On the Structure of the Space of Lattices in a Class of Simply Connected, 2-Step Solvable Real Lie Groups and Genus Sets of Certain Spaces

by

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ABSTRACT

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We classify all the lattices for a class of 2-step solvable simply connected Lie groups G. These are semi-direct products of $\mathbf{R}$ acting on $\mathbf{R}^n$ via a 1-parameter subgroup whose infinitesimal generator, Δ, is real upper triangular and of trace zero. We show that the lattices of G are all of the forms $\mathcal{L}(A, \sigma)$, where $A \in SL(n, \mathbf{Z})$, $\sigma \in GL(n, \mathbf{R})$ and $\sigma^{-1} A \sigma$ equals $\exp t\Delta$ for some real $t \neq 0$. The lattice $\mathcal{L}(A, \sigma)$ is then given by $\sigma^{-1}\mathbf{Z}^n \times t\mathbf{Z}$. Furthermore, two such lattices $\mathcal{L}(A, \sigma)$ and $\mathcal{L}(B, \tau)$ differ by a smooth automorphism of G if and only if A and B are *extendedly conjugate*. We then turn to some questions concerning the decomposition of the quasi-regular representation for such groups. We show that when $n = 2$ the representation decomposes into a direct sum of indecomposable subrepresentations in such a way that although each of these subrepresentation occurs with finite times, the multiplicity function itself is always unbounded. Then we turn to compute genus sets for quaternion projective spaces and wedges of finitely many spheres of the same dimension. In fact, we prove that both the Mislin genus set $\mathcal{G}(HP^n < 4 >)$ and the genus set $\hat{\mathcal{G}}_0(S^n \vee S^n < n >)$ (where $n > 2$) are uncountably large.

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§1. Introduction

Given a Lie group, it is often useful to have a parametrization of the set of all its lattices. In Euclidean space $\mathbf{R}^n$, for example, each lattice corresponds to a basis, and any lattice is equivalent to the standard integer lattice under an automorphism in $\mathrm{GL}(n, \mathbf{R})$. In the nilpotent case, the lattices of the Heisenberg groups are classified, up to automorphisms, by certain sequences of positive integers with divisibility conditions (see [1]). In [7], R. Mosak and M. Moskowitz studied the set of lattices in a class of simply connected, solvable, but not nilpotent Lie groups G. Their construction of G depends on a diagonal $n \times n$ matrix Δ with *distinct* non-zero eigenvalues, of trace 0. In this environment they define the 1-parameter subgroup $\eta(t) = e^{t\Delta}$ in $\mathrm{GL}(n, \mathbb{R})$ and G is the semi-direct product $\mathbb{R}^n \times_\eta \mathbb{R}$. The Lie group G is connected and simply connected, solvable, but not nilpotent. Because no d_i is 0, G has trivial center. Here, among other things, we shall consider a more general setting. Let Δ be an $n \times n$ upper triangular matrix of trace 0 in $\mathfrak{gl}(n, \mathbb{R})$, the full Lie algebra of real $n \times n$ matrices, with at least one non-zero element on the diagonal. As in [7] we can define a 1-parameter subgroup $\eta(t) = \exp(t\Delta)$ in $\mathrm{GL}(n, \mathbb{R})$ and construct a similar Lie group G. We define in G a class of distinguished lattices $\mathcal{L}(A, \sigma)$, for A in $\mathrm{SL}(n, \mathbb{Z})$, and for certain σ in $\mathrm{GL}(n, \mathbb{R})$. In [7] it was proven that up to commensurability, every lattice in G differs from one of these by an automorphism of G, and two such lattices, $\mathcal{L}(A, \sigma)$ and $\mathcal{L}(B, \tau)$, are equivalent by an automorphism of G if and only if A and B (or A and B^{-1}) are conjugate in $\mathrm{GL}(n, \mathbb{Z})$.

We will prove generalizations as well as strengthenings of the results of [7]. In fact, we shall prove that up to isomorphism, those lattices comprise *all* the lattices in G(whereas in [7] this was done only up to *commensurability*). Another result which we have strengthened is Corollary 5 in [7] in which a trichotomy of possibilities is significantly reduced. Here we also mention a conjecture which we have proven in case $n = 2$, but not in general.

We then turn to some related questions concerning the decomposition of the quasi-regular representation for such groups; that is where the group operates by right translation on the space of C^∞ functions on the homogeneous space G/Γ, Γ a lattice in G. Here we show that when $n = 2$ the quasi-regular representation decomposes into a direct sum of indecomposable subrepresentations in such a way that although each of these indecomposable subrepresentations occur with finite multiplicity, the multiplicity function itself is always unbounded.

Finally, the author would like to thank Prof. M. Moskowitz for proposing the problem and for various suggestions and advice given in the course of the preparation of this paper.

Then we turn to compute genus sets for quaternion projective spaces and wedges of finitely many spheres of the same dimension.

First let's recall various genus notions.

DEFINITION. Let X be a simply connected finite-type CW-complex.

1. $\hat{\mathcal{G}}(X)$ is the set of homotopy equivalence classes of finite type Y such that the profinite completion Y is equivalent to that of X.

2. $\hat{\mathcal{G}}_0(X)$ is the subset of $\hat{\mathcal{G}}(X)$ for which $X_{(0)} \simeq Y_{(0)}$. (Here $X_{(0)}$ and $Y_{(0)}$ are the rationalizations of X and Y.)

3. $\mathcal{G}(X)$ is the subset of $\hat{\mathcal{G}}(X)$ for which $Y_{(p)} \simeq X_{(p)}$ for any prime p.(Here $X_{(p)}$ and $Y_{(p)}$ are the p-localizations of X and Y.)

It is clear that $\mathcal{G}(X) \subset \hat{\mathcal{G}}_0(X) \subset \hat{\mathcal{G}}(X)$. $\hat{\mathcal{G}}(X)$ is called the completion genus, and $\mathcal{G}(X)$ is called the localization genus or Mislin genus. In [17], Wilkerson proves that the completion genus set(hence all 3 genus sets) of a simply connected finite CW-complex is always finite,while in [13], C. A. McGibbon and J. M. Moller surprisingly prove that the genus set $\mathcal{G}(S^{2n} < 2n >)(n > 1)$ is uncountably large, where $X < m >$ denotes the m-connective covering of X. Here we are going to prove that the genus set $\mathcal{G}(HP^n < 4 >)$ is uncountably large also. And we shall also prove that the genus set $\hat{\mathcal{G}}_0(S^n \vee S^n < n >)$ (where $n > 2$) is uncountably large.

Finally the author would like to thank Prof. Joseph Roitberg for suggesting to study these questions and for various advice.

§2. The Structure of Lattices

In order to construct these lattices in G, we make the following

DEFINITION. Let A be a matrix in $\mathrm{SL}(n, \mathbf{Z})$, and σ a matrix in $\mathrm{GL}(n, \mathbf{R})$. We shall say that the pair (A, σ) is Δ-*compatible* if $\sigma^{-1}A\sigma$ is upper triangular and there exists a number $g \in \mathbf{R}$ such that

$$\sigma^{-1}A\sigma = \exp(g\Delta)$$

To construct our lattice, let the pair (A, σ) be Δ-compatible. We denote by $\mathcal{L}(A, \sigma)$ the semi-direct product $\sigma^{-1}\mathbf{Z}^n \times_\eta \mathbf{Z}g$, which is a lattice in G. Indeed, $\eta(g) = \exp(g\Delta) = \sigma^{-1}A\sigma$ leaves $L = \sigma^{-1}\mathbf{Z}^n$ stable, since

$$\eta(g)L = \sigma^{-1}A\sigma L = \sigma^{-1}A\mathbf{Z}^n = \sigma^{-1}\mathbf{Z}^n = L.$$

Consequently $\mathcal{L}(A, \sigma)$ is a subgroup in G, and is obviously discrete and co-compact. Our first result is that up to isomophism, the lattices $\mathcal{L}(A, \sigma)$ are all the lattices in G. In order to prove this fact, we begin with the definition of the roots of a solvable Lie algebra. Let $\mathfrak{g}$ be a solvable Lie algebra over the complex number field. By Lie's Theorem, there exists a decreasing sequence

$\mathfrak{g}_0, \mathfrak{g}_1, \cdots, \mathfrak{g}_n$ of ideals of $\mathfrak{g}$, such that $\mathfrak{g}_0 = \mathfrak{g}, \mathfrak{g}_n = \{0\}, \dim \mathfrak{g}_{i-1} - \dim \mathfrak{g}_i = 1$. The representation of $\mathfrak{g}$ on $\mathfrak{g}_{i-1}/\mathfrak{g}_i$ induced by the adjoint representation of $\mathfrak{g}$ gives rise to linear forms λ_i over $\mathfrak{g}$. These n forms are called the roots of $\mathfrak{g}$. Now suppose $\mathfrak{g}$ is a real Lie algebra; the roots of $\mathfrak{g}$ are by definition the restriction to $\mathfrak{g}$ of the roots of its complexification $\mathfrak{g}^c$. Hence the values of $\lambda_1, \cdots, \lambda_n$ for X of $\mathfrak{g}$ are the eigenvalues of $\mathrm{ad}(X)$. Let G be a simply connected solvable real Lie group, and $\mathfrak{g}$ its Lie algebra. We call G of real type, if for any i, all values of λ_i are real. For such Lie groups, Saito proved the following: (see also Corollary 11 of [6] for a generalization)

THEOREM. Let G be a simply connected solvable real Lie group of real type, and Γ_1 and Γ_2 be two lattices in G. If they are isomorphic, then this isomorphism extends to an automorphism of G.

