij} which is a shortest path between x_{ij} and y_{ij} in M (c.f. [10][Prop 2.5.17]). And let γ^k_{ij} be the shortest path between x_{ij} and y_{ij} in $\bar{W}_k$.

We will estimate C_{ij} by curves in ∂W_j with controlled increase in the length. Denote the curve obtained in nth step by C^n_{ij} and let $C^0_{ij} = C_{ij}$. To obtain C^{n+1}_{ij} from C^n_{ij}, we proceed as follows: Suppose $\{\Omega^n_j\}_{n\in N}$ are the connected components met by C_{ij} (in more than one point). If C^n_{ij} does not intersect Ω^{n+1}_j, then we let $C^{n+1}_{ij} = C^n_{ij}$. If C^n_{ij} intersects Ω^{n+1}_j then define

$$t_1 = \inf\left\{t : C^n_{ij}(t) \in \Omega^{n+1}_j\right\}, \tag{4.150}$$

and

$$t_2 = \sup \left\{ t : C_{ij}^n(t) \in \Omega_j^{n+1} \right\}. \tag{4.151}$$

Since Ω_j^{n+1} is connected, we can replace the segment $C_{ij}^n[t_1, t_2]$ with a shortest path in Ω_j^{n+1}. The curve obtained in this way is our C_{ij}^{n+1}. Note that connectivity of the boundary components of $M \setminus W_j$ implies that if C_{ij}^n enters $M \setminus W_j$ through Ω_j^{n+1} at time t, then it has to intersect Ω_j^{n+1} again at time $t' > t$ in order to enter W_j.

This construction implies that for all n,

$$L(C_{ij}^n) \leq L(C_{ij}^0) + B_j, \tag{4.152}$$

hence, the sequence $\{C_{ij}^n\}_{n \in N}$ obtained in this way have uniform bounded length and as a result, we can apply the Arzela-Ascoli's theorem to obtain, after possibly passing to a subsequence, a limit C'_{ij} i.e. C'_{ij} is a curve with end points x_{ij}, y_{ij} and there are parametrizations of $\{C_{ij}^n\}_{n \in N}$ and C'_{ij} on the same domain such that $\{C_{ij}^n\}_{n \in N}$ uniformly converges to C'_{ij}. We claim that C'_{ij} is contained in $\bar{W}_j$. For any t, tracing the curve C_{ij}^0 back and forth from the point $C_{ij}^0(t)$, we reach two immediate components Ω_j^l and Ω_j^m and this means that for $n \geq \max\{l, m\}$, $C_{ij}^n(t) \in \bar{W}_j$ and since $C_{ij}^n(t) \to C'_{ij}(t)$ we must have $C'_{ij}(t) \in \bar{W}_j$. Furthermore, for i large enough (depending on j)

$$\begin{aligned}
\limsup_{i \to \infty} \left(L(\gamma_{ij}^k) - L(C_{ij}) \right) &\leq \limsup_{i \to \infty} \left(L(C'_{ij}) - L(C_{ij}) \right) \\
&\leq \limsup_{i \to \infty} \sum_{\beta} \operatorname{diam}^{g_i}_{\Omega_j^\beta} \left(\Omega_j^\beta \right) \\
&< B_j.
\end{aligned} \tag{4.153}$$

Also for all m,

$$\begin{aligned}\lim_{k\to\infty}\lambda_{ijk} &= \lim_{k\to\infty}\left(d_{(W_k,g_i)}\left(x_{ijk},y_{ijk}\right)-d_{(M,g_i)}\left(x_{ijk},y_{ijk}\right)\right)\\ &\leq \lim_{k\to\infty}\left(d_{(W_m,g_i)}\left(x_{ijk},y_{ijk}\right)-d_{(M,g_i)}\left(x_{ijk},y_{ijk}\right)\right)\\ &= L(\gamma_{ij}^m)-L(C_{ij}).\end{aligned} \tag{4.154}$$

Therefore, combining (4.153) and (4.154) we have

$$\lambda_j=\limsup_{i\to\infty}\lim_{k\to\infty}\lambda_{ijk}\leq B_j, \tag{4.155}$$

hence,

$$\limsup_{j\to\infty}\lambda_j\leq\lim_{j\to\infty}B_j=0. \tag{4.156}$$

□

Proof of Theorem 1.1.4:

Proof. The Lemma 4.3.1 combined with Theorem 3.2.2 completes the proof of Theorem 1.1.4. Recall Definition 3.2.1. □

Remark 4.3.2. *Example 4.1.5 demonstrates that the connectivity of the exhaustion in Theorem 1.1.4, hence in Theorems 1.1.5 and 4.3.5 is a necessary condition. Example 4.1.7 demonstrates that in Theorem 1.1.4, hence, in Theorems 1.1.5 and 4.3.5, the condition on the intrinsic diameter can not be replaced by the same condition on the extrinsic diameter. The necessity of other conditions follow as in Remark 4.2.3.*

Proof of Theorem 1.1.5:

Proof. The hypothesis $\mathrm{diam}(M_j)\leq D_0$ combined with the hypothesis including (1.22) and (1.23), allows us to apply Theorem 1.1.4. So (M_i,g_i) has an intrinsic flat limit and this intrinsic flat limit is the settled completion of $(M \setminus S, g_\infty)$. Thus by Proposition 3.4.1, the Gromov-Hausdorff and Intrinsic Flat limits agree. $\square$

Remark 4.3.3. *Example 4.1.3 proves the necessity of the the condition (1.20) in Theorem 1.1.5.*

Remark 4.3.4. *From [12], Ricci bounded below and* $d_{GH}(M_i, M) \to 0$ *imply* $\mathrm{Vol}(M_i) \to \mathrm{Vol}(M)$ *(conjectured by Anderson-Cheeger) and* M_i *are homeomorphic to* M *for* i *sufficiently large (later proved diffeomorphic in [40] .) This means that (1.23) in Theorem 1.1.5 is a necessary condition.*

Theorem 4.3.5. *Let* $M_i = (M, g_i)$ *be a sequence of Riemannian manifolds with a uniform linear contractibility function,* ρ*, which converges smoothly away from a closed singular set,* S*, with*

$$\mathrm{Vol}(M_i) \le V_0. \tag{4.157}$$

If there is a connected precompact exhaustion, W_j*, of* $M \setminus S$*, satisfying (1.8) such that each connected component of* $M \setminus W_j$ *has a connected boundary, satisfying (1.19)-(1.21), (1.2) and (1.16) then*

$$\lim_{j\to\infty} d_{GH}(M_j, N) = 0. \tag{4.158}$$

where N *is the metric completion of* $(M \setminus S, g_\infty)$*.*

Proof. By Lemma 3.3.1, we have

$$\mathrm{Vol}(M_i) \le V_0. \tag{4.159}$$

This combined with the uniform contractibility function allows us to apply the Greene-Petersen Compactness Theorem. In particular, we have a uniform upper bound on diameter

$$\mathrm{diam}(M_i) \leq D_0, \tag{4.160}$$

We may now apply Theorem 1.1.4 to obtain

$$\lim_{j\to\infty} d_{\mathcal{F}}(M_j, N') = 0. \tag{4.161}$$

We then apply Theorem 2.3.26 to see that the flat limit and Gromov-Hausdorff limits agree due to the existence of the uniform linear contractibility function and the fact that the volume is bounded below uniformly by the smooth limit. In particular, the metric and the settled completions agree. □

Chapter 5

Review of Ricci Flow

In many cases, singularities are unavoidable when one works with curvature flows. A geometric way to deal with singularities is by performing a geometric surgery thus removing the singular region (or a neighborhood of the singular region) and gluing in a suitable nonsingular object in a smooth way and then trying to continue the flow with this new initial condition. the famous examples of these geometric surgeries arise in Mean Curvature Flow and Ricci flow.

Brakke formulated a weak Mean Curvature Flow by replacing a family of manifolds with varifolds [9] and Brian White [52] proved that the Brakke flow is continuous with respect to the flat distance when considered as a sequence of integral current spaces. There are a few other useful formulations of Weak Mean Curvature Flow as well (see [9] [18] [14] [29]).

The Ricci flow with surgery was proposed by Richard Hamilton (see [27]) and was modified and made rigorous in the ground-breaking work of Grisha Perelman (see [39] and [38]) to prove the long lasting Poincare Conjecture.

As powerful and promising the geometric surgery in Ricci flow is, it depends on several parameters and by no means is a canonical procedure. This means that in contrast with Mean curvature Flow, there is no (known and complete) formulation of **weak Ricci flow** i.e. an evolving family of (possibly singular) metric spaces which decodes the Ricci flow and its formation and healing of singularities.

