Generalized Differential Galois Theory

by

Peter Landesman

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Date

Richard C. Churchill
Chair of Examining Committee

Date

Jozef Dodziuk
Executive Officer

Supervisory committee

Richard C. Churchill
Alex Heller
Raymond T. Hoobler
William Sit
Jerald J. Kovacic

THE CITY UNIVERSITY OF NEW YORK

Abstract

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A Galois theory of differential fields with parameters is developed in a manner that generalizes Kolchin's theory. It is shown that all connected differential algebraic groups are Galois groups of some appropriate differential field extension. Additionally, the functor of points on the category of differential schemes is shown to take differential group schemes to differential algebraic groups.

Preface

This thesis generalizes E. R. Kolchin's approach to the Galois Theory of differential fields, as presented in Chapter VI of his book *Differential Algebra and Algebraic Groups* [24]. Here differential algebraic groups, as developed in his book *Differential Algebraic Groups* [25], are shown to be Galois groups of certain types of differential field extensions. Elementary, but non-trivial, examples of this generalized Galois theory are discussed in the final Chapter.

Parts of this thesis are merely a straightforward generalization of Kolchin's work. When this is so, not only are the ideas of the proof that of Kolchin, but also many of the words. This is a testimony both to the quality of Kolchin's work and to the limitations of this author's ability to find a clearer means of expression. Although the words may be similar, the interpretations of the sentences are more general, and in that sense different.

Several other people have developed Galois theories of differential fields: Drach [8], Vessiot [38], Pommerat [33], Umemura [36] [37], Pillay [30] [31] [32] and Kovacic [21]. Although the Galois groups of Pillay's theory are differential groups, they are only algebraically finite dimensional. In general, a differential algebraic group may have infinite algebraic dimension even thought its differential dimension is finite. In this thesis, a Galois theory is developed in which any connected differential algebraic group is the Galois group of some differential field extension (Theorem 5.239).

To see how Kolchin's theory can be enriched, consider two commuting derivations D_t and D_x acting on a field $\mathfrak{F}$. Let $\mathfrak{U}$ be a universal differential extension field of $\mathcal{F}$ with respect to both D_t and D_x. Let $\mathcal{G}$ be a subfield of $\mathcal{U}$ containing $\mathcal{F}$ which is closed under the operation D_x. If $\mathcal{G}$ over $\mathcal{F}$ is a strongly normal extension of D_x-fields, in the sense of Kolchin, the set of D_x-isomorphisms of $\mathcal{G}$ into $\mathcal{U}$ over $\mathcal{F}$, when $\mathcal{U}$ is viewed as a universal differential extension of $\mathcal{F}$ with respect to D_x, has the structure of a C-group (algebraic group defined over C), where C is the field of D_x-constants of $\mathcal{F}$. All the D_x-isomorphisms are obtained by sending the D_x-generators of $\mathcal{G}$ to rational expressions in the D_x-generators of $\mathcal{G}$ and their D_x-derivatives, with D_x-constants in $\mathcal{U}$ as coefficients. These constants are not necessarily constants with respect to D_t. The generators of $\mathcal{G}$ may satisfy differential equations in D_t as well as D_x. A D_x-isomorphism of $\mathcal{G}$ into $\mathcal{U}$ will extend to a D_t-isomorphism of the D_t and D_x field $\mathcal{H}$ generated by D_t-derivatives of $\mathcal{G}$ if and only if it maps solutions of the system of differential equations in D_t and D_x to other solutions of the same system. The D_x-isomorphisms of $\mathcal{G}$ into $\mathcal{U}$ which are D_t and D_x isomorphisms form a subgroup H of G defined by differential equations with respect to D_t. If the D_x-constants of $\mathcal{F}$ equal the D_x-constants of D_t-extension $\mathcal{H}$ generated by $\mathcal{G}$, then it will be shown that H will be a Galois group for $\mathcal{H}$ over $\mathcal{F}$. That is: subfields of $\mathcal{H}$ closed under both D_t and D_x are in bijection with subgroups of H defined over C by D_t-equations.

An example will serve to elucidate the nature of the generalization herein. Let C be the complex numbers and $C[t, x]$ be a polynomial ring in two variables with standard derivations D_t and D_x. Consider $\mathcal{F} = C(t, x, \cos t, \sin t)$

and $\mathcal{G} = \mathcal{F}(\log x \sin t) = \mathcal{F}(\log x)$ as differential fields with respect to D_t and D_x. Let $\mathcal{C} = C(t, \cos t, \sin t)$ be the field of D_x-constants of $\mathcal{F}$ and let $\mathcal{U}^{D_x}$ be the same of $\mathcal{U}$. Note that in this example the D_t-field generated by $\mathcal{G}$ is $\mathcal{G}$, and the field of D_x-constants of $\mathcal{G}$ equals that of $\mathcal{F}$.

Let $\eta = \log x \sin t$, and $\zeta = \sin t/x \in \mathcal{F}$. Then η satisfies the equation $D_x\eta = \zeta$ and $\mathcal{G}$ as a D_x-extension over $\mathcal{F}$ is a strongly normal extension in the sense of Kolchin. The Galois group $\mathrm{Isom}^{D_x}(\mathcal{G}/\mathcal{F}) = \mathrm{Aut}^{D_x}(\mathcal{G}\mathcal{U}^{D_x}/\mathcal{F}\mathcal{U}^{D_x})$ is isomorphic to the additive group $\mathcal{U}^{D_x}$, is defined over $\mathcal{C}$, and will be denoted by G_a via this identification. More explicitly, consider $\sigma \in \mathrm{Aut}^{D_x}(\mathcal{G}/\mathcal{F})$. Then, in order for σ to commute with D_x, $\sigma\eta$ must again be a solution to this differential equation, and therefore $\sigma\eta$ must equal $\eta + \rho(\sigma)$, where $\rho(\sigma) \in \mathcal{U}^{D_x}$. There being no other algebraic conditions on $\rho(\sigma)$, the map $\rho : \sigma \mapsto \rho(\sigma)$ defines a group isomorphism between $\mathrm{Isom}^{D_x}(\mathcal{G}/\mathcal{F})$ and the full algebraic group G_a.

Let $\gamma = \cos t/\sin t \in \mathcal{F}$. Then η also satisfies the differential equation $D_t\eta - \gamma\eta = 0$. Indeed, in the (D_x, D_t)-differential polynomial ring $\mathcal{F}\{y\}$, it is easy to verify that the two differential polynomials $A(y) = D_x y - \zeta$ and $B(y) = D_t y - \gamma y$ form a characteristic set of a linear differential ideal $\mathfrak{P} = [A(y), B(y)]$ (relative to any ranking) with η as a generic zero over $\mathcal{F}$. Consider σ in the subgroup $H = \mathrm{Isom}^{D_t,D_x}(\mathcal{G}/\mathcal{F}) = \mathrm{Aut}^{D_t,D_x}(\mathcal{G}\mathcal{U}^{D_x}/\mathcal{F}\mathcal{U}^{D_x})$ of $\mathrm{Isom}^{D_x}(\mathcal{G}/\mathcal{F}) = G_a$. Then σ must map η to a generic solution of $\mathfrak{P}$. Thus $0 = B(\sigma(\eta)) = B(\eta+\rho(\sigma)) = B(\rho(\sigma))$, which implies that $\rho(\sigma) = c(\sigma) \sin t$, where $c(\sigma)$ is a constant with respect to D_t. But $\rho(\sigma)$ is a D_x-constant, and so $c(\sigma)$

must be one, too. Conversely, it is clear that given any D_t constant $k \in \mathcal{U}^{D_x}$, the map σ where $\sigma(\eta) = \eta + k \sin t$ is the unique isomorphism of $\mathcal{G}$ over $\mathcal{F}$ with $c(\sigma) = k$. Therefore, $\rho(H)$ is a differential algebraic subgroup of G_a defined over the D_x-constant field $\mathcal{D} = C\langle t, \cos t, \sin t\rangle$ of $\mathcal{F}$ by the prime differential ideal $[B(y)]$ in the D_t-differential polynomial ring $\mathcal{D}\{\, y \,\}$, or equivalently, the prime differential ideal $[D_x y, B(y)]$ in the (D_x, D_t)-differential polynomial ring $\mathcal{F}\{\, y \,\}$.

In 1967 I had the good fortune to take Professor Ellis R. Kolchin's Differential Algebra course at Columbia College. The following year, in Kolchin's Differential Algebra Seminar, I gave four two-hour lectures on the initial part of his Differential Galois Theory. I generalized the material from those lecture in this thesis. I wish I could thank Professor Kolchin for teaching me his special field of expertise.

I wish to acknowledge the assistance of Professors Phyllis Cassidy, Richard C. Churchill, Jerold Kovacic and William Sit, who sat through a series lectures spanning several years during which I explained my thesis. They gave me much advice and encouragement. I wish to thank Phyllis Cassidy for many discussions and for pointing out that there should be a functor from differential group schemes to Kolchin's differential algebraic groups, rather than from another category I was trying to use. William Sit and Jerold J. Kovacic read many rough drafts of various sections of my thesis, and suggested significant ways to improve it. Richard Churchill was my official thesis advisor. Because he is in the process of becoming an expert in differential algebra, he worked patiently with me every week and offered suggestions to help me solve mathematical problems and encouraged the development of topological pre-orders, which clarified the exposition of the first chapter. He also guided me through the administrative steps for the approval of my thesis.

Additionally, I would like to thank my son Aaron and my wife Susan for their support and love over many years.

Contents

1 Specializations of Isomorphisms

In this chapter, results and definitions from [24, Chapter 6, Section 1, p. 385] are recalled for the convenience of the reader. Here the perspective will be topological rather than algebraic.