Now we can state our results. From now on, we let $G = \mathbf{R}^n \times_\eta \mathbf{R}$, where $\eta(t) = \exp(t\Delta)$, and Δ is an upper triangular matrix in $\mathfrak{sl}(n, \mathbf{R})$ with at least one nonzero element on the diagonal. Note that G is solvable of real type but not nilpotent, since the roots of its Lie algebra $\mathfrak{g}$ are those elements on the diagonal of Δ together with 0. Hence all the roots of G are real and at least one root is nonzero, therefore G is not nilpotent.

THEOREM 1. *Every lattice $\mathcal{L}$ in G is isomorphic to $\mathcal{L}(A,\sigma)$, for some Δ-compatible pair (A,σ).*

PROOF. Suppose $\mathcal{L}$ is a lattice in G. Since $\mathbf{R}^n$ is the nilradical of G, $\mathcal{L}\cap\mathbf{R}^n$ is a lattice L in $\mathbf{R}^n$ (see [9]), and $\mathcal{L}/\mathcal{L}\cap\mathbf{R}^n \simeq \mathcal{L}\mathbf{R}^n/\mathbf{R}^n$ is a lattice of $G/\mathbf{R}^n \simeq \mathbf{R}$. Hence $\mathcal{L}/\mathcal{L}\cap\mathbf{R}^n$ is isomorphic to $\mathbf{Z}$. Let $g \in \mathcal{L}/\mathcal{L}\cap\mathbf{R}^n \subset \mathbf{R}$ be a generator of the group. Then we have a split short exact sequence

$$\{1\} \longrightarrow \mathcal{L}\cap\mathbf{R}^n \xrightarrow{\ i\ } \mathcal{L} \underset{s}{\overset{\pi}{\rightleftarrows}} \mathbf{Z}g \longrightarrow \{1\}$$

For any $(x_0,g) \in \mathcal{L}$, choose $s(g) = (x_0,g)$, then $s(g)^{-1} = (-\eta(-g)x_0,-g)$. For any $(x,0) \in \mathcal{L}$, we have

$$s(g)(x,0)s(g)^{-1} = (x_0,g)(x,0)(-\eta(-g)x_0,-g)$$

$$= (x_0+\eta(g)x,g)(-\eta(-g)x_0,-g) = (\eta(g)x,0).$$

Hence $(\eta(g)x,0) \in \mathcal{L}$. Similarly $(\eta(-g)x,0) \in \mathcal{L}$. Thus $\eta(g)L = L$. Since L is a lattice in $\mathbf{R}^n$, we can write $L = \sigma^{-1}\mathbf{Z}^n$, for some $\sigma \in \mathrm{GL}(n,\mathbf{R})$. So $A = \sigma\eta(g)\sigma^{-1}$ preserves $\mathbf{Z}^n$. Since Δ is upper triangular and $\operatorname{tr}\Delta = 0$, $\det(A) = \det(\eta(g)) = 1$, so A is in $\mathrm{SL}(n,\mathbf{Z})$, and by construction, (A,σ) is compatible with Δ. Thus $\mathcal{L} \simeq \mathcal{L}(A,\sigma)$. By Saito's rigidity theorem, these lattices must differ by an automorphism of G. □

The construction of the lattice $\mathcal{L}(A,\sigma)$ obviously depends on the choice of A, as well as the choice of diagonalizing matrix σ. However, if B in $\mathrm{SL}(n,\mathbf{Z})$ is conjugate to A or A^{-1} in $\mathrm{GL}(n,\mathbf{Z})$, and τ diagonalizes B, we shall show that the lattice $\mathcal{L}(B,\tau)$ differs from $\mathcal{L}(A,\sigma)$ by an automorphism of G. More precisely, we state:

DEFINITION. Let A and B be elements of $\mathrm{SL}(n,\mathbf{Z})$. We say that A and B are *extendedly conjugate* if B is conjugate to A or A^{-1} in $\mathrm{GL}(n,\mathbf{Z})$.

THEOREM 2. *Let (A,σ) and (B,τ) be Δ-compatible pairs, where*

$$\sigma A\sigma^{-1} = \exp(g\Delta), \quad \tau B\tau^{-1} = \exp(h\Delta)$$

Then the lattices $\mathcal{L}(A,\sigma)$ and $\mathcal{L}(B,\tau)$ differ by an automorphism of G iff A is extendedly conjugate to B in $\mathrm{GL}(n,\mathbf{Z})$.

PROOF. Suppose $\varphi \in \mathrm{Aut}(G)$, and $\varphi(\mathcal{L}(A,\sigma)) = \mathcal{L}(B,\tau)$. Since φ maps the nilradical $\mathbf{R}^n$ to itself, it takes $\sigma^{-1}\mathbf{Z}^n = \mathcal{L}(A,\sigma)\cap\mathbf{R}^n$ to $\tau^{-1}\mathbf{Z}^n = \mathcal{L}(B,\tau)\cap\mathbf{R}^n$ and the inverse φ^{-1} of φ takes $\tau^{-1}\mathbf{Z}^n$ to $\sigma^{-1}\mathbf{Z}^n$. Let $\alpha = \varphi|_{\sigma^{-1}\mathbf{Z}^n}$, then

$\alpha^{-1} = \varphi^{-1}|_{\tau^{-1}\mathbf{Z}^n}$. Hence α is an isomorphism and we have the following commutative diagram

$$\begin{array}{ccccccccc}
\{1\} & \longrightarrow & \sigma^{-1}\mathbf{Z}^n & \xrightarrow{\;i\;} & \mathcal{L}(A,\sigma) & \underset{s}{\overset{\pi}{\rightleftarrows}} & \mathbf{Z}g & \longrightarrow & \{1\}\,. \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\gamma} & & \\
\{1\} & \longrightarrow & \tau^{-1}\mathbf{Z}^n & \xrightarrow{\;i'\;} & \mathcal{L}(B,\tau) & \underset{s'}{\overset{\pi'}{\rightleftarrows}} & \mathbf{Z}h & \longrightarrow & \{1\}
\end{array}$$

It follows that γ must also be an isomorphism, and the splitting can be chosen so that this diagram is commutative also. Indeed, for any splitting s of π in the first row, we define $s'(\gamma(g)) = \varphi s(g)$. Then $\pi'(s'(\gamma(g))) = \pi'(\varphi s(g)) = (\pi'\varphi)s(g) = (\gamma\pi)s(g) = \gamma(\pi s)(g) = \gamma(g)$. Hence s' is indeed a cross section to π'. From the proof of Theorem 2, we know that $\eta(g)x = s(g)xs(g)^{-1}$. Hence we have

$$\alpha(\eta(g)x) = \varphi(s(g)xs(g)^{-1}) = \varphi(s(g))\varphi(x)\varphi(s(g))^{-1}$$

$$= s'(\gamma(g))\alpha(x)s'(\gamma(g))^{-1} = \eta(\gamma(g))\alpha(x).$$

Thus $\alpha\eta(g) = \eta(\gamma(g))\alpha$, and since γ is an isomorphism, $\gamma(g) = \pm h$. First we suppose that $\eta(g) = h$. Then $\alpha\eta(g) = \eta(h)\alpha$ and since $\eta(g) = \sigma^{-1}A\sigma$, and $\eta(h) = \tau^{-1}B\tau$, it follows that $\alpha\sigma^{-1}A\sigma = \tau^{-1}B\tau\alpha$, so $\tau\alpha\sigma^{-1}A = B\tau\alpha\sigma^{-1}$.