Angenent-Caputo-Knopf [3] have recently constructed a canonical smooth continuation of Ricci flow past a rotationally symmetric neckpinch which is called the smooth forward evolution of Ricci flow. The smooth forward evolution of Ricci flow is in fact a limit of the regularized metrics off the singular set that can be viewed as a minimally invasive (canonical) surgery performed by Ricci flow PDE. As one might expect, the smooth forward evolution of Ricci flow in fact breaks the evolving space into two disjoint parts.

In the second part of this thesis (The present Chapter and its sequel), we will study the smooth forward evolution of Ricci flow through the neckpinch singularity at the singular time T. Inspired by the work of B. White [52], we explain how one can view the resulting family of spaces (before, at and past the singular time) as a continuous flow of Integral current spaces.

In Section 5.1, we will briefly review some about the classic smooth Ricci flow that we will be using in the future sections. Section 5.2 is an account of Angenent-Knopf [4] Examples of rotationally symmetric non-degenerate neckpinch singularities on compact spheres. In Theorem 5.2.2, we have stated Angenent-Knopf's main theorem as has appeared in [4, Theorem 1.1]. This Theorem provides us with the asymptotic profile of the non-degenerate neckpinch as the flow develops a singularity. In Section 5.3, we review the Angenent-Knopf's result on diameter bound as has appeared in [5, Lemma 2] which will be used in Chapter 6. Section 5.4 is devoted to reviewing the Angenent-Caputo-Knopf's construction of the smooth forward evolution of Ricci flow through non-degenerate neckpinch singularities(see [3]); Theorem 5.4.1 states the Angenent-Caputo-Knopf's main Theorem as has appeared in [3].

In Chapter 6, we will present our results regarding the continuity of Ricci flow through neckpinch singularities with respect to Sormani-Wenger Intrinsic Flat Distance. In Section 6.1, we refine our estimates on the SWIF distance so that they are applicable in this new setting (see Theorem 6.1.6); In Section 6.3, we will explain how one can view the Ricci flow through neckpinch singularity as a family of integral current spaces and Sections 6.2, 6.4 and 6.5 provide the proof of Theorem 1.2.1 and Corollary 1.2.2 which claim the the continuity of the flow with respect to the SWIF distance (see Lemmas 6.2.1, 6.4.2 and 6.5.1).

5.1 Hamilton's Ricci flow

Ricci flow is an evolution equation of the metric on a Riemannian manifold, introduced for the first time by Richard Hamilton in [26] given by the following weakly parabolic equation:

$$\frac{d}{dt}g(t) = -2\mathrm{Ric}(t). \tag{5.1}$$

Hamilton proved that for the initial metric g_0 on a closed manifold M, the Ricci flow equation satisfies short time existence and uniqueness. [26]

Proposition 5.1.1. *Suppose M is a closed manifold and let $g(t)$ be a solution to the Ricci flow equation on the time interval $[0,T]$. If*

$$\|\operatorname{Rm}(t)\| \leq K \ \textit{ for all } t \in [0,T], \tag{5.2}$$

where $\|\operatorname{Rm}(t)\|$ is with respect to a fixed background metric g_0; then, for any $t_1 \leq t_2$ and $V \in TM$ we have the following:

$$e^{-\sqrt{n}K(t_2-t_1)}g(t_2)(V,V) \leq g(t_1)(V,V) \leq e^{\sqrt{n}K(t_2-t_1)}g(t_2)(V,V), \tag{5.3}$$

and therefore,

$$e^{-\sqrt{n}K(t_2-t_1)} \leq \frac{d_{M,g(t_2)}(x,y)}{d_{M,g(t_1)}(x,y)} \leq e^{\sqrt{n}K(t_2-t_1)} \tag{5.4}$$

Proof. c.f. [15]. □

The above result also holds locally, namely:

Proposition 5.1.2. *Suppose M is a closed manifold and let $g(t)$ be a solution to the Ricci flow equation on the time interval $[0,T]$ and $\Omega \subset M$ an open subset of the manifold then, if*

$$\sup_{x\in\Omega} \|\operatorname{Rm}(x,t)\| \leq K \ \textit{ for all } t \in [0,T], \tag{5.5}$$

where, $\|\operatorname{Rm}(x,t)\|$ is with respect to a fixed background metric g_0; then, for any $t_1 \leq t_2$ and $V \in T\Omega$ we have the following:

$$e^{-\sqrt{n}K(t_2-t_1)}g(t_2)(V,V) \leq g(t_1)(V,V) \leq e^{\sqrt{n}K(t_2-t_1)}g(t_2)(V,V) \tag{5.6}$$

Proof. c.f. [15]. □

5.2 Angenant-Knopf Neck Pinch

In this section, we will review the results about neckpinch singularity obtained by Angenent-Knopf [4][5] and Angenent-Caputo-Knopf [3]. We will repeat some of their Theorems and Lemmas from their work that we will be using later on in this thesis. A nondegenerate neckpinch is a local type I singularity (except for the round sphere shrinking to a point) is arguably the best known and simplest example of a finite-time singularity that can develop through the Ricci flow. A nonegenerate neckpinch is a type I singularity whose blow up limit is a shrinking cylinder soliton. more precisely,

Definition 5.2.1. *a solution $(M^{n+1}, g(t))$ of Ricci flow develops a neckpinch at a time $T < \infty$ if there exists a time-dependent family of proper open subsets $U(t) \subset M^{n+1}$ and diffeomorphisms $\phi(t) : \mathbb{R} \times \mathbb{S}^n \to U(t)$ such that $g(t)$ remains regular on $M^{n+1} \setminus U(t)$ and the pullback $\phi(t)^* (g(t))$ on $\mathbb{R} \times \mathbb{S}^n$ approaches the* ***shrinking cylinder soliton*** *metric*

$$ds^2 + 2(n-1)(T-t)g_{can} \tag{5.7}$$

For the first time, Angenent-Knopf in [4] rigorously proved the existence of nondegenerate neckpinch on the sphere $\mathbb{S}^{n+1}$ in [4]. Their main result in [4] is as follows:

Theorem 5.2.2. *If $n > 2$, there exists an open subset of the family of metrics on $\mathbb{S}^{n+1}$ possessing $\mathrm{SO}(n+1)$ symmetries such that the Ricci fow starting at any metric in this set develops a neckpinch at some time $T < 1$. The singularity is rapidly-forming (Type I), and any sequence of parabolic dilations formed at the developing singularity converges to a shrinking cylinder soliton.*

$$ds^2 + 2(n-1)(T-t)g_{can}. \tag{5.8}$$

This convergence takes place uniformly in any ball of radius

$$o\left(\sqrt{(T-t)\log\frac{1}{T-t}}\right), \tag{5.9}$$

centered at the neck.

Furthermore, there exist constants $0 < \delta, C < \infty$ *such that the radius* ψ *of the sphere at distance* σ*from the neckpinch is bounded from above by*

$$\psi \leq \sqrt{2(n-1)(T-t)} + \frac{C\sigma^2}{-\log{(T-t)}\sqrt{T-t}}, \tag{5.10}$$

for $|\sigma| \leq 2\sqrt{(T-t)\log{(T-t)}}$*, and,*

$$\psi \leq C\frac{\sigma}{\sqrt{-\log{(T-t)}}}\sqrt{\log\frac{\sigma}{-(T-t)\log{(T-t)}}}, \tag{5.11}$$

for $2\sqrt{(T-t)\log{(T-t)}} \leq \sigma \leq (T-t)^{\frac{1}{2}-\delta}$

The class of initial metrics for which we establish ”neckpinching” is essentially described by three conditions: (i) the initial metric should have positive scalar curvature, (ii) the sectional curvature of the initial metric should be positive on planes tangential to the spheres $\{x\} \times \mathbb{S}^n$, and (iii) the initial metric should be ” sufficiently pinched”.

Lemma 5.2.3. *There is a constant* C *depending on the solution* $g(t)$ *such that:*

$$\|\mathrm{Rm}\| \leq \frac{C}{\psi^2} \tag{5.12}$$

Proof. See the Lemma 7.1 in [4]. ☐

5.3 Angenant-Knopf Diameter Bound

The following diameter bound argument is necessary before we can talk about the intrinsic flat convergence. Proposition 5.3.1 in below is taken from [5]. We are also including the proof of this Proposition from [5] for completeness of exposition because we need the estimates in the proof as well as the result itself.