Central to all that follows is the category of differential rings as developed by Ritt and Kolchin. To define this category fix a set $\Delta = \{\delta_1, \ldots, \delta_m\}$. The objects, called Δ-*rings* or *differential rings*, are rings on which the set Δ acts as commuting derivations. The morphisms, called Δ-*homomorphisms* or *differential homomorphisms*, are ring homomorphisms that commute with the action of Δ. Many terms of algebra, such as ideal, field and extension, have straightforward interpretations the category of Δ-rings and are indicated by the modifier "Δ" or "differential". However, "Δ-embeddings" are referred to as "Δ-isomorphisms", and the now standard term "radical ideal" is used in place of Kolchin's "perfect ideal".

Henceforth, all rings are assumed to have characteristic zero. Throughout this chapter, the set of commuting derivations $\Delta = \{\delta_1, \ldots, \delta_m\}$ is fixed, and $\mathcal{F}$ is a Δ-field.

Standard notation will now be reviewed from [24]. The Δ-polynomial algebra $\mathcal{F}\{y_1, \ldots, y_n\}_\Delta$ over $\mathcal{F}$ in Δ-indeterminates $y_1, \ldots, y_n$ is the polynomial ring over $\mathcal{F}$ having one indeterminate for each derivative of $y_1, \ldots, y_n$ on which Δ operates in the expected manner. (For details see Kolchin [24, pages 69-71].) If S is a subset of $\mathcal{F}\{y_1, \ldots, y_n\}_\Delta$, the Δ-ideal generated by

S is denoted by $[\mathrm{S}]_\Delta$ (or $[s_1, \dots, s_n]_\Delta$ if $\mathrm{S} = \{s_1, \dots, s_n\}$), and the radical Δ-ideal generated by S will be denoted by $\{\mathrm{S}\}_\Delta$. Let $\mathcal{G}$ be a Δ-field that is a Δ-extension of $\mathcal{F}$, and let T be a subset of $\mathcal{G}$. The Δ-ring generated by T over $\mathcal{F}$ is denoted by $\mathcal{F}\{\mathrm{T}\}_\Delta$ (or $\mathcal{F}\{t_1, \dots, t_n\}_\Delta$ if $\mathrm{T} = \{t_1, \dots, t_n\}$), and the Δ-field generated by T over $\mathcal{F}$ is denoted by $\mathcal{F}\langle \mathrm{T}\rangle_\Delta$ (or $\mathcal{F}\langle t_1, \dots, t_n\rangle_\Delta$ if $\mathrm{T} = \{t_1, \dots, t_n\}$). If T is a finite set, the Δ-ring $\mathcal{F}\{t_1, \dots, t_n\}_\Delta$ and the Δ-field $\mathcal{F}\langle t_1, \dots, t_n\rangle_\Delta$ are said to be *finitely Δ-generated by* T *over $\mathcal{F}$*, or, for simplicity, *Δ-$\mathcal{F}$-finitely generated.* If R is any Δ-ring, the symbol R^Δ denotes the constants of R with respect to Δ, i.e. the elements α of R such that $\delta\alpha = 0$ for every $\delta \in \Delta$.

A Δ-field $\mathcal{U}$ containing a Δ-subfield $\mathcal{F}$ is called *Δ-universal over $\mathcal{F}$* if the following conditions hold: for each Δ-field $\mathcal{G}$ of $\mathcal{U}$ finitely Δ-generated over $\mathcal{F}$ and for each Δ-field $\mathcal{H}$ (not necessarily contained in $\mathcal{U}$) finitely Δ-generated over $\mathcal{G}$, there exists a Δ-isomorphism of $\mathcal{H}$ into $\mathcal{U}$ over $\mathcal{G}$. The existence of Δ-universal Δ-extension of any Δ-field is established by Kolchin in [24, Theorem 2, page 134]. Such an extension contains all the solutions to differential equations over $\mathcal{F}$ necessary in Kolchin's work.

Let $\mathcal{G}$ be a Δ-field that is a Δ-extension of $\mathcal{F}$. This chapter contains a description of a topology on the set $X = \mathrm{Isom}^\Delta_\mathcal{F}(\mathcal{G}, \mathcal{U})$ of Δ-isomorphisms of $\mathcal{G}$ over $\mathcal{F}$ defined by a pre-order associated to algebraic specializations of isomorphisms. If $\mathcal{G}$ is finitely Δ-generated over $\mathcal{F}$, Proposition 1.49 shows that X is Noetherian, and the irreducible components are disjoint. If $\mathcal{G}$ over $\mathcal{F}$ is strongly normal in the sense of Kolchin (Definition 2.128 or [24, Chapter

6, Section 3, p. 393]), then X may be identified with the set of $\mathcal{U}^{\Delta}$-points of an algebraic $\mathcal{F}^{\Delta}$-group which is the Galois group of $\mathcal{G}$ over $\mathcal{F}$. In later chapters these concepts will be generalized to differential algebraic groups.

1.1 The Pre-Order Topology

Let X be a non-empty set with a pre-order, i.e. a reflexive, transitive relation $\mathcal{R} \subset X \times X$. Write $p \longrightarrow q$ instead of $(p, q) \in \mathcal{R}$. All such pre-ordered sets form the objects of the *category of pre-ordered sets.* A morphism is a map $f : X \longrightarrow Y$ such that $p \longrightarrow q$ implies $f(p) \longrightarrow f(q)$.

Any set $Y \subseteq X$ inherits the *relative pre-order* $\mathcal{R} \cap (Y \times Y)$, also denoted by arrows.

When $p \longrightarrow q$, the element p is said to *specialize* to q and q is a *specialization of* p. If $p \longrightarrow q$ and $q \longrightarrow p$, denoted by $p \longleftrightarrow q$, then q will be said to be a *generic specialization* of p. For each $p \in X$, let $C(p) = \{x \in X : p \longrightarrow x\}$.

Observation 1.1 *For $p \in X$ and $q \in X$, the following three statements are clearly equivalent:*

1. $p \longrightarrow q$,
2. $q \in C(p)$, *and*
3. $C(q) \subseteq C(p)$.

The *pre-order topology* on a pre-ordered set X is the topology generated by closed sets of the form $C(p)$ for $p \in X$. Let X_T denote the set X with this topology.

Lemma 1.2 *The closure of $p \in X_T$ is $C(p)$.*

Proof: If not, $C(p)$ has an element q not contained in the closure C of p. Since C is closed in the pre-order topology, $C = \cap_i(\cup_j C(p_{i,j}))$ such that the intersection is of a possibly infinite number of sets and the unions are of a finite number of sets. If $q \notin C$, then there exists i_0 such that $q \notin \cup_j C(p_{i_0,j})$. Because $p \in C$, for every i, $p \in \cup_j C(p_{i,j})$ and, in particular for $i = i_0$, $p \in \cup_j C(p_{i_0,j})$. Therefore there exists j_0 such that $p_{i_0,j_0} \longrightarrow p$. Since $p \longrightarrow q$, by transitivity $p_{i_0,j_0} \longrightarrow q$. This contradicts $q \notin \cup_j C(p_{i_0,j})$. □

A morphism $f : X \longrightarrow Y$ of pre-ordered sets does not necessarily induce a continuous morphism $f : X_T \longrightarrow Y_T$, as the following example illustrates. Let Y be the pre-ordered set $[0,1] \subset \mathbb{R}$ with the pre-order: $1 \longrightarrow x'$ if $x' = 1$ or $x' \in [0, 1/2]$ and $x \longrightarrow x$ for all $x \neq 1$. Note that $C(1) = [0, 1/2] \cup \{1\}$ is closed by Lemma 1.2. Let $X = [0, 1) \subset Y$ be the pre-ordered set with the restricted pre-order. Then the inclusion map $f : X_T \longrightarrow Y_T$ is not continuous because the proper closed subsets of X_T consist of finite subsets and $f^{-1}(C(1)) = [0, 1/2]$ does not have this property. The next lemma shows that in certain circumstances continuity may be deduced.

Lemma 1.3 *Let $f : X \longrightarrow Y$ be a morphism of pre-ordered sets. Each of the following conditions implies $f : X_T \longrightarrow Y_T$ is continuous.*

1. *The morphism f is an inclusion, and the topology of X_T is the relative topology.*

2. The morphism f is a bijection, and $x \to x'$ if and only if $f(x) \to f(x')$.

3. Endow the set $f(X) \subseteq Y$ with the relative pre-order and assume that:

  (a) the pre-order topology of $f(X)$ is the relative topology, and

  (b) the morphism f is a injection, and $x \to x'$ if and only if $f(x) \to f(x')$.

Proof: The first item is the condition of continuity for an inclusion. The second item is equivalent to $f(C(x)) = C(f(x))$ for all x in X. So that f_T is continuous because the closed sets $C(f(x))$ generate the topology on Y_T and $C(x) = f_T^{-1}(C(f(x)))$ is closed. In the third item, the morphism f is the composite of a bijection followed by an inclusion. So that items 1 and 2 may be applied to conclude continuity. □

Example 1.4 (*Pre-orders on $\mathfrak{U}^r$*) *Let $\mathfrak{F}$ be a Δ-field, and $\mathfrak{U}$ be a Δ-extension of $\mathfrak{F}$ that is Δ-universal over $\mathfrak{F}$. Let $\mathfrak{G}$ be a Δ-extension of $\mathfrak{F}$ in $\mathfrak{U}$ over which $\mathfrak{U}$ is universal. For $\eta = (\eta_1, \ldots, \eta_r)$ and $\xi = (\xi_1, \ldots, \xi_r)$ in $\mathfrak{U}^r$, define the pre-order by $\eta \underset{\mathfrak{G}}{\to} \xi$, called* Δ-specialization over $\mathfrak{G}$ *or* Δ-$\mathfrak{G}$-specialization, *if there exists a Δ-$\mathfrak{G}$-homomorphism of $\mathfrak{G}\{\eta_1, \ldots, \eta_r\}_\Delta$ to $\mathfrak{G}\{\xi_1, \ldots, \xi_r\}_\Delta$ over $\mathfrak{G}$ taking η_i to ξ_i for $i = 1, \ldots, r$.*

Example 1.5 (*The pre-order on* $\operatorname{diffspec} R$) *Let R be a Δ-ring. Let* $\operatorname{diffspec} R$ *be the set of prime Δ-ideals of R. For $\mathfrak{P}$ and $\mathfrak{Q}$ in* $\operatorname{diffspec} R$ *define the pre-order by $\mathfrak{P} \to \mathfrak{Q}$ if $\mathfrak{Q} \supseteq \mathfrak{P}$.*

When Δ is empty, these two examples are well-known in commutative algebra.

