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HSU, Michael Bing-sun, 1933-
FRAMED DOME DYNAMICS AND LOWER BOUND TO THE FUNDAMENTAL FREQUENCY OF STRUCTURAL SYSTEMS.

The City University of New York, Ph.D., 1970
Engineering, civil

University Microfilms, Inc., Ann Arbor, Michigan

FRAMED DOME DYNAMICS AND LOWER BOUND TO THE FUNDAMENTAL FREQUENCY OF STRUCTURAL SYSTEMS

BY

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A dissertation submitted to the
Graduate Faculty in Engineering in partial
fulfillment of the requirements for the
degree of Doctor of Philosophy, The City
University of New York.

1969

This manuscript has been read and accepted for the Graduate Faculty in Engineering in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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ABSTRACT

A new method for obtaining a lower bound to the smallest eigenvalue of a real, finite, linear system is developed. The computation of this lower bound involves only elementary operations on the elements of the coefficient matrix of the system. The method is used for finding a lower bound to the fundamental frequency of structural systems.

Matrix formulation of dynamic analysis has been carried out for structural systems composed of one dimensional elements. The formulation is valid for both lumped parameter and distributed parameter analyses.

The free vibrations of a dome-type frame structure are studied. The analysis assumes that the masses of the member elements of the dome are lumped at the joints of the dome. Both natural frequencies and mode shapes of a 60 degrees-of-freedom dome are studied.

A dynamic experiment of the framed dome is designed and performed in order to study the accuracy of natural frequencies obtained from lumped parameter analysis. Within the limitation of the experimental accuracy, the analytical and experimental results are in good agreement.

ACKNOWLEDGEMENTS

The author wishes to thank Professor M. L. Pei, his advisor, for his help and advice to prepare this dissertation.

He also wishes to thank Professors H. Appelgate, J. E. Benveniste, N. C. Jen and S. B. Menkes, members of his doctoral guidance committee, for their many suggestions at critical junctures of this work.

PREFACE

Natural frequency is the most basic information in time domain dynamic analysis of an elastic structural system. Its primary importance has long been recognized. It is known that to study this type of problem one has to derive the equations of motion of the system and seek the characteristic values which yield homogeneous solutions for the equations. Generally, the natural frequencies of the system can not be obtained directly from that of member elements in the system. Karnopp(15) divided a vibrating system into subsystems and hoped to learn the behavior of the complex system from the behavior of the simpler subsystem. He found that even when the natural frequencies of all the subsystems were the same, it was not always possible to derive the natural frequencies of the coupled system.

In the case of a linearstructural system composed of one-dimensional member elements, the equations of motion consist of a set of simultaneous differential equations. Depending upon the mass distribution of member elements, these may be ordinary or partial differential equations, or a combination of both. By specifying unknown displacements at a finite number of discrete points in the system (5,14,19,21), one obtains a set of finite simultaneous homogeneous algebraic equations in terms of unknown displacements. The free vibration frequency equation of the system can be obtained by setting the determinant of coefficient matrix to zero. Many aspects of this characteristic-value problem have been very extensively studied and reported. If the frequency equation is given in an explicit form, the roots of the equation can be obtained by existing methods. However, in most of the cases this is not possible, and various iterative techniques(7,9,16) must be used to locate the roots of the coefficient matrix. The amount of computation required is quite large when the order of the matrix is fairly large. In this dissertation a new method to obtain a lower bound to the root of the characteristic equation is presented. The method involves only elementary operations on the elements of the coefficient matrix of the system.

Bazley(2), Gould(10), Weinberger(28), Wing(29) and many other authors discussed the bounds to eigenvalues of an integral or a differential equation in great detail. A great number of results are reviewed by Marcus(18). Bellman(3) and Taussky(24) also gave summaries on this subject. Schneider(22) edited the more recent work by Brauer, Householder and Taussky. Fan and Hoffman(8) extended the study to find lower bounds for the rank of a matrix. Hoffman(12) established the conditions for the determinant of a real matrix to be positive or negative. He also obtained both lower and upper bounds to the roots of the real matrix. Shih and Wang(23) gave bounds for both determinant and roots for certain matrices. Washizu(27) discussed bounds to roots of a real symmetric matrix and gave a geometric interpretation to several theorems which provide bounds for eigenvalues. Literatures on the bounds for determinants (4,11,20) are also very extensive. However, a review of all these methods shows that in most cases the matrix must have very dominant main diagonal elements and the bounds so obtained are either trivial or very far away from the exact values.

This dissertation consists of four chapters. In chapter one, a method of obtaining a lower bound to the smallest eigenvalue of a real, finite, linear system is developed. Matrix formulation of dynamic analysis has been carried out in chapter two for structural systems composed of one dimensional elements. The study of free vibrations of a dome-type frame structure is included in chapter three. In chapter four, a dynamic experiment of the framed dome is designed and performed. The experimental results are used for verifying the analytical solution.

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CHAPTER 1 LOWER BOUND TO THE FUNDAMENTAL FREQUENCY

1.0 INTRODUCTION

The exact eigenvalue of the eigenvalue problem associated with physical systems are frequently not easily obtainable. It is therefore useful to have a procedure for finding upper and lower bound to the eigenvalues. Based on localization theorems and matrix inequalities, a new method for obtaining a lower bound to the smallest eigenvalue of linear discrete systems is developed in this chapter. The computation of this lower bound involves only elementary operations on the elements of the coefficient matrix of the system.

1.1 LOWER BOUND CRITERION

Consider a real, finite, linear, homogeneous system of parameter ω and unknown vector $\{X\}$

$$\left(L(\omega)\right)\left\{X\right\} = 0 \qquad (1.1\text{-}1)$$

with the following assumptions:

1. The parameter ω is real, continuous and positive. $\omega \geq 0$.
2. $L(\omega)$ is an n-square real, symmetric matrix
3. $L(0)$, the matrix $L(\omega)$ evaluated at $\omega = 0$, is positive definite.

The characteristic equation of the system is

$$\det \left| L(\omega) \right| = 0. \qquad (1.1\text{-}2)$$

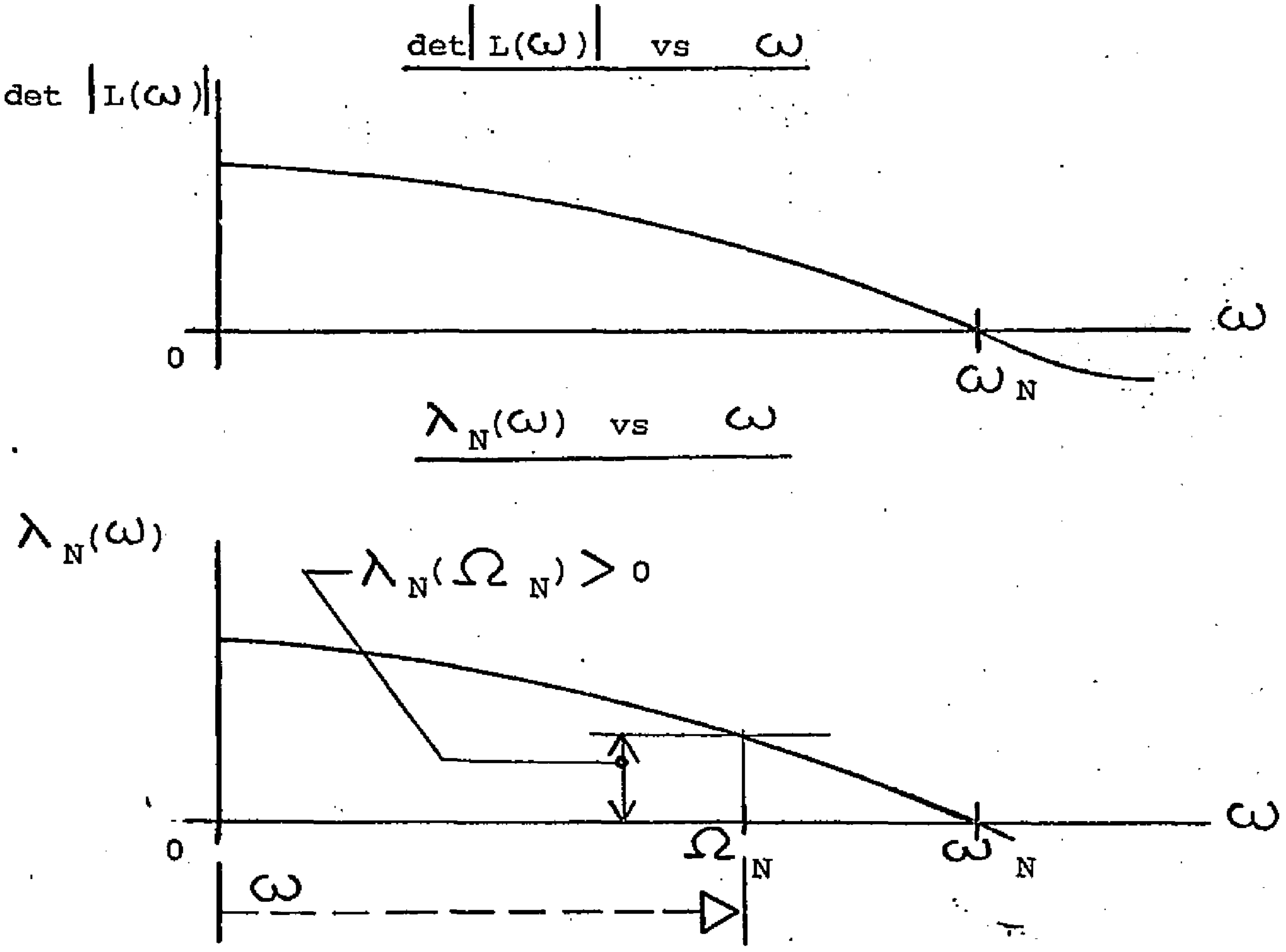
Let the smallest root of the characteristic equation (1.1-2) be ω_N ; the characteristic roots of the matrix $L(\omega)$ and the matrix $L(0)$ be $\lambda_1(\omega) \geq \lambda_2(\omega) \geq \cdots \geq \lambda_N(\omega)$ and $\lambda_1(0) \geq \lambda_2(0) \geq \ldots. \geq \lambda_N(0)$ respectively. Because the matrix $L(0)$ is positive definite and ω_N is a root of equation (1.1-2), we have

$$\lambda_N(0) > 0 \qquad (1.1\text{-}3)$$

$$\lambda_N(\omega_N) = 0. \qquad (1.1\text{-}4)$$

The function $\lambda_N(\omega)$ is continuous for all $\omega \geq 0$. Consequently, if ω is continuously increased from $\omega = 0$ and the function $\lambda_N(\omega)$ is such that $\lambda_N(\omega) > 0$ for $0 \leq \omega \leq \Omega_N$, then the value of Ω_N is bounded above, $0 \leq \Omega_N < \omega_N$.

This is a criterion for obtaining a lower bound, Ω_N, to the smallest root, ω_N, of equation (1.1-2). A graphic representation of this criterion is shown in Fig. 1.1.



det $|L(\omega)|$ vs ω

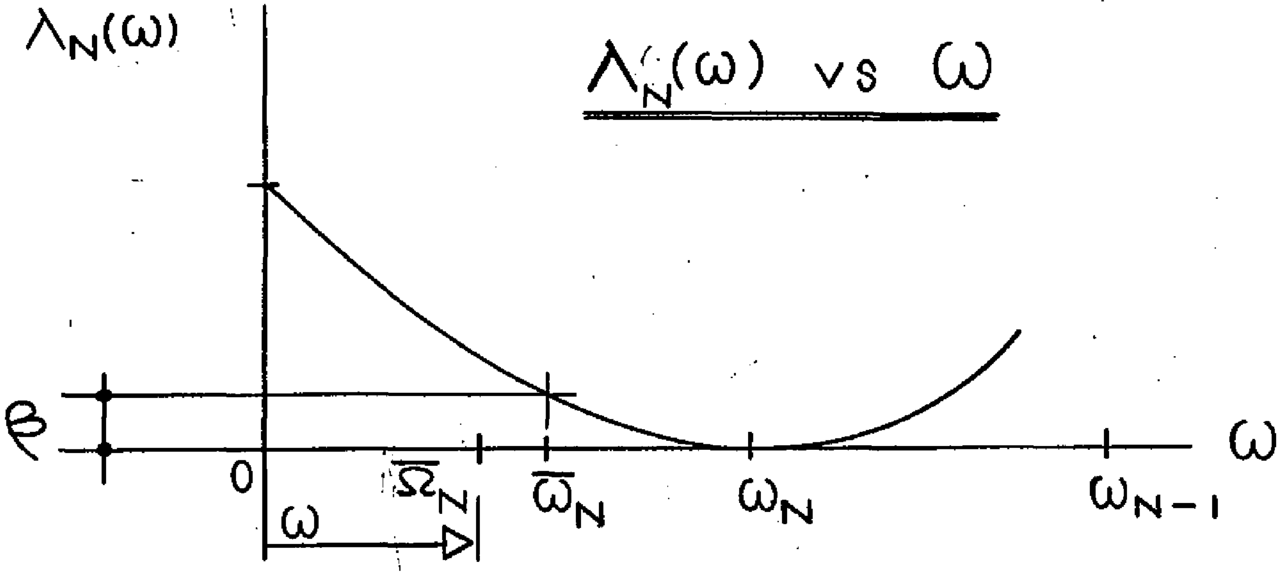
$\lambda_N(\omega)$ vs ω

ω_N: Smallest root of the matrix $L(\omega)$.
Ω_N: A lower bound to ω_N.

Fig.1.1 Graphic Representation of Lower Bound Criterion.

In the above discussion, it is assumed that the value of $\lambda_N(\omega)$ is negative for $\omega_N < \omega < \omega_{N-1}$ (see Fig. 1.1). If ω_N is a single root, this is assured. If $\lambda_N(\omega) > 0$ for $\omega_N < \omega < \omega_{N-1}$, then the criterion for obtaining a lower bound to the smallest root of equation (1.1-2) is modified as follows.

Continuously increase ω from $\omega = 0$. If $\lambda_N(\omega) - \beta > 0$ (where β is a small positive number) for $0 \leq \omega \leq \bar{\Omega}_N$, then the value of $\bar{\Omega}_N$ is bounded above, $0 \leq \bar{\Omega}_N < \bar{\omega}_N < \omega_N$.



In the following discussions, $\lambda_N(\omega)$ is assumed to be less than zero in the interval $\omega_N < \omega < \omega_{N-1}$.

1.2 DEVELOPMENT OF THE METHOD.

Consider the system of equations (1.1-1)

$$\left(L(\omega)\right)\left\{X\right\} = 0. \tag{1.1-1}$$

Let the real, positive parameter ω be expressed as the product of two positive real numbers q and ϵ.

$$\omega = q\,\epsilon. \tag{1.2-1}$$

The values of q and ϵ are restricted.

$$\begin{aligned} q &\geq 0, \\ 0 < \epsilon &\leq 1. \end{aligned} \tag{1.2-2}$$

Define a new matrix $E(q,\epsilon)$ as follows

$$E(q,\epsilon) = \left(1-\frac{1}{\epsilon}\right)L(0) + \frac{1}{\epsilon}L(q\epsilon). \tag{1.2-3}$$

The coefficient matrix $L(\omega)$ may be rewritten as

$$\begin{aligned} L(\omega) &= \epsilon\left[\left(1-\frac{1}{\epsilon}\right)L(0) + \frac{1}{\epsilon}L(\omega)\right] + (1-\epsilon)L(0) \\ &= \epsilon\,E(q,\epsilon) + (1-\epsilon)L(0). \end{aligned} \tag{1.2-4}$$

Let the characteristic roots be:

$\lambda_1(\omega) \geq \lambda_2(\omega) \geq \cdots \geq \lambda_N(\omega)$ for matrix $L(\omega)$

$\lambda_1(0) \geq \lambda_2(0) \geq \cdots \geq \lambda_N(0)$ for matrix $L(0)$

and $e_1(q,\epsilon) \geq e_2(q,\epsilon) \geq \cdots \geq e_N(q,\epsilon)$ for matrix $E(q,\epsilon)$.

The characteristic roots $e_i(q,\epsilon)$ are functions of q and ϵ.

1.2-1 Using Fan's theorem (Appendix 1, Theorem 2), we have

$$\lambda_N(\omega) + \lambda_{N-1}(\omega) + \cdots + \lambda_k(\omega) \geq$$

$$\epsilon\left[e_N + e_{N-1} + \cdots + e_k\right] + (1-\epsilon)\left[\lambda_N(0) + \cdots + \lambda_k(0)\right]$$

$$K = 1, \cdots, N. \qquad (1.2\text{-}5)$$

Since $\lambda_i(\omega)$ $(i = 1,\ldots,N)$ are arranged in descending order, it follows immediately that

$$(N-k+1)\quad \lambda_k(\omega) > \epsilon\left[e_N+e_{N-1}+\ldots+e_k\right] + (1-\epsilon)\left[\lambda_N(0)+\ldots+\lambda_k(0)\right]$$

$$k=1,\ldots,N, \qquad (1.2\text{-}6)$$

(1.2-6) yields the following inequality:

$$\lambda_k(\omega) > 0 \quad \text{if}$$

$$\left[e_N+e_{N-1}+\ldots+e_k\right] > -\frac{(1-\epsilon)}{\epsilon}\left[\lambda_N(0)+\lambda_{N-1}(0)+\cdots+\lambda_k(0)\right]$$

$$k=1,\ldots,N, \qquad (1.2\text{-}7)$$

Let k=N in (1.2-7). We have

$$\lambda_N(\omega) > 0$$

if $$e_N(q,\epsilon) > -\frac{(1-\epsilon)}{\epsilon}\lambda_N(0), \qquad (1.2\text{-}8)$$

Let $\alpha(q,\epsilon)$ be a lower bound to e_N. We have

$$e_N > \alpha(q,\epsilon), \qquad (1.2\text{-}9)$$

Substitute (1.2-9) into (1.2-8)

$$\lambda_N(\omega) > 0 \quad \text{if}$$

$$\alpha(q,\epsilon) > -\frac{(1-\epsilon)}{\epsilon}\left[\lambda_N(0)\right] \qquad (1.2\text{-}10)$$

or

$$\lambda_N(\omega) > 0 \quad \text{if}$$

$$\alpha(q,\epsilon) + \frac{(1-\epsilon)}{\epsilon}\lambda_N(0) > 0 \qquad (1.2\text{-}11)$$

Consider the case where $0 \leq \omega \leq \Omega_N$. If the inequality (1.2-11) is valid, then the criterion described in section 1.1 is satisfied and Ω_N is proved to be alower bound.

$$\Omega_N < \omega_N \qquad (1.2\text{-}12)$$

<u>1.2-2</u> Let C be a real number and

$$C > \frac{1}{\lambda_N(0)} \qquad (1.2\text{-}13)$$

Multiplying (1.2-4) by C and adding an identity matrix I to both sides of the equation, we have

$$[I + C\,L(\omega)] = [I + C\,\epsilon E(q,\epsilon)] + C(1-\epsilon)L(0). \qquad (1.2\text{-}14)$$

Note that both $\left[I + C\epsilon E(q,\epsilon)\right]$ and $\left[C\,(1-\epsilon)L(0)\right]$ are symmetric matrices. The application of Theorem 3 (Appendix 1) to (1.2-14) yields the following result:

If the matrix $\left[I + C\epsilon E(q,\epsilon)\right]$ is positive definite then,

$$1 + C\lambda_k(\omega) > C(1-\epsilon)\lambda_k(0) \qquad (1.2\text{-}15)$$

$k=1,\ldots,N$.

For k=N we have

$c\,\lambda_N(\omega) > -1 + c(1-\epsilon)\,\lambda_N(0)$, if the matrix $\left[I + C\epsilon E(q,\epsilon)\right]$ is positive definite.

(1.2-16)

The condition of $\left[I + C\epsilon E(q,\epsilon)\right]$ being positive definite can be obtained from the following:

Let $\alpha^*(q,\epsilon;C)$ be a lower bound to the smallest root of the real symmetric matrix $\left[I + C\epsilon E(q,\epsilon)\right]$

$$\alpha^*(q,\epsilon;C) < 1 + C\epsilon e_N(q,\epsilon). \qquad (1.2\text{-}17)$$

Obviously, if $\alpha^*(q,\epsilon;C) > 0$ then, the matrix $\left[I + C\epsilon E(q,\epsilon)\right]$ is positive definite. Therefore, from (1.2-16) and (1.2-17) we obtain

$$\lambda_N(\omega) > 0 \quad , \text{if } \alpha^*(q,\epsilon;C) > 0 \qquad (1.2\text{-}18a)$$

$$q > 0 \qquad (1.2\text{-}18b)$$

$$0 < \epsilon < 1 - \frac{1}{C\lambda_N(0)} \qquad (1.2\text{-}18c)$$

Note that if the smallest root of L(0) is greater than one:

$$\lambda_N(0) > 1 \qquad (1.2\text{-}19)$$

then, we can choose $C = 1$ and obtain the following inequalities:

$$\lambda_N(\omega) > 0 \quad , \text{if } \alpha^*(q,\epsilon) > 0 \qquad (1.2\text{-}20a)$$

$$q > 0 \qquad (1.2\text{-}20b)$$

$$0 < \epsilon < 1 - \frac{1}{\lambda_N(0)} \qquad (1.2\text{-}20c)$$

where $\alpha^*(q,\epsilon)$ is a lower bound to the smallest root of the matrix $\left[I + \epsilon E(q,\epsilon)\right]$ and is a function of q and ϵ only.

Thus we may conclude that if the values of q, ϵ and C are such that the inequalities (1.2-18a) through (1.2-18C) (or, (1.2-20a) through (1.2-20C) in the case of $\lambda_N(0) > 1$) are satisfied, then the product is a lower bound to the smallest root of $L(\omega)$.

1.2 - 3

Let $\lambda_1(\omega) \geq \lambda_2(\omega) \geq \cdots \geq \lambda_N(\omega)$ be N characteristic roots of the coefficient matrix $L(\omega)$ in (1.1-1) and $\alpha(\omega)$ be a lower bound to the smallest characteristic root $\lambda_N(\omega)$. We have

$$\alpha(\omega) < \lambda_N(\omega) \qquad \omega \geq 0. \tag{1.2-21}$$

The lower bound $\alpha(\omega)$ can be obtained from (1.2-22) (Appendix 1, Theorem 1)

$$\alpha(\omega) = \underset{i}{\mathrm{MIN}} \left(l_{ii}(\omega) - \sum_{\substack{j=1 \\ j \neq i}}^{N} \left| l_{ij}(\omega) \right| \right)$$

$$\omega \geq 0$$

$$i = 1, \cdots, N. \tag{1.2-22}$$

if $\omega = 0$,

$$\alpha(0) = \underset{i}{\mathrm{MIN}} \left(l_{ii}(0) - \sum_{\substack{j=1 \\ j \neq i}}^{N} \left| l_{ij}(0) \right| \right)$$

$$i = 1, \cdots, N. \tag{1.2-23}$$

The lower bound $\alpha(0)$ for roots of the matrix L(0) can be positive, negative, or equal to zero. It can be shown from the following discussion that if $\alpha(0) > 0$, that a lower bound for ω_N (the smallest root of the characteristic equation (1.1-2)) can be obtained.

Since ω_N is the smallest root of (1.1-2)

$$\det \left| L(\omega_N) \right| = 0 \qquad (1.2\text{-}24)$$

But

$$\det \left| L(\omega_N) \right| = \lambda_N(\omega_N) . \lambda_{N-1}(\omega_N) \ldots \lambda_1(\omega_N)$$

we obtain

$$\lambda_N(\omega_N) = 0 \qquad (1.2\text{-}25)$$

From (1.2-21), we have

$$\alpha(\omega_N) < 0. \qquad (1.2\text{-}26)$$

If $$\alpha(0) > 0 \qquad (1.2\text{-}27)$$

then, from (1.2-26) and (1.2-27) we conclude that there must exist at least a value in the interval $0 \leq \omega \leq \omega_N$, say Ω_N, such that

$$\alpha(\Omega_N) = 0 \qquad (1.2\text{-}28)$$

Therefore, in order to find a lower bound to ω_N, we can compute $\alpha(\omega)$, for values of ω increasing continuously from $\omega=0$. We repeat this computation so long as the value of $\alpha(\omega) > 0$. If we find a $\omega=\Omega_N$, such that

$$\begin{aligned} &\alpha(\Omega_N - \Delta) > 0 \\ &\alpha(\Omega_N + \Delta) < 0 \\ &|\Delta| \ll 1 \end{aligned} \tag{1.2-29}$$

then, the number Ω_N is a lower bound to ω_N.

Note that if $\alpha(0)=0$, we obtain a trivial lower bound for ω_N, i.e. $0 < \omega_N$.

If $\alpha(0) < 0$, we can not use this method for obtaining a lower bound to ω_N.

1.2 - 4 Special Case:

Assume that the coefficient matrix $L(\omega)$ in (1.1-1) can be expressed as

$$L(\omega) = A - \omega B \qquad (1.2\text{-}30)$$
$$\omega \geq 0$$

where A and B are real symmetric matrices and A is positive definite. Let us, as we did before, express the parameter ω as the product of a pair of real numbers q and $\in$, $q > 0$, $0 < \in \leq 1$. We obtain

$$L(\omega) = A - (q\in)B. \qquad (1.2\text{-}31)$$

Define a new matrix $D(q)$, $q > 0$

$$D(q) = A - qB, \qquad (1.2\text{-}32)$$

and rewrite (1.2-31)

$$\begin{aligned} L(\omega) &= \in \left[A-qB\right] + (1-\in)A \\ &= \in D(q) + (1-\in)A. \qquad (1.2\text{-}33) \\ 0 &< \in \leq 1 \end{aligned}$$

Let the characteristic roots be

$$\begin{array}{lllllll} \lambda_1(\omega) \geq & \lambda_2(\omega) \geq & \ldots & \geq \lambda_N(\omega) & \text{for } L(\omega) \\ a_1 \geq & a_2 \geq & \ldots & \geq a_N & \text{for } A \\ d_1 \geq & d_2 \geq & \ldots & \geq d_N & \text{for } D(q) \end{array}$$

the characteristic roots $d_i = d_i(q)$ are functions of q. Using Fan's theorem (Appendix 1, Theorem 2) it can be shown that

$$\lambda_N(\omega) + \lambda_{N-1}(\omega) + \ldots + \lambda_k(\omega) \geq \epsilon\left[d_N + d_{N-1} + \ldots + d_k\right] + (1-\epsilon)\left[a_N + a_{N-1} + \ldots + a_k\right]$$

$$k=1,\ldots,N, \tag{1.2-34}$$

or

$$(N-k+1)\lambda_k(\omega) > \epsilon\left[d_N + d_{N-1} + \ldots + d_k\right] + (1-\epsilon)\left[a_N + a_{N-1} + \ldots + a_k\right]$$

$$k=1,\ldots,N, \tag{1.2-35}$$

and

$$\lambda_k(\omega) = \lambda_k(q\epsilon) > 0 \quad \text{if}$$

$$(d_k + d_{N-1} + \ldots + d_k) > -\frac{(1-\epsilon)}{\epsilon}(a_N + a_{N-1} + \ldots + a_k)$$

$$k=1,\ldots,N, \tag{1.2-36}$$

Let k=N in (1.2-36). We have

$$\lambda_N(q\epsilon) > 0 \quad \text{if}$$

$$d_N > -\frac{(1-\epsilon)}{\epsilon} a_N \tag{1.2-37}$$

$$0 < \epsilon \leq 1$$

Let $\alpha(q)$ be a lower bound to d_N and we obtain the following inequalities

$$\lambda_N(q\epsilon) > 0 \qquad \text{if}$$

$$\alpha(q) > -\frac{(1-\epsilon)}{\epsilon}(a_N) \tag{1.2-38}$$

$$0 < \epsilon \leq 1$$

or

$$\lambda_N(q\epsilon) > 0 \qquad \text{if}$$

$$\alpha(q) + \frac{(1-\epsilon)}{\epsilon} a_N > 0 \tag{1.2-39}$$

$$0 < \epsilon \leq 1$$

Using the similar argument, if the value of q and ϵ are such that the second inequality in (1.2-39)holds, then, the product $(q\epsilon)$ is a lower bound of the smallest root of the coefficient matrix $L(\omega)$.

1.3 DISCUSSION OF THE METHOD

In the previous section we developed a method of obtaining a lower bound to the smallest root of the characteristic equation (1.1-2). Four versions of the method are given in this section. Step by step descriptions, the applications, restrictions and extensions of the method are also given.

1.3-1 Description of the method (Version 1)

Step 1:	Given an n-square matrix $L(\omega)$	$L(\omega)=l_{ij}(\omega)$
Step 2:	Properly choose two real numbers q and ϵ	$q\ ;\ \epsilon$ $q > 0;\ 0 < \epsilon \leq 1$
Step 3:	Evaluate the matrix $L(\omega)$ at $\omega=0$	$L(0)$
Step 4:	Compute the smallest root of the matrix $L(0)$	$\lambda_N(0)$
Step 5:	Formulate a new matrix $E(q, \epsilon)$	$E(q, \epsilon)$ See (1.2-3)
Step 6:	Compute a lower bound to the smallest root of the matrix $E(q, \epsilon)$	$\alpha(q, \epsilon)$
Step 7:	Compute a lower bound, Ω_N, to the smallest root of the given matrix $L(\omega)$	So long as the following inequality holds $\lvert\alpha(q,\epsilon)\rvert < \left\lvert\frac{(1-\epsilon)}{\epsilon}\lambda_N(0)\right\rvert$ the product of q and ϵ is a lower bound of the smallest root of the given matrix $L(\omega)$. i.e. $\Omega_N = q\epsilon$

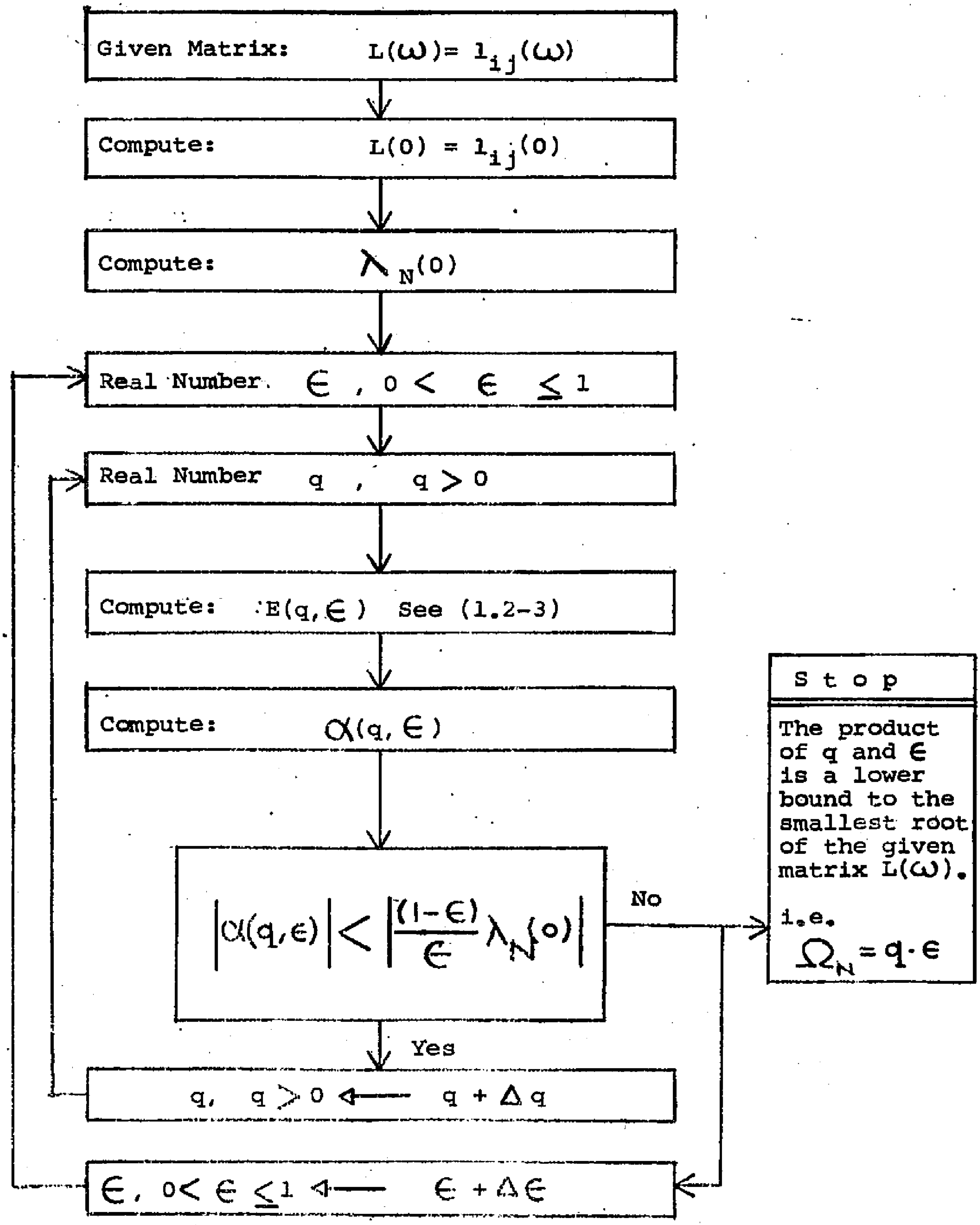


Fig.1.2 Flowchart Diagram of the Method (Version 1)

1.3 - 2 Description of the method (version 2)

Step 1:	Given an n-square matrix $L(\omega)$	$L(\omega) = l_{ij}(\omega)$
Step 2:	Evaluate the matrix $L(\omega)$ at $\omega = 0$	$L(0)$
Step 3:	Compute the smallest root of the matrix $L(0)$	$\lambda_N(0)$
Step 4:	Choose a proper real number C and compute $c\lambda_N(0)$ NOTE: if $\lambda_N(0) \leq 1$, the value of C is such that $c\lambda_N(0) > 1$ if $\lambda_N(0) > 1$, the value of C is 1.	$c\,\lambda_N(0)$
Step 5:	Properly choose two real numbers q and ϵ.	$q\ ,\ \epsilon$ $q > 0;\ 0<\epsilon<1-\dfrac{1}{c\lambda_N(0)}$
Step 6:	Formulate a new matrix $E(q, \epsilon)$	$E(q, \epsilon)$ See (1.2-3)
Step 7:	Compute the matrices addition $I + c\epsilon E(q,\epsilon)$	$\left[I + c\epsilon E(q, \epsilon)\right]$
Step 8:	Compute a lower bound to the smallest root of the matrix $I + c\,\epsilon\, E(q, \epsilon)$	$\alpha^*(q, \epsilon\ ;c)$
Step 9:	Compute a lower bound Ω_N, to the smallest root of the given matrix $L(\omega)$	If $\alpha^*(q,\epsilon\ ;\ c) > 0$ the product of q and ϵ is a lower bound of the smallest root of the given matrix $L(\omega)$ i.e. $\Omega_N = q\,\epsilon$

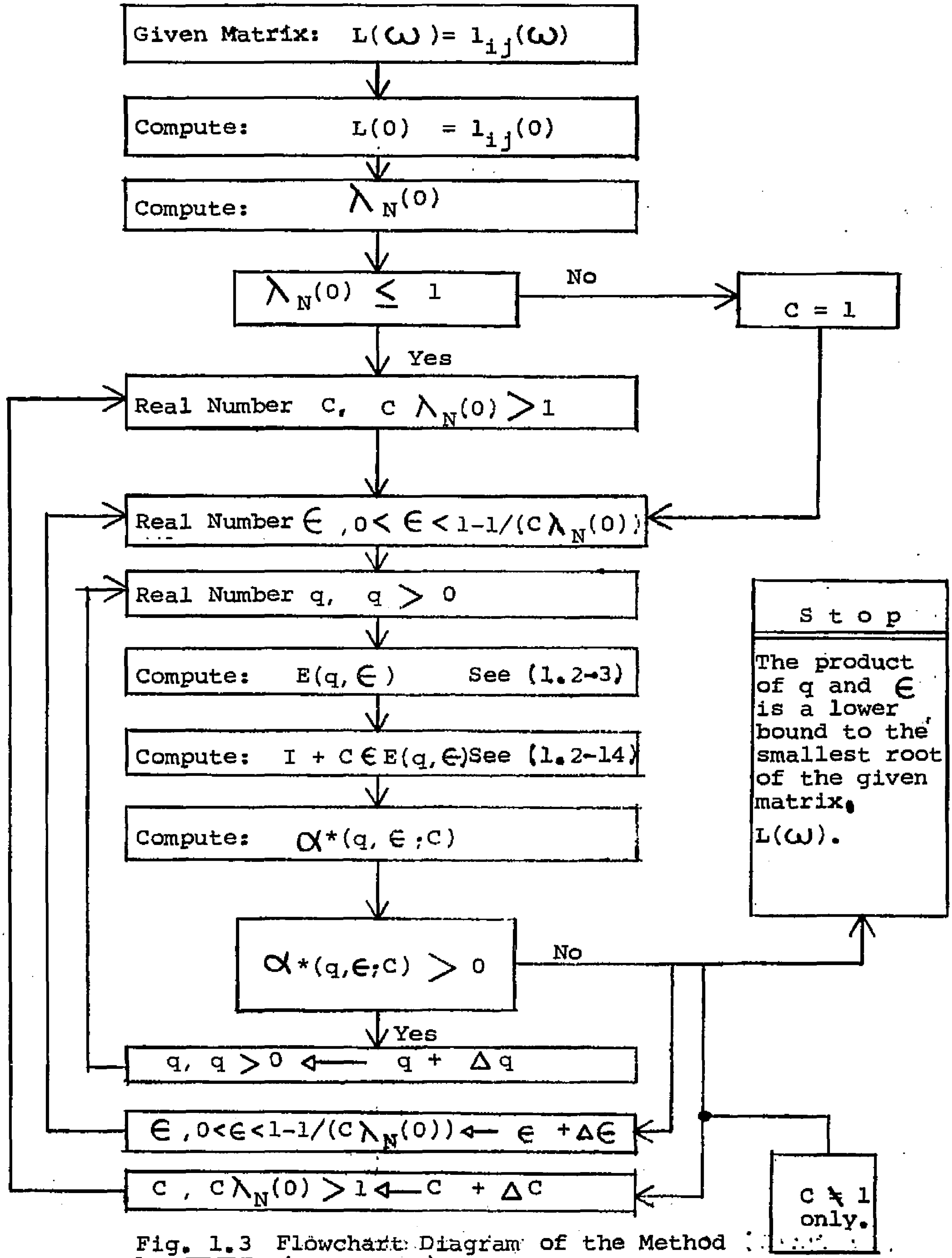


Fig. 1.3 Flowchart Diagram of the Method (Version 2)

1.3 - 3 Description of the method (version 3)

Step 1:	Given an n-square matrix $L(\omega)$	$L(\omega) = l_{ij}(\omega)$
Step 2:	Evaluate the matrix $L(\omega)$ at $\omega = 0$	$L(0)$
Step 3:	Add up the absolute values of row elements (excluding the element on main diagonal) for each row of $L(0)$	$P_i(0) = \sum_{\substack{j=1 \\ j \neq i}}^{N} \lvert l_{ij}(0) \rvert$ $i = 1, 2, \ldots, N$
Step 4:	Compute the differences $D_i(0)$	$D_i(0) = l_{ii}(0) - P_i(0)$ $i = 1, 2, \ldots, N$
Step 5:	Search for the minimum of $D_i(0)$	$\alpha(0) = \underset{i}{\text{Min}} \, (D_i(0))$

NOTE: If the minimum, $\alpha(0)$, obtained from Step 5 is greater than zero then, continue the following steps and compute a lower bound to the smallest root of the given matrix $L(\omega)$. If the minimum, $\alpha(0)$, is less than or equal to zero then, this method is not applicable.