Proposition 5.3.1 ([5]). *Let* $(\mathbb{S}^{n+1}, g(t))$ *be any* $\mathrm{SO}(n+1)$ *invariant solution of the Ricci Flow such that* $g(0)$ *has positive scalar curvature and positive sectional curvature on planes tangential to the spheres* $x \times \mathbb{S}^n$*, assume that in the language of [4], each* $g(t)$ *has at least two bumps for all* $t < T$*. Let* $x = a(t)$ *and* $y = b(t)$ *be the locations of the left- and right- most bumps, and assume that for all* $t < T$*, one has* $\psi(a(t), t) \geq c$ *and* $\psi(b(t), t) \geq c$ *for some constant* $c > 0$*. If* $g(t)$ *becomes singular at* $T < \infty$*, then diam*$(\mathbb{S}^{n+1}, g(t))$ *remains bounded as* $t \nearrow T$*.*

Proof. ([5]) By Proposition 5.4 of [4], the limit profile $\psi(\ . \ , T)$ exists. let $a(t) \to a(T)$ and $b(t) \to b(T)$. By lemma 5.6 of [4], the Ricci curvature is positive (and so the distances are decreasing) on $(-1, a(t)]$ and $[b(t), 1)$. Hence it will suffice to bound $d_{(M,g(t))}(x_1, x_2)$ for arbitrary $x_1 < x_2$ in $(a(T) - \epsilon, b(T) + \epsilon) \subset (-1, 1)$.

Equations (5) and (11) of [4] imply that

$$\begin{aligned}\frac{d}{dt}d_{M,g(t)}(x_1,x_2) &= \frac{d}{dt}\int_{x_1}^{x_2} \phi(x,t)\,\mathrm{d}x \\ &= n\int_{s(x_1)}^{s(x_2)} \frac{\psi_{ss}}{\psi}\,\mathrm{d}s \qquad (5.13)\\ &= n\left\{\frac{\psi_s}{\psi}\bigg|_{s(x_1)}^{s(x_2)} + \int_{s(x_1)}^{s(x_2)} \left(\frac{\psi_s}{\psi}\right)^2\,\mathrm{d}s\right\}.\end{aligned}$$

Proposition 5.1 of [4], bounds ψ_s uniformly, while lemma 5.5 shows that the number of bumps and necks are non-increasing in time. It follows that:

$$\begin{aligned}\int_{s(x_1)}^{s(x_2)} \left(\frac{\psi_s}{\psi}\right)^2\,\mathrm{d}s &\leq C\int_{s(x_1)}^{s(x_2)} \frac{|\psi_s|}{\psi^2}\,\mathrm{d}s \\ &\leq C\left[\frac{1}{\psi_{min}(t)} - \frac{1}{\psi_{max}(t)}\right] \qquad (5.14)\\ &\leq \frac{C}{\psi_{min}(t)}.\end{aligned}$$

Hence lemma 6.1 of [4] lets us conclude that:

$$\left|\frac{d}{dt}d_{M,g(t)}(x_1,x_2)\right| \leq \frac{C}{\sqrt{T-t}}, \tag{5.15}$$

which is obviously integrable. □

Lemma 5.3.2. *If the diameter of the solution* $g(t)$ *stays bounded as* $t \nearrow T$ *then* $\psi(s,T) > 0$ *for all* $0 < s < D/2$*, where* D *is defined as* $D = \lim_{t\nearrow T}\psi(x_*(t),t)$ *in which* $x_*(t)$ *denotes the location of the right bump.*

Proof. See the Lemma 10.1 of [4]. □

5.4 Angenant-Caputo-Knopf Smooth Forward evolution

In this section we will review the results obtained by Angenent, Caputo and Knopf in [3] about the neckpinch on the sphere in any dimension and their attempt to find a canonical way to perform surgery at the singular time (in this case, finding a limit for surgeries whose scale of the surgery is going to zero).

Consider the degenerate metric $g(T)$ resulted from the Ricci Flow neckpinch on $\mathbb{S}^{n+1}$ as described earlier. Angenent-Caputo-Knopf in [3] construct the smooth forward evolution of Ricci flow by regularizing the pinched metric in a small neighborhood of the pinched singularity (of scale ω) and hence producing a smooth metric g_ω. Notice that performing surgery at a small scale ω produces two disjoint Ricci Flows. For simplicity, we only consider one of these resulting spheres and then we assume that the north pole is the future of the neckpinch point singularity. By the short time existence of Ricci flow, for any small scale ω, the flow exists for a short time depending on ω. Using the asymptotics for Ricci flow neckpinch derived in Angenent-Knopf [4], they find a lower bound for the maximal existence time T_ω of the Ricci flows $g_\omega(t)$ with initial metrics g_ω. Of course, as $\omega \to 0$, one has $g_\omega \to g(T)$ away from the point singularity. Now the question is if also the resulting Ricci flow solutions $g_\omega(t)$ admit a limit as $\omega \to 0$. They prove that this is in fact the case by proving bounds on the curvature off the singularity and then proving a compactness theorem. This limit flow is called the smooth forward evolution of Ricci flow out of a neckpinch singularity.

Angenent-Caputo-Knopf [3] show that a smooth forward evolution of Ricci flow out of a neckpinch singularity comes from (via a change of variable) a positive solution of the following quasilinear PDE:

$$v_t = vv_{rr} - \frac{1}{2}v^2 + \frac{n-1-v}{r}v_r + \frac{2(n-1)}{r^2}(v - v^2) \tag{5.16}$$

with the singular initial data:

$$v_{init}(r) = [1 + o(1)]v_0(r) \text{ as } r \searrow 0, \tag{5.17}$$

where

$$v_0(r) \doteqdot \frac{\frac{1}{4}(n-1)}{-\log r}. \tag{5.18}$$

One notices that away from the point singularity, any smooth forward evolution of (5.17) has to satisfy

$$\lim_{t \searrow 0} v(r,t) = v_{init}(r) \tag{5.19}$$

They prove that the only way, a solution to this equation can be complete is if v satisfies the smooth boundary condition $v(0,t) = 1$ which is incompatible with the fact that $\lim_{r \searrow 0} v_{init}(r) = 0$. Roughly speaking, this means that for any forward evolution of Ricci flow, v immediately jumps at the singular hypersurface $\{0\} \times \mathbb{S}^n$, yielding a compact forward evolution that replaces the singularity with a smooth n-ball by performing a surgery at scale 0. See Figure 5.1.

For given small $\omega > 0$, they split the manifold $\mathbb{S}^{n+1}$ into two disjoint parts, one of which is the small neighborhood $\mathcal{N}_\omega$ of the north pole in which $\psi_T(s) < \rho_* \sqrt{\omega}$ (See [3] for details about this construction).

They keep the metric unchanged on $\mathbb{S}^{n+1} \setminus \mathcal{N}_\omega$. Within $\mathcal{N}_\omega$, they take g_ω to be a metric of the form

$$g_\omega = (ds)^2 + \psi_\omega(s)^2 g_{can}, \tag{5.20}$$

where ψ_T is a monotone when $\psi_\omega(s) \leq \rho_* \sqrt{\omega}$. Monotonicity of ψ in $\mathcal{N}_\omega$, allows them to perform the change of variables $r = \psi(s)$. In this new coordinate, one sees that g_ω is of the form:

$$g_\omega = \frac{dr^2}{v_\omega(r)} + r^2 g_{can}. \tag{5.21}$$

Angenent-Caputo-Knopf then proceed to apply a maximum principle to find sub- and supersolutions of this equation which bound all the positive solutions. A detailed analysis of these bounds enables them to prove curvature estimates that are required in their compactness theorem (See [3] for further details).

The main theorem of Angenent-Caputo-Knopf's work about the smooth forward evolution of the Ricci flow past the singularity time is as follows:

Theorem 5.4.1 ([3]). *For $n > 2$, let g_0 denote a singular Riemannian metric on $\mathbb{S}^{n+1}$ arising as the limit as $t \nearrow T$ of a rotationally symmetric neckpinch forming at time T. Then there exists a complete smooth forward evolution*

$$\big(S^{n+1}, g(t)\big) \quad \text{for } T < t < T_1, \tag{5.22}$$

of $g(T)$ by Ricci Flow. Any complete smooth forward evolution is compact and satisfies a unique asymptotic profile as it emerges from the singularity. In a local coordinate $0 < r < r_* \ll 1$ such that the singularity occurs at $r = 0$

and the metric is

$$g(r,t) = \frac{dr^2}{v(r,t)} + r^2 g_{can} \tag{5.23}$$

This asymptotic profile is as follows:

Outer Region: *For* $c_1\sqrt{t-T} < r < c_2$*, one has:*

$$v(r,t) = [1+o(1)]\frac{n-1}{-4\log r}\left[1+2(n-1)\tfrac{t-T}{r^2}\right] \text{ uniformly as } \quad t \searrow T. \tag{5.24}$$

Parabolic Region: *Let* $\rho = \frac{r}{\sqrt{t-T}}$ *and* $\tau = \log(t-T)$*; then for* $\frac{c_3}{\sqrt{-\tau}} < \rho < c_4$*, one has:*

$$v(r,t) = [1+o(1)]\frac{n-1}{-2\tau}\left[1+\frac{2(n-1)}{\rho^2}\right] \text{ uniformly as } \quad t \searrow T. \tag{5.25}$$