1.2 NG-Spaces

In this section, two axioms on a topological space are introduced to ensure that an associated pre-order topology has a finite number of points of which any point in the space is a specialization. The first is the Noetherian condition which insures that a topological space has a finite number of irreducible components. The second is the assumption that every irreducible closed set has a generic point, which implies that each point of the space is a specialization of one of finitely many generic points. Initially, some lemmas which are useful for verifying these axioms are presented.

Recall that a topological space X is called *Noetherian* if it satisfies the descending chain condition on closed sets and *irreducible* if it is not empty and the union of two closed sets distinct from X is always distinct from X [3, page 94]. A closed set W of X has a *generic point* $\zeta \in X$ if the closure of ζ in X, denoted by $\overline{\{\zeta\}}$, is equal to W.

Lemma 1.6 *Let X have a pre-order. Consider the topological space X_T, i.e., the set X with the pre-order topology. Let W be a closed set of X_T, and let p be a point in W. Then the following are equivalent:*

1. *p is a generic point of W,*
2. *$C(p) = W$ and*

3. $p \longrightarrow q$ *for all* $q \in W$.

Proof: If p is a generic point of W, then by the definition of generic point W equals the closure $\overline{\{p\}}$. By Lemma 1.2, $C(p) = \overline{\{p\}}$. Therefore, by the definition of $C(p)$, $p \longrightarrow q$ for all $q \in W$. This argument is reversible. $\square$

Lemma 1.7 *Let* $f : X \longrightarrow Y$ *be a continuous map between topological spaces. Let* E *be an irreducible closed set of* X. *Then* $f(E)$ *and* $\overline{f(E)}$ *are irreducible. If* p *is a generic point of* E, *then* $f(p)$ *is a generic point of* $\overline{f(E)}$.

Proof: By [3, Proposition 4, page 95], $f(E)$ is irreducible. By [3, Proposition 2, page 95], $\overline{f(E)}$ is also irreducible. Let p be a generic point of E so that $E = \overline{\{p\}}$. By the continuity of f, the set $f^{-1}(\overline{\{f(p)\}})$ is closed. Since $p \in f^{-1}(\overline{\{f(p)\}})$, the closure $\overline{\{p\}} \subset f^{-1}(\overline{\{f(p)\}})$. By applying f, $f(p) \in f(\overline{\{p\}}) \subset \overline{\{f(p)\}}$. Since $f(\overline{\{p\}}) = f(E)$, $f(E) \subset \overline{\{f(p)\}}$, and $\overline{f(E)} \subset \overline{\{f(p)\}}$. Since $f(p) \in f(E)$, clearly $\overline{f(E)} \supset \overline{\{f(p)\}}$. Thus $\overline{f(E)} = \overline{\{f(p)\}}$, which shows that $f(p)$ is a generic point of $\overline{f(E)}$. $\square$

Definition 1.8 *A topological space has property* $\boldsymbol{G}$ (*resp.* $\boldsymbol{G}_0$) *if each irreducible closed set has a* (*resp. unique*) *generic point.*

The following example shows that a Noetherian topology defined by a pre-order need not satisfy property **G**.

Example 1.9 *Let* X *be the set of positive integers with the pre-order* $m \rightarrow n$ *if* $m \geq n$. *For* $m \in X$, $C(m)$ *is the set of positive integers less than or equal* *to m. Each $C(m)$ is closed and irreducible by Lemma 1.10 below. The whole space X is clearly closed and irreducible, but there is no generic point for X.*

After a brief review of some properties of irreducible closed subsets, a lemma about properties $\mathbf{G}$ and $\mathbf{G}_0$ will be established.

Lemma 1.10 *Let X be a topological space, and let ζ be a point of X. Then $W = \overline{\{\zeta\}}$ is an irreducible closed set of X with generic point ζ. If U is an open subset of X such that $U \cap W$ is non-empty, the generic point ζ is contained in U* (as well as in W).

Proof: By definition, W is closed. Since the singleton set $\{\zeta\}$ is irreducible and the closure of an irreducible subset is irreducible by [3, Proposition 2, page 95], W is irreducible. If the generic point $\zeta \notin U$, then $X \setminus U$ would be a closed set that contains both ζ and the minimal closed set W containing ζ. Therefore, $U \cap W$ would be empty. ☐

The following Proposition is from Bourbaki [3, Proposition 7, page 96].

Proposition 1.11 *Let U be an open subset of a topological space X. The mapping $V \mapsto \overline{V}$ (closure in X) is a bijection of the set of irreducible subsets of U which are closed in U onto the set of irreducible subsets of X which are closed in X and meet U; the inverse bijection is $Z \mapsto Z \cap U$. In particular, this bijection maps the set of irreducible components of U onto the set of irreducible components of X which meet U.*

The next lemma shows that, if property **G** (*resp.* $\mathbf{G}_0$) is true for all open sets in a cover of the topological space X, then property **G** (*resp.* $\mathbf{G}_0$) is true for X.

Lemma 1.12 *Let X be a topological space. A sufficient condition for any open subset U of X to have property* **G** (*resp.* $\mathbf{G}_0$) (*in the relative topology*) *is for X to have property* **G** (*resp.* $\mathbf{G}_0$). *Let $\{U_i\}_{i\in I}$ be an open cover of X. Then X has property* **G** (*resp.* $\mathbf{G}_0$) *if and only if each of the U_i has property* **G** (*resp.* $\mathbf{G}_0$).

Proof: To show the sufficiency, let U be any open subset of X, and let W be an irreducible closed set of U (non-empty). The closure $\overline{W}$ of W in X is irreducible, and $\overline{W} \cap U = W$ (Proposition 1.11). Because X is assumed to have property **G**, there exists a point ζ in X such that $\overline{W} = \overline{\{\zeta\}}$. By Lemma 1.10, $\zeta \in U$, and the closure of ζ in U is W. If ζ and ζ' are two generic points of W in U, then they are also generic points of $\overline{W}$ in X. If X has property $\mathbf{G}_0$, then $\zeta = \zeta'$.

Let $\{U_i\}_{i\in I}$ be an open cover of X. Assume each U_i has property **G**. Let W be a closed irreducible subset of X. Since an irreducible subset is non-empty there is a point $\beta \in W$, and β is contained in some U_i. Then $W \cap U_i$ is non-empty and irreducible in U_i by Proposition 1.11. Because U_i satisfies property **G**, there is a point $\zeta \in U_i$ such that the closure of ζ in U_i is $W \cap U_i$. The closure of ζ in X is the closure of $W \cap U_i$ in X which is W by Proposition 1.11. Therefore, ζ is a generic point of W in X. If ζ and ζ' are two generic points of W, then by Lemma 1.10 both points are in U_i and are generic points of $U_i \cap W$ in U_i. Thus, if U_i satisfies property $\mathbf{G_0}$, then $\zeta = \zeta'$. If X has property $\mathbf{G}$ (*resp.* $\mathbf{G}_0$), each open U_i of the cover has property $\mathbf{G}$ (*resp.* $\mathbf{G}_0$) by the first part of the Lemma. □

This Lemma will be applied to E-schemes in the sense of Kovacic (Proposition 2.63) and to $\mathrm{Isom}^{\Delta}_{\mathcal{F}}(\mathcal{G}, \mathcal{U})$ (Corollary 2.60).

Definition 1.13 *A topological space X will be called an NG-space if it is Noetherian and satisfies hypothesis* $\mathbf{G}$ (*Definition* 1.8).

Examples of NG-spaces include algebraic varieties in the sense of Weil with the Zaiski topology and Noetherian schemes. Further examples will be given in later sections.

Now some properties of NG-spaces are assembled for future reference.

Proposition 1.14 *Let X be an NG-space. For any closed subset $W \subseteq X$, there exists a finite subset $\Phi \subset W$ such that each of the irreducible components of W has a generic point in Φ.*

Proof: Any closed subset $W \subset X$ is the union of a finite number of irreducible closed components $W_1, \ldots, W_r$ by the Noetherian hypothesis [3, Proposition 5, page 95], and each $W_i = \overline{\{\zeta_i\}}$ for some $\zeta_i \in W$ by hypothesis $\mathbf{G}$. Take $\Phi = \{\zeta_1, \ldots, \zeta_r\}$. □

Proposition 1.15 *Let X be a topological space, and let $\{U_i\}_{i \in I}$ be a finite open cover. Then X is an NG-space if and only if each U_i is an NG-space.*

Proof: This is true by Lemma 1.12 and the fact that X is Noetherian if and only if each of the U_i is Noetherian space [3, Proposition 8, page 97]. $\square$

Any closed subset V of an NG-space X is an NG-space because V is Noetherian and any irreducible closed subset W of V is also irreducible in X with a generic point contained in $W \subset V$. In the next Lemma, it is observed that certain other subsets of X are also NG-spaces.