Step 6:	Compute lower bound by iterations: Let $\omega = \Omega_s$	Ω_s
Step 7:	Compute row sums of the matrix $L(\Omega_s)$	$P_i(\Omega_s) = \sum_{\substack{j=1 \\ j \neq i}}^{N} \lvert l_{ij}(\Omega_s) \rvert$ $i = 1, 2, \ldots, N$
Step 8:	Compute the differences $D_i(\Omega_s)$	$D_i(\Omega_s) = l_{ii}(\Omega_s) - P_i(\Omega_s)$ $i = 1, 2, \ldots, N$

Step 9:	Search for the minimum, $\alpha(\Omega_s)$ of $D_i(\Omega_s)$	$\alpha(\Omega_s)=\min\limits_i (D_i(\Omega_s))$
Step 10:	Compute an improved lower bound to the smallest root of the given matrix $L(\omega)$	If $\alpha(\Omega_s)>0$, an improved lower bound to the smallest root of the given matrix $L(\omega)$ can be obtained by repeating the computation from Step 6 and using a new value of $\omega=\Omega_s+\Delta\omega$. If $\alpha(\Omega_s)\leq 0$ then, $\omega=\Omega_s$ is a lower bound to the smallest root of the given matrix $L(\omega)$.

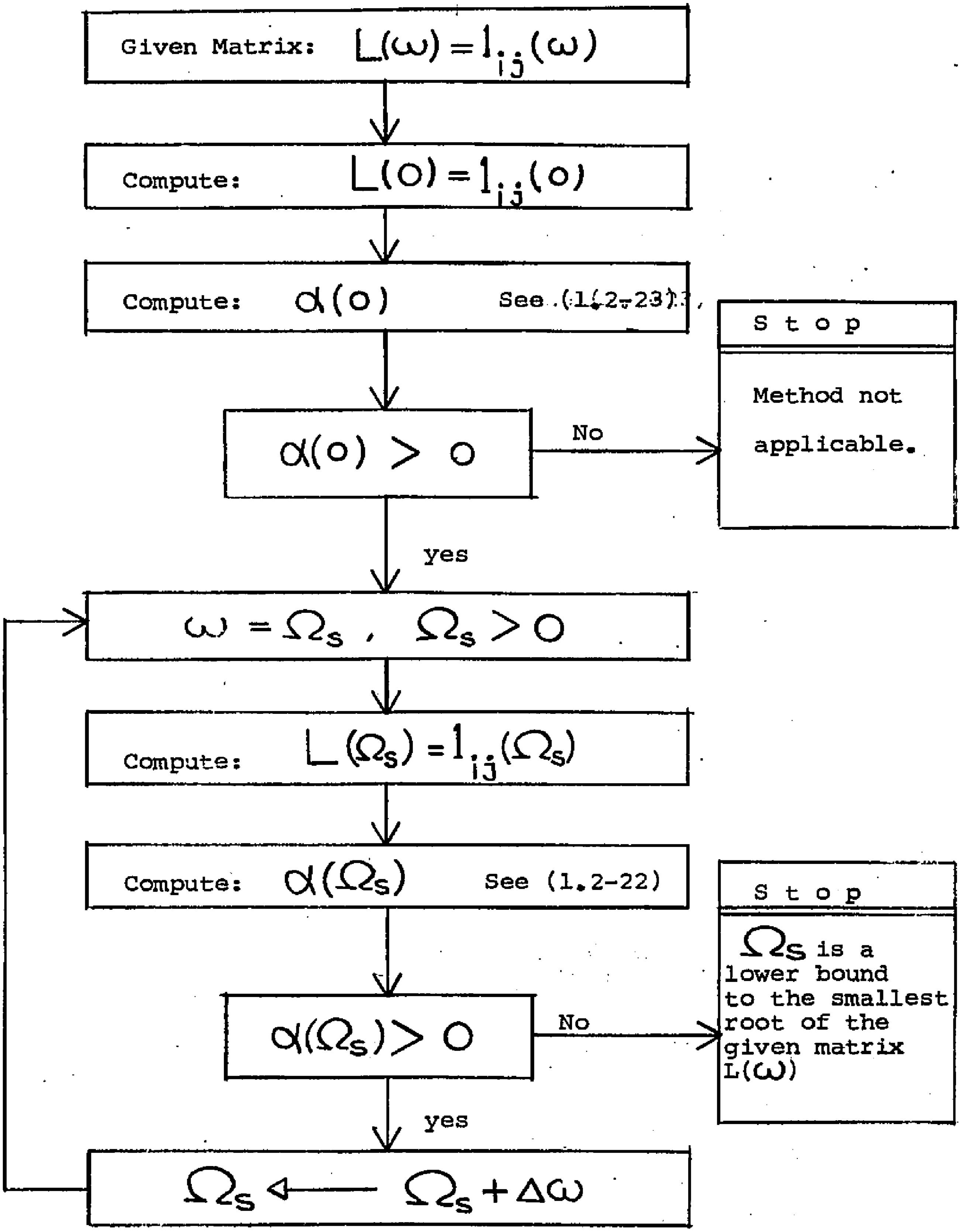


Fig. 1.4 Flowchart Diagram of the Method (Version 3)

1.3 - 4 Description of the method (Version 4)

If the given matrix $L(\omega)$ can be expressed as $L(\omega)=A-\omega B$ then, the computation of a lower bound to the smallest root of the matrix $L(\omega)$ can be simplified as follows:

Step 1:	Given an n-square matrix $L(\omega)$	$L(\omega)=A-\omega B$
Step 2:	Properly choose a real number q	$q, \ q>0$
Step 3:	Compute the smallest root of matrix A	a_N
Step 4:	Formulate a new matrix $D(q)$	$D(q)=A-qB$
Step 5:	Compute a lower bound to the smallest root of the matrix $D(q)$	$\alpha(q)$
Step 6:	Compute the value of ϵ	$\epsilon = \dfrac{1}{1+\lvert\alpha(q)\rvert/a_N}$
Step 7:	Compute a lower bound, Ω_N, to the smallest root of the given matrix $L(\omega)$	$\Omega_N = q\,\epsilon$

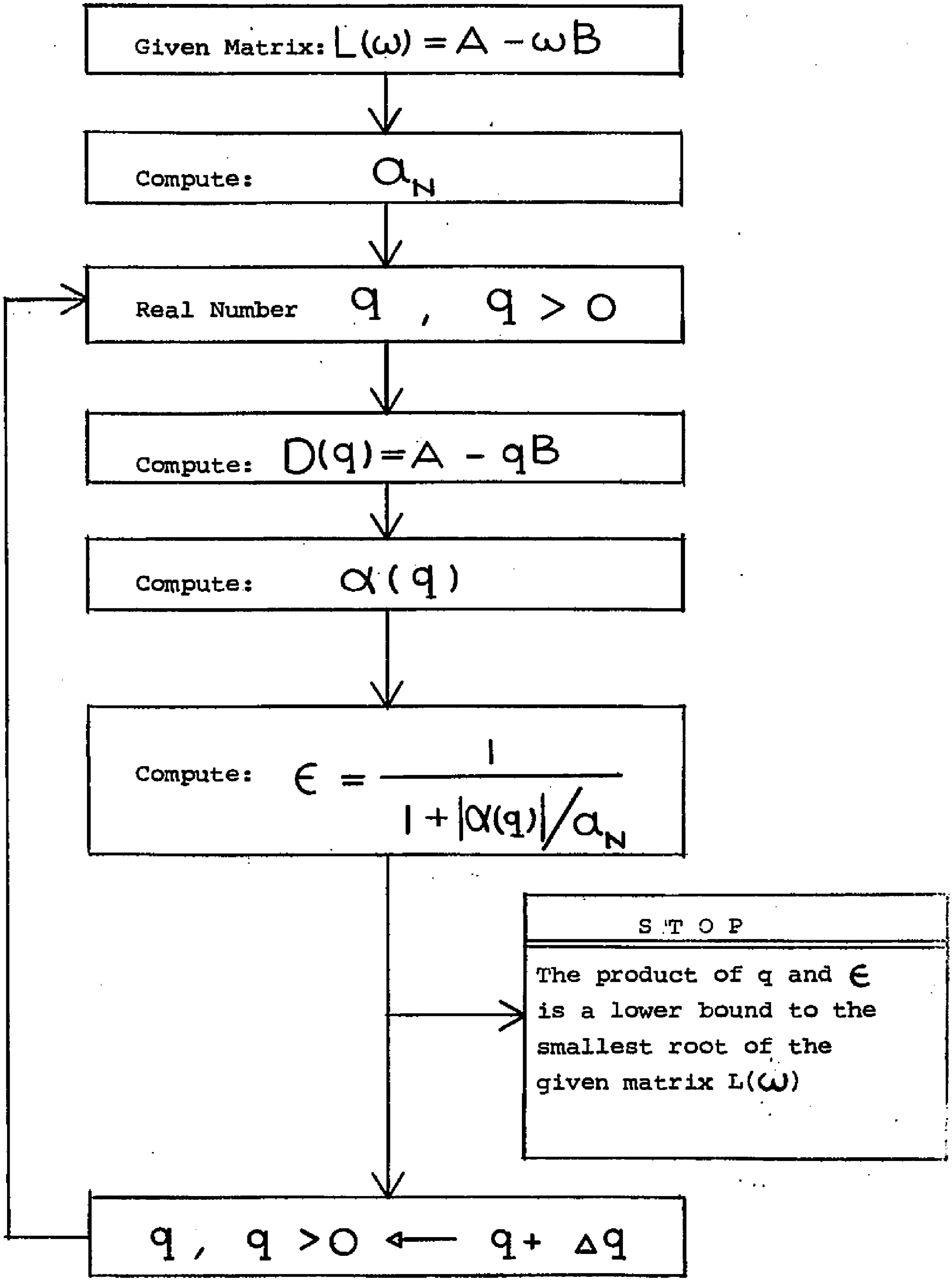


Fig. 1.5 Flowchart Diagram of the Method (Version 4)

1.3 - 5 Summary of the method

(a) The matrix $L(\omega)$ is real, symmetric and $L(0)$ positive definite. The matrices A and B in the special case are real, symmetric, and positive definite.

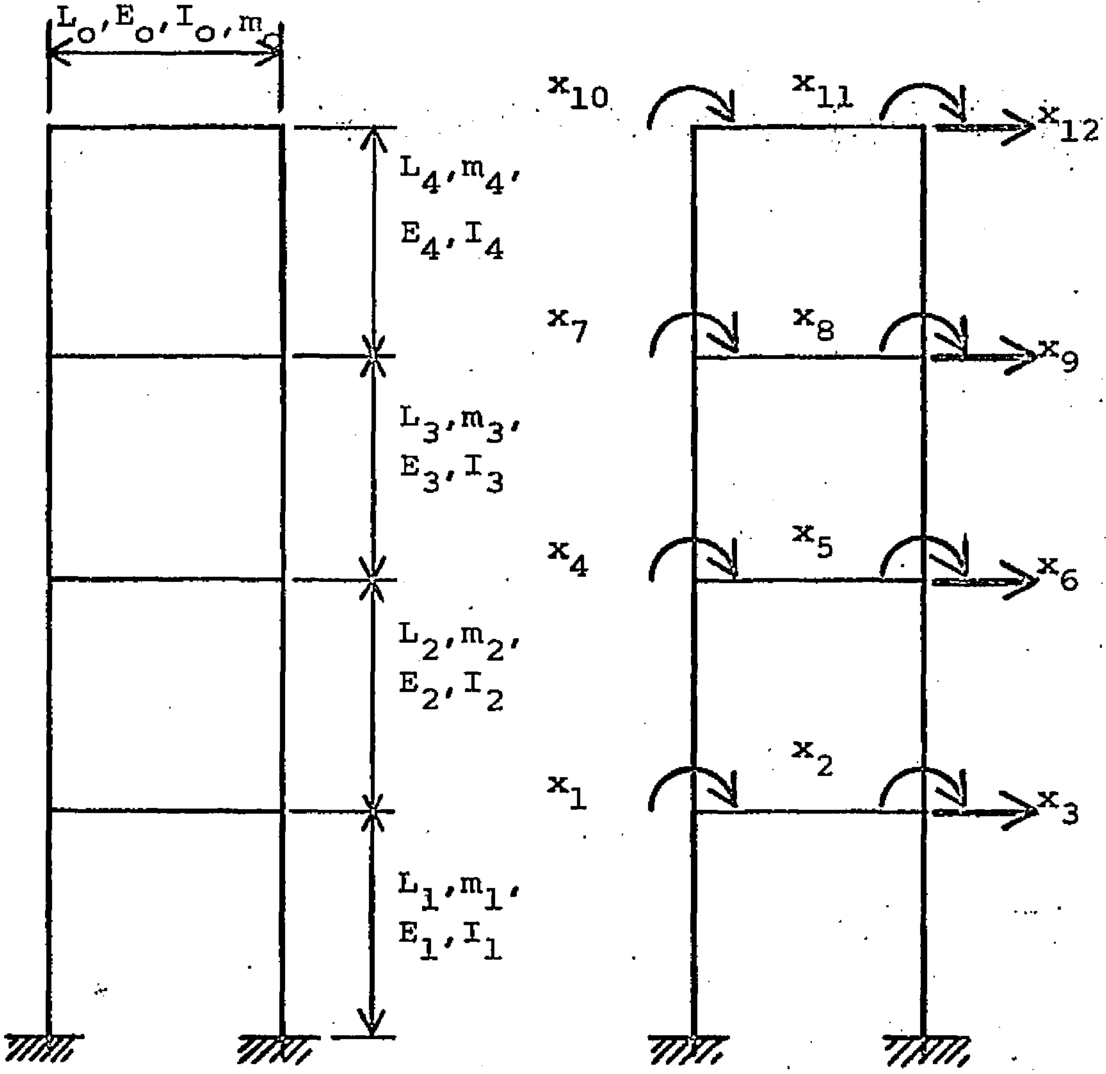
(b) In case when the matrix $L(\omega)$ can be expressed as $L(\omega)=A-\omega B$, where A and B are real symmetric and positive definite matrices, it is possible to obtain an upper bound to the greatest root ω_1 of (1.1-2) also. This can be achieved by formulating a new matrix $L'(\omega)=B-\omega A$, and computing the lower bound to the smallest root of the characteristic equation det $\left| L'(\omega) \right| =0$. The inverse of this lower bound is an upper bound to ω_1.

(c) Version 3 of the method is restricted to the case $\alpha(0) > 0$.

(d) In all cases, an improved lower bound can be obtained by iterations.

1.4 ILLUSTRATIVE EXAMPLE - A MULTI-STORY RIGID FRAME.

Consider the multi-story rigid frame as shown in Fig. 1.6. For the purpose of illustrating the method derived in section 1.2, a lower bound to the fundamental frequency of the frame will be calculated. The structure is analysed as a distributed mass system with 12 degrees of freedom.



Multi-story Frame Joint displacements

Fig. 1.6 Multi-story Frame

1.4.1 DEVELOPMENT OF L(ω)

Let $\{x\}$ be the joint displacement vector of the system (see Fig. 1.6)

$$\{x\} = \{x_1, x_2, \ldots, x_{12}\} \qquad (1.4\text{-}1)$$

The equilibrium equations yield

$$\left[L(\omega) \right] \{x\} = 0 \qquad (1.1\text{-}1)$$

The coefficient matrix $L(\omega) = l_{ij}(\omega)$ is a 12 x 12 real, symmetric matrix. The elements $l_{ij}(\omega)$ can be expressed as follows:

$$l_{ij}(\omega) = l_{ji}(\omega) \qquad i, j = 1, 2, \ldots, 12.$$

$$l_{ij}(\omega) = 0, \text{ except}$$

$$\begin{aligned}
l_{11}(\omega) &= \alpha_o L_o^2 A_o + \alpha_1 L_1^2 A_1 + \alpha_2 L_2^2 A_2 \\
l_{12}(\omega) &= \beta_o L_o^2 A_o \\
l_{13}(\omega) &= \gamma_1 L_1 A_1 - \gamma_2 L_2 A_2 \qquad (1.4\text{-}2) \\
l_{14}(\omega) &= \beta_2 L_2^2 A_2 \\
l_{16}(\omega) &= \eta_2 L_2 A_2 \\
l_{22}(\omega) &= l_{11}(\omega) \\
l_{23}(\omega) &= l_{13}(\omega) \\
l_{25}(\omega) &= l_{14}(\omega) \\
l_{26}(\omega) &= l_{16}(\omega) \\
l_{33}(\omega) &= 2\mu_1 A_1 + 2\mu_2 A_2
\end{aligned}$$

$$l_{34}(\omega) = -l_{16}(\omega)$$
$$l_{35}(\omega) = -l_{16}(\omega)$$
$$l_{36}(\omega) = -2\xi_2 A_2$$
$$l_{44}(\omega) = \alpha_0 L_0^2 A_0 + \alpha_2 L_2^2 A_2 + \alpha_3 L_3^2 A_3 \qquad (1.4\text{-}2)$$
$$l_{45}(\omega) = l_{12}(\omega)$$
$$l_{46}(\omega) = \gamma_2 L_2 A_2 - \gamma_3 L_3 A_3$$
$$l_{47}(\omega) = \beta_3 L_3^2 A_3$$
$$l_{49}(\omega) = \eta_3 L_3 A_3$$
$$l_{55}(\omega) = l_{44}(\omega)$$
$$l_{56}(\omega) = l_{46}(\omega)$$
$$l_{58}(\omega) = l_{47}(\omega)$$
$$l_{59}(\omega) = l_{49}(\omega)$$
$$l_{66}(\omega) = 2\mu_2 A_2 + 2\mu_3 A_3$$
$$l_{67}(\omega) = -l_{49}(\omega)$$
$$l_{68}(\omega) = -l_{49}(\omega)$$
$$l_{69}(\omega) = -2\xi_3 A_3$$
$$l_{77}(\omega) = \alpha_0 L_0^2 A_0 + \alpha_3 L_3^2 A_3 + \alpha_4 L_4^2 A_4$$
$$l_{78}(\omega) = l_{12}(\omega)$$
$$l_{79}(\omega) = \gamma_3 L_3 A_3 - \gamma_4 L_4 A_4$$
$$l_{7,10}(\omega) = \beta_4 L_4^2 A_4$$
$$l_{7,12}(\omega) = \eta_4 L_4 A_4$$
$$l_{88}(\omega) = l_{77}(\omega)$$

$$
\begin{aligned}
l_{89}(\omega) &= l_{79}(\omega) \\
l_{8,11}(\omega) &= l_{7,10}(\omega) \\
l_{8,12}(\omega) &= l_{7,12}(\omega) \\[1em]
l_{99}(\omega) &= 2\mu_3 A_3 + 2\mu_4 A_4 \\
l_{9,10}(\omega) &= -l_{7,12}(\omega) \\
l_{9,11}(\omega) &= -l_{7,12}(\omega) \\
l_{9,12}(\omega) &= -2\delta_4 A_4 \\[1em]
l_{10,10}(\omega) &= \alpha_0 L_0^2 A_0 + \alpha_4 L_4^2 A_4 \\
l_{10,11}(\omega) &= l_{12}(\omega) \\
l_{10,12}(\omega) &= \gamma_4 L_4 A_4 \\[1em]
l_{11,11}(\omega) &= l_{10,10}(\omega) \\
l_{11,12}(\omega) &= l_{10,12}(\omega) \\[1em]
l_{12,12}(\omega) &= 2\mu_4 A_4
\end{aligned}
\tag{1.4-2}
$$

Where,

$$\alpha_i = \frac{u_i(\text{Sin}u_i.\text{Cosh}u_i - \text{Cos}u_i.\text{Sinh}u_i)}{1 - \text{Cos}u_i.\text{Cosh}u_i}$$

$$\beta_i = \frac{u_i(\text{Sinh}u_i - \text{Sin}u_i)}{1 - \text{Cos}u_i.\text{Cosh}u_i}$$

$$\gamma_i = \frac{u_i^2(\text{Sin}u_i.\text{Sinh}u_i)}{1 - \text{Cos}u_i.\text{Cosh}u_i}$$

$$\eta_i = \frac{u_i^2(\text{Cosh}u_i - \text{Cos}u_i)}{1 - \text{Cos}u_i.\text{Cosh}u_i} \qquad (1.4\text{-}3)$$

$$\mu_i = \frac{u_i^3(\text{Sin}u_i.\text{Cosh}u_i + \text{Cos}u_i.\text{Sinh}u_i)}{1 - \text{Cos}u_i.\text{Cosh}u_i}$$

$$\varsigma_i = \frac{u_i^3(\text{Sin}u_i + \text{Sinh}u_i)}{1 - \text{Cos}u_i.\text{Cosh}u_i}$$

And

$$A_i = \frac{E_i I_i}{L_i^3}, \qquad u_i^4 = \frac{m_i L_i^4 \omega^2}{E_i I_i}$$

$i = 0,1,2,3,4.$

For simplicity the following physical properties and the dimensions of the member elements are assumed

$$
\begin{aligned}
&m_o=m_1=m_2=m_3=m_4=1.0\\
&L_0=L_1=L_2=L_3=L_4=1.0\\
&E_0=E_1=E_2=E_3=E_4=1.0 \qquad (1.4\text{-}4)\\
&I_o=I_1=I_2=I_3=I_4=1.0
\end{aligned}
$$

1.4.2 FUNDAMENTAL FREQUENCY BY DETERMINANT EVALUATION

It is evident that the form of $L(\omega)$ is quite complicated and further expansion is useless. To determine the fundamental frequency by the present practice, the following procedure is used.

(a) Assume a series of trial values of ω.

(b) Substitute into Eq. (1.4-2)

(c) Evaluate the determinant, det $|L(\omega)|$

(d) A result is obtained when det $|L(\omega)| = 0$.

This procedure was carried out as summarized in Table 1.1, to provide a check value for the new method.

Substitute (1.4-4) into (1.4-2) and evaluate the determinant, det $|L(\omega)|$, of the coefficient matrix of equation (1.1-1). for successive trial values of ω. The fundamental frequency of the structural system has been found between 0.81 and 0.90 RAD/SEC. as shown in Table 1.1.

Table 1.1 ω vs det $|L(\omega)|$ (Multi-Story Frame).

ω	det $\lvert L(\omega) \rvert$
0.01	5.77×10^{12}
0.04	5.79×10^{12}
0.09	5.74×10^{12}
0.16	5.59×10^{12}
0.25	5.29×10^{12}
0.36	4.75×10^{12}
0.49	3.88×10^{12}
0.64	2.61×10^{12}
0.81	9.01×10^{11}
0.90	-1.19×10^{11}
1.00	-1.23×10^{12}
1.21	-3.69×10^{12}

The fundamental frequency of the multi-story frame is:

$$0.81^{\mathrm{RAD/SEC}} < \omega < 0.90^{\mathrm{RAD/SEC}}$$

or:

$$0.129^{\mathrm{CYC/SEC}} < f < 0.143^{\mathrm{CYC/SEC}} \qquad (1.4\text{-}5)$$

1.4.3 STEP BY STEP PROCEDURE

We now outline briefly the principal steps of the new method.

Step 1. COMPUTE THE VALUE OF $\lambda_N(0)$

Substituting $\omega = 0$ into (1.4-2) and (1.4-3) we obtain the matrix L(0)

$$L(0) = \begin{bmatrix} 12 & 2 & 0 & 2 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 12 & 0 & 0 & 2 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 48 & -6 & -6 & -24 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -6 & 12 & 2 & 0 & 2 & 0 & 6 & 0 & 0 & 0 \\ 0 & 2 & -6 & 2 & 12 & 0 & 0 & 2 & 6 & 0 & 0 & 0 \\ 6 & 6 & -24 & 0 & 0 & 48 & -6 & -6 & -24 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -6 & 12 & 2 & 0 & 2 & 0 & 6 \\ 0 & 0 & 0 & 0 & 2 & -6 & 2 & 12 & 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 6 & 6 & -24 & 0 & 0 & 48 & -6 & -6 & -24 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -6 & 8 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -6 & 2 & 8 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 6 & -24 & 6 & 6 & 24 \end{bmatrix} \tag{1.4-6}$$

The smallest root of the matrix L(0) is computed by the usual method.

$$\lambda_N(0) = 1.084 \tag{1.4-7}$$

Step 2. CHOOSE INITIAL VALUES OF ω AND ϵ.

The initial values of ω and ϵ are chosen and q is calculated

$$\omega = 0.25$$
$$\epsilon = 0.03$$
$$q = \frac{\omega}{\epsilon} = 8.34$$

Since 0 is the trivial lower bound of ω_N, one can normally choose a reasonably low value which is less than the ω_N. (Should the chosen value be too high, the procedure will indicate that it is not a lower bound. A second value can then be chosen.)

The method requires that $0 < \epsilon \leq 1$. Experience has shown that for small values of ϵ, the computations are insensitive to the choice of ϵ.

Step 3. Calculate $\frac{(1-\epsilon)}{\epsilon} \Lambda_N(0)$.

Step 4. Compute $\alpha(q, \epsilon)$

Step 5. Verify whether a lower bound is obtained.

If $\alpha > -\frac{1-\epsilon}{\epsilon} \Lambda_N(0)$, then the trial value chosen is proven to be a lower bound. One may either stop at this point, or repeat steps 1 - 5, with a higher trial value.

If $\alpha < -\frac{1-\epsilon}{\epsilon} \Lambda_N(0)$, the trial value has not been proved to be a lower bound. Choose a small trial value and repeat step 1 - 5.

The numerical computations are summarized in Table 1.2.

TABLE 1.2 - LOWER BOUND OF FUNDAMENTAL FREQUENCY
MULTI-STORY FRAME.

ϵ	ω	q	α	$\frac{(1-\epsilon)}{-\epsilon}\Lambda_N(0)$	Lower Bound
0.03	0.25	8.34	-28.42	-35.04	0.25
	0.36	12.00	-33.17		0.36
	0.49	16.33			—
0.01	0.36	36.00	-51.52	-107.31	0.36
	0.49	49.00	-75.00		0.49
	0.56	56.00	-91.22		0.56
	0.64	64.00			—
0.006	0.56	90.34	-136.04	-179.58	0.56
	0.64	106.67	-169.07		0.64
	0.81	135.00			—
0.002	0.64	320.00	-459.21	-540.91	0.64
	0.81	405.00			—

To illustrate the effect of ϵ on the computation, four different values of ϵ were used and the results given in table 1.2. These appear to indicate that better results are obtainable with small values of ϵ. We note that the method does not require us to progress from large ϵ to small ϵ .

1.4.4 COMPUTATION DETAILS

As further illustration, we give some additional details of one set of computations.

Step 1. $\Lambda_N(0) = 1.084$

Step 2. Choose ω and ϵ .

$\omega = 0.64$

$\epsilon = 0.006$

$q = \frac{\omega}{\epsilon} = 106.67$

Step 3. $-\frac{1-\epsilon}{\epsilon}\Lambda_N(0) = -179.58$

Step 4. The matrices L(0.64) and E(106.67,0.006) are developed as given in (1.4-8),(1.4-9). respectively. Next we calculate α , which is a lower bound to the smallest root of E(106.67, 0.006). The method described in Section 1.2-3 is used. The row sums are calculated:

$$\text{DIFF}(i) = e_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{12} |e_{ij}|$$

The twelve row sums are found as below:

DIFF(1) = -3.-43091
DIFF(2) = -3.043091
DIFF(3) = -111.268448
DIFF(4) = -13.646236
DIFF(5) = -13.646236
DIFF(6) = -169.071320
DIFF(7) = -13.646236
DIFF(8) = -13.646236
DIFF(9) = -169.071320
DIFF(10)= -8.814468
DIFF(11)= -8.814468
DIFF(12)= -89.379654

$$
L(0.64) = \left[\begin{array}{cccccccccccc}
1.198829\text{E }01 & 2.002929\text{E }00 & 0. & 2.002929\text{E }00 & 0. & 6.012690\text{E }00 & 0. & 0. & 0. & 0. & 0. & 0. \\
2.002929\text{E }00 & 1.198829\text{E }01 & 0. & 0. & 2.002929\text{E }00 & 6.012690\text{E }00 & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 4.739121\text{E }01 & -6.012690\text{E }00 & -6.012690\text{E }00 & -2.410544\text{E }01 & 0. & 0. & 0. & 0. & 0. & 0. \\
2.002929\text{E }00 & 0. & -6.012690\text{E }00 & 1.198829\text{E }01 & 2.002929\text{E }00 & 0. & 2.002929\text{E }00 & 0. & 6.012690\text{E }00 & 0. & 0. & 0. \\
0. & 2.002929\text{E }00 & -6.012690\text{E }00 & 2.002929\text{E }00 & 1.198829\text{E }01 & 0. & 0. & 2.002929\text{E }00 & 6.012690\text{E }00 & 0. & 0. & 0. \\
6.012690\text{E }00 & 6.012690\text{E }00 & -2.410544\text{E }01 & 0. & 0. & 4.739121\text{E }01 & -6.012690\text{E }00 & -6.012690\text{E }00 & -2.410544\text{E }01 & 0. & 0. & 0. \\
0. & 0. & 0. & 2.002929\text{E }00 & 0. & -6.012690\text{E }00 & 1.198829\text{E }01 & 2.002929\text{E }00 & 0. & 2.002929\text{E }00 & 0. & 6.012690\text{E }00 \\
0. & 0. & 0. & 0. & 2.002929\text{E }00 & -6.012690\text{E }00 & 2.002929\text{E }00 & 1.198829\text{E }01 & 0. & 0. & 2.002929\text{E }00 & 6.012690\text{E }00 \\
0. & 0. & 0. & 6.012690\text{E }00 & 6.012690\text{E }00 & -2.410544\text{E }01 & 0. & 0. & 4.739121\text{E }01 & -6.012690\text{E }00 & -6.012690\text{E }00 & -2.410544\text{E }01 \\
0. & 0. & 0. & 0. & 0. & 0. & 2.002929\text{E }00 & 0. & -6.012690\text{E }00 & 7.992193\text{E }00 & 2.002929\text{E }00 & 5.978544\text{E }00 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 2.002929\text{E }00 & -6.012690\text{E }00 & 2.002929\text{E }00 & 7.992193\text{E }00 & 5.978532\text{E }00 \\
0. & 0. & 0. & 0. & 0. & 0. & 6.012690\text{E }00 & 6.012690\text{E }00 & -2.410544\text{E }01 & 5.978532\text{E }00 & 5.978532\text{E }00 & 2.369560\text{E }01
\end{array}\right]
\tag{1.4-8}
$$

E(106.67,0.006) =

```
 1.004814E 01      2.488087E 00      0.                2.488087E 00
 2.488087E 00      1.004814E 01      0.                0.
 0.                0.               -5.346558E 01     -8.115059E 00
 2.488087E 00      0.               -8.115059E 00      1.004814E 01
 0.                2.488087E 00     -8.115059E 00      2.488087E 00
 8.115059E 00      8.115059E 00     -4.157275E 01      0.
 0.                0.                0.                2.488087E 00
 0.                0.                0.                0.
 0.                0.                0.                8.115059E 00
 0.                0.                0.                0.
 0.                0.                0.                0.
 0.                0.                0.                0.


 0.                8.115059E 00      0.                0.
 2.488087E 00      8.115059E 00      0.                0.
-8.115059E 00     -4.157275E 01      0.                0.
 2.488087E 00      0.                2.488087E 00      0.
 1.004814E 01      0.                0.                2.488087E 00
 0.               -5.346558E 01     -8.115059E 00     -8.115059E 00
 0.               -8.115059E 00      1.004814E 01      2.488087E 00
 2.488087E 00     -8.115059E 00      2.488087E 00      1.004814E 01
 8.115059E 00     -4.157275E 01      0.                0.
 0.                0.                2.488087E 00      0.
 0.                0.                0.                2.488087E 00
 0.                0.                8.115059E 00      8.115059E 00


 0.                0.                0.                0.
 0.                0.                0.                0.
 0.                0.                0.                0.
 8.115059E 00      0.                0.                0.
 8.115059E 00      0.                0.                0.
-4.157275E 01      0.                0.                0.
 0.                2.488087E 00      0.                8.115059E 00
 0.                0.                2.488087E 00      8.115059E 00
-5.345668E 01     -8.115059E 00     -8.115059E 00     -4.157275E 01
-8.115059E 00      6.698761E 00      2.488087E 00      2.421997E 00
-8.115059E 00      2.488087E 00      6.698761E 00      2.421997E 00
-4.157275E 01      2.421997E 00      2.421997E 00     -2.673279E 01
```

(1.4-9)

The minimum DIFF(i), is a lower bound to the smallest root of the matrix E(106.67,0.006).