Inner Region: *Let* $\sigma = \sqrt{-\tau}\rho = \sqrt{\frac{-\tau}{t-T}}r$*; then for* $0 < \sigma < c_5$*, one has:*

$$v(r,t) = [1+o(1)]\mathcal{B}\left(\frac{\sigma}{n-1}\right) \text{ uniformly as } \quad t \searrow T. \tag{5.26}$$

where $\frac{d\sigma^2}{\mathcal{B}(\sigma)} + \sigma^2 g_{can}$ *is the Bryant soliton metric.*

Remark 5.4.2. *In the work of Angenent-Caputo-Knopf [3], assumptions on the singular initial metric* $g(T) = ds^2 + \psi_T(s)^2 g_{can}$ *is as follows:*

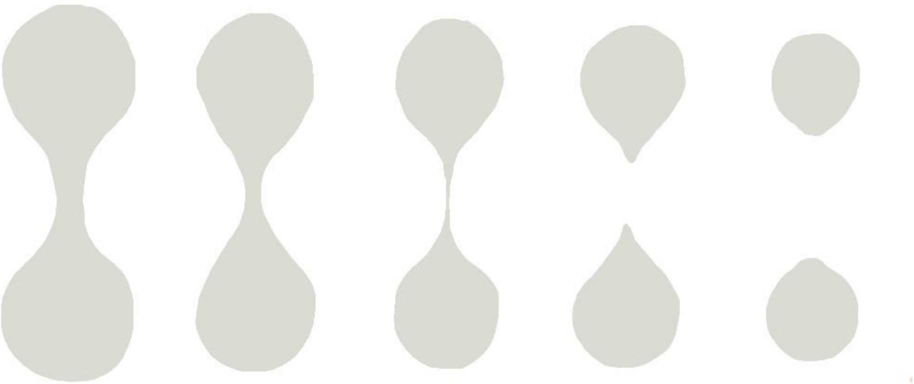

Figure 5.1: Angenent-Caputo-Knopf Ricci Flow Through Neckpinch Singularity

Credits: Illustration by Penelope Chang

$$
\begin{aligned}
&(M1)\ \ \psi_T(s) > 0\ \ \textit{for all}\ \ s \in J \\
&(M2)\ \ \psi_T(0) = \psi_T(l) = 0 \\
&(M3)\ \ \psi_T'(l) = -1 \\
&(M4)\ \ \psi_T(s)^2 = \left(\frac{n-1}{4} + o(1)\right)\frac{s^2}{-\log s}\ \ (s \searrow 0) \\
&(M5)\ \ \psi_T(s)\psi_T'(s) = \left(\frac{n-1}{4} + o(1)\right)\frac{s}{-\log s}\ \ (s \searrow 0) \\
&(M6)\ \ |\psi_T'(s)| \le 1\ \ (0 < s < l) \\
&(M7)\ \ \exists r_\# > 0, \psi_T'(s) \neq 0\ \ \textit{whenever}\ \ \psi_T(s) < 2r_\# \\
&(M8)\ \ \exists \mathcal{A}\ \forall s \in J, |a_T(s)| \le \mathcal{A}\ (\textit{where}\ a_0(s) = \psi_T'\psi_T'' - \psi_T'^2 + 1)
\end{aligned}
\tag{5.27}
$$

See [4] and [3] for details.

Chapter 6

SWIF Continuity of the ACK Ricci flow

In this chapter we present our research work regarding the SWIF continuity of Ricci flow through non-degenrate neckpinch singularities. We will give a proof of Theorem 1.2.1.

6.1 Adapted estimates

Since at the post surgery times our space is an integral current space rather than a manifold, we can not apply the Theorem 3.1.1 right away. The integral current space we study, possesses nice properties that will allow us to apply a refined version of the Theorem 3.1.1. This section is devoted to prove this refined version of the estimate on the SWIF distance.

Definition 6.1.1. *Let (M, g) be a Riemannian manifold (possibly disconnected and with boundary). Let $d : M \times M \to \mathbb{R}$ be a metric on M. We say* *that the metric d is* ***related*** *to the Riemannian metric g if for any smooth curve $C : (-r, r) \to M$, we have*

$$g\left(C'(0), C'(0)\right) = \left(\frac{d}{dt}\big|_{t=0}\, d\left(C(t), C(0)\right)\right)^2. \tag{6.1}$$

Lemma 6.1.2. *Let (M, g) be a Riemannian manifold (possibly disconnected and with boundary). Let $d : M \times M \to \mathbb{R}$ be a metric on M. If d is related to g, then for any smooth curve $C : (-r, r) \to M$, we have*

$$L_{g_i}(C) = L_{d_i}(C). \tag{6.2}$$

Proof. Consult any standard text on Metric Geometry for example [10]. □

Definition 6.1.3. *Let $D > 0$ and M, M' are geodesic metric spaces. We say that $\varphi : M \to M'$ is a* ***D-geodesic embedding*** *if for any smooth minimal geodesic, $\gamma : [0, 1] \to M$, of length $\leq D$ we have*

$$d_{M'}(\varphi(\gamma(0)), \varphi(\gamma(1))) = L(\gamma). \tag{6.3}$$

Proposition 6.1.4. *Given a manifold M with Riemannian metrics g_1 and g_2 and $D_1, D_2, t_1, t_2 > 0$. Let $M' = M \times [t_1, t_2]$ and let $\varphi_i : M_i \to M'$ be defined by $\varphi_i(p) = (p, t_i)$. If a metric g' on M' satisfies*

$$g' \geq dt^2 + \cos^2((t - t_i)\pi/D_i)g_i \text{ for } |t - t_i| < D_i/2 \tag{6.4}$$

and

$$g' = dt^2 + g_i \text{ on } M \times \{t_i\} \subset M' \tag{6.5}$$

then any geodesic, $\gamma : [0, 1] \to M_i$, of length $\leq D_i$ satisfies (6.3). If, the diameter is bounded, $diam_{g_i}(M) \leq D_i$, then φ_i is an isometric embedding.

Furthermore, for $q_1, q_2 \in M$, we have

$$d_{M'}(\varphi_1(q_1), \varphi_2(q_2)) \geq d_{M_i}(q_1, q_2). \tag{6.6}$$

Proposition 6.1.5. *Suppose* $M_1 = (M, g_1)$ *and* $M_2 = (M, g_2)$ *are diffeomorphic oriented precompact Riemannian manifolds and suppose there exists* $\epsilon > 0$ *such that*

$$g_1(V,V) < (1+\epsilon)^2 g_2(V,V) \text{ and } g_2(V,V) < (1+\epsilon)^2 g_1(V,V) \qquad \forall\, V \in TM. \tag{6.7}$$

Then for any

$$a_1 > \frac{\arccos(1+\epsilon)^{-1}}{\pi} diam(M_2) \tag{6.8}$$

and

$$a_2 > \frac{\arccos(1+\epsilon)^{-1}}{\pi} diam(M_1), \tag{6.9}$$

there is a pair of isometric embeddings $\varphi_i : M_i \to M' = \bar{M} \times [t_1, t_2]$ *with a metric as in Proposition 6.1.4 where* $t_2 - t_1 \geq \max\{a_1, a_2\}$.

Thus the Gromov-Hausdorff distance between the metric completions is bounded,

$$d_{GH}(\bar{M}_1, \bar{M}_2) \leq a := \max\{a_1, a_2\}, \tag{6.10}$$

and the intrinsic flat and scalable intrinsic flat distances between the settled completions are bounded,

$$d_{\mathcal{F}}(M_1', M_2') \leq a\,(V_1 + V_2 + A_1 + A_2), \tag{6.11}$$

$$d_{s\mathcal{F}}(M_1', M_2') \leq (a(V_1+V_2))^{1/(m+1)} + (a(A_1+A_2))^{1/(m)} \tag{6.12}$$

where $V_i = \mathrm{Vol}_m(M_i)$ *and* $A_i = \mathrm{Vol}_{m-1}(\partial M_i)$.

