Homotopy Batalin-Vilkovisky algebras, trivializing circle actions, and moduli space

by

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A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

2010

This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirements for the degree of Doctor of Philosophy.

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Abstract

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This thesis comprises two main results, one topological, one algebraic. The topological result is that an action of the framed little disks operad and a trivialization of the circle action within it determine an action of the Deligne-Mumford compactification of the moduli space of genus zero curves. The algebraic result is a description of the structure of minimal homotopy Batalin-Vilkovisky algebras and the the theorem that in the case that the Batalin-Vilkovisky operator and its higher homotopies are trivial, the remaining algebraic structure is a minimal homotopy hypercommutative algebra. These results are related to one another because the algebraic structures involved are representations of the homology of, respectively, the framed little disks and the Deligne-Mumford compactification.

Acknowledgements

I would like to thank my advisor, John Terilla. He led me to the right path, showed me how to follow it, and kept me from taking too many detours. His guidance and clarity of vision were key in the formulation of my results and my understanding of their applications. I would also like to thank him for his encouragement, engagement, and patience. I am proud to be his first student, and I hope that I reflect well on him.

I am also grateful for my conversations with Dennis Sullivan, Kevin Costello, and Bruno Vallette. Each of them taught me a great deal about the topics covered in this document; without any of them, my knowledge and understanding would be far weaker.

I would also like to thank Nathaniel Rounds, Scott Wilson, Joseph Hirsh, Kate Poirier, and Travis Waddington. They each helped me on many occasions.

Most of all, I would like to thank Angela Long for her constant love and support. She was the source of my strength throughout this process.

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Chapter 1

Introduction

1.1 Overview

This dissertation is an investigation of the homotopy theory of operads arising from the moduli space of genus zero Riemann surfaces and their representations. It comprises two principal results, one topological, one algebraic, characterizing those representations where a particular suboperad acts trivially.

In the topological setting, the configuration space of multiple disks in the standard disk, each with a marked point on its boundary, forms a topological operad known as the framed little disks operad. In many representations of this operad which arise in geometry and physics, the suboperad corresponding to the configuration space of just one disk, the framed little annuli, acts trivially. The first result of this dissertation is the statement that any representation of the framed little disks operad satisfying this triviality condition has an induced representation of another well-known topological operad, namely the Deligne-Mumford compactification of the moduli space of genus zero surfaces with marked points. The proof involves simple categorical notions and combinatorics of trees. Specifically, the Deligne-Mumford operad is proved to be weakly homotopy equivalent as a topological operad to a categorical pushout in the category of topological operads of the framed little disks operad and a particular trivial operad along the framed little annuli.

On the algebraic side, the second main result of this dissertation is the homotopy-algebraic clarification and extension of a construction of Barannikov and Kontsevich. Their construction begins with a representation of the homology of the framed little disks operad, called a Batalin-Vilkovisky algebra structure, on a chain complex, which satisfies special conditions which form an example of what might be called a homotopy trivialization, and ends with a representation of the homology of the Deligne-Mumford operad, called a hypercommutative algebra structure, on the homology of the chain complex. The results herein explain this construction as follows: there is a well-understood procedure to transfer a representation of any operad on a chain complex to a representation of the homotopy version of that operad on its homology. So in the context of the Barannikov-Kontsevich construction, there is a transferred homotopy Batalin-Vilkovisky algebra structure on the homology of the chain complex. In the case where the original Batalin-Vilkovisky algebra satisfies their special conditions, this homotopy Batalin-Vilkovisky algebra structure induces a homotopy hypercommutative algebra structure, which can be truncated to the hypercommutative algebra of the original Barannikov-Kontsevich construction.

This result accomplishes three main things. First, it refines the results of Barannikov and Kontsevich by weakening their special conditions. Second, it brings their construction into the realm of homotopy operadic algebra, thus clarifying it. Third, it generalizes the construction by allowing it to stop at the level of the homotopy hypercommutative algebra structure, keeping data which is destroyed by the truncation to the hypercommutative algebra structure. The result is a consequence of a theorem which constructs an explicit map of operads from the homotopy Batalin-Vilkovisky operad to the homotopy hypercommutative operad and identifies its kernel.

This dissertation is organized into two main body chapters, one topological, one algebraic, followed by two appendices. The first appendix gives some background on operads, trees, and related concepts, and the second is a brief primer on the Deligne-Mumford compactification of the genus zero moduli space.

1.2 Connections to Other Work

The work in the topological section is a rigorous presentation of one version of an idea of Kontsevich. In lectures at Max Planck [18], he suggested that a picture like this should be possible, that is, that a version of the Deligne-Mumford operad would be some sort of pushout in the category of topological operads. To the author's knowledge, this was never described in more detail.

The motivation for the algebraic section, again, goes back to the paper of Barannikov and Kontsevich [3] in which they studied the Dolbeaut resolution of the exterior powers of the holomorphic tangent sheaf of a Calabi-Yau manifold. In the paper, they described a Frobenius manifold structure on the homology of this complex. One of the multiple ways to describe a Frobenius manifold is as a hypercommutative algebra with a compatible metric. They viewed the Dolbeaut complex as a differential graded Lie algebra, proved its formality, and then used a particular choice of universal solution of the master equation to construct the hypercommutative structure on the homology of the Lie algebra. This construction was accomplished "by hand" and is arguably opaque.

Barannikov and Kontsevich very briefly described their construction in general terms. This discussion was expanded and made explicit by Manin [20]. In particular, the algebraic structure used in the Barannikov-Kontsevich construction was that of a Batalin-Vilkovisky algebra satisfying several other special conditions. Some of the conditions were used for the formality argument, while other, independent conditions were used to obtain the metric. Park [26] and Terilla [28] further generalized this construction by successively weakening the "homotopy triviality" conditions necessary to achieve the formality and construct the special solution of the master equation.

One way to view a universal solution of the master equation in a differential graded Lie algebra is as at isomorphism from the transferred L_∞ structure on its homology (see, for example, [17]). This indicates that the Barannikov-Kontsevich construction "belongs to" the realm of homotopy algebra, and that it should be viewed in some way as a transfer of an algebraic structure.

The algebraic results presented here do precisely that. We identify the transferred structure on the homology of any Batalin-Vilkovisky algebra, not necessarily one satisfying Barannikov and Kontsevich's special conditions, and show that this structure can be truncated to a hypercommutative algebra whenever the weakest homotopy triviality conditions hold. It is clear from the algebra that one could even construct examples where the homotopy triviality conditions did not hold but the transferred structure had the hypercommutative truncation.

These results accomplish several useful generalizations to previous work. First, they generalize and extend the conditions of Barannikov-Kontsevich, Manin, Park, and Terilla. In particular, one can transfer a Batalin-Vilkovisky algebra to its homology without any triviality conditions. This is because these results describe the construction as an example of a transfer of structure, a captured part of homotopy algebra. Another consequence is that the hypercommutative algebra constructed by Barannikov and Kontsevich is a truncation of a richer homotopy hypercommutative algebra structure. That is, forgetting the higher operations in the transferred structure actually loses information. Further, because these results are part of homotopy algebra, they indicate precisely how one can recover that lost information and how it can be organized in a meaningful and coherent way.

Sergei Merkulov, motivated by the construction of Barannikov and Kontsevich, considered a kind of generalization of the notion of Frobenius manifold which he called an F_∞ manifold [24, 25]. He showed how to obtain an F_∞ manifold from a homotopy Gerstenhaber algebra. Batalin-Vilkovisky algebras are Gerstenhaber algebras with extra structure; similarly, a homotopy hypercommutative algebra (with compatible metric) induces a special kind of F_∞ manifold. Therefore, the algebraic results of this thesis can be viewed as a further refinement of Merkulov's work as well as that of Barannikov and Kontsevich.

The calculations in the algebraic section rely on a particular model of the homotopy Batalin-Vilkovisky operad due to Gálvez-Carrillo, Tonks, and Vallette [9]. We use their construction to calculate the minimal model for the homotopy Batalin-

Vilkovisky operad. There is not a logical reliance on their results, because the same calculation could be applied to a different model, but there is a psychological reliance, because their presentation, by virtue of its parsimony, made certain features of the algebra more apparent. There is a good discussion in Section 2.4 of that reference of the relationship of algebras over a cofibrant replacement for the Batalin-Vilkovisky operad with various other notions of homotopy Batalin-Vilkovisky algebra that exist in the literature.

Finally, many of the explicit calculations in the algebraic section were performed by Getzler in two papers [11, 12]. He described the homotopy hypercommutative operad and in so doing analyzed the behavior of a differential on the Gerstenhaber operad. That same differential, in a twisted dual guise, is at the core of the description of the minimal Batalin-Vilkovisky operad.

Chapter 2

Trivializing topological circle actions

2.1 Overview

The purpose of this chapter is to provide a picture, in the realm of topological operads, of the trivialization of the circle action in an representation of the framed little disks operad. This is a topological model for the algebraic passage from Batalin-Vilkovisky to hypercommutative algebras discussed in the next chapter. These species of algebra are algebras over operads which are the homology groups of topological spaces, the framed little disks operad $\mathcal{D}$ for Batalin-Vilkovisky and the genus zero Deligne-Mumford compactification of the moduli space of genus zero surfaces $\overline{\mathcal{M}}$ for hypercommutative. The idea here is to describe the topological operad which is the pushout of the following diagram:

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{D} \\ \downarrow & & \vdots \\ \mathcal{A}_t & \cdots\!\!> & \mathcal{D}_t \end{array}$$

Here $\mathcal{A}_t$ is a trivial operad and $\mathcal{A}$ is the operad of framed annuli. This means that whenever you have an action of the framed little disks where the annuli act trivially (this could be called "filling in the circle action" or "triviality of the BV operator"), you have an action of $\mathcal{D}_t$.

We will show, as our main theorem, that this pushout is homotopy equivalent to $\overline{\mathcal{M}}$, with an inclusion map $\overline{\mathcal{M}} \to \mathcal{D}_t$ which is a map of topological operads.

Once one has this, one can apply various functors. By passing to homology one obtains the regular passage from BV algebras to hypercommutative algebras. However, one can apply instead the singular cubical chain functor and obtain a different statement that contains some homotopy information. In the next chapter, we shall take an algebraic approach to isolate all of the homotopy information and strip out everything extraneous. It would be nice to present this pushout operad with a cellular model which had nice compatibility with the minimal algebraic formulae of Chapter 3.

This chapter begins in Section 2.2 with a review of the affine group of $\mathbb{C}$, which is intimately involved in the operad compositions for most of the operads involved in this pushout, then describes, in Section 2.3, all of these operads explicitly. Sections 2.4 and 2.5 establish the pushout operad, first in the category of sets and then in the category of spaces. Finally, Section 2.6 shows that the pushout is homotopy equivalent to $\overline{\mathcal{M}}$.

2.2 $\mathrm{Aff}\,\mathbb{C}$ and friends

Most of the operads in this chapter can be viewed as configuration spaces of disks, possibly of zero radius, with marked points on their boundary. The composition laws are all similar to one another in flavor, and both the configuration spaces and their composition laws are descended directly from the group $\mathrm{Aff}\,\mathbb{C}$ of conformal automorphisms of the complex plane.

Definition 2.2.1. The group $\mathrm{Aff}\,\mathbb{C}$ is $\mathbb{C} \ltimes GL_1\mathbb{C} \cong \mathbb{C} \ltimes \mathbb{C}^*$, which acts on the complex plane $\mathbb{C}$ by translation, dilation, and rotation.

In coordinates, if the action is taken to be rotation and dilation first, followed by translation, then then the group action is

$$(c_1, r_1) \odot (c_2, r_2) = (c_1 + r_1 c_2, r_1 r_2).$$

The inverse of (c, r) is $(-\frac{c}{r}, \frac{1}{r})$ and the identity is $(0, 1)$.

$\mathrm{Aff}\,\mathbb{C}$ can also be identified with the configuration space of a disk with a marked point on its boundary in the plane; The element (c, r) in this presentation corresponds to the disk centered at c with a marked point on its boundary at $c + r$. In this context, the composition map involves scaling a disk up or down and rotating it. This will become clearer in a slightly more restricted setting soon.

We immediately generalize this definition to deal with both degenerate disks of radius zero and their degenerate actions. As in $\mathrm{Aff}\,\mathbb{C}$, we will represent a disk

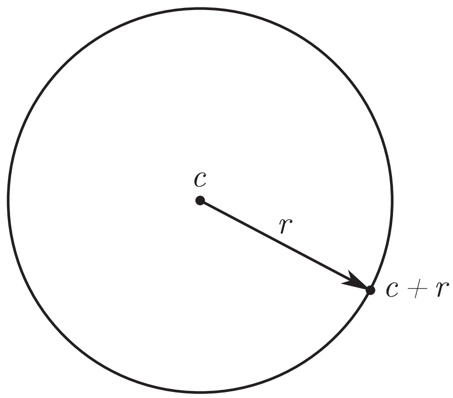


Figure 2.1: An element of $\mathrm{Aff}\,\mathbb{C}$

in the plane as two points in $\mathbb{C}$; the first will be the center point of the disk and the second a radius vector to the marked point. In this general situation, we will allow the center and radius to be any points in the complex plane.

Definition 2.2.2. 1. Let S be a finite set. The *configuration space of S disks in the plane* is:

$$\mathfrak{C}(S) = \{(c_s, r_s)_{s \in S} \in (\mathbb{C} \times \mathbb{C})^S : |c_s - c_{s'}| > |r_s| + |r_{s'}|\}.$$

The condition ensures that every two disks in the plane are disjoint.

2. Let $\odot$ be the composition map

$$(\mathbb{C} \times \mathbb{C}) \times (\mathbb{C} \times \mathbb{C}) \to (\mathbb{C} \times \mathbb{C})$$

given by

$$(c_1, r_1) \odot (c_2, r_2) = (c_1 + r_1 c_2, r_1 r_2).$$

In particular cases below, we will use $\odot$ to denote as well its restriction to various subspaces of $\mathbb{C} \times \mathbb{C}$, such as $\operatorname{Aff}\mathbb{C}$.

Lemma 2.2.3. *$\odot$ is an open map.*

Proof. First, let $(c_{\mathrm{Aff}}, r_{\mathrm{Aff}}) \in \operatorname{Aff}\mathbb{C} \subset \mathbb{C} \times \mathbb{C}$. Then

$$\operatorname{Aff}\mathbb{C} \times (c_{\mathrm{Aff}}, r_{\mathrm{Aff}}) \xrightarrow{\odot} \operatorname{Aff}\mathbb{C} \subset \mathbb{C} \times \mathbb{C}$$

is a homeomorphism onto its open image so an open map. Then on a product of open sets, the image is a union of open sets, so we can conclude $\odot$ is an open map at least on $\operatorname{Aff}\mathbb{C} \times \operatorname{Aff}\mathbb{C}$.

Now fix $c_0 \in \mathbb{C}$ and $\epsilon > 0$ and let

$$U_\epsilon^{c_0} = \{(c, r) \in \mathbb{C} \times \mathbb{C} : |c - c_0|, |r| < \epsilon\}.$$

Such sets, for arbitrarily small choices of ϵ, form a basis for the topology of $\mathbb{C} \times \mathbb{C}$ around $(c_0, 0)$. Composing $(c_{\mathrm{Aff}}, r_{\mathrm{Aff}}) \in \operatorname{Aff}\mathbb{C}$ with $U_\epsilon^{c_0}$ gives directly

$$(c_{\mathrm{Aff}}, r_{\mathrm{Aff}}) \odot U_\epsilon^{c_0} = U_\delta^c$$

where $\delta = r_{\mathrm{Aff}}\epsilon$ and $c = c_{\mathrm{Aff}} + r_{\mathrm{Aff}}c_0$.

Composing in the other direction gives

$$U_\epsilon^{c_0} \odot (c_{\mathrm{Aff}}, r_{\mathrm{Aff}}) = \{(c + rc_{\mathrm{Aff}}, rr_{\mathrm{Aff}}) \in \mathbb{C} \times \mathbb{C} : |c - c_0|, |r| < \epsilon\}.$$

Reparameterizing this set by the homeomorphism $(z, w) \mapsto (z - \frac{c_{\text{Aff}}}{r_{\text{Aff}}} w, w)$, we get the open set

$$\{(z, w) \in \mathbb{C} \times \mathbb{C} : |z - c_0| < \epsilon, |w| < \frac{\epsilon}{r_{\text{Aff}}}\}.$$

Finally, we would like to see that we get an open set by composing two neighborhoods of radius zero disks. For fixed c_1 and δ, define

$$V_\delta^{c_1} = \{(c_1, r) \in \mathbb{C} \times \mathbb{C} : |r| < \delta\}.$$

Then

$$U_\epsilon^{c_0} = \bigcup_{|c_1 - c_0| < \epsilon} V_\epsilon^{c_1}.$$

So it suffices to show that for any c_0, c_1, ϵ, and δ, $U_\epsilon^{c_0} \odot V_\delta^{c_1}$ is an open set. We can write $V_\delta^{c_1}$ as $(c_1, 1) \odot V_\delta^0$, and then by the associativity of $\odot$, we have

$$U_\epsilon^{c_0} \odot V_\delta^{c_1} = (U_\epsilon^{c_0} \odot (c_1, 1)) \odot V_\delta^0.$$

We know from the previous argument that $U_\epsilon^{c_0} \odot (c_1, 1)$ is open, so we can restrict to a basic open set inside of it. Now we have reduced the problem to verifying that $U_{\epsilon'}^{c_0'} \odot V_\delta^0$ is open. But this set is precisely

$$\{(c, r) \in \mathbb{C} \times \mathbb{C} : |c - c_0'| < \epsilon', r < \epsilon' \delta\}$$

which is open. $\square$

2.3 Operads of configuration spaces

The operads in this section are versions of the little disks operad described by Boardman and Vogt [5]. Our presentation, which has useful features for our analysis, is similar to the presentation of the little disks operad used by Getzler [12].

Definition 2.3.1. The topological operad $\mathcal{A}$ of *framed little annuli* consists of the following spaces: $\mathcal{A}(1)$ is the subset of $\mathfrak{C}(1) = \mathbb{C} \times \mathbb{C}$ such that:

1. $|r| > 0$,
2. $|c| + |r| \leq 1$, and
3. this inequality is strict unless $c = 0$.

$\mathcal{A}(n)$ is empty for $n \neq 1$. A point in the space $\mathcal{A}(1)$ describes an annulus of outer radius one and inner radius $|r|$ centered at c with a marked point on the inner boundary circle at $c + r$. There are also the special annuli allowed of form $(0, r)$ with $r \in S^1$.

The composition maps $\circ_1$ or $\diamond$ are just $\odot$. In pictures, this composition is given by scaling down the second annulus so that its outer radius is the same as the inner radius $|r_1|$ of the first annulus, and then gluing them together, rotating the second annulus so that the fixed point at 1 on its boundary is mapped to $c_1 + r_1$. The identity is $(0, 1)$.

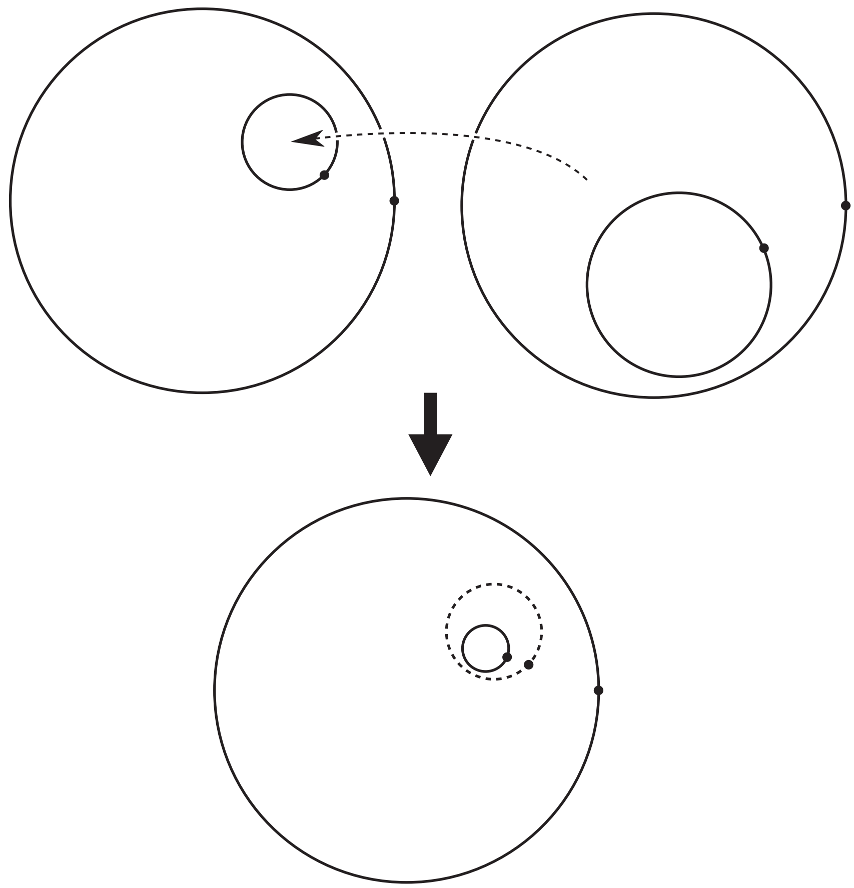


Figure 2.2: Composition in $\mathcal{A}$

Remark. There is some awkwardness to the inclusion of the identity, which shall be repeated multiple times. There are several ways to address this awkwardness, all equivalent up to homotopy. One can deal with operads without the identity, one can deal more generally with degenerate annuli where the two boundary circles can intersect, or one can include the identity as a special case. We have chosen this particular approach because it streamlines certain proofs—at the expense of the simplicity of some definitions, including this one.

Definition 2.3.2. The topological operad $\mathcal{A}_t$ of *trivialized annuli* consists of the following spaces: $\mathcal{A}_t(1)$ is the subset of $\mathfrak{C}(1)$ such that:

1. $|c| + |r| \leq 1$, and

2. this inequality is strict unless $c = 0$.

$\mathcal{A}_t(n)$ is empty for $n \neq 1$.

A point in $\mathcal{A}_t(1)$ describes an annulus of outer radius one and inner radius $|r|$ (which can degenerate to zero) centered at c with a marked point on the inner boundary circle at $c + r$. Composition is again $\odot$; the space is contractible by the homotopy $H(t, (c, r)) = (tc, tr)$.

Definition 2.3.3. The topological operad $\mathcal{D}$ of framed little disks consists of the following spaces: for $n \neq 0$, $\mathcal{D}(n)$ is the subset of $\mathfrak{C}(n)$ such that:

1. $0 < |r_i|$,

2. $|c_i| + |r_i| \leq 1$, and

3. this inequality is strict unless $c_i = 0$.

$\mathcal{D}(0)$ is empty. Note that for $n > 1$, the second and third condition together imply rather that $|c_i| + |r_i| < 1$.

Geometrically, the c_i correspond to the centers of disjoint little disks in the large disk of radius one in the plane, each having radius $|r_i|$ and with a marked point at $c_i + r_i$.

The $\mathbb{S}_n$ action is on the factors of $\mathfrak{C}(n)$, and the composition map $\circ_j$ is derived from $\odot$; explicitly, using the shorthand d_i to refer to the pair (c_i, r_i):

$$(d_1, \ldots, d_m) \circ_i (d'_1, \ldots, d'_n) = (d_1, \ldots, d_{i-1}, d_i \odot d'_1, \ldots, d_i \odot d'_n, d_{i+1}, \ldots, d_m).$$

So we compose $((c_i, r_i))$ into the factor (c, r) by replacing (c, r) with the factors $(c, r) \odot (c_i, r_i)$.

Clearly $\mathcal{A}$ injects into $\mathcal{A}_t$ and $\mathcal{D}$; let us describe the pushout of these two maps. This can be done quite concretely because of some features of these operads. We will need a few intermediate definitions.

2.4 The pushout in sets

Definition 2.4.1. Let D and P be finite sets at least one of which is nonempty. The *root set* $R(D, P)$ is a right $\mathbb{S}_D \times \mathbb{S}_P$-module in the category of sets. It is the subset of $\mathfrak{C}(D \sqcup P)$ such that:

1. $0 < |r_d|$ if r_d is in a factor (c_d, r_d) indexed by D,

2. $r_p = 0$ if r_p is in a factor indexed by P,

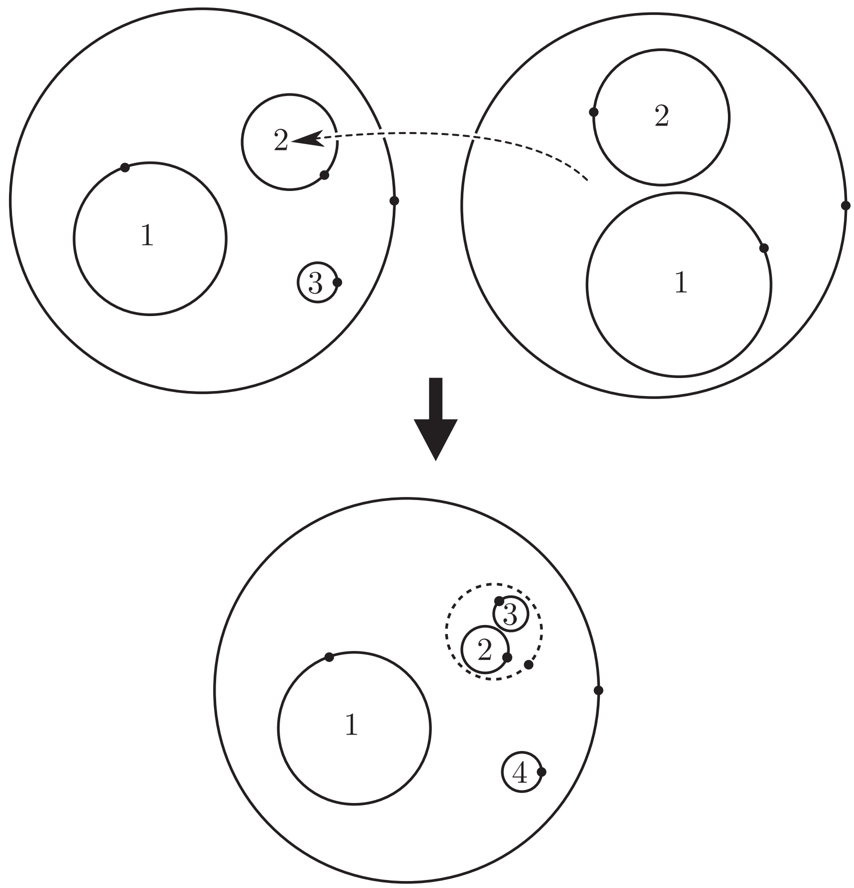


Figure 2.3: Composition in $\mathcal{D}$

3. $|c| + |r| \leq 1$ for every factor in the product, and

4. this inequality is strict unless $c = 0$.

This is the configuration space of disjoint little disks indexed by the set D in the disk centered at c_d of radius $|r_d|$ with one marked point on the boundary circle at $c_d + r_d$ along with additional disjoint marked "disks" of radius zero at c_p indexed by P.

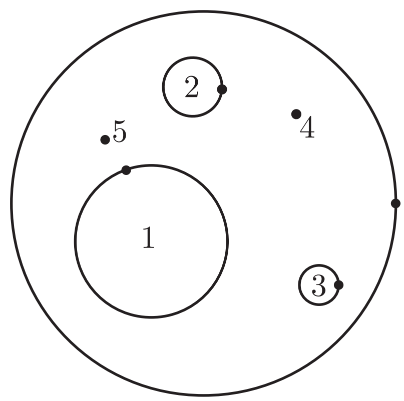


Figure 2.4: A configuration in $R(\{1, 2, 3\}, \{4, 5\})$

Definition 2.4.2. Let D and P be finite sets such that either D is nonempty or P contains at least two elements. The *branch set* $B(D, P)$ is a right $\mathbb{S}_D \times \mathbb{S}_P$-module in the category of sets. It is the subset of $\mathfrak{C}(D \sqcup P)/\operatorname{Aff}\mathbb{C}$ such that:

1. $0 < |r_d|$ if r_d is in a factor (c_d, r_d) indexed by D, and
2. $r_p = 0$ if r_p is in a factor indexed by P.

Here, the $\operatorname{Aff}\mathbb{C}$ action is on all factors of $\mathfrak{C}(D \sqcup P)$ by left multiplication with $\odot$. The two conditions are stable up to the action of $\operatorname{Aff}\mathbb{C}$, so it makes sense to talk about them as defining a subset of the quotient.

So this is the configuration space of D-indexed disjoint little disks with a marked point on each little boundary circle and P-indexed additional disjoint marked points in the plane up to conformal equivalence. The limitations of at least one disk or at least two points should be viewed as a stability requirement, so that the affine group action is free. If D is empty and P contains fewer than two elements, let $B(D, P)$ be the empty set.

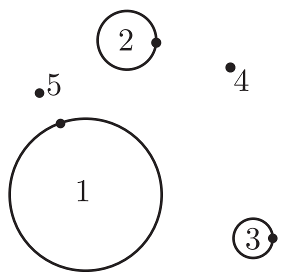


Figure 2.5: A configuration in $B(\{1,2,3\},\{4,5\})$

Definition 2.4.3. A vertex of a tree $T = (L, V, N)$ is called *nearly stable* (see Appendix A for the notation used for trees) if it is either:

1. the root vertex,
2. a leaf vertex, or
3. at least trivalent.

A vertex is called *stable* if it is at least trivalent. A vertex which is not nearly stable is called *unstable*.

A tree is called nearly stable (or stable) if it is not trivial and every vertex is nearly stable (or stable).

Definition 2.4.4. Let $T = (L, V, N)$ be a nontrivial tree. For $v \in V$, R_v denotes $R(in^e(v), in^i(v))$; similarly, B_v denotes $B(in^e(v), in^i(v))$. To make this uniform, if v is the root of the tree, M_v denotes R_v; if it is any other vertex, then M_v denotes B_v. We call M_v the *marking of* v.

Definition 2.4.5. The trivialized little disks operad $\mathcal{D}_t$ is an operad of decorated trees where the decorations on the vertex v of an underlying tree T is M_v. For now we will consider it as an operad in the category of sets; it will be topologized in Section 2.5. We will specify a tree-like composition as part of the proof of Proposition 2.4.6.

Remark. In words, the root vertex of a nontrivial tree is decorated by a configuration space of disjoint points (corresponding to internal incoming edges) and disks with marked point (corresponding to external incoming edges) in the complex *disk*, with no equivalence relation. The other vertices are decorated by a configuration space of disjoint points (internal incoming edges) and disks with marked point (external incoming edges) in the complex *plane* up to conformal automorphism. Since there is no $B(D, P)$ for D empty and P of order less than two, we only use nearly stable trees as underlying trees in our operad. We omit the trivial

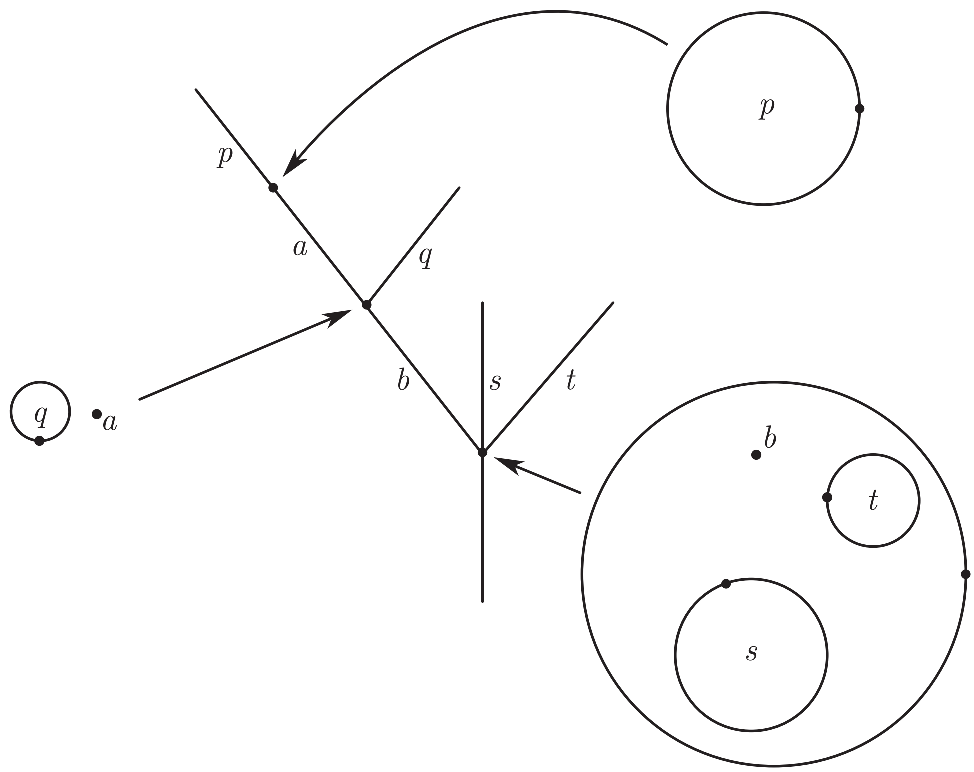


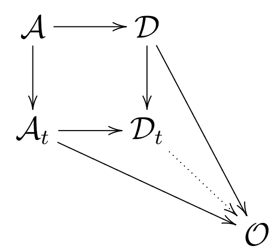
Figure 2.6: An element of $\mathcal{D}_t(4)$

tree as well; we will get the unit for our operad composition elsewhere.

The proof of the following two propositions occupies the rest of this section:

Proposition 2.4.6. *$\mathcal{D}_t$ carries a well-defined operad structure in the category of sets which accepts operad embeddings from the operads of trivialized annuli and framed little disks which agree on the images of the operad of framed little annuli in each.*

Proposition 2.4.7. *The trivialized little disks operad $\mathcal{D}_t$ is the categorical pushout of the framed little disks and the trivialized annuli along the framed little annuli as operads in the category of sets; that is, for every operad $\mathcal{O}$ which fits into this diagram there is a unique induced map from $\mathcal{D}_t$:*



Remark. Before beginning the proof of Proposition 2.4.6, let us describe the basic idea of the composition maps of $\mathcal{D}_t$. To glue a disk into a little disk in the plane up to conformal automorphism, scale the disk and glue it in using $\odot$. This is independent of the conformal representative. The other vertices of the tree involved do not change any decorations. If the resulting configuration in the plane is unstable (a single point) then forget the corresponding vertex of the tree.

Proof of Proposition 2.4.6. We will specify a tree-like operad structure on $\mathcal{D}_t$ (see the remark at the end of section A.3). The operadic composition maps are not exactly the grafting operation on trees, but descend from grafting by edge contraction. To be precise, let $x_m \in \mathcal{D}_t(m)$ have underlying tree T_m and let $x_n \in \mathcal{D}_t(n)$ have underlying tree T_n. Then the underlying tree of $x_m \circ_i^T x_n$ is obtained by first grafting T_n to T_m along the vertex i, as expected, but then contracting the grafting edge to obtain the tree $\tilde{T}$. If the new vertex created by contracting the grafting edge is unstable, we will forget it. This gives the underlying tree T of the composition.

Specifically, if the leaf vertex of the leaf i in T_m and the root of T_n are both bivalent, then the contraction vertex is bivalent. Such a vertex is unstable if it is not the root or a leaf vertex of $\tilde{T}$. No other vertex in the graph has its valence changed by the grafting and contraction; nor does any other leaf or root vertex lose its status, so after possibly forgetting this vertex, the resultant tree is in fact nearly stable.

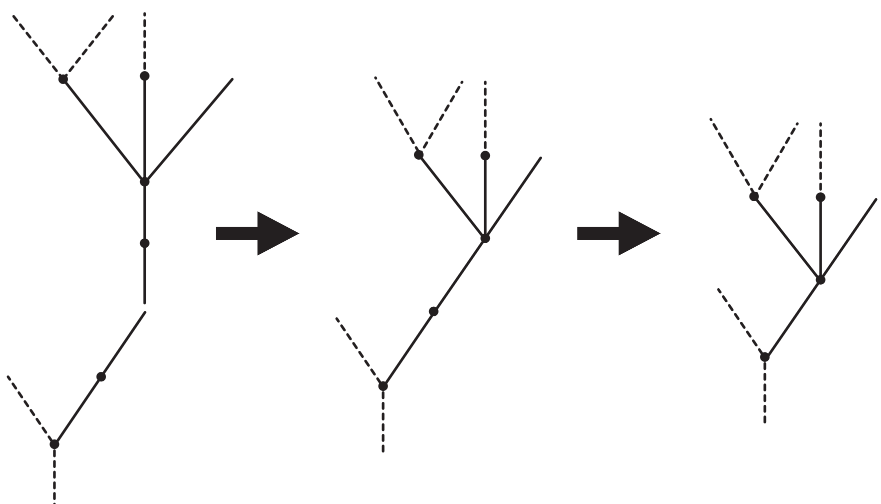


Figure 2.7: Forgetting an unstable contraction vertex

Let us describe the decorations on the vertices of the composition tree $T = (m+n-1, V, N)$. Most of the vertices are vertices of either T_m or T_n; these vertices will preserve their decoration up to the standard permutation of their inputs coming from the construction of T out of T_m and T_n. Let ω be the contraction vertex of $\tilde{T}$. It is the equivalence class of the leaf vertex of i in T_m and the root of T_n; call these two vertices μ and ν respectively for brevity.