Let $\rho = \tau\alpha\sigma^{-1} \in \mathrm{GL}(n, \mathbf{R})$. Then $\rho A = B\rho$, since $\rho : \mathbf{Z}^n \to \mathbf{Z}^n$ is an isomorphism, we have $\rho \in \mathrm{GL}(n, \mathbf{Z})$. Similarly, if $\gamma(g) = -h$, we can prove that there exists an ρ' such that $\rho' A = B^{-1}\rho'$, $\rho' \in \mathrm{GL}(n, \mathbf{Z})$.

On the other hand, if $A = \rho B \rho^{-1}$, where $\rho \in \mathrm{GL}(n, \mathbf{Z})$, then $\sigma\eta(g)\sigma^{-1} = \rho\tau\eta(h)\tau^{-1}\rho^{-1}$. Let $\alpha = \tau^{-1}\rho^{-1}\sigma$, then $\alpha : \sigma^{-1}\mathbf{Z}^n \to \tau^{-1}\mathbf{Z}^n$ is an isomorphism, and $\alpha\eta(g) = \eta(h)\alpha$. Hence for any $k \in \mathbf{Z}$, we have $\alpha\eta(kg) = \eta(kh)\alpha$. Thus we can define a homomorphism $\varphi : \sigma^{-1}\mathbf{Z}^n \times_\eta \mathbf{Z}g \to \tau^{-1}\mathbf{Z}^n \times_\eta \mathbf{Z}h$, by $\varphi(x, kg) = (\alpha(x), kh)$. Indeed for $(x_1, k_1 g)$ and $(x_2, k_2 g)$ in $\sigma^{-1}\mathbf{Z}^n \times_\eta \mathbf{Z}g$, we have

$$\varphi((x_1, k_1 g)(x_2, k_2 g)) = \varphi(x_1 + \eta(k_1 g)(x_2), (k_1 + k_2)g)$$

$$= (\alpha(x_1) + \alpha\eta(k_1 g)(x_2), (k_1 + k_2)h))$$

$$= (\alpha(x_1) + \eta(k_1 h)\alpha(x_2), (k_1 + k_2)h),$$

and

$$\varphi(x_1, k_1 g)\varphi(x_2, k_2 g) = (\alpha(x_1), k_1 h)(\alpha(x_2), k_2 h)$$

$$= (\alpha(x_1) + \eta(k_1 h)\alpha(x_2), (k_1 + k_2)h) = \varphi((x_1, k_1 g)(x_2, k_2 g)).$$

We have proven that φ is a homomorphism. It is easy to see that φ is actually an isomorphism. Thus $\mathcal{L}(A,\sigma) \simeq \mathcal{L}(B,\tau)$. If $A = \rho B^{-1}\rho^{-1}$, we proceed similarly. Again, by using Saito's rigidity theorem, these lattices must differ by an automorphism of G. □

For a lattice $\mathcal{L}$ in a Lie group G, we shall denote by $C(\mathcal{L})$ the *commensuralizer* of $\mathcal{L}$, that is, the set of $x \in G$ such that the conjugate $\mathcal{L}^x$ is commensurable with $\mathcal{L}$, i.e. $\mathcal{L} \cap \mathcal{L}^x$ is also a lattice of G. From now on we will assume that all the elements on the diagonal of Δ are nonzero. In this more general setting, Theorem 4 in [7] remains true. Furthermore we can use its proof verbatim to get Theorem 3 and Corollary 4 below.

THEOREM 3. *Let $\mathcal{L}$ be a lattice in G. Then*

1. The normalizer $N(\mathcal{L})$ of $\mathcal{L}$ is also a lattice.

2. If $(\mathcal{L}_i)_{i>1}$ denotes the increasing sequence of normalizers, $\mathcal{L}_i = \mathcal{L}_{i-1}$, with $\mathcal{L}_0 = \mathcal{L}$, then

$$\cup \mathcal{L}_i \subset C(\mathcal{L}).$$

3.If $C(\mathcal{L})$ is discrete, then G contains a lattice which is its own normalizer.

PROOF. *1.* We observe first that in the usual embedding of G into the affine group of $\mathbf{R}^n$, G becomes a linear group all of whose eigenvalues are real; thus the conditions for the density theorem of [5] are satisfied. Now the identity component $N(\mathcal{L})_0$ of the closed subgroup $N(\mathcal{L})$ normalizes $\mathcal{L}$, and since $\mathcal{L}$ is discrete it must therefore centralize $\mathcal{L}$. Hence by the density theorem of [5], $N(\mathcal{L})_0 \subset Z(G)$, the center of G. This means that $N(\mathcal{L})_0 = (1)$ and so $N(\mathcal{L})$ is discrete. Since it contains $\mathcal{L}$ it is a lattice.

2. Since $N(\mathcal{L})$ is commensurable with $\mathcal{L}$ it follows that $\mathcal{C}(\mathcal{L}) = \mathcal{C}(N(\mathcal{L}))$, and hence that $\mathcal{C}(\mathcal{L}) = \mathcal{C}(\mathcal{L}_i)$ for all i. Because each $\mathcal{C}(\mathcal{L}_i) \supset \mathcal{L}_i$, (2) follows.

3. If $\mathcal{C}(\mathcal{L})$ is discrete then since it contains $\mathcal{L}$ it is a lattice. By (2) the indexes $[\mathcal{L}_i : \mathcal{L}]$ are all bounded by $[\mathcal{C}(\mathcal{L}) : \mathcal{L}]$. As the sequence $\mathcal{L}_i$ is increasing, it must stabilize at some point $\mathcal{L}_{i_0}$. □

COROLLARY 4. *Let $\mathcal{L}$ be a lattice in G, and let $C = \mathcal{C}(\mathcal{L})$ be its commensuralizer. Then either C is discrete, or it is dense in G, or the closure of C in the Euclidean topology is the semi-direct product of the nil-radical, $\mathbf{R}^n$ of G with a discrete subgroup of G.*

PROOF. If C is not discrete then denote its closure by H. Now $\mathcal{L}$ normalizes the identity component H_0, and so $\mathrm{Ad}(\mathcal{L})$ normalizes the Lie algebra $\mathfrak{h}$. Since

Ad($\mathcal{L}$) is Zariski dense in Ad(G) we see that $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ and therefore H_0 is normal in G. If C is neither discrete nor dense, then H_0 is neither trivial nor all of G, so we must have $H_0 = N$. Since H_0 is open in H, the manifold $H = N \times D$ where D is a discrete subgroup of $\mathbb{R}$. □

Actually Corollary 4 can be strengthened by eliminating one possibility, namely the discrete case can not occur when Δ is digonal. In fact we have the following

THEOREM 5. *Let $\mathcal{L}$ be a lattice in G, and $C = C(\mathcal{L})$ be its commensuralizer. Then either C is dense in G, or the closure of C in the Euclidean topology is the semi-direct product of the nil-radical $\mathbb{R}^n$ of G with a discrete subgroup of $\mathbb{R}$. Furthermore, if n=2, C must be dense in G.*

PROOF. First, by Theorem 1, we may suppose that $\mathcal{L} = \mathcal{L}(A,\sigma)$, where the pair (A,σ) is Δ-compatible, i.e. $\sigma^{-1}A\sigma = \mathcal{D}(\lambda_1,\cdots,\lambda_n)$. Now $\mathcal{L}(A,\sigma) = \sigma^{-1}\mathbb{Z}^n \times_\eta \mathbb{Z}g$, so for any $(x_1,\cdots,x_n) \in \sigma^{-1}\mathbb{Z}^n$ and $k \in \mathbb{Z}$,

$$\eta(kg)(x_1,\cdots,x_n) = (\lambda_1^k x_1,\cdots,\lambda_n^k x_n) \in \sigma^{-1}\mathbb{Z}^n$$

also. For any $(x_1,\cdots,x_n) \in \sigma^{-1}\mathbb{Z}^n$ and $k,l \in \mathbb{Z}$ with $k \neq 0$, we claim that

$$X = (\frac{\lambda_1^l x_1}{1-\lambda_1^k}, \cdots, \frac{\lambda_n^l x_n}{1-\lambda_n^k}, 0) \in \mathcal{C}.$$

Since for any $m \in \mathbf{Z}$, $(x,0)(y,mkg)(x,0)^{-1} = (x - \eta(mkg)(x) + y, mkg)$, it is easy to see that $\mathcal{L}^X \supset \sigma^{-1}\mathbf{Z}^n \times_\eta \mathbf{Z}kg$. Hence

$$\mathcal{L} \cap \mathcal{L}^X \supset \sigma^{-1}\mathbf{Z}^n \times_\eta \mathbf{Z}kg,$$

and $X \in \mathcal{C}$. For any given $\epsilon > 0$, we can choose l sufficiently large so that $|\lambda_i^l x_i| < \epsilon$, if $\lambda_i < 1$. Then we can choose k large enough, so that $\lambda_i^{l-k} < \epsilon$ if $\lambda_i > 1$ and $\lambda_i^k < \epsilon$ if $\lambda_i < 1$. It follows that $(0, \cdots, 0)$ can be approximated by points in $\mathcal{C}$. Thus the discrete case can not occur. By Corollary 4, we have proven the first part of this theorem.