Theorem 6.1.6. *Given a pair of geodesic metric spaces* (Y_i, d_i), $i = 1, 2$, *containing integral current spaces* (X_i, d_i, T_i) , $i = 1, 2$ *with restricted metrics*

d_i, suppose there are precompact subregions $U_i \subset \mathrm{set}(X_i)$ (possibly disconnected) that are Riemannian manifolds (possibly with boundary) with metrics g_i such that the induced integral current spaces are

$$(U_i, d_i, T_i)\,, \quad i = 1, 2, \tag{6.13}$$

where, the metric d_i on U_i is restricted from d_i on X_i and,

$$T_i = \int_{U_i} \quad, \quad i = 1, 2. \tag{6.14}$$

and such that the metric d_i is ***related*** *(see Definition 6.1.1) to the Riemannian metric g_i for $i = 1, 2$.*

Assume there exist diffeomorphisms $\psi_i : U \to U_i$ such that

$$\psi_1^* g_1(V, V) < (1+\epsilon)^2 \psi_2^* g_2(V, V) \qquad \forall\, V \in TU \tag{6.15}$$

and

$$\psi_2^* g_2(V, V) < (1+\epsilon)^2 \psi_1^* g_1(V, V) \qquad \forall\, V \in TU. \tag{6.16}$$

We take the following ***extrinsic*** *diameters,*

$$D_{U_i} = \sup\left\{ diam_{X_i}(W) : W \text{ is a connected component of } U_i \right\} \le diam(X_i), \tag{6.17}$$

and define a hemispherical width,

$$a > \frac{\arccos(1+\epsilon)^{-1}}{\pi} \max\{D_{U_1}, D_{U_2}\}. \tag{6.18}$$

Let the distance distortion with respect to the outside integral current spaces be

$$\lambda = \sup_{x,y \in U} \left| d_{X_1}(\psi_1(x), \psi_1(y)) - d_{X_2}(\psi_2(x), \psi_2(y)) \right|, \tag{6.19}$$

we define heights,

$$h = \sqrt{\lambda(\max\{D_{U_1}, D_{U_2}\} + \lambda/4)} \tag{6.20}$$

and

$$\bar{h} = \max\{h, \sqrt{\epsilon^2 + 2\epsilon}\, D_{U_1}, \sqrt{\epsilon^2 + 2\epsilon}\, D_{U_2}\}. \tag{6.21}$$

Then, the SWIF distance between the settled completions are bounded above as follows:

$$\begin{aligned} d_{\mathcal{F}}(X_1', X_2') \;\le\; & \left(2\bar{h} + a\right)\left(\mathrm{Vol}_m(U_1) + \mathrm{Vol}_m(U_2) + \mathrm{Vol}_{m-1}(\partial U_1) + \mathrm{Vol}_{m-1}(\partial U_2)\right) \\ & + \|T_1\|\,(X_1 \setminus U_1) + \|T_2\|\,(X_2 \setminus U_2). \end{aligned} \tag{6.22}$$

Proof. The theorem begins exactly as in the proof of [35, Theorem 4.6] with a construction of an ambient space Z.

For every pair of corresponding diffeomorphic connected components U_i^β of U_i, we can create a hemispherically defined filling bridge X_β' diffeomorphic to $U_i^{\beta_i} \times [0, a]$ with metric g_β' satisfying (6.3) by applying Proposition 6.1.4 and Proposition 6.1.5 using the $a_i = a_i(\beta)$ defined there for the particular connected component, U_i^β and $D_i = D_{U_i}$. Observe that all $a_i \le a$, so $|t_1 - t_2| = a$ will work for all the connected components. Any minimal geodesic $\gamma : [0,1] \to U_i^\beta$ of length $\le D_{U_i} \le \mathrm{diam}_{X_i}(U_i)$ satisfies (6.3).

Let X' be the disjoint unions of these bridges. X' has a metric g' satisfying

$$g'(V,V) \le dt^2(V,V) + g_1(V,V) + g_2(V,V) \qquad \forall\, V \in TM'. \tag{6.23}$$

The boundary of X' is $(U,g_1)\cup(U,g_2)\cup(\partial U\times[0,a],g')$. Therefore,

$$\begin{aligned}\mathrm{Vol}_m(X') &= \sum_\beta \mathrm{Vol}_m(X'_\beta)\\ &\le \sum_\beta a(\mathrm{Vol}_m(U_1^\beta)+\mathrm{Vol}_m(U_2^\beta)) \qquad (6.24)\\ &\le a(\mathrm{Vol}_m(U_1)+\mathrm{Vol}_m(U_2)),\end{aligned}$$

and

$$\mathrm{Vol}_m\left(\partial X'\setminus(\varphi_1(U_1)\cup\varphi_2(U_2)\right)\le a\left(\mathrm{Vol}_{m-1}(\partial U_1)+\mathrm{Vol}_{m-1}(\partial U_2)\right) \qquad (6.25)$$

as in Proposition 6.1.5.

Since our regions are not necessarily convex, we cannot directly glue X_i to X' in order to obtain a distance preserving embedding. We first need to glue isometric products $U^\beta\times[0,\bar h]$ with cylinder metric dt^2+g_i to both ends of the filling bridges, to have all the bridges extended by an equal length on either side. This creates a Lipschitz manifold,

$$X''=(U_1\times[0,\bar h])\sqcup_{U_1}X'\sqcup_{U_2}(U_2\times[0,\bar h]). \qquad (6.26)$$

We then define $\varphi_i:U_i\to X''$ such that

$$\begin{aligned}\varphi_1(x) &= (x,0)\in U_1\times[0,\bar h] \qquad (6.27)\\ \varphi_2(x) &= (x,\bar h)\in U_2\times[0,\bar h] \qquad (6.28)\end{aligned}$$

Then by (6.24) and (6.25), we have

$$\begin{aligned}\mathrm{Vol}_{m+1}(X'') &= \mathrm{Vol}_{m+1}(X')+\bar h(\mathrm{Vol}_m(U_1)+\mathrm{Vol}_m(U_2))\\ &\le (a+\bar h)(\mathrm{Vol}_m(U_1)+\mathrm{Vol}_m(U_2)) \qquad (6.29)\end{aligned}$$

and $\mathrm{Vol}_m\left(\partial X'' \setminus (\varphi_1(U_1) \cup \varphi_2(U_2)\right) =$

$$\begin{aligned} &= \mathrm{Vol}_m\left(\partial X' \setminus (\varphi_1(U_1) \cup \varphi_2(U_2)\right) + \bar{h}(\mathrm{Vol}_{m-1}(\partial U_1) + \mathrm{Vol}_{m-1}(\partial U_2)) \\ &\leq (a + \bar{h})(\mathrm{Vol}_{m-1}(\partial U_1) + \mathrm{Vol}_{m-1}(\partial U_2)). \end{aligned} \tag{6.30}$$

Finally we glue Y_1 and Y_2 to the far ends of X'' along $\varphi_i(U_i)$ to create a connected length space. This is possible since U_i's are manifolds and Y_is are geodesic spaces.

$$Z = \bar{X}_1 \sqcup_{U_1} X'' \sqcup_{U_2} \bar{X}_2 \tag{6.31}$$

As usual, distances in Z are defined by taking the infimum of lengths of curves. See Figure 6.1. Each connected component, X''_β of X'' will be called the filling bridge corresponding to U^β.

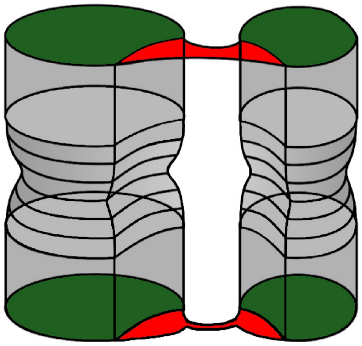

Figure 6.1: Creating Z for Theorem 6.1.6. Green: Regions That Are Lipschitz Close ; Red: Regions without Lipschitz Comparison

Credits: Illustration by Penelope Chang

In the proof of Theorem 4.6 in [35], it is proven that $\varphi_1 : Y_1 \to Z$ mapping Y_1 into its copy in Z is a distance preserving embedding. The proof there is given for manifolds but it can be easily adapted to our case since it only relies on the fact that our spaces are geodesic spaces and the fact that g_i and d_i are **related** (see Definition 6.1.1) on U_is and both conditions are satisfied in our case. The same argument shows that $\varphi_2 : Y_2 \to Z$ is also a distance preserving embedding.

In order to bound the SWIF distance, we take $B^{m+1} = X''$ to be the filling current. Then the excess boundary is

$$A^m = \varphi_1(X_1 \setminus U_1) \cup \varphi_2(X_2 \setminus U_2) \cup \partial X'' \setminus (\varphi_1(U_1) \cup \varphi_2(U_2)). \tag{6.32}$$

Using appropriate orientations we have

$$\varphi_{1\#}(T_1) - \varphi_{2\#}(T_2) = B^{m+1} + A^m. \tag{6.33}$$

Notice that (6.33) is true since the set $(\varphi_{i\#}(T_i)) = \varphi_i(X_i)$.