Step 5: As the criterion

$$\alpha(q,\epsilon) > -\frac{(1-\epsilon)}{\epsilon} \cdot \Lambda_N(0)$$

is satisfied, we conclude that $\omega = 0.64$ is a lower bound.

A lower bound to the fundamental frequency of the multi-story frame structure is found.

$$\Omega_N = 0.64 \text{ RAD/SEC}$$
$$= 0.101 \text{ CPS}.$$

1.5 CONCLUSION

The determination of the fundamental frequency of real elastic structural system is a lengthy process, especially if the structure is represented as a distributed mass system. In this chapter, a new method for the lower bound is presented. The method has certain advantages:

(1) The method applies to lumped mass and distributed mass system.

(2) The results are guaranteed to be lower bounds to the fundamental frequency.

(3) The method is extremely fast, as compared to other known methods.

There are certain shortcomings inherent in the method also. Some of these are listed below:

(1) The method does not calculate higher frequencies.

(2) Although the method can be used iteratively to obtain successively better lower bounds, it can not be used to calculate the fundamental frequency exactly. Thus the method does not give an indication of the accuracy of the result.

CHAPTER 2 MATRIX FORMULATION OF DYNAMIC ANALYSIS

2.0 INTRODUCTION

In the time domain dynamical analysis of an elastic structural system, natural frequencies of the system are of fundamental importance. In this chapter we discuss the matrix formulation of this problem. The basic approach is to generate the total system stiffness matrix and the system mass matrix by the synthesis of that of the member elements of the system. This is valid for both lumped parameter and distributed parameter analyses. The matrix formulation is not new, and is presented for the sake of convenience and completeness.

2.1 IDEALIZED SYSTEM vs REAL SYSTEM

The real structural system considered can be any space or plane structure composed of one dimensional elements. We assume that the member elements of the system are of the straight slender-beam type and possess a doubly-symmetric cross section. The size of the member elements need not be constant along the axis of the element. It is assumed that the material of the structural system obeys Hooke's law. The joint connections of member elements can be either completely rigid or of the frictionless pin-joint type. The motions of the system are expressed in terms of the motions of its joints and are measured from the position of static equilibrium. There is no rigid-body motion as the structural system is well supported. In the lumped parameter analysis, the masses of the member elements of the system are further assumed to be lumped at discrete points (the joint of the system).

2.2 STATE VECTORS

Let AB be the j^{th} member of the system (See Fig. 2.1). For this member a set of right-handed rectangular coordinates x,y,z is attached as member coordinates. The x coordinate axis always coincides with the longitudinal axis of the member. The principal axes of the cross section are along y and z axes. Let vector $\left\{u\right\}_j = (u_1,u_2,\ldots,u_{12}\}$ be specified as follows: u_1,u_2,u_3, and u_7,u_8,u_9 are the translational displacements of end-point A and B of the element along x,y and z axes respectively; and u_4,u_5,u_6 and u_{10},u_{11},u_{12} are the rotational displacements at A and B. The position of the member AB is completely defined in the space once the vector $\left\{u\right\}_j$ is specified. We refer to $\left\{u\right\}_j$ as the element displacement vector of member, AB. At the i^{th} joint of the structural system a displacement vector $\left\{q\right\}_i = (q_1,q_2,\ldots,q_6)_i$ can be specified with respect to the global coordinates X,Y,Z. The components q_1,q_2,q_3 and q_4,q_5,q_6 are translational and rotational displacements of the joint in the direction of X,Y,Z respectively. Let the structural system consist of R member elements and N joints, we then have the displacement vector for the entire system as:

In element coordinate system

$$\left\{u\right\} = \left(\left\{u\right\}_1 \middle| \left\{u\right\}_2 \middle| \ldots \middle| \left\{u\right\}_R \right) \qquad (2.2\text{-}1)$$

In global coordinate system

$$\{q\} = (\{q\}_1 | \{q\}_2 | \ldots | \{q\}_N) \quad (2.2\text{-}2)$$

Note that the $\{u\}$ is a (12R x 1) column vector and $\{q\}$ is a (6N x 1) column vector. Since the joint movements are not restricted $\{q\}$ is a generalized coordinate system. On the other hand, the u's are constrained displacements because of the equilibrium conditions at each joint.

The force vector of the system thus can be written as:

$$\{F\} = (\{F\}_1 | \{F\}_2 | \ldots | \{F\}_R) \quad (2.2\text{-}3)$$

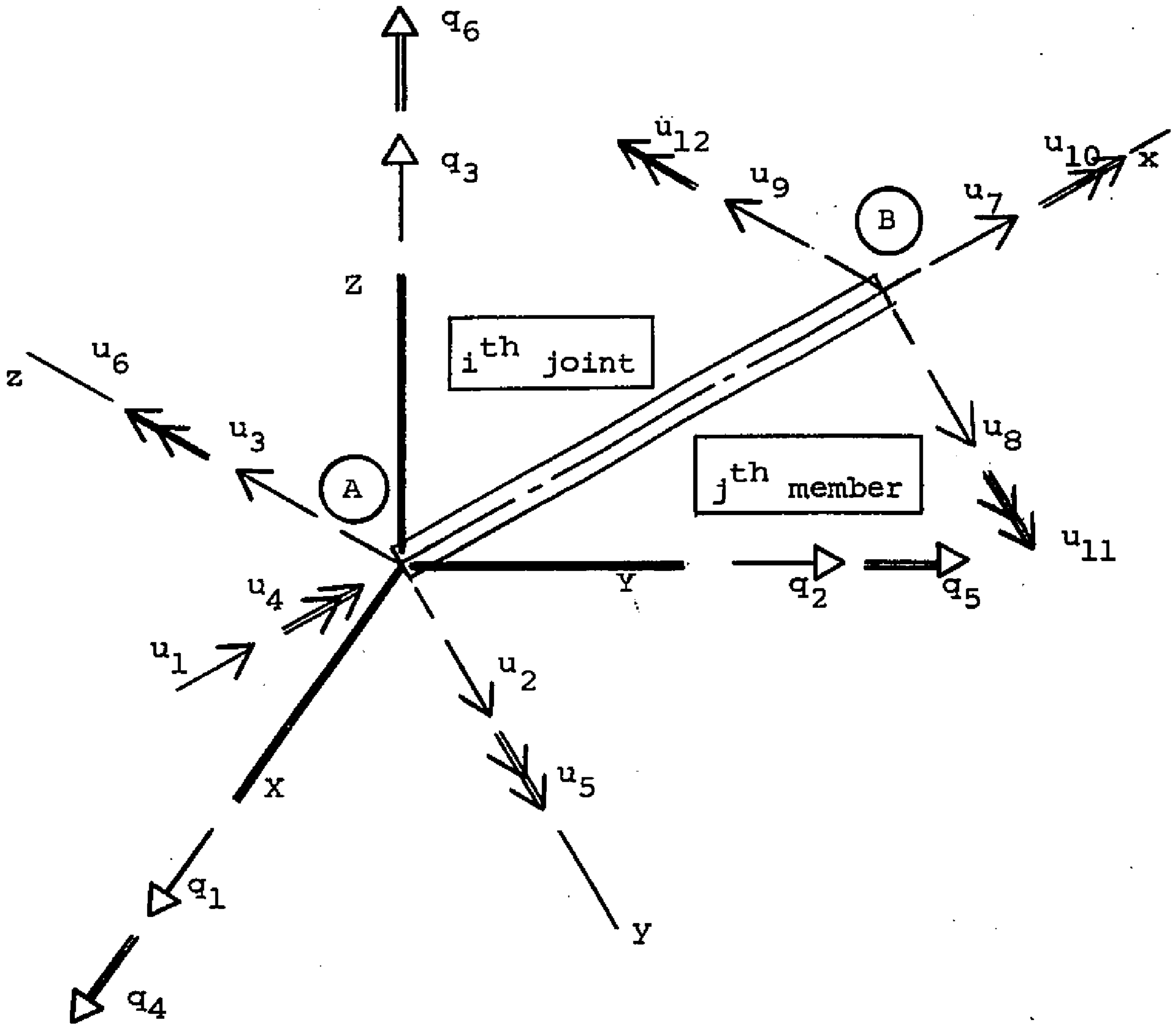
and

$$\{Q\} = (\{Q\}_1 | \{Q\}_2 | \ldots | \{Q\}_N) \quad (2.2\text{-}4)$$

where

$$\{F\}_i = (F_1, F_2, \ldots, F_{12})_i \quad , \{Q\}_j = (Q_1, Q_2, \ldots, Q_6)_j$$

Both $\{F\}_i$ and $\{Q\}_j$ are shown in Fig. 2.1.



Element displacement vector $\{u\}_j$

$$\{u\}_j = (u_1, u_2, \ldots, u_{12})_j$$

Joint displacement vector $\{q\}_i$

$$\{q\}_i = (q_1, q_2, \ldots, q_6)_i$$

Fig.2.1 Element coordinate system x,y,z and Global coordinate system X,Y,Z.

2.3 DIFFERENTIAL EQUATIONS OF MOTION

Let $\{q\}$ and $\{\ddot{q}\}$ be the joint displacement and joint acceleration vectors of the structural system, respectively. Because of the possible existence of concentrated masses and the assumption on distribution of the masses of the member elements of the structural system, we present the differential equations of motion in matrix form for both distributed and lumped parameter analysis with or without concentrated masses as follows:

Case 1. Distributed parameter analysis. Member elements of the system have distributed masses; no concentrated masses in the system.

$$K\ (\omega)\ \{q\} = 0 \qquad (2.3\text{-}1)$$

Case 2. Distributed parameter analysis. Member elements of the system have distributed masses with concentrated masses at nodal points.

$$\overline{M}\ \{\ddot{q}\} + K\ (\omega)\ \{q\} = 0 \qquad (2.3\text{-}2)$$

Case 3. Lumped parameter analysis. Masses of the member elements of the system are assumed to be lumped at nodal points with or without additional concentrated masses at nodal points.

$$(\ \overline{M} + M\ )\ \{\ddot{q}\} + K\ \{q\} = 0 \qquad (2.3\text{-}3)$$

Note that in (2.3-1), (2.3-2) and (2.3-3)

1. All the matrices are formulated in the global coordinate system.
2. $\overline{M}$ = diagonal mass matrix of concentrated masses.
   M = mass matrix of the system.
3. K = stiffness matrix of the system.
4. The entries of stiffness matrix K (ω) are transcendental functions of the real variable $\omega \geq 0$.
5. All matrices are square, real symmetric matrices; $\overline{M}$, M and K are positive definite, K (ω) is positive definite at $\omega = 0$.
6. The matrices in this section are given from (2.5-27) and (2.5-28).

2.4 FREQUENCY EQUATION

Let $q(t) = q\, e^{i\omega t}$ (2.4-1)

Substitute (2.4-1) into (2.3-1), (2.3-2) and (2.3-3) to obtain the frequency equation for the structural system:

Case 1. $\det \left| K(\omega) \right| = 0$ (2.4-2)

Case 2. $\det \left| K(\omega) - \omega^2 \overline{M} \right| = 0$ (2.4-3)

Case 3. $\det \left| K - \omega^2 (\overline{M} + M) \right| = 0$ (2.4-4)

The matrices in (2.4-2), (2.4-3) and (2.4-4) satisfy the required conditions in section 1.1 and 1.2. Therefore the methods developed in Chapter 1 can be used to obtain the lower and upper bounds of the natural frequencies of an elastic structural system.

2.5 STIFFNESS MATRIX AND MASS MATRIX ASSOCIATED WITH THE SYSTEM

(A) Stiffness matrix and mass matrix of a member element.

(1) Stiffness matrix in element coordinate system.

Let AB be the j^{th} member element of the system. The associated end force and end displacement vector are respectively $\{F\}_j = (F_A, F_B)$ and $\{u\}_j = (u_A, u_B)$. The force displacement relationship of AB can be written as

$$\left\{\frac{F_A}{F_B}\right\} = \left[k_u\right]\left\{\frac{u_A}{u_B}\right\} \qquad (2.5\text{-}1)$$

The stiffness matrix (k_u) member AB in element coordinate system is

$$(k_u)=\begin{bmatrix} a & \cdot & \cdot & \cdot & \cdot & \cdot & -a & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & b & \cdot & \cdot & \cdot & g & \cdot & -b & \cdot & \cdot & \cdot & g \\ \cdot & \cdot & c & \cdot & h & \cdot & \cdot & \cdot & -c & \cdot & h & \cdot \\ \cdot & \cdot & \cdot & d & \cdot & \cdot & \cdot & \cdot & \cdot & -d & \cdot & \cdot \\ \cdot & \cdot & h & \cdot & e & \cdot & \cdot & \cdot & -h & \cdot & j & \cdot \\ \cdot & g & \cdot & \cdot & \cdot & f & \cdot & -g & \cdot & \cdot & \cdot & k \\ -a & \cdot & \cdot & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -b & \cdot & \cdot & \cdot & -g & \cdot & b & \cdot & \cdot & \cdot & -g \\ \cdot & \cdot & -c & \cdot & -h & \cdot & \cdot & \cdot & c & \cdot & -h & \cdot \\ \cdot & \cdot & \cdot & -d & \cdot & \cdot & \cdot & \cdot & \cdot & d & \cdot & \cdot \\ \cdot & \cdot & h & \cdot & j & \cdot & \cdot & \cdot & -h & \cdot & e & \cdot \\ \cdot & g & \cdot & \cdot & \cdot & k & \cdot & -g & \cdot & \cdot & \cdot & f \end{bmatrix} \qquad (2.5\text{-}2)$$

See Appendix 2 for elements a,b,...,k.

Or, if we let

$$(R) = \begin{bmatrix} a & \cdot & \cdot \\ \cdot & b & \cdot \\ \cdot & \cdot & c \end{bmatrix} \qquad (S) = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & g \\ \cdot & h & \cdot \end{bmatrix}$$

$$(T) = \begin{bmatrix} d & \cdot & \cdot \\ \cdot & e & \cdot \\ \cdot & \cdot & f \end{bmatrix} \qquad (U) = \begin{bmatrix} -d & \cdot & \cdot \\ \cdot & j & \cdot \\ \cdot & \cdot & k \end{bmatrix}$$

Then (K_u) becomes

$$(k_u) = \begin{bmatrix} (R) & (S) & -(R) & (S) \\ (S)^T & (T) & -(S)^T & (U) \\ -(R) & -(S) & (R) & -(S) \\ (S)^T & (U) & -(S)^T & (T) \end{bmatrix} \qquad (2.5\text{-}3)$$

(2) Boundary conditions

If the end conditions of the member are specified then, owing to the existence of end constraint equations the order of the element stiffness matrix (k_u) is reduced.

a. Fixed end.

In case the end A of the member AB is completely fixed then.

$$u_1 = u_2 \ldots = u_6 = 0$$

$$\left\{ \frac{F_A}{F_B} \right\} = (k_u) \left\{ \frac{0}{u_B} \right\} \qquad (2.5\text{-}4)$$

thus

$$\left\{F_A\right\} = \begin{bmatrix} -(R) & (S) \\ -(S)^T & (U) \end{bmatrix} \left\{u_B\right\} \qquad (2.5\text{-}5)$$

$$\left\{F_B\right\} = \begin{bmatrix} (R) & -(S) \\ -(S)^T & (T) \end{bmatrix} \left\{u_B\right\}$$

Similarly if the end B is completely fixed, then

$$\left\{F_A\right\} = \begin{bmatrix} (R) & (S) \\ (S)^T & (T) \end{bmatrix} \left\{u_A\right\} \qquad (2.5\text{-}6)$$

$$\left\{F_B\right\} = \begin{bmatrix} -(R) & -(S) \\ (S)^T & (U) \end{bmatrix} \left\{u_A\right\}$$

Hence, the stiffnes matrix of member AB with one end fixed is:

end A fixed

$$(k_u) = \begin{bmatrix} (R) & -(S) \\ -(S)^T & (T) \end{bmatrix} \qquad (2.5\text{-}7a)$$

end B fixed

$$(k_u) = \begin{bmatrix} (R) & (S) \\ (S)^T & (T) \end{bmatrix} \qquad (2.5\text{-}7b)$$

b. <u>Simply supported end</u>

Let end B of the member AB be simply supported then,

$$u_7 = u_8 = u_9 = 0$$
$$F_{11} = F_{12} = 0 \qquad (2.5\text{-}8)$$

Consequently,

$$\left[\begin{array}{c} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_{10} \\ \hline 0 \\ 0 \\ \hline F_7 \\ F_8 \\ F_9 \end{array}\right] = \left[\begin{array}{ccccccc|cc|ccc} a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -a & \cdot & \cdot \\ \cdot & b & \cdot & \cdot & \cdot & g & \cdot & \cdot & g & \cdot & -b & \cdot \\ \cdot & \cdot & c & \cdot & h & \cdot & \cdot & h & \cdot & \cdot & \cdot & -c \\ \cdot & \cdot & \cdot & d & \cdot & \cdot & -d & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & h & \cdot & e & \cdot & \cdot & j & \cdot & \cdot & \cdot & -h \\ \cdot & g & \cdot & \cdot & \cdot & f & \cdot & \cdot & k & \cdot & -g & \cdot \\ \cdot & \cdot & \cdot & -d & \cdot & \cdot & d & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & h & \cdot & j & \cdot & \cdot & e & \cdot & \cdot & \cdot & -h \\ \cdot & g & \cdot & \cdot & \cdot & k & \cdot & \cdot & f & \cdot & -g & \cdot \\ \hline -a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a & \cdot & \cdot \\ \cdot & -b & \cdot & \cdot & \cdot & -g & \cdot & \cdot & -g & \cdot & b & \cdot \\ \cdot & \cdot & -c & \cdot & -h & \cdot & \cdot & -h & \cdot & \cdot & \cdot & c \end{array}\right] \left[\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_{10} \\ \hline u_{11} \\ u_{12} \\ \hline 0 \\ 0 \\ 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} (L)_{7 \times 7} & (H)^T \; 7 \times 2 & (V)^T \; 7 \times 3 \\ \hline (H)_{2 \times 7} & (P)_{2 \times 2} & (W)^T \; 2 \times 3 \\ \hline (V)_{3 \times 7} & (W)_{3 \times 2} & (R) \; 3 \times 3 \end{array}\right] \qquad (2.5\text{-}9)$$

Partition the stiffness matrix as shown in (2.5-9), we obtain

$$\begin{Bmatrix} F_A \\ F_{10} \end{Bmatrix} = (L) \begin{Bmatrix} u_A \\ u_{10} \end{Bmatrix} + (H)^T \begin{Bmatrix} u_{11} \\ u_{12} \end{Bmatrix} \qquad (2.5\text{-}10)$$

$$\begin{Bmatrix} F_7 \\ F_8 \\ F_9 \end{Bmatrix} = (V) \begin{Bmatrix} u_A \\ \\ u_{10} \end{Bmatrix} + (W) \begin{Bmatrix} u_{11} \\ \\ u_{12} \end{Bmatrix} \qquad (2.5\text{-}11)$$

$$\begin{Bmatrix} 0 \\ \\ 0 \end{Bmatrix} = (H) \begin{Bmatrix} u_A \\ \\ u_{10} \end{Bmatrix} + (P) \begin{Bmatrix} u_{11} \\ \\ u_{12} \end{Bmatrix} \qquad (2.5\text{-}12)$$

Premultiply $(P)^{-1}$ to (2.5-12) and solve $\begin{Bmatrix} u_{11} \\ u_{12} \end{Bmatrix}$ in terms of $\begin{Bmatrix} u_A \\ u_{10} \end{Bmatrix}$

$$\begin{Bmatrix} u_{11} \\ u_{12} \end{Bmatrix} = -(P)^{-1} \quad (H) \begin{Bmatrix} u_A \\ u_{10} \end{Bmatrix} \qquad (2.5\text{-}13)$$

Substitute (2.5-13) into (2.5-10) and (2.5-11)

$$\begin{Bmatrix} F_A \\ F_{10} \end{Bmatrix} = \quad (L) - (H)^T (P)^{-1} (H) \quad \begin{Bmatrix} u_A \\ u_{10} \end{Bmatrix} \qquad (2.5\text{-}14)$$

$$\begin{Bmatrix} F_7 \\ F_8 \\ F_9 \end{Bmatrix} = \quad (V) - (W)(P)^{-1}(H) \quad \begin{Bmatrix} u_A \\ u_{10} \end{Bmatrix} \qquad (2.5\text{-}15)$$

If the end A of AB is simply supported then,

$$\begin{aligned} u_1 = u_2 = u_3 &= 0 \\ F_5 = F_6 = \quad &\ 0 \end{aligned} \qquad (2.5\text{-}16)$$

The force displacement relationship is

$$\begin{bmatrix} F_7 \\ F_8 \\ F_9 \\ F_{10} \\ F_{11} \\ F_{12} \\ F_4 \\ \hline 0 \\ 0 \\ \hline F_1 \\ F_2 \\ F_3 \end{bmatrix} = \left[\begin{array}{ccccccc|cc|ccc}
a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -a & \cdot & \cdot \\
\cdot & b & \cdot & \cdot & \cdot & -g & \cdot & \cdot & -g & \cdot & -b & \cdot \\
\cdot & \cdot & c & \cdot & -h & \cdot & \cdot & -b & \cdot & \cdot & \cdot & -c \\
\cdot & \cdot & \cdot & d & \cdot & \cdot & -d & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -h & \cdot & e & \cdot & \cdot & j & \cdot & \cdot & \cdot & h \\
\cdot & -g & \cdot & \cdot & \cdot & f & \cdot & \cdot & k & \cdot & g & \cdot \\
\cdot & \cdot & \cdot & -d & \cdot & \cdot & d & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & -h & \cdot & j & \cdot & \cdot & e & \cdot & \cdot & \cdot & h \\
\cdot & -g & \cdot & \cdot & \cdot & k & \cdot & \cdot & f & \cdot & g & \cdot \\
\hline
-a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a & \cdot & \cdot \\
\cdot & -b & \cdot & \cdot & \cdot & g & \cdot & \cdot & g & \cdot & b & \cdot \\
\cdot & \cdot & -c & \cdot & h & \cdot & \cdot & h & \cdot & \cdot & \cdot & c
\end{array}\right] \begin{bmatrix} u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \\ u_4 \\ \hline u_5 \\ u_6 \\ \hline 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \left[\begin{array}{c|c|c}
(L)'_{7 \times 7} & (H')^T_{7 \times 2} & (V')^T_{7 \times 3} \\
\hline
(H)'_{2 \times 7} & (P)_{2 \times 2} & (W')^T_{2 \times 3} \\
\hline
(V)'_{3 \times 7} & (W)'_{3 \times 2} & (R)_{3 \times 3}
\end{array}\right] \qquad (2.5\text{-}17)$$

We obtain

$$\begin{Bmatrix} F_B \\ F_4 \end{Bmatrix} = (L)' - (H')^T (P)^{-1} (H)' \begin{Bmatrix} u_B \\ u_4 \end{Bmatrix} \qquad (2.5\text{-}18)$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = (V)' - (W)' (P)^{-1} (H)' \begin{Bmatrix} u_B \\ u_4 \end{Bmatrix} \qquad (2.5\text{-}19)$$

The stiffness matrix of the member AB with one end simply supported is

$$(k_u) = (L)' - (H')^T (P)^{-1} (H)' \qquad (2.5\text{-}20)$$

(3) <u>Stiffness matrix in joint coordinate system</u>

Let the (β') be transformation matrix between element displacement vector of AB, $\{u\}_j$, and joint displacement vector at joint A and joint B, $\{q_A , q_B\}$

$$\{u\}_j = (\beta') \{q_A , q_B\} \qquad (2.5\text{-}21)$$

The stiffness matrix of AB in joint coordinate system can be written as

$$(k_q) = (\beta')^T (k_u) (\beta') \qquad (2.5\text{-}22)$$

(4) <u>Element mass matrix in element coordinate and in joint coordinate system.</u> The derivation of element mass matrix is similar to that of element stiffness matrix.

a. In the element coordinate system, the mass matrix is.

$$(m_u) = \begin{bmatrix}
a^* & \cdot & \cdot & \cdot & \cdot & \cdot & f^* & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & b^* & \cdot & \cdot & \cdot & e^* & \cdot & g^* & \cdot & \cdot & \cdot & k^* \\
\cdot & \cdot & b^* & \cdot & e^* & \cdot & \cdot & \cdot & g^* & \cdot & k^* & \cdot \\
\cdot & \cdot & \cdot & c^* & \cdot & \cdot & \cdot & \cdot & \cdot & h^* & \cdot & \cdot \\
\cdot & \cdot & e^* & \cdot & d^* & \cdot & \cdot & \cdot & -k^* & \cdot & j^* & \cdot \\
\cdot & e^* & \cdot & \cdot & \cdot & d^* & \cdot & -k^* & \cdot & \cdot & \cdot & j^* \\
f^* & \cdot & \cdot & \cdot & \cdot & \cdot & a^* & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & g^* & \cdot & \cdot & \cdot & -k^* & \cdot & b^* & \cdot & \cdot & \cdot & -e^* \\
\cdot & \cdot & g^* & \cdot & -k^* & \cdot & \cdot & \cdot & b^* & \cdot & -e^* & \cdot \\
\cdot & \cdot & \cdot & h^* & \cdot & \cdot & \cdot & \cdot & \cdot & c^* & \cdot & \cdot \\
\cdot & \cdot & k^* & \cdot & j^* & \cdot & \cdot & \cdot & -e^* & \cdot & d^* & \cdot \\
\cdot & k^* & \cdot & \cdot & \cdot & j^* & \cdot & -e^* & \cdot & \cdot & \cdot & d^*
\end{bmatrix} \qquad (2.5\text{-}23)$$

See Appendix 2 for elements a*, b*,...,k*.

Let

$$(A) = \begin{bmatrix} a^* & \cdot & \cdot \\ \cdot & b^* & \cdot \\ \cdot & \cdot & b^* \end{bmatrix} \quad (B) = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & e^* \\ \cdot & e^* & \cdot \end{bmatrix} \quad (C) = \begin{bmatrix} f^* & \cdot & \cdot \\ \cdot & g^* & \cdot \\ \cdot & \cdot & g^* \end{bmatrix}$$

$$(D) = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & k^* \\ \cdot & k^* & \cdot \end{bmatrix} \quad (E) = \begin{bmatrix} c^* & \cdot & \cdot \\ \cdot & d^* & \cdot \\ \cdot & \cdot & d^* \end{bmatrix} \quad (G) = \begin{bmatrix} h^* & \cdot & \cdot \\ \cdot & j^* & \cdot \\ \cdot & \cdot & j^* \end{bmatrix}$$

Then

$$(m_u) = \begin{bmatrix} (A) & (B) & (C) & (D) \\ (B) & (E) & -(D) & (G) \\ (C) & -(D) & (A) & -(B) \\ (D) & (G) & -(B) & (E) \end{bmatrix} \qquad (2.5\text{-}24)$$

b. In the joint coordinate system

$$(m_q) = (\beta')^T (m_u) (\beta') \qquad (2.5\text{-}25)$$

(B) Stiffness matrix and mass matrix of the total structural system

The stiffness matrix and mass matrix of an elastic structural system can be obtained from the synthesis of that of the member elements of the system. Let R be the number of member elements in the system and (k_u^i), (m_u^i) $(i = 1,\ldots,R)$ be stiffness matrix and mass matrix of the i^{th} member respectively. It can be shown that

$$(K_u) = \begin{bmatrix} (k_u^1) & & & & & \\ & (k_u^2) & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & (k_u^R) \end{bmatrix} \qquad (2.5\text{-}26)$$

and

$$(M_u) \begin{bmatrix} (m_u^1) & & & & & \\ & (m_u^2) & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & (m_u^R) \end{bmatrix} \qquad (2.5\text{-}27)$$

The matrices (K_u) and $M_u)$ are stiffness matrix and mass matrix of the structural system in element coordinate system respectively. In joint coordinate system they can be written as

$$(K_q) = (\beta)^T (K_u) (\beta) \qquad (2.5\text{-}28)$$

and

$$(M_q) = (\beta)^T (M_u) (\beta) \qquad (2.5\text{-}29)$$

Where (β) is the transformation matrix between $\{u\}$ and $\{q\}$

$$\{u\} = (\beta) \{q\} \qquad (2.5\text{-}30)$$

CHAPTER 3 FRAMED DOME

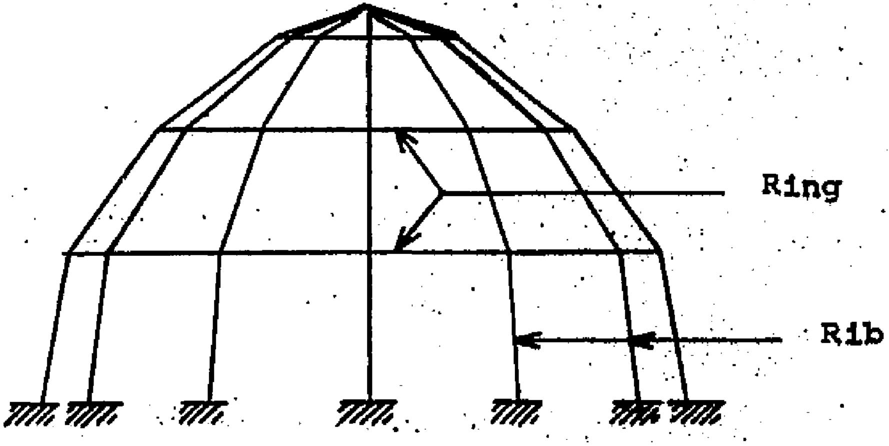
3.0 INTRODUCTION

In this chapter the free vibrations of a dome-type frame structure are studied. The structure is a solid-rib type dome as shown in Fig. 3.1. The analysis assumes that the masses of the member elements of the dome are lumped at the joints of the dome. The natural frequencies and mode shapes of a 60 degrees-of-freedom dome are calculated, and the numerical results are verified experimentally in chapter 4. The purpose of this study is to determine whether the lump-mass model is a satisfactory model of the real structure.

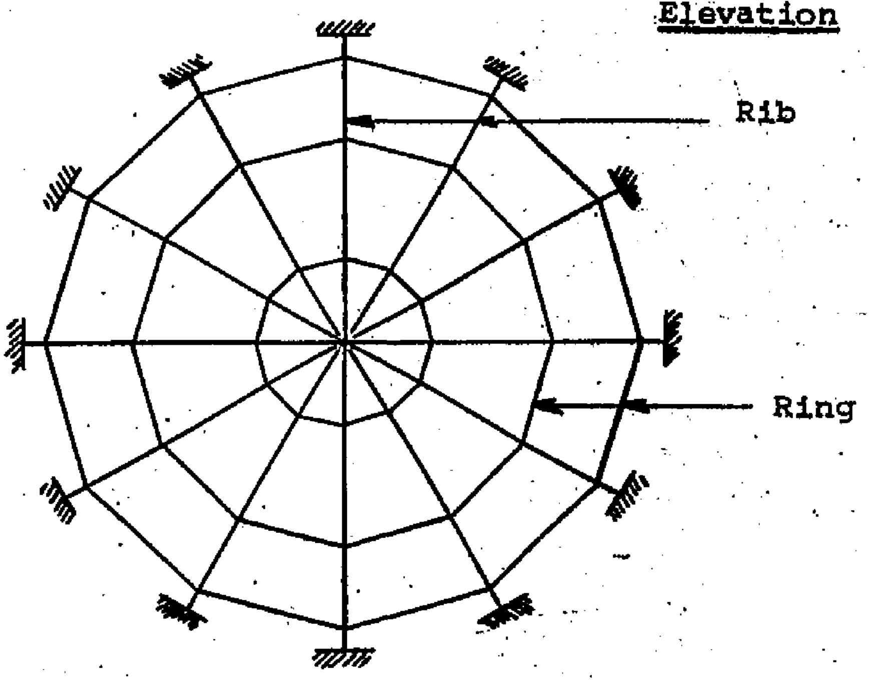
3.1 THE DOME

It is known that the metallic framed dome structure is one of the most efficient space structural systems. It is used for radomes, radio telescopes, roofs of auditoriums, fieldhouses, coliseums, and other large structures where intermediate supports are not permitted. The advantages of dome structures have been long recognized, but only since the recent development of structural analysis methods and construction techniques have large numbers of metallic domes been constructed economically.

A number of large metallic framed domes have been built in this country. The Astrodome in Houston, Tex. has a clear span of 642 ft. and an outer diameter of 710 ft.



Elevation



Plan

Fig. 3.1 SOLID-RIB DOME

The roof is 202 ft. above the ground level. The roof of the General Electric Company's pavilion at the 1964-1965 New York World's Fair is a dome 200 ft. in diameter and 78 ft. in height. The roof of the New York's World's Fair Pavilion is 176 ft. in diameter. The roof of the Pittsburgh Auditorium has a diameter of 417 ft. and a rise of 109 ft. The base diameter of the Charlotte, N.C. Coliseum is 332 ft., and its crown rises 120 ft. above the street level. All of these framed domes have an extremely large number of member elements and joints. Consequently, the dynamic behavior of dome structures is difficult to analyze.