Now we consider the case $n = 2$; with no loss of generality, we can suppose that $\Delta = \mathcal{D}(1,-1)$, (since $\Delta = \mathcal{D}(d,-d)$ for some $d \in \mathbf{R}$ and $d \neq 0$) and for any matrix

$$A = \begin{pmatrix} a & b \\ c & k-a \end{pmatrix} \in \mathrm{SL}(2,\mathbf{Z}),$$

with trace$(A) > 2$, we have a lattice $\mathcal{L} = \mathcal{L}(A,\sigma) = \sigma^{-1}\mathbf{Z}^2 \times_\eta \mathbf{Z}g$. Construct the matrices

$$A_m = \begin{pmatrix} \frac{2m(2a-k)+m^2+k^2-4}{m^2-k^2+4} & \frac{4mb}{m^2-k^2+4} \\ \frac{4mc}{m^2-k^2+4} & \frac{m^2+k^2-4-2m(2a-k)}{m^2-k^2+4} \end{pmatrix}$$

such that $A_m \in \mathrm{SL}(2,\mathbb{Q})$ and A commutes with all A_m. Let $\lambda^{\pm 1}$ be the eigenvalues of A, and $\lambda^{\pm h_m}$ the eigenvalues of A_m. We claim that the lattice $(0, h_m \ln\lambda)\mathcal{L}(0, h_m \ln\lambda)^{-1}$ is commensurable with $\mathcal{L}$. Since if $(x, kg) \in \mathcal{L}$, we know that

$$(0, h_m \ln\lambda)(x, kg)(0, h_m \ln\lambda)^{-1} = (\eta(h_m \ln\lambda)x, kg) = (\sigma^{-1}A_m\sigma x, kg).$$

Therefore $\eta(g)$ leaves $\sigma^{-1}A_m\sigma(\sigma^{-1}\mathbb{Z}^2) = \sigma^{-1}A_m\mathbb{Z}^2$ stable. Hence $\eta(g)$ leaves $\sigma^{-1}\mathbb{Z}^2 \cap \sigma^{-1}A_m\mathbb{Z}^2$, which is isomorphic to $\mathbb{Z}^2$, stable also. Hence

$$\sigma^{-1}\mathbb{Z}^2 \cap \sigma^{-1}A_m\mathbb{Z}^2 \times_\eta \mathbb{Z}g$$

is a lattice, which is contained in both $\mathcal{L}(A,\sigma)$ and $(0, h_m \ln\lambda)\mathcal{L}(0, h_m \ln\lambda)^{-1}$. We see easily that $\lim_{m\to\infty} h_m = 0$, so $\mathcal{C}$ is dense in G. □

From the above proof, it follows that the commensuralizer, $\mathcal{C}$, of a lattice $L(A, \sigma)$ for a compatible Δ-pair (A, σ) depends on those matrices over $\mathbb{Q}$ lying in the one parameter subgroup $\eta(t) = \exp(t\Delta)$; actually we have the following

PROPOSITION 6. *Let Δ be a matrix in $\mathfrak{sl}(n, \mathbb{R})$, with distinct nonzero elements in the diagonal, let $\eta(t) = \exp(t\Delta)$, and (A, σ) be a Δ-compatible pair. If $\sigma^{-1}A\sigma = \mathcal{D}(\lambda_1, \cdots, \lambda_n)$, and*

$$H = \{h \in \mathbb{R} : \lambda_i^h = a_0 + a_1\lambda_i + \cdots + a_i\lambda_i^{n-1}\},$$

where $a_i \in \mathbb{Q}$, for $0 < i \leq n$, then H is a subgroup of $\mathbb{R}$ containing $\mathbb{Z}$.

PROOF. First we know that $\lambda_1, \cdots, \lambda_n$ are distinct. Hence the representation of the expression in our proposition is unique. Let $f(x)$ be the characteristic polynomial of A, and $g(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$, where $a_i \in \mathbb{Q}$, $g(\lambda_i) = \lambda_i^h$ for any $1 \leq i \leq n$. Since $(f(x), g(x)) = 1$, there exist polynomials $u(x)$ and $v(x) \in \mathbb{Q}[x]$ and degree of $v(x) < n$ such that $f(x)u(x) + g(x)v(x) = 1$. So $\lambda_i^{-h} = v(\lambda_i)$, and $-h \in H$. If h_1 and h_2 in H, then there exist $u(x)$ and $v(x) \in \mathbb{Q}[x]$ such that $\lambda_i^{h_1} = u(\lambda_i)$ and $\lambda_i^{h_2} = v(\lambda_i)$.

Let $g(x)$ be the remainder of $u(x)v(x)$ divided by $f(x)$. Then $g(x) \in \mathbf{Q}(x)$ and $g(\lambda_i) = \lambda_i^{h_1+h_2}$. Hence $h_1 + h_2 \in H$, and H is a subgroup. □

REMARK. If we can prove that for a given A, H is not a discrete subgroup of $\mathbf{R}$, then the commensurilizer $\mathcal{C}$ of $\mathcal{L}(A,\sigma)$ would actually be dense in G.

As a consequence of our present results we now make the following:

CONJECTURE. The commensuralizer $\mathcal{C}(\mathcal{L})$ of any lattice $\mathcal{L}$ in G is dense in G.

As in [7], we have the following:

PROPOSITION 7. *If the pair (A,σ) is Δ-compatible, then $N(\mathcal{L}(A,\sigma))$ is of the form $\mathcal{L}(B,\tau)$ for some Δ-compatible pair (B,τ). More specifically, B can be chosen so that $B^p = A$ for some integer $p > 0$, and $\tau = (I - A)\sigma$.*

PROOF. As above, we have $\mathcal{L}(A,\sigma) = \sigma^{-1}\mathbf{Z}^n \times_\eta \mathbf{Z}g$, so a calculation shows that

$$\mathcal{N} = \mathcal{N}(\mathcal{L}(A,\sigma))$$

$$= \{(v,s) \in G : v - \eta(kg)v + \eta(s)\sigma^{-1}w \in \sigma^{-1}\mathbf{Z}^n$$

for all $w \in \mathbf{Z}^n$, $k \in \mathbf{Z}$. Setting $w = 0$ we see that v must satisfy the condition $v - \eta(kg)v \in \sigma^{-1}\mathbf{Z}^n$ for all k. But $v - \eta(kg)v$ can be written as a telescoping sum of terms of the form

$$\eta((j-1)g)v - \eta(jg)v = \eta(g)^{j-1}(v - \eta(g)v),$$

and $\eta(g) = \mathcal{D}(\lambda)$ leaves $\sigma^{-1}\mathbf{Z}^n$ invariant, so it suffices for v to satisfy $v - \eta(g)v \in \sigma^{-1}\mathbf{Z}^n$, or $v \in (\sigma(I - \eta(g)))^{-1}\mathbf{Z}^n$. Since $\sigma\eta(g) = A\sigma$, this is true iff

$$v \in ((I - A)\sigma)^{-1}\mathbf{Z}^n.$$

Now the condition on s reduces to $\sigma\eta(s)\sigma^{-1}\mathbf{Z}^n = \mathbf{Z}^n$, or

$$\sigma\eta(s)\sigma^{-1} \in \mathrm{SL}(n, \mathbf{Z}).$$

Clearly this condition defines a closed subgroup H of $\mathbf{R}$. Both the subgroup and its image in $\mathrm{SL}(n, \mathbf{Z})$ are nontrivial, since $\sigma\eta(g)\sigma^{-1} = A$. But the discreteness of $\mathrm{SL}(n, \mathbf{Z})$ then implies that H can't be all of $\mathbf{R}$, so H must be of the form $\mathbf{Z}h$, and since $g \in H$ we may choose h so that $g = ph$ for some