Lemma 1.16 *Let X be an NG-space. Let Y be a subset of X such that $\overline{\{x\}} \cap Y$ is empty for $x \in X$ not in Y. Then Y with the relative topology is an NG-space.*

Proof: As a subspace of a Noetherian space, Y is also Noetherian [3, Proposition 8(i), page 97]. Let W be an irreducible closed set of Y in the relative topology. Then $\overline{W} \subset X$ is an irreducible closed set of X [3, Proposition 2, page 95]. Because X is an NG-space, there is a point $w \in X$ which is a generic point of $\overline{W}$. Since W is closed in the relative topology of Y, $W = \overline{W} \cap Y = \overline{\{w\}} \cap Y$. If w were not in Y, this intersection would be empty by the assumption of the Lemma. But, by definition of irreducibility, W is non-empty. Therefore $w \in Y$. Hence $w \in W$, and w is a generic point of W. Thus Y is an NG-space. $\square$

1.3 The Pre-Order Topology of a Topological Space

Let X be a topological space. Define the *topological pre-order associated to (the topological space) X* to be the set X with the pre-order $p \underset{X}{\longrightarrow} q$ if

$\overline{\{q\}} \subseteq \overline{\{p\}}$.

Proposition 1.17 *Let $f : X \longrightarrow Y$ be a continuous map of topological spaces. Endow X and Y with their topological pre-orders* (Section 1.3). *For $x, x' \in X$, if $x \longrightarrow x'$, then $f(x) \longrightarrow f(x')$.*

Proof: By the definition of topological pre-order, $\overline{\{x'\}} \subseteq \overline{\{x\}}$. Then $f(\overline{\{x'\}}) \subseteq f(\overline{\{x\}})$ and $\overline{f(\overline{\{x'\}})} \subseteq \overline{f(\overline{\{x\}})}$. By Lemma 1.10, $\overline{\{x'\}}$ and $\overline{\{x\}}$ are irreducible with generic points x and x'. By Lemma 1.7, $\overline{f(\overline{\{x'\}})}$ and $\overline{f(\overline{\{x\}})}$ are irreducible with generic points $f(x')$ and $f(x)$. Therefore, $\overline{f(x')} \subseteq \overline{f(x)}$ and $f(x) \longrightarrow f(x')$. $\square$

Corollary 1.18 *Let $f : X \longrightarrow Y$ be a homeomorphism of topological spaces. Then f induces an isomorphism of topological pre-orders.*

Lemma 1.19 *Endow a pre-ordered set X with the pre-order topology. Then the topological pre-order is the given pre-order.*

Proof: Let $p \longrightarrow q$ be the given pre-order on X. By Lemma 1.2 and Observation 1.1, $p \underset{X}{\longrightarrow} q \Longleftrightarrow \overline{\{q\}} \subseteq \overline{\{p\}} \Longleftrightarrow C(q) \subseteq C(p) \Longleftrightarrow p \longrightarrow q$. $\square$

Let X be a topological space. The pre-order topology on X associated to the topological pre-order will be called the *pre-order topology* (*associated to the topological space*) X. Since pre-order topology on X is generated by the closed sets $C(p) = \overline{\{p\}}$ for $p \in X$ and, since each $C(p)$ is closed in the topology of X, the pre-order topology on X is contained in the given topology.

For an example in which the two topologies are different, let X be an infinite set with the discrete topology (every subset is closed). Then $p \longrightarrow q$ is equivalent to $p = q$. Because proper closed sets in the pre-order topology are arbitrary intersections of finite unions of sets of the form $C(p)$ for $p \in X$, the proper closed sets of X in the pre-order topology are the finite subsets.

The purpose of the next lemma is to present a condition that ensures that the two topologies on X are the same.

Lemma 1.20 *Let X be an NG-space. Then the pre-order topology is the given topology on X.*

Proof: As noted above, the topology of X contains the pre-order topology. Each closed set W of X is the finite union of irreducible closed sets V_i, and each V_i has a generic point p_i. Since $V_i = V(\overline{\{p_i\}})$, V_i and $W = \cup_{i=1}^{r} V_i$ are closed in the pre-order topology associated to X. $\square$

Let X be a pre-ordered set. An element $p \in X$ will be called *isolated* if $q \longrightarrow p$ implies $q \leftrightarrow p$. This is equivalent to $C(p)$ being maximal among all closed subsets in X of the form $C(q)$.

Proposition 1.21 *Let X be an NG-space, and let $\Phi = \{\zeta_1, \ldots, \zeta_d\}$ be a finite subset of X such that each of the irreducible components of X has a generic point in Φ. Then the ζ_i are isolated points with respect to the* *pre-order associated to the topology of X, and each point of X is the specialization of at least one of the ζ_i. Conversely, each isolated point $\zeta \in X$ is a generic point of one of the irreducible components of X.*

The existence of the finite set $\Phi \subset X$ was shown in Proposition 1.14.

Proof: By Lemma 1.20, the topology of X is the pre-order topology associated to X. By Lemma 1.6, $C(\zeta_i) = \overline{\{\zeta_i\}}$. Assume $\xi \longrightarrow \zeta_i$ for some $\xi \in X$ and for some i. Then $C(\xi) \supset C(\zeta_i) = \overline{\{\zeta_i\}}$. Since $\overline{\{\zeta_i\}}$ is an irreducible component and $C(\xi)$ is a closed set (Lemma 1.2) and irreducible (Lemma 1.7) , $C(\xi) = C(\zeta_i)$. This implies that $\xi \leftrightarrow \zeta_i$, and ζ_i is therefore isolated. Because each point x in X is in one of the components $C(\zeta_i) = \overline{\{\zeta_i\}}$, there exists at least one i such that $\zeta_i \longrightarrow x$.

For the converse, let $\zeta \in X$ be an isolated point. Then $\zeta_i \longrightarrow \zeta$ for one i. Since ζ is isolated $\zeta_i \leftrightarrow \zeta$, and ζ is a generic point of $C(\zeta_i)$. □

Proposition 1.22 *Let Y be a pre-ordered set. Suppose $X \subseteq Y_T$ with the relative topology is an NG-space. Then the relative topology on X is the pre-order topology of the relative pre-order.*

Proof: For any pre-ordered set Z with pre-order $p \longrightarrow q$ and for any $p \in Z$, let $C_Z(p) = \{q \in Z \mid p \longrightarrow q\}$. Since the topology of Y_T is generated by the irreducible closed sets $C_Y(p)$ for $p \in Y_T$, the relative topology on X is generated by the closed sets $C_Y(p) \cap X$ for $p \in Y$. Each closed set of X_T of the form $C_X(q)$ for $q \in X$ satisfies $C_X(q) = C_Y(q) \cap X$ because the pre-order on X is relative pre-order. Since the relative topology of X_T is generated by $C_X(q)$ for all $q \in X$, the closed sets of X_T are a subset of the closed sets of the relative topology on X.

It will now be shown that the closed sets of the relative topology on X are a subset of the closed sets of X_T. Because X is assumed to be an NG-space, the closed subset $C_Y(p) \cap X$ for $p \in Y$ of the relative topology on X is equal to $\bigcup_{i=1,\ldots,r} V_X(p_i)$ for $p_i \in X$ (Proposition 1.14) where $V_X(p_i)$ denotes the closure of p_i in the relative topology of X. Since the closed subsets $C_Y(r) \cap X$ for $r \in Y$ generate the relative topology, it is enough to show that $V_X(r) = C_X(r)$ for each $r \in X$. This follows from the sequence of equivalent statements below in which p and q are elements of X.

1. $q \in V_X(p)$.

2. $V_X(q) \subseteq V_X(p)$ where $V_X(p)$ is the closure of p in the induced topology on X. This is the definition of $p \longrightarrow q$ for the topological pre-order associated to the relative topology on X.

3. $V_Y(q) \subseteq V_Y(p)$ where $V_Y(p)$ is the closure of p in Y_T. This is the definition of $p \longrightarrow q$ for the topological pre-order associated to Y_T.

4. $p \longrightarrow q$ in the pre-order associated to Y by Lemma 1.19.

5. $p \longrightarrow q$ in the pre-order on X because $p \longrightarrow q$ on X is the restriction of the original $p \longrightarrow q$ on Y.

6. $q \in C_X(p)$ by Observation 1.1.

□

Corollary 1.23 *Let $f : X \longrightarrow Y$ be an injective morphism of pre-ordered sets such that, for $x, x' \in X$, $x \longrightarrow x'$ if and only $f(x) \longrightarrow f(x')$ and such that $f(X) \subseteq Y_T$ with relative topology is an NG-space. Then f is continuous.*

Proof: Condition 3a of Lemma 1.3 follows by considering $f(X)$ as a subset of Y and applying Proposition 1.22, and condition 3b is satisfied by that assumption in the proposition. □

1.4 The Δ-Zariski Topology on Affine Space

Let $\mathfrak{U}$ be Δ-universal over $\mathfrak{F}$, and let $\mathfrak{G}$ be a Δ-extension of $\mathfrak{F}$ contained in $\mathfrak{U}$. Now recall the definition of the Δ-$\mathfrak{G}$-Zariski topology on $\mathfrak{U}^r$ [24, Sections 3 and 4 of Chapter 4, pages 147-150]. For any subset Σ of $\mathfrak{G}\{y_1, \ldots, y_n\}_\Delta$ define $\mathfrak{Z}_\mathfrak{G}(\Sigma)$ to be the Δ-zeros of Σ in $\mathfrak{U}^r$. The subsets of $\mathfrak{U}^n$ of the form $\mathfrak{Z}_\mathfrak{G}(\Sigma)$, for a subset Σ of $\mathfrak{G}\{y_1, \ldots, y_n\}_\Delta$, are all the closed sets of a Noetherian topology called the Δ-$\mathfrak{G}$-*Zariski topology* [24, page 149]. These closed sets of $\mathfrak{U}^n$ are called $\mathfrak{G}$-*closed sets.*

For any subset M of $\mathfrak{U}^n$, define $\mathfrak{A}_\mathfrak{G}(\mathrm{M})$ to be the radical Δ-ideal of all Δ-polynomials $\mathfrak{G}\{y_1, \ldots, y_n\}_\Delta$ that vanish at every element of M. The map $\mathrm{M} \mapsto \mathfrak{A}_\mathfrak{G}(\mathrm{M})$ from $\mathfrak{G}$-closed sets of $\mathfrak{U}^n$ to radical Δ-ideals of $\mathfrak{G}\{y_1, \ldots, y_n\}_\Delta$ is an order reversing bijection with inverse $\Sigma \mapsto \mathfrak{Z}_\mathfrak{G}(\Sigma)$ [24, page 150]. Under this map irreducible $\mathfrak{G}$-closed sets correspond to prime Δ-ideals. If $\mathfrak{A}$ is any Δ-ideal of $\mathfrak{G}\{y_1, \ldots, y_n\}_\Delta$, then this correspondence restricts to a bijective mapping from $\mathcal{G}$-closed sets of $\mathfrak{Z}_{\mathcal{G}}(\mathcal{A})$ to radical Δ-ideals of $\mathcal{G}\{y_1, \ldots, y_r\}_\Delta$ containing $\mathfrak{A}$, and irreducible $\mathcal{G}$-closed subsets of $\mathfrak{Z}_{\mathcal{G}}(\mathfrak{A})$ correspond to prime Δ-ideals containing $\mathfrak{A}$.