The solid-rib type dome consists of concentric polygonal rings at different elevations, rigidly connected with inclined ribs at each corner of the ring. Under dead load, the uppermost ring is a compression ring and the lowest one is a tension ring. Elements of the dome are usually straight members composed of solid-web, rolled, or built-up sections. The height between two adjacent horizontal planes containing rings can be varied. The angles of inclination of the ribs with respect to horizontal plane may be different at different tiers. All the members of a polygonal ring are of the same length. The central angles extended over each member are equal. The dome is supported by foundation walls, columns or buttresses.

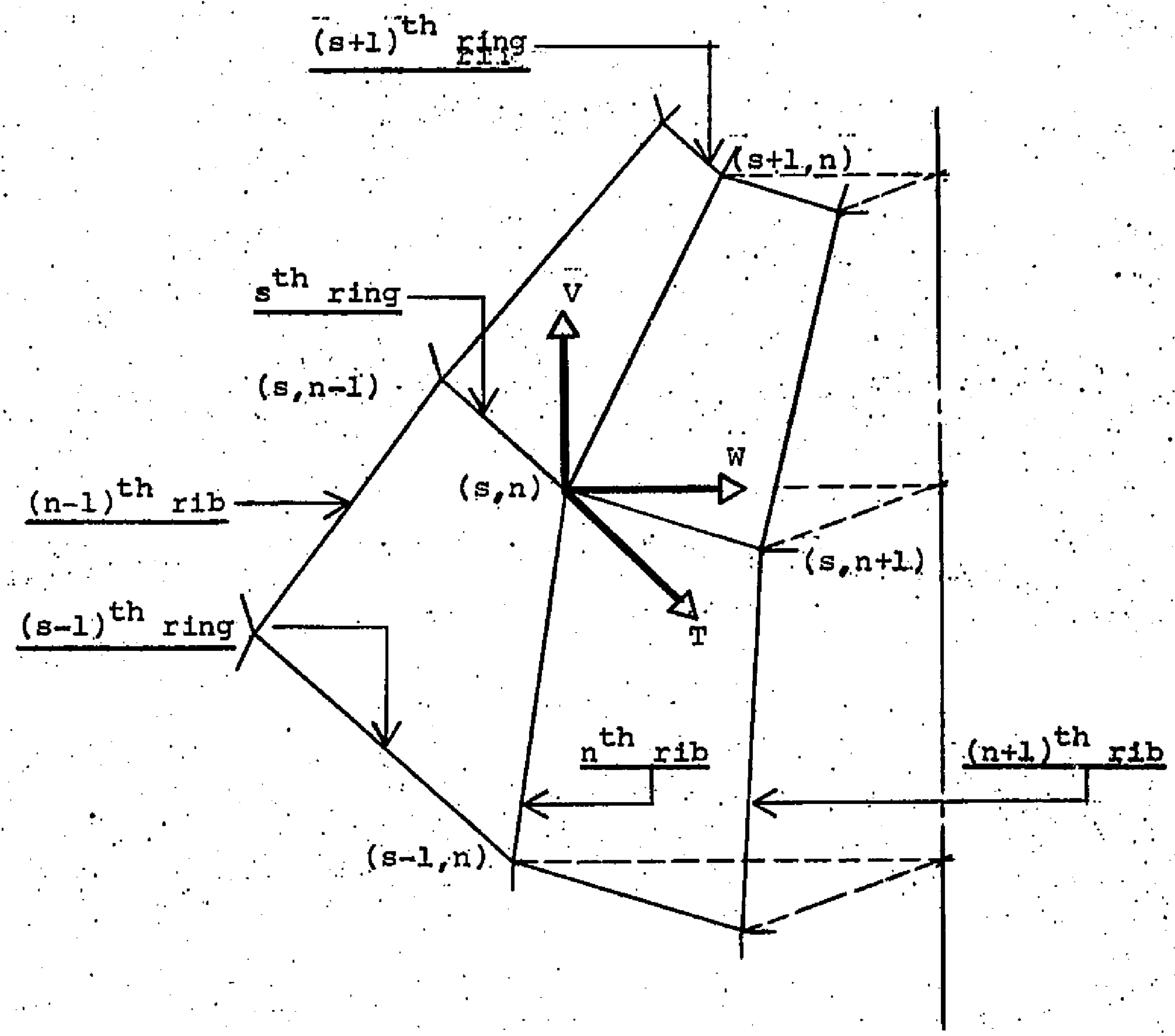
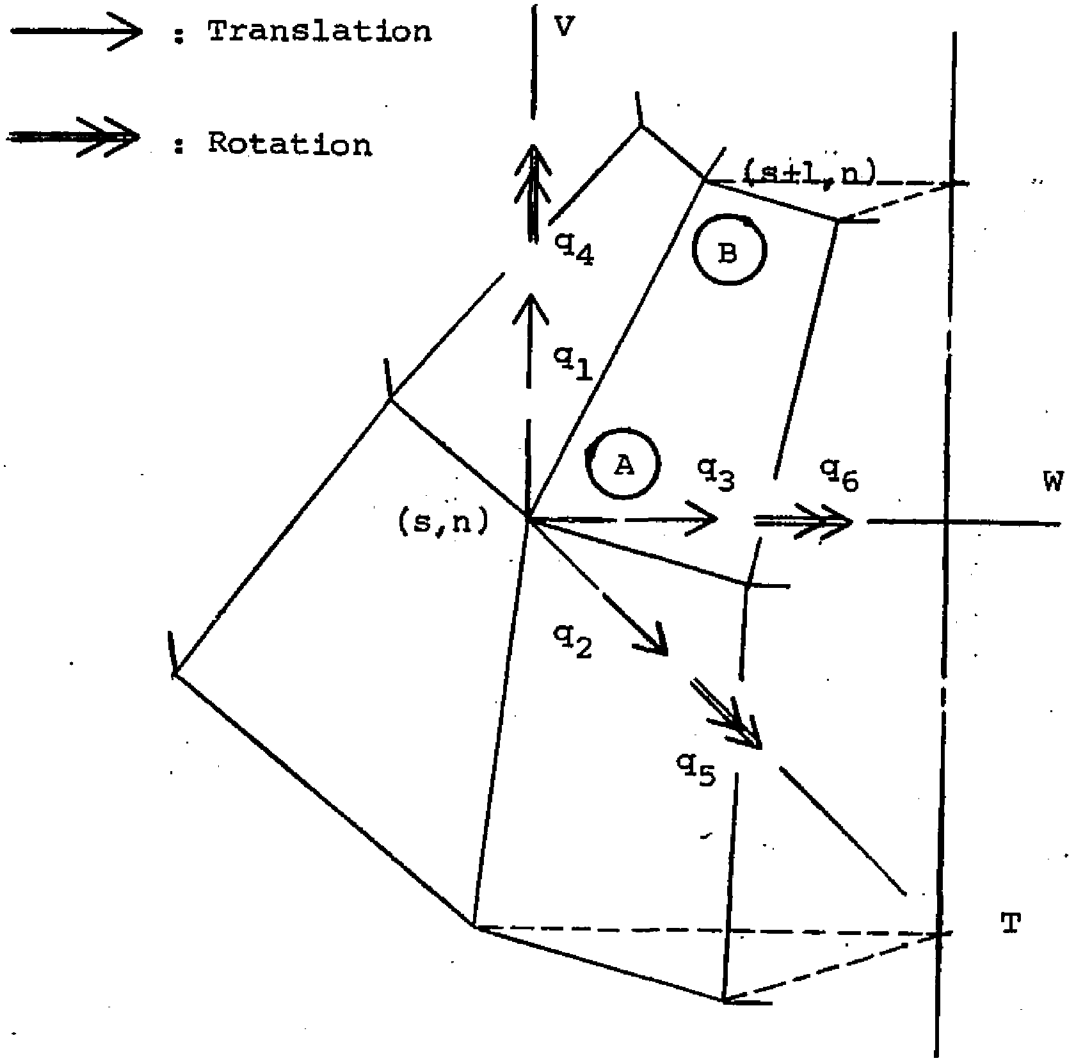


Fig. 3.2a Joint Coordinate System: V,T,W.

V: Vertical
T: Tangential
W: Radial (inward)

(s,n): A joint of the dome where ring s intersects rib n.

Fig. 3.2



<u>Fig. 3.2b</u> Joint Displacement Vector Components. Joint Displacement vector $\{q\}_{(s,n)}$

$$\{q\}_{(s,n)} = \{q_1, q_2, q_3, q_4, q_5, q_6\}_{(s,n)}$$

q_1, q_2, q_3: Translation in direction of V,T,W.

q_4, q_5, q_6: Rotation in direction of V,T,W;

<u>Fig. 3.2</u> (cont'd)

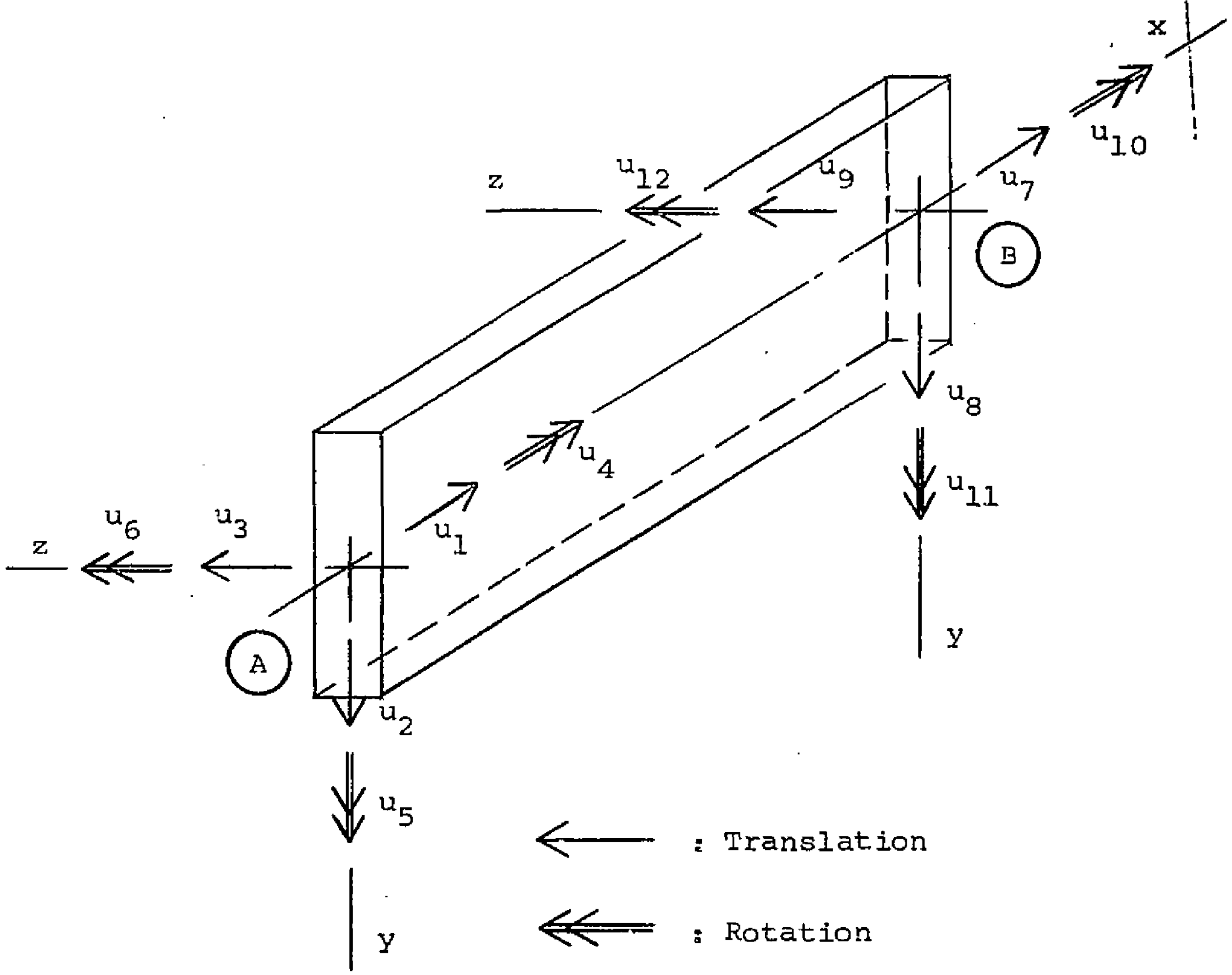


Fig. 3.2c Element Displacement Vector Components.

Element Displacement Vector $\{u\}_{AB}$

$$\{u\}_{AB} = \{ u^A, u^B \}$$

$$\{u^A\} = \{ u_1, u_2, u_3, u_4, u_5, u_6 \}$$

$$\{u^B\} = \{ u_7, u_8, u_9, u_{10}, u_{11}, u_{12} \}$$

Fig. 3.2 (cont'd)

3.2 ELEMENT AND JOINT COORDINATE SYSTEMS

As shown in Fig. 3.2, let us number the rings and ribs of the dome respectively by the numbers s and n (s = 1,2,3, ...; n = 1,2,3,...). The joint where the s^{th} ring and n^{th} rib intersect is designated as(s,n). At each joint of the dome we attach a right handed joint coordinate system V,T,W along the vertical, tangential and inward radial direction respectively. The joint displacement vector at a joint (s,n) is specified as

$$\left\{ q \right\}(s,n) = \left\{ q_1, q_2, q_3, q_4, q_5, q_6 \right\}_{(s,n)}$$

where q_1, q_2, q_3 are translational ; q_4, q_5, q_6 are rotational displacement components at joint (s,n) in the respective direction of V,T, and W. The coordinate system attached to any member element indicated as (s,n) - (s+1,n) or AB, is a right handed rectangular coordinate system x,y,z. The x direction is along the axis of the member; the y and z are along two principal directions of the cross section of the member is specified as $\left\{u^A\right\}$ or $\left\{u^B\right\}$ where

$$u^A = \left\{ u_1, u_2, u_3, u_4, u_5, u_6 \right\}$$

$$u^B = \left\{ u_7, u_8, u_9, u_{10}, u_{11}, u_{12} \right\}$$

Notice that u_1, u_2, u_3 and u_7, u_8, u_9 are translational displacements at ends A and B respectively; u_4, u_5, u_6 and u_{10}, u_{11}, u_{12} are rotational displacements at end A and B in direction of x, y and z.

Let N and R be the number of joints and the number of member elements of the dome respectively. We specify the following notation:

$\left\{F\right\}_i$: End force and end displacement vector of the i^{th} member element of the dome, i=1,...,R

$\left\{u\right\}_i$:

$\left\{Q\right\}_j$: Joint force and joint displacement vector of the j^{th} joint of the dome, j = 1,...N

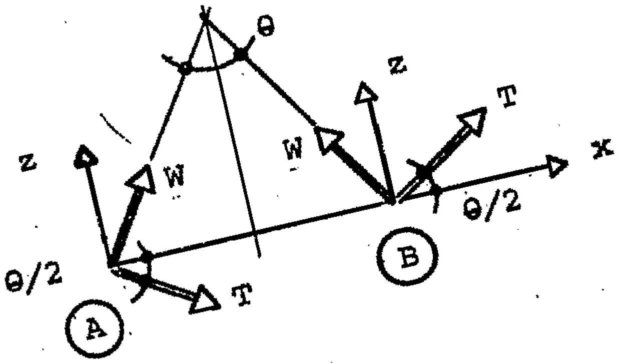
$\left\{q\right\}_i$:

It is clear from Fig. 3.1 that the geometric configuration at each joint is typical throughout the structural system. As a result, the coordinate transformation matrix between coordinate systems F-u and Q-q can be obtained from the study of a joint.

As shown in Fig. 3.2 and Fig. 3.3, the direction cosines between system and element coordinate systems can be expressed as

For vertical member at both ends

	x	y	z
V	$\sin\psi$	$-\cos\psi$	0
T	0	0	-1
W	$\cos\psi$	$\sin\psi$	0



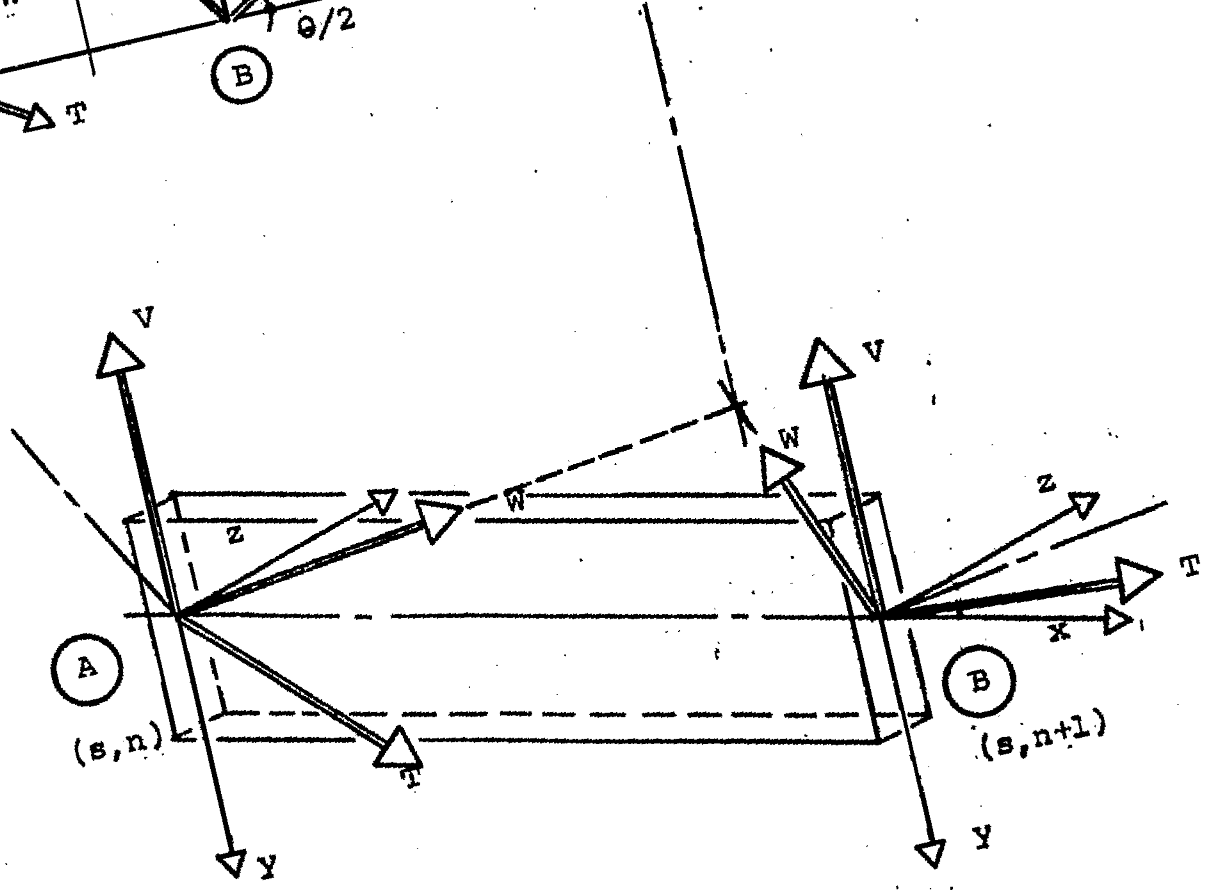
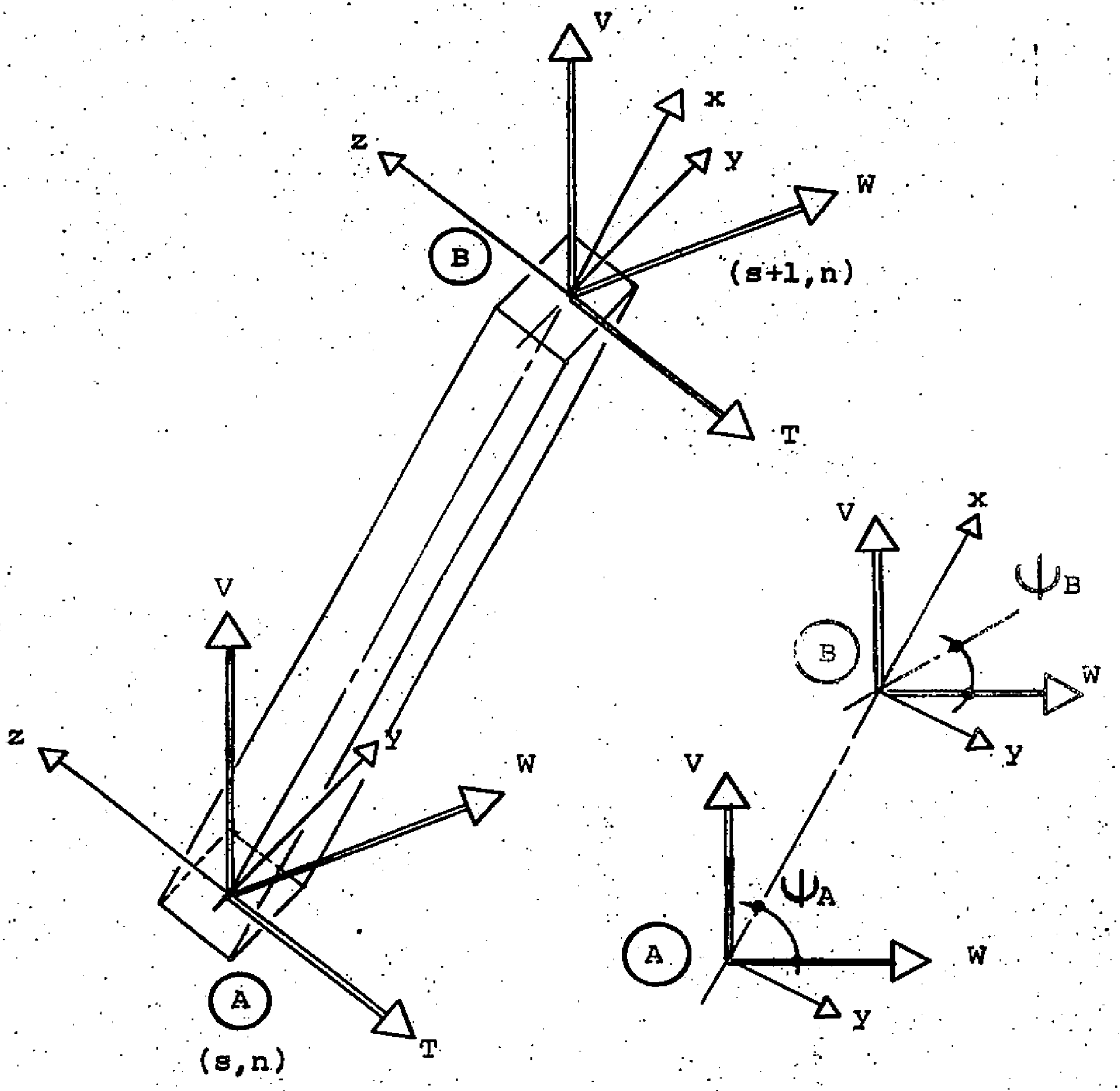


Fig.3.3a Horizontal Member

θ: Central Angle

Fig.3.3 Joint and Element Coordinate System

Element Coordinate System: x,y,z.
Joint Coordinate System : V,T,W.



<u>Fig. 3.3b</u> Vertical Member

ψ: Angle of Inclination

<u>Fig. 3.3</u> (cont'd)

For horizontal member at end A

	x	y	z
V	0	-1	0
T	$\cos\theta/2$	0	$-\sin\theta/2$
W	$\sin\theta/2$	0	$\cos\theta/2$

For horizontal member at end B

	x	y	z
V	0	-1	0
T	$\cos\theta/2$	0	$\sin\theta/2$
W	$-\sin\theta/2$	0	$\cos\theta/2$

Where ψ is the angle of inclination of the vertical member with respect to the horizontal plane and θ is the central angle extended by the horizontal member AB.

At a typical joint, (s,n), of the dome (Fig. 3.2), there are four connected members. The element end displacement components at this joint are respectively

$$\left\{ u_1, u_2, u_3, u_4, u_5, u_6 \right\}_{(s,n)-(s+1,n)} = u^A_{(s,n)-(s+1,n)}$$

$$\left\{ u_7, u_8, u_9, u_{10}, u_{11}, u_{12} \right\}_{(s-1,n)-(s,n)} = u^B_{(s-1,n)-(s,n)}$$

for vertical members (s,n)-(s+1,n) and (s-1,n)-(s,n), and

$$\left\{ u_1, u_2, u_3, u_4, u_5, u_6 \right\}_{(s,n)-(s,n+1)} = u^A_{(s,n)-(s,n+1)}$$

$$\left\{ u_7, u_8, u_9, u_{10}, u_{11}, u_{12} \right\}_{(s,n-1)-(s,n)} = u^B_{(s,n-1)-(s,n)}$$

for horizontal members (s,n)-(s,n+1) and (s,n-1)-(s,n).

Let joint displacement vector at the joint (s,n) be

$$\left\{ q_1,\ q_2,\ q_3,\ q_4,\ q_5,\ q_6 \right\}_{(s,n)} = q_{(s,n)}$$

We botain

$$\left\{ \begin{array}{c} u^{A}_{(s,n)-(S+1,n)} \\ \\ \hline u^{B}_{(s-1,n)-(s,n)} \\ \\ \hline u^{A}_{(s,n)-(s,n+1)} \\ \\ \hline u^{B}_{(s,n-1)-(s,n)} \\ \\ \end{array} \right\}_{24\mathrm{x}1} = \left[\begin{array}{cc} (t)_{s,s+1} & (0) \\ (0) & (t)_{s,s+1} \\ \hline (t)_{s-1,s} & (0) \\ (0) & (t)_{s-1,s} \\ \hline (r) & (0) \\ (0) & (r) \\ \hline (n) & (0) \\ (0) & (n) \end{array} \right]_{24\mathrm{x}6} \left\{ q_{(s,n)} \right\}_{6\mathrm{x}1} \qquad (3.2\text{-}1)$$

where

$$(t) = \begin{bmatrix} \mathrm{Sin}\,\psi & 0 & \mathrm{Cos}\,\psi \\ -\mathrm{Cos}\,\psi & 0 & \mathrm{Sin}\,\psi \\ 0 & -1 & 0 \end{bmatrix}$$

$$(n) = \begin{bmatrix} 0 & \mathrm{Cos}\theta/2 & -\mathrm{Sin}\theta/2 \\ -1 & 0 & 0 \\ 0 & \mathrm{Sin}\theta/2 & \mathrm{Cos}\theta/2 \end{bmatrix}$$

$$(r) = \begin{bmatrix} 0 & \mathrm{Cos}\theta/2 & \mathrm{Sin}\theta/2 \\ -1 & 0 & 0 \\ 0 & -\mathrm{Sin}\theta/2 & \mathrm{Cos}\theta/2 \end{bmatrix}$$

3.3 ELEMENT MASS MATRIX AND ELEMENT STIFFNESS MATRIX

(3.2-1) is the relation between joint displacement vector $\{q_{(s,n)}\}$ and the components of end displacement vectors of all member elements connected at the joint. The transformation matrix is a 24x6 rectangular matrix. If we write out the relationship for all the joints of the dome, we obtain a 12Rx6N transformation matrix for the entire system between joint displacement vector $\{q\}$ and element displacement vector $\{u\}$ as

$$\{u\}_{12R \times 1} = (\beta'')_{12R \times 6N} \{q\}_{6N \times 1} \qquad (3.3\text{-}1)$$

Under the coordinate transformation in (3.3-1), we have the expressions for element stiffness matrix (k_{AB}) and element mass matrix (m_{AB}) in joint coordinate system as shown in (3.3-2) through (3.3-5). Notice that the values of a,b,...,k in (3.3-2) and (3.3-3), and that of a*,b*,...,k* in (3.3-4) and (3.3-5) are listed in Appendix 2.

Transformed element stiffness matrix for vertical member

$$\begin{bmatrix} k_{AB} \end{bmatrix} = (kv_{ij}) \qquad (3.3\text{-}2)$$

where $kv_{ij} = kv_{ji}$, $kv_{ij} = 0$ except

$kv_{1,1} = aA^2 + bB^2$
$kv_{1,3} = (a + b)AB$
$kv_{1,5} = -\ gB$
$kv_{1,7} = -aA^2 - bB^2$
$kv_{1,9} = -(a + b)AB$
$kv_{1,11} = -\ gB$
$kv_{2,2} = c$
$kv_{2,4} = hB$
$kv_{2,6} = hA$
$kv_{2,8} = -c$
$kv_{2,10} = hB$
$kv_{2,12} = hA$
$kv_{3,3} = aB^2 + bA^2$
$kv_{3,5} = -\ gA$
$kv_{3,7} = -(a + b)AB$
$kv_{3,9} = -aB^2 - bA^2$
$kv_{3,11} = -\ gA$
$kv_{4,4} = dA^2 + eB^2$
$kv_{4,6} = (d + e)AB$
$kv_{4,8} = -hB$
$kv_{4,10} = -dA^2 - jB^2$
$kv_{4,12} = (j - d)AB$
$kv_{5,5} = f$
$kv_{5,7} = gB$
$kv_{5,9} = gA$
$kv_{5,11} = k$
$kv_{6,6} = dB^2 + dA^2$
$kv_{6,8} = -\ hA$
$kv_{6,10} = (j - d)AB$
$kv_{6,12} = -dB^2 + jA^2$
$kv_{7,7} = aA^2 + bB^2$
$kv_{7,9} = (a + b)AB$

$kv_{7,11} = gB$
$kv_{8,8} = c$
$kv_{8,10} = -hB$
$kv_{8,12} = -hA$
$kv_{9,9} = aB^2 + bA^2$
$kv_{9,11} = gA$
$kv_{10,10} = dA^2 + eB^2$
$kv_{10,12} = (d + e)AB$
$kv_{11,11} = f$
$kv_{12,12} = dB^2 + eA^2$

Transformed element stiffness matrix for horizontal member

$$\begin{bmatrix} k_{AB} \end{bmatrix} = (kh_{ij}) \qquad (3.3\text{-}3)$$

where $kh_{ij} = kh_{ji}$, $kh_{ij} = 0$ except

$kh_{1,1} = b$
$kh_{1,5} = gP$
$kh_{1,6} = gQ$
$kh_{1,7} = -b$
$kh_{1,11} = gP$
$kh_{1,12} = gQ$
$kh_{2,2} = aQ^2 + cP^2$
$kh_{2,3} = (a + c)PQ$
$kh_{2,4} = -hP$
$kh_{2,8} = -aQ^2 - cP^2$
$kh_{2,9} = -(a + c)PQ$
$kh_{2,10} = -hP$
$kh_{3,3} = aP^2 + cQ^2$
$kh_{3,4} = -hQ$
$kh_{3,8} = -(a + c)PQ$
$kh_{3,9} = -aP^2 - cQ^2$
$kh_{3,10} = -hQ$
$kh_{4,4} = e$
$kh_{4,8} = hP$
$kh_{4,9} = hQ$
$kh_{4,10} = j$

$kh_{5,5} = dQ^2 + fP^2$
$kh_{5,6} = (d + f)PQ$
$kh_{5,7} = - gP$
$kh_{5,11} = kP^2 - dQ^2$
$kh_{5,12} = (k - d)PQ$
$kh_{6,6} = dP^2 + fQ^2$
$kh_{6,7} = - gQ$
$kh_{6,11} = (k - d)PQ$
$kh_{6,12} = kQ^2 - dP^2$
$kh_{7,7} = b$
$kh_{7,11} = - gP$
$kh_{7,12} = - gQ$
$kh_{8,8} = aQ^2 + cP^2$
$kh_{8,9} = (a + c)PQ$
$kh_{8,10} = hP$
$kh_{9,9} = aP^2 + cQ^2$
$kh_{9,10} = hQ$
$kh_{10,10} = e$
$kh_{11,11} = dQ^2 + fP^2$
$kh_{11,12} = (d + f)PQ$
$kh_{12,12} = dP^2 + fQ^2$

Transformed element mass matrix for vertical member

$$\left[m_{AB} \right] = (mv_{ij}) \tag{3.3-4}$$

where $mv_{ij} = mv_{ji}$, $mv_{ij} = 0$ except

$mv_{1,1} = a^*A^2 + b^*B^2$
$mv_{1,3} = (a^* + b^*)AB$
$mv_{1,5} = e^*B$
$mv_{1,7} = f^*A^2 + g^*B^2$
$mv_{1,9} = (f^* + g^*)AB$
$mv_{1,11} = - k^*B$
$mv_{2,2} = b^*$
$mv_{2,4} = - e^*B$
$mv_{2,6} = - e^*A$
$mv_{2,8} = g^*$
$mv_{2,10} = k^*B$
$mv_{2,12} = k^*A$
$mv_{3,3} = a^*B^2 + b^*A^2$

$$\begin{aligned}
mv_{3,5} &= e^*A \\
mv_{3,7} &= (f^* + g^*)AB \\
mv_{3,9} &= f^*B^2 + g^*A^2 \\
mv_{3,11} &= -k^*A \\
mv_{4,4} &= c^*A^2 + d^*B^2 \\
mv_{4,6} &= (c^* + d^*)AB \\
mv_{4,8} &= -k^*B \\
mv_{4,10} &= h^*A^2 - j^*B^2 \\
mv_{4,12} &= (h^* - j^*)AB \\
mv_{5,5} &= d^* \\
mv_{5,7} &= k^*B \\
mv_{5,9} &= k^*A \\
mv_{5,11} &= -j^* \\
mv_{6,6} &= c^*B^2 + d^*A^2 \\
mv_{6,8} &= -k^*A \\
mv_{6,10} &= (h^* - j^*)AB \\
mv_{6,12} &= h^*B^2 - j^*A^2 \\
mv_{7,7} &= a^*A^2 + b^*B^2 \\
mv_{7,9} &= (a^* + b^*)AB \\
mv_{7,11} &= -e^*B \\
mv_{8,8} &= b^* \\
mv_{8,10} &= e^*B \\
mv_{8,12} &= e^*A \\
mv_{9,9} &= a^*B^2 + b^*A^2 \\
mv_{9,11} &= -e^*A \\
mv_{10,10} &= c^*A^2 + d^*B^2 \\
mv_{10,12} &= (c^* + d^*)AB \\
mv_{11,11} &= d^* \\
mv_{12,12} &= c^*B^2 + d^*A^2
\end{aligned}$$

Transformed element mass matrix for horizontal member

$$\left[m_{AB}\right] = (mh_{ij}) \qquad (3.3\text{-}5)$$

where $mh_{ij} = mh_{ji}$, $mh_{ij} = 0$ except

$$\begin{aligned}
mh_{1,1} &= b^* \\
mh_{1,5} &= -e^*P \\
mh_{1,6} &= -e^*Q \\
mh_{1,7} &= g^*
\end{aligned}$$

$mh_{1,11} = k*P$

$mh_{1,12} = k*Q$

$mh_{2,2} = a*Q^2 + b*P^2$

$mh_{2,3} = (a* + b*)PQ$

$mh_{2,4} = e*P$

$mh_{2,8} = f*Q^2 + g*P^2$

$mh_{2,9} = (f* + g*)PQ$

$mh_{2,10} = - k*P$

$mh_{3,3} = a*P^2 + b*Q^2$

$mh_{3,4} = e*Q$

$mh_{3,8} = (f* + g*)PQ$

$mh_{3,9} = f*P^2 + g*Q^2$

$mh_{3,10} = - k*Q$

$mh_{4,4} = d*$

$mh_{4,8} = k*P$

$mh_{4,9} = k*Q$

$mh_{4,10} = j*$

$mh_{5,5} = c*Q^2 + d*P^2$

$mh_{5,6} = (c* + d*)PQ$

$mh_{5,7} = - k*P$

$mh_{5,11} = h*Q^2 - j*P^2$

$mh_{5,12} = (h* - j*)PQ$

$mh_{6,6} = c*P^2 + d*Q^2$

$mh_{6,7} = - k*Q$

$mh_{6,11} = (h* -j*)PQ$

$mh_{6,12} = h*P^2 - j*Q^2$

$mh_{7,7} = b*$

$mh_{7,11} = e*P$

$mh_{7,12} = e*Q$

$mh_{8,8} = a*Q^2 + b*P^2$

$mh_{8,9} = (a* + b*)PQ$

$mh_{8,10} = - e*P$

$mh_{9,9} = a*P^2 + b*Q^2$

$mh_{9,10} = - e*Q$

$mh_{10,10} = d*$

$mh_{11,11} = c*Q^2 + d*P^2$

$mh_{11,12} = (c* + d*)PQ$

$mh_{12,12} = c*P^2 + d*Q^2$

Note that $A=\mathrm{Sin}\psi$, $B=\mathrm{Cos}\psi$, $P=\mathrm{Sin}\theta/2$, and $Q=\mathrm{COs}\theta/2$.