$p \geq 1$. It follows that

$$\mathcal{N} = \tau^{-1}\mathbf{Z}^n \times_\eta \mathbf{Z}h,$$

with $\tau = (I - A)\sigma$. Now define

$$B = \tau\mathcal{D}(\lambda_1^{\frac{1}{p}}, \ldots, \lambda_n^{\frac{1}{p}})\tau^{-1},$$

so that B is in $\mathrm{SL}(n, \mathbf{R})$ with distinct positive eigenvalues, τ diagonalizes B, and the Δ-compatibility condition on the eigenvalues of B is satisfied. But from the fact that $\mathcal{N} = \tau^{-1}\mathbf{Z}^n \times_\eta \mathbf{Z}h$ is a subgroup of G, it follows that $\mathcal{D}(\lambda_1^{\frac{1}{p}}, \ldots, \lambda_n^{\frac{1}{p}}) = \eta(h)$ stabilizes $\tau^{-1}\mathbf{Z}^n$, so B must actually be in $\mathrm{SL}(n, \mathbf{Z})$. This proves that $\mathcal{N} = \mathcal{L}(B, \tau)$. Finally,

$$B^p = \tau\mathcal{D}(\lambda_1, \ldots, \lambda_n)\tau^{-1} = (I - A)A(I - A)^{-1} = A,$$

which completes the proof. □

By Theorem 2, it would be interesting to find when two matrices are conjugate by a matrix of $\mathrm{GL}(n, \mathbf{Z})$. First they must be similar. We consider the case $n = 2$. Here two such matrices are similar if and only if they have the same trace. It is natural to ask if they are also conjugate by a matrix in

GL(2, $\mathbb{Z}$). Unfortunately the answer to this question is no. For let

$$A = \begin{pmatrix} a & b \\ c & k-a \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & k \end{pmatrix}$$

be two matrices in SL(2, $\mathbb{Z}$) with $k > 2$. Then we have the following result.

THEOREM 8. *Let A and B be as above. If $b = \pm 1$ or $c = \pm 1$, then A and B are conjugate by a matrix in* GL(2, $\mathbb{Z}$).

PROOF. First we suppose that $b = \pm 1$. It is easy to check that

$$X = \begin{pmatrix} x & 2y \\ ax + 2cy & bx + 2(k-a)y \end{pmatrix}$$

satisfies $XA = BX$, and

$$\det(X) = bx^2 + 2(k-2a)xy - 4cy^2 = \frac{(bx + (k-2a)y)^2 - (k^2-4)y^2}{b}.$$

We know that the Pell equation $x^2 - (k^2 - 4)y^2 = 1$ always has infinitely many integer solutions (see [8]). Hence there is a $X \in$ GL(2, $\mathbb{Z}$) such that $XA = BX$, i.e. A and B are conjugate by a matrix in GL(2, $\mathbb{Z}$). For the case $c = \pm 1$, we proceed similarly, we only need to replace x by 2x and 2y by y. □

On the other hand, we can find two matrices with the same trace, which are not conjugate by any matrix in $\mathrm{GL}(2,\mathbb{Z})$. Let

$$A = \begin{pmatrix} 2m+1 & 2m \\ 2m+2 & 2m+1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 4m+2 \end{pmatrix},$$

where $m \in \mathbb{Z}$ and $m > 0$. We have the following:

PROPOSITION 9. *A and B are not extendedly conjugate.*

PROOF. Suppose that A and B are conjugate by a matrix

$$X = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}),$$

i.e. $AX = XB$. One see easily that $y = -(2m+1)x - 2mz$ and $w = -(2m+2)x-(2m+1)z$, so $\det(X) = -(2m+2)x^2 + 2mz^2$ is an even integer. Hence X is not in $\mathrm{GL}(2,\mathbb{Z})$. But

$$B^{-1} = \begin{pmatrix} 4m+2 & -1 \\ 1 & 0 \end{pmatrix},$$

and by Theorem 8, we know that B and B^{-1} are conjugate in $\mathrm{GL}(2, \mathbf{Z})$. Hence A and B^{-1} are not conjugate in $\mathrm{GL}(2, \mathbf{Z})$ either. $\square$

§3. The Quasi-regular Representations

We now turn to some related questions of representation theory for such groups. The representations R we shall consider are the so called quasi-regular representations. These are the ones where the group operates by right translation on some space of functions on the homogeneous space G/Γ, for a lattice Γ in G, as above. Thus $R_g(f)(x) = f(xg)$, where $g \in G$, $x \in G/\Gamma$ and f is a real or complex function on G/Γ. In the case that the function space is $L^2(G/\Gamma)$ such representations for nilpotent Lie groups were first studied by C.C. Moore in [4] and also by L. Corwin and F. Greenleaf in [3], where the latter two authors got a formula to compute the multiplicity of every irreducible subrepresentation. In the case the function space is $C^\infty(G/\Gamma)$ the problem of decomposition and calculation of multiplicity of the right regular representation was done for certain solvable groups by J. Brezin in [2]. Actually, there the author proves the multiplicity function (see below) is unbounded in certain cases by looking at its average. In what follows we shall extend this result of Brezin, but without averages, to all the groups we have been considering when $n = 2$ by estimating the multiplicity of an indecomposable representation.

On pg. 27 of [2] the author conjectures that the multiplicity function will be unbounded whenever the solvmanifold is non-degenerate, in the sense that its fundamental group does not contain any normal abelian subgroup of finite index. Since, if our lattice contains such an abelian subgroup, then this subgroup is also a lattice of G, in view of the density theorem of Mosak and Moskowitz see [5], G would be abelian. Therefore this non-degeneracy condition is always satisfied for the groups we consider. Hence, in our Theorem below, we are proving special cases of this conjecture. See below for the relevant definitions.

THEOREM 10. mult(ω) *is an unbounded function on* Ω.

Actually, our Theorem 10 is likely to be true for arbitrary n. The proof of such a result would depend, among other things, on generalizing Theorem 5 to higher n. This would, if true, yield a proof of the conjecture of Brezin in additional cases.

As a consequence of Theorem 1 above we can now replace our upper triangular matrix Δ by any matrix of trace 0 in $\mathfrak{sl}(k, \mathbf{R})$ with only real roots and henceforth we shall assume this has been done. Actually, a refinement of this statement can be made. It is the following

Proposition 11. *Let $\mathcal{L}(A,\sigma)$ be a lattice of $G = \mathbf{R}^k \times_\eta \mathbf{R}$, then there exists an isomorphism of G to G' such that G' has a lattice of $\mathbf{Z}^k \times_{\eta'} \mathbf{Z}$, where $\eta'(1) = A$.*

Proof. let $\eta'(t) = \sigma\eta(gt)\sigma^{-1}$, then we have the following commutative diagram

$$\begin{array}{ccc} \mathbf{R}^n & \xrightarrow{\sigma} & \mathbf{R}^n \\ \Big\downarrow{\scriptstyle \eta(tg)} & & \Big\downarrow{\scriptstyle \eta'(t)} \\ \mathbf{R}^n & \xrightarrow{\sigma} & \mathbf{R}^n \end{array}$$

and $\eta'(1) = A$. Let $G' = \mathbf{R}^k \times_{\eta'} \mathbf{R}$, then G' is isomorphisic to G, and $\mathbf{Z}^k \times_{\eta'} \mathbf{Z}$ is a lattice of G'. □

By Propositon 11, we can assume without loss of generality that Γ is the lattice, $\mathbf{Z}^k \times_\eta \mathbf{Z}$. Thus, from now on, we shall always consider Γ to be the integer points of our linear Lie group. We shall take for the space of functions $C^\infty(G/\Gamma)$, with the Frechet semi-norm topology. This means that a sequence of functions converges if all derivatives of all orders converge uniformly on the compact space G/Γ.