Note the following characterization of the incoming edges of ω:

$$in^i(\omega) = in^i(\mu) \sqcup in^i(\nu); \qquad in^e(\omega) = in^e(\mu)\backslash\{i\} \sqcup in^e(\nu).$$

The following describes how to construct a decoration for ω. The basic plan is to use $\odot$ to combine the decorations M_μ and M_ν to get the decoration M_ω. Here are the options:

1. ω may be unstable. In this case, T is obtained from $\tilde{T}$ by forgetting ω so we need not supply a decoration. This is the only case where T and $\tilde{T}$ differ.

2. ω may be the root vertex of T. For this to happen, μ must be the root vertex of T_m. In this case, $M_\mu = R_\mu$ and $M_\nu = R_\nu$. The incoming external edge (i, μ) of μ corresponds to a factor (c, r) of the decoration of μ; we can use $\odot$ to left multiply this by the decorations of R_ν to give us a decoration of R_ω, following the same pattern as in the equation defining $\circ_i$ in the framed little disks operad (see Definition 2.3.3).

3. ω may be any stable vertex other than the root vertex. In this case, μ is not the root vertex of T_m. In this case, because μ and ω are not the root vertices

of their respective trees, the map will be of the form $B_\mu \times R_\nu \to B_\omega$. It is also given by left multiplying the factor (c, r) of the decoration of μ by the decorations of R_ν to give a decoration in B_ω, but a little care must be taken because the factor (c, r) in B_μ is taken only up to the $\operatorname{Aff}\mathbb{C}$ action. However, B_ω is also only taken up to the $\operatorname{Aff}\mathbb{C}$ action; then the associativity of $\odot$ shows that this is independent of choice of representative. For instance, if $(z, w) \in \operatorname{Aff}\mathbb{C}$, (c, r) is the factor of the decoration of μ corresponding to its edge (i, μ), and (c', r') is a factor of the decoration of ν, then

$$((z, w) \odot (c, r)) \odot (c', r') = (z, w) \odot ((c, r) \odot (c', r'))$$

so that a different representative of the same class in B_μ just leads to a different representative of the same class in B_ω. This works on each factor of the composition simultaneously.

Note that, given this composition structure, the 1-corolla decorated by $(0, 1)$ is a unit for composition.

The composition product and the unit together suffice to define the operad structure. We must ensure that it is associative. Composition of trees is associative. Unstable vertices are a phenomenon that occurs within trees, not near the leaves or root, so a composition of trees is unstable at the composition vertex if and only if it is unstable when another tree is grafted on elsewhere. Then the two

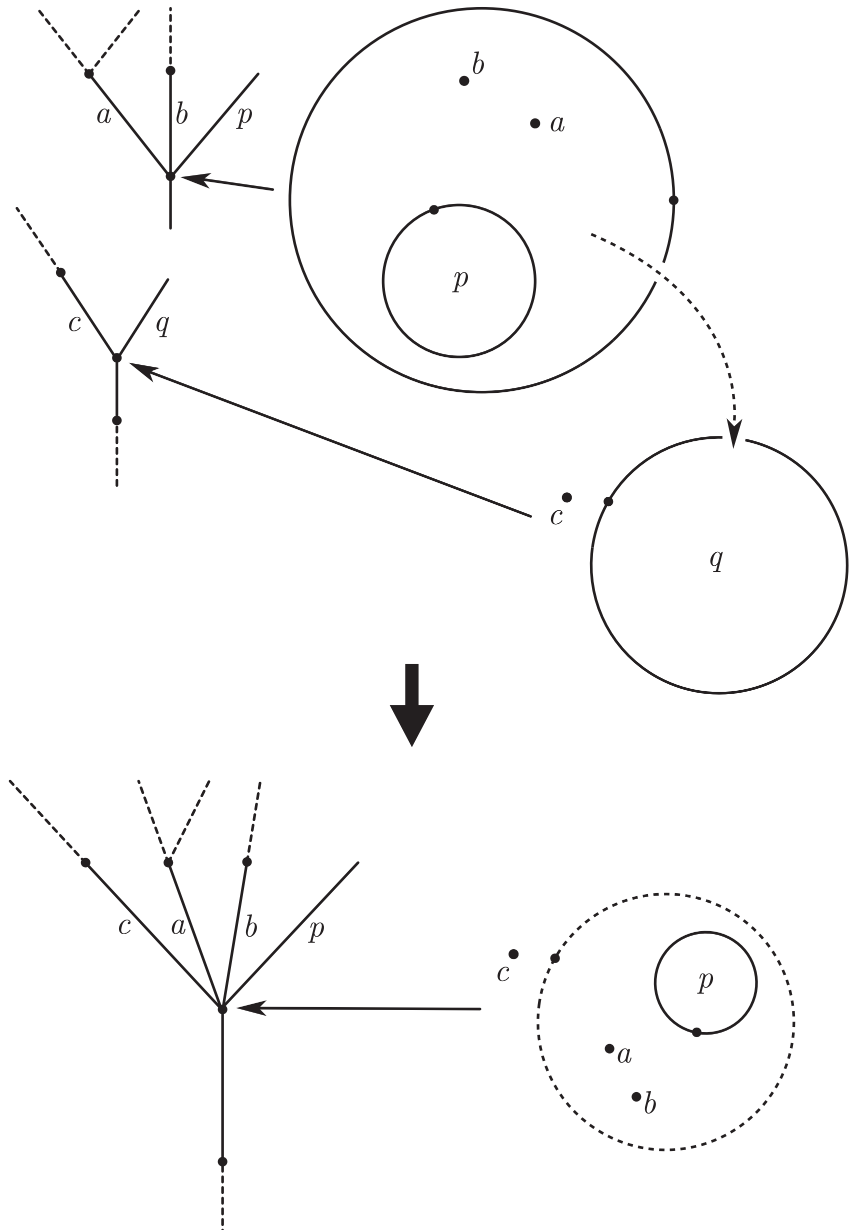


Figure 2.8: Stable composition in $\mathcal{D}_t$

compositions $(x \circ_i y) \circ_j z$ and $x \circ_i (y \circ_{j'} z)$ are equal if the decorations on the various vertices are equal. But all of these decorations are images under $\odot$, which is associative.

At this point, we are basically done with the proof. The space $\mathcal{D}(n)$ embeds as the decorations of the n-corollas; in fact, the configuration space in $\mathcal{D}_t$ for an n-corolla is in canonical bijection with $\mathcal{D}(n)$.

Trivialized annuli of positive radius embed in the same manner as framed little disks; annuli of radius zero embed along the unique 1-tree with two bivalent vertices. The root vertex is decorated by a configuration of a single point in the disk (the center of the trivial annulus) and the other vertex is decorated by a single little disk (c, r) in the plane up to $\mathrm{Aff}\,\mathbb{C}$. Note that this is $\mathrm{Aff}\,\mathbb{C}$ acting on itself by left multiplication; then it is transitive so the quotient space is a point and the decoration is unique.

These are set injections that agree on the annuli of positive radius. Let us see that they are operad maps. For the framed little disks, if we compose two elements of $\mathcal{D}_t$ with corollas as underlying trees, we obtain a corolla as the underlying tree of the composition. The composition map for the decoration of this new corolla agrees exactly with that of $\mathcal{D}$. This includes annuli of positive radius so all that remains to be seen is that composition agrees when annuli of radius zero are involved. There are three cases corresponding to the three cases above:

1. We compose two annuli of radius zero: $(c, 0) \odot (c', 0)$. In the trivialized annuli this yields $(c, 0)$; in $\mathcal{D}_t$, this corresponds to a grafting and edge contraction which has an unstable vertex that is contracted away. We obtain the same tree. The remaining markings are $(c, 0)$ for the root and the unique marking for the other vertex.

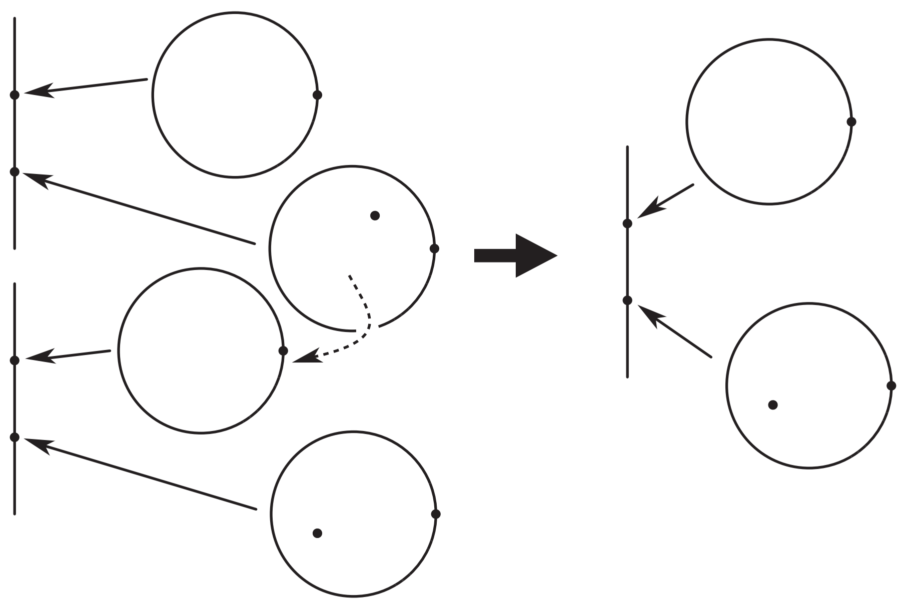


Figure 2.9: Composition of the images of $(c, 0)$ and $(c', 0)$ in $\mathcal{D}_t$. In each tree, the top vertex is decorated with a configuration of one disk in the plane up to $\mathrm{Aff}\,\mathbb{C}$. Every such disk is conformally equivalent to the standard disk. The bottom vertex is decorated with a configuration of one point in the standard disk.

2. We compose a positive radius annulus with a zero radius annulus: $(c, r) \odot (c', 0) = (c+rc', 0)$. In $\mathcal{D}_t$, composing with an element with underlying tree a 1-corolla will never change the underlying tree; in terms of decorations,

we are grafting our tree onto the 1-corolla and contracting. This is a grafting at the root vertex and so we use $\odot$ with no quotient, which yields the same result.

3. We compose a zero radius annulus with a positive radius annulus: $(c, 0) \odot (c', r') = (c, 0)$. In $\mathcal{D}_t$, the trees again match up, and the decoration modification will be along the map $B_\mu \times R_\nu \to B_\omega$. However, since ω is bivalent, B_ω is $\operatorname{Aff}\mathbb{C}/\operatorname{Aff}\mathbb{C}$, so is just a point, so there is a unique map to it. The decorations in the composite in $\mathcal{D}_t$ are then the unchanged decoration $(c, 0)$ on the root and the single point of B_ω on the second vertex.

□

Now we begin to build the background we will need to prove Proposition 2.4.7.

Definition 2.4.8. Let $x = ((c_1, r_1), \ldots, (c_n, r_n))$ be a point in $\mathcal{D}(n)$. The *annular image of* x is the set of little disks of the form

$$(c_{\text{out}}, r_{\text{out}}) \diamond x \diamond ((0, r'_1), \ldots, (0, r'_n))$$

for $(c_{\text{out}}, r_{\text{out}})$ and $(0, r'_i)$ in $\mathcal{A}(1)$. This composes x with annuli centered at 0 on the right and an arbitrary annulus on the left. If S is a subset of $\{1, \ldots, n, \text{out}\}$, the *annular image corresponding to* S is the subset of the annular image so that

$(c_i, r_i) = (0, 1)$ for indices not in the subset. That is, it is the composition with annuli only at the specified leaves (and possibly the root).

Lemma 2.4.9. *The image of $\mathcal{D}$ and the single trivial annulus $(0, 0) \in \mathcal{A}_t(1)$ generate $\mathcal{D}_t$ under operadic composition. In particular, every element of $\mathcal{D}_t(n)$ can be written as a composition of these generators along a tree $T = (n, V, N)$ with the following characteristics:*

1. *The root and leaf vertices are labeled with framed little disks,*
2. *if two vertices are adjacent in T, one is labeled with a framed little disk and the other is decorated with $(0, 0)$, and*
3. *if all vertices labeled with $(0, 0)$ are forgotten, then the resultant tree is nearly stable.*

Moreover, this decomposition is unique up to annuli. That is, any two such decompositions have the same underlying tree. If there are two decompositions with vertex labels x_v and y_v for $v \in V$, then $x_v = (0, 0)$ if and only if $y_v = (0, 0)$; if x_v is a label from the framed little disks, then y_v is as well and x_v and y_v share a configuration in their annular images corresponding to the internal edges of the vertex v.

On the other hand, if y_v is in the annular image of x_v corresponding to the internal edges of v for each vertex v decorated with a framed little disk configura-

tion, then the composition of the x_v along T and y_v along T is the same element of $\mathcal{D}_t$.

Proof. For brevity, we will not distinguish between the generators and their preimages in $\mathcal{D}$ and $\mathcal{A}_t$. Recall, however, that in $\mathcal{D}_t$ the image of $(0, 0)$ is a tree with two vertices, not just one. The bottom vertex is decorated with $(0, 0)$ and the top one with the unique marking in $B(1, 0)$.

Consider an element x of $\mathcal{D}_t$ with underlying tree T. That is, x is a product over $v \in V$ of M_v. Let T' be the tree obtained by inserting a vertex on every internal edge of T; we will describe x as a composition of the generators along T'. To do this, for each vertex in T', we must supply a label from our generating set. Each newly inserted vertex is bivalent; let these vertices be labeled with $(0, 0)$.

The base vertex of T in x is decorated by an configuration $\{(c_e, r_e)\}_{e \in in(v)}$ from the configuration set R_v. This configuration would be in $\mathcal{D}(in(v))$ if all the radii were nonzero. Replace every zero radius with an arbitrary nonzero one; because the conditions defining R_v are open, this is possible for sufficiently small $|r_e|$. Let the resultant element of $\mathcal{D}$ be the element labeling the corresponding vertex of T'.

For the other vertices of T, which are decorated with configurations from B_v, perform a similar procedure. First fix a representative under the $\operatorname{Aff}\mathbb{C}$ action which is contained in the interior of the unit disk; then again replace all zero radii with arbitrary nonzero radii which are sufficiently small to give the label for the corresponding vertex of T'.

Composing T' with the given labels, beginning upward from the root, the following occurs: at each step, we have a decorated tree T_k. When we compose $(0, 0)$ on the right, whether we are composing it into a decoration from R_v or B_v, the result is to shrink the corresponding radius to zero, keeping the center fixed, and to add a bivalent vertex to the appropriate leaf of the tree T_k. When composing the unique marking in $B(1, 0)$ on the left with an element y of $\mathcal{D}(m)$, this bivalent vertex is contracted away, and the resultant contraction vertex is marked with the class of y, included in the plane, under the $\mathrm{Aff}\,\mathbb{C}$ action. The entirety of this composition, then, results in the graph T' with the extra bivalent vertices forgotten. Because T was nearly stable to begin with, there are no unstable vertices created by the composition, and so this tree is just T. The radii of the appropriate little disks are zero, and every decoration is in the correct $\mathrm{Aff}\,\mathbb{C}$ class.

This argument shows more: if y_v is in the annular image of x_v corresponding to the appropriate edges, then y_v and x_v differ only by a choice of representative under $\mathrm{Aff}\,\mathbb{C}$ and choice of nonzero radii. Also, by the description of the composition it is clear that the tree T' is determined by the underlying nearly stable tree T.

The only thing remaining is to show uniqueness up to annuli. The vertices decorated with $(0, 0)$ are uniquely determined by the first and second conditions on the decomposition; it remains to show that any two labels from the framed little disks are equivalent up to annuli. We consider the decorations on the root vertex and all other vertices separately.

For the purposes of this calculation, let v be a vertex of T' and for convenience identify the internal incoming edges with the finite set m and external incoming edges with n. Let two possible labels for v be

$$(c_1, r_1), \ldots, (c_m, r_m), (d_1, s_1), \ldots, (d_n, s_n)$$

and

$$({c_1}', {r_1}'), \ldots, ({c_m}', {r_m}'), ({d_1}', {s_1}'), \ldots, ({d_n}', {s_n}')$$

where (c_i, r_i) and $({c_i}', {r_i}')$ correspond to internal incoming edges of the underlying tree while (d_i, s_i) and $({d_i}', {s_i}')$ correspond to external incoming edges.

1. Composing the decoration on the root vertex in the appropriate places with $(0, 0)$ yields the decoration (in the first case)

$$(c_1, 0), \ldots, (c_m, 0), (d_1, s_1), \ldots, (d_n, s_n)$$

on the root of T. For the two decorations on the root of T' to yield the same decoration on the root of T, the only thing that can differ between them is that r_i and ${r_i}'$ need not be equal. Let $\rho_i = \min\{r_i, {r_i}'\}$. Consider the decoration

$$(c_1, \rho_1), \ldots (c_m, \rho_m), (d_1, s_1), \ldots, (d_n, s_n).$$

It is in the annular image of both decorations of the root of T' corresponding to the appropriate edges, by the annuli $(0, \frac{\rho_i}{r_i})$ and $(0, \frac{\rho_i}{r_i'})$.

2. If the vertex v is not the root vertex, then we get a similar equation, but only up to $\mathrm{Aff}\,\mathbb{C}$. That is, for some element (c, r) of $\mathrm{Aff}\,\mathbb{C}$, we have

$$(c, r) \odot ((c_1, 0), \ldots, (c_m, 0), (d_1, s_1), \ldots, (d_n, s_n))$$
$$= (c_1', 0), \ldots, (c_m', 0), (d_1', s_1'), \ldots, (d_n', s_n').$$

If it were not for the element of $\mathrm{Aff}\,\mathbb{C}$, we could use the same annuli as before. Pick a real number z such that $z < 1$, $z < \frac{1}{2r}$, and $z < \frac{1}{2c}$ if $c \neq 0$. Then $(0, z) \odot (c, r) = (cz, rz)$; both cz and rz are less than $\frac{1}{2}$ so (cz, rz) is an annulus; also $(0, z)$ is an annulus. Then if we compose (cz, rz) on the left of

$$((c_1, r_1), \ldots, (c_m, r_m), (d_1, s_1), \ldots, (d_n, s_n))$$

and compose each factor (c_i, r_i) with $(0, \frac{\rho_i}{r_i})$ on the right, this gives the same configuration as the composition of $(0, z)$ on the left of

$$(c_1', r_1'), \ldots, (c_m', r_m'), (d_1', s_1'), \ldots, (d_n', s_n')$$

along with the composition of each $({c_i}', {r_i}')$ with $(0, \frac{\rho_i}{{r_i}'})$ on the right. Thus the two decorations share a common point in their annular images corresponding to internal edges, as desired.

□

Proof of Proposition 2.4.7. Let x be a point in $\mathcal{D}_t$, which by Lemma 2.4.9 can be written as a composition along a tree T of $(0, 0)$ and framed little disks. Such a composition must be taken by an operad map to a composition; because the induced map $f : \mathcal{D}_t \to \mathcal{O}$ is supposed to commute with the maps $\mathcal{D} \to \mathcal{O}$ and $\mathcal{A}_t \to \mathcal{O}$, this uniquely specifies where such a composition must be taken by the induced map, as long as it is well defined. That is, take the point x to the composition along T of the images of the appropriate points in $\mathcal{D}$ and the trivial annulus $(0, 0)$ in $\mathcal{A}_t$.

To see that this is well-defined, recall that the decomposition is unique up to composition of framed little disk components with annuli: composition of components other than the root by all annuli on the left and composition of inputs to be composed with $(0, 0)$ by annuli centered at zero on the right. We will consider making these two changes and show that the result in $\mathcal{O}$ is unchanged.

First, suppose that at some vertex other than the root, we make two different choices of framed little disk, x_v and $a \odot x_v$, where a is an annulus. Because this is not the root, it will be composed in the composition with $(0,0)$ on the left. If by abuse of notation we denote all the given maps into $\mathcal{O}$ by f, then we are comparing $f((0,0)) \circ_1 f(x_v)$ with $f((0,0)) \circ_1 f(a \odot x_v)$ Because $f : \mathcal{D} \to \mathcal{O}$ is a map of operads, the latter is equal to $f((0,0)) \circ_1 f(a) \circ_1 f(x_v)$; because the maps commute over annuli, this is the same as $f((0,0) \circ_1 a) \circ_1 f(x_v) = f((0,0)) \circ_1 f(x_v)$, as desired.

Similarly for the other type of ambiguity,

$$f(x_v \odot (0,r)) \circ_j f((0,0)) = f(x_v) \circ_j f((0,r)) \circ_1 f((0,0))$$

$$= f(x_v) \circ_j f((0,r) \odot (0,0)) = f(x_v) \circ_j f((0,0)).$$

It is necessary to verify that the induced map behaves well with respect to composition. Consider a pair of points x and y in $\mathcal{D}_t$. Write them as compositions of $(0,0)$ and framed little disks along trees T_x and T_y. Consider a particular composition $x \circ_k y$. We would like to verify:

$$f(x \circ_k y) = f(x) \circ_k f(y).$$

One of the following situations holds. The first through fourth are different kinds of stable composition, and are very easy to verify. The fifth case, of unstable composition, is still straightforward.

1. T_x and T_y each have one vertex so that x and y are in the image of $\mathcal{D}$. Then the condition holds because $f : \mathcal{D} \to \mathcal{O}$ is an operad map by assumption.

2. T_y has a single bivalent vertex, so that $y \in \mathcal{D}_t$ is the image of an annulus a_y of $\mathcal{A}(1)$, T_x has more than one vertex, and the leaf vertex of k in T_x is bivalent, labeled with an annulus a_x of $\mathcal{A}(1) = \mathcal{D}(1) \subset \mathcal{D}_t(1)$. In this case, we have

$$x \circ_k y = (\hat{x} \circ_k ((0,0) \circ_1 a_x)) \circ_k a_y$$

and

$$f(x) \circ_k f(y) = f(\hat{x} \circ_k ((0,0) \circ_1 a_x)) \circ_k f(a_x)$$

for some $\hat{x}$. Because the maps f are defined in terms of the compositions involved in this decomposition of x and $\mathcal{D}_t$ and $\mathcal{O}$ are operads, these equations yield

$$x \circ_k y = \hat{x} \circ_k ((0,0) \circ_1 (a_x \odot a_y))$$

and

$$f(x) \circ_k f(y) = f(\hat{x}) \circ_k f((0,0) \circ_1 (a_x \odot a_y)).$$

Now let us consider $f(x \circ_k y)$. To do so, we must write $x \circ_k y$ as a composition as in Lemma 2.4.9. This composition is along the tree T_x with all labels the same except that $a_x \odot a_y$ labels the vertex previously labeled by a_x. This composition satisfies all of the conditions of the lemma and presents $x \circ_k y$; its image under f gives the same result we got for $f(x) \circ_k f(y)$.

3. T_x has a single bivalent vertex labeled by the annulus a_x, T_y has more than

one vertex, and the root of T_y is bivalent and labeled by the annulus a_y. Just as in the last case, we can write y as $a_y \circ_1 (0,0) \circ_1 \hat{y}$, and the composition of x and y in $\mathcal{D}_t$ is

$$(a_x \odot a_y) \circ_1 (0,0) \circ_1 \hat{y}$$

(k must be 1). We can decompose this as in the lemma by using the tree T_y but changing the root label from a_y to $a_x \odot a_y$. The image of this decomposition under f is then $f(a_x \odot a_y) \circ_1 f((0,0)) \circ_1 f(\hat{y})$.

On the other hand, $f(x) \circ_1 f(y)$ is

$$f(a_x) \circ_1 f(a_y) \circ_1 f((0,0)) \circ_1 f(\hat{y}) = f(a_x \odot a_y) \circ_1 f((0,0)) \circ_1 f(\hat{y}).$$

These match up perfectly.

4. Both T_x and T_y have more than one vertex, and either the leaf k of T_x or the root of T_y has valence greater than two, so that its label is in $\mathcal{D}$ but not in $\mathcal{D}(1)$. Call the labels of these vertices d_x and d_y. As before, we can write

$$x \circ_k y = (\hat{x} \circ_k ((0,0) \circ_1 d_x)) \circ_k ((d_y \circ_i (0,0)) \circ_j \hat{y})$$
$$= \hat{x} \circ_k (((0,0) \circ_1 (d_x \circ_\ell d_y) \circ_m (0,0)) \circ_n \hat{y})$$

for some indices i, j, ℓ, m, n depending on d_x and d_y. As in the previous two examples, this indicates an easy presentation for $x \circ_k y$ in terms of Lemma 2.4.9. Namely, graft T_y to T_x along the leaf k and then contract the grafting edge. All vertices are labeled with the induced label of T_y or T_x except the vertex involved in the contraction; that vertex is the image of the leaf vertex in T_x and the root of T_y, and is labeled by the composition $d_x \circ_\ell d_y$ of their labels in $\mathcal{D}$.

As before, the same calculation can be done in $\mathcal{O}$, writing $f(x)$ and $f(y)$ as compositions, composing them, and then changing the order of the composition to compose $f(d_x)$ and $f(d_y)$ first; this is $f(d_x \circ_\ell d_y)$ since $\mathcal{D} \to \mathcal{O}$ is an operad map. So both calculations give the same thing.

5. Both T_x and T_y have more than one vertex and both the leaf vertex for the leaf k in T_x and the root of T_y are bivalent, labeled with annuli a_x and a_y of $\mathcal{A}(1)$. Again, we have:

$$\begin{aligned} x \circ_k y &= (\hat{x} \circ_k ((0,0) \circ_1 a_x)) \circ_k ((a_y \circ_1 (0,0)) \circ_1 \hat{y}) \\ &= \hat{x} \circ_k (((0,0) \circ_1 (a_x \odot a_y) \circ_1 (0,0)) \circ_1 \hat{y}). \end{aligned}$$

In this case, we cannot proceed precisely as before because the underlying tree of this composition, forgetting the vertices marked by $(0,0)$, is not nearly stable. To satisfy the conditions of Lemma 2.4.9, we need to condense a little further. Since $a_x \odot a_y \in \mathcal{A}(1)$, it is also in $\mathcal{A}_t(1)$, so it can be composed there and we can continue:

$$x \circ_k y = \hat{x} \circ_k (((0,0) \odot a_x \odot a_y \odot (0,0)) \circ_1 \hat{y}) = \hat{x} \circ_k ((0,0) \circ_1 \hat{y})$$

since $(0,0) \odot (c,r) = (0,0)$.

This indicates a suitable underlying tree for the composition: graft T_y to T_x along the kth leaf, preserving the decorations. Then forget the two bivalent vertices involved in the grafting: the leaf vertex of k in T_x and the root vertex of T_y. This leaves two adjacent vertices marked with $(0,0)$. Contract the edge between them, and decorate the contracted vertex with $(0,0)$. So

$$f(x \circ_k y) = f(\hat{x}) \circ_k (f((0,0)) \circ_1 f(\hat{y})).$$

On the other hand,

$$f(x) \circ_k f(y) = f(\hat{x} \circ_k ((0,0) \circ_1 a_x)) \circ_k f((a_y \circ_1 (0,0)) \circ_1 \hat{y})$$
$$= f(\hat{x}) \circ_k ((f((0,0)) \circ_1 f(a_x \odot a_y) \circ_1 f((0,0))) \circ_1 f(\hat{y}))$$

and by the same logic, since the map $\mathcal{A}_t \to \mathcal{O}$ is an operad map, this is

$$f(\hat{x}) \circ_k (f((0,0) \odot a_x \odot a_y \odot (0,0)) \circ_1 f(\hat{y}))$$
$$= f(\hat{x}) \circ_k (f((0,0)) \circ_1 f(\hat{y}))$$

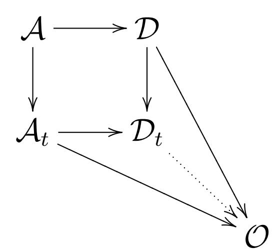
as desired.

□

2.5 The pushout in spaces

Proposition 2.5.1. *There are topologies on the spaces $\mathcal{D}_t(n)$, described below, which make $\mathcal{D}_t$ a topological operad and the embeddings of the trivialized annuli and the framed little disks continuous.*

Proposition 2.5.2. *Given these topologies, $\mathcal{D}_t$ is the pushout in the category of topological operads. That is, for every topological operad $\mathcal{O}$ which fits into this diagram there is a unique induced map from $\mathcal{D}_t$:*



Description of the topology of Proposition 2.5.1. We begin by describing a local basis for the topology on $\mathcal{D}_t(n)$, using the underlying topologies of each component and the composition maps just defined. To begin with, we shall contain the root set $R(D, P)$ and branch set $B(D, P)$ into larger ambient spaces.

Definition 2.5.3. Let D and P be finite sets, one of which is nonempty. The *augmented root space* $\hat{R}(D, P)$ is a right $\mathbb{S}_D \times \mathbb{S}_P$-module in the category of spaces. It is the subset of $\mathfrak{C}(D \sqcup P)$ such that:

1. $0 < |r_d|$ if r_d is in a factor (c_d, r_d) indexed by D,

2. $|c| + |r| \leq 1$ for every factor in the product, and

3. this inequality is strict unless $c = 0$.

The augmented root space contains the root set $R(D, P)$ as the subspace with $r_p = 0$ for each (c_p, r_p) indexed by P.

Definition 2.5.4. Let D and P be finite sets such that either D is nonempty or P has at least two elements. The *augmented branch space* $\hat{B}(D, P)$ is a right $\mathbb{S}_D \times \mathbb{S}_P$-module in the category of spaces. It is the subset of $\mathfrak{C}(D \sqcup P)/\operatorname{Aff}\mathbb{C}$ such that $0 < |r_d|$ if r_d is in a factor (c_d, r_d) indexed by D.

The augmented branch space contains the branch set $B(D, P)$ as the subspace with $r_p = 0$ for each (c_p, r_p) indexed by P.

Remark. In both cases there are two changes: allowing the points $(c_p, 0)$ to (possibly) expand into marked disks (c_p, r_p), and working in the category of spaces rather than sets.

Definition 2.5.5. For v a vertex of a tree T, let $\hat{R}_v$ and $\hat{B}_v$ denote, respectively, $\hat{R}(in^e(v), in^i(v))$ and $\hat{B}(in^e(v), in^i(v))$, as before. Let $\hat{M}_v$ denote $\hat{R}_v$ if v is the root of T and $\hat{B}_v$ if it is not.

Now fix a point $x \in \mathcal{D}_t(n)$ with underlying tree $T = (n, V, N)$; this is a product over $v \in V$ of points x_v in M_v. We will describe open sets containing x

in terms of basic open sets containing x in the product space

$$\prod_{v \in V} \hat{M}_v.$$

Our basic open sets will be products of open sets. So fix an open neighborhood U_v of x_v in $\hat{M}_v$. We will define a set map from $\prod U_v$ to $\mathcal{D}_t(n)$ mapping x to x; we will define its image to be an open set in $\mathcal{D}_t(n)$. When we need to refer to this basic open set of $\mathcal{D}_t(n)$, we will use the notation $\mathcal{N}_x^{\{U_v\}}$.

For a fixed tree $T = (n, V, N)$, the map $\prod \hat{M}_v \to \mathcal{D}_t(n)$ is given by the following. Let $y = \prod y_v$ be a point in $\prod \hat{M}_v$ with underlying tree T. For a given vertex v, the space $\hat{M}_v$ is a subspace of a product indexed by the incoming edges of v. Therefore, every internal edge of T appears as an index somewhere in the product $\prod y_v$. Let E_0 be the set of internal edges e of T so that the pair (c_e, r_e) indexed by e has nonzero radius vector r_e. The image of y will land in the part of $\mathcal{D}_t(n)$ with underlying tree T_{E_0}, the edge contraction of T along the edges in E_0.

The decorations of a point in the image of the map will then be unchanged on vertices not involved in any edge contraction; on other vertices, we must provide maps from a product of $\hat{M}_v$ to a single M_v. For each $y_v \in \hat{B}_v$, there is a unique representative $\tilde{y}_v$ in the $\mathrm{Aff}\,\mathbb{C}$ orbit of y_v with $c_1 = 0$, c_2 on the positive real line, and the Euclidean diameter of the union of the disks of center c_i and radius $|r_i|$ in the plane equal to $\frac{1}{2}$. There is a unique marking with center 0 and radius vector to 1 if v has only one incoming edge (which must then be external and have a nonzero radius). We can view $\tilde{y}_v$ as a configuration in the disk, rather than the plane. If $y_v \in \hat{R}_v$, then y_v is already a configuration in the disk; for consistency of notation, let $\tilde{y}_v$ denote y_v in this case. So the map from a product of $\hat{M}_v$ to M_v is just iterated $\odot$ of the $\tilde{y}_v$ along the factors of the product specified by the contraction edges. The associativity of $\odot$ ensures that this is independent of the order of contraction.

The point y has radius vector zero for all internal edges, so is taken to itself, as promised. □

Lemma 2.5.6. *The set* $\{\mathcal{N}_x^{\{U_v\}}\}$ *forms the basis for a topology.*

Proof of Lemma 2.5.6. To show this, it is necessary to show that the set covers $\mathcal{D}_t(n)$ and has the appropriate intersection property. Since x is contained in any $\mathcal{N}_x^{\{U_v\}}$, the set covers $\mathcal{D}_t(n)$. Now, $\mathcal{N}_x^{\{U_v\}} \cap \mathcal{N}_x^{\{U_v{}'\}}$ will contain $\mathcal{N}_x^{\{U_v \cap U_v{}'\}}$, so to show that this candidate for a basis satisfies the intersection property, it suffices to show that for any $y \in \mathcal{N}_x^{\{U_v\}}$ there exists a basis element $\mathcal{N}_y^{\{U_w\}}$ centered at y contained therein. We use the index w instead of v to emphasize that y and x have different underlying trees.

To see this containment property, let $\mathcal{N}_x^{\{U_v\}}$ and y be as above. Then x lives in a product over the tree $T = (n, V, N)$ and Y lives in the product over the tree T_{E_0}, where E_0 is some set of internal edges of T. We will denote T_{E_0} by (n, W, N_*). For each $U_v \in \hat{M}_v$, consider the open subset ${U_v}'$ specified to be those points of U_v so that factors (c, r) indexed by edges in E_0 have nonzero radius. We want to meaningfully collapse $\prod U_v$ along the edges in E_0, so in order to do so, we first prepare by getting rid of all the points where disks indexed by edges in E_0 have 0 radius.

Now we must define open sets $U_w \in \hat{M}_w$ for each $w \in W$. Such a w is an equivalence classes of vertices in T, so the set $U_{\{v_{e_1},\ldots,v_{e_k}\}}$ (for v_i vertices of T) corresponds to the composition of U'_v by $\odot$ along the factors indexed by contraction edges. As before, we pick a unique normalized representative $\tilde{y}_v$ for the factors in $\hat{B}_v$ involved in a composition. This composition is well-defined because of the associativity of $\odot$.

It is an easy exercise to pass from $\odot$ being an open map on $\mathbb{C} \times \mathbb{C}$ (see Lemma 2.2.3) to being an open map on the sets $\hat{B}(S, T)$ and $\hat{Q}(S, T)$; moreover, by the rationale of the proof of that lemma, if we are talking about a subset where certain radii are fixed at zero and all other radii are never zero, it is much better than an open map: the image of an open set times any other set is still open. There are easy special cases corresponding to $\hat{M}(D, \emptyset)$ where D is a singleton.