If $\zeta \in \mathfrak{U}^n$, then trivially $\mathfrak{A}_{\mathcal{G}}(\{\zeta\})$ is a prime Δ-ideal. The converse is the following proposition. (See also [24, Proposition 1, page 146].)

Proposition 1.24 *Let $\mathcal{G} \subset \mathfrak{U}$ be a finitely Δ-generated extension of $\mathcal{F}$. For each prime Δ-ideal $\mathfrak{P}$ in $\mathcal{G}\{y_1, \ldots, y_n\}_\Delta$, there exists an element $\zeta \in \mathfrak{U}^n$ such that $\mathfrak{P} = \mathfrak{A}_{\mathcal{G}}(\{\zeta\})$.*

Proof: Let $\mathcal{E}$ be the quotient field of $\mathcal{G}\{y_1, \ldots, y_r\}_\Delta/\mathfrak{P}$. Because $\mathcal{E}$ is finitely Δ-generated over $\mathcal{G}$ and $\mathfrak{U}$ is Δ-universal over $\mathcal{F}$, there exists a Δ-$\mathcal{G}$-isomorphism $\sigma : \mathcal{E} \longrightarrow \mathfrak{U}$. Take $\zeta = (\sigma(y_1 + \mathfrak{P}), \ldots, \sigma(y_r + \mathfrak{P})) \in \mathfrak{U}^r$. Since ζ is a Δ-zero of $\mathfrak{P}$, $\mathfrak{P} \subseteq \mathfrak{A}_{\mathcal{G}}(\{\zeta\})$. If $f \in \mathfrak{A}_{\mathcal{G}}(\{\zeta\})$, then $\sigma(f + \mathfrak{P}) = f(\zeta) = 0$. Therefore $f \in \mathfrak{P}$. $\square$

Corollary 1.25 *The topological space $\mathfrak{U}^n$ with the Δ-$\mathcal{G}$-Zariski topology is an NG-space.*

Proof: It has already been noted that $\mathfrak{U}^n$ is Noetherian. Also, each irreducible $\mathcal{G}$-closed set W of $\mathfrak{U}^n$ in the Δ-$\mathcal{G}$-Zariski topology is the set of Δ-zeros of a prime Δ-ideal $\mathfrak{P}$ of $\mathcal{G}\{y_1, \ldots, y_n\}_\Delta$. Since $\mathfrak{P}$ has a Δ-$\mathcal{G}$-generic Δ-zero ζ, by the lemma, ζ is a generic point of W. $\square$

For each $\zeta \in \mathfrak{U}^n$, define $\mathfrak{P}_{\zeta,\mathcal{G}} = \mathfrak{A}_{\mathcal{G}}(\{\zeta\}) \subseteq \mathcal{G}\{y_1, \ldots, y_n\}$. This is called the *defining Δ-$\mathcal{G}$-ideal of ζ*. Because $\mathfrak{P}_{\zeta,\mathcal{G}}$ is a prime Δ-ideal, $\mathfrak{Z}_{\mathcal{G}}(\mathfrak{P}_\zeta)$ is an irreducible $\mathcal{G}$-closed set of $\mathcal{U}^n$ containing ζ. The following observation relates the concept of Δ-$\mathcal{G}$-specialization to that of defining Δ-$\mathcal{G}$-ideal.

Observation 1.26 *For elements ζ and ξ of $\mathcal{U}^n$ following statements are equivalent.*

1. *$\zeta \longrightarrow \xi$ in the pre-order associated to the Δ-$\mathcal{G}$-Zariski topology on $\mathcal{U}^n$.*
2. *$\overline{\{\zeta\}} \supseteq \overline{\{\xi\}}$ where the closure operation refers to the Δ-$\mathcal{G}$-Zariski topology on $\mathcal{U}^n$.*
3. *$\mathfrak{P}_{\zeta,\mathcal{G}} \subseteq \mathfrak{P}_{\xi,\mathcal{G}}$ as prime Δ-ideals of $\mathcal{G}\{y_1, \ldots, y_n\}_\Delta$.*
4. *There is a Δ-$\mathcal{G}$-homomorphism $\rho_{\xi,\zeta} : \mathcal{G}\{\zeta_1, \ldots, \eta_n\}_\Delta \rightarrow \mathcal{G}\{\xi_1, \ldots, \xi_n\}_\Delta$ over $\mathcal{G}$ taking ζ_i to ξ_i for $i = 1, \ldots, n$.*

Definition 1.27 *When any of these conditions holds, ξ will be called a Δ-$\mathcal{G}$-specialization of ζ. This relation is a pre-order on $\mathcal{U}^n$. If W is a subset of $\mathcal{U}^n$, this pre-order restricted to W will be called the pre-order of Δ-$\mathcal{G}$-specialization on W.*

Corollary 1.28 *For ζ and ξ in $\mathcal{U}^n$, the following are equivalent:*

1. *$\zeta \leftrightarrow \xi$,*
2. *$\overline{\{\zeta\}} = \overline{\{\xi\}}$,*
3. *$\mathfrak{P}_{\zeta,\mathcal{G}} = \mathfrak{P}_{\xi,\mathcal{G}}$,*

4. *the Δ-homomorphism $\rho_{\xi,\zeta} : \mathfrak{G}\{\zeta_1, \dots, \zeta_n\}_\Delta \to \mathfrak{G}\{\xi_1, \dots, \xi_n\}_\Delta$ over $\mathfrak{G}$ taking ζ_i to ξ_i for $i = 1, \dots, n$ is a Δ-isomorphism,*

5. *$\mathfrak{P}_{\zeta,\mathfrak{G}}$ is the defining Δ-ideal of ξ in $\mathfrak{G}\{y_1, \dots, y_n\}_\Delta$.*

Suppose a prime Δ-ideal $\mathfrak{P} \subset \mathfrak{G}\{y_1, \dots, y_n\}_\Delta$ is given. An element $\zeta \in \mathfrak{U}^n$ such that $\mathfrak{P} = \mathfrak{P}_{\zeta,\mathfrak{G}}$ is called a *generic Δ-zero over $\mathfrak{G}$* or a *$\mathfrak{G}$-generic Δ-zero* of the prime Δ-ideal $\mathfrak{P}$. By Proposition 1.24, each prime Δ-ideal $\mathfrak{P}$ in $\mathfrak{G}\{y_1, \dots, y_n\}_\Delta$ has a $\mathfrak{G}$-generic Δ-zero.

Lemma 1.29 *Consider $\mathfrak{U}^n$ with the Δ-$\mathfrak{G}$-Zariski topology. Let $\mathfrak{P}$ be a prime Δ-ideal of $\mathfrak{G}\{y_1, \dots, y_n\}_\Delta$. An element $\zeta \in \mathfrak{U}^n$ is a $\mathfrak{G}$-generic Δ-zero of $\mathfrak{P}$ if and only if ζ is a generic point of $\mathfrak{Z}_\mathfrak{G}(\mathfrak{P})$.*

Proof: Let ζ be a $\mathfrak{G}$-generic Δ-zero of $\mathfrak{P}$, i.e. $\mathfrak{P} = \mathfrak{P}_{\zeta,\mathfrak{G}}$. Then $\mathfrak{Z}_\mathfrak{G}(\mathfrak{P}) = \mathfrak{Z}_\mathfrak{G}(\mathfrak{P}_{\zeta,\mathfrak{G}}) = \overline{\{\zeta\}}$ by [24, page 149]. Therefore ζ is a generic point of W.

Conversely, let ζ be a generic point of $\mathfrak{Z}_\mathfrak{G}(\mathfrak{P})$, i.e. $\overline{\{\zeta\}} = \mathfrak{Z}_\mathfrak{G}(\mathfrak{P})$. Applying $\mathfrak{A}_\mathfrak{F}$ gives $\mathfrak{A}_\mathfrak{G}(\overline{\{\zeta\}}) = \mathfrak{A}_\mathfrak{G}(\mathfrak{Z}_\mathfrak{G}(\mathfrak{P}))$, which equals $\mathfrak{P}$ since $\mathfrak{A}_\mathfrak{G}(\mathfrak{Z}_\mathfrak{G}(\mathfrak{P})) = \mathfrak{P}$ ([24, page 149]). Therefore $\mathfrak{A}_\mathfrak{G}(\overline{\{\zeta\}}) = \mathfrak{P}$. Also, $\mathfrak{A}_\mathfrak{G}(\overline{\{\zeta\}}) = \bigcap_{\zeta \to \xi} \mathfrak{P}_{\xi,\mathfrak{G}} = \mathfrak{P}_{\zeta,\mathfrak{G}}$ because, by Observation 1.26, $\mathfrak{P}_{\xi,\mathfrak{G}} \supseteq \mathfrak{P}_{\zeta,\mathfrak{G}}$ if and only if $\zeta \to \xi$. Therefore $\mathfrak{P} = \mathfrak{P}_{\zeta,\mathfrak{G}}$, and ζ is a $\mathfrak{G}$-generic Δ-zero of $\mathfrak{P}$. $\square$

Let W be a $\mathfrak{G}$-closed set of $\mathfrak{U}^n$ in the Δ-$\mathfrak{G}$-Zariski topology. Then the relative topology on W is called the *Δ-$\mathfrak{G}$-Zariski topology on W*.