Since the group $G = \mathbf{R}^k \times_\eta \mathbf{R}$, if $f \in C^\infty(G/\Gamma)$, then for each fixed $t \in \mathbf{R}$, the function $\underline{u} \to f(\underline{u}, t)$ on $\mathbf{R}^k$ is periodic with respect to $\mathbf{Z}^k$. This is because

$\underline{u} \in \mathbf{R}^k$, $\underline{n} \in \mathbf{Z}^k$, and $t \in \mathbf{R}$ imply

$$f(\underline{u}+\underline{n},t) = f((\underline{n},0)\cdot(\underline{u},t) = f(\underline{u},t).$$

Hence, for each fixed $t \in \mathbf{R}$, we can expand $f(\cdot,t)$ in a Fourier series

$$f(\underline{u},t) = \sum_{\underline{n}\in\mathbf{Z}^k} a_{\underline{n}}(t)e(<\underline{n},\underline{u}>),$$

in which $<\underline{n},\underline{u}>= n_1u_1 + n_2u_2 + \cdots + n_ku_k$ and e is the function $t \to \exp(2\pi it)$ on $\mathbf{R}$. For the functions $a_{\underline{n}}(t)$, we have the following

PROPOSITION 12. *The function $a_{\underline{n}}$ on $\mathbf{R}$ is a bounded C^∞ function. Furthermore,*

$$a_{\underline{n}\eta(m)}(t) = a_{\underline{n}}(t+m)$$

for all $m \in \mathbf{Z}$ and $t \in \mathbf{R}$.

PROOF. We write $a_{\underline{n}}$ in terms of f as an integral over a fundamental domain,

$$a_{\underline{n}}(t) = \int_0^1 \cdots \int_0^1 f(\underline{u},t)e(\underline{n},-\underline{u})du_1\cdots du_n.$$

It follows that $a_{\underline{n}}$ is C^∞ and that $||a_{\underline{n}}||_\infty \le ||f||_\infty$.

The relation between $a_{\underline{n}}$ and $a_{\underline{n}\eta(m)}(t)$ follows from the periodicity of f with respect to the elements $(\underline{0}, m)$ of Γ. Because of left invariance we have

$$f(\underline{u}, t) = f((\underline{0}, m)(\underline{u}, t)) = f(\eta(m)\underline{u}, t + m),$$

which in turn implies that

$$\sum_{\underline{n}} a_{\underline{n}}(t + m)e(< \underline{n}, \eta(m)\underline{u} >) = \sum_{\underline{n}} a_{\underline{n}}(t)e(\underline{n}, \underline{u}). \qquad (1)$$

Since $< \underline{n}, \eta(m)\underline{u} >=< \underline{n}\eta(m), \underline{u} >$, we get our formula, by comparing the coefficients of both sides of (1). $\square$

As a consequence of Proposition 12, if $a_{\underline{n}} = 0$, then $a_{\underline{m}} = 0$, whenever $\underline{m}$ is in the $\eta(\mathbb{Z})$-orbit of $\underline{n}$. Let Ω denote the family of all the $\eta(\mathbb{Z})$ orbits of points of $\mathbb{Z}^k$. For each $S \subset \Omega$, let $\mathcal{Z}(S)$ denote those $f \in C^\infty(G/\Gamma)$ for which $a_{\underline{m}} = 0$ whenever $\underline{m}$ is not in S. As in [2], we have the following

PROPOSITION 13. *$\mathcal{Z}(S)$ is an R–invariant subspace of $C^\infty(G/\Gamma)$.*

PROOF. Let $f \in \mathcal{Z}(S)$, and $g = (\underline{v}, t)$, then

$$(R_g f)(\underline{u}, s) = f(\underline{u} + \eta(s)\underline{v}, s + t).$$

By the Fourier expansion, we have

$$(R_g f)(\underline{u}, s) = \sum_{\underline{n}\in S} a_{\underline{n}}(s+t)e < \underline{n}, \underline{u} + \eta(s)\underline{v} >= \sum_{\underline{n}\in S} b_{\underline{n}}(s) < \underline{n}, \underline{u} >,$$

where $b_{\underline{n}} = a_{\underline{n}}(s+t)e < \underline{n}, \eta(s)\underline{v} >$. Hence if $\underline{n}$ is not in S, then $b_{\underline{n}} = 0$. Hence $R_g f \in \mathcal{Z}(S)$. □

Now among the subspaces $\mathcal{Z}(S)$, those of the form $\mathcal{Z}(\omega)$ for some $\omega \in \Omega$ are minimal. These minimal subspaces also span $C^\infty(G/\Gamma)$ in the sense that their closed linear span in the C^∞ topology gives everything.

In the following proposition f_ω will denote the partial sum

$$\sum_{\underline{n}\in\omega} a_{\underline{n}}(t)e(< \underline{n}, \underline{u} >). \tag{2}$$

PROPOSITION 14. *The quasi regular representation decomposes uniquely as a direct sum*

$$C^\infty(G/\Gamma) = \sum \oplus_{\omega\in\Omega}\mathcal{Z}(\omega),$$

in the sense that every $f \in C^\infty(G/\Gamma)$ can be written in precisely one way as a convergent sum $\sum_{\omega\in\Omega} f_\omega$ with $f_\omega \in \mathcal{Z}(\omega)$.

PROOF. It is obvious that the summands f_ω are unique, if they exist. The main point is to check that the function f_ω is in $\mathcal{Z}(\omega)$ and that the series $\sum_\omega f_\omega$ converges to f (with any ordering of Ω). Now f_ω is in $\mathcal{Z}(\omega)$ because ω is a $\eta(\mathbf{Z})$-orbit in $\mathbf{Z}^k$. Since, for each fixed $t \in \mathbf{R}$, $f(.,t)$ is a C^∞ function on the k-torus T^k, the sum in (2) converges in the C^∞ topology on $C^\infty(T^k)$ and therefore, can be differentiated term by term with respect to the variables $u_1, \cdots, u_k$ as often as desired.

Recall that $a_{\underline{n}}$ is gotten by integrating:

$$a_{\underline{n}}(t) = \int_0^1 \cdots \int_0^1 f(\underline{u}, t) e < \underline{n}, \underline{u} > du_1 \cdots du_k. \tag{3}$$

Let Δ denote the Laplacian $\frac{\partial^2}{\partial_{u_1}} + \cdots + \frac{\partial^2}{\partial_{u_k}}$. Integrating by parts, we see that (3) implies that the l-th derivative $a_{\underline{n}}^{(l)}(t)$ of $a_{\underline{n}}$ with respect to t must satisfy

$$|a_{\underline{n}}^{(l)}(t)| \leq (4\pi^2(n_1^2 + \cdots + n_k^2))^{-h} \int_0^1 \cdots \int_0^1 |\partial_t^l \Delta^h f(\underline{u}, t)| du1 \cdots du_k, \tag{4}$$

for all integers $h \geq 0$. If we set $C_{h,l} = \sup_{0 \leq u_1, \cdots, u_k \leq 1} (4\pi^2)^{-h} |\partial_t^l \Delta^h f(\underline{u}, t)|$, then (4) can be rewritten as

$$\sup_{0\le t\le 1} |a_{\underline{n}}^{(l)}(t)| \le C_{h,l} < \underline{n}, \underline{n} >^{-h} . \qquad (5)$$

Using (5) with $h = a_1 + a_2 + \cdots + a_k + k$, we see that the mixed partial $\partial_t^j \partial_{u_1}^{a_1} \cdots \partial_{u_k}^{a_k}$ of the sum in (2) can be evaluated by term by term differentiation. Hence (2) defines a C^∞ function on G/Γ. If we let f_ω denote that function, then from (5) we have for some $C > 0$,

$$\sup_{0\le u_1,\cdots,u_k\le 1} |\frac{\partial^j f_\omega(\underline{u}, t)}{\partial_{u_1}^{a_1} \cdots \partial_{u_k}^{a_k} \partial_t^i}| \le C \sum_{\underline{n}} < \underline{n}, \underline{n} >^{j-h}, \qquad (6)$$

where $j = i + a_1 + \cdots + a_k$. Taking $h - j \ge \frac{m+1}{2}$ and summing (6) over all $\omega \in \Omega$, we see that, independent of the order of summation, the series $\sum_{\omega\in\Omega} f_\omega$ converges in the C^∞-topology to f. □

Since the representation we are considering is on a Frechet space, not on a Hilbert space, we can only decompose the representation as a direct sum of indecomposable sub-representations. Let's recall that an R-invariant subspace V of the space $C^\infty(G/\Gamma)$ is indecomposable if it can not be decomposed into two nontrival R-invariant subspaces. This is a weaker notion of irreducible. (An irreducible subspace does not have any nontrivial R-

invariant subspaces.) By Corollary 1.39 in [2], $\mathcal{Z}(\omega)$ is an indecomposable R-invariant subspace, when $n = 2$.