3.4 NUMERICAL APPLICATION

The numerical example presented in this section is a 20-member 60 degrees-of-freedom framed dome model (Fig. 3.4). It has two pentagonal rings supported by inclined ribs. The angles of inclination of ribs with respect to horizontal plane are 60 degrees for lower tier and 45 degrees for upper tier. Each joint of the dome has been numbered as (1,1),...,(3,5). The joints (1,1),...,(1,5) are assumed to be fixed with foundations. Components of joint displacement vector are indicated by number 1,2,...,60. At each joint the first three numbers are translational displacement components and the last three numbers the rotational displacement components of the joint. For instance, at joint (3,1) 31,32 and 33 are translational and 34,35,36 are rotational displacement components in respective direction of V,T and W (Fig. 3.2a). Masses of the member elements are assumed to be lumped at each joint of the domes and indicated as small circles as shown in Fig. 3.4. The dimensions and physical properties of each member as well as the Code Numbers for the synthesis of dome stiffness and dome mass matrices are listed in the following table.

Table 3.1 Dimension and Physical Properties of Dome Model

MEMBER	LENGTH	CODE NUMBERS												ψ
(1,1)-(2,1)	L=5.0"	0	0	0	0	0	0	1	2	3	4	5	6	= 60 degrees
(1,2)-(2,2)		0	0	0	0	0	0	7	8	9	10	11	12	
(1,3)-(2,3)		0	0	0	0	0	0	13	14	15	16	17	18	
(1,4)-(2,4)		0	0	0	0	0	0	19	20	21	22	23	24	
(1,5)-(2,5)		0	0	0	0	0	0	25	26	27	28	29	30	
(2,1)-(3,1)	L=4.949"	1	2	3	4	5	6	31	32	33	34	35	36	= 45 degrees
(2,2)-(3,2)		7	8	9	10	11	12	37	38	39	40	41	42	
(2,3)-(3,3)		13	14	15	16	17	18	43	44	45	46	47	48	
(2,4)-(3,4)		19	20	21	22	23	24	49	50	51	52	53	54	
(2,5)-(3,5)		25	26	27	28	29	30	55	56	57	58	59	60	
(2,1)-(2,2)	L=7.641"	1	2	3	4	5	6	7	8	9	10	11	12	FOR ALL MEMBERS: AREA=(9/256) in^2 UNIT WT.=0.04114 $\#/in^3$ E=4.5 x 10^5 psi G=1.8 x 10^5 psi θ= 72 degrees
(2,2)-(2,3)		7	8	9	10	11	12	13	14	15	16	17	18	
(2,3)-(2,4)		13	14	15	16	17	18	19	20	21	22	23	24	
(2,4)-(2,5)		19	20	21	22	23	24	25	26	27	28	29	30	
(2,5)-(2,1)		25	26	27	28	29	30	1	2	3	4	5	6	
(3,1)-(3,2)	L=3.526"	31	32	33	34	35	36	37	38	39	40	41	42	
(3,2)-(3,3)		37	38	39	40	41	42	43	44	45	46	47	48	
(3,3)-(3,4)		43	44	45	46	47	48	49	50	51	52	53	54	
(3,4)-(3,5)		49	50	51	52	53	54	55	56	57	58	59	60	
(3,5)-(3,1)		55	56	57	58	59	60	31	32	33	34	35	36	

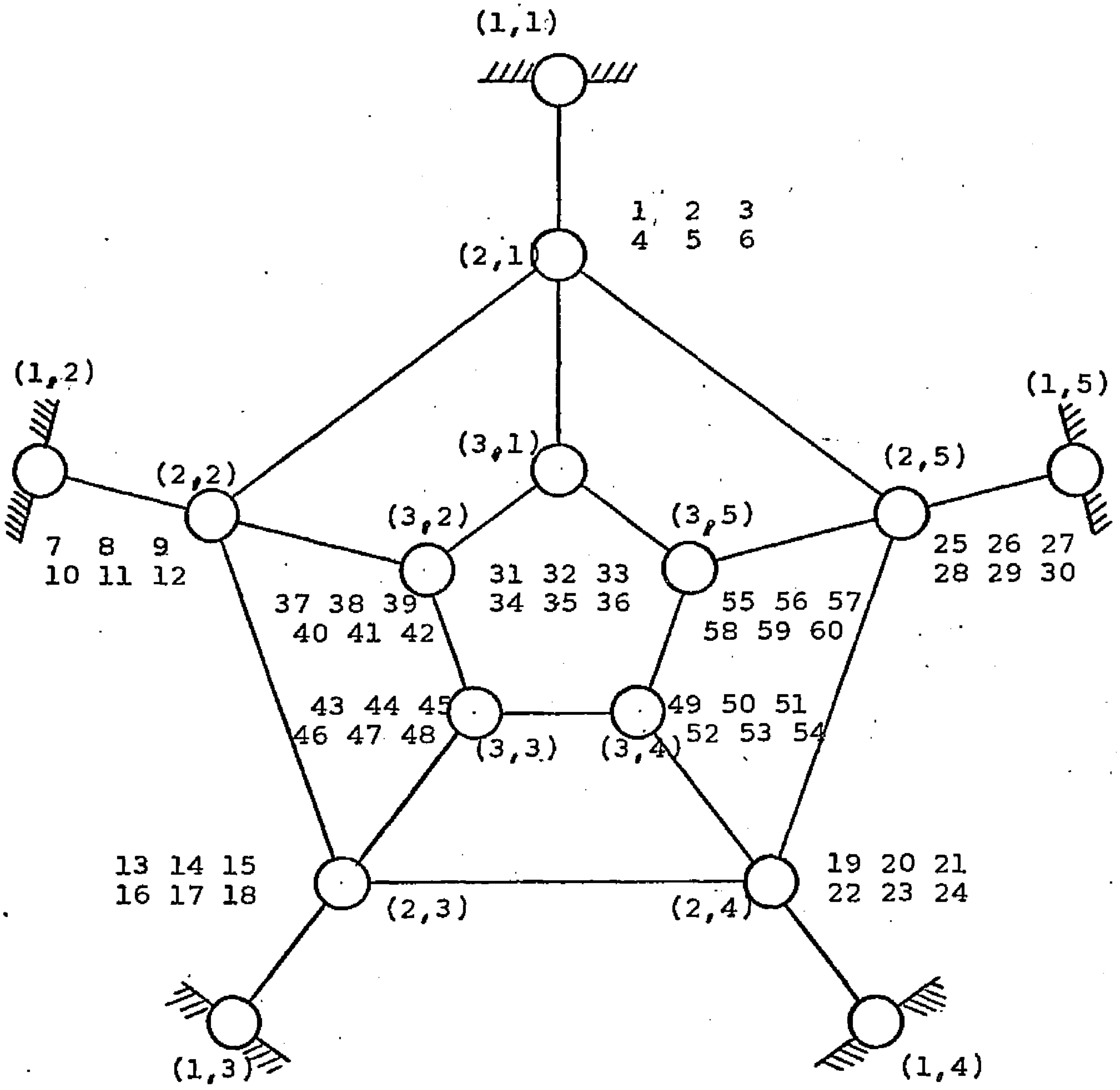


Fig. 3.4 Mass points and Joint displacement vector components of the dome model.

○ : Point mass (lumped from adjacent members).

(2,1): A joint of the dome where ring 2 intersects rib 1.

31,32,33 : Displacement components of the joint
34,35,36 (3,1)

31,32,33: Translation
34,35,36: Rotation

in resepctive direction of V,T, and W.
(Fig.3.2)

3.5 NATURAL FREQUENCIES AND MODE SHAPES OF THE DOME MODEL

Both mass matrix and stiffness matrix of the 60 degrees-of-freedom dome model were generated numerically (see appendix 3). Computer programs were written in FORTRAN (see appendix 4). Table 3.2 presents the calculated values of the natural frequencies(7). There are 21 multiplicities of order two. Each of the multiplicity corresponds to a symmetric and an anti-symmetric mode of vibration at the same frequency. Calculated mode shapes of the first and second mode of vibration are shown in Fig. 3.5, Fig. 3.6 and Fig. 3.7. Note that the mode shapes as shown in Fig. 3.6 and Fig. 3.7 are obtained from a linear combination of the two eigenvectors corresponding to the same frequency of 55.78 cps. The eigenvectors are given in Table 3.3d and Table 3.3e. The computations of the joint displacement component q_1 from the eigenvectors are given below.

(1) Decomposition of the eigenvector component $\left\{eq_1\right\}$.

$$\left\{\begin{matrix} 0.063 \\ -0.117 \\ 0.137 \\ 0.033 \\ 0.158 \\ 0.074 \\ -0.136 \\ -0.158 \\ 0.038 \\ 0.182 \end{matrix}\right\} = \left\{\begin{matrix} 0.048 \\ -0.128 \\ -0.128 \\ 0.048 \\ 0.158 \\ 0.056 \\ -0.147 \\ -0.147 \\ 0.056 \\ 0.182 \end{matrix}\right\}_{\text{sym.}} + \left\{\begin{matrix} 0.015 \\ 0.009 \\ -0.009 \\ -0.015 \\ 0.0 \\ 0.018 \\ 0.011 \\ -0.011 \\ -0.018 \\ 0.0 \end{matrix}\right\}_{\text{antisym.}}$$

$$\left\{\begin{matrix} -0.146 \\ -0.103 \\ 0.083 \\ 0.154 \\ 0.013 \\ -0.170 \\ -0.119 \\ 0.095 \\ 0.178 \\ 0.015 \end{matrix}\right\} = \left\{\begin{matrix} 0.004 \\ -0.010 \\ -0.010 \\ 0.004 \\ 0.013 \\ 0.004 \\ -0.012 \\ -0.012 \\ 0.004 \\ 0.015 \end{matrix}\right\}_{\text{sym.}} + \left\{\begin{matrix} -0.150 \\ -0.093 \\ 0.093 \\ 0.150 \\ 0.0 \\ -0.174 \\ -0.107 \\ 0.107 \\ 0.174 \\ 0.0 \end{matrix}\right\}_{\text{antisym.}}$$

(2) Linear combination of the eigenvectors.

a. <u>Symmetric Mode</u>

$$\left\{\begin{matrix} 0.052 \\ -0.138 \\ -0.138 \\ 0.052 \\ 0.171 \\ 0.060 \\ -0.159 \\ -0.159 \\ 0.060 \\ 0.197 \end{matrix}\right\} = \left\{\begin{matrix} 0.048 \\ -0.128 \\ -0.128 \\ 0.048 \\ 0.158 \\ 0.056 \\ -0.147 \\ -0.147 \\ 0.056 \\ 0.182 \end{matrix}\right\} + \left\{\begin{matrix} 0.004 \\ -0.010 \\ -0.010 \\ 0.004 \\ 0.013 \\ 0.004 \\ -0.012 \\ -0.012 \\ 0.004 \\ 0.015 \end{matrix}\right\}$$

b. <u>Antisymmetric Mode</u>

$$\left\{\begin{matrix} -0.135 \\ -0.084 \\ 0.084 \\ 0.135 \\ 0.0 \\ -0.156 \\ -0.096 \\ 0.096 \\ 0.156 \\ 0.0 \end{matrix}\right\} = \left\{\begin{matrix} 0.015 \\ 0.009 \\ -0.009 \\ -0.015 \\ 0.0 \\ 0.018 \\ 0.011 \\ -0.011 \\ -0.018 \\ 0.0 \end{matrix}\right\} + \left\{\begin{matrix} -0.150 \\ -0.093 \\ 0.093 \\ 0.150 \\ 0.0 \\ -0.174 \\ -0.107 \\ 0.107 \\ 0.174 \\ 0.0 \end{matrix}\right\}$$

TABLE 3.2 Natural Frequencies of the Dome Model

CPS	MULTIPLICITY
53.85	1
55.78	2
88.16	2
116.69	1
119.58	2
186.18	2
216.71	2
250.95	1
280.52	2
295.15	1
310.02	2
330.71	2
419.79	1
426.12	2
451.55	2
579.67	1
651.67	2
662.50	2
703.61	2
709.90	2
728.87	1
935.29	2
1062.80	1
1160.79	1
1301.90	2
1338.61	1
1445.37	2
1514.25	2
1777.29	2
2055.88	2
2715.99	1
2970.07	1
2973.87	1
3040.93	2
4003.90	1
4070.69	1
4071.86	1
4630.68	1
4644.70	1

TABLE 3.3a Joint Displacement Components
(First Mode f=53.85 cps)

q_1	q_2	q_3	q_4	q_5	q_6
0.000	0.347	0.000	0.043	0.000	0.048
0.000	0.347	0.000	0.043	0.000	0.048
0.000	0.347	0.000	0.043	0.000	0.048
0.000	0.347	0.000	0.043	0.000	0.048
0.000	0.347	0.000	0.043	0.000	0.048
0.000	0.260	0.000	0.083	0.000	0.006
0.000	0.260	0.000	0.083	0.000	0.006
0.000	0.260	0.000	0.083	0.000	0.006
0.000	0.260	0.000	0.083	0.000	0.006
0.000	0.260	0.000	0.083	0.000	0.006

TABLE 3.3b Joint Displacement Components
(Second Mode f=55.78, Symmetric Mode)

q_1	q_2	q_3	q_4	q_5	q_6
0.053	-0.283	-0.092	0.022	0.014	-0.010
-0.139	-0.175	0.241	0.014	-0.037	-0.007
-0.139	0.175	0.241	-0.014	-0.037	0.007
0.053	0.283	-0.092	-0.022	0.014	0.010
0.171	0.000	-0.298	0.000	0.045	0.000
0.061	-0.309	-0.100	0.005	-0.012	0.055
-0.160	-0.190	0.262	0.003	0.033	0.034
-0.160	0.190	0.262	-0.003	0.033	-0.034
0.061	0.309	-0.100	-0.005	-0.012	-0.055
0.198	0.000	-0.325	0.000	-0.040	0.000

TABLE 3.3c Joint Displacement Components
(Second Mode f=55.78, Antisymmetric Mode)

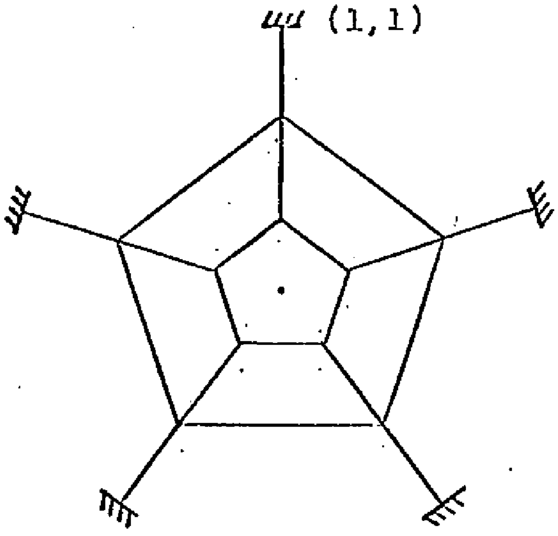
q_1	q_2	q_3	q_4	q_5	q_6
-0.135	-0.076	0.234	0.006	-0.036	-0.003
-0.083	0.199	0.144	-0.016	-0.022	0.008
0.083	0.199	-0.144	-0.016	0.022	0.008
0.135	-0.076	-0.234	0.006	0.036	-0.003
0.000	-0.246	0.000	0.019	0.000	-0.009
-0.155	-0.083	0.255	0.001	0.032	0.014
-0.096	0.217	0.157	-0.003	0.019	-0.039
0.096	0.217	-0.157	-0.003	-0.019	-0.039
0.155	-0.083	-0.255	0.001	-0.032	0.014
0.000	-0.268	0.000	0.004	0.000	0.048

TABLE 3.3d Joint Displacement Components
(Second Mode f=55.78, First Eigenvector)

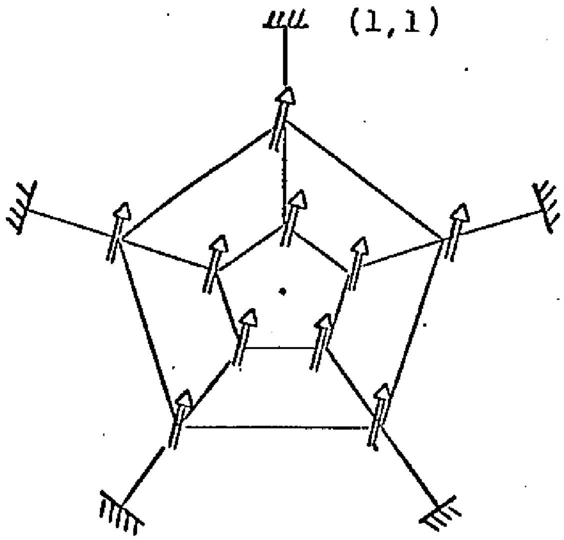
eq_1	eq_2	eq_3	eq_4	eq_5	eq_6
0.064	-0.252	-0.112	0.020	0.017	-0.010
-0.118	-0.184	0.205	0.014	-0.031	-0.007
-0.137	0.138	0.239	-0.011	-0.036	0.005
0.033	0.270	-0.057	-0.021	0.008	0.010
0.158	0.028	-0.274	-0.002	0.042	0.001
0.074	-0.275	-0.122	0.047	-0.015	0.049
-0.136	-0.201	0.223	0.034	0.028	0.036
-0.158	0.150	0.260	-0.026	0.032	-0.027
0.038	0.294	-0.062	-0.051	-0.007	-0.052
0.182	0.031	-0.299	-0.005	-0.037	-0.005

TABLE 3.3e Joint Displacement Components
(Second Mode f=55.78, Second Eigenvector)

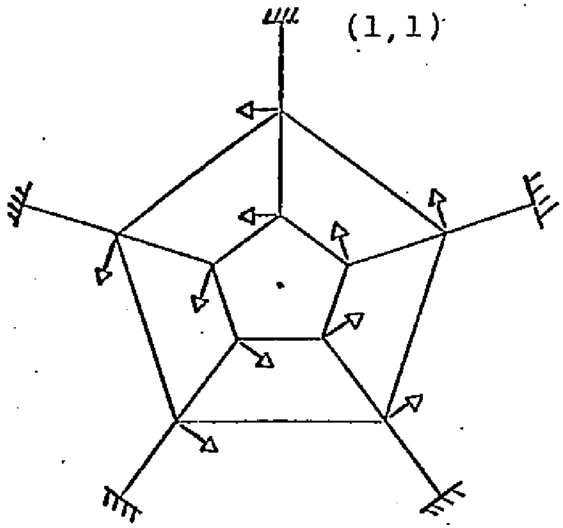
eq_1	eq_2	eq_3	eq_4	eq_5	eq_6
-0.146	-0.107	0.254	0.008	-0.039	-0.004
-0.104	0.208	0.180	-0.016	-0.027	0.008
0.082	0.236	-0.142	-0.019	0.022	0.009
0.155	-0.062	-0.269	0.005	0.041	-0.002
0.013	-0.275	-0.023	0.022	0.003	-0.010
-0.169	-0.116	0.277	0.002	0.034	0.020
-0.120	0.227	0.196	-0.003	0.024	-0.040
0.094	0.257	-0.155	-0.004	-0.019	-0.046
0.178	-0.068	-0.293	0.001	-0.036	0.012
0.015	-0.300	-0.025	0.005	-0.003	0.054



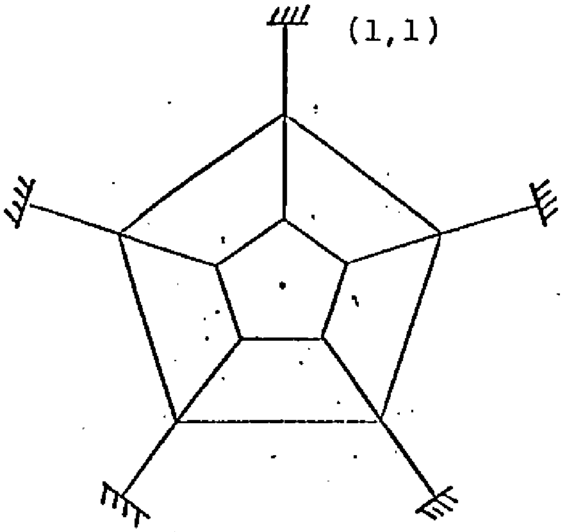
<u>q_1:</u> (Vertical,V)



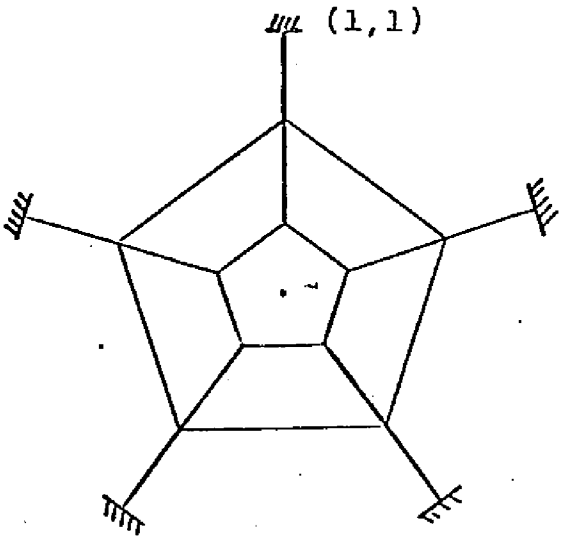
<u>q_4:</u> (Vertical,V)



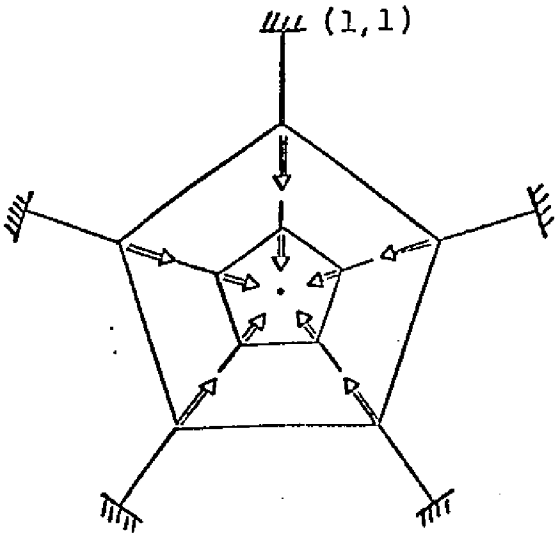
<u>q_2:</u> (Tangential,T)



<u>q_5:</u> (Tangential,T)

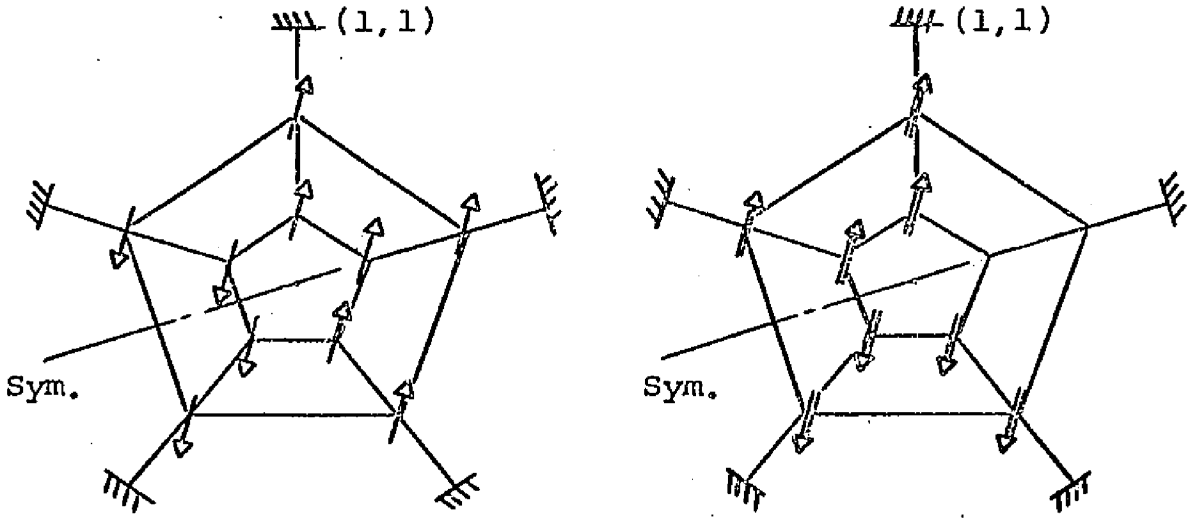


<u>q_3:</u> (Radial,W)

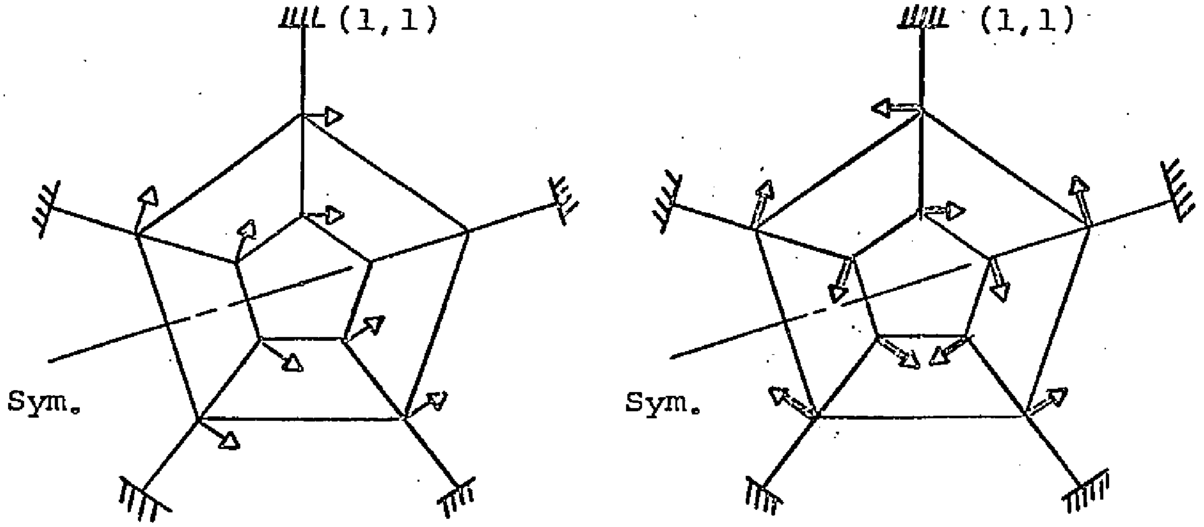


<u>q_6:</u> (Radial,W)

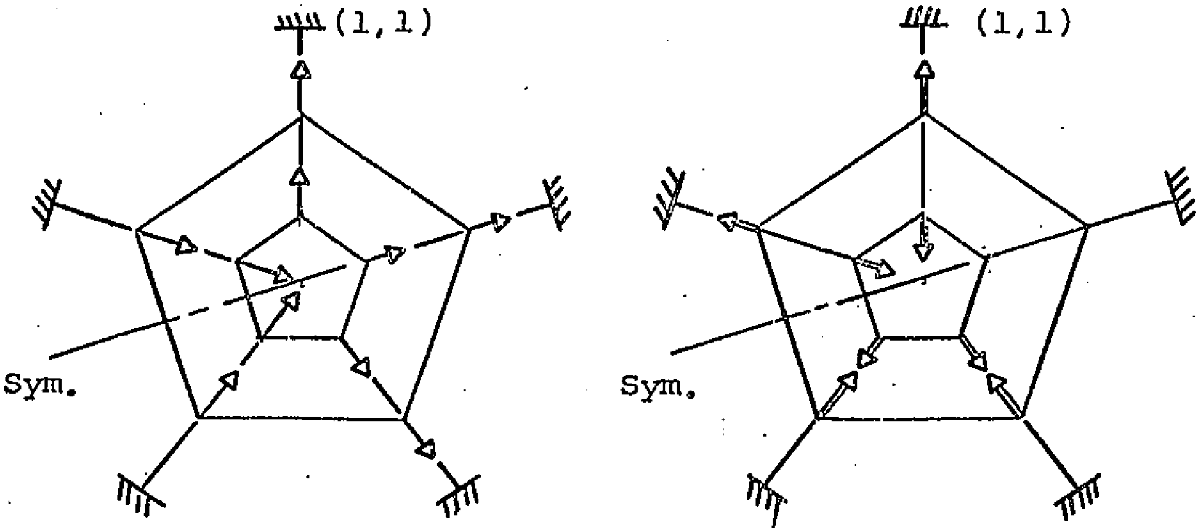
<u>Fig.3.5</u> Calculated Mode Shape (1st Mode)



q_1: (Vertical,V) q_4: (Vertical, V)

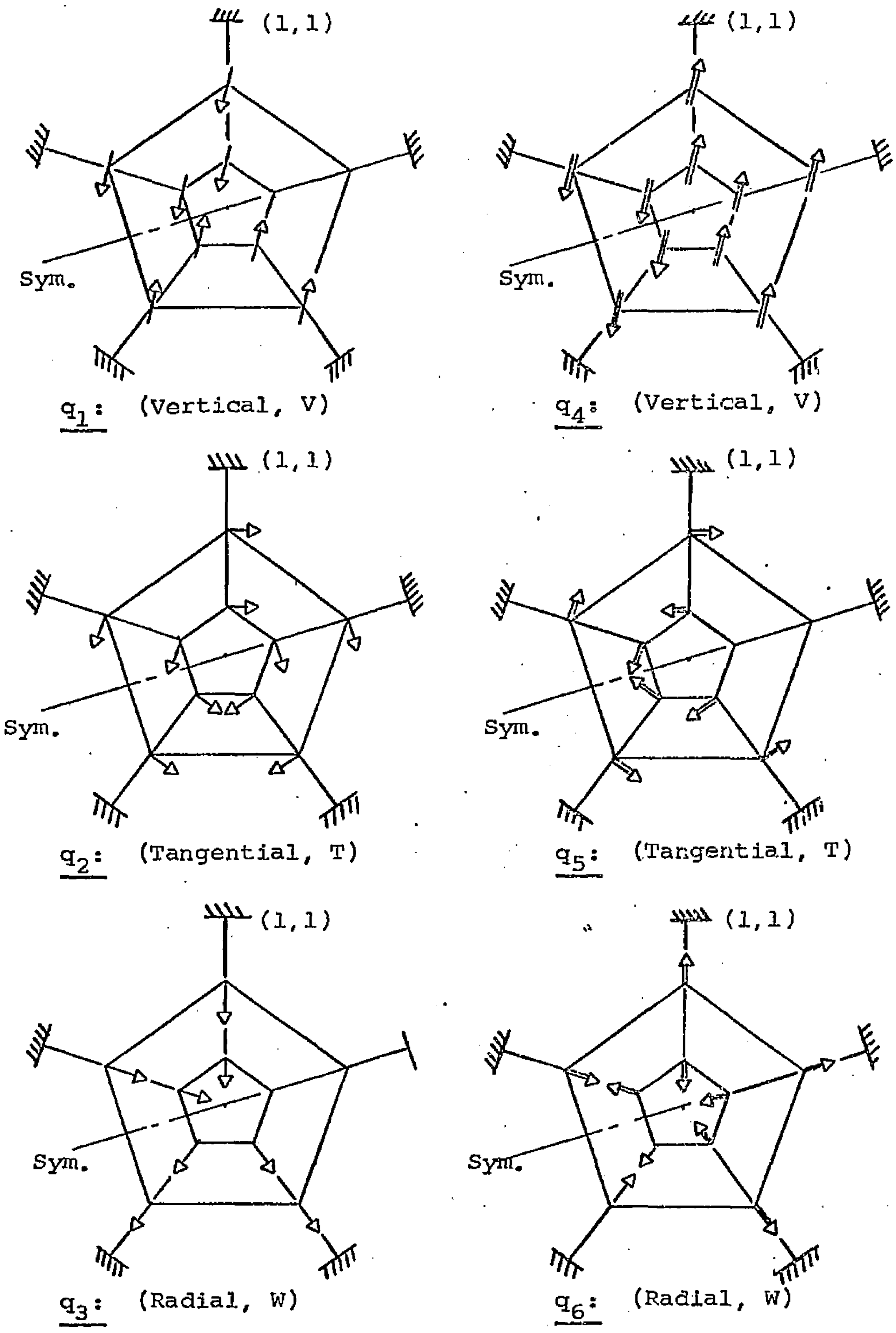


q_2: (Tangential, T) q_5: (Tangential, T)



q_3: (Radial, W) q_6: (Radial W)

Fig. 3.6 Calculated mode Shape (2nd Mode)
(Symmetric Mode)



q1: (Vertical, V)

q4: (Vertical, V)

q2: (Tangential, T)

q5: (Tangential, T)

q3: (Radial, W)

q6: (Radial, W)

Fig.3.7 Calculated Mode Shape (2^{nd} Mode) (Anti-symmetric Mode)

3.6 LOWER BOUND TO THE FUNDAMENTAL FREQUENCY OF THE DOME MODEL.

The method developed in Chapter 1 is used for finding a lower bound to the fundamental frequency of the dome model. Since the dome model is analyzed using lumped mass analysis, the matrix $L(\omega)$ used for computing a lower bound is

$$L(\omega) = (K) - \omega^2(M). \qquad (3.6\text{-}1)$$

Where (K) = stiffness matrix of the dome model,

(M) = mass matrix of the dome model,

ω = frequencies of the dome model.

The matrix D(q) is thus defined as

$$D(q) = (K) - q(M) \qquad (3.6\text{-}2)$$

The procedure described in section 1.3-4 (version 4 of the method) is used for the computations. The lower bound to the fundamental frequency of the dome model is found to be 22 cps. The computational result are given in Table 3.4 and Fig. 3.8 .