Lemma 1.30 *Let W be a $\mathfrak{G}$-closed set of $\mathfrak{U}^n$. The relative pre-order on W of the topological pre-order on $\mathfrak{U}^r$ is the topological pre-order on W.*

Proof: Because W is closed, the closure of a point of W is the same with respect to the Δ-$\mathcal{G}$-Zariski topology on $\mathcal{U}^n$ and with respect the Δ-$\mathcal{G}$-Zariski topology on W. Since the associated topological pre-orders are determined by the closures of points (Observation 1.26(2)), the topological pre-orders on W associated to both topologies are the same. ☐

The following proposition follows immediately from the observation that a closed subset of an NG-space is an NG-space and the application of Lemma 1.20.

Proposition 1.31 *Let W be a $\mathcal{G}$-closed set of $\mathcal{U}^n$. Then W is an NG-space. Also, the pre-order topology on W associated to Δ-$\mathcal{G}$-Zariski topology on W is the Δ-$\mathcal{G}$-Zariski topology on W.*

Corollary 1.32 *Let W be a $\mathcal{G}$-closed set of $\mathcal{U}^n$. The Δ-$\mathcal{G}$-Zariski topology on W is the pre-order topology associated to Δ-$\mathcal{G}$-specialization on W.*

Proof: By Observation 1.26, the topological pre-order on $\mathcal{U}^n$ associated to the Δ-$\mathcal{G}$-Zariski topology is that of Δ-$\mathcal{G}$-specializations (Definition 1.27). By Lemma 1.30, the topological pre-order on W associated to the Δ-$\mathcal{G}$-Zariski topology is Δ-$\mathcal{G}$-specializations on W. Since W is an NG-space (Proposition 1.31), the Δ-$\mathcal{G}$-Zariski topology on W is the pre-order topology associated to Δ-$\mathcal{G}$-specialization on W (Lemma 1.20). ☐

Corollary 1.33 *Let W be a $\mathcal{G}$-closed set of $\mathcal{U}^n$. The closure of any $\zeta \in W$, with respect to the Δ-$\mathcal{G}$-Zariski topology, is the set of all Δ-$\mathcal{G}$-specializations of ζ.*

Proof: By Corollary 1.32, the closure of $\zeta \in W$ with respect to the Δ-$\mathcal{G}$-Zariski topology is the closure of ζ in the pre-order topology of Δ-$\mathcal{G}$-specializations on W. By Lemma 1.2, this is the set $C(\zeta)$ of all Δ-$\mathcal{G}$-specializations of ζ in the pre-order of Δ-$\mathcal{G}$-specializations on W. □

1.5 The Space of Isomorphisms

In this section, the topological space of isomorphisms $X = \mathrm{Isom}^{\Delta}_{\mathcal{F}}(\mathcal{G}, \mathcal{U})$ is shown to be an NG-space. Kolchin descibes some other properties of X in [25, Secction 1, pages 157–159]. In later chapters, X will be the set of elements of a generalized Galois group.

Throughout this section, let $\eta = (\eta_1, \ldots, \eta_n)$ be a finite family of Δ-generators for $\mathcal{G}$ over $\mathcal{F}$. Also let $\mathfrak{P} = \mathfrak{P}_{\eta,\mathcal{F}}$, and let $\mathfrak{P}_\xi = \mathfrak{P}_{\xi,\mathcal{G}}$ for $\xi \in \mathfrak{Z}_{\mathcal{G}}(\mathcal{G}\mathfrak{P})$.

Definition 1.34 (Δ-$\mathcal{G}$-*Specialization of* Δ-$\mathcal{F}$-*Isomorphisms*) *On the set* $X^r = X \times \cdots \times X$ *define a pre-order* $\underset{\mathcal{G}}{\rightarrow}$ *(or, for simplicity,* $\rightarrow$*) called* Δ-$\mathcal{G}$-*specialization (of elements of* X^r*) as follows: for* $\sigma = (\sigma_1, \ldots, \sigma_r)$ *and* $\tau = (\tau_1, \ldots, \tau_r) \in X^r$, $\sigma \underset{\mathcal{G}}{\rightarrow} \tau$ *if there exists a* Δ-$\mathcal{G}$-*homomorphism* $\phi : \mathcal{G}\{\sigma_1\mathcal{G} \cup \ldots \cup \sigma_r\mathcal{G}\}_\Delta \rightarrow \mathcal{G}\{\tau_1\mathcal{G} \cup \ldots \cup \tau_r\mathcal{G}\}_\Delta$ *such that* $\phi(\alpha) = \alpha$ *and* $\phi(\sigma_i\alpha) = \tau_i\alpha$ *for all* α *in* $\mathcal{G}$ *and* $i = 1, \ldots, r$.

Henceforth, if X is considered as a topological space, it will be with the pre-order topology associated to Δ-$\mathcal{G}$-specialization (Definition 1.34 with $r = 1$).

In the above definition, note that the Δ-rings $\mathcal{G}\{\sigma_1\mathcal{G} \cup \ldots \cup \sigma_r\mathcal{G}\}_\Delta \subset \mathcal{U}$ and $\mathcal{G}\{\tau_1\mathcal{G} \cup \ldots \cup \tau_r\mathcal{G}\}_\Delta \subset \mathcal{U}$ are the same as the rings $\mathcal{G}[\sigma_1\mathcal{G} \cup \ldots \cup \sigma_r\mathcal{G}]$ and $\mathcal{G}[\tau_1\mathcal{G} \cup \ldots \cup \tau_r\mathcal{G}]$. So that ϕ is in fact a Δ-$\mathcal{G}$-homomorphism from $\mathcal{G}[\sigma_1\mathcal{G}\cup\ldots\cup\sigma_r\mathcal{G}]$ to $\mathcal{G}[\tau_1\mathcal{G}\cup\ldots\cup\tau_r\mathcal{G}]$. Also, since $\tau_i \circ \sigma_i^{-1}$ is a Δ-$\mathcal{F}$-isomorphism for all i, ϕ is a Δ-$\mathcal{G}$-homomorphism if and only if it is a $\mathcal{G}$-homomorphism(See [24, Lemma 1, page 385]).

Lemma 1.35 *For $\sigma, \tau \in X^r$, $\sigma \underset{\mathcal{G}}{\rightarrow} \tau$ as in Definition 1.34 if and only if $(\ldots, \sigma_i\eta_j, \ldots) \underset{\mathcal{G}}{\rightarrow} (\ldots, \tau_i\eta_j, \ldots)$ as in Example 1.4 or Definition 1.27.*

Proof: (See [24, Lemma 2, page 386] for a statement of the same lemma without a proof.) If $\sigma \underset{\mathcal{G}}{\rightarrow} \tau$, the Δ-$\mathcal{G}$-homomorphism $\phi : \mathcal{G}\{\sigma_1\mathcal{G} \cup \ldots \cup \sigma_r\mathcal{G}\}_\Delta \rightarrow \mathcal{G}\{\tau_1\mathcal{G} \cup \ldots \cup \tau_r\mathcal{G}\}_\Delta$ of Definition 1.34 restricts to a Δ-$\mathcal{G}$-homomorphism $\rho : \mathcal{G}\{\ldots, \sigma_i\eta_j, \ldots\}_\Delta \rightarrow \mathcal{G}\{\ldots, \tau_i\eta_j, \ldots\}_\Delta$, and $(\ldots, \sigma_i\eta_j, \ldots) \underset{\mathcal{G}}{\rightarrow} (\ldots, \tau_i\eta_j, \ldots)$.

On the other hand, if $(\ldots, \sigma_i\eta_j, \ldots) \underset{\mathcal{G}}{\rightarrow} (\ldots, \tau_i\eta_j, \ldots)$, then there is a Δ-$\mathcal{G}$-homomorphism $\rho : \mathcal{G}\{\ldots, \sigma_i\eta_j, \ldots\}_\Delta \rightarrow \mathcal{G}\{\ldots, \tau_i\eta_j, \ldots\}_\Delta$. Let $\mathfrak{I}$ be the kernel of ρ. Since the image of ρ is in $\mathcal{U}$ and, therefore, an integral domain, $\mathfrak{I}$ is a prime Δ-ideal. Let $\mathcal{G}\{\ldots, \sigma_i\eta_j, \ldots\}_{\Delta,\mathfrak{I}}$ be the localization of $\mathcal{G}\{\ldots, \sigma_i\eta_j, \ldots\}_\Delta$ at $\mathfrak{I}$, and let the induced Δ-$\mathcal{G}$-homomorphism of $\mathcal{G}\{\ldots, \sigma_i\eta_j, \ldots\}_{\Delta,\mathfrak{I}}$ into the quotient field of $\mathcal{G}\{\ldots, \tau_i\eta_j, \ldots\}_\Delta$ be

$$\overline{\rho} : \quad \mathcal{G}\{\ldots, \sigma_i\eta_j, \ldots\}_{\Delta,\mathfrak{I}} \rightarrow QF(\mathcal{G}\{\ldots, \tau_i\eta_j, \ldots\}_\Delta).$$

The Δ-$\mathcal{G}$-homomorphism $\overline{\rho}$ restricted to $\mathcal{F}\{\sigma_i(\eta_1), \ldots, \sigma_i(\eta_n)\}_\Delta$ is the Δ-$\mathcal{F}$-isomorphism $\tau_i \circ \sigma_i^{-1} : \sigma_i(\mathcal{G}) \rightarrow \tau_i(\mathcal{G})$ restricted to $\mathcal{F}\{\sigma_i(\eta_1), \ldots, \sigma_i(\eta_n)\}_\Delta$.