DEFINITION. Let ω_1 and ω_2 be two elements in Ω; if there exists an $f \in \mathrm{Hom}_R(\mathcal{Z}(\omega_1), \mathcal{Z}(\omega_2))$, and f is an isomorphism as linear map, then we call $\mathcal{Z}(\omega_1)$ and $\mathcal{Z}(\omega_2)$ isomorphic. For any $\omega \in \Omega$, we define the multiplicity of $\mathcal{Z}(\omega)$, mult(ω), as the number of all the subspaces $\mathcal{Z}(\omega')$ isomorphic to $\mathcal{Z}(\omega)$.

REMARK. By Proposition 1.44 of [2], we see that mult(ω) is finite when $k = 2$.

Our purpose now is to show that when $n = 2$, the multiplicities mult(ω) in $Z(\omega)$ are unbounded. A very particular case of this result was proven in [2]. More precisely, there only the case where $A = \eta(1)$ is of the form

$$X = \begin{pmatrix} a & b \\ Db & a \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}),$$

where D is a square-free positive integer congruent to 2 or to 3 (mod 4) and $a^2 - Db^2 = 1$ is treated. We now turn to the proof of Theorem 10.

PROOF OF THEOREM 10. Here we use the same notation as in the proof of Theorem 5. $A = \eta(1)$,

$$\begin{pmatrix} \lambda^g & 0 \\ 0 & \lambda^{-g} \end{pmatrix} = \sigma\eta(g)\sigma^{-1},$$

where σ diagonalizes A and A_m and h_m have the same meaning as in the proof of Theorem 5. Since $h_m \to 0$, for any given positive integer N, there exists an m such that

$$0 < |Nh_m| < |g|.$$

We claim that

$$((m^2 - k^2 + 4)^N, (m^2 - k^2 + 4)^N)\eta(ih_m)\eta(\mathbb{Z}) \subset \mathbb{Z}^2 (0 \leq i \leq N)$$

are disjoint orbits under the action of $\eta(\mathbb{Z})$. Indeed, if

$$((m^2 - k^2 + 4)^m, (m^2 - k^2 + 4)^m)\eta(ih_m)\eta(\mathbb{Z})$$

and

$$((m^2-k^2+4)^m,(m^2-k^2+4)^m))\eta(jh_m)\eta(\mathbf{Z})$$

are in the same $\eta(\mathbf{Z})$ orbit, then there exists an integer k such that

$$((m^2-k^2+4)^m,(m^2-k^2+4)^m)\eta(kg)\eta(ih_m)=$$

$$((m^2-k^2+4)^m,(m^2-k^2+4)^m)\eta(jh_m).$$

Since

$$\begin{pmatrix} \lambda^g & 0 \\ 0 & \lambda^{-g} \end{pmatrix} = \sigma\eta(g)\sigma^{-1},$$

we have $kg=(j-i)h_m(-N\leq j-i\leq N)$. But, this is true only when $k=0$, and $i=j$. However, these are obviously in the same $\eta(\mathbf{R})$ orbit. Hence, by Corollary 1.35 of [2], the multiplicity of $((m^2-k^2+4)^N,(m^2-k^2+4)^N)$ is at least $N+1$. □

§4. Genus sets

We begin with the space HP^n. We have the following

THEOREM 15. *The Mislin genus set $\mathcal{G}(HP^n < 4 >)$ is uncountably large.*

REMARK. Actually we can use the same method to prove the corresponding results for more general spaces. Indeed, let $X = S^{2n} \cup e^{4n} \cdots \cup e^{2kn}$, where $n \geq 2$ and $k > 0$ be a space with truncated polynomial rational cohomology algebra, then the genus set of $X < 2n >$ is uncountably large also.

To prove this Theorem, we need some lemmas. Let W denote $HP^n < 4 >$, then we have

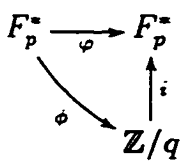
LEMMA 16. *The image of $Aut(W_p)$ in $Aut((W_p)_0)$ is contained in the $(n+1)$-power of the p-adic units.*

PROOF. Considering the Sullivan's minimal model of HP^n, plus Corollary 1.1 of [13], we know that the image of $Aut(W_p)$ in $Aut((W_p)_0)$ is contained in the $(n+1)$-power of the p-adic units. □

Now we turn to some facts in number theory. All known results we need can be found in Serre [16]. First we fix an odd prime number q. Then for all those prime numbers $p \equiv 1 (\mathrm{mod}\, q)$, we define a homomorphism $\chi_{p,q} : F_p^* \rightarrow U_q \subset \mathbf{C}$, where U_q is the multiplicative group of the q-th roots of unity, like the Legendre symbol. First we define a homomorphism φ from F_p^* to F_p^* by

$$x \rightarrow x^{\frac{p-1}{q}}$$

Since F_p^* is a cyclic group of order $p-1$, and $q|p-1$, we have the following lifting diagram



and we use the standard homomorphism $v : \mathbb{Z}/q :\rightarrow U_q$, defined by $v(x) = e^{2\pi i x}$. Setting $\chi_{p,q} = v\phi$, it is obvious that $\chi_{p,q}$ is a homomorphism. It is easy to see that we have the following

LEMMA 17. *$\chi_{p,q}$ is an epimorphism and $\chi_{p,q}(x) = 1$ if and only if $x = y^q$, for some $y \in F_p^*$.*

Now we turn to the proof of Theorem 15.

PROOF OF THEOREM 15. First note that W is rationally equivalent to an odd sphere, hence W is a rational H-space. Hence we can use Wilkerson's double coset formula (see [17]), i.e. we have

$$\mathcal{G}(W) = i_* Aut(W_0) \backslash C Aut(\bar{W}) / j_* Aut(\hat{W})$$

(Note that since W is rationally equivalent to an odd sphere, it can be seen that $\mathcal{G}(W) = \hat{\mathcal{G}}_0(W)$. In general, the double coset formula is valid only for $\hat{\mathcal{G}}_0$.)

In the formula, W_0 is the rationalization of W, $\hat{W}$ is its profinite completion, and $\bar{W}$ is Sullivan's formal completion. i is the standard inclusion $\hat{W} \to \bar{W}$, j first rationalizes and then identifies $(\hat{W})_0$ with $\bar{W}$, and $CAut(\bar{W})$ is all those automorphisms of $\bar{W}$ whose induced homotopy group automorphisms are $\hat{\mathbf{Q}}$-module isomorphisms, where $\hat{\mathbf{Q}} = \mathbf{Q} \otimes \hat{\mathbf{Z}}$.

If n is odd, then the image of

$$Aut(W_p) \to Aut((W_p)_0)$$

is contained in $\mathcal{U}_p^{n+1}$, which is a subgroup of $\mathcal{U}_p^2$, hence we have the following surjective map

$$\Delta(\mathbb{Q}^*)\backslash\hat{\mathbb{Q}}^*/\Pi_p\mathcal{U}_p^{n+1} \to \Delta(\mathbb{Q}^*)\backslash\hat{\mathbb{Q}}^*/\Pi_p\mathcal{U}_p^2$$

Hence the genus set $\mathcal{G}(HP^n < 4 >)$ is uncountably large by the result of McGibbon and Moller.

If n is even, let q be an odd prime dividing $n+1$. Let $\dot{\underline{\mathbb{Q}}} = \mathbb{Q} \otimes \Pi_{p\equiv 1(\mathrm{mod}\, q)}\mathbb{Z}_p$, and $\underline{\mathbb{Q}^*}$ denote the units of $\underline{\mathbb{Q}}$. Then we can construct a map

$$\Phi : \Delta(\mathbb{Q}^*)\backslash\underline{\mathbb{Q}^*}/\Pi_{p\equiv 1(\mathrm{mod}\, q)}\mathcal{U}_p^q \to \Pi_{p\equiv 1(\mathrm{mod}\, q)}(\mathbb{Z}/q)$$

Like McGibbon and Moller, we define Φ by sending the p-th component to its image under $\chi_{p,q}$. Then it is an epimophism and its kernel is $\Pi_{p\equiv 1(\mathrm{mod}\, q)}\mathcal{U}_p^q$. Since there are infinitely many primes $p \equiv 1(\mathrm{mod}\, q)$ (by Dirichlet's Theorem), the following set

$$\Delta(\mathbb{Q}^*)\backslash\underline{\mathbb{Q}^*}/\Pi_{p\equiv 1(\mathrm{mod}\, q)}\mathcal{U}_p^q$$

is uncountably large. It is obvious that we have the following surjective map

$$\Delta(\mathbb{Q}^*)\backslash\hat{\mathbb{Q}}^*/\Pi_p\mathcal{U}_p^q \to \Delta(\mathbb{Q}^*)\backslash\underline{\mathbb{Q}^*}/\Pi_{p\equiv 1(\mathrm{mod}\, q)}\mathcal{U}_p^q.$$

Therefore, the set $\mathcal{G}(HP^n < 4 >)$, which has the same cardinality as $\Delta(\mathbb{Q}^*)\backslash\hat{\mathbb{Q}}^*/\Pi_p\mathcal{U}_p^q$, is uncountable also. Hence, we have proved Theorem 15.