The volumes of the Lipschitz manifold parts have been computed in (6.30) and (6.29). So we get:

$$\begin{aligned} d_{\mathcal{F}}(X_1, X_2) \leq\ & \mathrm{Vol}_m(U_1)\left(\bar{h} + a\right) + \mathrm{Vol}_m(U_2)\left(\bar{h} + a\right) \\ & + \left(\bar{h} + a\right)\mathrm{Vol}_{m-1}(\partial U_1) + \left(\bar{h} + a\right)\mathrm{Vol}_{m-1}(\partial U_2) \qquad (6.34) \\ & + \|T_1\|\,(X_1 \setminus U_1) + \|T_2\|\,(X_2 \setminus U_2). \end{aligned}$$

□

6.2 Smooth Ricci Flow

Here we estimate the Intrinsic Flat distance between two times of a compact smooth Ricci flow defined on $[0, T)$ which will be easily derived from Theorem 3.1.1. Since as $t \to t_0 \in [0, T)$ we have

$$g(t) \to g(t_0) \tag{6.35}$$

uniformly in smooth norm, it is not surprising that we must also have:

$$d_{\mathcal{F}}\Big(\,(M,g(t))\,,(M,g(t_0))\,\Big)\to 0. \tag{6.36}$$

as $t\to t_0$. In fact, we have

Lemma 6.2.1. *Suppose* $(M^n,g(t))$ *is a smooth solution of Ricci flow on a closed manifold* M^n *defined on the time interval* $[0,T)$. *Then, for any* $t_1,t_2\in[0,T)$

$$\begin{aligned} &d_{\mathcal{F}}\Big(\,(M,g(t_1))\,,(M,g(t_2))\,\Big)\le \\ &\qquad\frac{\arccos\sqrt{e^{\sqrt{n}C(t_1-t_2)}}}{\pi}\max\left\{\mathit{diam}(M,g(t_1)),\,\mathit{diam}(M,g(t_2))\right\}\end{aligned} \tag{6.37}$$

where C *is a uniform upper bound for* $\|\mathrm{Rm}\|$.

Proof. Since the flow is smooth on $[0,T)$, for any compact sub-interval $J\subset[0,T)$ we have

$$\sup_{M^n\times J}\|\mathrm{Rm}\|\le C=C(J). \tag{6.38}$$

with respect to the initial metric g_0 on M^n.

By applying Theorem 5.1.1, we get

$$e^{-\sqrt{n}C(t_2-t_1)}g(t_2)(V,V)\le g(t_1)(V,V)\le e^{\sqrt{n}C(t_2-t_1)}g(t_2)(V,V), \tag{6.39}$$

Let $\epsilon=\sqrt{e^{\sqrt{n}C(t_2-t_1)}}-1$ then (6.39) gives

$$g(t_1)(V,V)\le(1+\epsilon)^2\,g(t_2)(V,V), \tag{6.40}$$

and

$$g(t_2)(V,V)\le(1+\epsilon)^2\,g(t_1)(V,V). \tag{6.41}$$

Finally using Proposition 6.1.5, we get:

$$d_{\mathcal{F}}\Big((M, g(t_1)) , (M, g(t_2)) \Big) \leq \tag{6.42}$$
$$\frac{\arccos\sqrt{e^{\sqrt{n}C(t_1-t_2)}}}{\pi} \max\{\mathrm{diam}(M, g(t_1)), \mathrm{diam}(M, g(t_2))\}$$

□

6.3 Ricci Flow Through the Singularity as an Integral Current Space

Let $(M, g(t))$ be the Angenent-Knopf's example. At any time $t < T$, $(M, g(t))$ is a Riemannian manifold. As before, taking the current structure

$$T = \int_M \tag{6.43}$$

on M, one can think of $(M, g(t))$ as an integral current space.

It is well-known that any point p in any Riemmanian manifold M is asymptotically Euclidean hence

$$\Theta_*(p) = 1, \tag{6.44}$$

and as a result, $\mathrm{set}(M) = M$.

At the singular time $t = T$, the metric $g(T)$ is degenerate at the level set $\{0\} \times \mathbb{S}^n$ but nonetheless still gives rise to the distance metric d on the pinched sphere by minimizing the length $L(\gamma) = \int_\gamma (g(\gamma'(s), \gamma'(s))^{\frac{1}{2}}\, \mathrm{d}s$ along curves as usual. The pinched sphere $(M, g(T))$ is again an integral current space. One way to see this is that one observes that the pinched sphere is

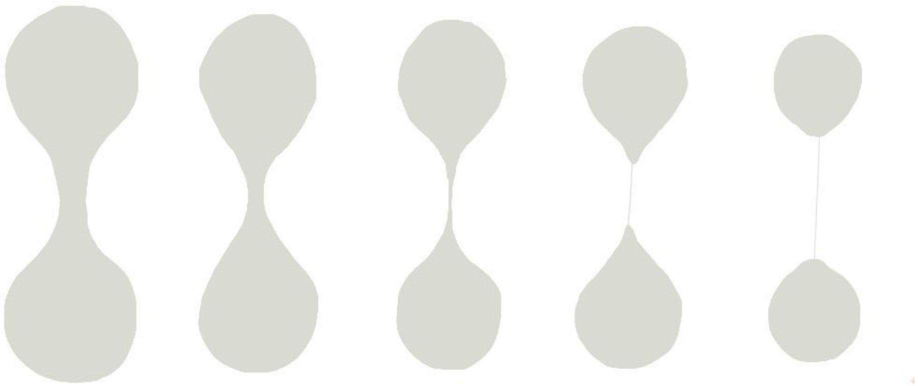

Figure 6.2: The Ambient Space for Angenent-Caputo-Knopf's Ricci Flow Through Neckpinch Singularity (Not *Settled*).

Credits: Illustration by Penelope Chang

a union of two C^1 - manifolds and the singular point which is of course of measure 0. See Figure 6.2.

The caveat here is that when considering the singular $(M, g(T))$ as an integral current space, by definition, we need to only consider the settled completion i.e. the points with positive density (See Figure 6.3). The Lemma 6.3.1 below computes the settled completion.

Lemma 6.3.1. *Let p be the singular point in the pinched sphere $(M, g(T))$*

then,

$$\mathrm{set}(M, g(T)) = M \setminus \{p\}. \tag{6.45}$$

Proof. According to [3, Table 1] (or Remark 5.4.2 in this thesis), at the singular time T, for the singular metric

$$g(T) = ds^2 + \psi_T(s) g_{can} \tag{6.46}$$

we have:

$$\psi_T(s) \sim s\,|\ln s|^{-\frac{1}{2}} \quad as \quad s \to 0 \tag{6.47}$$

therefore, one computes

$$\begin{aligned} \Theta_*(p) &= \liminf_{r\to 0} \frac{\mathrm{Vol}(B(p,r))}{r^{n+1}} \\ &\leq C \liminf_{r\to 0} \frac{\int_0^r s\left(s\,|\ln s|^{-\frac{1}{2}}\right)^n}{r^{n+1}} \\ &\leq C \liminf_{r\to 0} \frac{r^{n+1}\left(|\ln r|^{-\frac{1}{2}}\right)^n}{r^{n+1}} \\ &= C \liminf_{r\to 0} \left(|\ln r|^{-\frac{1}{2}}\right)^n \\ &= 0, \end{aligned} \tag{6.48}$$

which means that the singular point $p \notin \mathrm{set}(M, g(T))$.

For all the regular points $x \in M \setminus \{p\}$, we again have

$$\Theta_*(x) = 1. \tag{6.49}$$

This concludes the proof. □

At the post surgery times $t > T$, the Flow is a result of the smooth forward evolution as in [3] and hence consists of two separate smooth pointed

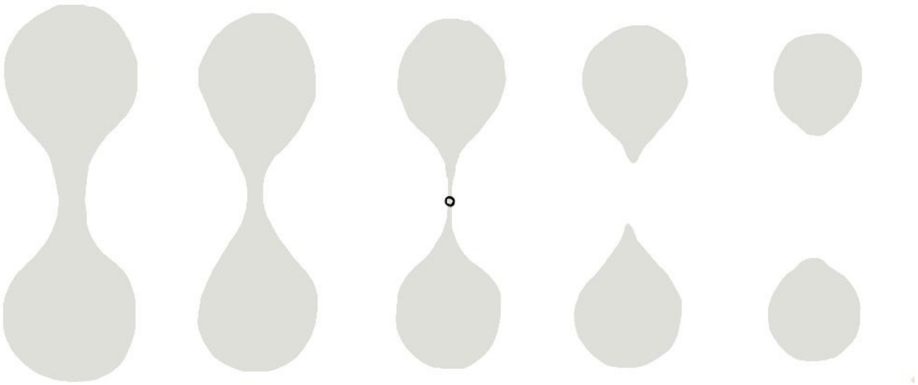

Figure 6.3: Angenent-Caputo-Knop's Ricci Flow Through Neckpinch Singularity Viewed as a *Settled* Flow of Integral Current Spaces.