Therefore, $\overline{\rho}$ restricted to $\mathfrak{F}\{\sigma_i(\eta_1),\ldots,\sigma_i(\eta_n)\}_\Delta$ is an Δ-$\mathfrak{F}$-isomorphism. Consequently, $\mathfrak{F}\{\sigma_i(\eta_1),\ldots,\sigma_i(\eta_n)\}_\Delta \cap \mathfrak{I} = \{0\}_\Delta$, and the nonzero elements of $\mathfrak{F}\{\sigma_i(\eta_1),\ldots,\sigma_i(\eta_n)\}_\Delta$ are invertible in $\mathfrak{G}\{\ldots,\sigma_i\eta_j,\ldots\}_{\Delta,\mathfrak{I}}$, i.e. $\sigma_i(\mathfrak{G}) \subseteq \mathfrak{G}\{\ldots,\sigma_i\eta_j,\ldots\}_{\Delta,\mathfrak{I}}$ for all i. Since η Δ-generates $\mathfrak{G}$ over $\mathfrak{F}$, $\overline{\rho}$ and $\tau_i \circ \sigma_i^{-1}$ coincide on $\sigma_i(\mathfrak{G}) \subseteq \mathfrak{G}\{\ldots,\sigma_i\eta_j,\ldots\}_{\Delta,\mathfrak{I}}$. Therefore $\overline{\rho}$ restricted to $\mathfrak{G}\{\sigma_1\mathfrak{G}\cup\ldots\cup\sigma_r\mathfrak{G}\}_\Delta$ is a Δ-$\mathfrak{G}$-homomorphism $\phi : \mathfrak{G}\{\sigma_1\mathfrak{G}\cup\ldots\cup\sigma_r\mathfrak{G}\}_\Delta \longrightarrow \mathfrak{G}\{\tau_1\mathfrak{G}\cup\ldots\cup\tau_r\mathfrak{G}\}_\Delta$ such that $\phi(\alpha) = \alpha$ and $\phi(\sigma_i\alpha) = \tau_i\alpha$ for all α in $\mathfrak{G}$ and $i = 1,\ldots,r$. Thus, ρ may be extended to the Δ-$\mathfrak{G}$-homomorphism ϕ. By the definition of Δ-$\mathfrak{G}$-specialization of elements of X^r (Definition 1.34), $\sigma \underset{\mathfrak{G}}{\longrightarrow} \tau$. □

There is a bijective correspondence between X and $\mathfrak{F}$-generic Δ-zeros of $\mathfrak{P}$ in $\mathfrak{U}^n$ which associates to each Δ-$\mathfrak{F}$-isomorphism σ of $\mathfrak{G}$ the $\mathfrak{F}$-generic Δ-zero $\sigma\eta = (\sigma\eta_1,\ldots,\sigma\eta_n)$ of $\mathfrak{P}$ in $\mathfrak{U}^n$ and to each $\mathfrak{F}$-generic Δ-zero $\xi = (\xi_1,\ldots,\xi_n)$ in $\mathfrak{U}^n$ of $\mathfrak{P}$ the Δ-$\mathfrak{F}$-isomorphism σ_ξ of $\mathfrak{G}$ defined by $\sigma_\xi(\eta) = \xi$.

Note that the Δ-ideal $\mathfrak{G}\mathfrak{P} = \mathfrak{G}\{y_1,\ldots,y_n\}_\Delta\mathfrak{P}$ is the Δ-ideal generated by $\mathfrak{P}$ in $\mathfrak{G}\{y_1,\ldots,y_n\}_\Delta$. Since each $\mathfrak{F}$-generic Δ-zero ξ of $\mathfrak{P}$ is also a Δ-zero of the ideal $\mathfrak{G}\mathfrak{P} \subset \mathfrak{G}\{y_1,\ldots,y_n\}_\Delta$, such a ξ can be regarded as an element of the $\mathfrak{G}$-closed subset $\mathfrak{Z}_\mathfrak{G}(\mathfrak{G}\mathfrak{P})$, the Δ-zeros of $\mathfrak{G}\mathfrak{P}$ in $\mathfrak{U}^n$. Combining this observation with that of the last paragraph, one obtains a map $\phi_\eta : X \rightarrow \mathfrak{Z}_\mathfrak{G}(\mathfrak{G}\mathfrak{P})$ such that $\phi_\eta(\sigma) = \sigma(\eta)$ for every $\sigma \in X$. This map is clearly injective since, for $\sigma, \tau \in X$, $\sigma(\eta) = \tau(\eta)$ implies that $\sigma = \tau$ because $\mathfrak{G}$ over $\mathfrak{F}$ is Δ-generated by η.

Corollary 1.36 *For* $\sigma, \tau \in X$, $\sigma \longrightarrow \tau$ *if and only if* $\phi_\eta(\sigma) \longrightarrow \phi_\eta(\tau)$. *Also,* *if* $\psi_\eta : \phi_\eta(X) \to X$ *is the inverse of* ϕ_η*, then* $\zeta \to \zeta'$ *if and only if* $\psi_\eta(\zeta) \to \psi_\eta(\zeta')$ *for* $\zeta, \zeta' \in \phi_\eta(X)$*.*

Proof: For the first part, take r=1 in the Lemma 1.35. For the second part, apply the definition of inverse to the first part. ☐

The next two lemmas establish a characterization of $\phi_\eta(X)$ as a subset of $\mathfrak{Z}_{\mathfrak{G}}(\mathfrak{G}\mathfrak{P})$.

Lemma 1.37 *Let* $\xi \in \mathfrak{Z}_{\mathfrak{G}}(\mathfrak{G}\mathfrak{P})$*. Then* $\mathfrak{P} \subset \mathfrak{P}_\xi \cap \mathfrak{F}\{y_1, \ldots, y_m\}_\Delta$*, and there exists a* Δ*-*$\mathfrak{F}$*-homomorphism* $\rho_{\xi,\eta} : \mathfrak{F}\{\eta_1, \ldots, \eta_m\}_\Delta \to \mathfrak{F}\{\xi_1, \ldots, \xi_m\}_\Delta$ *taking each* η_i *to* ξ_i*.*

Proof: Because $\xi \in \mathfrak{Z}_{\mathfrak{G}}(\mathfrak{G}\mathfrak{P})$, it follows that $\mathfrak{G}\mathfrak{P} \subset \mathfrak{P}_\xi$. Consequently, $\mathfrak{P} \subset \mathfrak{G}\mathfrak{P} \cap \mathfrak{F}\{y_1, \ldots, y_m\}_\Delta \subset \mathfrak{P}_\xi \cap \mathfrak{F}\{y_1, \ldots, y_m\}_\Delta$, and therefore the Δ-$\mathfrak{F}$-homomorphism $\rho_{\xi,\eta} : \mathfrak{F}\{\eta_1, \ldots, \eta_m\}_\Delta \to \mathfrak{F}\{\xi_1, \ldots, \xi_m\}_\Delta$ taking each η_i to ξ_i exists. ☐

Lemma 1.38 *Let* $\xi \in \mathfrak{Z}_{\mathfrak{G}}(\mathfrak{G}\mathfrak{P})$*. The following assertions are equivalent:*

1. $\xi \in \phi_\eta(X)$*,*

2. ξ *is an* $\mathfrak{F}$*-generic* Δ*-zero of* $\mathfrak{P}$*,*

3. $\rho_{\xi,\eta}$ *is a* Δ*-*$\mathfrak{F}$*-isomorphism of rings,*

4. $\mathfrak{P} = \mathfrak{P}_\xi \cap \mathfrak{F}\{y_1, \ldots, y_m\}_\Delta$*.*

Proof: Items 1 and 2 are equivalent by the definition of ϕ_η, which maps elements σ of $X = \mathrm{Isom}^{\Delta}_{\mathcal{F}}(\mathcal{G}, \mathcal{U})$ to the $\mathcal{F}$-generic Δ-zeros of $\mathfrak{P}$ in $\mathcal{Z}_{\mathcal{G}}(\mathcal{G}\mathfrak{P})$. In general the defining Δ-ideal of ξ in $\mathcal{F}\{y_1, \ldots, y_m\}_\Delta$ is $\mathfrak{P}_\xi \cap \mathcal{F}\{y_1, \ldots, y_m\}_\Delta$. Therefore 2 is equivalent to 3 or 4 by Lemma 1.37. ☐

Proposition 1.39 *The subset $\phi_\eta(X)$ of $\mathfrak{Z}_{\mathcal{G}}(\mathcal{G}\mathfrak{P})$ with the induced topology is an NG-space.*

Proof: By Lemma 1.16, since $\mathfrak{Z}_{\mathcal{G}}(\mathcal{G}\mathfrak{P})$ is an NG-space (Proposition 1.31), it suffices to prove $\overline{\{\xi\}} \cap \phi_\eta(X)$ is empty for all $\xi \in \mathfrak{Z}_{\mathcal{G}}(\mathcal{G}\mathfrak{P})$ not in $\phi_\eta(X)$. Since $\xi \notin \phi_\eta(X)$, $\mathfrak{P}_\xi \cap \mathcal{F}\{y_1, \ldots, y_m\}_\Delta$ properly contains $\mathfrak{P}$ by Proposition 1.38. By Corollary 1.33, the closure $\overline{\{\xi\}}$ of ζ with respect to the Δ-$\mathcal{G}$-Zariski topology of $\mathfrak{Z}_{\mathcal{G}}(\mathcal{G}\mathfrak{P})$ is the set $C(\zeta)$ of all Δ-$\mathcal{G}$-specializations of ζ. Let ξ' be a Δ-$\mathcal{G}$-specialization of ξ. Then $\mathfrak{P}_{\xi'} \supseteq \mathfrak{P}_\xi$ by Observation 1.26(3). Therefore $\mathfrak{P}_{\xi'} \cap \mathcal{F}\{y_1, \ldots, y_m\}_\Delta \supseteq \mathfrak{P}_\xi \cap \mathcal{F}\{y_1, \ldots, y_m\}_\Delta$, and $\mathfrak{P}_{\xi'} \cap \mathcal{F}\{y_1, \ldots, y_m\}_\Delta$ properly contains $\mathfrak{P}$ also. So ξ' is not in $\phi_\eta(X)$. ☐

The proof of the last proposition shows that the complement of $\phi_\eta(X)$ in $\mathfrak{Z}_{\mathcal{G}}(\mathcal{G}\mathfrak{P})$ is the union of a possibly infinite number of closed subsets. However, Example 2.150 shows that, in general, $\phi_\eta(X)$ is not an open subset of $\mathfrak{Z}_{\mathcal{G}}(\mathcal{G}\mathfrak{P})$.