Because $\odot$ has this openness property the resultant sets U_w are open, and a point in $\mathcal{N}_y^{\{U_w\}}$ contained in the image of a point z in $\prod U_w$ is also in $\mathcal{N}_x^{\{U_v\}}$, corresponding to the preimage of z under the identification along the edges of E_0. □

Proof of Proposition 2.5.1. Now that the topology is defined, it should be shown that the structure maps of the operad are continuous. Let x be a point in the open set $\mathcal{N}_x^{\{U_v\}}$ with underlying tree $T_x = (n_x, V_x, N_x)$. So x is of the form $\prod x_v$, with the product taken over $v \in V_x$. Each x_v is a product over the incoming edges e of v of pairs (c_e, r_e). There is no loss of generality to assume that U_v is of the form $\{(c_e{}', r_e{}') : |c_e - c_e{}'| + |r_e - r_e{}'| < \epsilon\}$ for some uniform $\epsilon > 0$. This, of course, should be taken up to $\mathrm{Aff}\,\mathbb{C}$ around a fixed representative in the appropriate cases. This forms a basis of the configuration spaces $\hat{M}_v$.

Now pick a point $y \times z$ in the preimage of x via the operad composition map $\circ_k$; say y has underlying tree $T_y = (n_y, V_y, N_y)$ and $T_z = (n_z, V_z, T_z)$. We shall show that the preimage of $\mathcal{N}_x^{\{U_v\}}$ contains an open set around $y \times z$. There are three cases, corresponding to the three cases in the composition map. Recall that μ denotes the leaf vertex of the leaf k in T_y and ν the denotes the root vertex of T_z, while ω refers to the contraction vertex in T_x formed by the identification of μ and ν in the nearly stable cases. Here are the options:

1. If the composition $x = y \circ_k z$ is unstable, then μ and ν are bivalent; the vertices of T_x are all other vertices of T_y and T_z and the edges are all edges

not incident on either of the two omitted vertices along with an edge from $N_z^{-1}(\nu)$ to $N_y(\mu)$.

So in such a composition, there is a determined special edge of T_x along which the composition and collapse has occurred. The special edge must be internal because otherwise the composition would be stable. For fixed T_x, different T_y and T_z yield different such edges, and every choice of an internal edge gives rise to such a pair of trees, by cutting the tree along the given edge and then inserting a vertex on the cut edge on both trees created by the cut.

So if x arises from an unstable composition of y and z, then the vertices of T_y and T_z are in correspondence with the vertices of T_x, along with two extra vertices, μ and ν. We will choose open sets in M_v for each vertex v of T_y and T_z. If v is in T, take the open set U_v; at the vertex μ (which is not the root by virtue of the instability of the composition) take $\hat{B}((k, \mu), \emptyset)$ (which is a point), and at the vertex ν take all of $\hat{R}(\emptyset, (N_z^{-1}(\nu), \nu))$.

Now let us verify that the product of open sets $\mathcal{N}_y^{\{U_v\}}$ and $\mathcal{N}_z^{\{U_v\}}$ just constructed lands under the composition map in $\mathcal{N}_x^{\{U_v\}}$. So choose $y' \in \mathcal{N}_y^{\{U_v\}}$ and $z' \in \mathcal{N}_z^{\{U_v\}}$. We want to show that $y' \circ_k z'$ sits in $\mathcal{N}_k^{\{U_v\}}$.

For $v \in V_x$, let $x_v{}'$ be as follows:

(a) if $v = N(\mu)$, let ${x_v}' = y_v \circ_k z_\nu$;

(b) otherwise, let ${x_v}'$ be whichever of ${y_v}'$ or ${z_v}'$ is appropriate.

Then we claim that $\prod_v {x_v}'$ sits in $\mathcal{N}_k^{\{U_v\}}$ and that it is equal to $y' \circ_k z'$. Clearly, in the second case ${x_v}'$ is in U_v; we need to show that this is also true for $v = N(\mu)$. But if the factor (c, r) of $y_{N(\mu)}$ corresponding to the edge e satisfies $|c - c_e| + |r| < \epsilon$ (since the original radius of this decoration in y is zero), and if the decoration on ν is $(c', r') \in \operatorname{Aff}\mathbb{C}$, then $(c, r) \odot (c', r') = (c + rc', rr')$ and we get

$$|c + rc' - c_e| + |rr'| \leq |c - c_e| + |r|(|c'| + |r'|) \leq |c - c_e| + |r| < \epsilon.$$

This shows that $\prod {x_v}'$ is in $\mathcal{N}_x^{\{U_v\}}$. To see that it is the same as $y' \circ_k z'$, consider that the only difference is that the unique marking of μ is missing and the marking of ν has been shifted to the other side of the composition. But the marking of μ is (up to $\operatorname{Aff}\mathbb{C}$) just $(0, 1)$, the identity, and cannot change the composition. A little checking of the cases where the radius of the factor of the decoration of $N(\mu)$ corresponding to the edge from μ is zero and where the radius of the decoration of ν is zero verifies that this is in fact $y' \circ_k z'$.

2. In both of the stable cases, the tree T_x is obtained by grafting T_y and T_z along the leaf k of T_y and then contracting the grafting edge. This yields

the special vertex ω of T_x; on the other hand, for each vertex of T, one can obtain such T_y, T_z, and k by splitting the incoming edges incident on the vertex into two sets, the edges incident on μ and those on ν, the second of which is nonempty. μ has one new leaf (k) and the corresponding incoming edge. Take the preimage under the map $\circ_k = \odot$ from $\hat{M}_\mu \times \hat{R}_\nu \to \hat{M}_\omega$ of U_ω; this preimage is an open set so around any point we can find a basic open set which is a product of open sets U_μ and U_ν.

Keeping the open sets the same for all other vertices, the sets $\mathcal{N}_y^{\{U_v\}}$ and $\mathcal{N}_z^{\{U_v\}}$ are evidently open and in the preimage of $\mathcal{N}_x^{\{U_v\}}$.

Finally to finish the proof of Proposition 2.5.1, we must check that the embeddings of the framed little disks and trivialized annuli are continuous. For the framed little disks, there is nothing to check; the tree describing a point in the image of the framed little disks has no internal edges, so the topology on the image of $\mathcal{D}(n)$ is just the subspace topology in $(\mathbb{C} \times \mathbb{C})^n$, just as it is in the framed little disks. For the annuli, it must be checked that the preimage of an open set around the image of a radius zero annulus is open in the operad of trivialized annuli. Since such an annulus is described by a tree with one leaf and two vertices, the root $*$ and another vertex, say, v. there is only one internal edge e and one leaf edge. The decoration on v is unique, so the space $\hat{R}_* \times \hat{B}_v$ is the subset of $\mathfrak{C}(\{e\})$ so that

1. $|c| + |r| \leq 1$, and

2. this inequality is strict unless $c = 0$.

A basis for the open sets around the trivial annulus $(c, 0)$ in this space are formed by the subsets

$$\{(c + x, y) \in \mathfrak{C}(\{e\}) : |c + x| + |y| < 1, |x|, |y| < \epsilon\}.$$

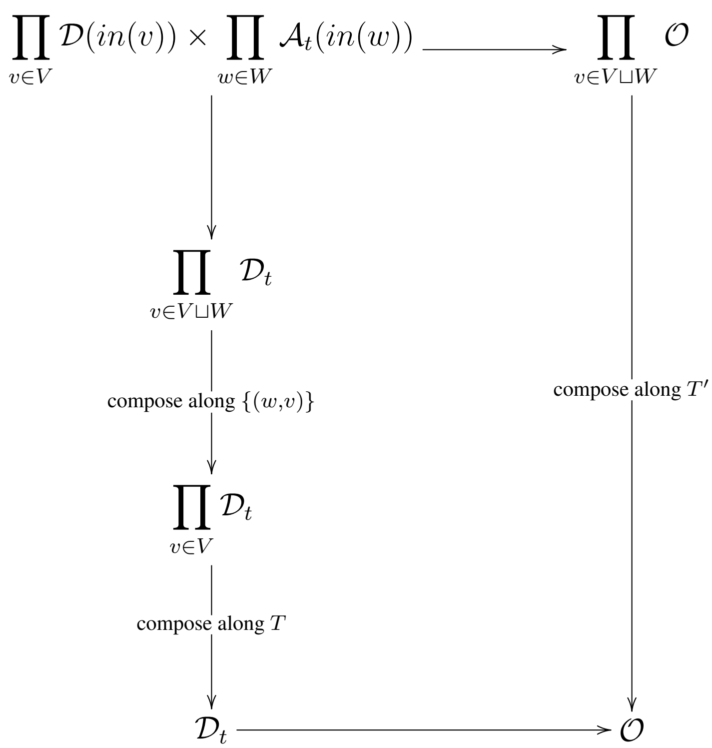
Fix a point $(c + x, y) \in \hat{R}_* \times \hat{B}_v$. If $y = 0$ then the corresponding point in $\mathcal{D}_t(1)$ is the image of the trivial annulus of radius zero centered at $c + x$; if $y \neq 0$ then the corresponding point is the image of the annulus $(c + x, y)$. Then the preimage is precisely

$$\{(c + x, y) \in \mathfrak{C}(1) : |c + x| + |y| < 1, |x|, |y| < \epsilon\}$$

which are open in the trivialized annuli. □

Proof of Proposition 2.5.2. Let $\mathcal{O}$ be a topological operad that accepts topological operad maps from $\mathcal{A}_t$ and $\mathcal{D}$ that agree on $\mathcal{A}$. Proposition 2.4.7 indicates that there is a unique morphism of operads of sets from $\mathcal{D}_t \to \mathcal{O}$ which factors both of these maps. To prove the proposition, we must show that the induced set map is continuous.

Let $y \in \mathcal{O}$ and $x \in \mathcal{D}_t$ in its preimage under the induced set map. Then x has an underlying tree $T = (n, V, N)$ and can be written as a composition along the tree T' obtained from by inserting a vertex on every internal edge of T, where the new vertices are labeled by $(0, 0)$ and the old vertices are labeled with minor modifications of the decorations of the corresponding vertices of T, as in Lemma 2.4.9. Call the new vertices W, so that $T' = (n, V \sqcup W, N_*)$. The following diagram commutes:



Let U be an open set containing y. The composition along the top and right side is continuous by assumption, so we can come up with open sets in the preimage of U in the product of $\mathcal{D}$ and $\mathcal{A}_t$. Then we can push those maps down the left side of the diagram. We will show that the composition of the first two vertical maps is a product of open maps. Then we will have a collection of open sets U_v for $v \in V$

containing x_v, and $\mathcal{N}_x^{\{U_v\}}$ will be contained in the preimage of y.

So what we want to show is that composing an open set in $\mathcal{D}$ with open sets around $(0, 0)$ in $\mathcal{A}_t$ yields an open set in $\mathcal{D}_t$. Pick $x \in \mathcal{D}(n)$; pick an ϵ and take the ϵ-neighborhood around x and around $(0, 0)$. Take the tree T formed by inserting vertices on some of the leaf edges of the n-corolla. Replace the factors (c, r) in x corresponding to the inserted vertices with $(c, 0)$; call the result $\hat{x}$. The image of the ϵ neighborhoods of x and $(0, 0)$ under composition along T contains the neighborhood of the composition of x and $(0, 0)$ determined by unique decorations on the leaf vertices and the ϵ-neighborhood of $\hat{x}$ as the decoration in R_* on the root vertex. □

2.6 The main theorem

The main theorem of this chapter connects the pushout operad $\mathcal{D}_t$, topologized as above, to the genus zero Deligne-Mumford operad, as defined in Appendix B.

Theorem 2.6.1. *The constituent spaces of the genus zero Deligne-Mumford operad $\overline{\mathcal{M}}(n)$, are deformation retracts of $\mathcal{D}_t(n)$. The inclusion maps $\overline{\mathcal{M}}(n) \hookrightarrow \mathcal{D}_t(n)$ form a map of topological operads.*

The proof will follow from three propositions related to a suboperad $\mathcal{D}_{tDM}$ of the pushout operad $\mathcal{D}_t$. After defining $\mathcal{D}_{tDM}$, we shall prove:

Proposition 2.6.2. *For $n > 1$, $\mathcal{D}_{tDM}(n)$ is locally homeomorphic to $\mathbb{R}^{2n-4}$.*

Proposition 2.6.3. *$\mathcal{D}_{tDM}(n)$ is a deformation retract of $\mathcal{D}_t(n)$.*

Proposition 2.6.4. *There is a bijective map of topological operads:*

$$\varphi : \mathcal{D}_{tDM} \to \overline{\mathcal{M}}.$$

Proof of Theorem 2.6.1. $\overline{\mathcal{M}}(n) = \overline{\mathcal{M}}_{0,n+1}$ is a manifold of dimension $2(n+1) - 6 = 2n - 4$ as well (see Appendix B). Invariance of domain says that an injective continuous map from a space modelled locally by $\mathbb{R}^{2n-4}$ to a manifold of the same dimension is a homeomorphism onto its image. Therefore φ is an isomorphism of topological operads. This combined with Proposition 2.6.3 gives the deformation retract. The inclusion map is a composition of operad maps, so it is an operad map.

Definition 2.6.5. $\mathcal{D}_{tDM}(n)$ is the subspace of $\mathcal{D}_t(n)$ characterized by the following:

1. If $n = 1$, then $\mathcal{D}_{tDM}(n)$ is just the identity configuration $(0, 1)$ in $\mathcal{D}_t(1)$,
2. If $n > 1$, then every leaf vertex of the underlying tree of an element of $\mathcal{D}_{tDM}(n)$ is bivalent, and
3. If $n > 1$, then the root vertex of the underlying tree of an element of $\mathcal{D}_{tDM}(n)$ is bivalent, and is decorated with the fixed configuration $(0, 0)$.

Remark.

1. Note that for $n > 1$ there must be a root vertex, and specifying that it is bivalent implies that it is distinct from any leaf vertices.

2. Recall that there is a unique decoration on a bivalent vertex other than the root.

3. There is nothing special about the configuration $(0, 0)$; any fixed choice $(c_0, 0)$ would work just as well.

Lemma 2.6.6. *$\mathcal{D}_{tDM}$ is a suboperad of $\mathcal{D}_t$.*

Proof. We must prove that the composition of two elements of $\mathcal{D}_{tDM}$ is again in $\mathcal{D}_{tDM}$. If either one is the identity in $\mathcal{D}_{tDM}(1)$, this is trivial. If neither is the identity, then the composition starts with a grafting of a bivalent root vertex (whose incoming edge is not a leaf) to a bivalent leaf vertex (not the root) and then a contracting of the intermediate edge. This necessarily gives an unstable tree; forgetting the unstable contraction vertex yields the underlying tree of the composition. Its root is just the root of the first factor of the composition, decorated with the same decoration (it was not involved in the contraction) and the leaf vertices of the composition are all the leaf vertices of both factors except for the one involved in the grafting. None of these vertices were involved in the contraction, and all of them are still bivalent. Thus the composition is still in $\mathcal{D}_{tDM}$. □

Proof of Proposition 2.6.2. Let $n > 1$. We will provide charts near every point of $\mathcal{D}_{tDM}(n)$. Recall that there is a base of open sets of $\mathcal{D}_t$ of the form $\mathcal{N}_x^{\{U_v\}}$; we will investigate their intersection with $\mathcal{D}_{tDM}$ and describe one which gives a local embedding to $\mathbb{R}^{2n-4}$.

First consider a point $x \in \mathcal{D}_{tDM}(n)$, along with such a basic open set around it. Let us identify which points in $\prod U_v$ correspond to points in $\mathcal{D}_{tDM}$. We must preserve the bivalence of all root and leaf vertices, so these vertices could only be involved in an edge contraction with another bivalent vertex. That other bivalent vertex is either a root or leaf vertex, in which case the contraction would violate the conditions defining $\mathcal{D}_{tDM}$, or it is an internal vertex, in which case the tree is unstable, violating the conditions defining $\mathcal{D}_t$. Therefore, the root and leaf vertices cannot be involved in an edge contraction. If v is a bivalent leaf vertex v, then the radius of the factor (c, r) corresponding to $(v, N(v))$ in $N(v)$, the vertex *below* this bivalent one, must be zero.

As for the root, its decoration must be precisely our fixed decoration $(0, 0)$; changing the center or radius will change the decoration in the image or the bivalence of the root in $\mathcal{N}_x^{\{U_v\}}$.

On the other hand, as long as these two restrictions are satisfied, the image sets will all have bivalent root and leaf vertices with the specified decoration on the root. To state this concretely, let us consider one final modification of the branch space $B(D, P)$.

Definition 2.6.7. Let D and P be finite sets whose disjoint union contains at least two elements. The *Deligne-Mumford branch space* $\overline{B}(D, P)$ is a right $\mathbb{S}_D \times \mathbb{S}_P$-module in the category of spaces. It is the subset of $\mathfrak{C}(D \sqcup P)/\operatorname{Aff}\mathbb{C}$ such that $r_d = 0$ if r_d is in a factor (c_d, r_d) indexed by D.

Let $T = (n, V, N)$ be an n-tree and let v be a vertex of V. Let D_v be the incoming edges of v from bivalent vertices; let P_v be the incoming edges of v from vertices which are not bivalent. Then let $\overline{B}_v$ denote $\overline{B}(D_v, P_v)$.

Remark. The Deligne Mumford branch space is not contained in the augmented branch space $\hat{B}(D, P)$. However, because the D and P for $\overline{B}_v$ differ from the D and P used for $\hat{B}_v$ and B_v, the Deligne-Mumford branch space $\overline{B}_v$ of a vertex with no external edges is contained in $\hat{B}_v$, and in fact in the branch set B_v.

With this definition and the argument that came before it, we have:

Lemma 2.6.8. *Let x be in $\mathcal{D}_{tDM}(n)$ with $n > 1$, and let x have the underlying tree (n, V, N) with decoration x_v on the vertex $v \in V$. If v is at least trivalent, let U_v be an open set in $\overline{B}_v$ containing x_v. If v is a bivalent root vertex, let $U_v = \{(0, 0)\}$; if v is a bivalent leaf vertex, then let U_v be the decoration space for v, which is a point.*

Then the set

$$\prod_{v \in V} U_v$$

is an open neighborhood of x in $\mathcal{D}_{tDM}(n)$ and sets of this form make up a basis for the topology of $\mathcal{D}_{tDM}(n)$.

We are still trying to prove Proposition 2.6.2. To do so, we will move back and forth between two normalizations of the decorations of vertices in $\mathcal{D}_{tDM}$.

Definition 2.6.9. Consider the configuration $((c_1, r_1), \ldots (c_k, r_k))$ of $k > 1$ disjoint points and/or disks in the plane. There are (different) unique elements in Aff $\mathbb{C}$ which:

1. take c_1 to 0, c_2 to the positive real line, and take the diameter of the set of points and disks $\{c_i + \epsilon_i : \epsilon_i < |r_i|\}$ to $\frac{1}{2}$, and
2. take c_1 to 0 and c_2 to 1.

Call these normalizations the $\mathbb{R}_+$ *normalization* and the 1-*normalization*, respectively. Let T be an n-tree; let v be a vertex of T which is at least trivalent. if $\prod(c_e, r_e)$ (with the product taken over $e \in in(v)$) is a configuration of disjoint points and disks, we can induce an order on $in(v)$ from the order of the leaves. For example, let $e_1 < e_2$ if $\min\{\ell \in n : \ell \gg e_1\} < \min\{\ell \in n : \ell \gg e_2\}$. So we will refer to the $\mathbb{R}_+$ normalization and 1-normalization in this context without further comment.

The unique elements of $\mathrm{Aff}\,\mathbb{C}$ can be given explicitly. If D is the diameter, then they are, respectively:

1. $$(\frac{-c_1 e^{-i\arg(c_2-c_1)}}{2D}, \frac{e^{-i\arg(c_2-c_1)}}{2D}) \text{ and}$$

2. $$(\frac{-c_1}{c_2-c_1}, \frac{1}{c_2-c_1}).$$

Uniqueness can be verified directly by considering the stabilizer of each condition. Then a point x in $\mathcal{D}_{tDM}(n)$ with underlying tree $T = (n, V, N)$ can be described uniquely by 1-normalized decorations of points only on each vertex with valence greater than two. Let the decoration on the vertex $v \in V$ be x_v; now consider varying its parameters in a neighborhood of x_v in $\overline{B}_v$ (preserving the 1-normalization). Forget the bivalent vertices; then D_v corresponds to incoming external edges of v and P_v to incoming internal edges, as usual. So there are $2in^i(v) + in^e(v) - 2$ complex parameters for the decoration in a neighborhood of x_v with the specified normalization. The total number $in^i(v)$ summed over all vertices is the number of internal edges, which is one less than the number of vertices. The total number of $in^e(v)$ is n. So we have a total dimension of

$$\sum_v 2in^i(v) + in^e(v) - 2 = n + 2(v-1) - 2v = n - 2.$$

This has real dimension $2n - 4$. If we pick a small enough neighborhood in $\mathbb{R}^{2n-4}$, it will be contractible and parameterize an open set of $\mathcal{D}_{tDM}(n)$. For instance, let ϵ be the minimum Euclidean distance $|c - c'|$ between a disjoint pair $(c, 0)$ and $(c', 0)$ in any of the 1-normalized configurations and then take the ball of supremum radius $\frac{\epsilon}{4}$ in $\mathbb{R}^{2n-4}$ around the product of the decorations, keeping the factors corresponding to leaves. That is, a factor of form $(c_0, 0)$ can vary to (c, r) where $|c - c_0|, |r| < \epsilon$. If the factor corresponds to an edge coming from a bivalent leaf vertex, then its radius stays fixed at 0.

Consider the image in $\mathcal{D}_{tDM}(n)$ of such a basic open set, or any basic open set contained within it. It is automatically open.

Lemma 2.6.10. *The map from the ball in $\mathbb{R}^{2n-4}$ to $\mathcal{D}_{tDM}(n)$ is injective.*

This gives us a bijection between any small open set U in $\mathbb{R}^{2n-4}$ around a particular point and a corresponding open set U' around a particular point in $\mathcal{D}_{tDM}(n)$. The lemma also indicates that the restriction of this bijection to a smaller open set is an open set. On the other hand, any open set in U' can be built out of basic sets intersected with U', whose preimages are basic sets intersected with U. So this is a homeomorphism, proving the proposition. $\square$

This same argument, with a modification of the counting, also shows that

Proposition 2.6.11. *For $n > 1$, $\mathcal{D}_t(n)$ is locally homeomorphic to $\mathbb{R}^{4n}$.*

Proof of Lemma 2.6.10. We want to show that a configuration in $\mathcal{D}_{tDM}$ coming from the open ϵ-ball in $\mathbb{R}^{2n-4}$ uniquely determines the decorations on the vertices that were identified to make it. Fix $x \in \mathcal{D}_{tDM}(n)$ with underlying tree $T = (n, V, N)$. Let $\tilde{x}$ be a point in the ϵ-ball in $\mathbb{R}^{2n-4}$ around x considered with the parameterization above. Then $\tilde{x}$ determines an edge contraction set E. Let $S = \{v_1, \ldots, v_m\}$ be a vertex in the contracted tree T_E, or, equivalently, an equivalence class of vertices of T, ordered in a way compatible with the induced ordering by $\gg$. we would like to recover the decorations on the vertices v_i involved in the contraction from the decoration on the equivalence class. We will proceed downward through the equivalence class, starting with the topmost vertices of S.

Assume we have recovered the decoration on every vertex above the vertex v; we wish to recover the decoration x_v of v from the contraction vertex decoration $\tilde{x}_S$. Essentially, we know that the decoration on S is equal to a composition

$$x_{v_1} \circ_{k_1} (\cdots \circ_{k_{j-1}} (x_{v_j} \circ_{k_j} (\cdots) \cdots)) \cdots)$$

for specified k_j so by inverting $\circ_{k_j} = \odot$ where we can, we will be able to recover the marking on v.

By assumption, we know all the markings above v, so we can compose the appropriate representatives to get a configuration x_v^e of points in the disk for each incoming internal edge e of v coming from S. There are no little disks because at the top level, every disk must have radius 0. Consider each x_v^e with the $\mathbb{R}_+$ normalization, which is used for composition. Because c_1 in each representative is 0, the configuration x_v^e has a marked point at 0. It has another marked point at $c_1 \neq 0$. In order that $(c, r) \odot x_v^e = (c', r') \odot x_v^e$, we must have $c + 0r = c' + 0r$ so $c = c'$ and $c + rc_1 = c' + r'c_1$ so $r = r'$. This shows that we can uniquely factor $\tilde{x}_S$ as the composition of $\tilde{x}'_S$ with the various x_v^e. This gives us a center and radius vector in $\tilde{x}'_S$ for each incoming edge of v. Now we want to decompose $\tilde{x}'_S = \tilde{x}''_S \circ_k x_v$. We know which centers and radii in x' sitting in the standard disk came from x_v, including c_0 and c_1, the image of 0 and a positive real. We can also measure the Euclidean diameter D of the union of the disks involved. So setting $(c, r) \odot x_v$ to be the points and distances we know, we get the equations

$$c + 0r = c_0;\ c_1 \in c + r\mathbb{R}_+;\ D = \frac{|r|}{2}.$$

These uniquely specify c and $r \neq 0$ and we can compose on the left by the inverse to (c, r) to obtain x_v. □

Proof of Proposition 2.6.3. For $n = 1$, the map is just the map to the point, and $\mathcal{D}_t(1)$ is a disk bundle over the disk, so contractible. So assume $n > 1$. We first define the retraction map and homotopy, and then prove they are continuous. Fix a point x in $\mathcal{D}_t(n)$ with underlying tree $T = (n, V, N)$. To obtain a new tree, insert a vertex on each external edge of T which does not have a bivalent vertex as one of its vertices. Each new vertex and each vertex with an incident edge involved in a vertex insertion must have its decoration changed, because in $\mathcal{D}_t$, the decoration corresponding to a leaf is of form (c, r) and the decoration corresponding to an internal edge is of form $(c, 0)$; what is more, the root vertex decoration is a configuration in the disk while the other vertices are decorated with configurations in the plane up to the $\operatorname{Aff}\mathbb{C}$ action. The requisite changes are accomplished as follows:

1. Any new vertices inserted on leaf edges are not the root vertex and have a unique decoration.
2. If there is a new root vertex, it is decorated with the fixed decoration $(0, 0)$.
3. A vertex other than the root of T which has had vertices inserted into its incoming edges needs a new decoration. If the previous decoration had the factor (c, r) corresponding to an incoming external edge, this is replaced with $(c, 0)$ on the new incoming internal edge from the inserted vertex.
4. The root vertex of T may have had some bivalent vertices inserted into incoming external edges. All corresponding decorations (c, r) are replaced with $(c, 0)$, as in the previous case.
5. If the root vertex was not already bivalent, it had a vertex inserted on its

outgoing edge, so now the configuration of little disks in the standard disk is taken to its image in the plane and then quotiented by the action of the affine group. The new vertex is decorated with $(0,0)$.

6. If the root vertex was bivalent but decorated with $(c,0)$ for $c \neq 0$, it is redecorated with $(0,0)$.

This defines the retract. Because it does not change a tree whose leaf and root vertices are bivalent and whose root vertex is decorated with $(0,0)$, it is the identity on $\mathcal{D}_{tDM}(n)$.

The homotopy is a map $H : \mathcal{D}_t(n) \times [0,1] \to \mathcal{D}_t(n)$; for time $t = 0$ it is defined to be the retract map; for other times, it makes the following changes to the decorations of each vertex:

1. Away from the root, $(c,r) \times t \mapsto (c,rt)$.

2. At the root, $(c,r) \times t \mapsto (ct, rt^2)$.

This is clearly the identity at $t = 1$. On a point of $\mathcal{D}_{tDM}(n)$, the only nonzero radii are at bivalent vertices, and have a unique marking, so that $(c,rt) \overset{\mathrm{Aff}\,\mathbb{C}}{\sim} (c,t)$; at the root, $(c,r) = (0,0)$ so that $(ct, rt^2) = (0,0)$ as well.

It only remains to prove that the homotopy is continuous. This will imply continuity of the retract because it is the range restriction of the composition $x \mapsto (x,0) \mapsto H(x,0)$.

Choose a point x in $\mathcal{D}_t(n)$ with underlying tree $T = (n, V, N)$, a point $(\tilde{x}, t)$ in its preimage under H, and a basic open set containing x. We will find a neighborhood of $(\tilde{x}, t)$ in $\mathcal{D}_t(n) \times [0, 1]$ whose image is contained in the basic open set.

Focus on a particular vertex v of T. Our basic open set should have a factor $U_v \subset \hat{M}_v$ corresponding to v. If v is not the root, consider its decoration with the 1-normalization. Let ϵ be chosen so that each U_v contains the ϵ neighborhood of the decoration x_v of x on the vertex v (for vertices which are at least trivalent). Let R be the maximum radius of any disk in any of the decorations x_v, for v at least trivalent and, if not the root vertex, in the 1-normalization.

First, let us assume that $t \neq 0$ (the easy part). H only changes the underlying tree at $t = 0$, so $\tilde{x}$ and x have the same underlying tree and x can be obtained from $\tilde{x}$ by shrinking radii by a factor of t throughout the decorations (with the appropriate modification on the root vertex). That is, if a decoration on $v \neq *$ is given by (c, r) in x, then the corresponding decoration in $\tilde{x}$ is $(c, \frac{r}{t})$. There is a similar formula for the root vertex.

By assumption, $\tilde{x}$ is in $\mathcal{D}_t(n)$, so the disks defined by the decorations are all disjoint:

$$|c_a - c_b| > |\frac{r_a}{t}| + |\frac{r_b}{t}|.$$

There are finitely many of these, so now we can pick an δ which satisfies:

1. $\delta < \frac{\epsilon}{3}$,

2. $\delta^2 + (1+R)t\delta < \epsilon$,

3. δ is less than a fourth of the minimum of $|c_a - c_b| - |\frac{r_a}{t}| + |\frac{r_b}{t}|$ taken over all pairs of decorations of each vertex, and

4. δ is also less than t.

We can now use δ to define a neighborhood of $\tilde{x}$. Over every vertex, we allow each decoration (in the 1-normalization away from the root) to change by at most δ; we also allow the time to vary by at most δ. The third condition ensures that as we vary the decorations, we remain in $\mathcal{D}_t(n)$. The first and second conditions together ensure that once we map via H, all of the points in the image have decorations differing from those on x by no more than ϵ. The first condition suffices for the root vertex; a weaker version of the first condition along with the second condition is necessary for the non-root vertices. The calculation is similar to the calculation below for $t = 0$ but slightly easier.

Now we turn to the case when $t = 0$. Now the underlying trees of $\tilde{x}$ and x may not be the same. Specifically, the tree underlying $\tilde{x}$ is obtained from that of x by forgetting some bivalent vertices. The decorations on $\tilde{x}$ away from the root agree with the corresponding decorations on x except that where a bivalent vertex has been forgotten, the decoration on x is $(c, 0)$ and the decoration on $\tilde{x}$ is (c, r) (for some arbitrary r). At the root, the decoration on $\tilde{x}$ is either

1. a single point $(c, 0)$, or

2. a configuration in the disk of pairs (c_e, r_e) for $e \in in(*)$ (only if there are at least two incoming edges) such that the conformal class of $\prod(c_e, 0)$ in the plane is the decoration on the unique vertex immediately above the root of the tree underlying x.

Let ϵ and R be as before; let $\lambda = |c_2 - c_1|$ if the root is at least trivalent. Now pick a δ which satisfies:

1. $\delta < \frac{1}{5}$,

2. $\delta < \frac{\lambda}{2}$ (if the root is at least trivalent),

3. $\delta < \frac{\epsilon}{4}$,

4. $\delta^2 + R\delta < \epsilon$,

5. $\delta < \frac{\epsilon\lambda^2}{6+2\epsilon}$ (if the root is at least trivalent),

6. $\delta < \frac{\epsilon\lambda}{1+\epsilon}$ (if the root is at least trivalent), and

7. δ is less than a fourth of the minimum of $|\frac{r_a}{s}| + |\frac{r_b}{s}| - |c_a - c_b|$ taken over all pairs of decorations of each vertex.

We will show that the ϵ-neighborhood of x described above contains the image of a δ-neighborhood of $\tilde{x}$ as described above. Parts of this calculation will be performed explicitly. To begin, consider a time $0 < t < \delta$ along with decorations on the underlying tree of $\tilde{x}$ which differ from those of $\tilde{x}$ by less than δ.

First, consider the decoration on a vertex other than the root with fixed normalization. Any center c is within $\delta < \epsilon$ of its position c_0 in the tree underlying $\tilde{x}$. Any nonzero radius r is within δ of the corresponding radius r_0 in the tree underlying $\tilde{x}$; then H applied to this radius at time t is tr which satisfies

$$|tr| \leq |tr - tr_0| + |tr_0| < \delta^2 + \delta R$$

which is less than ϵ by condition 4.

Now consider the decoration $\prod(c_e, r_e)$ on the root vertex of the tree underlying $\tilde{x}$, assuming the root is at least trivalent (the easier case where the root is bivalent is omitted). This is a configuration in the disk, so all the radii and centers are bounded above by 1. We will denote the image of these decorations at time t by c_e^t and r_e^t. The diameter D of the set formed by the disks and points described by the configuration $\prod(c_e^t, r_e^t)$ is at most 2δ since the original diameter was at most 2. Call e_1 the incoming edge with the leaf 1 somewhere above it. We also have $(c_{e_1}^t) < \delta$. So there is a disk centered at $c_{e_1}^t$ of radius $2D \leq 4\delta$ containing all of the disks and points described by the various (c_e^t, r_e^t). By condition 1 this disk is contained in the standard disk. Then we will show that the decoration we are considering is in the part of our ϵ-neighborhood of x in $\mathcal{D}_t$ where the root vertex is decorated by the pair $(c^t_{e_1}, 2D)$. By condition 3 these coordinates are within ϵ of $(0,0)$, which is the decoration of the root in x.

Now consider the interior of the radius $2D$ disk, recentered at the origin. This will be the model for our decoration on the vertex immediately above the root in x; we must verify that the centers and radii of this decoration are within ϵ of the corresponding decorations in x, which are just the conformal class of the decoration on the root of $\tilde{x}$.

In order to compare this decoration to that of $\tilde{x}$, we must move, stretch and rotate them so that $c_1 = 0$ and $c_2 = 1$ to address them within our normalization. The real factor involved in this stretching is $|c^t_2 - c^t_1|$. To recenter it is necessary to subtract c^t_1. So beginning with c_j, we get eventually to

$$\frac{tc_j - tc_1}{tc_2 - tc_1} = \frac{c_j - c_1}{c_2 - c_1}.$$

We want to compare this to the corresponding points in $\tilde{x}$, which are up to δ away. Since $|c_2 - c_1| > \lambda - 2\delta$, a little calculating shows that the total difference is bounded above by

$$\frac{6\delta}{\lambda(\lambda - 2\delta)}$$

which is less than ϵ precisely when condition 5 holds. Condition 6 is used in a similar manner to show that when the radii of the root decoration of $\tilde{x}$ and our chosen point are within δ of each other, the image of our chosen point has radii less than ϵ.

There is one loose end to tie up, namely that the same conditions suffice when $t = 0$. When $t = 0$, all radii in all decorations other than bivalent leaf vertices are set to zero, and bivalent leaf vertices are added where appropriate. In addition, if the root is already bivalent, the marking (c, r) is moved to $(0, 0)$. If the root is not bivalent, then there is a new root vertex grafted in underneath with marking $(0, 0)$, and the root marking is taken up to conformal equivalence on the vertex immediately above the root. In every case, the fact that $\delta < \epsilon$ ensures the desired behavior away from the root and condition 5 again ensures that the decorations on the root play nicely after being taken up to equivalence. □

Remark. The homotopy does not respect the operad structure in any intermediate stage. Its purpose is just to ensure that the induced map from the homology of $\mathcal{D}_{tDM}$ to that of $\mathcal{D}_t$ is an isomorphism.

Proof of Proposition 2.6.4. We will construct the bijective map of topological operads $\varphi : \mathcal{D}_{tDM} \to \overline{\mathcal{M}}$ explicitly. First of all, $\mathcal{D}_{tDM}(1)$ and $\overline{\mathcal{M}}(1)$ are both a single point, the identity. For $n > 1$, a point x in $\mathcal{D}_{tDM}(n)$ is a decorated tree, as is a point in $\overline{\mathcal{M}}(n)$. We will describe first how to modify the underlying tree of x

to obtain the underlying tree for its image; then we will describe how to map the decorations.

The tree T underlying x is nearly stable, but it has bivalent root and leaf vertices. Forget the bivalent root and leaf vertices, and the resulting tree T' is stable. This will be the tree underlying $\varphi(x)$. Now we shall describe a map, also called φ, from the possible decorations on the vertex v of T with valence more than 2 in $\mathcal{D}_t$ to the possible decorations on the vertex v in $\overline{\mathcal{M}}$.