TABLE 3.4 LOWER BOUND TO THE DOME FUNDAMENTAL FREQUENCY.

q	α	LOWER BOUND(cps)
10^5	7.491×10^3	0.8
10^6	7.507×10^3	2.6
10^7	7.669×10^3	8.0
10^8	1.003×10^4	22.0
10^9	1.014×10^5	22.0
10^{10}	1.015×10^6	22.0
10^{11}	1.015×10^7	22.0
10^{12}	1.015×10^8	22.0
10^{13}	1.015×10^9	22.0
10^{14}	1.015×10^{10}	22.0
10^{15}	1.015×10^{11}	22.0
10^{16}	1.015×10^{12}	22.0
10^{17}	1.015×10^{13}	22.0
10^{18}	1.015×10^{14}	22.0
10^{19}	1.015×10^{15}	22.0

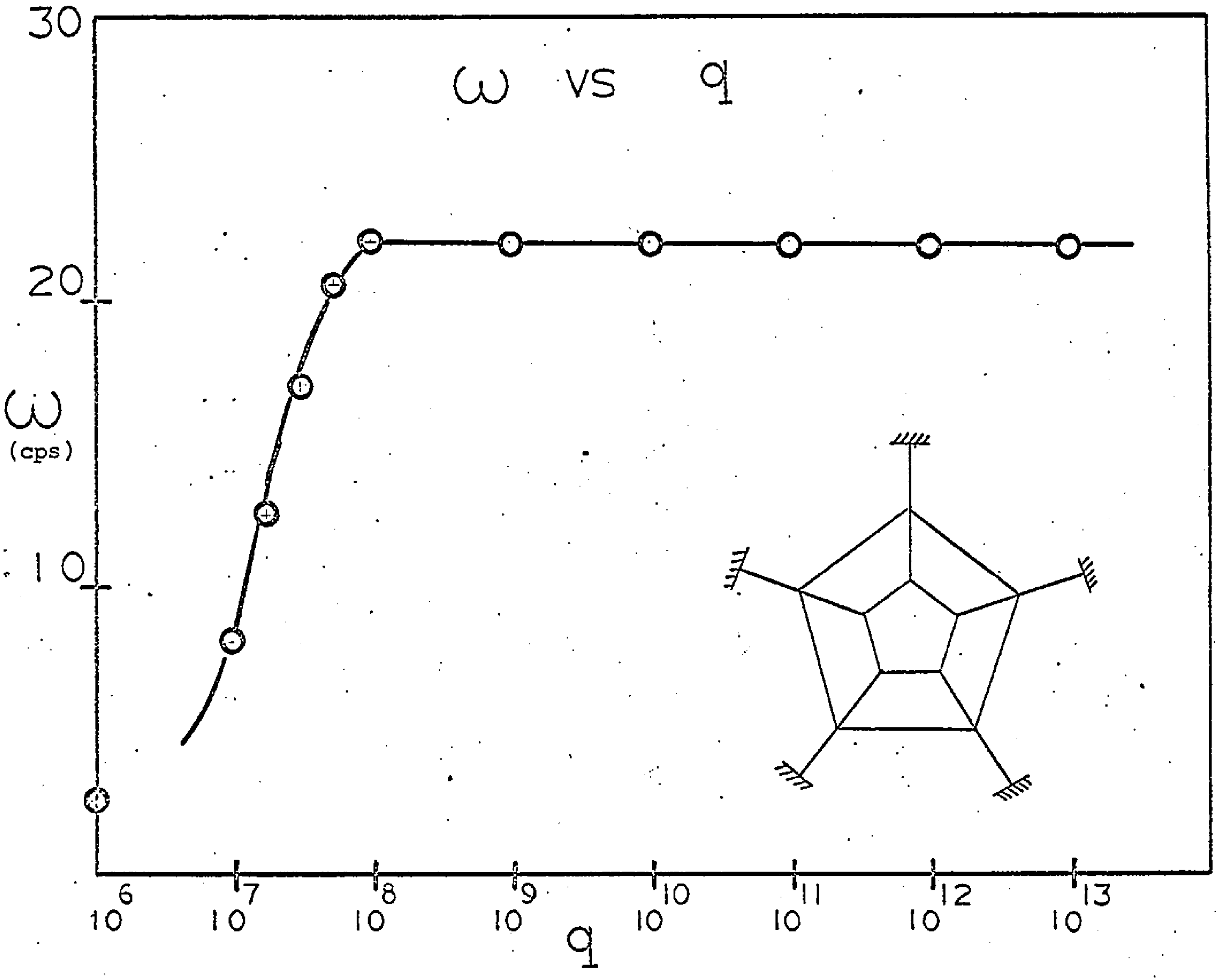


Fig.3.8 LOWER BOUND TO THE DOME FUNDAMENTAL FREQUENCY.

CHAPTER 4 DYNAMIC EXPERIMENT OF THE FRAMED DOME

4.0 INTRODUCTION

In order to study the accuracy of natural frequencies obtained from lumped parameter analysis, an experiment was designed and performed in the Engineering Materials Laboratory of the School of Engineering. A small model is built and mounted on a dynamic shake table. The dynamic stresses are observed by means of strain gages. By varying the frequency of the excitation force, the natural frequency of the model is obtained.

The instrumentation was calibrated by tests, on an aluminum cantilever beam. The dynamic modulus of elasticity of lucite was obtained by tests of lucite beams. Finally, a dome-shaped frame, made of lucite was constructed and tested (see Fig. 4.5a).

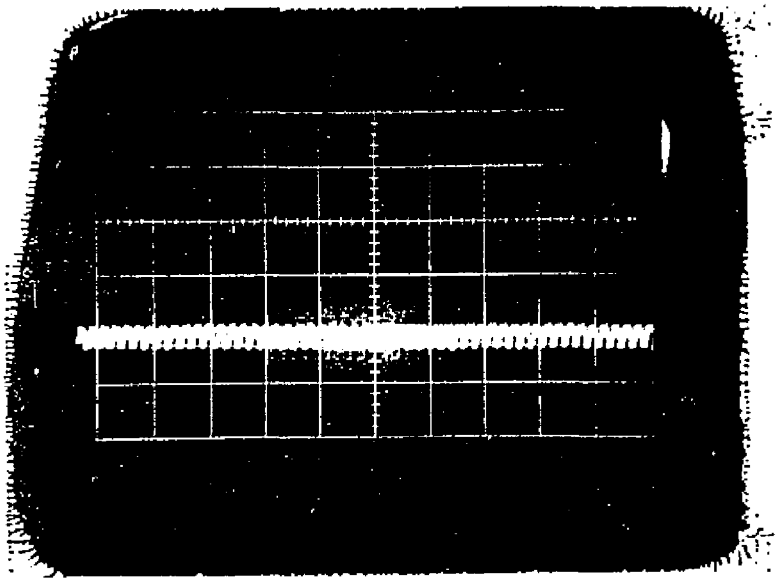


Fig. 4.1a

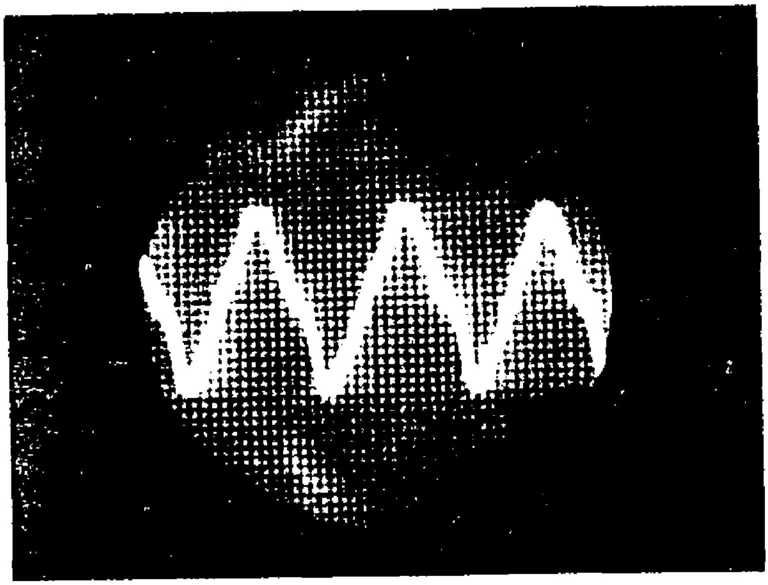


Fig. 4.1b (enlarged scale)

Fig. 4.1 Waveform on the oscillograph screen Before and After Resonance Frequencies

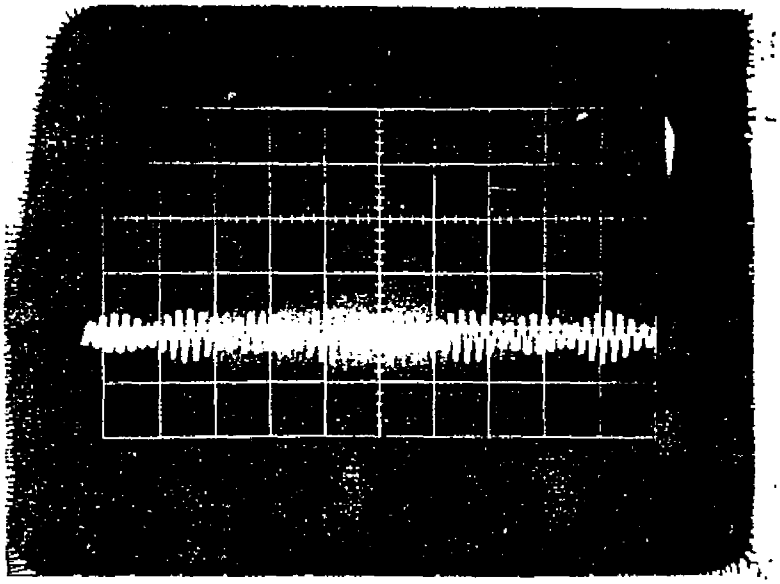


Fig. 4.2a

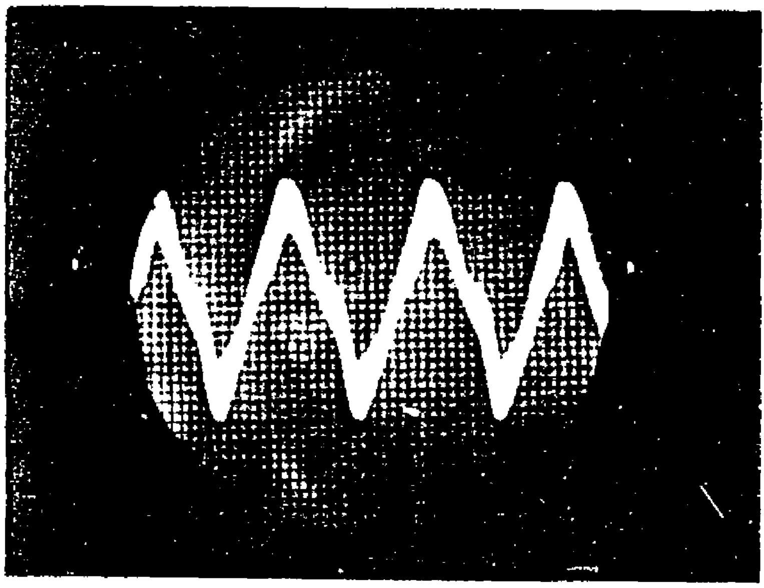


Fig. 4.2b (enlarged scale)

Fig. 4.2 Wave-form on the oscillograph screen in the Neighborhood of and at Resonance Frequencies.

4.1 DESCRIPTION OF THE EXPERIMENT

The experiment consisted of the excitation of vibration and the observation of resonance frequencies. The vibration was generated by a MB model C11-D vibration pickup calibrator and MB model T-112531 control, the resonance frequencies were observed with a Baldwin SR-4 type N portable strain indicator and Du Mont type 304 cathode-ray oscillograph. The experiment was designed according to the capacity of the vibration pickup calibrator, which had a calibration range from 5 to 2000 cycles per second and a max. exciter table load of 5 lbs. The model tested was made of Lucite (polymerized methyl-mathacrylate), and was pasted with SR-4 type A-7 strain gages for dynamic strain observations. The power supply for the experiment was 110 volts 50-60 cps single-phase current.

4.2 THE MODEL

A twenty-member framed dome-model was constructed as shown in Fig. 4.3. Because of the limited exciter table load and for the convenience of fabrication, it was made of Lucite. A square cross-section, 3/16"x3/16", was chosen for all elements in the model. Members were cut from a 3/16" thick Lucite plate. They were sanded smooth, mitered at both ends and cemented with Plexite No.11 (a solvent type cement).

The connecting pieces were cleaned and fitted properly, then immersed in the cementing solution for a few minutes and pressed firmly together. Cemented joints were left overnight under room temperature for curing. Plexite No. 11 was brushed on it with a swab whenever immersion was impossible. Gusset plate 1/16" thick was placed at each joint of the horizontal members for the purpose of increasing the rigidity of the model as well as making connection with inclined vertical members possible. The model was fixed to a plywood base which was securely fastened on the exciter table of vibration pickup calibrator. SR-4 type A-7 strain gages were pasted on the members of the dome with Duco Cement (a product of DuPont Co.). The modulus of elasticity of Lucite was found to be 4.5×10^5 psi at room temperature (See section 4.3). Assuming a Poisson's ratio of 0.25, the shear modulus was calculated as $G = 1.8 \times 10^5$ psi. The unit weight is 0.04114 (lbs per cu.in.). Three measurements of depth and width (one at each end and one at mid-point) were taken for each member. The average was used for the calculation of the cross-sectional area and the moment of inertia. The length of a member was measured along its center line.

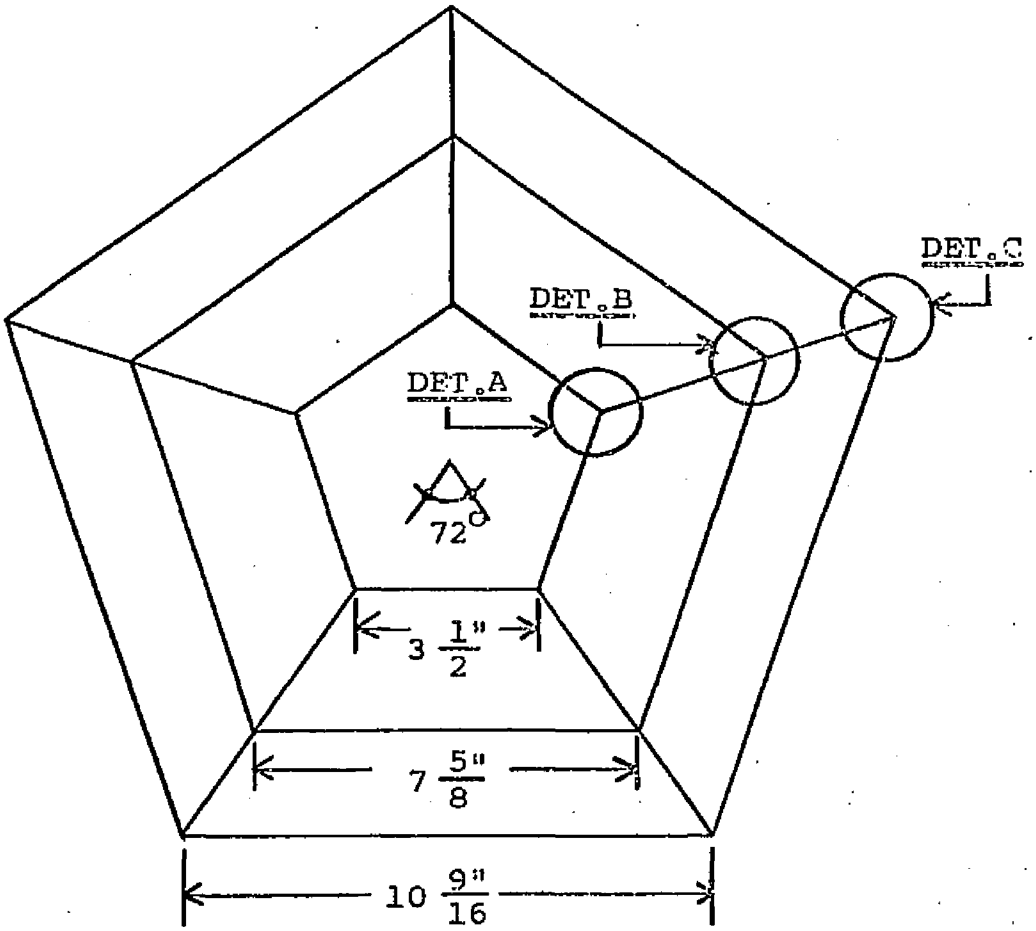


Fig.4.3a PLAN

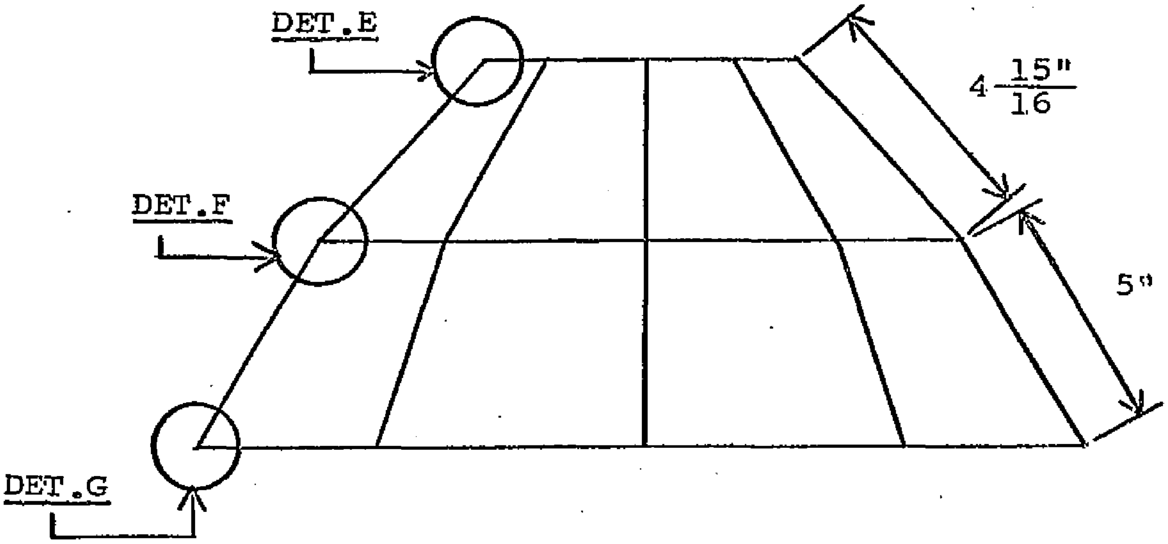
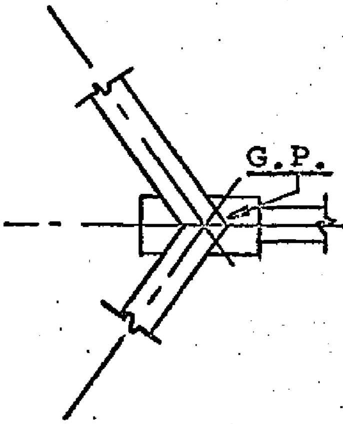
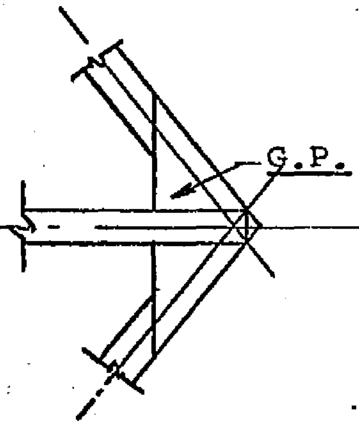


Fig.4.3b ELEVATION

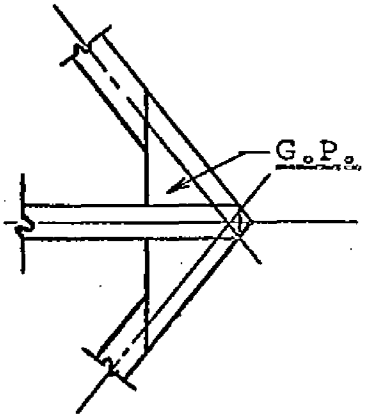
Fig.4.3 Model



DET.A

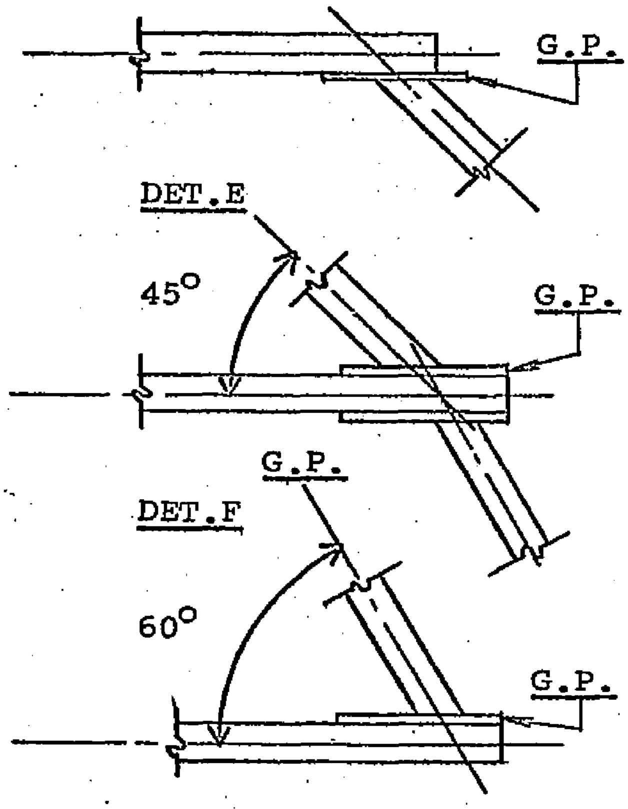


DET.B

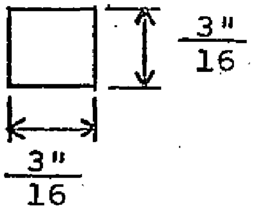


DET.C

NOTE: G.P. indicates 1/16" thick gusset plate



DET.G



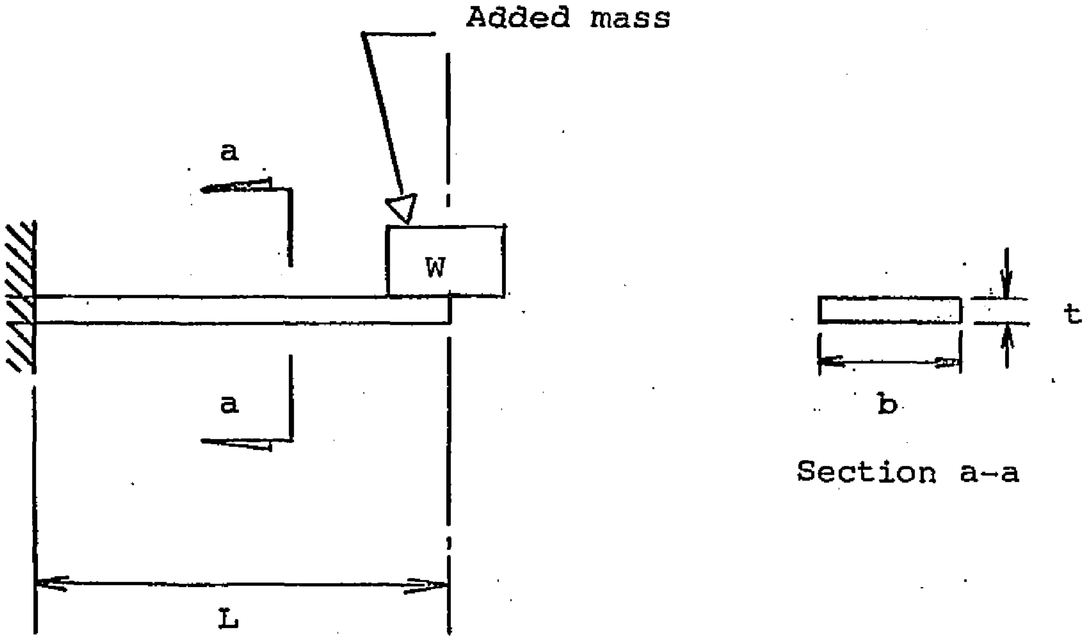
TYPICAL CROSS SECTION OF MODEL ELEMENT
Fig.4.3d

Fig.4.3c CORNER DETAILS

Fig.4.3 (cont'd)

4.3 MODULUS OF ELASTICITY OF LUCITE

The modulus of elasticity of lucite generally varies from $4\text{x}10^5$ to $5\text{x}10^5$ psi based on static flexural tests.(6) These values are in agreement with the static tests performed in the Engineering Material Laboratory. Since a more



b=1.098"
t=0.251"
$I=1.446\text{x}10^{-3}$ in^4
L= 6" and 8"

Unit weight of Lucite= 0.04114 #/in^3
Weight of beam per inch= 0.01136 #/in
Added weight= from 0.11# to 1.1#

Fig. 4.8 LUCITE CANTILEVER REAM WITH ADDED MASS AT FREE END.

TABLE 4.3 OBSERVED FIRST MODE FREQUENCIES (CPS) OF LUCITE CANTILEVER BEAMS WITH ADDED MASS AT FREE END.

L: Beam length (in)
W: Added weight at free end (gram)
f: Observed fundamental frequency (cps)

W =	L = 6						
	50	100	150	200	250	300	500
f =	26.7	19.3	16.0	14.2	12.6	11.4	8.7
	26.4	19.4	15.9	13.7	12.6	11.6	9.0
	26.6	19.2	16.3	14.1	12.5	11.4	9.2
	26.4	19.3	16.3	14.0	12.5	11.4	8.5
	26.4	19.3	16.5	14.0	12.8	11.7	9.1
Ave	26.5	19.3	16.2	14.0	12.6	11.5	8.9

W =	L = 8						
	50	100	150	200	250	300	500
f =	16.5	12.3	10.5	8.9	8.3	7.6	6.0
	17.0	12.8	10.2	9.3	8.1	7.3	5.6
	16.9	12.6	10.6	8.9	8.3	7.1	5.8
	16.6	12.6	10.1	9.1	8.2	7.5	5.7
	16.5	12.7	10.6	9.3	8.1	7.5	5.9
Ave	16.7	12.6	10.4	9.1	8.2	7.4	5.8

TABLE 4.4 MODULUS OF ELASTICITY OF LUCITE

L=6		L=8	
f(CPS)	$E(x10^5)$	f(CPS)	$E(x10^5)$
26.5	4.44	16.7	4.35
19.3	4.42	12.6	4.49
16.2	4.57	10.4	4.53
14.0	4.50	9.1	4.56
12.6	4.53	8.2	4.59
11.5	4.51	7.4	4.46
8.9	4.46	5.8	4.51

$$(E)_{average} = 4.5x10^5 \text{ psi}$$

accurate value of the modulus of elasticity is needed for verification of analytical results a series of experiments were carried out as follows. Two lucite cantilever beams with added mass at free end (Fig.4.8) were tested dynamically. First mode natural frequencies were recorded for different added masses. The results indicate that the lucite material used has a modulus of elasticity of $4.5x10^5$ psi. The experiment is shown schematically in Fig. 4.8 and the results are summarized in Table 4.4.

4.4 TEST EQUIPMENT

A schematic drawing (see Fig.4.4) shows the arrangement of equipment used in this experiment. The MB model C11-D vibration pickup calibrator and its associated MB model T-112531 control were manufactured by MB Electronics, a Division of Extron Electronics, Inc., New Haven, Connecticut. The model C11-D vibration exciter consisted of two parts: the body structure and the moving element assembly on which the model was placed. Readings on the oscillator frequency vernier setting dial mounted on the front panel of model T-112531 control indicated the frequencies of the exciter table. Baldwin SR-4 type N portable strain indicator was used for the measuring of strains in SR-4 type A-7 strain gages pasted on the model. Proper connection was made between Du Mont type 304 cathode-ray oscillagraph and the Scope Jack Plug on the

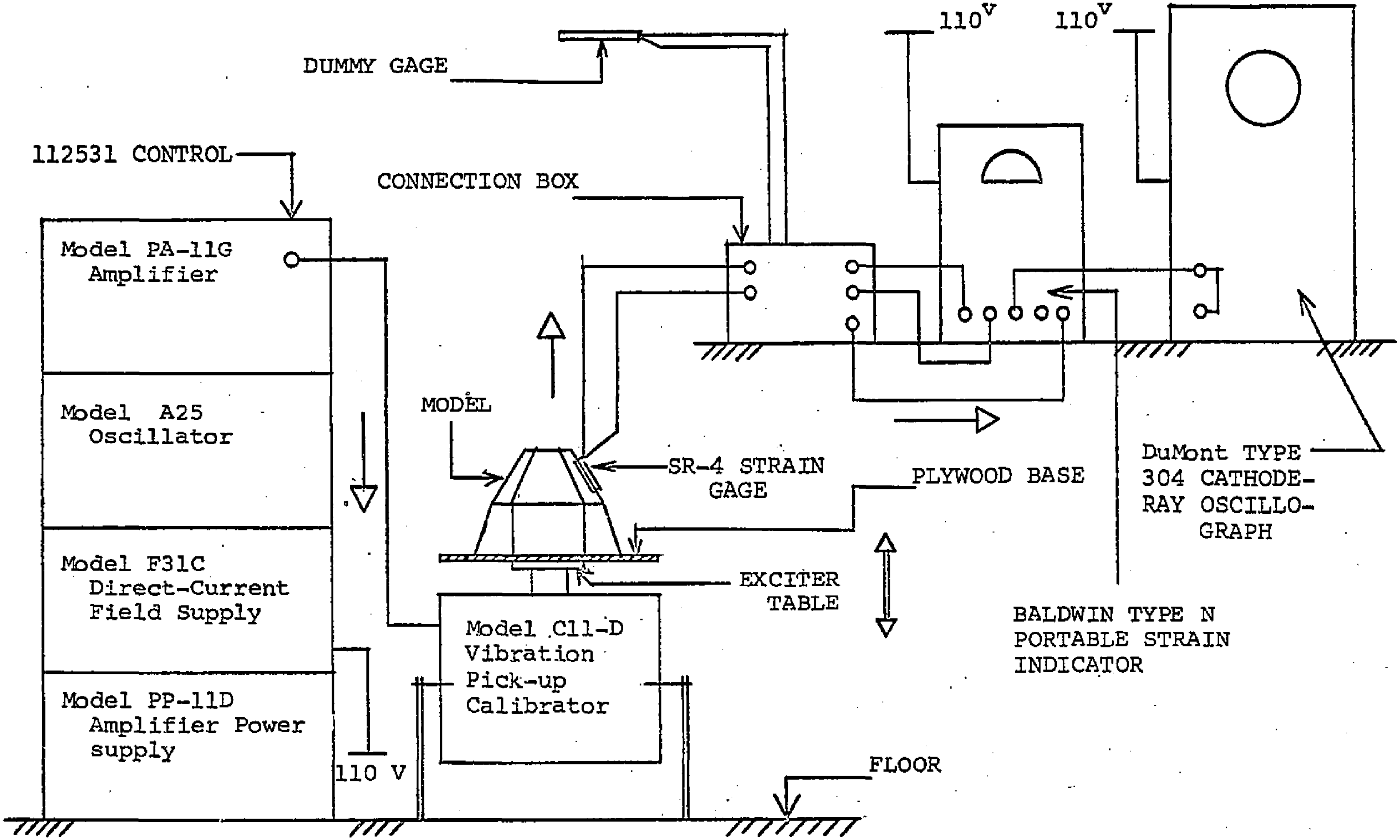


Fig. 4.4 SCHEMATIC ARRANGEMENT OF TEST APPARATUS

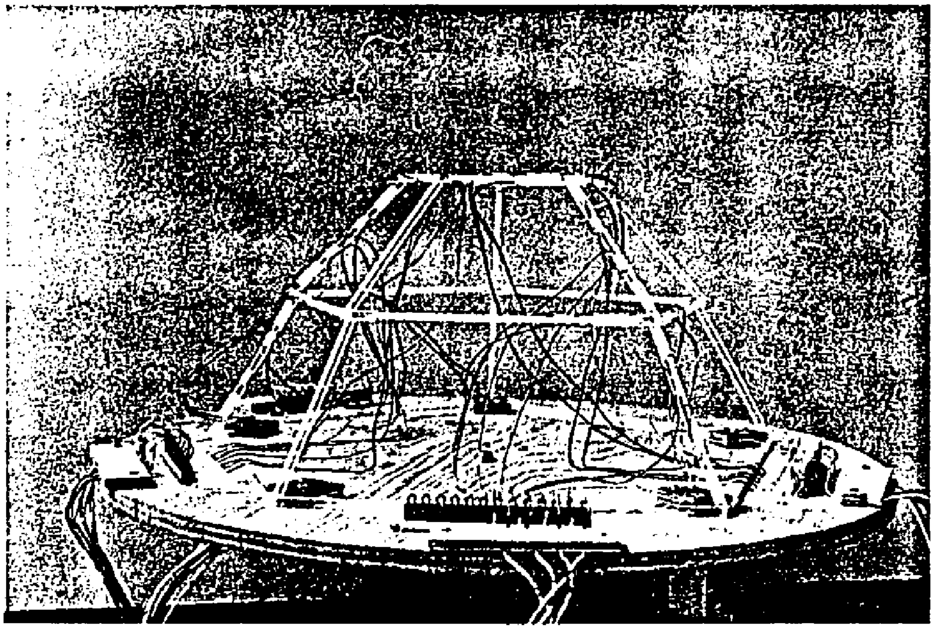


Fig. 4.5a Model

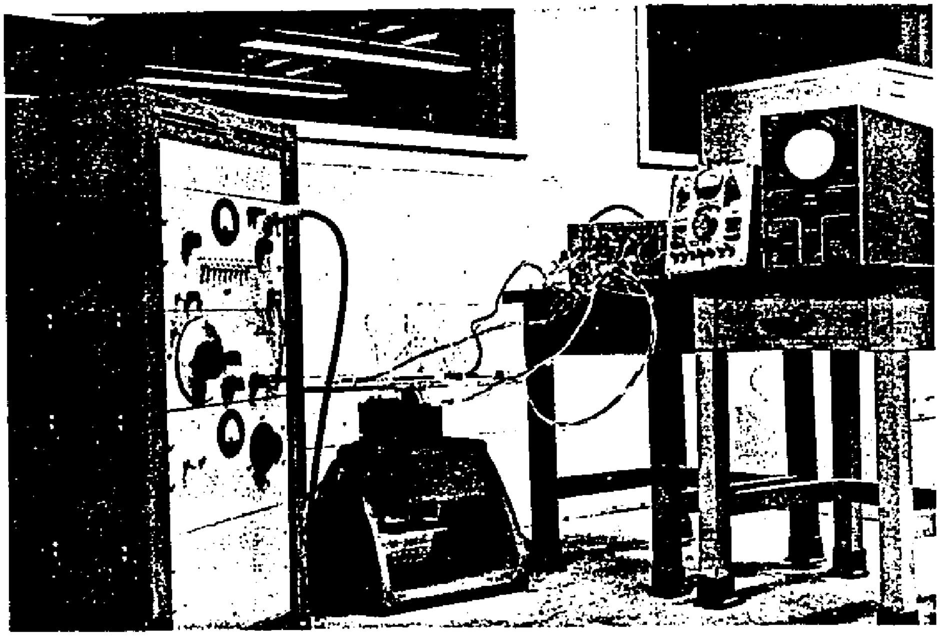


Fig. 4.5b Test Apparatus

Fig. 4.5 Photographs

strain indicator, so that variation of strains during vibration could be seen from the screen of oscillograph. Both vibration pickup calibrator and T-112531 control were mounted on the laboratory floor, and all other equipments were placed on the working desk, Pictures of waveforms seen from the oscillograph screen were taken with a polaroid camera.