Credits: Illustration by Penelope Chang

Ricci Flows $(M_i, g_i(T), p_i)$ $i = 1, 2$ obtained by regularizing the metric at the singular time. Points p_i are just the future of the singular point p. Again in order to work in the framework of integral current spaces, we first need to make the post surgery flow into a metric space and also define an appropriate current structure on it. One way to make a current space out of the disjoint union $M_1 \sqcup M_2$ as suggested by Knopf is to attach them by a thread. Another way is to define the metric using techniques from optimal transport which is described in the work of Author with Munn [34]. Here, we will focus on the thread approach.

We should clarify that the added thread joins p_1 to p_2, has length $L(t)$ which is continuous with respect to t and satisfies

$$\lim_{t \searrow T} L(t) = 0. \tag{6.50}$$

6.4 Continuity and Volume Convergnce as $t \nearrow T$

Consider the neckpinch on the sphere with bounded diameter. Our goal is to use Theorem 6.1.6 to find an estimate on the intrinsic flat distance between the sphere prior to the singular time and the pinched sphere.

Lemma 6.4.1. *Let $(M, g(t))$ be the Ricci flow on the $n+1$-sphere with a neckpinch singularity at time T. Then we have the following metric distortion estimate:*

$$\left| d_{(M,g(T))}(x_1, x_2) - d_{(M,g(t))}(x_1, x_2) \right| \leq C\sqrt{T-t}. \tag{6.51}$$

Proof. From the proof of Proposition 5.3.1, we have:

$$\left| \frac{d}{dt} d_{(M,g(t))}(x_1, x_2) \right| \leq \frac{C}{\sqrt{T-t}}. \tag{6.52}$$

Hence a simple integration shows that

$$\left| d_{(M,g(T))}(x_1, x_2) - d_{(M,g(t))}(x_1, x_2) \right| \leq C\sqrt{T-t}. \tag{6.53}$$

□

Lemma 6.4.2. *For the neckpinch on the $n+1$-sphere, we have:*

$$\lim_{t \nearrow T} d_{\mathcal{F}}\Big((M, g(t)), (M, g(T)) \Big) = 0. \tag{6.54}$$

Proof. Notice that From lemma 5.3.2, we know that the pinching occurs only at the equator given by $x = 0$. Let $M = \mathbb{S}^{n+1}$ and S be the $n+1$ sphere, the singular set $\{x = 0\}$ respectively. Let U_j be the exhaustion of $M \setminus S$ defined by:

$$U_j = \{(x, \theta) \in M : |x| \geq 1/j\}\,. \tag{6.55}$$

Each U_j consists of two connected components U_j^β , $\beta = 1, 2$. Fix j, then from Lemma 5.2.3, There is constant C_j depending on j and the solution $g(t)$ such that :

$$\sup_{U_j} \|\mathrm{Rm}\| \leq C_j. \tag{6.56}$$

Therefore by Proposition 5.1.2, we have:

$$e^{-\sqrt{n}C_j(t_2-t_1)}g(t_2)(V, V) \leq g(t_1)(V, V) \leq e^{\sqrt{n}C_j(t_2-t_1)}g(t_2)(V, V), \tag{6.57}$$

and letting $t_2 \nearrow T$, we get:

$$e^{-\sqrt{n}C_j(T-t)}g(T)(V, V) \leq g(t)(V, V) \leq e^{\sqrt{n}C_j(T-t)}g(T)(V, V), \tag{6.58}$$

where $V \in TU_j$.

Therefore, in the setting of the Theorem 6.1.6, we let $\epsilon_{tj} = e^{\sqrt{n}C_j(T-t)} - 1$. We also need to compute the distortion $\lambda_{t,j}$ between these two length spaces which is defined as:

$$\lambda_{tj} = \sup_{x,y\in U_j} \left| d_{(M,g(t))}(\psi_1(x), \psi_1(y)) - d_{(M,d_T)}(\psi_2(x), \psi_2(y)) \right|. \tag{6.59}$$

Let $x \in U_j^1$ and $y \in U_j^2$ then from Proposition 5.3.1, we obtain the following estimates on the distortion of distances which is independent of j; i.e.

$$\lambda_{tj} \leq C\sqrt{T - t}. \tag{6.60}$$

As in Theorem 6.1.6, let

$$h_{tj} = \sqrt{\lambda_{tj}(\max\{D_{U_j^1}, D_{U_j^2}\} + \lambda_{tj}/4)}\,, \tag{6.61}$$

and,

$$\bar{h}_{tj} = \max\{h_{tj}, \sqrt{\epsilon_{tj}^2 + 2\epsilon_{tj}}\; D_{U_j^1}, \sqrt{\epsilon_{tj}^2 + 2\epsilon_{tj}}\; D_{U_j^2}\}. \tag{6.62}$$

For fixed j, as $t \nearrow T$, we have:

$$\epsilon_{tj} \to 0 \;\text{ and }\; \lambda_{tj} \to 0. \tag{6.63}$$

Therefore for all j:

$$\lim_{t \nearrow T} d_{\mathcal{F}}\left((M, g(t)), (M, d_T)\right) \le \mathrm{Vol}_{g(t)}(M \setminus U_j) + \mathrm{Vol}_{g(T)}(M \setminus U_j), \tag{6.64}$$

Also since the diameter stays bounded as $t \nearrow T$, one sees that as $j \to \infty$,

$$\mathrm{Vol}_{g(t)}(M \setminus U_j) \;\text{ and }\; \mathrm{Vol}_{g(T)}(M \setminus U_j) \to 0. \tag{6.65}$$

Therefore,

$$\lim_{t \nearrow T} d_{\mathcal{F}}\left((M, g(t)), (M, g(T))\right) = 0. \tag{6.66}$$

□

Remark 6.4.3. *Since the diameter stays bounded as $t \nearrow T$, (6.65) implies the volume convergence*

$$\mathrm{Vol}\,(M, g(t)) \to \mathrm{Vol}\,(M, g(T)) \tag{6.67}$$

6.5 Continuity and Volume Convergence as $t \searrow T$

To complete the proof of the continuity of the Smooth Forward Evolution of the Ricci Flow out of neckpinch singularity, we need to also prove the continuity as the time approaches the singular time from the post surgery times. For simplicity we let $T = 0$ then, post surgery times will correspond to positive values of t.

Our flow at the positive time $t > 0$ consists of two pointed smooth Ricci flows $(M_1, g_1(t), p_1)$ and $(M_2, g_2(t), p_2)$ both modeled on the $n+1$-sphere and a thread of length $L(t)$ joining p_1 to p_2. As before, we let $(M, g(t))$ denote the pre-surgery Ricci flow and $(M, g(T))$ to be the singular space at the singular time T. Also we let $X = (M_1 \cup M_2, D(t), T(t))$ be the current space associated to the post surgery time t, where

$$D(t)(x,y) = \begin{cases} d_1(t)(x,y) & x, y \in M_1 \\ d_2(t)(x,y) & x, y \in M_2 \\ L(t) + d_1(t)(x,p_1) + d_2(t)(y,p_2) & x \in M_1 \;\; and \;\; y \in M_2, \end{cases}$$

where, $d_i(t)$ is the metric induced by the Riemannian metric $g_i(t)$ on M_i.

Lemma 6.5.1. *If $(M_i, g_i(t), p_i)$ $i = 1, 2$ represent the two parts of the post-surgery Ricci flow ($t > T = 0$) obtained by smooth forward evolution out of a neckpinch singularity and if $L(t)$ is the length of the thread joining p_1 and p_2 at time t with*

$$\lim_{t \searrow 0} L(t) = 0, \tag{6.68}$$

then, letting $X = M_1 \cup M_2$, we have:

$$\lim_{t \searrow 0} d_{\mathcal{F}}\Big((X, D(t), T(t)), (M, g(T)) \Big) = 0. \tag{6.69}$$

Proof. Similar to the proof of the continuity for pre-surgery times, we need to find proper diffeomorphic open subsets. For fixed small $\omega > 0$ consider the open subsets $\mathcal{N}^i_\omega \subset M_i$ for as defined in Section 5.4. Let U_1 be the open subset of M defined by:

$$U_1 = M \setminus \left(\bar{\mathcal{N}}^1_\omega \cup \setminus \bar{\mathcal{N}}^2_\omega\right), \tag{6.70}$$

therefore,U_1 is comprised of two connected components U_1^β for $\beta = 1, 2$.