Because x is in $\mathcal{D}_{tDM}$, such a vertex is not the root vertex of T, so its decoration is a configuration of disks and points in the plane up to $\mathrm{Aff}\,\mathbb{C}$. Further, all of the incoming edges of v in T are internal edges, so the disks are all degenerate, of radius zero. Therefore, the decoration on v is a product of of disjoint points in the plane up to simultaneous $\mathrm{Aff}\,\mathbb{C}$ action indexed by $e \in in(v)$.

From such a configuration $\prod(c_e, 0)$, consider its complement as a subset of the plane, $\mathbb{C}\backslash\{c_e\}$. This inherits a Riemann surface structure from $\mathbb{C}$ which is invariant under the action of $\mathrm{Aff}\,\mathbb{C}$, so such a configuration gives a well-defined sphere with $E(v)$ marked points (we mark the $out(v)$ marked point at ∞). This is precisely the kind of decoration that we would put on v in T' to specify a point of $\overline{\mathcal{M}}(n)$.

So we have defined φ on a point in $\mathcal{D}_{tDM}(n)$; it remains to show that φ is an operad map, that it is bijective, and that it is continuous.

First, to see that φ is an operad map, recall the composition map of $\mathcal{D}_{tDM}$. It grafts underlying trees, forgetting the two internal bivalent vertices so created, and preserves all other decorations. This corresponds to merely grafting underlying trees in $\overline{\mathcal{M}}$, which is how composition works there. The identity of $\mathcal{D}_{tDM}$ was is taken to the identity in $\overline{\mathcal{M}}$ so this demonstrates that φ is an operad map.

To show that φ is a bijection, we shall construct an inverse explicitly. Fix a point $y \in \overline{\mathcal{M}}(n)$ with underlying tree T'. Insert a vertex on each external edge to obtain the tree T. This gives the underlying tree of $\varphi^{-1}(y)$. The decorations on the leaf vertices are unique, and the root must be decorated with $(0,0)$ for $\varphi^{-1}(y)$ to be in $\mathcal{D}_{tDM}(n)$. We will define φ^{-1} on a vertex, and showing it is inverse to φ there will show that it is inverse on all of $\overline{\mathcal{M}}(n)$.

So consider the decoration on a particular vertex v with valence at least three. This is an $E(v)$-marked sphere. Consider a representative Σ_v of the conformal class $[\Sigma_v]$ decorating v. The $E(v)$-marked sphere can have its punctures filled, so there exists a conformal embedding of Σ_v in a Riemann surface homeomorphic to the sphere. Every Riemann surface homeomorphic to the sphere is conformally equivalent to the standard sphere, so there is an embedding of Σ_v in $\overline{\mathbb{C}}$ so that the complement of its image is $|E(v)|$ points. By composing with a conformal automorphism of the $\overline{\mathbb{C}}$, there is an embedding of $\Sigma_v \in \overline{\mathbb{C}}$ where the puncture corresponding to the outgoing edge is completed at the point ∞ on the sphere. Then there is a conformal embedding of Σ_v in $\mathbb{C}$ taking the puncture corresponding to the outgoing edge to the puncture at ∞. The coordinates of the other punctures give disjoint points in $\mathbb{C}$ indexed by $in(v)$.Any other conformal representative Σ'_v is conformally equivalent to Σ_v so can also be mapped into $\mathbb{C}$ conformally taking the punctures to the same points.

However, this is not a well-defined map because there may be many choices of conformal embedding of Σ_v in $\mathbb{C}$. However, any two embeddings ι and ι' are embeddings of conformally equivalent surfaces, so there is a conformal equivalence between $\iota(\Sigma)$ and $\iota'(\Sigma)$. This can be completed to a conformal equivalence of their partial completions, so a conformal automorphism of $\mathbb{C}$. This shows that the embedding in $\mathbb{C}$ is unique up to the action of $\mathrm{Aff}\,\mathbb{C}$. Therefore, the conformal class of Σ gives rise to a well-defined configuration of disjoint points in $\mathbb{C}$ indexed by $in(v)$ up to the simultaneous action of $\mathrm{Aff}\,\mathbb{C}$, precisely the necessary data for a decoration on v in T to make it an element of $\mathcal{D}_{tDM}$.

Let us see that these maps are inverses. In one direction, φ^{-1} takes the conformal class $[\Sigma_v]$ to the configuration of the complement of Σ_v in the plane up to $\mathrm{Aff}\,\mathbb{C}$, and then φ takes that to the conformal class of the image of *its* complement, Σ_v, which is exactly $[\Sigma_v]$. In the other direction, if $\{c_e\}$ are disjoint points in the plane indexed by $in(v)$, then their complement, $\mathbb{C}\backslash\{c_e\}$ forms a representative of its own conformal class, so we can choose it as Σ_v and the inclusion of $\mathbb{C}\backslash\{c_e\}$ in the plane is a conformal embedding with $out(v)$ at ∞ whose complement is again $\{c_e\}$. This concludes the proof that φ is bijective.

The last thing to show is that φ is continuous. Consider a point $x \in \mathcal{D}_{tDM}(n)$ and the corresponding point $\varphi(x)$ in $\overline{\mathcal{M}}(n)$. To show that the map is continuous, we must show that each basic neighborhood of $\varphi(x)$ contains the image of a neighborhood of x (for details on the topology of $\overline{\mathcal{M}}(n)$, see Appendix B). To fix a basic open set, we must pick a representative $\widetilde{\varphi(x)}$ of $\varphi(x)$. This entails picking a representative of the conformal class of the decoration on each vertex. We can use x to do this. That is, each vertex of x (other than the root and leaf vertices) is decorated with a configuration x_v of at least two points in the plane up to $\mathrm{Aff}\,\mathbb{C}$. Take the $\mathbb{R}_+$ normalization of each configuration; this gives a representative of the marked conformal class of the corresponding vertex of $\varphi(x)$.

Now, to specify a neighborhood of $\varphi(x)$, we choose a neighborhood $\mathcal{N}$ of the nodal points of $\widetilde{\varphi(x)}$, and an $\epsilon > 0$. For the marked sphere decorating the vertex v, viewed in the completed plane as x_v in the $\mathbb{R}_+$ normalization, let $d_{min,v}$ be a number less than half the minimal distance between any two marked points. Note that $\mathcal{N}$ gives a neighborhood $\mathcal{N}_e$ of each marking c_e corresponding to an internal edge e, and a neighborhood $\mathcal{N}_{\infty,v}$ of ∞ except on the bottommost vertex. For a vertex v, let $R_v > 1$ be a number so that the disk of radius R_v centered at the origin contains the complement of $\mathcal{N}_{\infty,v}$ and let r_e be a number so that the disk of radius r_e centered at c_e is contained in $\mathcal{N}_e$. Choose a δ satisfying the following conditions:

1. $\delta < \frac{\epsilon}{d_{min,v}(2+\epsilon)}$ for all v and
2. for the edge $e = (a, b)$ between two stable vertices, let $\delta < \frac{r_e}{1+R_a}$. In particular, this implies that $\delta < \frac{r_e}{2}$.

We will show that the neighborhood determined by moving the c_e and expanding radii corresponding to internal edges of T' by amounts less than δ (preserving the normalization) is contained in $U_{N,\epsilon}$. To do this, we will construct a map from a representative surface in this neighborhood which is $(1+\epsilon)$-quasiconformal outside of the preimage of $\overline{N}$, a homeomorphism outside the preimage of the nodes of $\widetilde{\varphi(x)}$, and a point or simple closed curve in the preimage of a node.

We will define the map little by little. Consider a representative surface $\tilde{y}$ in the neighborhood. We can write $\tilde{y}$ as the image of a point in a product of $\tilde{B}_v$. From this point of view, a vertex has a decoration which is a configuration of pairs $({c_e}', {r_e}')$ for incoming edges from vertices with valence three or higher and $({c_e}', 0)$ for incoming edges from bivalent vertices. Call this configuration $\tilde{y}_v$. We will compare this to the corresponding surface Σ_v in $\widetilde{\varphi(x)}$, which is a configuration of pairs $(c_e, 0)$. The δ condition means that $|c_e - {c_e}'| < \delta$ and $|{r_e}'| < \delta$. We can cut a disk of radius r_e around each point c_e corresponding to an internal edge of T';

then this disk completely contains the disk of radius $|r_e{}'|$ centered at $c_e{}'$. Then by Lemma B.2.4, there is a $(1+\epsilon)$-quasiconformal map of the complement of these disks in the plane to their complement in Σ_v which takes each $c_e{}'$ corresponding to an external edge to c_e corresponding to the same external edge and is the identity outside of small disks around each such c_e.

This is not quite good enough for the following reason: this vertex has the outgoing edge $(v, N(v))$, and if the factor of the decoration of $N(v)$ corresponding to this edge has nonzero radius, then the actual surface we will construct will compose the disk of radius 1 in $\tilde{y}_v$ into the factor corresponding to the edge $(v, N(v))$ in the surface decorating $N(v)$. Therefore we do not yet have a map that is $(1+\epsilon)$-quasiconformal on the complement of $\overline{\mathcal{N}_{\infty,v}}$, because we are only able to use the part of $\tilde{y}_v$ that is contained within the unit disk.

However, in this case we are gluing the unit disk in $\tilde{y}_v$ into a disk (c', r') of radius $|r'| < \delta$, and then the disk of center c' and radius $|R_v r'| < r_{(v,N(v))} - \delta$ (by our assumptions on δ) is contained in the disk of radius $r_{(v,N(v))}$ centered at c which we cut out. Therefore we can consider the unit disk in $\tilde{y}_v$ as a subset of the disk of radius $|R_v r'|$ centered at c' in the surface decorating $N(v)$; this is conformally equivalent to the disk of radius R_v in $\tilde{y}_v$. Now we can legitimately use the $(1+\epsilon)$-quasiconformal map defined on Σ' to map the interior of the disk of radius R_v centered at c' with the disks of radius r_e cut out around each c_e

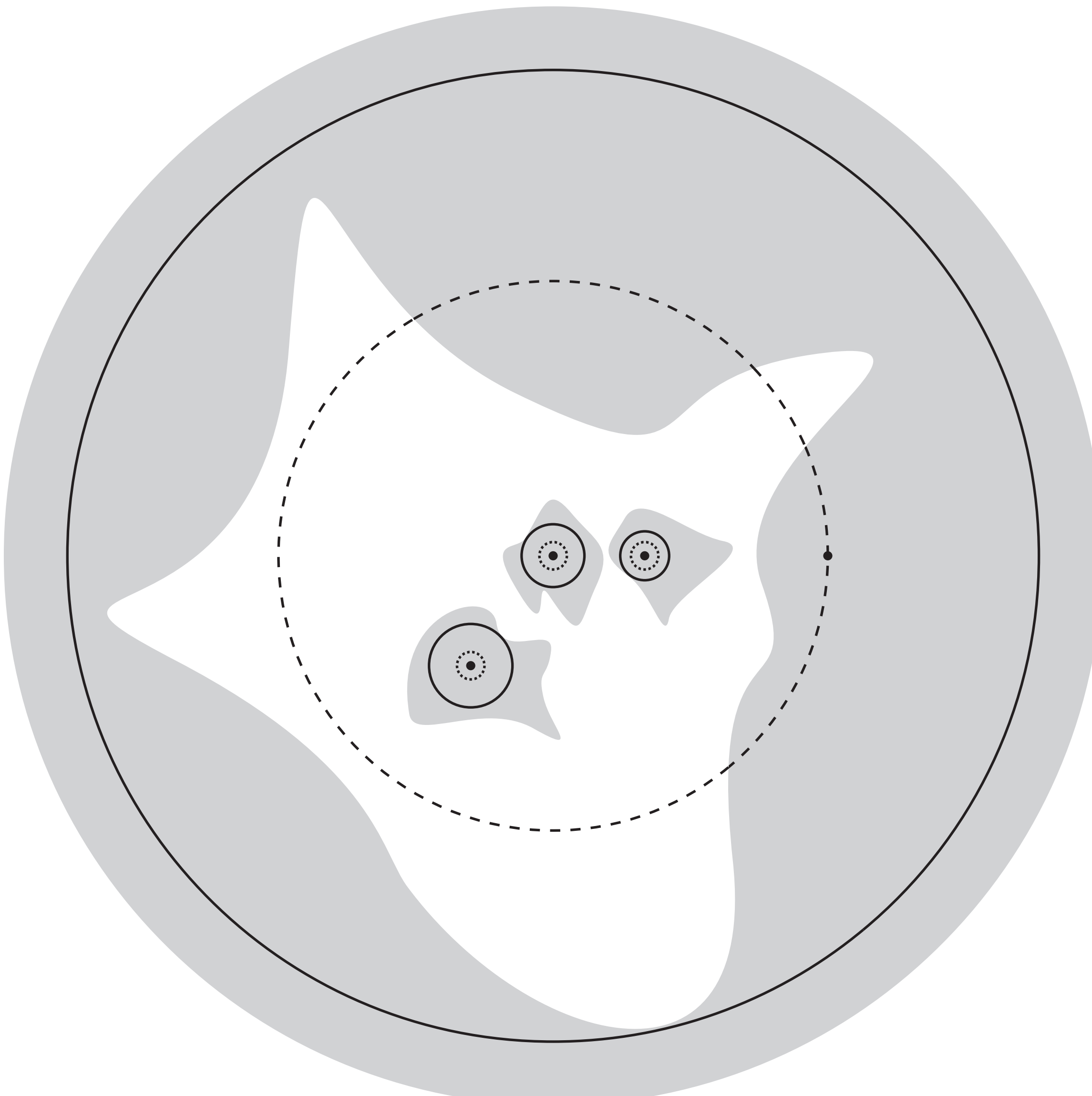


Figure 2.10: A configuration of three points in the $\mathbb{R}_+$ normalization, with $\mathcal{N}$ indicated in grey. The disks centered at each of the three points of radius r_e (solid circle) and δ (dotted circle) are shown. Also shown are the standard disk in the plane (dashed circle) and the disk of radius r_v centered at the origin (solid circle).

corresponding to an internal edge in the decoration of v into Σ.

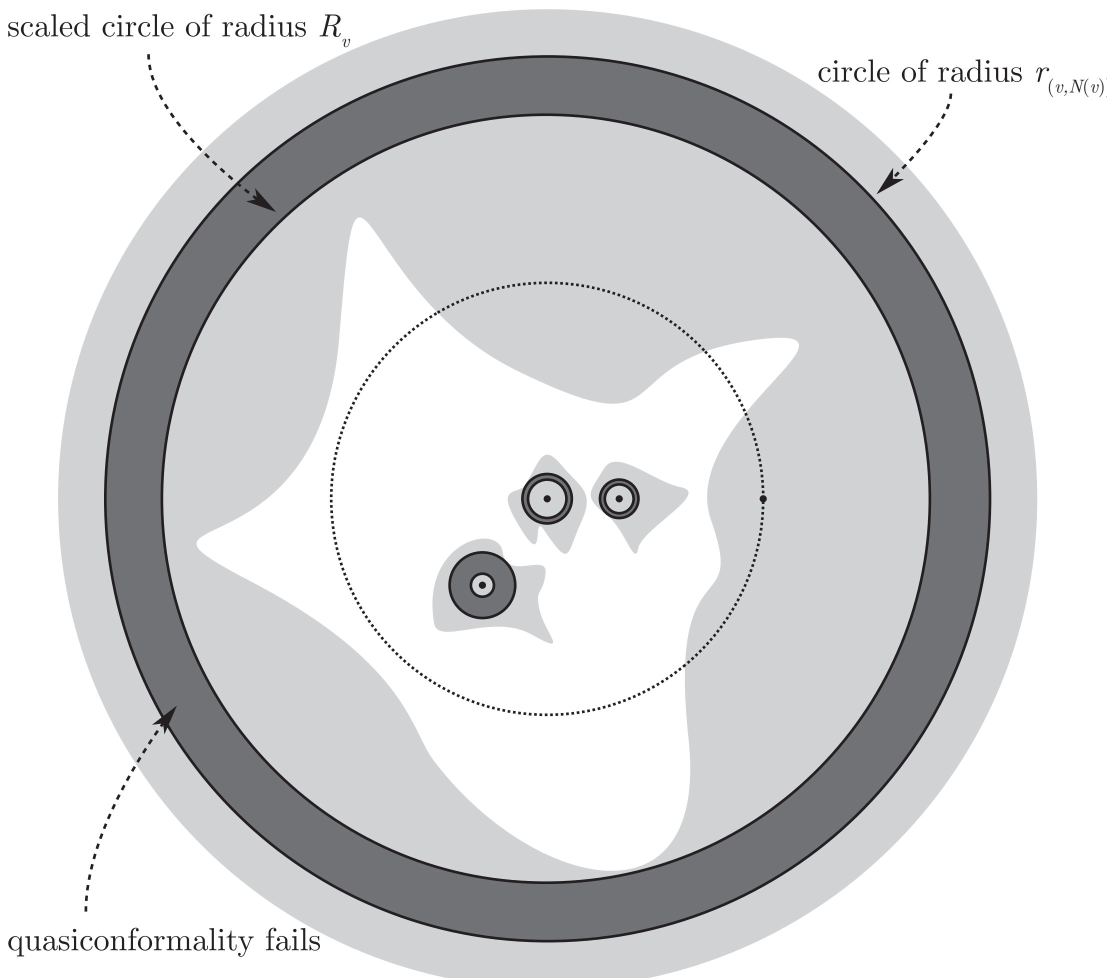


Figure 2.11: The configuration from Figure 2.10 decorating the vertex v has been composed with the decoration of $N(v)$. The standard disk in the configuration decorating v has been shrunk down and glued into the dotted disk of radius less than δ in the configuration decorating $N(v)$. Because δ is so small, the shrunk disk of radius r_v fits inside the disk of radius $r_{(v,N(v))}$. Therefore, this configuration can be mapped to $\widetilde{\varphi(x)}$ by a map which is $(1+\epsilon)$-quasiconformal away from the dark grey regions.

This defines a $(1+\epsilon)$-quasiconformal embedding from a subsurface of $\tilde{y}$ to a subsurface of $\widetilde{\varphi(x)}$ which contains $\widetilde{\varphi(x)}\backslash\mathcal{N}$. The complement in $\tilde{y}$ is a disjoint union of annuli and nodes; the complement in $\widetilde{\varphi(x)}$ is a disjoint union of nodes; the quasiconformal part of the map gives us a unique continuous extension from the boundary of an annulus or node in $\tilde{y}$ to the boundary of the corresponding node in $\widetilde{\varphi(x)}$. Then we can extend our map via a map from the annulus to the node that collapses a medial circle to the nodal point and is elsewhere a homeomorphism, or map the node to the node homeomorphically. This completes the construction of the required map.

□

Chapter 3

Trivializing algebraic circle actions

3.1 Overview

In the previous chapter, we proved that the genus zero Deligne-Mumford operad $\mathcal{M}$ was homotopy equivalent to the pushout of the framed little disks operad along an operad that trivialized the framed little annuli. In a representation, one could say that an action of the framed little disks along with a trivialization of the induced annulus action gives an action of $\mathcal{M}$. This chapter presents a minimal homotopy-invariant algebraic version of the same theorem.

The homology functor $\mathbf{H}$ takes topological operads to operads in graded vector spaces, and so we can take a homology version of the same pushout diagram:

$$\begin{array}{ccc} \mathbf{H}\mathcal{A} & \longrightarrow & \mathbf{H}\mathcal{D} \\ \downarrow & & \downarrow \\ \mathbf{H}\mathcal{A}_t & \longrightarrow & \mathbf{H}\overline{\mathcal{M}} \end{array}$$

Let us switch to an algebraic notation, as each operad here is also an object of interest in the purely algebraic setting. $\mathbf{H}\mathcal{A}$ is the nilpotent algebra $\mathcal{N}il = \mathbf{k}[D]/(D^2)$, and its representations are graded vector spaces with two commuting differentials of degree 1 and -1. $\mathbf{H}\mathcal{A}_t$ is the trivial operad since $\mathcal{A}_t(1)$ is contractible. $\mathbf{H}\mathcal{D}$ is the Batalin-Vilkovisky operad $\mathcal{BV}$ and the inclusion of $\mathcal{N}il$ into it gives the $\mathcal{BV}$ operator. Finally, $\mathbf{H}\overline{\mathcal{M}}$ is the hypercommutative operad $\mathcal{H}yc$. All of these will be defined more carefully in Section 3.3. So with our new notation, the diagram looks like:

$$\begin{array}{ccc} \mathcal{N}il & \longrightarrow & \mathcal{BV} \\ \downarrow & & \downarrow \\ \mathbf{k} & \longrightarrow & \mathcal{H}yc \end{array} \tag{3.1}$$

In general, the homology functor loses a lot of information. However, in this case, each of the operads involved is formal, so we might hope that this diagram is still a pushout. In fact, it is not. But all is not lost. After replacing each of the operads involved with its minimal cofibrant replacement, we obtain a pushout:

$$\begin{array}{ccc} \mathcal{N}il_\infty & \longrightarrow & \mathcal{BV}_\infty \\ \downarrow & & \downarrow \\ \mathbf{k} & \longrightarrow & \mathcal{H}yc_\infty \end{array}$$

Since the lower left corner is a trivial operad, we can rewrite this to say that

Theorem 3.1.1.

$$\mathcal{BV}_\infty/\mathcal{N}il_\infty \to \mathcal{H}yc_\infty$$

is an isomorphism.

What's more, because $\mathcal{N}il_\infty \to \mathcal{BV}_\infty$ is a cofibration, the diagram is a homotopy pushout so it is the correct homotopy invariant concept.

Here is what this means at the level of representations. If (A, d) is a $\mathcal{BV}$ algebra in the category of chain complexes, then the homotopy theory of operads tells us that we can transfer the $\mathcal{BV}$ structure to the homology of (A, d) as a $\mathcal{BV}_\infty$ structure. The theorem tells us what happens if it transfers in such a way that the subrepresentation of $\mathcal{N}il_\infty$ is trivial. To wit:

Corollary 3.1.2. *Let A be a $\mathcal{BV}_\infty$ algebra (for example, A can be a $\mathcal{BV}$ algebra). Then a strong deformation retract of A onto its homology so that the $\mathcal{BV}$ operator and its higher homotopies transfer as zero induces the structure of a $\mathcal{H}yc_\infty$ algebra on the homology of A. This structure can be (forgetfully) truncated to a $\mathcal{H}yc$ algebra on the homology of A.*

As described in Chapter 1, this generalizes the result of Barannikov and Kontsevich that a $\mathcal{BV}$ algebra structure on A satisfying stronger conditions than these induces the structure of a $\mathcal{H}yc$ algebra on homology.

Section 3.2 will present an high level picture of the proof, and then most of the chapter will be spent "under the hood."

3.2 Outline of the argument

For the statement of the theorem to have meaning, we need cofibrant replacements $\mathcal{BV}_\infty \to \mathcal{BV}$ and $\mathcal{H}yc_\infty \to \mathcal{H}yc$. Getzler [12] gave a presentation of the minimal cofibrant replacement $\mathcal{H}yc_\infty$, but only recently have relatively small cofibrant replacements for the $\mathcal{BV}$ operad become available. We distill a minimal $\mathcal{BV}_\infty$ using the language of ∞-cooperads. Once things are minimal, they are relatively rigid, and at this point we can construct by hand maps among generating spaces of these cofibrant replacements which induce the map of Theorem 3.1.1.

Let us be more specific. In [6], we present a general framework for transferring the structure of a chain cooperad to homology, where it becomes an ∞-*cooperad*. This is modeled after the transfer of associative algebra structures on a chain complex to A_∞ algebras on homology, and is motivated by Sullivan's theory of minimal models [27]. In particular, a strict cooperad is an example of an ∞-cooperad, and any ∞-cooperad on a chain complex with zero differential can be truncated to form a strict cooperad which is in general inequivalent. This truncation is the wrong tactic, just as in the case of the algebra of cochains or differential forms:

Theorem 3.2.1. *Let* $(\mathbf{C}, d)$ *be a cooperad in the category of chain complexes over a field. There exists a transferred* ∞*-cooperad structure on the homology of* $\mathbf{C}$ *and maps of* ∞*-cooperads in both directions which induce the identity isomorphism* *on homology.*

The other cogent result for our purposes here is:

Theorem 3.2.2. *There exists a cobar functor Ω_∞ from (nilpotent, coaugmented) ∞-cooperads to cofibrant operads satisfying the following conditions:*

1. *If $\mathbf{C}$ is an ∞-cooperad, the underlying operad in graded vector spaces of $\Omega_\infty\mathbf{C}$ is $\mathrm{F}(\Sigma^{-1}\overline{U\mathbf{C}})$. This is the same as the underlying vector space operad of the usual cobar functor,*

2. *if $\mathbf{C}$ is a strict cooperad, the differential of $\Omega_\infty\mathbf{C}$ coincides with that of $\Omega\mathbf{C}$, and*

3. *if $\mathbf{C}$ and $\mathbf{C}'$ satisfy sufficient connectedness conditions, and $f : \mathbf{C} \to \mathbf{C}'$ is a map of ∞-cooperads which induces an isomorphism on homology, then $\Omega_\infty f$ induces an isomorphism on homology.*

Remark. If $\mathcal{O}$ is sufficiently connected and $(\mathbf{C}, d) \to (B\mathcal{O}, d_\diamond)$ is a Koszul dual to $\mathcal{O}$, then we know $\Omega\mathbf{C} \to \mathcal{O}$ gives a resolution of $\mathcal{O}$. However, by applying Theorems 3.2.1 and 3.2.2, we find that $\Omega_\infty\mathbf{H}(\mathbf{C}, d)$ gives a *minimal* resolution of $\mathcal{O}$, that is, a resolution where the differential always increases the grading in terms of the number of vertices in the underlying operad $\mathrm{F}\mathbf{H}(\mathbf{C}, d)$. Minimal resolutions of sufficiently connected cooperads are unique up to ∞-cooperad isomorphism.

Note that in the case where $\mathcal{O}$ is Koszul and has no internal differential, $(\mathcal{O}^{¡}, 0) \to (B\mathcal{O}, d_{\diamond})$ is a Koszul dual so the ∞-cooperad structure is strict and $\Omega\mathcal{O}^{¡}$ is the minimal resolution, so this gives nothing new.

We would like to apply this machinery to our pushout square (Diagram 3.1). $\mathcal{N}il$ and $\mathcal{H}yc$ are Koszul, but $\mathcal{BV}$ is not. In [9], a Koszul dual, called $\mathcal{BV}^{¡}$, for the $\mathcal{BV}$ operad was constructed (since $\mathcal{BV}$ is not quadratic, this notation does not directly clash with that of Definition A.5.15). It is of the form $\mathcal{BV}^{¡} = (q\mathcal{BV}^{¡}, d_{\varphi})$ for a quadratic operad $q\mathcal{BV}$.

The main technical theorem is the following:

Theorem 3.2.3. *The (truncated) strict cooperad* $\mathrm{H}(q\mathcal{BV}^{¡}, d_{\varphi})$ *is strictly isomorphic to the cooperad* $\mathcal{H}yc^{¡} \oplus \mathcal{N}il^{¡}$.

Its accompanying proposition is easier:

Proposition 3.2.4. *The full* ∞*-cooperad structure on the quotient of* $\mathrm{H}(q\mathcal{BV}^{¡}, d_{\varphi})$ *by* $\mathcal{N}il^{¡}$ *is a strict cooperad structure.*

Theorem 3.1.1 now follows from the shape of the differential in Ω_{∞}, which comes as a sum of derivations induced by the structure maps of the ∞-cooperad.

Now, let us prove Theorem 3.2.3. Using distributive isomorphisms, we can bigrade $(q\mathcal{BV}^{¡}, d_{\varphi})$. That is, there is a cooperad $\mathcal{G}e^{¡}$ with differential d_{CE} so that

$$(q\mathcal{BV}^{¡}, d_{\varphi}) \cong (\mathcal{G}e^{¡} \otimes \mathcal{N}il^{¡}, d_{CE} \otimes \delta^{-1}).$$

We can prove that $\mathcal{G}e^{\text{i}}$ is contractible, a classical result presented here with a proof which is, so far as we know, new. This gives us a presentation of $\mathbf{H}(q\mathcal{BV}^{\text{i}}, d_\varphi)$ as a subset of $q\mathcal{BV}^{\text{i}}$ which splits into two parts: $\mathbf{1} \otimes \mathcal{N}il^{\text{i}}$ and $\mathcal{G}e^{\text{i}}/\operatorname{im} d_{CE} \otimes \mathbf{1}$. We reframe work of Getzler [11, 12] in a dual context to show that this is isomorphic to $\mathcal{H}yc^{\text{i}}$, the Koszul dual of $\mathcal{H}yc$, which concludes the proof.

The arguments can be somewhat opaque in the less familiar dual picture of cooperads and cofree things, but the proofs themselves are no more difficult or even longer, except in that some old results must be repackaged in a dual language. In some cases, such as the construction of the contracting homotopy for $\mathcal{G}e^{\text{i}}$, the cooperadic language makes the combinatorics significantly easier.

Next, in Section 3.3, we introduce the various operads and cooperads that will be the major players in the chapter. In Section 3.4, we begin relating them to one another by distributive isomorphisms. Nothing in these two sections is new. In Section 3.5 we construct the contracting homotopy to compute the homology of $\mathcal{BV}^{\text{i}}$, and in the final sections we ensure that Getzler's identifications respect the structures involved in this different context.

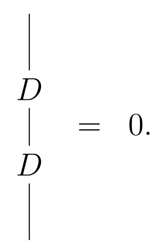
3.3 $\mathcal{BV}$, $\mathcal{H}yc$, and related operads and cooperads

This section consists mainly of definitions and characterizations of various operads in the category of chain complexes. The operads will be presented as quotients of free operads on generating spaces of operations. We will refer to sets of generators, by which we mean the **k** vectors space on the sets. We will call a generating operation *type* (n, k) to mean that it is of arity n and degree k with a trivial $\mathbb{S}_n$ action.

Definition 3.3.1. The *nilpotent algebra*, $\mathcal{N}il$, is an operad generated by the operation D of type $(1, 1)$ subject to the nilpotence relation:

$$D \circ_1 D = 0$$

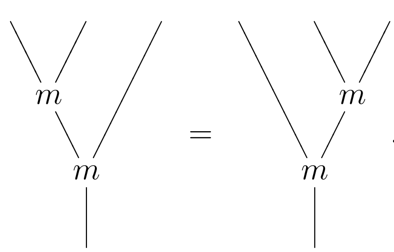
or, in trees:



Definition 3.3.2. The *commutative operad*, $\mathcal{C}om$, is generated by a product m of type $(2, 0)$ subject to the associativity relation:

$$m \circ_1 m = m \circ_2 m$$

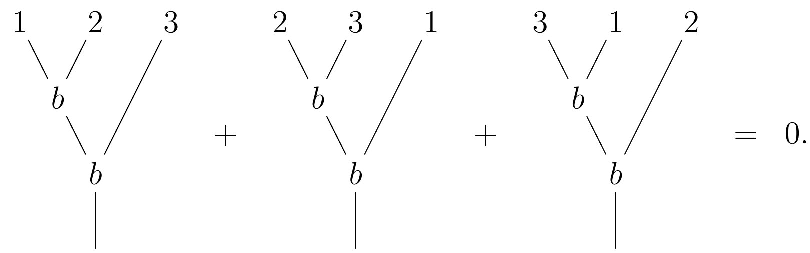
or, in trees:



Definition 3.3.3. The *odd symmetric Lie operad*, $s\mathcal{L}ie$, is generated by a bracket b of type $(2,1)$ subject to a Jacobi relation. Let σ be of order three in $\mathbb{S}_3$. Then:

$$\sum_{i=1}^{3}(b \circ_1 b)\sigma^i = 0$$

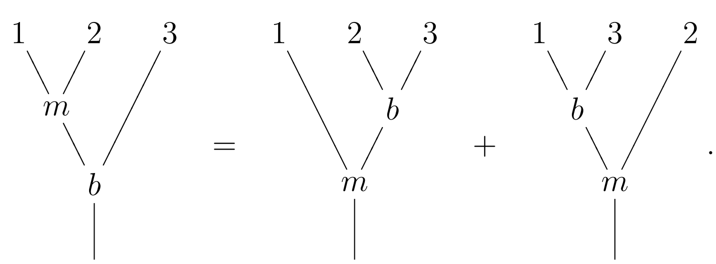
or, in trees:



Definition 3.3.4. The *Gerstenhaber operad*, $\mathcal{G}e$, is generated by a product m of type $(2,0)$ and a bracket b of type $(2,1)$ subject to the relations of the commutative and odd Lie operads, along with a Leibniz relation. Let $\tau \in \mathbb{S}_3$ interchange 2 and 3. Then:

$$b \circ_1 m = m \circ_2 b + (m \circ_1 b)\tau$$

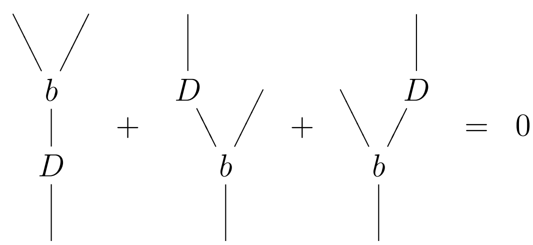
or, in trees:



Definition 3.3.5. The Batalin Vilkovisky operad, or $\mathcal{BV}$ *operad*, $\mathcal{BV}$, is generated by a product m of type $(2,0)$, a bracket b of type $(2,1)$, and a $\mathcal{BV}$ operator D of type $(1,1)$ subject to the relations of the nilpotent algebra and the Gerstenhaber operad, along with compatibilities between the $\mathcal{BV}$ operator and the other two generators. Namely:

$$D \circ_1 b + b \circ_1 D + b \circ_2 D = 0$$

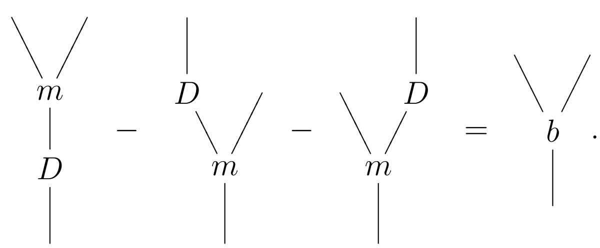
or, in trees:



and

$$D \circ_1 m - m \circ_1 D - m \circ_2 D = b$$

or, in trees:



The $\mathcal{BV}$ operad is sometimes presented instead without b as a generator and with a "seven term relation" expressing the fact that D is a second order differential operator with respect to m.

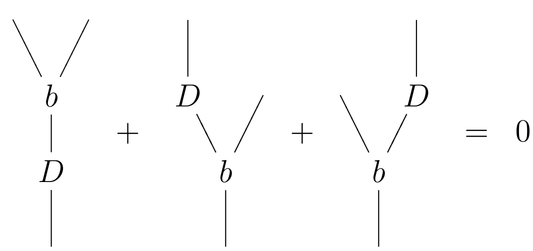
Remark. This agrees with the conventions of [9], and disagrees by a sign with other conventions for the bracket in a $\mathcal{BV}$ algebra. This approach has the disadvantage of making the bracket seem unfamiliar by virtue of being symmetric rather than skew symmetric, but is easier to describe with the operadic formalism. The two definitions are equivalent.

Theorem 3.3.6. *[10]. $\mathcal{BV}$ is the homology operad of $\mathcal{D}$.*

Definition 3.3.7. The *quadratic $\mathcal{BV}$ operad*, $q\mathcal{BV}$, is generated by operations m of type $(2,0)$, b of type $(2,1)$ and a $\mathcal{BV}$ operator D of type $(1,1)$, as before, subject to the relations of the nilpotent algebra and the Gerstenhaber operad, along with compatibilities between the $\mathcal{BV}$ operator and the other two generators. Namely:

$$D \circ_1 b + b \circ_1 D + b \circ_2 D = 0$$

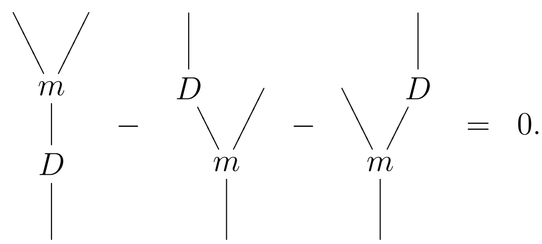
or, in trees:



and

$$D \circ_1 m - m \circ_1 D - m \circ_2 D = 0$$

or, in trees:



This differs only in the last relation from the definition of the $\mathcal{BV}$ operad. There, the relation had the linear term b on the right side; here the relations are generated by operations which are homogeneous quadratic in the generators.