□

Now we consider the spaces $S^3 \vee S^3$ and $S^4 \vee S^4$. We have the following

THEOREM 18. *The genus sets* $\hat{\mathcal{G}}_0(S^3 \vee S^3 < 3 >)$ *and* $\hat{\mathcal{G}}_0(S^4 \vee S^4 < 4 >)$ *both are uncountably large.*

PROOF. We prove the two cases simultaneously. Let $W = S^3 \vee S^3 < 3 >$(or $S^4 \vee S^4 < 4 >$, respectively). Like the proof of Theorem 15, we use Wilkerson's double coset formula

$$\hat{\mathcal{G}}_0(W) = i_*Aut(W_0)\backslash CAut(\bar{W})/j_*Aut(\hat{W})$$

where we use the same notation as in the proof of Theorem 15. By Felix and Halperin's mapping theorem (see [11] and also [12] for James's elegant proof of it). we know that the rational category $cat_0(W)$ of W is 1, hence W is a co-H_0 space, i.e. W is rationally equivalent to a wedge of certain spheres. Suppose that $W_0 = (\vee_{n>1} S_n)_0$, where S_n is a wedge of spheres of dimension $2n+1$ (or $3n+1$ respectively), let k_n be the number of spheres S^{2n+1} (or S^{3n+1}) in S_n. Hence $k_2 = 1$ (or $k_2 = 3$, respectively) and $k_3 = 2$. And also $cat_0(\bar{W}) = 1$. Hence $\bar{W} = K(\hat{\mathbf{Q}}, 5) \vee K(\hat{\mathbf{Q}}, 7) \vee K(\hat{\mathbf{Q}}, 7) \vee V_1$ (or $K(\hat{\mathbf{Q}}, 7) \vee K(\hat{\mathbf{Q}}, 7) \vee K(\hat{\mathbf{Q}}, 7) \vee V_2$ respectively), where V_1 and V_2 are wedges of spheres of dimension bigger than 7. Since S_n is a retract of $\bar{W}$, $CAut(S_n)$, which is isomorphic to $\mathrm{GL}(k_n, \hat{\mathbf{Q}})$, is a split subgroup of $CAut(\bar{W})$. Let P_i be the projection of $CAut(\bar{W})$ to $\mathrm{GL}(k_i, \hat{\mathbf{Q}})$, then we have the following surjective map

$$i_* Aut(W_0) \backslash CAut(\bar{W}) / j_* Aut(\hat{W}) \to$$

$$P_i(i_* Aut(W_0)) \backslash P_i(CAut(\bar{W})) / P_i(j_* Aut(\hat{W})).$$

defined by the obvious map. Hence it is enough to prove the latter set is uncountably large. Now let's consider the group $j_* Aut(\hat{W})$. By Neisendorfer's Theorem, we know that any map from W_p to itself comes from a self map of $(S^3 \vee S^3)_p$(or $(S^4 \vee S^4)_p$ respectively), whose automorphism group is exactly $\mathrm{GL}(2, \mathbf{Z}_p)$. Let ι_1, ι_2 be generators of $\pi_3(S^3 \vee S^3)$ (or $\pi_4(S^4 \vee S^4)$ respectively), then for any self-equivalence of $(S^3 \vee S^3)_p$(or $(S^4 \vee S^4)_p$ respectively), we can determine it by

$$(\iota_1)_p \to a(\iota_1)_p + b(\iota_2)_p, \quad (\iota_2)_p \to c(\iota_1)_p + d(\iota_2)_p$$

where $a, b, c, d \in \mathbf{Z}_p$, and $\delta = ad - bc \in \mathbf{Z}_p^*$. We know that up to rationalization, $\pi_7(S_p^3 \vee S_p^3)$ is generated by $[[(\iota_1)_p, (\iota_2)_p], (\iota_1)_p]$,and $[[(\iota_1)_p, (\iota_2)_p], (\iota_2)_p]$, hence the above map induces the following map

$$[[(\iota_1)_p, (\iota_2)_p], (\iota_1)_p] \to \delta a[[(\iota_1)_p, (\iota_2)_p], (\iota_1)_p] + \delta b[[(\iota_1)_p, (\iota_2)_p], (\iota_2)_p],$$

$$[[(\iota_1)_p, (\iota_2)_p], (\iota_2)_p] \to \delta c[[(\iota_1)_p, (\iota_2)_p], (\iota_1)_p] + \delta d[[(\iota_1)_p, (\iota_2)_p], (\iota_1)_p].$$

for $(S^4 \vee S^4)_p$, it induces the following map

$$[(\iota_1)_p, (\iota_1)_p] \to a^2[(\iota_1)_p, (\iota_1)_p] + 2ab[(\iota_1)_p, (\iota_2)_p] + b^2[(\iota_2)_p, (\iota_2)_p]$$

$$[(\iota_1)_p, (\iota_2)_p] \rightarrow ac[(\iota_1)_p, (\iota_1)_p] + (ad + bc)[(\iota_1)_p, (\iota_2)_p] + bd[(\iota_2)_p, (\iota_2)_p]$$

$$[(\iota_2)_p, (\iota_2)_p] \rightarrow c^2[(\iota_1)_p, (\iota_1)_p] + 2cd[(\iota_1)_p, (\iota_2)_p] + d^2[(\iota_2)_p, (\iota_2)_p]$$

Therefore $P_3(j_*Aut(\hat{W}))$ (for the former case) consists of all such matrices

$$\begin{pmatrix} \delta a & \delta b \\ \delta c & \delta d \end{pmatrix}$$

and (for the latter case) $P_2(j_*Aut(\hat{W}))$ consists of all matrices

$$\begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}$$

where $a, b, c, d \in \mathbb{Z}_p$ and $\delta = ad - bc \in \mathbb{Z}_p^*$. These matrices have determinant δ^3. Hence we can construct the following obvious onto map (for the former case)

$$P_3(i_*Aut(W_0))\backslash P_3(CAut(\bar{W}))/P_3(j_*Aut(\hat{W}))$$

or (for the latter case)

$$P_2(i_*Aut(W_0))\backslash P_2(CAut(\bar{W}))/P_2(j_*Aut(\hat{W}))$$

$$\to \mathbb{Q}^*\backslash\hat{\mathbb{Q}}^*/\Pi_p\mathcal{U}_p^3$$

induced by sending an element of $\mathrm{GL}(k_3, \mathbb{Q}_p)$ (or $\mathrm{GL}(k_2, \mathbb{Q}_p)$, respectively) to its determinant. We know that the set $\mathbb{Q}^*\backslash\hat{\mathbb{Q}}^*/\Pi_p\mathcal{U}_p^3$ is uncountably large by the proof of Theorem 15. Hence $\hat{\mathcal{G}}_0(W)$ is uncountably large. □

REMARK. It is obvious that the genus set of $S^n \vee S^n < n > (n > 2)$ is uncountably large, via the above method. And also the above map induces a map on $\pi_9(S^3 \vee S^3)_p$ can be represented by the matrix

$$\begin{pmatrix} \delta a^2 & \delta ab & \delta b^2 \\ \delta c^2 & \delta cd & \delta d^2 \\ 2\delta ac & \delta(ad+bc) & 2\delta bd \end{pmatrix}$$

with respect to a suitable basis. This matrixhas determinant δ^6. Therefore for any $2 < k < 9$, the genus set $\hat{\mathcal{G}}_0(S^3 \vee S^3 < k >)$ is uncountably large. More generally, the genus set of $S^n \vee S^n < k >$(where $n - 1 < k < 4(n-1)+1$), and n is an odd number bigger than 3) is uncountably large.

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