4.5 PROCEDURE

The waveform of initial reading of a strain gage is seen on the screen of oscillograph. It has a band shape as shown in Fig.4.1. This waveform can be focused, centered on the screen and adjusted to a desirable amplitude (approximately 2"). Before and after resonance frequencies, this waveform remained in its original shape. However, in the neighborhood of and at the resonance frequencies, its shape distorted completely as can be seen in Fig.4.2. At the instant of waveform distortion on the screen the natural frequency of the model was read from T-112531 control. The equipment was not effective at higher frequencies.

4.6 OBSERVED NATURAL FREQUENCIES OF THE MODEL

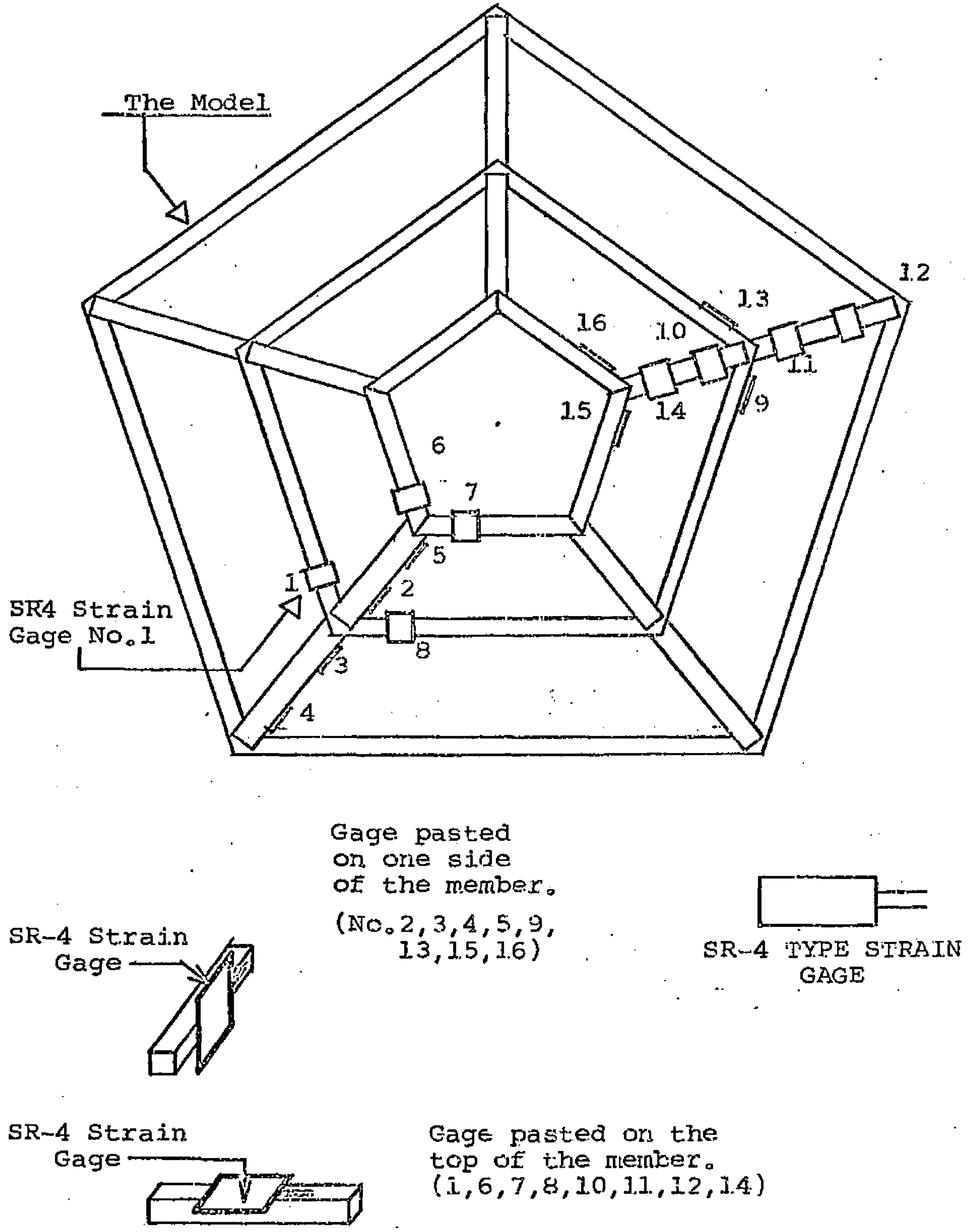
A total of sixteen SR-4 type strain gages placed at different locations as shown in Fig.4.6 were used for recording resonance frequencies of the model. Waveform of the strain in each strain gage on oscillograph screen has been observed for different frequencies. The resonance frequencies of

TABLE 4.1 NATURAL FREQUENCIES OF THE MODEL OBSERVED AT SR-4 TYPE STRAIN GAGES.

Gage No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Avg.
f_1, cps	52	*	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52.0
f_2, cps	93	*	99	94	94	94	94	95	93	94	94	92	95	96	95	98	94.67
f_3, cps	135	*	135	135	135	135	133	133	133	*	*	*	131	*	133	*	133.8

* Readings were not obtainable.

NOTE: See Fig.4.6 for location of strain gages.



Gage pasted on one side of the member. (No.2,3,4,5,9, 13,15,16)

Gage pasted on the top of the member. (1,6,7,8,10,11,12,14)

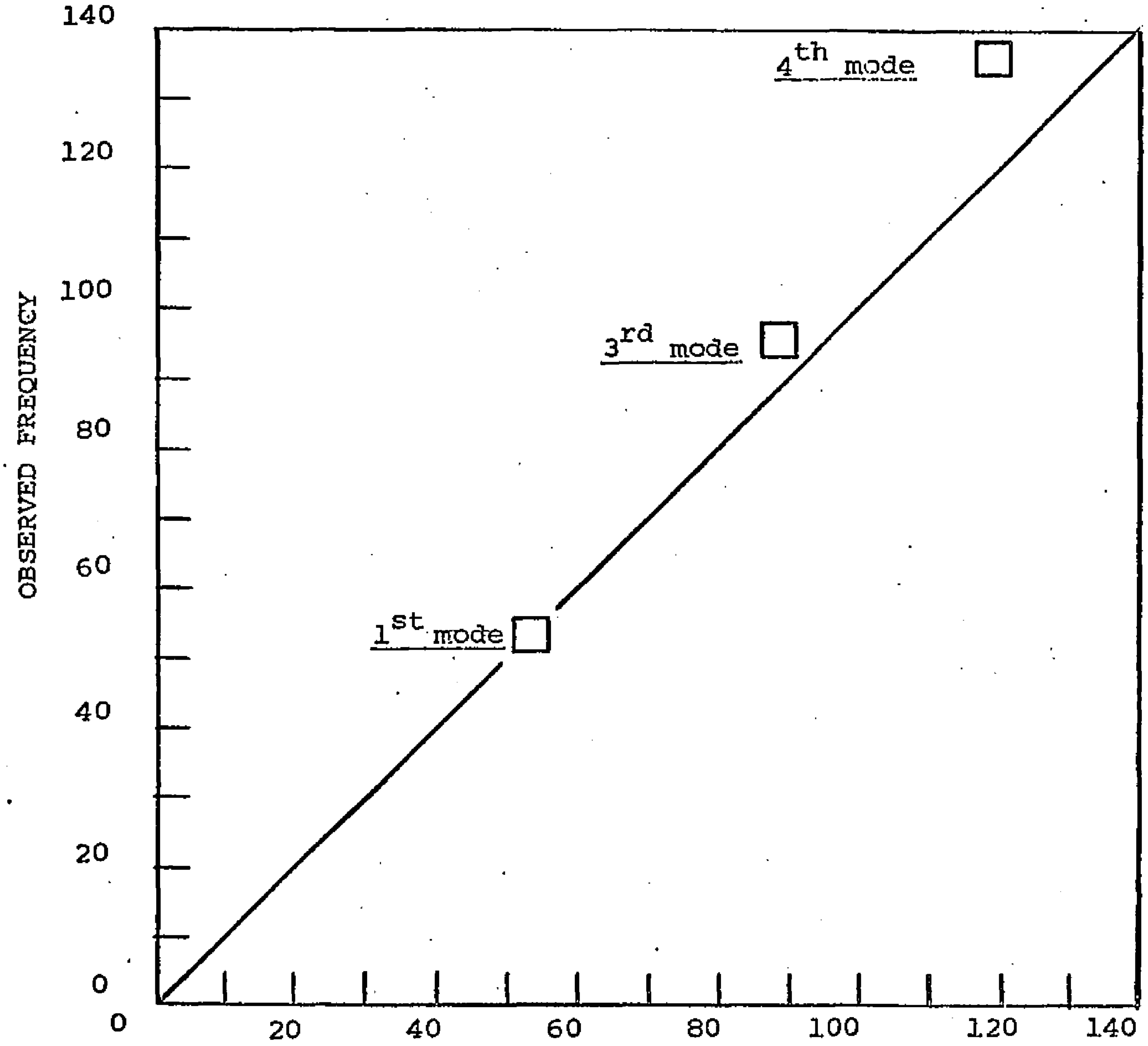
Fig. 4.6 LOCATION OF SR-4 STRAIN GAGES ON THE MODEL the model were recorded at each gage. The readings were very consistant as can be seen in Table 4.1.

Comparison of the observed values with the first five calculated frequencies (see Table 4.2) show good agreements. The calculated second and fifth mode frequencies are very close to that of the first and fourth mode respectively. These small differences can not be observed on the oscillator frequency vernier of model T-112531 control. It appears reasonable to assume that the observed second and fifth frequencies of the model are 52 and 133.80 CPS respectively.

TABLE 4.2 OBSERVED AND CALCULATED FIRST FIVE MODE NATURAL FREQUENCIES OF THE MODEL

Mode	Natural Frequency (CPS)	
	Observed	Calculated
1	52.00	53.85
2		55.78
3	94.67	88.16
4	133.80	116.69
5		119.58

A plot of the observed frequencies versus calculated frequencies is given in Fig. 4.7.



CALCULATED FREQUENCY
(Lumped mass analysis)

Fig. 4.7 OBSERVED FREQUENCIES vs CALCULATED FREQUENCIES (CPS)

4.7 CONCLUSION

The simple instrumentation employed in the experiment appeared to give good results in the measurements of natural frequencies of model structures. When the frequencies of two normal modes are nearly equal, as in the first and second mode of the model, the instrument can not distinguish the two.

Due to the small excitation force of the dynamic shaker, high frequency vibrations can not be observed.

Within the limitation of the experimental accuracy, the results of lumped-mass analysis are in good agreement. Whether the lumped mass analysis is sufficiently accurate for framed domes in general must await more extensive experimental investigation.

CHAPTER 5 CONCLUSION

A new method of finding a lower bound to the smallest eigenvalue of a linear system is developed. The results are guaranteed to be lower bounds to the smallest eigenvalue of the system. Since the computational effort required for finding this lower bound is much less than that for existing methods, the method developed in this dissertation is a very useful tool to the engineer who wishes to obtain some information about the smallest eigenvalue of a linear system. The new method is developed on the basis of the study of locations of eigenvalues and matrix inequalities.

Further research in this area, in the opinion of the author, will be benificial to the engineering profession. Topics which are open for further study may include the following:

(1) Improvement and refinement of numerical methods.

(2) Lower bound to the smallest eigenvalue of unsymmetric matrices.

(3) Lower bound to the smallest eigenvalue of complex matrices.

(4) Loer bounds to higher eigenvalues.

APPENDIX 1 THEOREM OF GERSGORIN AND FAN

We state without proof three theorems which are used in the dissertation.

THEOREM 1. (Gersgorin)(18)

The characteristic roots of an N-square complex matrix $C=(c_{ij})$ lie in the closed region of the complex plane consisting of all the disks

$$|z - c_{ii}| = P_i$$

$$P_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} |c_{ij}|$$

$i=1,\ldots,N.$

THEOREM 2. (Fan)(3)

Define the matrix function

$$S_k(A) = \lambda_N + \lambda_{N-1} + \ldots + \lambda_k$$

for a real symmetric matrix A. Then

$$S_k\left[\in A + (1-\in)B\right] \geq \in S_k(A) + (1-\in)S_k(B)$$

for $0 \leq \in \leq 1$, $k=1,2,\ldots,N$.

Note that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$, and λ_i $i=1,\ldots,N$ are roots of A.

THEOREM 3. (Fan)$^{(3)}$

Let A and B be symmetric matrices, with B non-negative definite. Then

$$\lambda_k(A+B) \geq \lambda_k(A) \qquad k = 1,2,\ldots,N.$$

If B is positive definite, then

$$\lambda_k(A+B) > \lambda_k(A) \qquad K = 1,2,\ldots,N.$$

Note that $\lambda_i(A)$ $i=1,\ldots,N$ are roots of A and $\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_N(A)$.

APPENDIX 2 MATRIX ELEMENTS OF ELEMENT STIFFNESS AND ELEMENT MASS MATRIX

The elements a,b,...,k in matrix (2.5-2) are the following:

$a = AE/L$

$b = \mu_z EI_z / L^3$

$c = \mu_y EI_y / L^3$

$d = GI_x / L$

$e = \alpha_y EI_y / L$

$f = \alpha_z EI_z / L$

$g = -\gamma_z EI_z / L^2$

$h = -\gamma_y EI_y / L^2$

$j = \beta_y EI_y / L$

$k = \beta_z EI_z / L$

where

A: Area of the member.

L: Length of the member.

I_x, I_y, I_z : Moment of inertia about respective x,y, and z axis.

E: Modulus of elasticity.

G: Shear Modulus.

and

$\alpha_y = \alpha_z = 4$

$\beta_y = \beta_z = 2$

$\gamma_y = \gamma_z = 6$

$\mu_y = \mu_z = 12.$

The elements in (2.5-23) can be expressed as follows: (1)

$a^* = mL/3$

$b^* = 156mL / 420$

$c^* = \rho^2 mL / 3$

$d^* = 4mL^3 / 420$

$e^* = -22mL^2 / 420$

$f^* = mL / 6$

$g^* = 54mL / 420$

$h^* = \rho^2 mL / 6$

$j^* = -3mL^3 / 420$

$k^* = 13mL^2 / 420.$

where m: Mass per unit length of the member.

ρ : The radius of gyration of mass m (Δx) about the inertial axis.

APPENDIX 3 STIFFNESS MATRIX AND MASS MATRIX OF THE DOME MODEL.

The partial stiffness matrix and mass matrix of the dome model are presented herein. These matrices were generated by computer programs from basic data which are given in Table 3.1. The programs are listed in Appendix 4.

SYSTEM MASS MATRIX

```
                         M(1,1)...M(60,60)

 3.419191E-05      0.                -6.615727E-07      0.
 0.                0.                 3.976384E-06      5.473023E-06
 0.                0.                 0.                0.
-3.677453E-06      0.                 0.                0.
-3.528482E-07      0.                -2.006521E-06      0.
 0.                0.                 0.                0.
 0.                0.                 0.                0.
 0.                0.                 0.                0.

 0.                3.365316E-05      -0.                1.440335E-05
 4.015608E-06      3.976384E-06       0.                0.
 0.                0.                 0.                0.
 0.               -1.849558E-06      -4.015608E-06      3.976384E-06
 0.                2.006521E-06       0.               -2.006520E-06
 0.                0.                 0.                0.
 0.                0.                 0.                0.
 0.                0.                 0.                0.

-6.615727E-07     -0.                 3.379549E-05     -0.
-7.599427E-07     -5.473023E-06       0.                0.
 0.                0.                 0.                0.
 0.                4.015608E-06      -7.599427E-07      5.473023E-06
-2.734573E-06      0.                 2.006520E-06      0.
 0.                0.                 0.                0.
 0.                0.                 0.                0.
 0.                0.                 0.                0.

 0.                1.440335E-05      -0.               -3.513193E-05
 5.473023E-06      1.192945E-05       0.                0.
 0.                0.                 0.                0.
 0.                3.976384E-06      -5.473023E-06      1.192945E-05
 0.                1.611362E-06       0.               -1.629452E-06
 0.                0.                 0.                0.
 0.                0.                 0.                0.
 0.                0.                 0.                0.

-1.440335E-05      0.                -8.493099E-07      0.
 0.                0.                -[illegible].139805E-06     -5.659508E-06
 0.                0.                 0.                0.
 3.976384E-06      0.                 0.                0.
-2.006520E-06      0.                 3.240813E-06      0.
 0.                0.                 0.                0.
 0.                0.                 0.                0.
 0.                0.                 0.                0.

 0.                8.493099E-07       0.               -4.056153E-06
 0.                0.                 5.659508E-06      7.817578E-06
 0.                0.                 0.                0.
 5.473023E-06      0.                 0.                0.
 0.               -1.629452E-06       0.                1.611361E-06
 0.                0.                 0.                0.
 0.                0.                 0.                0.
 0.                0.                 0.                0.
```

SYSTEM STIFFNESS MATRIX

```
                    K(1,1)...K(60,60)

  3.977279E 03    0.                    2.964193E 03    0.
 -0.             -0.                   -2.799371E 00   -3.853003E-00
  0.              0.                    0.              0.
 -1.246519E-00   -0.                   -0.             -0.
 -1.596040E 03    0.                    8.028562E 00    0.
 -0.              0.                   -0.             -0.
  0.              0.                    0.              0.
 -0.             -0.                   -0.             -0.

 -0.             -2.720012E 03         -0.             -8.065469E 00
  9.850989E 02   -2.799371E 00          0.              0.
 -0.             -0.                   -0.             -0.
  0.             -1.354626E 03         -9.850989E 02   -2.799371E 00
 -0.             -8.028562E 00          0.             -8.028559E 00
  0.              0.                    0.              0.
 -0.             -0.                   -0.             -0.
  0.              0.                    0.              0.

  2.964193E 03    0.                    3.827186E 03   -0.
 -7.144697E 02   -3.853003E-00         -0.             -0.
  0.              0.                    0.              0.
 -0.             -9.850989E-02          7.144697E 02   -3.853003E 00
 -1.600629E 03    0.                   -8.028559E 00    0.
 -0.             -0.                   -0.             -0.
 -0.             -0.                   -0.             -0.
  0.              0.                    0.              0.

  0.             -8.065469E 00          0.              8.583178E 01
 -3.853003E-00   -1.213091E-01         -0.             -0.
  0.              0.                    0.              0.
 -0.             -2.799371E-00         -3.853003E-00   -1.213091E-01
  0.              5.619148E 00          0.             -1.311134E 01
 -0.             -0.                   -0.             -0.
  0.              0.                    0.              0.
 -0.             -0.                   -0.             -0.

 -8.065469E-00   -0.                   -1.604807E-00   -0.
  0.              0.                   -7.367040E 00   -3.461154E 00
 -0.             -0.                   -0.             -0.
 -2.799371E 00    0.                    0.              0.
 -8.028559E-00   -0.                   -1.873048E-01   -0.
  0.              0.                    0.              0.
 -0.             -0.                   -0.             -0.
  0.              0.                    0.              0.

 -0.             -1.604807E 00          0.             -2.782888E 01
 -0.             -0.                   -3.461154E-00    9.616235E-00
  0.              0.                    0.              0.
 -3.853003E 00   -0.                   -0.             -0.
  0.             -1.311134E 01          0.              5.619139E 00
 -0.             -0.                   -0.             -0.
  0.              0.                    0.              0.
 -0.             -0.                   -0.             -0.
```

APPENDIX 4 COMPUTER PROGRAMS.