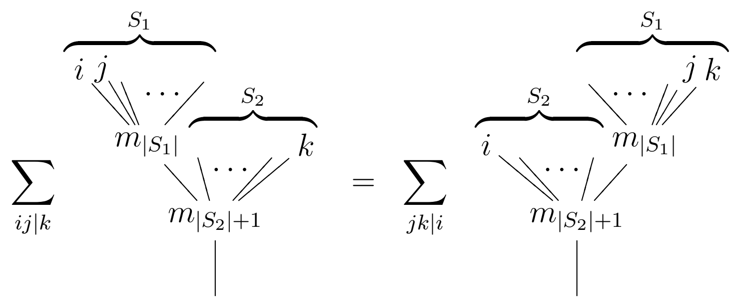
To describe the hypercommutative operad, it will be helpful to have some notation for the necessary symmetric group elements.

Definition 3.3.8. Let i, j and k be distinct elements of n. An *ij,k-partition* of n is an ordered partition $S = S_1 \sqcup S_2$ so that $i, j \in S_1$ and $k \in S_2$. We denote the set of ij,k-partitions of n by $ij|k$. Let p be an ij,k partition. $|p|$ denotes $|S_1|$, and the *p-unshuffle* σ_p in $\mathbb{S}_n$ is the unique permutation which sends S_1 to $\{1,\ldots,|p|\}$ and S_2 to $\{|p|+1,\ldots,n\}$, preserving the order of each.

Definition 3.3.9. The *hypercommutative operad*, $\mathcal{H}yc$, is generated by operations m_n of type $(n, 2n-4)$ for $n \geq 2$. These satisfy relations that generalize associativity:

$$\sum_{p \in ij|k} (m_{n-|p|+1} \circ_1 m_{|p|})\sigma_p - \sum_{p \in jk|i} (m_{n-|p|+1} \circ_1 m_{|p|})\sigma_p = 0$$

or, in trees:



3.3.1 Koszul duals

Now, we will discuss some of the naive dual cooperads to the operads just defined. Excepting $\mathcal{BV}$, which is not quadratic, all of the operads of the last section are Koszul, so each of the naive duals is a Koszul dual. The background material for these descriptions is in Sections A.5 and A.6 of the appendix. As a quick refresher, remember that for the presented quadratic operad $\mathcal{O} = \mathrm{F}(X)/(R)$, the naive dual $\mathcal{O}^{¡}$ is the subset of the cofree cooperad $\mathrm{F}^{\mathrm{c}}(\Sigma X)$ on the shifted generators characterized by being the kernel of the differential $d_\diamond$ induced by the composition map $\diamond$ of $\mathcal{O}$. Because we are dealing with multiple different operads, we will refer to $d_\diamond$ as $d_{\mathcal{O}}$.

$\mathrm{F}^{\mathrm{c}}(\Sigma X)/\mathcal{O}^{¡}$ is cogenerated by its further projection to $\mathrm{F}^{\mathrm{c}(2)}(\Sigma X)/\mathcal{O}^{¡(2)}$. This is one of the tools we will use, so we will describe $\mathcal{O}^{¡(2)}$ explicitly in terms of shifted generators.

Our naming convention is that the generators of operads are Latin letters, while their shifts are corresponding Greek letters. So μ will denote an operation of type $(2,1)$, β an operation of type $(2,2)$, δ an operation of type $(1,2)$, and μ_n an operation of type $(n, 2n-3)$.

Description 3.3.10. $\mathcal{N}il^{¡}$ is the subcooperad of the cofree cooperad generated by δ characterized by being the kernel of the differential induced by the composition of $\mathcal{N}il$. However, the composition of $\mathcal{N}il$ is zero, so $\mathcal{N}il^{¡}$ is all of $\mathrm{F}^{\mathrm{c}}(\delta)$. Because δ is arity one, this is just a cofree coalgebra on a generator δ in degree two. It can be identified linearly with $\mathbf{k}[\delta]$ with decomposition map

$$\delta^n \mapsto \sum_{i+j=n} \delta^i \otimes \delta^j.$$

Description 3.3.11. $q\mathcal{BV}^{¡}$ is a subcooperad of the cofree cooperad generated by μ, β, and δ. $q\mathcal{BV}^{¡(2)}$ consists of the shifted relations of $q\mathcal{BV}$. These are the linear span of:

$$\overbrace{\delta \circ_1 \delta}^{\text{nilpotence}}\ ,\ \overbrace{\mu \circ_1 \mu - \mu \circ_2 \mu}^{\text{associativity}},\ \overbrace{\sum_{i=1}^{3} (\beta \circ_1 \beta)\sigma^i}^{\text{Jacobi}}\ ,\ \overbrace{\beta \circ_1 \mu + \mu \circ_2 \beta + (\mu \circ_1 \beta)\tau}^{\text{Leibniz}},$$
$$\underbrace{\delta \circ_1 \beta + \beta \circ_1 \delta + \beta \circ_2 \delta, \text{ and } \delta \circ_1 \mu + \mu \circ_1 \delta + \mu \circ_2 \delta}_{\text{compatibilities}}.$$

Here σ is as in Definition 3.3.3 and τ is as in Definition 3.3.4. Note that while most of the shifted relations keep the same sign as the original relations, this is not true for the relations involving μ and β together or μ and δ together. The signs necessary to shift the relations coherently force the signs to change slightly in $q\mathcal{BV}^{¡}$. This is convenient, as it turns most of the negative signs into positive signs.

Because the relations are homogeneous in the number of vertices decorated with each of μ, β, and δ, we can add additional gradings to $q\mathcal{BV}^{¡}$, where we grade by the number of vertices decorated by any one of these.

We will mainly be concerned with the splitting into summands $q\mathcal{BV}^{¡}_{(k)}$ which have exactly k vertices decorated by δ.

Lemma 3.3.12. *$q\mathcal{BV}^{¡}_{(0)}$ is isomorphic to $\mathcal{G}e^{¡}$.*

Proof. $\mathcal{G}e^{¡}$ is the subcooperad of $\mathrm{F}^{c}(\beta,\mu)$ which is the kernel of $d_{\mathcal{G}e}$. But $d_{\mathcal{G}e}$ and $d_{q\mathcal{BV}}$ coincide on $\mathrm{F}^{c}(\beta,\mu)$. □

Remark. The same sort of reasoning applies to $\mathcal{C}om^{¡}$ and $s\mathcal{L}ie^{¡}$; they are graded components of $q\mathcal{BV}^{¡}$ and of $\mathcal{G}e^{¡}$, and there are projection maps to them.

Lemma 3.3.13. *[14]. $\mathcal{C}om^{¡}(n)$ is of dimension $(n-1)!$*

Definition 3.3.14. Let Γ denote the shifted generators of $\mathcal{H}yc$, that is,

$$\{\mu_n : n \geq 2\}.$$

Then $\mathcal{H}yc^{¡\,(2)}$ consists of decorated trees of the form

$$\sum_{p\in ij|k}(\mu_{n-|p|+1}\circ_1\mu_{|p|})\sigma_p - \sum_{p\in jk|i}(\mu_{n-|p|+1}\circ_1\mu_{|p|})\sigma_p = 0.$$

Theorem 3.3.15. *[12].* $\mathcal{H}yc$ *is Koszul.*

Remark. Getzler shows this by exhibiting $\mathcal{H}yc^{\text{i}}(n)$ (rather, its linear dual) as the shifted homology of the moduli space $\mathcal{M}_{0,n+1}$. Because the dimensions of the homology of this space are not hard to calculate, he also obtains:

Corollary 3.3.16. *The dimension of* $\mathcal{H}yc^{i}(n)$ *is* $\frac{n!}{2}$.

Remark. There is a different proof of the same corollary in [11].

3.4 Presentations and distributive isomorphisms

Lemma 3.4.1. *[13, 21, 9, 29]. The natural maps*

$$q\mathcal{BV}^{i} \to q\mathcal{BV}^{i} \boxtimes q\mathcal{BV}^{i} \to \mathcal{N}il^{i} \boxtimes \mathcal{G}e^{i}$$

and

$$\mathcal{G}e^{i} \to \mathcal{G}e^{i} \boxtimes \mathcal{G}e^{i} \to s\mathcal{L}ie^{i} \boxtimes \mathcal{C}om^{i}$$

that come from the decomposition map and projections are isomorphisms of collections. We will refer to both of these maps as ρ*.*

Definition 3.4.2. An element of $\mathrm{F}^{\mathrm{c}}(\beta, \mu)$ is a sum of decorated trees, with decorations μ and β. Define a map of collections

$$\hat{\rho} : \mathcal{N}il^{\text{i}} \boxtimes \mathrm{F}^{\mathrm{c}}(\beta, \mu) \to \mathrm{F}^{\mathrm{c}}(\beta, \mu, \delta)$$

as follows. We will describe the image of $\delta^m \boxtimes x$ where x has underlying tree T. Let κ range over functions from $E(T)$ to $\mathbb{N}$ such that

$$\sum_{e \in E(T)} \kappa(e) = m.$$

Then the image of $\delta^m \boxtimes x$ has underlying tree T' which is obtained from T by inserting $\kappa(e)$ vertices on each edge e. The decorations of the vertices from T remain the same; the new bivalent vertices are decorated by κ.

Lemma 3.4.3. *The restriction of $\hat{\rho}$ to $\mathcal{N}il^{¡} \boxtimes \mathcal{G}e^{¡}$ is the inverse to $\rho : q\mathcal{BV}^{¡} \to \mathcal{N}il^{¡} \boxtimes \mathcal{G}e^{¡}$.*

Proof. Consider $\rho\hat{\rho}(\delta^m \boxtimes x)$. Because ρ first decomposes and then projects, it is zero on any tree decorated by β, δ, and μ unless all of the vertices decorated by δ are below all of the other vertices. There is precisely one summand in the sum defining $\hat{\rho}$ which satisfies this condition. That is the summand corresponding to κ with κ of the outgoing edge of the root equal to m and κ of every other edge equal to zero. ρ splits this into two levels and then projects; the only way for the projection to be nonzero is for it to split with δ^m as the bottom level; then $\rho\hat{\rho}(\delta^m \boxtimes x) = (\delta^m \boxtimes x)$.

Now it is necessary to verify that $\hat{\rho}(\delta^n \boxtimes x)$ is in $q\mathcal{BV}^{¡}$, which will prove that $\hat{\rho} = \rho^{-1}$ because Lemma 3.4.1 implies that a one-sided inverse is an inverse. To do this, we will postcompose with $d_{q\mathcal{BV}}$, which will give zero precisely if the image of $\hat{\rho}$ is in $q\mathcal{BV}^{\text{i}}$.

Consider $d_{q\mathcal{BV}}\hat{\rho}(\delta^m \boxtimes x)$. $\hat{\rho}$ inserts vertices decorated by δ, and $d_{q\mathcal{BV}}$ composes pairs of adjacent vertices. The sum involved in applying $d_{q\mathcal{BV}}$ includes compositions involving 0, 1, and 2 vertices decorated by δ. Each of these vanishes for a different reason.

1. The insertion of a vertex decorated with δ commutes up to sign with compositions that do not involve it, so inserting m vertices decorated with δ and then contracting an edge whose vertices are decorated by μ or β is the same as contracting the edge first and then inserting vertices decorated with δ. But since $d_{\mathcal{G}e}$ coincides with $d_{q\mathcal{BV}}$ on the δ^0 component of $q\mathcal{BV}^{\text{i}}$, and we are starting in the kernel of $d_{\mathcal{G}e}$ to begin with, this summand is zero.

2. Contracting an edge whose vertices are both decorated by δ gives a bivalent vertex whose decoration is $\Sigma(D \circ_1 D)$, which is zero since $D \circ_1 D = 0$ in $q\mathcal{BV}$.

3. Finally, consider contracting an edge between a vertex v decorated by a μ or β and an adjacent vertex decorated by a δ. Let κ' be a map from the edges of T to the natural numbers so that the sum of the images adds to $m-1$. There are precisely three choices of κ with a δ adjacent to v which

can be forgotten to yield an element whose underlying tree is T with vertices inserted according to κ'. The sum of the three contractions with v associated to κ' together make up a relation of $q\mathcal{BV}$. No signs are necessary because the shifts on δ give everything the right sign.

□

Lemma 3.4.4. *1. The graded component $s\mathcal{L}ie^{i}$ of $\mathcal{G}e^{i}$ consisting of nontrivial trees with vertices decorated only by β is isomorphic as a collection of graded vector spaces to $\Sigma\mathbf{\Gamma}$. Call the isomorphism ζ.*

2. The graded component consisting of trees with precisely one vertex decorated by μ is dimension $\binom{n}{2}$ in arity n, spanned by the classes described below.

Proof. Let S_n be the set of trivalent n-trees. Then define β_n in $\mathrm{F^c}(\beta) \subset \mathrm{F^c}(\beta, \mu)$ to be the sum of decorated trees where the trees vary over S_n and the decorations are all β. We claim that this is closed under $d_\diamond$ and so is in $s\mathcal{L}ie^{\mathrm{i}} \subset \mathcal{G}e^{\mathrm{i}}$. To see this, consider the set $\hat{S}_n$ of n-trees with one vertex of valence four and every other vertex of valence three. This is the only type of tree that can be obtained from S_n via a single edge contraction, and every tree in $\hat{S}_n$ is the edge contraction of precisely three distinct trees in S_n. $d_\diamond\beta_n$ is a sum over $\hat{S}_n$; each tree in $\hat{S}_n$ appears precisely three times. The decoration on the unique 4-valent vertex varies over the three possible compositions of b with b in $s\mathcal{L}ie$, so the sum of the three is the Jacobi relation and the composition is then zero in $s\mathcal{L}ie$.

The construction of β_n makes it clearly $\mathbb{S}_n$-invariant; its degree is $2n-2$ because it is must have $n-1$ vertices to have n leaves. This is in precise correspondence with $\Sigma\mu_n \in \Sigma\mathbf{\Gamma}$. It remains to show that the β_n exhaust $s\mathcal{L}ie^{\text{i}}$.

Suppose x is an element in $s\mathcal{L}ie^{\text{i}}(n)$. Further, suppose the coefficient in x of the decorated tree with decorations β on the underlying tree $T \in S(n)$ is 1. Fix an internal edge e of T. In $d_\diamond x$, the contracted tree T_e with β decorating each trivalent vertex and the appropriate coefficient of $\beta \circ_k \beta$ decorating the 4-valent vertex gets coefficient 1 from T. For this to yield zero in the larger sum defining $d_\diamond x$, the two other trees in $S(n)$ which contract to give T_e must have coefficient 1 in x (with all vertices decorated by β, of course). Then every trivalent tree that can be obtained from T by a sequence of moves which constitute an edge contraction followed by the inverse image of an edge collapse of the same contraction vertex must have the same coefficient as T in x.

On a trivalent tree, such a move takes a pair of trivalent vertices connected by an edge $v_1 \xrightarrow{N} v_2$ so that $N(s_i) = N(s_j) = v_1$ and $N(s_k) = v_2$ for some leaves and/or vertices s_i, s_j, and s_k, and moves them around so that $N(s_k) = v_1$ and one of s_i or s_j is taken to v_2 by N.

A standard inductive argument shows that all trivalent n-trees can be obtained

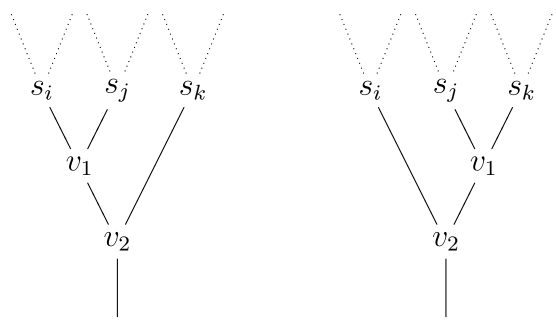


Figure 3.1: Two trivalent trees with the same edge contraction

from one another by a sequence of such moves. This shows that the coefficient of every tree in $S(n)$ must be the same in x, which shows it is a multiple of β_n.

For the second part of the lemma, for $1 \leq i < j \leq n$, define μ_n^{ij} as a sum over the same trees. In this case, for each tree $T \in S(n)$, there will be precisely one vertex decorated by μ. In any n-tree there is a unique vertex v_{ij} with i above one incoming edge and j above the other. That vertex is labelled by μ and all other vertices are labelled by β. Since β is even, and there is only one vertex decorated by μ, which is odd, again the ordering does not matter and we do not need signs.

We can make a similar argument as in the first part of the lemma to show that μ_n^{ij} is closed under $d_\diamond$. As there, $d_\diamond \mu_n^{ij}$ is a sum over $\hat{S}_n$, n-trees with precisely one 4-valent vertex and all other vertices trivalent. Here there are two cases, depending on whether the 4-valent vertex has i and j above distinct incoming edges or not. If not, then we can proceed exactly as for β_n: this tree arises three times in the sum, corresponding to the three terms of the Jacobi identity. On the other hand, if the 4-valent vertex has i and j above distinct incoming edges, we must worry about the order of the labels on the two trivalent vertices that were contracted together to form that vertex in the sum. There are again three isomorphism classes of trivalent tree which so contract; one of the three has a μ decorating the upper vertex and the other two have μ decorating the lower vertex. The sum of these three compositions is exactly the shifted Leibniz relation.

This shows that all of the μ_n^{ij} are closed under $d_\diamond$. To show that they are linearly independent for different choices $\{i,j\}$, consider for a given $\{i,j\}$ an isomorphism class of tree for which the special vertex with i above one input and j above the other has no other vertices above it, just the two leaves i and j. If this tree occurs in the sum which defines $\mu_n^{k\ell}$ then k must be above one input edge and ℓ above the other, meaning that $\{k,\ell\}$ must be the same as $\{i,j\}$.

Finally, they form a spanning set because they are of the appropriate dimension. Lemma 3.4.1 tells us that $\mathcal{G}e^{\text{¡}}$ is isomorphic to $s\mathcal{L}ie^{\text{¡}}\boxtimes\mathcal{C}om^{\text{¡}}$. The component of $\mathcal{G}e^{\text{¡}}(n)$ with one μ is isomorphic to the component of $s\mathcal{L}ie^{\text{¡}}\boxtimes\mathcal{C}om^{\text{¡}}(n)$ with one μ. This must have a $s\mathcal{L}ie^{\text{¡}}$ element of arity $n-1$; the first part of this lemma shows that $s\mathcal{L}ie^{\text{¡}}(n-1)$ is one dimensional, spanned by β_{n-1}. So this component of $\mathcal{G}e^{¡}(n)$ is isomorphic to

$$\beta_{n-1} \otimes_{\mathbb{S}_{n-1}} \left[(\mu \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{n-2}) \otimes_{\mathbb{S}_2 \times \mathbb{S}_1 \times \cdots \times \mathbb{S}_1} \mathbb{S}_n \right].$$

The elements of $\mathbb{S}_{n-1}$ fixing μ on the left in this quotient are exactly $\mathbb{S}_{n-2}$ acting on the rightward factors. Then there are $n!$ elements of $\mathbb{S}_n$ and we are quotienting by commuting actions of $\mathbb{S}_{n-2}$ and $\mathbb{S}_2$, yielding a count of $\frac{n!}{(n-2)!2!} = \binom{n}{2}$.

We should think of these elements of $s\mathcal{L}ie^{¡} \boxtimes \mathcal{C}om^{¡}$ as having an $(n-1)$-corolla from $s\mathcal{L}ie^{¡}$ with a single μ from $\mathcal{C}om^{¡}$ grafted somewhere on top, with i and j the leaves of μ. □

3.4.1 Extending the isomorphisms to the differential context

Lemma 3.4.5. *1. The degree -1 map $q\mathcal{BV}^{¡}_{(2)} \to q\mathcal{BV}^{¡}_{(1)}$ which takes $\delta \circ_1 \mu$ to β and every other decorated tree to zero can be extended to a square zero coderivation d_φ of $q\mathcal{BV}$.*

2. The degree 1 map $\mathcal{G}e^{¡}_{(1)} \to \mathcal{G}e^{¡}_{(1)}$ given by

$$\mu \mapsto \beta \mapsto 0$$

can be extended to a square zero coderivation d_{CE} of $\mathcal{G}e^{¡}$, and

3. ρ respects these differentials so that it is an $\mathbb{S}$-equivariant isomorphism of chain complexes between $(q\mathcal{BV}^{¡}, d_\varphi)$ and $(\mathcal{N}il^{¡} \boxtimes \mathcal{G}e^{¡}, \delta^{-1} d_{CE})$.

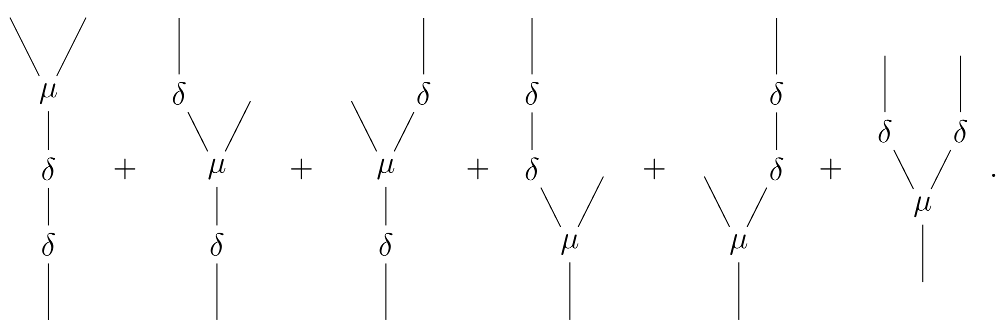
Proof. There is a conceptual proof of the first part in [9] relying on [30]. We prove things by brute force. For the first two parts, we can apply Lemmas A.6.6, A.6.8, and A.6.9. In both cases, we are considering $\mathcal{O}^{¡} \subset \mathrm{F^c}(\Xi)$, and $\mathrm{F^c}(\Xi)/\mathcal{O}^{¡}$ is cogenerated by the projection onto $\mathrm{F^c}_{(2)}(\Xi)/\mathcal{O}^{¡}_{(2)}$. Therefore, it suffices to consider the composition $\mathcal{O}^{¡} \to \mathrm{F^c}(\Xi) \to \mathrm{F^c}(\Xi) \to \mathrm{F^c}(\Xi)/\mathcal{O}^{¡} \to \mathrm{F}^{c(2)}(\Xi)/\mathcal{O}^{¡(2)}$. If this composition is zero, then the corresponding coderivations on $\mathrm{F^c}(\Xi)$ descend to the subcooperad $\mathcal{O}^{¡}$. Now, because these maps are homogeneous in terms of the number of vertices, we have only a small set to check.

To show that d_φ is well-defined on $q\mathcal{BV}^{¡}$, we need to check that

$$d_\varphi(q\mathcal{BV}^{¡(3)}) \subset q\mathcal{BV}^{¡(2)}.$$

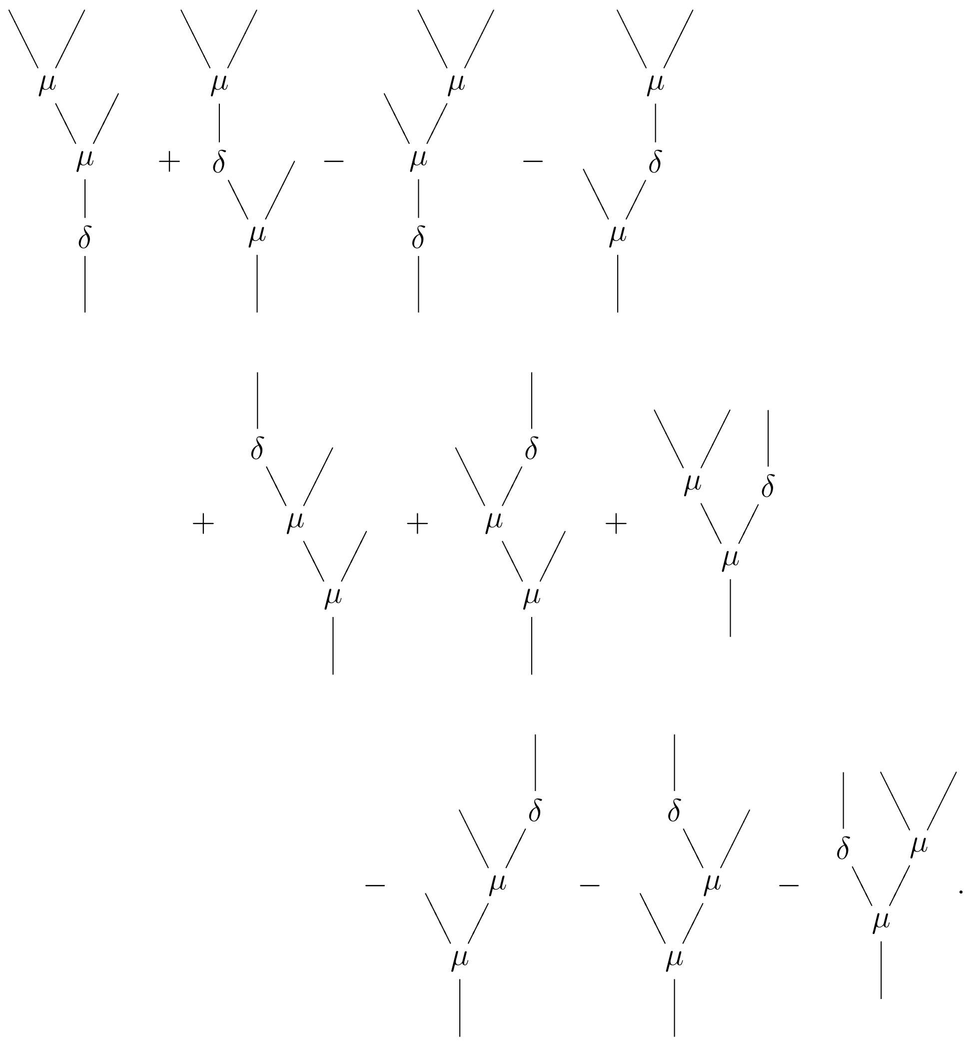
In particular, since d_φ only acts on vertex pairs decorated with δ and μ, we can restrict our attention even further, to three cases divided up by how the third vertex is marked. In each of these, Lemma 3.4.1 tells us the dimensions involved:

1. The component of $q\mathcal{BV}^{¡(3)}$ with vertices decorated by two δ and a μ is one dimensional, spanned by the following six term sum:



d_φ kills the last three terms of this sum; the first three become the shifted Leibniz relation between δ and β.

2. The component decorated by one δ and two μ. This is two dimensional, spanned by the cyclic permutations of the following ten term sum:



d_φ kills the last six trees, and the remaining four terms become a difference of two shifted Leibniz relations between β and μ.

3. The component decorated by one δ, one μ, and one β. This is three dimensional, spanned by the cyclic permutations of a similar fifteen term sum. Just as in the last examples, only the terms with δ directly below μ contribute. There are three of these, and under d_φ they yield the shifted Jacobi relation.

A similar argument shows that $d_{CE}(\mathcal{G}e^{\mathrm{i}\,(2)}) \subset \mathcal{G}e^{\mathrm{i}\,(2)}$. We have a good characterization of $\mathcal{G}e^{\mathrm{i}\,(2)}$ as the space of shifted generators of $\mathcal{G}e$, so it is spanned by the shifted versions of the associators, Jacobi, and the Leibniz relation. d_{CE} takes the associator to a sum of Leibniz relations, the Leibniz relation to Jacobi, and Jacobi to zero.

Now, to show that ρ respects the two differentials, we shall use the characterization of ρ^{-1} from Lemma 3.4.3, and check that

$$\rho d_\varphi \rho^{-1}(\delta^n \boxtimes x) = \delta^{n-1} \boxtimes d_{CE} x.$$

Recall that ρ^{-1} inserts n vertices decorated by δ throughout the decorated trees whose sum constitutes x. Then d_φ acts on one of these trees along with a μ directly above it and ρ projects back. Since ρ kills any decorated tree which has a vertex decorated by δ above vertices decorated by β or μ, it suffices to check that this is true on the summands of the image under ρ^{-1} which have all or all but one of the vertices decorated by δ inserted along the outgoing edge of the root, and on the summands of d_φ which only act on the highest of these δ-decorated vertices. This means we have a fixed δ^{n-1} below the rest of the tree which we will not acted on with d_φ and then the sum over possible insertions of one vertex decorated by δ in all possible locations in the rest of the tree. In total, this means it suffices to check this fact for $\delta \boxtimes x$.

In this case, consider a summand x_T of x with underlying tree T. In $\rho^{-1}x_T$, for each edge there is a summand with a vertex decorated by δ inserted in that edge. By Description A.6.7, d_φ acts on this sum by edge contraction; because of the specifics of the map defining the coderivation, it can only act on the edge immediately above the inserted vertex. Then the composition of these two maps acts on each vertex marked by μ by first inserting δ on the outgoing edge and then contracting the edge between them, turning the μ into a β. This is exactly the action of d_{CE}.

Both coderivations square to zero for strictly formal reasons, because applying, say, d_{CE} twice acts on two distinct vertices, and acting on the two choices of which order to act on the vertices come with opposite signs because μ is odd. □

3.5 Homology of $q\mathcal{BV}^{\text{i}}$

Proposition 3.5.1. *The homology of* $(\mathcal{G}e^{\text{i}}, d_{CE})$ *is one dimensional, represented by the trivial tree.*

Corollary 3.5.2. *Grading $q\mathcal{BV}^{i}$ by the number of δs in each cooperation, its homology is isomorphic to:*

$$\mathbf{H}(q\mathcal{BV}^{i})_{(n)} = \begin{cases} \textit{one dimensional, spanned by } \delta^n & : \quad n > 0 \\ \mathcal{G}e^{i} / \operatorname{im} d_{CE} & : \quad n = 0. \end{cases}$$

Proof. By Lemmas 3.4.1 and 3.4.5, we can decompose $q\mathcal{BV}^{¡}$ by the grading induced by the number of vertices decorated by δ, and we get the following commutative diagram:

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{d_\varphi} & q\mathcal{BV}^{¡}_{(n)} & \xrightarrow{d_\varphi} & \cdots & \xrightarrow{d_\varphi} & q\mathcal{BV}^{¡}_{(1)} & \xrightarrow{d_\varphi} & q\mathcal{BV}^{¡}_{(0)} & \longrightarrow & 0 \\ & & \| & & & & \| & & \| & & \\ \cdots & \xrightarrow{d_{CE}} & \mathcal{G}e^{¡} & \xrightarrow{d_{CE}} & \cdots & \xrightarrow{d_{CE}} & \mathcal{G}e^{¡} & \xrightarrow{d_{CE}} & \mathcal{G}e^{¡} & \xrightarrow{d_{CE}} & 0 \end{array}$$

The homology is then one dimensional by Proposition 3.5.1 everywhere except at $q\mathcal{BV}^{¡}_{(0)}$, where it everything is in the kernel so it is just the quotient by the image of d_{CE}. ☐

Corollary 3.5.3.

The dimension of the n-ary component of $H(q\mathcal{BV}^{i}, d_\varphi)$, for $n > 1$, is $\frac{n!}{2}$.

Proof. By Corollary 3.5.2, it suffices to count the dimension of $\mathcal{G}e^{¡}(n) / \operatorname{im} d_{CE}$. Because the homology of $\mathcal{G}e^{¡}(n)$ is 0 for $n > 1$, this is precisely half the rank of $\mathcal{G}e^{¡}(n)$ itself. By Lemma 3.4.1, this is the same as the rank of $s\mathcal{L}ie^{¡} \boxtimes \mathcal{C}om^{¡}(n)$. An element of $s\mathcal{L}ie^{¡} \boxtimes \mathcal{C}om^{¡}$ has an arity m element of $s\mathcal{L}ie^{¡}$ (the space of these is dimension one by Lemma 3.4.4), a partition of the leaf set n into m unordered nonempty subsets $S_1, \dots, S_m$, and an element of $\mathcal{C}om^{\text{!`}}(S_i)$ for each $i$. The rank of $\mathcal{C}om^{\text{!`}}(j)$ is $(j-1)!$ by Lemma 3.3.13, so this is:

$$F(n) = \sum_{n=\sqcup S_i} \prod_{i=1}^{m} (|S_i| - 1)!$$

We prove that this sum is $n!$ by induction. For $n = 1$ there is only one partition, $S_1 = \{1\}$, and the calculation yields 1. Now assume F is the factorial for integers up to n and consider the formula defining it for $n+1$. In any given partition, $n+1$ must appear in some S_i. Suppose $|S_i| = k$. Then there are $k-1$ choices of which other elements of $\{1, \dots, n\}$ are in S_i. The remainder of the numbers, which are in bijection with $(n-k+1)$, run over all possible partitions of $n-k+1$. So the sum for $n+1$ becomes

$$\underbrace{n!}_{\text{if } k=n+1} + \sum_{k=1}^{n} F(n-k+1) \binom{n}{k-1} \underbrace{(k-1)!}_{\text{rank of } \mathcal{C}om^{\text{!`}}(k)}$$

$$= n! + \sum_{k=1}^{n} (n-k+1)! \frac{n!}{(n-k+1)!(k-1)!} (k-1)! = n! + n(n!) = (n+1)!$$

□

Remark. Another corollary to this proposition is the well-known fact that the Lie and commutative operads are Koszul, and that, equivalently, the Chevalley-Eilenberg homology of the free Lie algebra is trivial.

To prove Proposition 3.5.1, we will construct an explicit contracting homotopy. To do this, we will need a combinatorial factor.

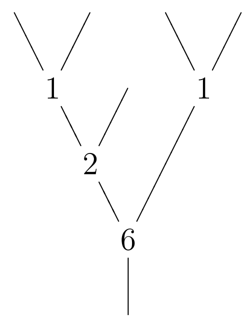


Figure 3.2: A trivalent tree with the weight ω indicated at each vertex

Definition 3.5.4. Let $T = (L, V, N)$ be a tree with rank greater than one. The *weight* of the vertex $v \in V$, $\omega(v)$, is the sum over pairs of incoming edges of v of the product of the number of leaves above the two incoming edges (see Figure 3.2).

Lemma 3.5.5.

$$\sum_{v \in V} \omega(v)$$

depends only on the rank of T, and in fact is $\binom{rk\, T}{2}$.

Proof. Every leaf is taken to the root vertex by some power of N; then for every pair of leaves ℓ and ℓ', there is a first vertex which is in both the image of ℓ under powers of N and the image of ℓ' under powers of N. This is the only vertex for which the pair ℓ and ℓ' are above different incoming edges of the same vertex, and so this is the only time that they contribute to the sum. Each such pair contributes the product 1, so the total sum is the number of such pairs, namely $\binom{rk\, T}{2}$. □

Definition 3.5.6. Let x in $\mathrm{F}^{\mathrm{c}}(\beta, \mu)$ be a decorated tree with underlying tree T. Choose an order of the vertices of T; then x can be written up to a scalar as $x_{v_1} \otimes \cdots \otimes x_{v_m}$, where each x_v is either a β or a μ. As usual, two orders on the tensor product are equivalent up to the sign induced by the permutation of the vertices decorated by μ (which are odd). Let ϵ_i be $(-1)^{|x_{v_i}|}$, and let h act on the vector space spanned by β and μ by taking β to μ and μ to zero.

We define a chain homotopy H of degree -1 on $\mathrm{F}^{\mathrm{c}}(\beta, \mu)$ by

$$H\left(x_{v_1} \otimes \cdots \otimes x_{v_m}\right) = \sum_{i=1}^{m} \epsilon_1 \cdots \epsilon_{i-1} \frac{\omega(v_i)}{\binom{rk\, T}{2}} x_{v_1} \otimes \cdots \otimes h(x_{v_i}) \otimes \cdots \otimes x_{v_m}.$$

In words, H is the sum over all ways of changing a β to a μ, but comes with a sign (the regular Koszul sign) and a combinatorial factor.

Lemma 3.5.7. *By abuse of notation, let d_{CE} denote the unique coderivation of $\mathrm{F}^{\mathrm{c}}(\beta, \mu)$ extending the map taking μ to β and β to zero. Then $d_{CE}H + Hd_{CE} = \mathrm{id}$ on the reduced collection $\overline{\mathrm{F}^{\mathrm{c}}(\beta, \mu)}$.*

Proof. Consider a proper tree in $\mathrm{F}^{\mathrm{c}}(\beta, \mu)$. H acts on it by taking the signed and weighted sum of replacing each β with a μ; d_{CE} acts by taking the signed sum of replacing each μ with a β. To act first with one and then with the other means that either

1. H and d_{CE} act on two distinct vertices of the tree, or

2. they act on the same vertex, changing it first from β to μ or vice versa and then back, ending up with the same tree, possibly with a sign and combinatorial factor.