And let U_2 be the open subset of $M_1 \cup L(t) \cup M_2$ defined as

$$U_2 = \left(M_1 \setminus \bar{\mathcal{N}}^1_\omega\right) \cup \left(M_2 \setminus \bar{\mathcal{N}}^2_\omega\right). \tag{6.71}$$

Then obviously, these two open sets are diffeomorphic through diffeomorphisms between their correspondng connected components:

$$\psi_1 : W_1 \to M_1 \setminus \bar{\mathcal{N}}^1_\omega \tag{6.72}$$

$$\psi_2 : W_2 \to M_2 \setminus \bar{\mathcal{N}}^2_\omega. \tag{6.73}$$

Now let $\epsilon(t)$ be the smallest positive number for which

$$\psi_i^* g_i(t)(V, V) < (1 + \epsilon(t))^2 g(T)(V, V) \qquad \forall V \in TW_i \tag{6.74}$$

and

$$\psi_i^* g_i(t)(V, V) < (1 + \epsilon(t))^2 g(T)(V, V) \qquad \forall V \in TW_i. \tag{6.75}$$

then, by the construction of the Smooth Forward Evolution as seen in Section 5.4, as $t \searrow 0$, the metrics $\psi_i^* g_i(t)$ smoothly converge to $g(T)$ on W_i therefore,

$$\lim_{t \searrow 0} \epsilon(t) = 0. \tag{6.76}$$

Now let $\omega_j > 0$ be a sequence for which

$$\lim_{j \to \infty} \omega_j = 0 \tag{6.77}$$

and consider the length distortions:

$$\lambda_{tj} = \sup_{x,y\in U_1} \left| d_{(X,D(t))}(\psi_1(x),\psi_1(y)) - d_{(M,d_T)}(x,y) \right| . \tag{6.78}$$

Then,

$$\lambda_{tj} \leq L(t) + \Big((1+\epsilon(t))^2 - 1\Big)\Big(\mathrm{diam}\,(M_1, g_1(t)) + \mathrm{diam}\,(M_2, g_2(t))\Big) \tag{6.79}$$

As in Theorem 3.1.1, we define

$$h_{tj} = \sqrt{\lambda_{tj}(\max\{D_{U_j^1}, D_{U_j^2}\} + \lambda_{tj}/4)}\,, \tag{6.80}$$

and,

$$\bar{h}_{tj} = \max\{h_{tj}, \sqrt{\epsilon_{tj}^2 + 2\epsilon_{tj}}\; D_{U_j^1}, \sqrt{\epsilon_{tj}^2 + 2\epsilon_{tj}}\; D_{U_j^2}\}. \tag{6.81}$$

For fixed j, as $t \searrow T$, we have:

$$\epsilon_{tj} \to 0 \;\text{ and }\; \lambda_{tj} \to 0. \tag{6.82}$$

Therefore for all j:

$$\lim_{t\searrow T} d_{\mathcal{F}}\left((X, D(t)), (M, g(T)\right) \leq \mathrm{Vol}_{g(t)}(X \setminus U_2) + \mathrm{Vol}_{g(T)}(M \setminus U_1), \tag{6.83}$$

Also since the diameter stays bounded as $t \searrow T$, one sees that as $j \to \infty$,

$$\mathrm{Vol}_{g(T)}(M \setminus U_1) \;\text{ and }\; \mathrm{Vol}_{g(t)}(X \setminus U_2) \to 0. \tag{6.84}$$

Therefore,

$$\lim_{t\searrow T} d_{\mathcal{F}}\Big(\,(X, D(t))\,, (M, g(T))\,\Big) = 0. \tag{6.85}$$

□

Remark 6.5.2. *Notice that since the diameter is bounded as $t \searrow T$, (6.84) gives*

$$\mathrm{Vol}\,(M_1, g_1(t)) + \mathrm{Vol}\,(M_2, g_2(t)) \to \mathrm{Vol}\,(M, g(T)) \tag{6.86}$$

By applying Theorem 6.1.6, we can also find an estimate on the Intrinsic Flat distance between two post surgery integral current spaces at times $0 < t_1 < t_2$.

Theorem 6.5.3. *Suppose* $(X, D(t), T(t))$ *is as before, and the smooth flows* $(M_1, g_1(t))$ *and* $(M_1, g_1(t))$ *do not encounter singularities on* $(0, T)$*, then as* $t \to t_0 \in (0, T)$*, we have*

$$\lim_{t\to t_0} d_{\mathcal{F}}\Big((X, D(t)), (X, D(t)) \Big) = 0. \tag{6.87}$$

Proof. Since the post surgery flows do not encounter any other singularity on $(0, T)$, we have

$$\sup_{M_i\times[t_0-\delta,t_0+\delta]} \|\mathrm{Rm}\| \le C = C(\delta) \tag{6.88}$$

with respect to a fixed background metric g_0 on $M_1 \sqcup M_2$.

Let $\epsilon(t) = \sqrt{e^{\sqrt{n}C|t-t_0|}} - 1$ and let $\omega_j > 0$ be a sequence with

$$\lim_{j\to\infty} \omega_j = 0. \tag{6.89}$$

Define U_i as in Lemma 6.5.1. Then, we have the following estimate on the metric distortion

$$\begin{aligned} \lambda_{tj} &= \sup_{x,y\in U_1} \left| d_{(X,D(t_0))}(x,y) - d_{(X,D(t))}(x,y) \right| \\ &\le |L(t) - L(t_0)| \\ &\quad + \Big((1+\epsilon(t))^2 - 1 \Big) \Big(\mathrm{diam}\,(M_1, g_1(t)) + \mathrm{diam}\,(M_2, g_2(t)) \Big) \end{aligned} \tag{6.90}$$

We observe that as $t \to t_0$, $\epsilon(t) \to 0$ and $L(t) \to L(t_0)$ due to continuity therefore, $\lambda_{tj} \to 0$. The rest of the proof is the same as in Lemma 6.5.1.

□

Chapter 7

Future Directions

Here I will briefly mention the possible future directions for my postdoctoral research.

7.1 Reduced Distance as $t \searrow T$ and Length of the Thread

In [32], I proved the continuity of the Ricci flow through rotationally symmetric neckpinch singularity (see Theorem 1.2.1). We joined the two smooth Ricci flows at post surgery times by a thread of length $L(t)$. Now, one may ask what $L(t)$ should be? One possible candidate is to use Perelman's reduced distance as a measure of the space-time length. Since the scalar curvature is assumed to be positive through a rotationally symmetric neckpinch singularity, the resulting reduced distance will also be positive which is what we expect from a proper notion of length. One issue in this approach is that the reduced distance is defined on a regular portion of the space-time.

Enders in [17] generalizes the notion of reduced length up to the singular time T. In order to use the reduced length for measuring distances through the neckpinch singularity, we need to also define the reduced distance from the singular time T to any post surgery time $t > T$. This is being currently investigated by the author.

7.2 Intrinsic Flat Convergence into the General Neckpinch Singularity

For a general neckpinch singularity without rotational symmetry, a canonical Ricci flow with surgery is still missing (if it exists). One can still talk about the continuity of the flow into the singular time. One of my possible future direction of research is to investigate the following question:

Question 1. *Suppose* $(M, g(t))$ *flows into a neck pinch singularity (not necessarily rotationally symmetric) that is a point or a finite interval then is there intrinsic flat convergence into the singular time? And what happens if we remove the diameter bound?*

Although it is not clear what the weak flow at post surgery times should be, it is natural to think that a canonical Ricci flow through singularities must be a limit of a sequence of Ricci flows with surgery at scale s_i as $s_i \to 0$. One might ask the following question:

Question 2. *Is the limit of this hypothetical sequence continuous with respect to the Intrinsic Flat distance?*

We hope to provide an affirmative answer to the questions 1 and 2.

7.3 Size of the Neckpinch Singularity and Optimal Transport

In [4] and [6], Angenent-Knopf prove that if $(M, g(t))$ is a rotationally and reflection symmetric Ricci flow with one neck that developes a neckpinch singularity in finite time, then the neckpinch happens at only one point. They first prove that if the diameter stays bounded, then there is only one point pinching and then they prove that the diameter actually stays bounded.

Currently, I am exploring the application of the Topping-McCann's definition of weak Ricci flow to the rotationally symmetric neckpinch in order to generalize the *only one point pinching* result of Angenent-Knopf to the cases without reflection symmetry or even to a general neckpinch.

7.4 Kähler Ricci Flow and More

Understanding and finding Kähler-Einstein metrics (when they exist) has been the center of many important research works in the past few decades. In many cases, Ricci flow on Kähler manifolds (or the corresponding Kähler-Ricci flow) has proven to be a proper means of deforming Kähler metrics to a Kähler-Einstein metric on Kähler-Einstein manifolds (for example see [48] and references therein). Also there is a notion of Kähler Ricci flow through singularities established by Song-Tian in [44]. There are many directions to be explored in this context especially regarding the degeneration of Kähler Ricci Flow and the geometry of the singular limit. In this setting, one general question is to investiate the continuity of these flows with respect to different notions of convergence; we are hoping to be able to answer these questions using methods similar to what has been presented in this thesis.

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