A number of computer programs used for analytical solutions are presented herein. These were written in FORTRAN. The program consists of a main program which controls a series of subroutine programs. The listings of some less important programs have been omitted from the Appendix. All programs have been executed on the IBM 360 at the City College Computation Center.

```
C****************************************************************************
C*                                                                          *
C*        LOWER BOUND TO THE FUNDAMENTAL FREQUENCY OF STRUCTURAL            *
C*        SYSTEM                                                            *
C*                                                                          *
C****************************************************************************
C
C
C
C         *****  M A I N    P R O G R A M *****
C         H - L O W E R     B O U N D
C         LUMPED    AND    DISTRIBUTED  MASS   SYSTEM
C
C         MBERP=0  PHYSICAL PROPERTIES OF ALL MEMBERS ARE DIFFERENT
C         MBERP=1  PHYSICAL PROPERTIES OF ALL MEMBERS ARE IDENTICAL
C         IPRINT=0  NO  OUTPUT PRINT
C         IPRINT=1  OUTPUT PRINT REQUIRED
C         LUMP = 0  DISTRIBUTED MASS SYSTEM
C         LUMP = 1  LUMPED MASS SYSTEM
C         METHOD = 2   H-LOWER BOUND    (ONE)
C         METHOD = 3   H-LOWER BOUND    (TWO)
C         METHOD = 4   EXACT FREQUENCY
C
C
C
      COMMON /AA/ RRL( 50),RRE( 50),RRIY( 50),RRMASS( 50),CODE( 50,4)
      DIMENSION  SYSK(3,3),SYSM(3,3),P(3,3)
      DIMENSION TIT(20)
C
C
 9999 CONTINUE
      CALL  TIME(ITIME)
      TSTART = ITIME
      READ (1,800) (TIT(I),I=1,20)
      WRITE(3,801) (TIT(I),I=1,20)
  800 FORMAT (20A4)
  801 FORMAT (1H1,20A4)
      READ (1,101)  NBDOF,NUELMT,MBERP,IPRINT,LUMP
  101 FORMAT (5I5)
C
      WRITE (3,208)  NBDOF,NUELMT,MBERP,IPRINT,LUMP
  208 FORMAT (///
     1 1X,'NO.OF DEGREES-OF-FREEDOM                  ='I10/
     2 1X,'NO.OF ELEMENTS                            ='I10/
     3 1X,'MEMBER PROPERTIES                         ='I10/
     4 1X,'OUT-PUT PRINT                             ='I10/
     5 1X,'MASS SYSTEM                               ='I10)
C
      IF   (MBERP.EQ.0)
     1READ (1,202)  (N,RRL(N),RRE(N),RRIY(N),RRMASS(N),N=1,NUELMT)
  202 FORMAT (I5,4F10.0)
      IF   (IPRINT.EQ.1 .AND. MBERP.EQ.0)   GO   TO   16
      GO   TO    15
```

```
   16 WRITE(3,204)  (N,RRL(N),RRE(N),RRIY(N),RRMASS(N),N=1,NOELMT)
  204 FORMAT (1H0, '   NO.EL       L            E           IY            M   '//
     1(I5,4F10.2))
C
   15 CONTINUE
      READ (1,203)  ((CODE(N,I),I=1,4),N=1,NOELMT)
  203 FORMAT (16I5)
       WRITE (3,205) ((CODE(N,I),I=1,4),N=1,NOELMT)
  205 FORMAT (1H0, '       CODE NO. '//(4I5))
C
C
      IF  (LUMP)  12,12,11
C
C
C
   11 CONTINUE
C
C
C        LUMPED MASS SYSTEM,     H-LOWER BOUND
C        SYSK = STIFFNESS MATRIX,  SYSM = MASS MATRIX
C
C
      READ (1,102)   METHOD
  102 FORMAT (I15)
      IF  ( METHOD.EQ.4 )  GO  TO  24
      IF  (METHOD-2)  21,22,23
   21 CONTINUE
C
   22 CONTINUE
C        METHOD = 2   H-LOWER BOUND    (ONE)
      READ (1,103)  AMIN,AINC,NBINC
  103 FORMAT (2F10.0,I10)
      WRITE (3,206)  AMIN,AINC,NBINC
  206 FORMAT (1H0,'  QMIN='F10.2,2X,'QINC='F10.2,2X,'NBINC='I10)
      CALL  SYSTMX(SYSM,SYSK,NBDOF,NOELMT,MBERP,IPRINT,1,0.0)
      REWIND  4
      WRITE (4)   ((SYSK(I,J),J=1,NBDOF),I=1,NBDOF)
      CALL  EIGSYM(SYSK,NBDOF,AN,P)
      REWIND  4
      READ  (4)   ((SYSK(I,J),J=1,NBDOF),I=1,NBDOF)
      CALL LMPBD1(SYSM,SYSK,NBDOF,AMIN,AINC,NBINC,AN,P)
      GO  TO  99
C
   23 CONTINUE
C        METHOD = 3   H-LOWER BOUND    (TWO)
      READ (1,103)  AMIN,AINC,NBINC
      WRITE (3,206)  AMIN,AINC,NBINC
      CALL  SYSTMX(SYSM,SYSK,NBDOF,NOELMT,MBERP,IPRINT,1,0.0)
      REWIND  4
      WRITE (4)   ((SYSK(I,J),J=1,NBDOF),I=1,NBDOF)
      CALL  EIGSYM(SYSK,NBDOF,AN,P)
      REWIND  4
      READ  (4)   ((SYSK(I,J),J=1,NBDOF),I=1,NBDOF)
```

```
      CALL LMPBD2
      GO  TO  99
C
C
   12 CONTINUE
C
C
C          DISTRIBUTED MASS SYSTEM,      H-LOWER BOUND
C          SYSM = STATIC STIFFNESS MATRIX
C          SYSK = DYNAMIC STIFFNESS MATRIX
C
C
      READ (1,102)  METHOD
      IF (METHOD.EQ.4)   GO  TO  34
      IF  (METHOD-2)  31,32,33
   31 CONTINUE
C
   32 CONTINUE
C          METHOD = 2   H-LOWER BOUND    (ONE)
      READ (1,103)  AMIN,AINC,NBINC
      READ  (1,103)  EMIN,EINC,MBINC
      WRITE (3,209)  AMIN,AINC,NBINC
      WRITE (3,207)  EMIN,EINC,MBINC
  207 FORMAT (1H0,'   EMIN='F10.2,2X,'EINC='F10.2,2X,'MBINC='I10)
  209 FORMAT (1H0,'  OMGMIN='F10.2,2X,'OMGINC='F10.2,2X,'MBINC='I10)
      CALL  SYSTMX(SYSK,SYSM,NBDOF,NOELMT,MBERP,IPRINT,1,1.0)
      REWIND  4
      WRITE (4)  ((SYSM(I,J),J=1,NBDOF),I=1,NBDOF)
      CALL EIGSYM (SYSM,NBDOF,AN,P)
      REWIND  4
      READ  (4)  ((SYSM(I,J),J=1,NBDOF),I=1,NBDOF)
      CALL  DISBD1(SYSM,SYSK,NBDOF,NOELMT,MBERP,IPRINT,
     1                            AMIN,AINC,NBINC,EMIN,EINC,MBINC,AN,P)
      GO  TO  99
C
   33 CONTINUE
C          METHOD = 3   H-LOWER BOUND    (TWO)
      READ (1,103)  AMIN,AINC,NBINC
      WRITE (3,209)  AMIN,AINC,NBINC
      CALL  SYSTMX(SYSK,SYSM,NBDOF,NOELMT,MBERP,IPRINT,1,1.0)
      REWIND  4
      WRITE (4)  ((SYSM(I,J),J=1,NBDOF),I=1,NBDOF)
      CALL EIGSYM (SYSM,NBDOF,AN,P)
      REWIND  4
      READ  (4)  ((SYSM(I,J),J=1,NBDOF),I=1,NBDOF)
      CALL  DISBD2
      GO  TO  99
C
C
C
C
C          METHOD = 4   EXACT FREQUENCY
   24 CONTINUE
```

```
C           LUMPED MASS SYSTEM
C
      CALL   SYSTMX(SYSM,SYSK,NBDOF,NOELMT,MBERP,IPRINT,1,1.0)
      DO  41   I = 1,NBDOF
      DO  41   J = 1,NBDOF
   41 P(I,J) = 0.0
C
C
      CALL   INV (SYSK,NBDOF,FLAG)
      WRITE (3,401)  FLAG
  401 FORMAT (1H1,//'  INVERSION OF STIFFNESS MATRIX  FLAG = ' F10.2)
C
      CALL   FRQLMP (SYSM,SYSK,NBDOF)
      GO  TO  99
C
   34 CONTINUE
C          DISTRIBUTED MASS SYSTEM
C
      READ (1,103)  AMIN,AINC,NBINC
      CALL   FRODIS(SYSM,SYSK,NBDOF,NOELMT,MBERP,IPRINT,0,AMIN,AINC,
     1 NBINC,P)
      GO  TO  99
C
C
C
   99 CONTINUE
      CALL   TIME(ITIME)
      TEND = ITIME
      TTOTAL =(TEND - TSTART)/60.
      WRITE (3,301)  TTOTAL
  301 FORMAT (//////'   TOTAL TIME IN MIN = ' F10.5)
      GO  TO  9999
      END
      SUBROUTINE   SYSTMX(SYSM,SYSK,NBDOF,NOELMT,MBERP,IPRINT,LUMP,
     1OMEGA2)
C
C
C          MBERP=0  PHYSICAL PROPERTIES OF ALL MEMBERS ARE DIFFERENT
C          MBERP=1  PHYSICAL PROPERTIES OF ALL MEMBERS ARE IDENTICAL
C          IPRINT=0  NO  OUTPUT PRINT
C          IPRINT=1  OUTPUT PRINT REQUIRED
C          LUMP = 0  DISTRIBUTED MASS SYSTEM
C          LUMP = 1  LUMPED MASS SYSTEM
C
C
C
      COMMON /AA/ RRL( 50),RRE( 50),RRIY( 50),RRMASS( 50),CODE( 50,4)
      DIMENSION SYSK(NBDOF,NBDOF),SYSM(NBDOF,NBDOF),ELK(4,4),ELM(4,4)
      INTEGER        CODE,CODENO(4),COUNT
C
C
      DO  4   I=1,NBDOF
      DO  4   J=1,NBDOF
```

```
      SYSK(I,J)=0.0
      SYSM(I,J)=0.0
    4 CONTINUE
C
      IF  (MBERP-1)10,11,10
   11 DO  1  I=1,NOELMT
      RRL(I)=1.0
      RRE(I)=1.0
      RRIY(I)=1.0
    1 RRMASS(I)=1.0
C
C
C
   10 CONTINUE
      DO  2  COUNT=1,NOELMT
      RL=RRL(COUNT)
      RE=RRE(COUNT)
      RIY=RRIY(COUNT)
      RMASS=RRMASS(COUNT)
C
      DO  3  J=1,4
    3 CODENO(J)=CODE(COUNT,J)
C
      BL=1.0
C
C
C
      IF  (LUMP-1)  18,17,18
C
C        LUMPED MASS SYSTEM, STIFFNESS AND MASS MATRIX
C
   17 CONTINUE
      CALL  ELMTMK (ELK,RL,RE,RIY,RMASS,BL,0)
      CALL  ELMTMK (ELM,RL,RE,RIY,RMASS,BL,1)
C
C
      IF  (IPRINT)  16,15,16
   16 WRITE (3,206)
      CALL  PRNTMF(ELK,4,4)
      WRITE (3,207)
      CALL  PRNTMF(ELM,4,4)
  206 FORMAT (1H1,'ELEMENT STIFFNESS MATRIX')
  207  FORMAT (1H1,'ELEMENT MASS MATRIX')
   15 CONTINUE
C
C
      CALL  ASMBLE (SYSK,ELK,NBDOF,CODENO)
      CALL  ASMBLE (SYSM,ELM,NBDOF,CODENO)
      GO  TO  2
C
C
C        DISTRIBUTED MASS SYSTEM,DYNAMIC STIFFNESS MATRIX
C
```

```
   18 CONTINUE
      CALL          DYNMCK (ELK  ,RL,RE,RIY,RMASS,BL,OMEGA2)
      IF  (IPRINT)  21,22,21
   21 WRITE  (3,206)
      CALL  PRNTMF (ELK,4,4)
   22 CALL  ASMBLE (SYSK,ELK,NBDOF,CODEND)
    2 CONTINUE
C
C
      IF  (IPRINT)  13,14,13
   13 WRITE (3,204)
      CALL  PRNTMF (SYSK,NBDOF,NBDOF)
      WRITE (3,205)
      CALL  PRNTMF (SYSM,NBDOF,NBDOF)
  204 FORMAT (1H1,'SYSTEM STIFFNESS MATRIX')
  205 FORMAT (1H1,'SYSTEM MASS MATRIX')
   14 CONTINUE
C
C
C
      RETURN
      END
      SUBROUTINE   DISBD1(SYSM,SYSK,NBDOF,NOELMT,MBERP,IPRINT,
     1  AMIN,AINC,NBINC,EMIN,EINC,MBINC,AN,STOR)
C
      DIMENSION SYSM(NBDOF,NBDOF),SYSK(NBDOF,NBDOF),STOR(NBDOF,NBDOF)
      COMMON /AA/ RRL( 50),RRE( 50),RRIY( 50),RRMASS( 50),CODE( 50,4)
C
C
      WRITE  (3,103)
      EPSLON = EMIN
      DO  1     ICOUNT = 1,MBINC
      AAAA = 1. - 1./EPSLON
      BBBB = 1./EPSLON
      HBOND = (1.-EPSLON)*AN/EPSLON
      OMEGA2 = AMIN
      DO  2     JCOUNT = 1,NBINC
      OMEGA = SQRT(OMEGA2)
      CALL  SYSTMX(STOR,SYSK,NBDOF,NOELMT,MBERP,IPRINT,0    ,OMEGA2)
      DO  11   I = 1,NBDOF
      DO  11   J = 1,NBDOF
   11 STOR(I,J) = AAAA*SYSM(I,J) + BBBB*SYSK(I,J)
      CALL  HLBOND(STOR,NBDOF,BOND)
      IF  (BOND .GT. 0.0)   GO  TO  21
      IF  (ABS(BOND)-ABS(HBOND))  21,21,22
   22 B = ABS(BOND)
      H = ABS(HBOND)
      WRITE (3,101)  B,H
  101 FORMAT (1H0,'  BOND GREATER THAN HBOND,   BOND ='E13.5,10X,
     1'HBOND ='E13.5)
      GO  TO  1
   21 B = ABS(BOND)
      H = ABS(HBOND)
```

```
      FRQ = SQRT(OMEGA2)/(2.0*3.14159)
      Q = OMEGA /EPSLON
      WRITE (3,102)  EPSLON,Q,B,H,FRQ
  102 FORMAT (1X,4E13.5,E25.5)
C
    2 OMEGA2 = OMEGA2 + AINC
    1 EPSLON = EPSLON + EINC
C
  103 FORMAT (1H1,'   EPSLON              Q              BOND              HBOND
     1LOWER BOND FUNDAMENTAL FREQUENCY CYC/SEC'//)
C
C
      RETURN
      END
      SUBROUTINE   LMPBD1(SYSM,SYSK,NBDOF,AMIN,AINC,NBINC,AN,STOR)
C
C
      DIMENSION  SYSM(NBDOF,NBDOF),SYSK(NBDOF,NBDOF),STOR(NBDOF,NBDOF)
C
C
C
      OMEGA2 = AMIN
      DO  1   ICOUNT = 1,  NBINC
      DO  2   I = 1,NBDOF
      DO  2   J = 1,NBDOF
    2 STOR(I,J) = SYSK(I,J) - OMEGA2*SYSM(I,J)
C
      CALL    HLBOND(STOR,NBDOF,BOND)
C
      EPS = 1./(1.+ABS(BOND)/AN)
      WRITE (3,104)  OMEGA2,EPS
      WRITE (3,103)  BOND,EPS
  103 FORMAT(1H0,10X,'BOND='E13.5,10X,'EPS='E13.5)
  104 FORMAT(1H0,10X,' Q  ='E13.5,10X,'EPS='E13.5)
      OMGA = OMEGA2*EPS
      FRQ = SQRT(OMGA)
      WRITE (3,101)  ICOUNT,FRQ
  101 FORMAT (1H0, 10X,'TRIAL NO. =' I10,10X,'LOWER BOUND FREQUENCY RAD/
     1SEC ='E13.5/)
      FRQ = FRQ/(2.0*3.14159)
      WRITE (3,102)  FRQ
  102 FORMAT (1H , 10X,'                ' 10X,10X,'LOWER BOUND FREQUENCY CYC/
     1SEC ='E13.5////)
    1 OMEGA2 =OMEGA2 + AINC
C
C
      RETURN
      END
      SUBROUTINE   FRQLMP(SYSM,SYSK,NBDOF)
C
      DIMENSION SYSM(NBDOF,NBDOF),SYSK(NBDOF,NBDOF),P(3,3)
      DO  1   I = 1,NBDOF
      DO  1   J = 1,NBDOF
```

```
    1 P(I,J) = 0.0
C
C
C
C
C
      DO  2   I = 1,NBDOF
      DO  2   J = 1,NBDOF
      DO  2   L = 1,NBDOF
    2 P(I,J) = P(I,J) + SYSK(I,L)*SYSM(L,J)
C
      DO  5   I = 1,NBDOF
      DO  5   J = 1,NBDOF
    5 SYSK(I,J) = P(I,J)
C
      CALL  EIGENU (SYSK,NBDOF,20,0.001,SYSM,II)
C
C
      DO  3   I = 1,NBDOF
    3 SYSK(I,I) = SQRT(1./SYSK(I,I))
      WRITE (3,102)  (SYSK(I,I),   I = 1,NBDOF)
  102 FORMAT (1H0,//'   NATURAL FREQUENCIES IN  RAD/SEC'//(20X,1E13.5))
C
      DO  4   I = 1,NBDOF
    4 SYSK(I,I) = SYSK(I,I)/(2.0*3.14159)
      WRITE (3,103)  (SYSK(I,I),   I = 1,NBDOF)
  103 FORMAT (1H0,//'   NATURAL FREQUENCIES IN  CYC/SEC'//(20X,1E13.5))
C
C
C
      RETURN
      END
      SUBROUTINE   FRODIS(SYSM,SYSK,NBDOF,NOELMT,MBERP,IPRINT,LUMP,
     1 AMIN,AINC,NBINC,P)
C
C
      COMMON /AA/ RRL( 50),RRE( 50),RRIY( 50),RRMASS( 50),CODE( 50,4)
      DIMENSION  SYSM(NBDOF,NBDOF),SYSK(NBDOF,NBDOF),P(NBDOF,NBDOF)
C
C
      WRITE (3,102)
      OMEGA2 = AMIN
      DO  1   ICOUNT =1,NBINC
      OMEGA = SQRT(OMEGA2)
      CALL  SYSTMX(SYSM,SYSK,NBDOF,NOELMT,MBERP,IPRINT,0   ,OMEGA2)
      CALL  DETM(SYSK,NBDOF,DET,SYSM,P)
      FRQ = SQRT(OMEGA2)/(2.0*3.14159)
      WRITE (3,101)  ICOUNT,DET,FRQ
  101 FORMAT (I20,10X,E13.5,10X,E13.5)
    1 OMEGA2 = OMEGA2 + AINC
C
  102 FORMAT (1H1,' NO. OF   COMPUTATION          VALUE OF DET.
     1  FUNDAMENTAL FREQUENCY CYC/SEC ')
```

```
C
C
C
      RETURN
      END
      SUBROUTINE   HLBOND (A,N,BOND)
      DIMENSION  A(N,N)
      DIMENSION TEMP(100),DIFF(100)
      DO  5  I=1,N
    5 TEMP(I)=0.0
      DO  1  I=1,N
      DO  1  J=1,N
      TEMP(I)=TEMP(I)+ABS(A(I,J))
    1 CONTINUE
      DO  2  I=1,N
      IF  (A(I,I))   11,11,10
   10 DIFF(I)=2.*A(I,I) - TEMP(I)
      GO  TO  2
   11 DIFF(I)=-TEMP(I)
    2 CONTINUE
      BOND =DIFF(1)
      DO  3  I=1,N
      IF  (BOND .GT. DIFF(I))   BOND=DIFF(I)
    3 CONTINUE
C
C
      RETURN
      END
      SUBROUTINE   ASMBLE (A,AAA,N,CODENO)
C
C
      INTEGER RAT,HEN,DOG,CAT,CODENO(1)
      DIMENSION A(N,N),AAA(4,4)
C
      DO  1  RAT=1,4
      DO  2  HEN=1,4
      IF (CODENO(RAT) .NE. 0.AND.CODENO(HEN).NE.0) GO  TO  11
      GO  TO  2
   11 DOG=RAT
      CAT=HEN
      I=CODENO(RAT)
      J=CODENO(HEN)
      A(I,J)=A(I,J)+AAA(DOG,CAT)
    2 CONTINUE
    1 CONTINUE
C
C
      RETURN
      END
      SUBROUTINE   ELMTMK (AAA,RL,RE,RIY,RMASS,BL,ITYPE)
C
C
C  ITYPE=0    ELEMENT STIFFNESS MATRIX
```

```
C  ITYPE=1     ELEMENT MASS MATRIX
C
      DIMENSION  AAA(4,4)
      REAL   MUEY,LO
C
C
           LO=BL
           ALPHAY=4.
           BATAY=2.
           GAMAY=6.
           EATAY=6.
           MUEY=12.
           ROWY=12.
C
C
      IF  (ITYPE) 11,11,12
   11 CONTINUE
           BLUE=RE*RIY/(RL*RL*RL)
           AAA(1,1)=MUEY*BLUE
           AAA(1,2)=-GAMAY*BLUE*RL *LO
           AAA(1,3)=-ROWY*BLUE
           AAA(1,4)=-EATAY*BLUE*RL *LO
           AAA(2,1)=AAA(1,2)
           AAA(2,2)=ALPHAY*BLUE*RL*RL *LO*LO
           AAA(2,3)=EATAY*BLUE*RL *LO
           AAA(2,4)=BATAY*BLUE*RL*RL *LO*LO
           AAA(3,1)=AAA(1,3)
           AAA(3,2)=AAA(2,3)
           AAA(3,3)=MUEY*BLUE
           AAA(3,4)=GAMAY*BLUE*RL *LO
           AAA(4,1)=AAA(1,4)
           AAA(4,2)=AAA(2,4)
           AAA(4,3)=AAA(3,4)
           AAA(4,4)=ALPHAY*BLUE*RL*RL *LO*LO
      GO TO  13
C
C
   12 CONTINUE
           RED=(RMASS*RL)/420.
           AAA(1,1)=156.*RED
           AAA(1,2)=-22.*RL*RED *LO
           AAA(1,3)=54.*RED
           AAA(1,4)=13.*RL*RED *LO
           AAA(2,1)=AAA(1,2)
           AAA(2,2)=4.*RL*RL*RED *LO*LO
           AAA(2,3)=-13.*RL*RED *LO
           AAA(2,4)=-3.*RL*RL*RED *LO*LO
           AAA(3,1)=AAA(1,3)
           AAA(3,2)=AAA(2,3)
           AAA(3,3)=156.*RED
           AAA(3,4)=22.*RL*RED *LO
           AAA(4,1)=AAA(1,4)
           AAA(4,2)=AAA(2,4)
```

```
       AAA(4,3)=AAA(3,4)
       AAA(4,4)=4.*RL*RL*RED *LO*LO
C
C
  13 CONTINUE
    RETURN
    END
    SUBROUTINE   DYNMCK (AAA ,RL,RE,RIY,RMASS,BL,OMEGA2)
C
    DIMENSION AAA(4,4)
    REAL   MUEY,LO
C
    IF (OMEGA2.EQ.0.0)  OMEGA2=0.1
       LO=BL
       UYO=SQRT(OMEGA2)
       UYO= SQRT(UYO)
       UY4=RMASS*RL*RL*RL*RL/(RE*RIY)
       UY=SQRT(SQRT(UY4))*UYO
       SINUY=SIN (UY)
       COSUY=COS (UY)
       SHUY=SINH (UY)
       CHUY=COSH (UY)
       UYUY=1.-COS (UY)*COSH (UY)
C
       ALPHAY=UY*(SINUY*CHUY-COSUY*SHUY)/UYUY
       BATAY=UY*(SHUY-SINUY)/UYUY
       GAMAY=UY*UY*(SINUY*SHUY)/UYUY
       EATAY=UY*UY*(CHUY-COSUY)/UYUY
       MUEY=UY*UY*UY*(SINUY*CHUY+COSUY*SHUY)/UYUY
       ROWY=UY*UY*UY*(SINUY+SHUY)/UYUY
C
       BLUE=RE*RIY/(RL*RL*RL)
       AAA(1,1)=MUEY*BLUE
       AAA(1,2)=-GAMAY*BLUE*RL *LO
       AAA(1,3)=-ROWY*BLUE
       AAA(1,4)=-EATAY*BLUE*RL *LO
       AAA(2,1)=AAA(1,2)
       AAA(2,2)=ALPHAY*BLUE*RL*RL *LO*LO
       AAA(2,3)=EATAY*BLUE*RL *LO
       AAA(2,4)=BATAY*BLUE*RL*RL *LO*LO
       AAA(3,1)=AAA(1,3)
       AAA(3,2)=AAA(2,3)
       AAA(3,3)=MUEY*BLUE
       AAA(3,4)=GAMAY*BLUE*RL *LO
       AAA(4,1)=AAA(1,4)
C
       AAA(4,4)=ALPHAY*BLUE*RL*RL *LO*LO
       AAA(4,3)=AAA(3,4)
       AAA(4,2)=AAA(2,4)
    RETURN
    END
    SUBROUTINE   DISBD2(SYSM,SYSK,NBDOF,NOELMT,MBERP,IPRINT,
   1  AMIN,AINC,NBINC,EMIN,EINC,MBINC,AN,STOR)
```

```
C
      DIMENSION SYSM(NBDOF,NBDOF),SYSK(NBDOF,NBDOF),STOR(NBDOF,NBDOF)
      COMMON /AA/ RRL( 50),RRE( 50),RRIY( 50),RRMASS( 50),CODE( 50,4)
C
C
C
      EMIN=0.0
      EINC=0.0
      WRITE  (3,103)
      IF  (AN-1.0)  31,31,32
   32 C = 1.0
      GO  TO  33
   31 C = 1.0/AN + 0.2
   33 EPSMAX = 1.0 - 1.0/(C*AN)
      MBINC = 10
      EPSLON = EPSMAX
C
C
      DO  1     ICOUNT = 1,MBINC
      AAAA = 1. - 1./EPSLON
      BBBB = 1./EPSLON
      OMEGA2 = AMIN
      DO  2     JCOUNT = 1,NBINC
      CALL  SYSTMX(STOR,SYSK,NBDOF,NOELMT,MBERP,IPRINT,0    ,OMEGA2)
      DO  11  I = 1,NBDOF
      DO  11  J = 1,NBDOF
   11 STOR(I,J) = AAAA*SYSM(I,J) + BBBB*SYSK(I,J)
      DO  4  I=1,NBDOF
      DO  5  J=1,NBDOF
      STOR(I,J)=STOR(I,J)*EPSLON*C
    5 CONTINUE
      STOR(I,I)=STOR(I,I) + 1.0
    4 CONTINUE
      CALL  HLBOND(STOR,NBDOF,BOND)
      IF   (BOND)   14,14,15
   14 WRITE (3,104)  EPSLON,OMEGA2
  104 FORMAT (///'  TRIAL FAILED'10X,'EPSLON='E13.5,10X,'OMEGA2='E13.5)
      GO  TO  1
   15 FRQ = SQRT(OMEGA2)/(2.0*3.14159)
      Q = OMEGA2/EPSLON
      WRITE (3,102)  EPSLON,Q,FRQ
  102 FORMAT (1X,2E13.5,26X,E25.5)
C
    2 OMEGA2 = OMEGA2 + AINC
    1 EPSLON = EPSLON/2.0
C
  103 FORMAT (1H1,'   EPSLON            Q
     1LOWER BOND FUNDAMENTAL FREQUENCY CYC/SEC'//)
C
C
      RETURN
      END
      SUBROUTINE   LMPBD2(SYSM,SYSK,NBDOF,AMIN,AINC,NBINC,AN,STOR)
```

```
C
C
      DIMENSION  SYSM(NBDOF,NBDOF),SYSK(NBDOF,NBDOF),STOR(NBDOF,NBDOF)
C
C
C
      IF  (AN-1.0)  11,11,12
   12 C = 1.0
      GO  TO  13
   11 C = 1.0/AN  + 0.2
   13 EPSMAX = 1.0 - 1.0/(C*AN)
      MBINC=10
      EPS=EPSMAX
      Q=AMIN
C
C
      DO  3  JCOUNT=1,MBINC
      DO  1  ICOUNT = 1,  NBINC
C
      DO  2  I = 1,NBDOF
      DO  2  J = 1,NBDOF
    2 STOR(I,J) = SYSK(I,J) - Q*SYSM(I,J)
      DO  4  I=1,NBDOF
      DO  5  J=1,NBDOF
      STOR(I,J)=STOR(I,J)*EPS*C
    5 CONTINUE
      STOR(I,I)=STOR(I,I) + 1.0
    4 CONTINUE
C
      CALL    HLBOND(STOR,NBDOF,BOND)
C
      WRITE (3,104)  EPS,Q
  104 FORMAT (1H0,10X,'EPS='E13.5,10X,'Q='E13.5)
      WRITE (3,103)  BOND,EPS
  103 FORMAT(1H0,10X,'BOND='E13.5,10X,'EPS='E13.5)
      IF  (BOND)  14,14,15
   14 WRITE (3,105)
      GO  TO  3
  105 FORMAT (///'  TRIAL   FAILED')
   15 OMGA=Q*EPS
      FRQ = SQRT(OMGA)
      WRITE (3,101)  ICOUNT,FRQ
  101 FORMAT (1H0, 10X,'TRIAL NO. =' I10,10X,'LOWER BOUND FREQUENCY RAD/
     1SEC ='E13.5/)
      FRQ = FRQ/(2.0*3.14159)
      WRITE (3,102)  ICOUNT,FRQ
  102 FORMAT (1H0, 10X,'TRIAL NO. =' I10,10X,'LOWER BOUND FREQUENCY CYC/
     1SEC ='E13.5////)
    1 Q=Q+ AINC
    3 EPS = EPS/2.0
C
C
      RETURN
      END
```

```
C***********************************************************************
C*                                                                     *
C*       F R A M E D    D O M E                                        *
C*       SYSTEM STIFFNESS AND SYSTEM MASS MATRIX                       *
C*                                                                     *
C***********************************************************************
C
C
C        IDENTI = 0  HORIZONTAL MEMBER
C        IDENTI = 1  VERTICAL MEMBER
C
C
C
C
      NAMELIST/A1/ NODOF,THETA
      NAMELIST/A2/   DENO,L,CSI,IDENTI
      REAL K(60,60),M(60,60),KEVERT(12,12),KEHORI(12,12),MVERT(12,12),
     1MHORI(12,12),IX,IY,IZ,MUEY,MUEZ,L,JJ,KK,MMM,JJJ,KKK
      INTEGER DOG,CAT,RAT,HEN,DUMMY,CODENO(12),DENO(12)
C
C
      DO 1  I = 1,60
      DO 1  J = 1,60
      K(I,J) = 0.0
    1 M(I,J) = 0.0
C
      READ (1,A1)
      WRITE (3,1015)  NODOF,THETA
 1015 FORMAT (1H1,'NO.OF DEGREES-OF-FREEDOM ='I10/' THETA ='F20.10///)
C
           E#450000.
           G#180000.
           IX=81./(6.*16.**4)
           IY=81./(12.*16.**4)
           IZ#IY
           AREA#9./256.
           UNIWT#0.04114
           P=SIN(THETA/2.)
           Q=COS(THETA/2.)
           ALPHAY#4.
           ALPHAZ#4.
           BATAY#2.
           BATAZ#2.
           GAMAY#6.
           GAMAZ#6.
           MUEY#12.
           MUEZ#12.
           DUMMY=0
C
C
C
      ICOUNT = 0
   10 READ (1,A2)
```

```
      DO  2  I =1,12
      CODENO(I) = DENO(I)
      DO  2  J=1,12
      KEVERT(I,J)=0.0
      KEHORI(I,J)=0.0
      MVERT(I,J)=0.0
      MHORI(I,J)=0.0
    2 CONTINUE
      WRITE (3,1020) (CODENO(I),I=1,12),L,CSI,IDENTI
 1020 FORMAT (1H0,'CODE ='(12I5),/' L ='F10.6,5X,'CSI ='F10.6,5X,'IDENTI
     1 ='I10)
      ICOUNT = ICOUNT + 1
      IF (ICOUNT .EQ. 20)  DUMMY = 100
          IF (DUMMY.EQ.100)  GO  TO  20
          A=SIN(CSI)
          B=COS(CSI)
          AA#AREA*E/L
          BB#MUEZ*IZ*E/%L*L*L□
          CC#MUEY*IY*E/%L*L*L□
          DD#G*IX/L
          EE#ALPHAY*IY*E/L
          FF#ALPHAZ*IZ*E/L
          GG#-GAMAZ*IZ*E/%L*L□
          HH#-GAMAY*IY*E/%L*L□
          JJ#BATAY*IY*E/L
          KK#BATAZ*IZ*E/L
C
          ROW=SQRT(IX/AREA)
          W#UNIWT *AREA*12.
          MMM#W/%12.*12.*32.2□
          AAA#MMM*L/3.
          BBB#156.*MMM*L/420.
          CCC#MMM*ROW*ROW*L/3.
          DDD#4.*MMM*L*L*L/420.
          EEE#22.*MMM*L*L/420.
          FFF#MMM*L/6.
          GGG#54.*MMM*L/420.
          HHH#MMM*ROW*ROW*L/6.
          JJJ#3.*MMM*L*L*L/420.
          KKK#13.*MMM*L*L/420.
C
C
          IF (IDENTI-1)  30,31,30
   31     CONTINUE
          KEVERT%1,1□#AA*A*A&BB*B*B
          KEVERT%1,3□#%AA-BB□*A*B
          KEVERT%1,5□#- GG*B
          KEVERT%1,7□#- %AA*A*A&BB*B*B□
          KEVERT%1,9□#- %AA-BB□*A*B
          KEVERT%1,11□#- GG*B
          KEVERT%2,2□#CC
          KEVERT%2,4□# HH*B
          KEVERT%2,6□#- HH*A
```

```
KEVERT%2,8□# - CC
KEVERT%2,10□# HH*B
KEVERT%2,12□#- HH*A
KEVERT%3,3□#AA*B*B&BB*A*A
KEVERT%3,5□#& GG*A
KEVERT%3,7□#-%AA-BB□*A*B
KEVERT%3,9□#-%AA*B*B&BB*A*A□
KEVERT%3,11□#  GG*A
KEVERT%4,4□# DD*A*A&EE*B*B
KEVERT%4,6□#%DD-EE□*A*B
KEVERT%4,8□#- HH*B
KEVERT%4,10□#-DD*A*A&JJ*B*B
KEVERT%4,12□#%-DD-JJ□*A*B
KEVERT%5,5□#FF
KEVERT%5,7□# GG*B
KEVERT%5,9□#-GG*A
KEVERT%5,11□# KK
KEVERT%6,6□#DD*B*B&EE*A*A
KEVERT%6,8□#     HH*A
KEVERT%6,10□#%-DD-JJ□*A*B
KEVERT%6,12□#-DD*B*B&JJ*A*A
KEVERT%7,7□#AA*A*A&BB*B*B
KEVERT%7,9□#%AA-BB□*A*B
KEVERT%7,11□# GG*B
KEVERT%8,8□# CC
KEVERT%8,10□#- HH*B
KEVERT%8,12□#  HH*A
KEVERT%9,9□#AA*B*B&BB*A*A
KEVERT%9,11□#-GG*A
KEVERT%10,10□# DD*A*A&EE*B*B
KEVERT%10,12□#%DD-EE□*A*B
KEVERT%11,11□# FF
KEVERT%12,12□#DD*B*B&EE*A*A
```

C

```
MVERT%1,1□#AAA*A*A & BBB*B*B
MVERT%1,3□#%AAA-BBB□*A*B
MVERT%1,5□#-EEE*B
MVERT%1,7□#-FFF*A*A-GGG*B*B
MVERT%1,9□#%GGG-FFF□*A*B
MVERT%1,11□#-KKK*B
MVERT%2,2□#BBB
MVERT%2,4□#  EEE*B
MVERT%2,6□# -EEE*A
MVERT%2,8□#-GGG
MVERT%2,10□# KKK*B
MVERT%2,12□#-KKK*A
MVERT%3,3□#AAA*B*B & BBB*A*A
MVERT%3,5□# EEE*A
MVERT%3,7□#%GGG-FFF□*A*B
MVERT%3,9□#-FFF*B*B-GGG*A*A
MVERT%3,11□#  KKK*A
MVERT%4,4□#CCC*A*A & DDD*B*B
MVERT%4,6□#%CCC-DDD□*A*B
```

```
      MVERT%4,8□# -KKK*B
      MVERT%4,10□#-HHH*A*A&JJJ*B*B
      MVERT%4,12□#-%HHH&JJJ□*A*B
      MVERT%5,5□# DDD
      MVERT%5,7□# KKK*B
      MVERT%5,9□#-KKK*A
      MVERT%5,11□#   JJJ
      MVERT%6,6□#CCC*B*B & DDD*A*A
      MVERT%6,8□#  KKK*A
      MVERT%6,10□#-%HHH&JJJ□*A*B
      MVERT%6,12□#-HHH*B*B&JJJ*A*A
      MVERT%7,7□#AAA*A*A & BBB*B*B
      MVERT%7,9□#%AAA-BBB□*A*B
      MVERT%7,11□#  EEE*B
      MVERT%8,8□#BBB
      MVERT%8,10□#-EEE*B
      MVERT%8,12□#EEE*A
      MVERT%9,9□#AAA*B*B & BBB*A*A
      MVERT%9,11□# -EEE*A
      MVERT%10,10□#CCC*A*A & DDD*B*B
      MVERT%10,12□#%CCC-DDD□*A*B
      MVERT%11,11□# DDD
      MVERT%12,12□#CCC*B*B &DDD*A*A
      GO  TO  40
30    CONTINUE
      KEHORI%1,1□#BB
      KEHORI%1,5□#-GG*P
      KEHORI%1,6□#GG*Q
      KEHORI%1,7□# - BB
      KEHORI%1,11□# GG*P
      KEHORI%1,12□# GG*Q
      KEHORI%2,2□# AA*Q*Q&CC*P*P
      KEHORI%2,3□#%AA-CC□*P*Q
      KEHORI%2,4□#  HH*P
      KEHORI%2,8□# - AA*Q*Q&CC*P*P
      KEHORI%2,9□#  %AA&CC□*P*Q
      KEHORI%2,10□#  HH*P
      KEHORI%3,3□#AA*P*P&CC*Q*Q
      KEHORI%3,4□# -HH*Q
      KEHORI%3,8□# -%AA&CC□*P*Q
      KEHORI%3,9□#   AA*P*P-CC*Q*Q
      KEHORI%3,10□# -HH*Q
      KEHORI%4,4□# EE
      KEHORI%4,8□# HH*P
      KEHORI%4,9□# HH*Q
      KEHORI%4,10□# JJ
      KEHORI%5,5□# DD*Q*Q&FF*P*P
      KEHORI%5,6□# %DD-FF□*P*Q
      KEHORI%5,7□#   GG*P
      KEHORI%5,11□# -DD*Q*Q-KK*P*P
      KEHORI%5,12□#% DD-KK□*P*Q
      KEHORI%6,6□#DD*P*P&FF*Q*Q
      KEHORI%6,7□# -GG*Q
```

```
      KEHORI%6,11□#%-DD&KK□*P*Q
      KEHORI%6,12□#   DD*P*P&KK*Q*Q
      KEHORI%7,7□# BB
      KEHORI% 7,11□# - GG*P
      KEHORI%7,12□# -GG*Q
      KEHORI%8,8□#AA*Q*Q&CC*P*P
      KEHORI%8,9□#%CC-AA□*P*Q
      KEHORI%8,10□# HH*P
      KEHORI%9,9□#AA*P*P&CC*Q*Q
      KEHORI%9,10□# HH*Q
      KEHORI%10,10□# EE
      KEHORI%11,11□#DD*Q*Q&FF*P*P
      KEHORI%11,12□#%FF-DD□*P*Q
      KEHORI%12,12□#DD*P*P&FF*Q*Q
C
      MHORI%1,1□# BBB
      MHORI%1,5□# -FEE*P
      MHORI%1,6□#  EEE*Q
      MHORI%1,7□#-GGG
      MHORI%1,11□# KKK*P
      MHORI%1,12□# KKK*Q
      MHORI%2,2□#AAA*Q*Q & BBB*P*P
      MHORI%2,3□#%AAA-BBB□*P*Q
      MHORI%2,4□# EEE*P
      MHORI%2,8□#-FFF*Q*Q&GGG*P*P
      MHORI%2,9□#%FFF&GGG□*P*Q
      MHORI%2,10□#  KKK*P
      MHORI%3,3□#AAA*P*P & BBB*Q*Q
      MHORI%3,4□#-EEE*Q
      MHORI%3,8□#-%FFF&GGG□*P*Q
      MHORI%3,9□#FFF*P*P - GGG*Q*Q
      MHORI%3,10□# -KKK*Q
      MHORI%4,4□# DDD
      MHORI%4,8□# KKK*P
      MHORI%4,9□# KKK*Q
      MHORI%4,10□# JJJ
      MHORI%5,5□#CCC*Q*Q & DDD*P*P
      MHORI%5,6□#%CCC-DDD□*P*Q
      MHORI%5,7□#  KKK*P
      MHORI%5,11□#-HHH*Q*Q-JJJ*P*P
      MHORI%5,12□#%HHH-JJJ□*P*Q
      MHORI%6,6□#CCC*P*P & DDD*Q*Q
      MHORI%6,7□# -KKK*Q
      MHORI%6,11□#%JJJ-HHH□*Q*P
      MHORI%6,12□#HHH*P*P&JJJ*Q*Q
      MHORI%7,7□# BBB
      MHORI%7,11□#-EEE*P
      MHORI%7,12□#-EEE*Q
      MHORI%8,8□#AAA*Q*Q & BBB*P*P
      MHORI%8,9□#%BBB-AAA□*P*Q
      MHORI%8,10□#  EEE*P
      MHORI%9,9□#AAA*P*P & BBB*Q*Q
      MHORI%9,10□#  EEE*Q
```

```
             MHORI%10,10¤# DDD
             MHORI%11,11¤#CCC*Q*Q & DDD*P*P
             MHORI%11,12¤#%DDD-CCC¤*P*Q
             MHORI%12,12¤#CCC*P*P & DDD*Q*Q
   40        CONTINUE
C
C
      DO   11   DOG = 1,12
      IA = DOG + 1
      DO   11   CAT = IA,12
             KEVERT%CAT,DOG¤#KEVERT%DOG,CAT¤
             KEHORI%CAT,DOG¤#KEHORI%DOG,CAT¤
             MVERT%CAT,DOG¤#MVERT%DOG,CAT¤
             MHORI%CAT,DOG¤#MHORI%DOG,CAT¤
   11 CONTINUE
C
C
      DO   13   RAT = 1,12
      DO   14   HEN = 1,12
      IF  (CODENO(RAT).NE.0 .AND. CODENO(HEN).NE.0)  GO TO 12
      GO  TO  14
   12 DOG = RAT
      CAT = HEN
      I = CODENO(RAT)
      J = CODENO(HEN)
      IF  (IDENTI)  15,16,15
   15 CONTINUE
             K%I,J¤# K%I,J¤ & KEVERT%DOG,CAT¤
             M%I,J¤#M%I,J¤ & MVERT%DOG,CAT¤
      GO  TO  14
   16 CONTINUE
             M%I,J¤#M%I,J¤ & MHORI%DOG,CAT¤
             K%I,J¤# K%I,J¤&KEHORI%DOG,CAT¤
   14 CONTINUE
   13 CONTINUE
      GO  TO  10
C
   20 CONTINUE
C
C
   50 WRITE (3,1005)((K(NB,I),I=1,6),NB=1,60)
   55 WRITE (3,1010)((M(NB,I),I=1,6),NB=1,60)
 1005 FORMAT (1H1,'STIFFNESS MATRIX'//(1X,6E13.5))
 1010 FORMAT (1H1,'MASS      MATRIX'//(1X,6E13.5))
C
      END
```

```
C*********************************************************************************
C*                                                                               *
C*        M A S S - S P R I N G     S Y S T E M                                  *
C*                                                                               *
C*********************************************************************************
C
C     SPRING MASS SYSTEM    EXACT SOLUTION
C
      REAL  M(2000),K(2000),P(2000),VA(2000),VB(2000),SPG(10),ELMAS(10)
      INTEGER  DEL,DUMY,I,J,R,S,W,ROWP,COUNT,MAXITR
C
   99 CONTINUE
      WRITE (3,705)
  705 FORMAT (1H1,'**************************** START *********************')
C
      CALL  TIME (ITIME)
      TSTART = ITIME
C
      INDEX = 1
      WRITE (3,303)  INDEX
  303 FORMAT (///'  EIGEN VALUE COMPUTATION RUN NO. ' I10)
      READ (1,101) NBDOF,DUMY,DEL,MAXITR,(SPG(L),L=1,10),(ELMAS(L),L=1,
     110)
  101 FORMAT (4I5,/10F5.0/10F5.0)
      DO  123  I=1,NBDOF
  123 VA(I) = 1.0
C
      DO  133  I=1,DEL
      LL2=I+1*DEL
      LL3=I+2*DEL
      LL4=I+3*DEL
      LL5=I+4*DEL
      LL6=I+5*DEL
      LL7=I+6*DEL
      LL8=I+7*DEL
      LL9=I+8*DEL
      LL10=I+9*DEL
      K(I) = SPG(1)
      K(LL2    ) = SPG(2)
      K(LL3    ) = SPG(3)
      K(LL4    ) = SPG(4)
      K(LL5    ) = SPG(5)
      K(LL6    ) = SPG(6)
      K(LL7    ) = SPG(7)
      K(LL8    ) = SPG(8)
      K(LL9    ) = SPG(9)
  133 K(LL10   ) = SPG(10)
C
C
   88 CONTINUE
      COUNT = 0
      IF  (DUMY)   11,10,11
   10 DO  143  I=1,DEL
```

```
      LL2=I+1*DEL
      LL3=I+2*DEL
      LL4=I+3*DEL
      LL5=I+4*DEL
      LL6=I+5*DEL
      LL7=I+6*DEL
      LL8=I+7*DEL
      LL9=I+8*DEL
      LL10=I+9*DEL
      M(I) = ELMAS(1)
      M(LL2     ) =ELMAS(2)
      M(LL3     ) =ELMAS(3)
      M(LL4     ) =ELMAS(4)
      M(LL5     ) =ELMAS(5)
      M(LL6     ) =ELMAS(6)
      M(LL7     ) =ELMAS(7)
      M(LL8     ) =ELMAS(8)
      M(LL9     ) =ELMAS(9)
  143 M(LL10    ) =ELMAS(10)
      GO  TO  1000
   11 DO  153  I=1,NBDOF
      VA(I) =1.0
  153 M(I)=1.0
C
 1000 CONTINUE
      DO  1  L=1,2000
    1 VB(L) = 0.0
      COUNT = COUNT +1
      IF  (COUNT .GT. MAXITR)  GO TO  2000
      DO   1010  ROWP = 1,NBDOF
      DO   1011  I = 1,NBDOF
      IF  ( I - ROWP) 12,12,13
   12 STIF = 0.0
      DO  1020  J=1,I
 1020 STIF = STIF + 1.0/K(J)
   13 P(I) = M(I)*STIF
 1011 CONTINUE
      DO  1030  R=1,NBDOF
 1030 VB(ROWP) = VB(ROWP) +P(R)*VA(R)
 1010 CONTINUE
C
      EIGEN = VB(1)
  301 FORMAT(10X,'NO.OF ITERATIONS ='I5,5X,'EIGEN VALUE ='E13.5)
      DO  1040  S=1,NBDOF
 1040 VB(S) = VB(S)/EIGEN
      DO  1060  S=1,NBDOF
      IF((ABS(VB(S)) - ABS(VA(S))) .GT. 0.001)  GO  TO  14
      GO  TO  1060
   14 DO  1050  W = 1,NBDOF
      VA(W) = VB(W)
 1050 CONTINUE
      GO TO  1000
 1060 CONTINUE
```

```
C
 2000 CONTINUE
      IF  (INDEX.EQ.2)  GO  TO  31
      FRQNCY = SQRT(ABS(1.0/EIGEN))/(2.0*3.14159)
      WRITE (3,302)  COUNT, EIGEN, FRQNCY
  302 FORMAT (///' NO.OF ITERATIONS ='I5,2X,'EIGEN VALUE ='E13.5,2X,'FUN
     1DAMENTAL FREQUENCY IN CYC/SEC ='E13.5)
      IF  (INDEX - 2)  30,31,30
   30 DUMY = 1
      INDEX = 2
      WRITE (3,303)   INDEX
      GO  TO  88
   31 CONTINUE
C
C
C
C
C
C
C          H-LOWER BOUND, MASS-SPRING SYSTEM
C
      DIMENSION  TEMP(2000),STOR(2000)
      EQUIVALENCE (TEMP,VA),(STOR,VB)
      DO  801  I=1,DEL
      LL2=I+1*DEL
      LL3=I+2*DEL
      LL4=I+3*DEL
      LL5=I+4*DEL
      LL6=I+5*DEL
      LL7=I+6*DEL
      LL8=I+7*DEL
      LL9=I+8*DEL
      LL10=I+9*DEL
      M(I) = ELMAS(1)
      M(LL2     ) =ELMAS(2)
      M(LL3     ) =ELMAS(3)
      M(LL4     ) =ELMAS(4)
      M(LL5     ) =ELMAS(5)
      M(LL6     ) =ELMAS(6)
      M(LL7     ) =ELMAS(7)
      M(LL8     ) =ELMAS(8)
      M(LL9     ) =ELMAS(9)
  801 M(LL10    ) =ELMAS(10)
C
      DO  501  I = 1,2000
  501 TEMP(I) = 0.0
      OMEGA2 = 1000000.0
      DO  502  I = 1,NBDOF
      DO  503  II=1,2000
  503 P(II) = 0.0
      IF  (I.EQ.1)  GO TO  601
      IF  (I.GT.1 .AND. I.LT.NBDOF)  GO  TO  602
      IF  (I.EQ.N)  GO TO  603
```

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