The first type come in pairs, one from $d_{CE}H$ and one from Hd_{CE}, with the same combinatorial factors. They have the opposite sign, because the sign conventions for H and d_{CE} are the same, and in one of the cases, there is one more or fewer μ than the other in a position that induces a sign. This means all of these terms cancel.

For the second type, note first of all that the induced signs from H and d_{CE} will be the same sign acting on whichever vertex we have chosen, and every vertex will be acted on by precisely one of Hd_{CE} and $d_{CE}H$ nontrivially, depending on whether it begins decorated by β or μ, so the final result of so acting will be the sum over all vertices of $T = (L, V, N)$:

$$\sum_{v \in V} \frac{\omega(v)}{\binom{rk\ T}{2}} x_{v_1} \otimes \cdots \otimes x_{v_n}.$$

By Lemma 3.5.5, the sum of the coefficients over all of V is exactly one, which yields the desired result. □

Lemma 3.5.8. $H(\mathcal{G}e^i) \subset \mathcal{G}e^i$.

Proof of Proposition 3.5.1. Consider the counit map $\eta : \mathbf{I} \to \mathrm{F^c}(\beta, \mu)$ and the coaugmentation map $\mathbf{I} \to \mathrm{F^c}(\beta, \mu)$ which takes $\mathbf{I}$ to the trivial tree in $\mathrm{F^c}(\beta, \mu)$. These are chain maps with respect to the zero differential on $\mathbf{I}$ and d_{CE} on $\mathrm{F^c}(\beta, \mu)$ because d_{CE} kills the trivial tree, but the trivial tree is not in the image of d_{CE}. H also kills the trivial tree in $\mathrm{F^c}(\beta, \mu)$. Then $d_{CE}H + Hd_{CE}$ kills the trivial tree but acts as the identity on its complement in $\mathrm{F^c}(\beta, \mu)$. So H is a contracting homotopy between $(\mathrm{F^c}(\beta, \mu), d_{CE})$ and $(\mathbf{I}, 0)$. By Lemma 3.5.8, this is true in the subcooperad $(\mathcal{G}e^{\mathrm{i}}, d_{CE})$ as well. □

It would be nice to use something like Lemma A.6.3 or Lemma A.6.8 to prove Lemma 3.5.8, but because H is neither a map of cooperads nor a coderivation, it is not immediately clear that this approach will work. In essence, the combinatorial factors involved in H are dependent on the global structure of the decorated tree on which it acts, not just local structure, so we can't directly prove the result by looking at the local picture provided by the cogenerators. It will turn out that a local calculation will suffice, but only after some careful analysis. Let us build a little bit of terminology that we will only use for the proof.

Definition 3.5.9. Let S be a finite set; for convenience, let us assume that S is ordered. Then let $\Lambda(S)$ be the set of decorated trees in $\mathrm{F^c}(\beta, \mu)(S)$. If λ is a decorated tree from $\Lambda(S)$, let $T(\lambda)$ denote the underlying tree of λ. Every vertex

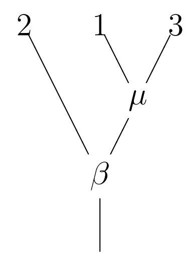


Figure 3.3: A picture of (2^{μ}_{β})

of $T(\lambda)$ must be trivalent. An element λ of $\Lambda(S)$ has a degree induced by the degrees of β and μ in $\mathrm{F}^{\mathrm{c}}(\beta, \mu)$.

Description 3.5.10. We shall want to discuss $\Lambda(3)$ in more detail. A trivalent 3-tree must have two vertices, v and $*$, with $N(v) = *$. One of the three leaves also shares a edge with the root; this gives us precisely three trivalent 3-trees. Then because there are two possible decorations for each vertex, there are twelve elements of $\Lambda(3)$. We will refer to them with a number indicating which leaf shares an edge with the root and upper and lower indices indicating their decorations as in Figure 3.3. If λ is in $\Lambda(3)$, then let $\lambda_\diamond$ be the decorated 3-corolla whose decoration is the shifted composition of the two decorations of λ in $\mathcal{G}e$ along the tree $T(\lambda)$.

Proof of Lemma 3.5.8. Recall that $\mathcal{G}e^{\mathrm{i}}$ is characterized as the kernel of $d_\diamond$ on

$\mathrm{F}^{\mathrm{c}}(\beta, \mu)$. A typical element of $\mathrm{F}^{\mathrm{c}}(\beta, \mu)(S)$ looks like

$$x = \sum_{\lambda \in \Lambda(S)} a(\lambda)\lambda$$

for structure constants $a(\lambda)$. When we apply $d_{\diamond}$, this is a sum over contractions along internal edges, with a decoration assigned by composition. Then

$$d_{\diamond}x = \sum_{\lambda \in \Lambda(S)} a(\lambda) \sum_{e \in E(T(\lambda))} \lambda_e$$

where λ_e is the decorated tree obtained from λ by contracting e. The decoration on the arity 3 contraction vertex comes from $\mathcal{G}e(3)$ and is induced via composition in $\mathcal{G}e$ of the decorations on the two vertices of the contraction edge.

So we can see that the image of $\mathrm{F}^{\mathrm{c}}(\beta, \mu)(S)$ under $d_{\diamond}$ is spanned by decorated S-trees whose underlying trees have precisely one 4-valent vertex, v, and all other vertices trivalent and decorated by either μ or β. By cutting at all the edges of v, we can recover four trivalent trees with vertices decorated by μ or β. The leaf set of all four trees together is $S \sqcup \{v\}$. Together, this gives us most, but not all, of the information about λ. That is, if T' is such a contracted tree, then there are precisely three trivalent trees that could give rise to T' by an edge contraction. Therefore, we can rewrite the sum according to the possibilities for a two-vertex trivalent tree.

Let $\{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ range over decorated trees in various Λ so that the total disjoint union of their leaf sets is $S \sqcup \{v\}$. We can assume that $\{v\}$ is a leaf of λ_0

and that the other three are ordered in some way descending from the ordering of S. Call P the set of all such sets of decorated trees.

Let λ be a decorated 3-tree. If we have a fixed $p \in P$, then $\widehat{\lambda}$ will indicate the decorated n-tree obtained by grafting λ to λ_0 along the leaf v and grafting λ_i to this tree along the leaf i of λ. Then we can rewrite the above sum as:

$$d_\diamond x = \sum_{p \in P} \sum_{\lambda \in \Lambda(3)} a(\widehat{\lambda})\widehat{\lambda_\diamond}.$$

What this means concretely is that for this sum to be zero, it must be zero for each $p \in P$. Since we have an enumeration of $\Lambda(3)$, and know the relations of $\mathcal{G}e$, we can say even more. Let $\{i, j, k\} = \{1, 2, 3\}$. Then for any fixed p we must have:

$$a(\widehat{(1^\beta_\beta)}) = a(\widehat{(2^\beta_\beta)}) = a(\widehat{(3^\beta_\beta)}) \text{ (Jacobi)}, \tag{3.2}$$

$$a(\widehat{(i^\beta_\mu)}) = a(\widehat{(j^\mu_\beta)}) + a(\widehat{(k^\mu_\beta)}) \text{ (Leibniz), and} \tag{3.3}$$

$$a(\widehat{(1^\mu_\mu)}) + a(\widehat{(2^\mu_\mu)}) + a(\widehat{(3^\mu_\mu)}) = 0 \text{ (associative)}. \tag{3.4}$$

Now let us assume that $d_\diamond x = 0$, so that it satisfies these conditions. We will show that $d_\diamond Hx = 0$ as well. Beginning as before, we have

$$d_\diamond Hx = \sum_{\lambda \in \Lambda(S)} a(\lambda) \sum_{e \in E(T(\lambda))} (H\lambda)_e.$$

We can rewrite H as the sum of h over vertices in $T(\lambda)$We want to rewrite this sum as we did in the analysis of $d_\diamond x$, in terms of a sum over P. We have two options, depending on whether v is in one of the trees λ_i in our partition P, or if it is in the tree $\lambda \in \Lambda(3)$ where the contraction happens.

If v is in one of the λ_i, then $(h_v\lambda)_e$ and $h_v\lambda_e$ differ only by a sign which does not change as we change λ in a homogeneous component of $\Lambda(3)$. Our assumption that x began in $\ker d_\diamond$ implies that these terms vanish, as we are performing the same contractions as before (yielding all zero coefficients) and then modifying some vertex or other elsewhere in the tree.

Therefore we can disregard these terms and assume, instead, that we will sum over h acting only on the vertices of the contraction edge. Let us apply h to such vertices. We began with an x that satisfied the conditions 3.2–3.4. We will see that we will end up satisfying the same conditions.

We need to apply h not to the vertices of $\lambda \in \Lambda(3)$, but to $\widehat{\lambda}$ for fixed $p \in P$. This matters because the combinatorial factors of h depend on p. So, in $\widehat{\lambda}$, let n_i be the rank of the underlying tree $T(\lambda_i)$. For $i > 0$, this is the number of leaves in $\widehat{\lambda}$ above the grafting edge of λ_i to the leaf i of λ. Then the if λ is the symbol (i^γ_α) then its lower vertex has weight $n_i(n_j + n_k)$ and its upper vertex has weight $n_j n_k$.

Then applying h to vertices decorated with two β is as follows:

$$(i^\beta_\beta) \mapsto n_i(n_j + n_k)(i^\beta_\mu) + n_j n_k (i^\mu_\beta).$$

So assume that we begin with a system of coefficients satisfying condition 3.2.

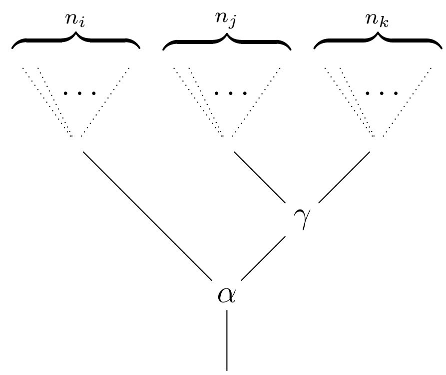


Figure 3.4: The part of $\widehat{\lambda}$ which comes from λ

Then after applying h we will get a symmetrized system of coefficients that satisfies condition 3.3, since $n_i(n_j + n_k) = n_i n_j + n_i n_k$.

Applying h to vertices decorated with one β and one μ takes

$$(i^{\mu}_{\beta}) \mapsto n_i(n_j + n_k)(i^{\mu}_{\mu}); \qquad (i^{\beta}_{\mu}) \mapsto -n_j n_k (i^{\mu}_{\mu}).$$

We get an overall sign of -1 on the right because h must pass over the bottom vertex, which is decorated by μ. Again, applying this to a system of coefficients satisfying condition 3.3, we get exactly condition 3.4.

Of course, applying h to vertices decorated only with μ gives zero.

This shows, then, that H preserves the kernel of $d_\diamond$, as desired.

□

Proposition 3.2.4 now follows from the shape of H.

Proof of Proposition 3.2.4. The formulas that define all of the higher ∞ cooperations on $\mathbf{H}(q\mathcal{BV}^{¡}, d_\varphi)$ involve applying a contracting homotopy for the strong deformation retract

$$\mathbf{H}(q\mathcal{BV}^{¡}, d_\varphi) \rightleftarrows (q\mathcal{BV}^{¡}, d_\varphi).$$

We can choose for our homotopy a version of H which also increases the δ grading by one. Then in the image of any higher cooperation, there will be an operation involving a δ. If we quotient by $\mathcal{N}il^{¡}$, killing all the δ, then all of these higher operations collapse. □

3.6 The explicit isomorphism

In [11], Getzler introduced the *gravity operad* and identified it as the kernel of the derivation on $\mathcal{G}e$ extending the map on generators defined by $m \mapsto b \mapsto 0$. Later, he showed that the gravity operad had a dual relationship with $\mathcal{H}yc$ [12]. We are working with cooperads instead of operads, but the calculations are similar. We have a derivation, d_{CE}, on $\mathcal{G}e^{¡}$, extending the map on cogenerators defined by $\mu \mapsto \beta \mapsto 0$; we will show that the kernel of this coderivation is isomorphic to $\mathcal{H}yc^{¡}$, the dual cooperad to $\mathcal{H}yc$. Many of the calculations of this section are a translation of Getzler's calculations into cooperadic language. Therefore, the rest of this section will be devoted to showing that:

Theorem 3.6.1. *The cooperad $\mathcal{G}e^{i} / \operatorname{im} d_{CE}$ is isomorphic to $\mathcal{H}yc^{i}$.*

Remark. The main theorem, theorem 3.2.3, follows directly, after a verification that the cooperation on $\mathcal{N}il^{\text{i}}$ splits in $\mathbf{H}(q\mathcal{BV}^{\text{i}}, d_\varphi)$.

Definition 3.6.2. Define a map $\psi : q\mathcal{BV}^{\text{i}} \to \mathbf{\Gamma}$ as follows: project to $\mathcal{G}e^{\text{i}}$, apply d_{CE}, project to $s\mathcal{L}ie^{\text{i}}$, and then identify $s\mathcal{L}ie^{\text{i}}$ with $\mathbf{\Gamma}$ via the identification of Lemma 3.4.4. That is:

$$q\mathcal{BV}^{\text{i}} \longrightarrow \mathcal{G}e^{\text{i}} \xrightarrow{d_{CE}} \mathcal{G}e^{\text{i}} \longrightarrow \overline{s\mathcal{L}ie^{\text{i}}} \xrightarrow{\Sigma^{-1}\zeta} \mathbf{\Gamma}.$$

Lemma 3.6.3. *ψ is a map of collections of chain complexes from $q\mathcal{BV}^{i}$ to $\mathbf{\Gamma}$.*

Proof. ψ is certainly $\mathbb{S}_n$-equivariant. The degree of d_{CE} is 1 while the degree of Σ^{-1} is -1, so it is degree preserving. Since $\mathbf{\Gamma}$ has no differential, to see that this is a map of differential $\mathbb{S}$-modules, we must show that the image of d_φ is killed by ψ. But $d_\varphi = (\delta^{-1} \boxtimes d_{CE})$ and so ψd_φ begins with a composition of ${d_{CE}}^2$, which is zero. □

Proposition 3.6.4. *The induced map of differential cooperads*

$$\mathrm{F}^{\mathrm{c}}(\psi) : (q\mathcal{BV}^{i}, d_\varphi) \to \mathrm{F}^{\mathrm{c}}(\mathbf{\Gamma})$$

can be range restricted to $\mathcal{H}yc^{i} \subset \mathrm{F}^{\mathrm{c}}(\mathbf{\Gamma})$.

Lemma 3.6.5. *The restriction of the induced map from $\mathbf{H}(q\mathcal{BV}^{i}, d_\varphi) \to \mathcal{H}yc^{i}$ to $\mathcal{G}e^{i} / \operatorname{im} d_{CE}$ is injective.*

Proof of Theorem 3.6.1. Since the induced map $\mathcal{G}e^{\mathsf{i}}/\operatorname{im} d_{CE} \to \mathcal{H}yc^{\mathsf{i}}$ is an injective map between spaces of the same finite dimension (these were calculated in Corollaries 3.3.16 and 3.5.2), the induced map is a bijective map of cooperads. □

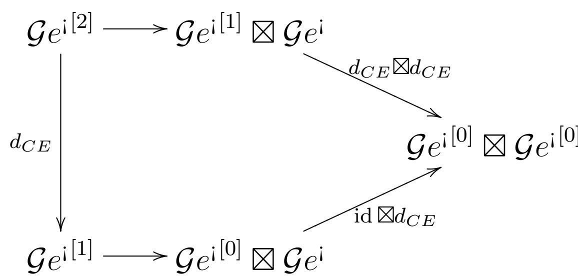
Proof of Proposition 3.6.4. For $\mathrm{F}^{\mathrm{c}}(\psi)$ to range restrict to $\mathcal{H}yc^{\mathsf{i}} = \ker d_{\diamond}$ means that $q\mathcal{BV}^{\mathsf{i}} \to \mathrm{F}^{\mathrm{c}}(\mathbf{\Gamma}) \to \mathrm{F}^{\mathrm{c}}(\mathbf{\Gamma})/\mathcal{H}yc^{\mathsf{i}}$ is zero. By Lemma A.6.3 and Lemma A.6.9, it suffices to show that the further projection to the cogenerators is zero:

$$q\mathcal{BV}^{\mathsf{i}} \longrightarrow \mathrm{F}^{\mathrm{c}}(\mathbf{\Gamma}) \longrightarrow \mathrm{F}^{\mathrm{c}}(\mathbf{\Gamma})/\mathcal{H}yc^{\mathsf{i}} \longrightarrow {\mathrm{F}^{\mathrm{c}}}_{(2)}(\mathbf{\Gamma})/\mathcal{H}yc^{\mathsf{i}}_{(2)}.$$

ψ is only nonzero on elements of $q\mathcal{BV}^{\mathsf{i}}$ made up of trees with exactly one vertex decorated by μ (and none by δ). By homogeneity arguments, only the component of $\mathrm{F}^{\mathrm{c}}(\psi)$ made up of decorated trees with two vertices decorated by μ lands in ${\mathrm{F}^{\mathrm{c}}}_{(2)}(\mathbf{\Gamma})$. For this proof, let a superscript $[k]$ denote the total number of vertices decorated by μ. Then we are looking at the map:

$$\mathcal{G}e^{\mathsf{i}\,[2]} \longrightarrow \mathcal{G}e^{\mathsf{i}\,[1]} \boxtimes \mathcal{G}e^{\mathsf{i}} \longrightarrow \mathbf{\Gamma} \boxtimes \mathbf{\Gamma} \xrightarrow{\ddagger} {\mathrm{F}^{\mathrm{c}}}_{(2)}(\mathbf{\Gamma})$$

where the second factor in $\mathcal{G}e^{\mathsf{i}\,[1]} \boxtimes \mathcal{G}e^{\mathsf{i}}$ has precisely one nontrivial factor. We can factor the first part of this map:



It is necessary to verify that this pentagon commutes. Applying d_{CE} to $\mathcal{G}e^{\mathsf{i}[2]}$ and then splitting it with Δ is the same as splitting with Δ and then applying $\mathrm{id} \boxtimes d_{CE} + d_{CE} \boxtimes \mathrm{id}$. Projecting the former onto $\mathcal{G}e^{\mathsf{i}[0]} \boxtimes \mathcal{G}e^{\mathsf{i}}$ corresponds, in the latter, to only considering splittings into

1. $\mathcal{G}e^{\mathsf{i}[1]} \boxtimes \mathcal{G}e^{\mathsf{i}}$, acted on in the first factor by $d_{CE} \boxtimes \mathrm{id}$, and

2. splittings into $\mathcal{G}e^{\mathsf{i}[0]} \boxtimes \mathcal{G}e^{\mathsf{i}}$, acted on in the second factor by $\mathrm{id} \boxtimes d_{CE}$.

Further composition with $\mathrm{id} \boxtimes d_{CE}$ kills all of the summands of the second type. Then this total composition corresponds to first splitting into $\mathcal{G}e^{\mathsf{i}[1]} \boxtimes \mathcal{G}e^{\mathsf{i}}$, then acting by $d_{CE} \boxtimes \mathrm{id}$, and finally acting by $\mathrm{id} \boxtimes d_{CE}$.

Now, applying d_{CE} to $\mathcal{G}e^{\mathsf{i}[2]}$ lands us in the kernel of d_{CE} and also in $\mathcal{G}e^{\mathsf{i}[1]}$. We have a description, from Lemma 3.4.4, of $\mathcal{G}e^{\mathsf{i}[1]}(n)$; it is spanned by μ_n^{ij}. From the description of μ_n^{ij}, it is clear that its image under d_{CE} is β_n. Therefore, differences $\mu_n^{ij} - \mu_n^{k\ell}$ span the kernel of d_{CE} in $\mathcal{G}e^{\mathsf{i}[1]}$. In fact, differences $\mu_n^{ij} - \mu_n^{jk}$ span the kernel, since

$$\mu_n^{ij} - \mu_n^{k\ell} = (\mu_n^{ij} - \mu_n^{jk}) + (\mu_n^{jk} - \mu_n^{k\ell}).$$

Let us now look to what happens when we split μ_n^{ij}, project into $\mathcal{G}e^{\mathrm{i}\,[0]} \boxtimes \mathcal{G}e^{\mathrm{i}}$, and then apply $\mathrm{id} \boxtimes d_{CE}$. We will get zero on summands where the vertex decorated by μ is in the lower component of the splitting. On the other components, we will get some sort of composition $\beta_r \circ_\ell \beta_{n-r+1}$; in these components, since the vertex marked by μ had to be in the upper component of the splitting, the leaves i and j will both share their edges with the upper vertex of the splitting. Depending on the particulars of the tree, any other combination of leaves is possible. Then if $S(ij)$ varies over subsets of n containing i and j, and σ_{ij} is the unshuffle corresponding to the partition $n = S(ij) \sqcup n \backslash S(ij)$, then we get the total sum

$$\sum_{S(ij)} (\beta_{n-|S(ij)|+1} \circ_1 \beta_{|S(ij)|})\sigma_{ij}.$$

In looking at the difference between this sum and the corresponding sum for $S(jk)$, we see that the terms where S contains i, j, and k cancel, and so yields precisely

$$\sum_{p \in ij|k} (\beta_{n-|p|+1} \circ_1 \beta_{|p|})\sigma_p - \sum_{p \in jk|i} (\beta_{n-|p|+1} \circ_1 \beta_{|p|})\sigma_p.$$

Changing the β_ℓ to μ_ℓ completes the map into $\mathrm{F}^{\mathrm{c}(2)}(\mathbf{\Gamma})$. Further projection to the cogenerators kills this difference, since it is a shifted relation from $\mathcal{H}yc$. □

Proof of Lemma 3.6.5. We shall show that the following square commutes for $n > 1$:

$$\begin{array}{ccc} \mathcal{G}e^{\mathsf{i}}/\operatorname{im} d_{CE} & \xrightarrow{d_{CE}} & \mathcal{G}e^{\mathsf{i}} \\ {\scriptstyle \psi_*}\downarrow & & \downarrow \\ \mathcal{H}yc^{\mathsf{i}} & \longrightarrow & s\mathcal{L}ie^{\mathsf{i}} \boxtimes \mathrm{F}^{\mathsf{c}}(\mu) \end{array}$$

d_{CE} is injective on $\mathcal{G}e^{\mathsf{i}}/\operatorname{im} d_{CE}$ since $\operatorname{im} d_{CE} = \ker d_{CE}$, and the map going down the right side is an isomorphism followed by an inclusion. Therefore the map going along the left and bottom is injective. Then at least the map going along the left, namely ψ_*, must be injective.

Let us verify that the square commutes. We will replace $\mathcal{G}e^{\mathsf{i}}/\operatorname{im} d_{CE}$ with $\mathcal{G}e^{\mathsf{i}}$; since the former is a quotient of the latter, the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{G}e^{\mathsf{i}} & \xrightarrow{d_{CE}} & \mathcal{G}e^{\mathsf{i}} \\ {\scriptstyle \psi}\downarrow & & \downarrow \\ \mathcal{H}yc^{\mathsf{i}} & \longrightarrow & s\mathcal{L}ie^{\mathsf{i}} \boxtimes \mathrm{F}^{\mathsf{c}}(\mu) \end{array}$$

will suffice.

Now, because μ is not in the image of d_{CE}, we can commute the application of d_{CE} with the splitting map and know that d_{CE} must act on the first factor:

$$\begin{array}{ccccc} \mathcal{G}e^{\mathsf{i}} & \xrightarrow{d_{CE}} & \mathcal{G}e^{\mathsf{i}} & \longrightarrow & \mathcal{G}e^{\mathsf{i}} \boxtimes \mathcal{G}e^{\mathsf{i}} \\ \downarrow & & & & \downarrow \\ \mathcal{G}e^{\mathsf{i}} \boxtimes \mathcal{G}e^{\mathsf{i}} & & & & \\ {\scriptstyle d_{CE}\boxtimes \mathrm{id}}\downarrow & & & & \\ \mathcal{G}e^{\mathsf{i}} \boxtimes \mathcal{G}e^{\mathsf{i}} & & \longrightarrow & & s\mathcal{L}ie^{\mathsf{i}} \boxtimes \mathrm{F}^{\mathsf{c}}(\mu) \end{array}$$

In the second factor here, we are merely projecting $\mathcal{G}e^{\text{i}} \to \mathcal{C}om^{\text{i}} \subset \mathrm{F}^{\mathrm{c}}(\mu)$. Equivalently, we could do the following thing: we could apply d_{CE} not as a derivation, but on every vertex; this would kill any summand that was not entirely decorated with μ and turn every μ to a β. Then we could use $\Sigma^{-1}\zeta$ to turn every β to a μ_2 and include the tree decorated by μ_2 into $\mathrm{F}^{\mathrm{c}}(\mu)$.

On the other hand, let us follow the map $\mathcal{G}e^{\text{i}} \to \mathcal{H}yc^{\text{i}} \to s\mathcal{L}ie^{\text{i}} \boxtimes \mathrm{F}^{\mathrm{c}}(\mu)$. This map begins by splitting x in $\mathcal{G}e^{\text{i}}$ into many levels and applies d_{CE} to all of them before turning them into μ_n. However, because we are eventually projecting on the right into $\mathrm{F}^{\mathrm{c}}(\mu)$ which is the image of $\mathrm{F}^{\mathrm{c}}(\mu_2)$, the only important summands in the splittings are those where every factor but the leftmost are just trivalent vertices in ${\mathcal{G}e^{\text{i}}}^{(1)}$. Further, because we are applying d_{CE} to these vertices, they should be decorated by μ.

Then the map is given by application of d_{CE} to each factor in the splitting, followed by some relabelling: on the right, β becomes μ_2 becomes μ and on the left μ_n^{ij} becomes β_n becomes μ_n becomes β_n.

These two processes are thus the same. □

Appendices

Appendix A

Operadic algebra

The purpose of this appendix is to give a concise, general, overview of the basics of operadic algebra from the ground up. Section A.1 describes the structure in a background category which is used to build a category of operads. This is followed by section A.2 which deals with categorical group actions in a monoidal category, and defines operads and cooperads. The historical thread, and more detailed exposition, can be found in the references [23, 22, 19]. Section A.3 defines trees and shows how they form an operad in the category of sets. Section A.4 describes some sufficient conditions on functors between background categories for them to carry operads to operads. The last two sections focus on operads in the category of chain complexes. Section A.5 recalls Koszul duality in operads and section A.6 gives some algebraic propositions in the world of cooperads. These propositions are dual to familiar propositions about operads and algebras, but the "co" versions may seem quite alien. Notationally, throughout, if n is a natural number, the set n will refer to $\{1, \ldots, n\}$ (0 refers to the empty set). This should cause no confusion.

A.1 Monoidal categories

Definition A.1.1. A *monoidal category* is given by the data $(\mathcal{C}, \bullet, I, \alpha, \lambda, \rho)$ where

1. $\mathcal{C}$ is a category,
2. $\bullet$ is a functor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$,
3. I is an object of $\mathcal{C}$,
4. α is an natural isomorphism $\bullet \circ (\bullet \times \mathrm{id}) \to \bullet \circ (\mathrm{id} \times \bullet)$,
5. λ is a natural isomorphism $\bullet \circ (I \times \mathrm{id}) \to \mathrm{id}$, and
6. ρ is a natural isomorphism $\bullet \circ (\mathrm{id} \times I) \to \mathrm{id}$.

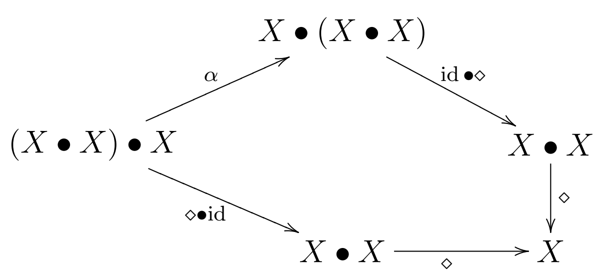
These must satisfy coherence axioms which serve to ensure that there is at most one morphism between an object built out of monoidal products via compositions of products of α, λ, ρ, and the identity.

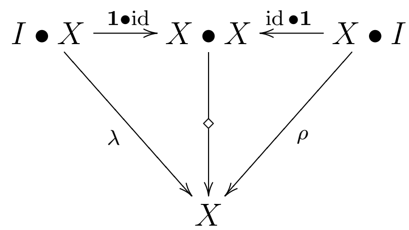
Definition A.1.2. A *monoid* in a monoidal category is a triple $(X, \diamond, \mathbf{1})$ where:

1. X is an object of $\mathcal{C}$,
2. The product $\diamond$ is a morphism $X \bullet X \to X$, and

3. The unit $\mathbf{1}$ is a morphism $I \to X$

so that the following diagrams commute:





Thus a monoid has a unital associative product defined on it.

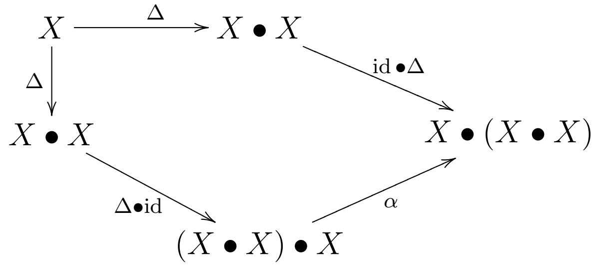
Definition A.1.3. Dually, a *comonoid* in a monoidal category is a triple (X, Δ, η) where:

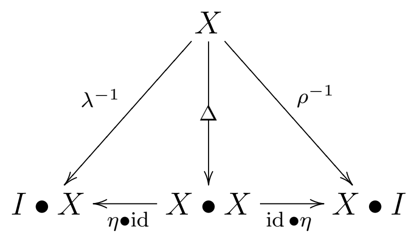
1. X is an object of $\mathcal{C}$,

2. The coproduct Δ is a morphism $X \to X \bullet X$, and

3. The counit η is a morphism $X \to I$

so that the following diagrams commute:





Thus a comonoid has a counital coassociative coproduct defined on it.

Definition A.1.4. A *distributive symmetric monoidal category* is given by the data $(\mathcal{C}, \bullet, I, \sqcup, \alpha, \lambda, \rho, \pi)$ where

1. $(\mathcal{C}, \bullet, I, \alpha, \lambda, \rho)$ is a monoidal category with finite limits, including
2. a choice of functorial coproduct $\sqcup : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
3. π is a natural isomorphism $\bullet \to \bullet \circ \sigma$ where σ acts on $\mathcal{C} \times \mathcal{C}$ by permuting factors, and
4. the canonical distributivity map

   $$(A \bullet B) \sqcup (A \bullet C) \to A \bullet (B \sqcup C)$$

   is an isomorphism.

These must satisfy even more coherence conditions, which serve to ensure that there is at most one way to get between one object built out of monoidal products and coproducts to another such object via compositions of products of α, λ, ρ, π, σ, canonical distributivity maps (and their inverses) and the identity.

In the following examples, we will specify the data $(\mathcal{C}, \bullet, I, \sqcup)$ and suppress the natural isomorphisms. When we use $\bullet$ without parenthesization, we mean, for concreteness, the parenthesization from left to right; in fact, any choice would be uniquely isomorphic to this one via α.

Lemma A.1.5. *The following are distributive symmetric monoidal categories:*

1. $(\mathbf{Set}, \times, \{*\}, \sqcup)$.

2. $(\mathbf{Top}, \times, \{*\}, \sqcup)$. *Here* $\mathbf{Top}$ *could be any of a number of topological categories containing the singleton which is closed under finite colimits and Cartesian products.*

3. $(\mathbf{grVect_k}, \otimes, \mathbf{k}_0, \oplus)$*,* $\mathbb{Z}$*-graded vector spaces over a field* $\mathbf{k}$ *of characteristic zero, with tensor product and direct sum. Here* $\mathbf{k}_0$ *is the one dimensional vector space concentrated in degree* 0*.*

4. $(\mathbf{Chain_k}, \otimes, \mathbf{k}_0, \oplus)$*,* $\mathbb{Z}$*-graded chain complexes over* $\mathbf{k}$*.*

Remark (Koszul signs). The permutation isomorphism π in graded vector spaces and chain complexes is not $x \otimes y \mapsto y \otimes x$, but rather the signed permutation $x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$, where $|x|$ and $|y|$ denote the grading of x and y.

A.2 Groups

Definition A.2.1. Let $\mathcal{C}$ be a category and G a group, viewed as a category with one element and morphisms G. A *left G-module* in $\mathcal{C}$ is a functor $G \to \mathcal{C}$. That is, the data is given by an object X in $\mathcal{C}$ and a homomorphism from G into the automorphism group of X. A *right G-module* is a left G-module over the group

G^{op}; a *G-H-bimodule* is a left G-module over the group $G \times H^{op}$. Often we will suppress the homomorphism and refer to a G-module by the object X.

G-modules form a category whose morphisms are natural transformations. In this context, a natural transformation between G-modules X and Y is a $\mathcal{C}$-morphism $X \to Y$ which is equivariant with respect to G.

Definition A.2.2. If $\mathcal{C}$ is a category with finite colimits, G is a finite group, and X is the object of the left G-module $G \to \mathcal{C}$, then the *quotient* X/G is the colimit of the functor $G \to \mathcal{C}$. In practice, this object is a model for the quotient of X by the action of G. It accepts a morphism $X \to X/G$ so that the compostion $X \xrightarrow{g_*} X \to X/G$ is the same morphism for all $g \in G$.

Example. If $\mathcal{C}$ is one of the categories of Lemma A.1.5, then a G-module is a set, space, vector space, or chain complex with a group action, and X/G is modeled by the ordinary quotient by that group action.

Example. Let $(\mathcal{C}, \bullet, I, \sqcup)$ be a distributive symmetric monoidal category. Then $\underbrace{X \bullet \cdots \bullet X}_{n}$ is a left $\mathbb{S}_n$-module in $\mathcal{C}$. The structure morphisms are given by combinations of π and α that permute the factors of the monoidal product according to the permutations of $\mathbb{S}_n$.

For the rest of this section, let $(\mathcal{C}, \bullet, I, \sqcup)$ be a distributive symmetric monoidal category, and let S be a finite set.

Definition A.2.3. Suppose S is a G-H-bimodule in the category of finite sets. Then the object $\widetilde{S}$ of $\mathcal{C}$ defined as

$$\widetilde{S} = \coprod_S I$$

is a G-H-bimodule in $\mathcal{C}$. The structure is provided by a homomorphism from $G \times H^{op}$ to the automorphisms of $\widetilde{S}$. Let g, h, and s be elements of G, H, and S, respectively. This homomorphism takes $(g, h) \in G \times H^{op}$ to the automorphism which takes the factor I_s in the coproduct to the factor I_{gsh} via the identity map on I. In particular, the group G is a G-G-bimodule in the category of sets, so $\widetilde{G}$ is a G-G-bimodule in $\mathcal{C}$.

Definition A.2.4. Let X be a right G-module and Y be a left G-module in $\mathcal{C}$. Then $X \bullet Y$ is a left G-module; the action is given by acting on X by g^{-1} and Y by g. Define

$$X \bullet_G Y = (X \bullet_G Y)/G.$$

If X is a H-G-bimodule and Y a G-K-bimodule in $\mathcal{C}$, then $X \bullet_G Y$ is an H-K-bimodule in $\mathcal{C}$.

Example. In the categories of Lemma A.1.5, $\widetilde{G}$ corresponds to the discrete group G or the group algebra $\mathbf{k}[G]$. The notion of $\bullet_G$ corresponds with the ordinary notion of the product over a group or the tensor product over the group algebra.

Definition A.2.5. The *symmetric group of* S, $\mathbb{S}_S$, is the group of automorphisms of S; $\mathbb{S}_S$ is noncanonically isomorphic to $\mathbb{S}_{|S|}$. $\mathbb{S}_S$ acts on the left by permutation on tuples $\{x_s\}_{s\in S}$ indexed by S. Let $\sigma \in \mathbb{S}_{|S|}$; then the element indexed by s in $\sigma\{x_s\}$ is $x_{\sigma^{-1}s}$. In particular, $\mathbb{S}_n$ acts on $(x_1, \ldots, x_n)$ by

$$\sigma(x_1, \ldots, x_n) = (x_{\sigma^{-1}1}, \ldots, x_{\sigma^{-1}n}).$$

Lemma A.2.6. *Let S' be of the same order as S. The set of isomorphisms from S' to S is an $\mathbb{S}_S$-$\mathbb{S}_{S'}$-bimodule in the category of sets. We call this bimodule* $\mathrm{Iso}(S', S)$.

Definition A.2.7. The *category of collections in* $\mathcal{C}$, $\mathcal{C}_{\mathbb{S}}$, has as objects collections $\{X(n)\}_{n\in\mathbb{N}}$ where $X(n)$ is a right $\mathbb{S}_n$-module in $\mathcal{C}$. A morphism is a collection of morphisms of $\mathbb{S}_n$-modules.

Definition A.2.8. Let $X = \{X(n)\}$ be a collection. Then $X(S)$ is the quotient

$$X(|S|) \bullet_{|S|} \mathrm{Iso}(|S|, S).$$

There is a similar construction on left $\mathbb{S}_n$ modules which is useful enough that we will record it explicitly:

Definition A.2.9. Let X be an object of $\mathcal{C}$. We define $X^{\bullet S}$ as the left $\mathbb{S}_S$-module

$$\left[\mathrm{Iso}\widetilde{(S, |S|)} \bullet_{\mathbb{S}_{|S|}} (\underbrace{X \bullet \cdots \bullet X}_{|S|})\right] / \mathbb{S}_{|S|}.$$

Remark. 1. $X^{\bullet S}$ is noncanonically isomorphic to $X^{\bullet |S|}$. This construction thus lets us take monoidal products over arbitrary finite sets.

2. If $F : S \to \mathcal{C}$ is a functor, we could consider a twisted version of this construction $F^{\bullet S}$; in this case, we quotient

$$\coprod_{\sigma \in \mathrm{Iso}(|S|,S)} (F\sigma 1 \bullet F\sigma 2 \bullet \cdots \bullet F\sigma |S|)$$

by a similar antidiagonal $\mathbb{S}_{|S|}$ action.

Definition A.2.10. Let $P_k(n)$ be the set of ordered partitions $p = (p_1, \ldots, p_k)$ of n of length k. Define the functor $\boxtimes : \mathcal{C}_{\mathbb{S}} \times \mathcal{C}_{\mathbb{S}} \to \mathcal{C}_{\mathbb{S}}$ as

$$(X \boxtimes Y)(n) = \coprod_{k \geq 0} X(k) \bullet_{\mathbb{S}_k} \left[\coprod_{p \in P_k(n)} \Big(Y(p_1) \bullet \cdots \bullet Y(p_n) \bullet_{\mathbb{S}_{p_1} \times \cdots \times \mathbb{S}_{p_k}} \widetilde{\mathbb{S}_n} \Big) \right].$$

Let $\mathbf{I}$ be the object of $\mathcal{C}_{\mathbb{S}}$ with $I(1) = I$ and $I(n) = 0$ otherwise.

For this definition to make sense, we need to discuss a few of the module structures involved. The right $\mathbb{S}_{p_i}$-module structures on $Y(p_i)$ induce the right $\mathbb{S}_{p_1} \times \cdots \times \mathbb{S}_{p_k}$-module structure on their monoidal product.

We can map $p_1 \sqcup \cdots \sqcup p_k$ to n bijectively. This induces an inclusion of groups $\mathbb{S}_{p_1} \times \cdots \times \mathbb{S}_{p_k} \hookrightarrow \mathbb{S}_n$; this inclusion gives $\widetilde{\mathbb{S}_n}$ the structure of a left $\mathbb{S}_{p_1} \times \cdots \times \mathbb{S}_{p_k}$-module (via left multiplication).

Finally, we need to describe the left $\mathbb{S}_k$-module structure. This will be built out of constituent pieces. One is given by taking the induced map that comes from

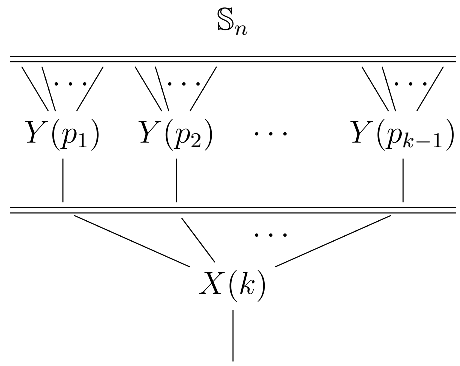


Figure A.1: A picture of a summand of $X \boxtimes Y$

$\mathbb{S}_k$ acting on the partition $(p_1, \ldots, p_k)$ by permuting its factors and on the product $Y(p_1) \bullet \cdots \bullet Y(p_k)$ by permuting its factors in the same manner. The other is given by using a map $\mathbb{S}_k \to \mathbb{S}_n$ and then acting on $\mathbb{S}_n$ by left multiplication. The map is given by acting on blocks of size p_1 through p_k in n.

Proposition A.2.11. $(\mathcal{C}_{\mathbb{S}}, \boxtimes, \mathbf{I})$ *is a monoidal category.*

Definition A.2.12. An *operad* $\mathcal{O}$ in the category $\mathcal{C}$ is a monoid in the monoidal category $\mathcal{C}_{\mathbb{S}}$. A *cooperad* $\mathbf{C}$ is a comonoid in $\mathcal{C}_{\mathbb{S}}$.

Proposition A.2.13. *The data of an operad $\mathcal{O}$ can be specified by giving*

1. *The operation objects $\mathcal{O}(n)$ with their right $\mathbb{S}_n$-module structures,*

2. *The morphism $\mathbf{1}$, which is specified by a $\mathcal{C}$-morphism $I \to \mathcal{O}(1)$, and*

3. *Partial composition maps $\circ_i : \mathcal{O}(m) \bullet \mathcal{O}(n) \to \mathcal{O}(m+n-1)$ for $1 \leq i \leq m$*

which should satisfy well-known associativity, equivariance, and identity constraints [22].

In practice, we will specify these data rather than the full structure maps $\diamond$ and $\mathbf{1}$ of the monoid.

A.3 Trees

Trees are a useful device in understanding operads and operad compositions. Compositions are indexed by trees and the free operad and cofree cooperad are modelled by operads of trees. There is a lot of vivid horticultural vocabulary involved in this section of the appendix, along with a few lemmas near the end made use of in the earlier chapters.

Definition A.3.1. A rooted *tree* $T = (L, V, N)$ (all trees will be rooted and the adjective rooted will be omitted) consists of the following data:

1. Finite sets L of leaves and V of vertices.
2. A surjective trunk or successor map $N : L \sqcup V \to V \sqcup \{\mathrm{out}\}$ satisfying the following:
   (a) for every vertex v there exists a height $h(v)$ so that $N^{h(v)}(v) = \mathrm{out}$
   (b) $N^{-1}(\mathrm{out})$ is a singleton, which will be called $*$ or the *root*.

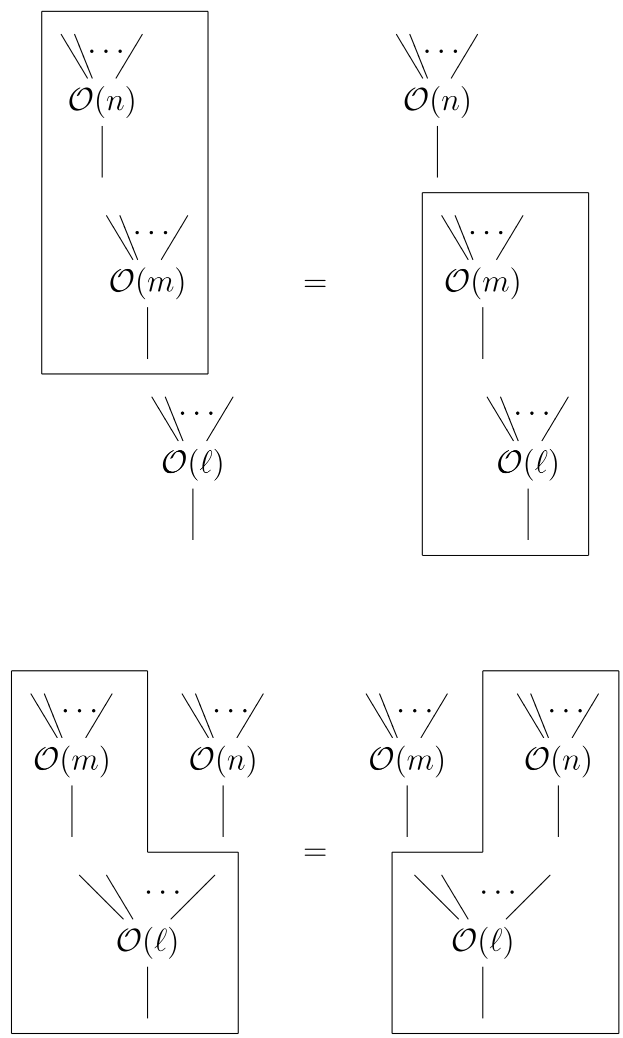


Figure A.2: Illustrations of both types of associativity relation for $\circ_i$

Trees with leaf set L are called L-trees.

Remark. Every L-tree is uniquely isomorphic to an L-tree where the set of vertices V_{fixed} is a subset of $\wp(L) \times \mathbb{N}$ constructed in the following manner. Let (L, V, N) be an L-tree. Let $\tilde{v} = (L_v, h(v))$, where $L(v)$ is the set of leaves $\ell \in L$ such that $N^k(\ell) = v$ for some k and $h(v)$ is the height of v. This turns out to give a unique pair to each vertex, and the bijection $v \leftrightarrow \tilde{v}$ induces a trunk map N. This convention allows us to avoid set theory problems. Modifying a tree or changing L may cause us to leave the universe we have constructed, but there is always a unique representative tree of this form uniquely isomorphic to the one we have created. Therefore, we will perform our constructions with impunity, suppressing the isomorphisms to our representatives.

Remark. Note that surjectivity of N implies that L is nonempty. Note also that there exist trees, called *trivial trees*, with one leaf and no vertices. All other trees are called proper, and $*$ is a vertex in a proper tree. Note also that the sets L and V can be recovered from $S = L \sqcup V$ and $N : S \to S \sqcup \{\text{out}\}$; L are the elements of S not in the image of N and V are the image of S with the point out removed.

Definition A.3.2. 1. An *edge* of a tree T is a pair (a, b) with $N(a) = b$. Here a and b are elements of $L \sqcup V \sqcup \{\text{out}\}$. The set of edges of T is denoted $E(T)$. If e is an edge, then $in(e) = a$ and $out(e) = b$.

2. An *internal edge* of a tree is an edge such that a and b are both vertices. All other edges are *external edges*.

3. An *incoming edge* of a vertex v is an edge (a, v); the (unique) *outgoing edge* is the edge $(v, N(v))$. The set of incoming (respectively incoming internal or external) edges is denoted $in(v)$ (respectively $in^i(v)$ or $in^e(v)$). The set of all incoming and outgoing edges of v is $E(v)$.

4. A *leaf vertex* is a vertex with at least one incoming external edge, that is, a vertex in the image of L under N. The leaf vertex of the leaf ℓ is $N(\ell)$.

5. The *rank* of a tree $rk(T)$ is the number of vertices.

6. The *arity* of a tree, $|T|$, is the number of leaves.

7. The *valence* $\mathrm{val}(v)$ of a vertex v of a tree is $|E(v)|$, the number of edges of v. The *arity* $|v|$ of v is $|in(v)|$, the number of incoming edges of v. The arity is always one less than the valence.

Remark. We will often draw pictures to describe trees, which will be something like the pictures of Figure A.3.

Definition A.3.3. 1. If $E = \{(a_e, b_e)\}$ is a set of internal edges of the tree $T = (L, V, N_T)$ then the *edge contraction* is the tree T_E with leaf set L, vertex set $V \backslash \{a_e\}$, and trunk map $N_{T_E}(v) = N_T^{k(v)}(v)$ where $k(v)$ is the

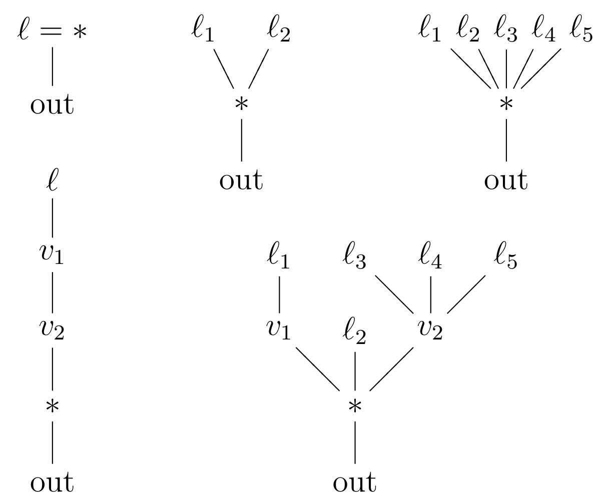


Figure A.3: Pictoral representations of five trees, including the trivial tree, the 2-corolla, and the 5-corolla. The trunk map N is indicated by the downward flow along line segments.

minimal positive integer such that $N_T^{k(v)}(v)$ is not in $\{a_e\}$. Alternatively, the vertices can be taken to be equivalence classes of vertices of T, where $v \sim v'$ if there is a sequence of edges in E connecting them. The vertices that are proper equivalence classes are called *contraction vertices*.

2. If W is a set of bivalent vertices of the tree $T = (L, V, N_T)$ which does not include the root, then the tree $T_W = (L, V_W, N_W)$ obtained by *forgetting the vertices* W is the edge contraction T_E where $E = \{(w, T(w))\}$ for all $w \in W$. If $W_* = W \sqcup \{*\}$ is a set of bivalent vertices including the root vertex then the tree T_{W_*} obtained by forgetting the vertices W_* has leaf set

L, vertices $V_W \backslash \{*\}$, and $N_{W_*} = N_W$ except that $N_{W_*}(N_W^{-1}(*)) = \mathrm{out}$. If v is a bivalent vertex, then T_v means $T_{\{v\}}$.

Definition A.3.4. To avoid confusion, we describe three partial orders having to do with trees. We will extensively use the first one, never use the second, and occasionally use the third.

1. For a fixed tree (L, V, N), there is a partial order on vertices and leaves induced by N, where $v \gg u$ if $N^k(v) = u$ for some $k \geq 0$. Comparisons for this partial order will use words like "higher," "lower," "above," and "below," even though it is not the same as:

2. The partial order on vertices (which can be extended to leaves) induced by the height function h, which is defined so that $h(v)$ is the number so that $N^{h(v)}(v) = \mathrm{out}$.

3. For a fixed set L, there is a partial order on trees (L, V, N) given by edge contraction. That is, $(L, U, M) \succ (L, V, N)$ if there is a finite sequence of edge contractions from (L, U, M) to (L, V, N).

Remark. Note that the order $\gg$ has a least element, the vertex (or leaf) $*$. For the third order, there is a least tree with leaf set L for $\succ$ up to isomorphism, represented by $(L, *, L \mapsto * \mapsto \mathrm{out})$, called the *L-corolla*. If $|L| = 1$ then this is only true if we ignore the trivial tree with leaf set L, which is incomparable with all other trees.

Definition A.3.5. Let the *stock* and *scion* be trees $T_{\text{stock}} = (L_{\text{stock}}, V_{\text{stock}}, N_{\text{stock}})$ and $T_{\text{scion}} = (L_{\text{scion}}, V_{\text{scion}}, N_{\text{scion}})$. The tree $T_{\text{graft}} = T_{\text{stock}} \ddagger_\ell T_{\text{scion}}$ obtained by *grafting* the scion to the stock at the leaf ℓ of T_{stock} has:

1. The leaves of T_{graft} are $L_{\text{stock}} \sqcup L_{\text{scion}} \backslash \{\ell\}$.

2. The vertices of T_{graft} are $V_{\text{stock}} \sqcup V_{\text{scion}}$.

3. The trunk map of T_{graft} takes the root of T_{scion} to $N_{\text{stock}}(\ell)$; on every other vertex and leaf, it is given by whichever of N_{stock} and N_{scion} is appropriate.

The edge from the root of T_{scion} to $N_{\text{stock}}(\ell)$ is called the *grafting edge*.

Definition A.3.6. There is a *cutting* operation on trees. Let $T = (L, V, N)$ be a tree and $E = \{(a_e, b_e)\}$ be a set of edges of T such that the a_e are pairwise incomparable under the partial order $\gg$. The stock tree $T_{\text{stock}} = (L_{\text{stock}}, V_{\text{stock}}, N_{\text{stock}})$ and scion trees $T^e_{\text{scion}} = (L^e_{\text{scion}}, V^e_{\text{scion}}, N^e_{\text{scion}})$ obtained by cutting T along E are as follows:

1. The leaves of T^e_{scion} are all the leaves above a_e in T, including a_e if it is a leaf.

2. The vertices of T^e_{scion} are all the vertices above a_e in T, including a_e if it is a vertex.

3. The trunk map of T^e_{scion} coincides with the trunk map of T except that $N_{\text{scion}}(a_e) = \text{out}$.

and

1. The leaves of T_{stock} are:
$$L_{\text{stock}} = L \backslash \left(\bigcup_e L^e_{\text{scion}} \right) \sqcup \bigsqcup_e \{a_e\}.$$
The a_e are called the *cut leaves*.

2. The vertices of T_{stock} are $V \backslash \bigcup V^e_{\text{scion}}$.

3. The trunk map of T_{stock} coincides with the trunk map of T except that $N_{\text{stock}}(a_e) = b_e$.

Lemma A.3.7. *Grafting and cutting are basically inverse operations:*

1. *If T is a tree with edge (a, b), then cutting at the single edge (a, b) and grafting the resulting scion to the resulting stock along the cut leaf yields T.*

2. *If T_{stock} and T_{scion} are trees, with ℓ a leaf of T_{stock}, then grafting T_{scion} to T_{stock} along ℓ and then cutting along the grafting edge gives T_{scion} as the*

scion of the cutting and a tree T'_{stock} as the stock which differs from T_{stock} only in that the leaf ℓ has been renamed $(L_{\mathrm{scion}}, h(\ell))$ by our vertex naming conventions.

Definition A.3.8. Let T be a tree. A set of edges $E = \{a_e, b_e\}$ is called a *pollard set* if

1. The a_e are pairwise incomparable under $\gg$, and

2. Every leaf is above or equal to some a_e under $\gg$.

Cutting allong a pollard set is called *pollarding*.

Definition A.3.9. The tree obtained by inserting k vertices on the edge (a, b) of the tree $T = (L, V, N)$ has:

1. Leaf set L

2. Vertex set $V \sqcup \{v_1, \ldots, v_k\}$

3. Trunk map which takes a to v_1, v_i to v_{i+1}, v_k to b, and acts as N otherwise.

This can also be described by cutting T along (a, b), and then grafting the scion to k copies of the 1-corolla and thence to the stock along the cut leaf.

Definition A.3.10. Let $\mathcal{O}$ be an operad and $T = (n, V, N)$ be an n-tree. *Composition along* T is a map

$$\prod_{v \in V} \mathcal{O}(in(v)) \to \mathcal{O}(n)$$

defined by iterated compositions in a pattern governed by T.

To be precise, let $M = \max\{h(i) : i \in n\}$ be the maximum height of any leaf of T. Let T' be the tree constructed by inserting $M - h(i)$ vertices on each leaf edge $(i, N(i))$. Recursively, beginning at height one, choose an order $(v_{k,1}, \ldots, v_{k,m_k})$ for the vertices of T' of height k. Choose this order so that a comes before b if $N(a)$ comes before $N(b)$. Extend this order to the leaves of T', which has been constructed so that all the leaves are at height M. This order induces an isomorphism of $in(v)$ with $|in(v)|$, which we can use to identify $\mathcal{O}(in(v))$ with $\mathcal{O}(|in(v)|)$. Now we can use the composition map $\diamond$ of the operad to compute the product in $\mathcal{O}$ of

$$\mathcal{O}(|v_{1,1}|) \bullet_{\mathbb{S}_{|v_{1,1}|}} \left[\mathcal{O}(|v_{2,1}|) \bullet \cdots \bullet \mathcal{O}(|v_{2,m_2}|) \bullet_{\mathbb{S}_{|v_{2,1}|} \times \cdots \times \mathbb{S}_{|v_{2,2}|}} \widetilde{\mathrm{id}}\right].$$

Recursively, we can continue up T', composing the resultant element of $\mathcal{O}$ with the vertices of the next height, until at the leaves we finally compose with the permutation of $\widetilde{\mathbb{S}_n}$ induced by comparing the leaves n, which have an intrinsic ordering, to the ordering they inherit from the ordering on the vertices.

Typically, $\mathcal{C}$ will be a concrete category, so $\mathcal{O}(in(v))$ will be a set with additional structure. In this case, if $x_v \in \mathcal{O}(in(v))$ for $v \in V$, then the *composition of* $\{x_v\}$ *along* T is the image of $\prod x_v$ under composition along T. We call x_v the *label of* v in the composition.

Lemma A.3.11. *Composition along* T *is independent of the choice of order.*

Definition A.3.12. The *operad of trees* $\mathcal{T}$ in the category of sets is given by the following data:

1. $\mathcal{T}(n)$ is the set of n-trees. Our vertex-naming convention makes this a set, rather than a proper class. The right $\mathbb{S}_n$-module structure is given as follows. Let $\sigma \in \mathbb{S}_n$. Then σ takes the tree (n, V, N) to the tree with leaves n, vertices V, and trunk map N on vertices and $N\sigma$ on leaves.

2. The composition map $\mathcal{T} \boxtimes \mathcal{T} \to \mathcal{T}$ is given by grafting: The image of $T_k \bullet_{\mathbb{S}_k} T_{p_1} \bullet \cdots \bullet T_{p_k}$ is an n-tree isomorphic to the tree formed by grafting each of T_{p_i} to T_k at the leaf i. The leaves of the grafted tree are identified with n by mapping p_1 to $\{1, \ldots, p_1\}$ in an order-preserving way, mapping p_2 to $\{p_1 + 1, \ldots, p_1 + p_2\}$ in an order preserving way, and so on.

3. the image of $\mathbf{1}$ is the trivial 1-tree.

This definition respects the various quotients and products involved in $\boxtimes$. As a simpler alternative to the second datum, we can define just the partial compositions $\circ_i$. Let $T_m \in \mathcal{T}(m)$, $T_n \in \mathcal{T}(n)$. Then $T_m \circ_i T_n$ is an $(m+n-1)$-tree isomorphic to $T_m \ddagger_i T_n$. The leaves $(m+n-1)$ are associated with the leaves of the grafted tree by identifying n with $\{i, \ldots, i+n-1\}$ and $\{1, \ldots, \hat{i}, \ldots, m\}$ with $\{1, \ldots, i-1, i+n, \ldots, m+n-1\}$ in an order preserving manner.

There is not a cooperad of trees in the category of sets, but there is a version in the categories of graded vector spaces or chain complexes over $\mathbf{k}$:

Definition A.3.13. The *cooperad of trees* $\mathbf{T}$ in the category of graded vector spaces or chain complexes is given by the following data:

1. $\mathbf{T}(n)$ is the free vector space on the set of all n-trees. The right $\mathbb{S}_n$-module structure is induced by the one on $\mathcal{T}(n)$.

2. The decomposition map can be described in terms of the pollarding operation. Let $T \in \mathbf{T}(m+n-1)$, and let E be a pollard set of T. Cutting along E yields a stock tree and some number of scion trees. We make a choice to put the scion trees in some order. For instance, we can build an order iteratively by choosing whichever scion contains the lowest numbered remaining leaf of $(m+n-1)$. Now we can refer to the scion trees as $L^1_{\text{scion}}, \ldots, L^k_{\text{scion}}$, where k is the number of edges in E. Rename the corresponding leaves of the stock, so that the edge j is the cut leaf corresponding to L^j_{scion}.

   Next, we can make another choice to identify the leaves of the scion tree

L^j_{scion} with their order $|T^j_{\text{scion}}|$. For instance, the leaves of L^j_{scion} are a subset of $(m+n-1)$ and we can map them in the unique order-preserving manner. Build an isomorphism σ_E of $(m+n-1)$ as follows. If the leaf i of T has been pollarded and renamed into leaf ℓ_i of the scion j_i, then

$$\sigma_E(i) = \ell_i + \sum_{j<j_i} |T^j_{\text{scion}}|.$$

Then we can describe one summand of ΔT as

$$T_{\text{stock}} \otimes_{\mathbb{S}_k} \left[\left(T^1_{\text{scion}} \otimes \cdots \otimes T^k_{\text{scion}}\right) \otimes_{\mathbb{S}_{|T^1_{\text{scion}}|} \times \cdots \times \mathbb{S}_{|T^k_{\text{scion}}|}} \sigma_E \right].$$

We made choices of the ordering of the scion trees and the reordering of their leaves. Having made a different choice would have led to a different construction of σ_E and would give a different representative for the same element of $\mathbf{T} \boxtimes \mathbf{T}$.

Now the full decomposition map is given by summing this formula over all pollard sets. That is, $\Delta(T)$ is given by:

$$\sum_{E \text{ a pollard set}} T_{\text{stock}} \otimes_{\mathbb{S}_k} \left[\left(T^1_{\text{scion}} \otimes \cdots \otimes T^k_{\text{scion}}\right) \otimes_{\mathbb{S}_{|T^1_{\text{scion}}|} \times \cdots \times \mathbb{S}_{|T^k_{\text{scion}}|}} \sigma_E \right].$$

3. The counit map η is projection onto the trivial 1-tree.

Definition A.3.14. Let $\mathcal{C}$ be a category. A *$\mathcal{C}$-decorated tree* is either

1. a pair $(T, \{X_v\})$ where $T = (L, V, N)$ and X_v is a right $\mathbb{S}_{in(v)}$-module in $\mathcal{C}$ for $v \in V$.

2. a pair $(T, \{X_v\})$ where $T = (L, V, N)$ is a tree and X_v is a right $\mathbb{S}_{in^e(v)} \times \mathbb{S}_{in^i(v)}$-module in $\mathcal{C}$ for $v \in V$.

The first type is an example of the second type, since every right $\mathbb{S}_{S \sqcup S'}$ action induces a right $\mathbb{S}_S \times \mathbb{S}_{S'}$ action. In either case, T is the *underlying tree* of the pair.

Lemma A.3.15. *Let $\mathcal{C}$ be a distributive symmetric monoidal category. Let $S \mapsto X_S$ (or $(S, S') \mapsto X_{S,S'}$) be an assignment from finite sets to right $\mathbb{S}_S$-modules (right $\mathbb{S}_S \times \mathbb{S}_{S'}$-modules) in $\mathcal{C}$ which is functorial with respect to isomorphisms of finite sets (or pairs of finite sets). Then decorated trees as described above can be given an operad structure that descends from the operad structure on trees.*

Proof. Let $\mathcal{T}_D(n)$ consist of decorated n-trees as described above. For ease, we will work with $\mathbb{S}_S$ modules; the case for $\mathbb{S}_S \times \mathbb{S}_{S'}$ modules is similar. More precisely,

$$\mathcal{T}_D(n) = \coprod_{n\text{-trees } (n,V,N)} \prod_{v \in V} X_{in(v)}.$$

where $\prod_{v \in V} X_{in(v)}$ is the monoidal product over the finite set V.

We can define an $\mathbb{S}_n$ action on $\mathcal{T}_D(n)$ by extending the $\mathbb{S}_n$ action on $\mathcal{T}(n)$. Namely, if $\sigma \in \mathbb{S}_n$, then σ induces an isomorphism σ_V of vertices (remember our vertex naming convention) as well as leaves, and thus induces an isomorphism $\sigma_* : in(v) \to in(\sigma_V v)$. By functoriality, this induces a morphism $X_{in(v)} \to X_{in(\sigma_V v)}$. □

Lemma A.3.16. *Let $\mathcal{C}$ is the category of graded vector spaces or graded chain complexes. Let $S \mapsto X_S$ be an assignment as in Lemma A.3.15. Then decorated trees as described above can be given a cooperad stucture that descends from the cooperad structure on trees.*

Proof. Just as in the previous lemma,

$$\mathbf{T}_D(n) = \bigoplus_{n\text{-trees } (n,V,N)} \bigotimes_{v \in V} X_{in(v)}.$$

with the same $\mathbb{S}_n$ action. Decomposition is by sums of pollardings, and the counit is projection onto the trivial 1-tree. □

Remark. In the category of graded vector spaces or graded chain complexes, there are signs involved in operads and cooperads of decorated trees. Technically, they have been dealt with by our Koszul sign rule on the permutation isomorphism and our definition of the product over an arbitrary finite set. In practice, one often chooses an arbitrary ordering of the vertices of a tree as a representative and then performs explicit sign calculations using the Koszul sign rule on an ordered tensor product.

Remark. In the linear categories, since the underlying vector spaces or chain complexes of the cooperad of decorated trees are the same as those of the operad of decorated trees, we can unambiguously refer to elements of the cooperad of decorated trees in terms of the composition map and partial composition maps of the operad of decorated trees.

Remark. Sometimes one wants to consider operad structures that modify the operad structure on trees in some way, for instance by restricting the isomorphism classes of trees allowed, weakening the functoriality of the assignment of decorations, or by changing the particulars of the composition map. To specify a *tree-like* operad structure on a collection whose underlying space $X(n)$ is a coproduct of decorated n-trees means to give a pseudo-composition map $\circ_i^T$ so that the composition of decorated trees with underlying trees $T_m \in \mathcal{T}(m)$ and $T_n \in \mathcal{T}(n)$ has leaves in bijection with the leaves of $T_m \ddagger_i T_n$. To define the actual operad composition map $\circ_i$, this should be relabeled to make it an $(m+n-1)$-tree by the bijection used in $\mathcal{T}$. It is assumed that the $\mathbb{S}_n$ action and the monoidal unit will be as in the category $\mathcal{T}_D$ unless otherwise specified.

A.4 Functors of operads

Definition A.4.1. A *monoidal functor* between two distributive symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a pair (F, ψ), where

1. F is a functor $\mathcal{C} \to \mathcal{D}$ which respects finite coproducts and the monoidal unit, and

2. ψ is a natural transformation $\bullet_{\mathcal{D}} \circ (F \times F) \to F \circ \bullet_{\mathcal{C}}$.

Example. Let C be the functor from $\mathbf{Top}$ to $\mathbf{Chain_k}$ which assigns to a space its singular cubical chain complex over $\mathbf{k}$. That is, a space X is taken by C to, in degree n, the free $\mathbf{k}$-module on maps from the standard n-cube $[0,1]^n$ into X which have some dependence on the final factor. A map $X \to Y$ of spaces composes with maps of $[0,1]^n \to X$ to give maps $[0,1]^n \to Y$. This respects coproducts because of basic properties of connectedness. Finally, $C_m(X) \otimes C_n(Y)$ maps naturally into $C_{m+n}(X \times Y)$ by

$$f \otimes g \mapsto f \times g.$$

Example. Let H be the homology functor on chain complexes. This respects $\oplus$ and $H(C \otimes D) \cong H(C) \otimes H(D)$ naturally over $\mathbf{k}$.

Any functor between two distributive symmetric monoidal categories induces a functor between their categories of collections. More is true.

Proposition A.4.2. *The induced functor respects* $\boxtimes$ *and* $\mathbf{I}$.

This means that the singular cubical chain complex of a topological operad is an operad in the category of chain complexes, and that the homology of a chain operad is an operad in the category of graded vector spaces.

A.5 Resolutions and Koszul duality

Koszul duality is a method for getting relatively small resolutions of "nice enough" operads. References for this section include [14, 13, 8, 19]. For this section and the next, let $\mathcal{C}$ be the category of chain complexes over $\mathbf{k}$.

Definition A.5.1. An *augmentation* of an operad $\mathcal{O}$ in $\mathcal{C}$ is a splitting $\eta : \mathcal{O} \to \mathbf{I}$ of the monoidal unit $\mathbf{1}$. A *coaugmentation* of a cooperad $\mathbf{C}$ is a splitting $\mathbf{1} : \mathbf{I} \to \mathbf{C}$ of the counit η.

For this section and the next, unless otherwise remarked, all operads will be augmented and all cooperads will be coaugmented, and morphisms will preserve the augmentation or coaugmentation.

Remark. This technical condition ensures that we cannot discuss operads which govern unital algebras of any kind. The restriction to the augmented and coaugmented case can be avoided [16] but assuming it streamlines the presentation.

Definition A.5.2. A *pointed collection X is a collection* $\{X(0), X(1), X(2), \ldots\}$ *equipped with the usual $\mathbb{S}_n$ action on X_n along with maps*

$$I \underset{\eta}{\overset{\mathbf{1}}{\rightleftarrows}} X(1)$$

so that $\eta\mathbf{1}$ is the identity.

The reduced collection $\overline{X}$ is the kernel of η.

Definition A.5.3. Let $\mathbf{C}$ be a cooperad. The reduced coproduct is

$$\overline{\Delta} = (\mathrm{id} \bullet \mathrm{id} - \eta \bullet \mathrm{id} - \mathrm{id} \bullet \eta + \eta \bullet \eta) \circ \Delta.$$

A cooperad is *conilpotent* if every element in it is killed by some iterated power of $\overline{\Delta}$.

Lemma A.5.4. *There is a forgetful functor* U *from the category of augmented operads (coaugmented conilpotent cooperads) to the category of pointed collections of graded vector spaces, which has a left adjoint* F *(right adjoint* F^{c}*) We will abuse notation, using the notation* $\mathcal{O}$ *and* $\mathbf{C}$ *to refer to both operads and cooperads and their underlying collections* $\mathrm{U}\mathcal{O}$ *and* $\mathrm{U}\mathbf{C}$*.*

Lemma A.5.5. *Let* X *be a pointed collection with* $X(0) = 0$*. Then the operad of decorated trees described in Lemma A.3.15 where the decoration on the vertex* v *is* $X(|v|) \bullet_{\mathbb{S}_{|v|}} \mathrm{Iso}(in(v), |v|)$ *is a free operad on* X*. Similarly, the cooperad of decorated trees from Lemma A.3.16 with the same decoration is a cofree cooperad on* X*.*

This lemma gives an internal grading to the free operad and cofree cooperad on X by the rank of the underlying tree. This is clearly respected by the composition map. So linearly,

$$\mathrm{F}X = \bigoplus_{n=0}^{\infty} \mathrm{F}^{(n)}X; \qquad \mathrm{F}^{\mathrm{c}}X = \bigoplus_{n=0}^{\infty} \mathrm{F}^{\mathrm{c}(n)}X.$$

Definition A.5.6. An operad in the category of chain complexes is called *quasifree* if it is a free operad in the category of graded vector spaces after forgetting the differential.

Definition A.5.7. Let $\mathcal{O}$ be an operad in the category of chain complexes. A *quasifree resolution* of $\mathcal{O}$ is a quasifree operad $\mathcal{O}_\infty$ with a surjective map $\mathcal{O}_\infty \to \mathcal{O}$ which induces an isomorphism on homology.

Remark. There is a model category structure on the category of operads in chain complexes (with certain restrictions) so that the quasifree resolutions are cofibrant replacements.

Let Σ denote the shift operator on chain complexes, so that $(\Sigma V)_n = V_{n-1}$.

Proposition A.5.8. *There are adjoint functors* Ω *(*cobar*, the left adjoint) and* B *(*bar*, the right adjoint) between augmented operads and conilpotent coaugmented cooperads. The underlying vector space of* $\mathrm{B}\mathcal{O}$ *is* $\mathrm{F}^c\Sigma\overline{\mathcal{O}}$ *and the differential is* $d = d_{\mathcal{O}} + d_{\diamond}$*. That is, it is the sum of a differential induced by the differential of* $\mathcal{O}$ *and a differential induced by the monoidal product of* $\mathcal{O}$*. Dually, the underlying vector space of* $\Omega\mathbf{C}$ *is* $\mathrm{F}\Sigma^{-1}\overline{\mathbf{C}}$ *and the differential is* $d = d_{\mathbf{C}} + d_{\Delta}$*, the sum of a differential induced by the differential of* $\mathbf{C}$ *and a differential induced by the monoidal coproduct of* $\mathbf{C}$.

Definition A.5.9. A cooperad $\mathbf{C}$ is *weakly* 1*-connected* if $\overline{\mathbf{C}}(1)$ is concentrated in degrees 2 and higher and $\overline{\mathbf{C}(n)}$ is concentrated in degrees 1 and higher.

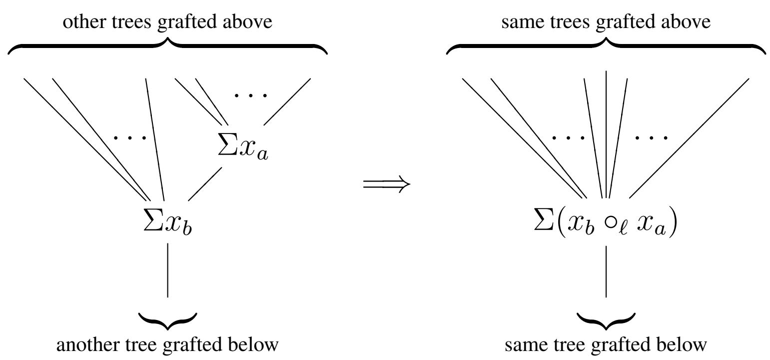
Proposition A.5.10. *The bar functor on operads preserves weak equivalences of chain complexes, that is, maps which induce an isomorphism on homology. The cobar functor does not preserve weak equivalences in general, but it does preserve weak equivalences between weakly* 1*-connected cooperads.*

Remark. There are known counterexamples to this proposition in the case of cooperads which are not weakly 1-connected.

Proposition A.5.11. *Let* $\mathcal{O}$ *be an operad. The natural map induced by the adjunction* $\Omega \mathrm{B}\mathcal{O} \to \mathcal{O}$ *is a quasifree resolution.*

Definition A.5.12. We will describe the induced differential $d_\diamond$ on the cofree cooperad $\mathrm{F}^{\mathrm{c}}\Sigma\overline{\mathcal{O}}$ using the model of the cooperad of decorated trees with decorations from $\Sigma\overline{\mathcal{O}}$. Pick a decorated tree with underlying tree T. Fix an internal edge (a, b) of T; then we will build a decorated tree whose underlying tree is T with the edge (a, b) contracted. Using the various group actions and cutting operations, we can describe T as some set of trees grafted to a tree with vertices a and b and leaf set k, grafted to some tree below b. We can further suppose that the tree with vertices a and b is given by grafting two corollas together along the leaf ℓ. We will replace the tree with vertices a and b with a k-corolla. Let the decorations on

a and b be Σx_a and Σx_b; then the decoration on the new vertex of the k-corolla is $(-1)^{|x_b|}\Sigma x_b \circ_\ell x_a$. All other decorations will be the same. Then $d_\diamond$ is the map obtained by summing this operation over every edge. The associativity of $\diamond$ and the signs of Σ make $d_\diamond$ square to zero.



Definition A.5.13. A *Koszul dual* cooperad for the operad $\mathcal{O}$ is a map of coaugmented conilpotent cooperads

$$\mathbf{C} \xrightarrow{\sim} \mathrm{B}\mathcal{O}$$

which induces an isomorphism on homology.

Remark. If the grading on $\mathbf{C}$ and $\mathcal{O}$ are suitably constrained, then $\Omega\mathbf{C} \to \Omega\mathrm{B}\mathcal{O}$ induces an isomorphism on homology. If the composition $\Omega\mathbf{C} \to \Omega\mathrm{B}\mathcal{O} \to \mathcal{O}$ is surjective, this realizes $\Omega\mathbf{C}$ as a quasifree resolution of $\mathcal{O}$.

Let us highlight a particular candidate for a Koszul dual cooperad in a restricted setting.

Definition A.5.14. A (presented) *quadratic operad* is the quotient of a free operad $\mathcal{O} = \mathrm{F}X$ by a relation ideal (R) generated by a subspace R in $\mathrm{F}^{(2)}X$.

Definition A.5.15. Let $\mathcal{O} = \mathrm{F}X/(R)$ be a quadratic operad. Then the map of collections $\Sigma X \hookrightarrow \Sigma\overline{\mathrm{U}\mathcal{O}}$ induces an inclusion of cooperads in graded vector spaces $\mathrm{F}^{\mathrm{c}}\Sigma X \hookrightarrow \mathrm{F}^{\mathrm{c}}\Sigma\overline{\mathrm{U}\mathcal{O}}$. The range of this map is the underlying graded vector space of $\mathrm{B}\mathcal{O}$. There is also an inclusion of collections of graded vector spaces $\ker d_\diamond \hookrightarrow \mathrm{B}\mathcal{O}$. The *naive dual* $\mathcal{O}^{¡}$ to $\mathcal{O}$ (given the presentation (X, R)) is $\mathrm{F}^{\mathrm{c}}\Sigma X \cap \ker d_\diamond$.

Proposition A.5.16. *$\mathcal{O}^{¡}$ is a subcooperad of* $\mathrm{B}\mathcal{O}$*.*

Lemma A.5.17. *$\mathcal{O}^{¡\,(2)} = \Sigma^2 R$.*

Proof. $\mathcal{O}^{¡\,(2)}$ are those sums of rank two trees decorated with ΣX which when composed along their unique internal edges using the composition of $\mathcal{O}$ yield zero. So they are in the ideal (R) (up to shift), but R is the only part of (R) which is contained in $\mathrm{F}^{(2)}(X)$. □

Definition A.5.18. An presented quadratic operad $\mathcal{O}$ is *Koszul* if the inclusion $(\mathcal{O}^{¡}, d_{\mathcal{O}}) \to (\mathrm{B}\mathcal{O}, d_{\mathcal{O}} + d_\diamond)$ is a Koszul dual.

Remark. The surjectivity of the map $\Omega\mathcal{O}^{¡} \to \mathcal{O}$ is automatic, so this rests only on the inclusion $\mathcal{O}^{¡} \to \mathrm{B}\mathcal{O}$ inducing an isomorphism on homology.

A.6 Cooperadic algebra

Here we collect several technical definitions and lemmas about cooperads that are almost obvious in the dual category of operads. For the remainder of this section, let $\mathbf{C}$ be a cooperad in the category of chain complexes. Recall that the structure maps of a cooperad are the decomposition map Δ and counit η.

Definition A.6.1. A $\mathbf{C}$-*comodule* is a collection M with structure map

$$\Delta_M : M \to \mathbf{C} \boxtimes M \boxtimes \mathbf{C}$$

which satisfy the following associativity and unit constraints:

$$\begin{array}{ccc}
M & \xrightarrow{\Delta_M} & \mathbf{C} \boxtimes M \boxtimes \mathbf{C} \\
\downarrow{\scriptstyle \Delta_M} & & \downarrow{\scriptstyle \Delta \boxtimes \mathrm{id} \boxtimes \Delta} \\
\mathbf{C} \boxtimes M \boxtimes \mathbf{C} & \xrightarrow[\mathrm{id} \boxtimes \Delta_M \boxtimes \mathrm{id}]{} & \mathbf{C} \boxtimes \mathbf{C} \boxtimes M \boxtimes \mathbf{C} \boxtimes \mathbf{C}
\end{array}$$

and

$$\begin{array}{ccc}
M & \xrightarrow{\Delta_M} & \mathbf{C} \boxtimes M \boxtimes \mathbf{C} \\
& \searrow{\scriptstyle (\lambda^{-1} \boxtimes \mathrm{id}) \circ \rho^{-1}} & \downarrow{\scriptstyle \eta \boxtimes \mathrm{id} \boxtimes \eta} \\
& & I \boxtimes M \boxtimes I
\end{array}$$

When the meaning is clear from context, we will refer to Δ_M without the subscript as Δ.

Remark. $\mathbf{C}$ is a $\mathbf{C}$-comodule with the decomposition map Δ, or more precisely, $(\Delta \boxtimes \mathrm{id}) \circ \Delta$. Henceforth, we will abuse the notation Δ even further, using it to refer to this map $\mathbf{C} \to \mathbf{C} \boxtimes \mathbf{C} \boxtimes \mathbf{C}$.

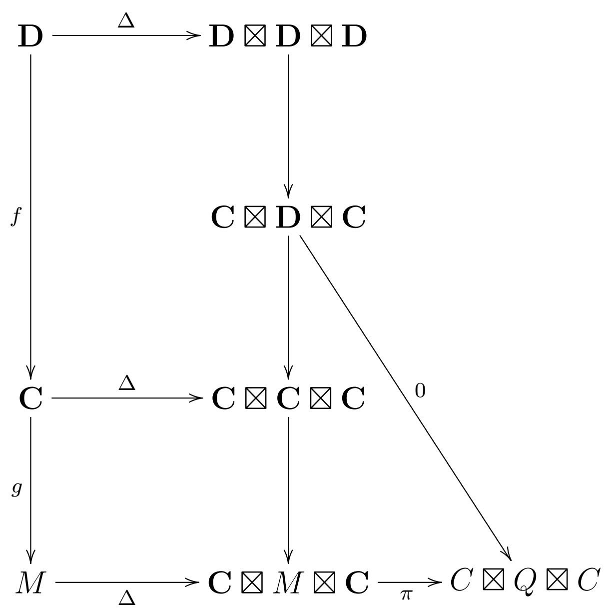
Definition A.6.2. Let M be a $\mathbf{C}$-comodule. A map of collections $\pi : M \to Q$ is said to cogenerate M if the composition

$$M \xrightarrow{\quad\Delta\quad} \mathbf{C} \boxtimes M \boxtimes \mathbf{C} \xrightarrow{\quad\pi\quad} \mathbf{C} \boxtimes Q \boxtimes \mathbf{C}$$

is injective.

Lemma A.6.3. *Let $\mathbf{D}$ be cooperad, f a cooperad map $\mathbf{D} \to \mathbf{C}$, M a $\mathbf{C}$-comodule cogenerated by Q, and $g : \mathbf{C} \to M$ be a map of $\mathcal{C}$-comodules. If $\pi g f = 0$ then $gf = 0$.*

Proof. By diagram.



The rectangle commutes because f is a map of cooperads, the square commutes because g is a map of comodules, and the triangle commutes by assumption. Then the composition that runs down the left side and across the bottom is equal to zero. By the definition of cogeneration, the composition along the bottom is injective, so the composition down the left side is zero, as desired. $\square$

Remark. The dual of this lemma is used so often it is beneath notice; namely to check that an ideal is killed by a map, it is enough to check that its generators are killed.

Lemma A.6.4. *Let* $\mathbf{D}$ *be a subcooperad of* $\mathbf{C}$*. The quotient* $\mathbf{C}/\mathbf{D}$ *has an induced* $\mathbf{C}$*-comodule structure.*

Proof. For $x \in \mathbf{C}$, define $\Delta(x + \mathbf{D})$ as the class in $\mathbf{C} \boxtimes \mathbf{C}/\mathbf{D} \boxtimes \mathbf{C}$ of $\Delta(x)$. This is well defined because if $s \in \mathbf{D}$ then $\Delta(x+s) \in \Delta(x) + \mathbf{C} \boxtimes \mathbf{D} \boxtimes \mathbf{C}$. The coassociativity and counit properties for $\mathbf{C}$ easily give coassociativity and counit for $\mathbf{C}/\mathbf{D}$. □

Definition A.6.5. A *coderivation* of $\mathbf{C}$ is an operator ∂ on the underlying collection of $\mathbf{C}$ which satisfies the dual Leibniz condition that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\Delta} & \mathbf{C} \boxtimes \mathbf{C} \boxtimes \mathbf{C} \\ \Big\downarrow{\scriptstyle \partial} & & \Big\downarrow{\scriptstyle \partial \boxtimes \mathrm{id} \boxtimes \mathrm{id} + \mathrm{id} \boxtimes \partial \boxtimes \mathrm{id} + \mathrm{id} \boxtimes \mathrm{id} \boxtimes \partial} \\ \mathbf{C} & \xrightarrow[\Delta]{} & \mathbf{C} \boxtimes \mathbf{C} \boxtimes \mathbf{C} \end{array}$$

Lemma A.6.6. *Let Ξ be a collection and let $\mathrm{F^c}(\Xi)$ be the cofree cooperad on it. Let φ be a morphism of collections $\mathrm{UF^c}(\Xi) \to \Xi$ between them. Then there exists a unique coderivation ∂_φ extending φ; that is, the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{F^c}(\Xi) & \xrightarrow{\partial_\varphi} & \mathrm{F^c}(\Xi) \\ & {\scriptstyle \varphi} \searrow & \Big\downarrow \\ & & \Xi \end{array}$$

where the vertical arrow is the projection of $\mathrm{F^c}(\Xi)$ *onto* $\mathrm{F^{c(1)}}(\Xi) \cong \Xi$.

Description A.6.7. If φ factors through the projection

$$\begin{array}{ccc} \mathrm{F}^{c}(\Xi) & \longrightarrow & \mathrm{F}^{c(1)}(\Xi) \\ & \searrow^{\varphi} & \downarrow \\ & & \Xi \end{array}$$

then ∂_φ acts on ξ with underlying tree $T = (n, V, N)$ as the sum over $v \in V$ of ξ_v, where ξ_v has the same underlying tree T and the same decorations as ξ except the decoration on v is given by $\varphi(v)$.

If φ factors through the projection

$$\begin{array}{ccc} \mathrm{F}^{c}(\Xi) & \longrightarrow & \mathrm{F}^{c(2)}(\Xi) \\ & \searrow^{\varphi} & \downarrow \\ & & \Xi \end{array}$$

then ∂_φ acts on ξ with underlying tree $T = (n, V, N)$ as a sum over the internal edges e of T of ξ_e. Let us define ξ_e. It will have the contraction tree T_e as its underlying tree T and the same decorations as ξ except for the decoration on the contraction vertex.

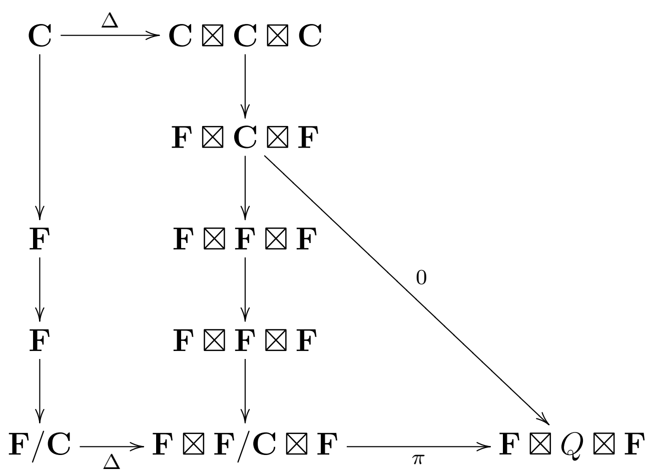
Let the contraction vertex be $\{a, b\}$ where e is the edge (a, b) of T. Then there is a rank two tree T_{ab} given by $(N^{-1}(a) \sqcup N^{-1}(b) \backslash \{a\}, \{a, b\}, N_*)$. We can make this a decorated tree by giving a and b their decorations from T. Then φ acts on this decorated tree, and gives an element of Ξ that we choose as the decoration of the contraction vertex $\{a, b\}$.

In both of these cases, because we are taking graded factors out of a tree to act on and then putting them back, we must choose a vertex ordering and use the Koszul sign rule.

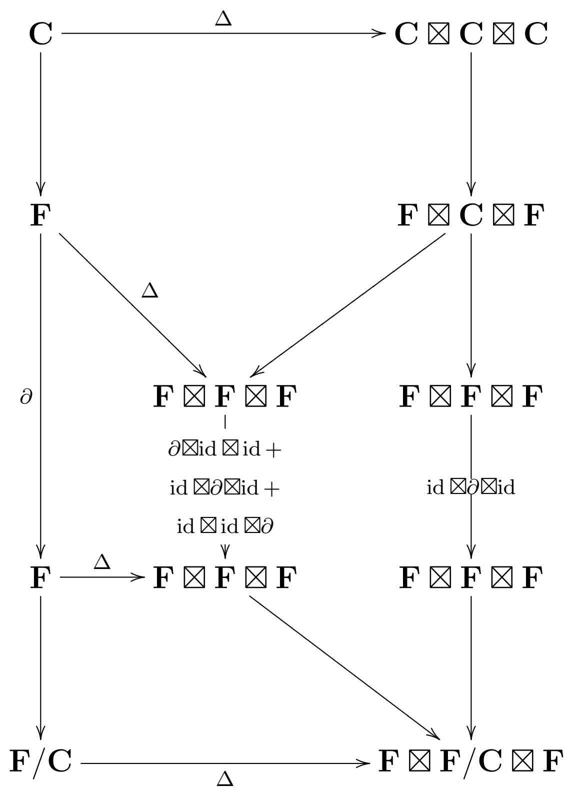
Lemma A.6.8. *Let* $\mathbf{C}$ *be a subcooperad of the cofree cooperad* $\mathrm{F}^{\mathrm{c}}(\Xi)$ *such that* $\mathrm{F}^{\mathrm{c}}(\Xi)/\mathbf{C}$ *is cogenerated by* $\pi : \mathrm{F}^{\mathrm{c}}(\Xi)/\mathbf{C} \to Q$*. Let* ∂ *be a coderivation of* $\mathrm{F}^{\mathrm{c}}(\Xi)$*. Then* ∂ *restricts to be a coderivation of* $\mathbf{C}$ *if and only if the composition* $\mathbf{C} \to \mathrm{F}^{\mathrm{c}}(\Xi) \xrightarrow{\partial} \mathrm{F}^{\mathrm{c}}(\Xi) \to \mathrm{F}^{\mathrm{c}}(\Xi)/\mathbf{C} \to Q$ *is zero.*

Proof. For this proof, let $\mathbf{F}$ denote $\mathrm{F}^{\mathrm{c}}(\Xi)$. If ∂ linearly restricts to $\mathbf{C}$, it automatically satisfies the coderivation property. For ∂ to restrict to $\mathbf{C}$, the kernel of the map $\mathbf{F} \to \mathbf{F}/\mathbf{C}$, it is necessary and sufficient for $\mathbf{C} \to \mathbf{F} \xrightarrow{\partial} \mathbf{F} \to \mathbf{F}/\mathbf{C}$ to be zero. This proves one direction of the implication.

Now assume that the composition $\mathbf{C} \to \mathbf{F} \xrightarrow{\partial} \mathbf{F} \to \mathbf{F}/\mathbf{C} \to Q$ is zero. We can follow a slightly modified version of the diagram used in Lemma A.6.3:



By assumption, the triangle commutes. If we can show the rectangle commutes, then the composition down the left side and along the bottom is zero. The composition along the bottom is injective, so this would mean that the composition down the left side yields zero, which would imply $\partial\mathbf{C} \subset \mathbf{C}$. To show commutativity of the rectangle, we introduce a couple more terms in the middle of it.



The upper pentagon commutes because $\mathbf{C}$ is a subcooperad of $\mathbf{F}$. The middle left quadrilateral commutes because ∂ is a coderivation. The right pentagon commutes because $\mathbf{C}$ is killed by quotienting it out, and the bottom quadrilateral commutes because the quotient is a comodule. □

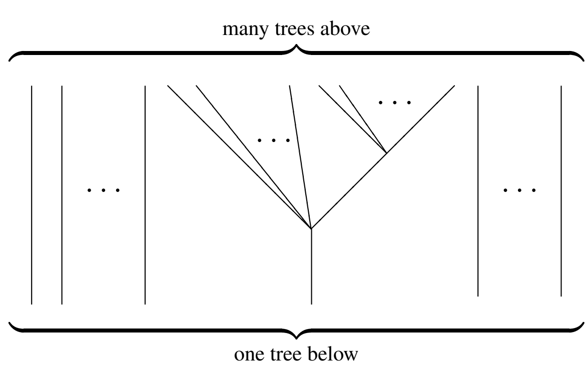
Lemma A.6.9. *Let* $\mathcal{O} = F(X)/(R)$ *be a quadratic operad. Denote by* Ξ *the shifted generators* ΣX*. The* $\mathrm{F^c}(\Xi)$*-comodule* $\mathrm{F^c}(\Xi)/\mathcal{O}^{i}$ *is cogenerated by its projection onto* $\mathrm{F}^{\mathrm{c}(2)}(\Xi)/\mathcal{O}^{i\,(2)}$.

Proof. The comodule property and a diamond isomorphism show that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{F^c}(\Xi) & \twoheadrightarrow & \mathrm{F^c}(\Xi)/\mathcal{O}^{\mathrm{i}} \\
\downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta} \\
\mathrm{F^c}(\Xi) \boxtimes \mathrm{F^c}(\Xi) \boxtimes \mathrm{F^c}(\Xi) & \longrightarrow & \mathrm{F^c}(\Xi) \boxtimes (\mathrm{F^c}(\Xi)/\mathcal{O}^{\mathrm{i}}) \boxtimes \mathrm{F^c}(\Xi) \\
\downarrow & & \downarrow \\
\mathrm{F^c}(\Xi) \boxtimes \mathrm{F}^{\mathrm{c}(2)}(\Xi) \boxtimes \mathrm{F^c}(\Xi) & \longrightarrow & \mathrm{F^c}(\Xi) \boxtimes (\mathrm{F}^{\mathrm{c}(2)}(\Xi)/\mathcal{O}^{\mathrm{i}(2)}) \boxtimes \mathrm{F^c}(\Xi)
\end{array}$$

The statement of the lemma is that the vertical composition on the right is injective. Let $\xi + \mathcal{O}^{\mathrm{i}} \in \mathrm{F^c}(\Xi)/\mathcal{O}^{\mathrm{i}}$. Assume that the vertical composition on the right takes $\xi + \mathcal{O}^{\mathrm{i}}$ to zero. Because the map across the top is surjective, $\xi + \mathcal{O}^{\mathrm{i}}$ can be lifted to $\xi \in \mathrm{F^c}(\Xi)$.

There is a direct summand of $\mathrm{F^c}(\Xi) \boxtimes \mathrm{F}^{\mathrm{c}(2)}(\Xi) \boxtimes \mathrm{F^c}(\Xi)$ in which there is exactly one nontrivial tree of $\mathrm{F}^{\mathrm{c}(2)}(\Xi)$. Taking the projection onto this summand gives a picture which has many trees above, one tree below, and several trivial trees along with one rank two tree in the middle.



This is precisely the decomposition one uses to apply $d_\diamond$.

Conducting this further projection also commutes with quotienting by $\mathcal{O}^{¡(2)}$:

$$\begin{array}{ccc} \mathrm{F}^{c}(\Xi)\boxtimes \mathrm{F}^{c(2)}(\Xi)\boxtimes \mathrm{F}^{c}(\Xi) & \longrightarrow & \mathrm{F}^{c}(\Xi)\boxtimes (\mathrm{F}^{c(2)}(\Xi)/\mathcal{O}^{¡(2)})\boxtimes \mathrm{F}^{c}(\Xi) \\ \downarrow & & \downarrow \\ \mathrm{F}^{c}(\Xi)\boxtimes \underbrace{\mathrm{F}^{c(2)}(\Xi)}_{\text{one nontrivial factor}} \boxtimes \mathrm{F}^{c}(\Xi) & \longrightarrow & \mathrm{F}^{c}(\Xi)\boxtimes \underbrace{(\mathrm{F}^{c(2)}(\Xi)/\mathcal{O}^{¡(2)})}_{\text{one nontrivial factor}} \boxtimes \mathrm{F}^{c}(\Xi) \end{array}$$

So if $\xi+\mathcal{O}^{¡}$ is taken to zero by the vertical composition on the right in the first diagram, then when we apply $d_\diamond$ to ξ, every place where we apply a composition is in $\mathcal{O}^{¡(2)}$. By Lemma A.5.17, every place where we apply a composition is in $\Sigma^2 R$, so composes to 0 in $\mathcal{O}$. Then ξ is in the kernel of $d_\diamond$, which implies that ξ is in $\mathcal{O}^{¡}$, so that $\xi+\mathcal{O}^{¡}$ is 0 in $\mathrm{F}^{c}(\Xi)/\mathcal{O}^{¡}$. This shows injectivity. □

Appendix B

The Deligne-Mumford compactification

In this appendix, we recall the necessary details of the moduli spaces of n-marked Riemann surfaces of genus zero and the Deligne-Mumford compactification of these moduli spaces. We begin in Section B.1 with a quick review of the spaces, viewed as sets. In Section B.2 we recall quasiconformal maps to the extent necessary to define the topology on the Deligne-Mumford compactification in section B.3. None of this material is original and the more important parts are presented without proof. The references for this appendix include [2, 4, 1, 12, 15, 7].

B.1 Riemann surfaces and their moduli

This background section is intended to fix notation and present the Deligne-Mumford compactification of the moduli space of genus zero Riemann surfaces with marked points as a set.

Definition B.1.1. A *Riemann surface* is a connected Hausdorff space equipped with an open covering of homeomorphic charts z_i from open sets U_i to domains in $\mathbb{C}$ so that the transition functions $z_i \circ z_j^{-1}$ are conformal maps where defined.

We will consider only Riemann surfaces with finitely presented fundamental group.

Definition B.1.2. A *neighborhood of a puncture* in a Riemann surface Σ is a conformal embedding ϕ of the standard punctured disk $\{z \in \mathbb{C} : 0 < |z| < 1\}$ in Σ which cannot be completed to an embedding of the disk. Two neighborhoods of punctures in Σ are equivalent if their intersection contains a neighborhood of a puncture. A *puncture* of Σ is the equivalence class of a neighborhood of a puncture in Σ.

Lemma B.1.3. *A puncture can be uniquely filled.*

Proof. To be more precise, consider a neighborhood of a puncture ϕ in Σ. We can consider the topological surface made by gluing the the standard disk to Σ along the image of ϕ. This surface, Σ', is topologically independent of the representative neighborhood we have chosen. As a set, it is the disjoint union of Σ and a point $*$. We can add a chart z_* to the conformal charts of Σ given by taking the inverse image of the inclusion of the disk into Σ'. Because ϕ was a conformal embedding, transition functions involving this chart will be conformal. $\square$

Definition B.1.4. Let S be a finite set. A *Riemann surface with* S *marked points* is a Riemann surface Σ along with an injective set map from S to the punctures of Σ. A conformal (continuous) map of S-marked Riemann surfaces is a conformal (continuous) map of Riemann surfaces that respects the S-marking, that is, so that for s in S, a representative of the image puncture of s contains a smaller representative which is taken to a representative of the image puncture of s.

Definition B.1.5. A *sphere with* S *marked points* is a S-marked Riemann surface so that when the marked punctures are filled, the resultant surface is homeomorphic to a sphere.

Remark. An S-marked sphere is the same thing as a Riemann surface homeomorphic to the sphere along with an injective set map S to it. In this point of view, a map of S-marked spheres is a map of spheres that respects the image of S.

Remark. If S has more than two elements, then the conformal automorphism group of the S-marked sphere is trivial.

Definition B.1.6. For a finite set S of order greater than two, the *set of moduli of* S*-marked spheres*, $\mathcal{M}_{0,S}$, consists of conformal isomorphism classes of S-marked spheres. We will consider $\mathcal{M}_{0,S}$ to be empty for sets of order two or less.

Definition B.1.7. The *operad of stable nodal spheres* (in the category of sets), $\overline{\mathcal{M}}$, is the operad of decorated trees in the category of sets where the decoration on a vertex v is $\mathcal{M}_{0,E(v)}$. So precisely:

$$\overline{\mathcal{M}}(n) = \coprod_{n\text{-trees } (n,V,N)} \prod_{v\in V} \mathcal{M}_{0,E(v)}.$$

Definition B.1.8. For $n > 2$, the *set of moduli of stable* n*-marked nodal spheres*, $\overline{\mathcal{M}}_{0,n}$, is $\overline{\mathcal{M}}(n-1)$.

Definition B.1.9. Let $x \in \overline{\mathcal{M}}(n)$ be in the summand of the disjoint union corresponding to the tree $T = (n, V, N)$. Write x as the product over $v \in V$ of the conformal classes $[\Sigma_v]$. A *realization* $\tilde{x}$ of x is the quotient of the topological disjoint union $\coprod \Sigma_v$ by the relation that for an internal edge (a, b) of T, the marked points on Σ_a and Σ_b corresponding to the edge (a, b) are identified. A realization is an example of a *nodal surface*, and any two realizations are connected by a unique homeomorphism which is conformal on each $\Sigma_v \subset \tilde{x}$.

Definition B.1.10. A *node* is a neighborhood of an identified point homeomorphic to two standard disks glued together at the origin, with conformal embeddings of the standard disk into each of the two disks taking the origin to the nodal point.

We will confuse $\overline{\mathcal{M}}(n)$ with the set of isomorphism classes of realizations.

Remark. Notice that $\overline{\mathcal{M}}(n)$ contains $\mathcal{M}_{0,\{1,\dots,n+1\}}$ naturally as the set decorating the n-corolla.

B.2 Quasiconformal maps

This section includes the definition of a quasiconformal map and a lemma about them. The definition is used to define the topology on the moduli space, and the lemma is used in the proof of Proposition 2.6.4.

Definition B.2.1. [2, 4]. Let f be a continuous map of S-marked Riemann surfaces which has locally square integrable distributional derivatives. Then f has partial derivatives almost everywhere. Let $K > 1$. We say f is K*-quasiconformal* if, almost everywhere,

$$|f_{\bar{z}}| \leq \frac{K-1}{K+1}|f_z|.$$

Some simple properties of quasiconformal maps are:

Lemma B.2.2. *1. If f is K-quasiconformal, and $K < K'$, then f is K'-quasiconformal,*

2. a 1-quasiconformal map is conformal and vice versa, and

3. a composition of a K-quasiconformal map with an K'-quasiconformal map is KK'-quasiconformal.

Proofs can be found in, for example, [2].

Lemma B.2.3. *Let $|z| < \frac{\epsilon}{2+\epsilon}$.There is a $(1+\epsilon)$-quasiconformal homeomorphism from the plane to itself which takes 0 to z and fixes the unit circle pointwise.*

Proof. It is easier to see a statement of this kind using the half-plane instead of the disk. The map $f : z \mapsto (1 + \frac{\epsilon}{2})z - \frac{\epsilon}{2}\bar{z}$ on the sphere (taking ∞ to itself) fixes the extended real line. f takes i to $(1 + \epsilon)i$ and is a C^1 homeomorphism. It satisfies:

$$\left|\frac{f_{\bar{z}}}{f_z}\right| = \frac{\frac{\epsilon}{2}}{1 + \frac{\epsilon}{2}} = \frac{\epsilon}{2 + \epsilon} = \frac{(1 + \epsilon) - 1}{(1 + \epsilon) + 1}.$$

This shows that f is K quasiconformal for $K > 1+\epsilon$. f can obviously be extended to the sphere as well.

So to construct the map of the lemma, we choose a conformal equivalence of the sphere ϕ taking the unit circle to the extended real line and the center of the circle to i. We apply ϕ, apply f, and then apply ϕ^{-1}. We can choose ϕ so that $\phi^{-1} f \phi$ moves 0 in any direction, that is, there is not a unique direction corresponding to the positive imaginary direction in the upper half-plane.

Explicitly, a map on the disk is given by, for example,

$$z \mapsto \frac{2|z|^2 + \epsilon(|z|^2 - 1) + 2\bar{z}}{2 - \epsilon(|z|^2 - 1) + 2z}$$

(with $-1 \mapsto -1$). When $|z| = 1$, this immediately simplifies to

$$\frac{2\bar{z} + 2}{2z + 2} = \frac{x + 1 - iy}{x + 1 + iy} = \frac{x^2 - y^2 + 2x + 1 - 2i(x + 1)y}{x^2 + y^2 + 2x + 1}$$

and using the fact that $x^2 + y^2 = 1$, this is

$$\frac{2x^2 + 2x - 2(x + 1)iy}{2 + 2x} = x + iy.$$

Also, 0 is taken by this map to $\frac{-\epsilon}{2+\epsilon}$. A calculation verifies that the partial derivatives of the function satisfy $\left|\frac{f_{\bar{z}}}{f_z}\right| = \frac{\epsilon}{2+\epsilon}$ except where both partials are zero. Again, by conjugating this map by rotations $z \mapsto e^{i\theta}z$, we can cause the point to move in any direction, not just along the real axis. □

Lemma B.2.4. *Let $(c_1, \ldots, c_k)$ be points in the plane. Choose a number $d_{\min}$ which is at most half the smallest distance between any pair c_i and c_j. Let $\delta = \frac{d_{\min}\epsilon}{(2+\epsilon)}$ and let $(c_1', \ldots, c_k')$ be points in the plane so that $|c_i - c_i'| < \delta$. Then there is a $(1+\epsilon)$-quasiconformal map of the plane to itself fixing ∞ and taking c_i to c_i'.*

Proof. We build this map by patching together maps modelled on the one used in Lemma B.2.3. That is, divide the plane into the disks of radius d_{min} centered at each c_i (these are disjoint because d_{min} is small) and the complement of the union of all these disks. The disk centered at c_i is conformally equivalent to the standard disk, and using the restriction of a map from Lemma B.2.3, we can define a $(1+\epsilon)$-quasiconformal map on the disk of radius d_{min} centered at c_i which takes c_i to c_i' and fixes the boundary of the disk pointwise. We can extend this map on the complement of all the disks by the identity. Away from the boundary circles (a set of measure zero), this homeomorphism is $(1+\epsilon)$-quasiconformal, so it is $(1+\epsilon)$-quasiconformal. □

B.3 The topology of moduli space

Theorem B.3.1. *The construction described below gives $\overline{\mathcal{M}}_{0,n}$ the structure of a topological space, and in fact a compact manifold of real dimension $2n-6$.*

Description of the topology of Theorem B.3.1. We describe a local basis of neighborhoods around a decorated tree $x \in \overline{\mathcal{M}}_{0,n}$ with underlying tree $T = (n, V, N)$, whose vertex $v \in V$ is decorated by the $E(v)$-marked sphere class $[\Sigma_v]$. Choose a representative $\tilde{x}$, and a neighborhood in Σ_v of each marked point corresponding to an internal edge of the tree; call the union of these neighborhoods $\mathcal{N}$. Also choose $\epsilon > 0$.

Now, the neighborhood $U_{\mathcal{N},\epsilon}$ of x consists of points y with a realization $\tilde{y}$ so that there exists a topological map $\tilde{y} \to \tilde{x}$ satisfying the following conditions:

1. The map respects the punctures corresponding to leaves and the root edge,
2. the preimage of the identified point corresponding to the edge (a, b) of T is a point or a simple closed curve,
3. outside the preimage of identified points, the map is a homeomorphism, and
4. outside of the closure of $\mathcal{N}$, the map is $(1+\epsilon)$-quasiconformal.

If we choose a different representative $\tilde{x}'$ then $\mathcal{N}$ is mapped uniquely to a neighborhood $\mathcal{N}'$ and $U_{\mathcal{N},\epsilon}$ and $U_{\mathcal{N}',\epsilon}$ are the same set. This lets us modify this slightly so that these neighborhoods form a set, rather than a proper class.

□

Proposition B.3.2. *The operad composition maps are continuous.*

Proof. The partial composition maps are of the form

$$\circ_i : \overline{\mathcal{M}}(m) \times \overline{\mathcal{M}}(n) \to \overline{\mathcal{M}}(m+n-1)$$

and are given by relabeled grafting. Choose a point $x \times y$ in the preimage of z, and choose a neighborhood $U_{\mathcal{N},\epsilon}$ of z (with realization $\tilde{z}$). Cutting $\tilde{z}$ along the composition edge yields realizations $\tilde{x}$ and $\tilde{y}$ of x and y.

Now N contains a neighborhood of each of the marked points of decorations of $\tilde{x}$ and $\tilde{y}$ corresponding to internal edges of their underlying trees; let the unions of these be $\mathcal{N}_x$ and $\mathcal{N}_y$, respectively. Then consider the product neighborhood $U_{\mathcal{N}_x,\epsilon} \times U_{\mathcal{N}_y,\epsilon}$. Take a point $x' \times y'$ in the neighborhood, and consider the composition z'.

The definition of the neighbohood $U_{\mathcal{N}_x,\epsilon}$ guarantees a map $\psi_x : \tilde{x}' \to \tilde{x}$ satisfying the various conditions, but in particular taking the leaf puncture i of $\tilde{x}'$ to the leaf puncture i of $\tilde{x}$. Similarly, there is a map $\psi_y : \tilde{y}' \to \tilde{y}$ which taking the root puncture of $\tilde{y}'$ to the root puncture of $\tilde{y}$. These two maps can be patched together to get a map $\tilde{z}' \to \tilde{z}$. This map must respect all the punctures corresponding to leaves of z because these were also leaves of x or y. An identified point could be in x or y, in which case its preimage is a point or simple closed curve because it was so in $\tilde{x}'$ or $\tilde{y}'$. It could also be the gluing point, in which case its preimage is the glued point of $\tilde{z}'$. Outside of the identified points, the map agrees with $\tilde{x}' \to \tilde{x}$ and $\tilde{y}' \to \tilde{y}$ so is a homeomorphism, and outside of the union of $\overline{\mathcal{N}_x}$, $\overline{\mathcal{N}_y}$, and the gluing point, the map is $(1+\epsilon)$-quasiconformal. Then certainly it is $(1+\epsilon)$-quasiconformal away from the larger set $\overline{\mathcal{N}}$. □

Definition B.3.3. By abuse of notation, the *genus zero Deligne-Mumford operad* $\overline{\mathcal{M}}$ is the operad of stable nodal spheres with the topology described above.

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