Corollary 1.40 *The induced topology on $\phi_\eta(X)$ as a subset of $\mathfrak{Z}_{\mathcal{G}}(\mathcal{G}\mathfrak{P})$ is the pre-order topology associated to Δ-$\mathcal{G}$-specialization on $\phi_\eta(X)$.*

Proof: Since the Δ-$\mathcal{G}$-Zariski topology on $\mathfrak{Z}_{\mathcal{G}}(\mathcal{G}\mathfrak{P})$ is the pre-order topology associated to Δ-$\mathcal{G}$-specialization (Corollary 1.32) and the subset $\phi_\eta(X)$

of $\mathfrak{Z}_{\mathfrak{G}}(\mathfrak{G}\mathfrak{P})$ with the induced topology is an NG-space (Proposition 1.39), Proposition 1.22 implies the corollary. □

Corollary 1.41 *Let $W_1, \ldots, W_d$ be the (distinct) irreducible components of $\mathfrak{Z}_{\mathfrak{G}}(\mathfrak{G}\mathfrak{P})$. Then $W_1 \cap \phi_\eta(X), \ldots, W_d \cap \phi_\eta(X)$ are the (distinct) irreducible components of $\phi_\eta(X)$. Furthermore, for every i, each generic element of $W_i \cap \phi_\eta(X)$ is a generic element of W_i, and each generic element of W_i is a generic element of $W_d \cap \phi_\eta(X)$.*

Proof: The prime Δ-ideals $\mathfrak{P}_1, \ldots, \mathfrak{P}_d \subset \mathfrak{G}\{y_1, \ldots, y_n\}_\Delta$ that are the prime components of $\mathfrak{G}\mathfrak{P} \subset \mathfrak{G}\{y_1, \ldots, y_n\}_\Delta$ satisfy $\mathfrak{P}_i \cap \mathfrak{F}\{y_1, \ldots, y_n\}_\Delta = \mathfrak{P}$ [24, Proposition 3(b), page 131]. The closed irreducible components of $\mathfrak{Z}_{\mathfrak{G}}(\mathfrak{G}\mathfrak{P})$ are $W_i = \mathfrak{Z}_{\mathfrak{G}}(\mathfrak{P}_i)$ for $i = 1, \ldots, d$ [24, page 148]. Let ζ_i be a generic point of W_i for each i (Proposition 1.14), and let ξ_j be a generic point of the distinct irreducible components V_j $(j = 1, \ldots, e)$ of the NG-space $\phi_\eta(X)$ (Proposition 1.39 and Proposition 1.21). The generic point ζ_i of W_i, is also a $\mathfrak{G}$-generic Δ-zero of $\mathfrak{P}_i$ (Lemma 1.29) and a $\mathfrak{F}$-generic Δ-zero of $\mathfrak{G}\mathfrak{P}_i \cap \mathfrak{F}\{y_1, \ldots, y_n\} = \mathfrak{P}$. Lemma 1.38 then implies that $\zeta_i \in \phi_\eta(X)$.

Since $\xi_j \in V_j \subset \phi_\eta(X) \subset \mathfrak{Z}_{\mathfrak{G}}(\mathfrak{G}\mathfrak{P})$, $\zeta_k \underset{\mathfrak{G}}{\to} \xi_j$ for some k (Proposition 1.21). Because ξ_j is isolated in $\phi_\eta(X)$ (Proposition 1.21), $\zeta_k \leftrightarrow \xi_j$. Since ζ_i is not a Δ-$\mathfrak{G}$-specialization of ζ_l for $i \neq l$, the same is true for the ξ_j and $d = e$. Renumber the V_j and the ξ_j so that $\zeta_i \underset{\mathfrak{G}}{\to} \xi_i$. Then clearly $V_i = W_i \cap \phi_\eta(X)$.

To prove the last statement, let ζ be a generic point of V_i for some i. Then $\zeta \leftrightarrow \zeta_i \iff \zeta \leftrightarrow \xi_i$. And ζ is a generic point of W_i. This argument is reversible. □

Because the generic points of each irreducible component of $\mathfrak{Z}_{\mathfrak{G}}(\mathfrak{G}\mathfrak{P})$ are contained in $\phi_\eta(X)$, the closure of $\phi_\eta(X)$ in $\mathfrak{Z}_{\mathfrak{G}}(\mathfrak{G}\mathfrak{P})$ is $\mathfrak{Z}_{\mathfrak{G}}(\mathfrak{G}\mathfrak{P})$, and $\phi_\eta(X)$ is dense in $\mathfrak{Z}_{\mathfrak{G}}(\mathfrak{G}\mathfrak{P})$.

Proposition 1.42 *The map $\phi_\eta : X \longrightarrow \mathfrak{Z}_{\mathfrak{G}}(\mathfrak{G}\mathfrak{P})$ is a homeomorphism onto its image $\phi_\eta(X)$*

Proof: Since, for $\sigma, \tau \in X$, $\sigma \longrightarrow \tau$ if and only if $\phi_\eta(\sigma) \longrightarrow \phi_\eta(\tau)$ by Corollary 1.36 and the subset $\phi_\eta(X)$ of $\mathfrak{Z}_{\mathfrak{G}}(\mathfrak{G}\mathfrak{P})$ with the induced topology is an NG-space by Lemma 1.39, ϕ_η is continuous (Lemma 1.3(3)).

Restrict the range of the injective map ϕ_η to $\phi_\eta(X)$, and let $\psi_\eta : \phi_\eta(X) \longrightarrow X$ be its inverse. For $\zeta, \zeta' \in \phi_\eta(X)$, again by Corollary 1.36, $\zeta \longrightarrow \zeta'$ if and only if $\psi_\eta(\zeta) \longrightarrow \psi_\eta(\zeta')$, and ψ_η is continuous by Lemma 1.3(2). □

Corollary 1.43 *The space of Δ-$\mathfrak{F}$-isomorphisms X is an NG-space.*

Proof: By the proposition, ϕ_η is a homeomorphism of X onto its image $\phi_\eta(X)$, which is an NG-space by Proposition 1.39. □

Corollary 1.44 *There exists $\Phi = \{\zeta_1, \ldots, \zeta_d\}$ be a finite subset of X such that each of the irreducible components of X has one generic point in Φ. Then the ζ_i are isolated points with respect to the pre-order associated to the topology of X, and each point of X is the specialization of at least one of the ζ_i. Conversely, each isolated point $\zeta \in X$ is a generic point of one of the irreducible components of X.*

Proof: Since X is an NG-space (Corollary 1.43), the proof follows from Proposition 1.21. $\square$

The purpose of the next three results is to establish a result about algebraic disjointness[1].

Lemma 1.45 *Let F be a field, and let $G = F(\eta)$ be a finitely generated field extension of F where $\eta = (\eta_1, \dots, \eta_r) \in U^r$ and U is universal over F (Δ-universal over F for Δ empty). Let $P_\eta \subset F[y_1, \dots, y_r]$ be the defining ideal of η. Let Q be a prime component of the radical ideal $GP_\eta \subset G[y_1, \dots, y_r]$, and let $\zeta = (\zeta_1, \dots, \zeta_r) \in U^r$ be a G-generic zero of Q. Then $G = F(\eta)$ is algebraically disjoint from $F(\zeta)$ over F.*

Proof: By [24, Proposition 7(a), page 25],

$$\operatorname{tr\,deg}_G G(\zeta) = \operatorname{tr\,deg}_F F(\eta).$$

Since η and ζ are both $\mathcal{F}$-generic zeros of P_η,

$$\operatorname{tr\,deg}_F F(\eta) = \operatorname{tr\,deg}_F F(\zeta).$$

So that

$$\operatorname{tr\,deg}_G G(\zeta) = \operatorname{tr\,deg}_F F(\zeta).$$

Assume that $F(\zeta)$ is not algebraically disjoint from $G = F(\eta)$ over F. Let $x_1, \dots, x_p$ be elements of $F(\zeta)$ algebraically independent over F and

---

[1] Let L and M be two fields containing the field K and contained within some larger field. Define M to be *algebraically disjoint* from L over K if any family of elements of M algebraically dependent over L is algebraically dependent over K [4, Proposition 13, page 112].

algebraically dependent over $F(\eta)$. Extend $x_1, \ldots, x_p$ to a transcendence basis $x_1, \ldots, x_p, y_1, \ldots, y_q$ of $F(\zeta)$ over F. Since $x_1, \ldots, x_p$ is algebraically dependent over $G = F(\eta)$,

$$\text{tr deg}_F\ F(x_1, \ldots, x_p) > \text{ tr deg}_G\ G(x_1, \ldots, x_p),$$

and

$$\text{tr deg}_F\ F(x_1, \ldots, x_p, y_1, \ldots, y_q) > \text{ tr deg}_G\ G(x_1, \ldots, x_p, y_1, \ldots, y_q).$$

Because $F(\zeta)$ is algebraic over $F(x_1, \ldots, x_p, y_1, \ldots, y_q)$, $G(\zeta)$ is also algebraic over $G(x_1, \ldots, x_p, y_1, \ldots, y_q)$, and

$$\text{tr deg}_F\ F(\zeta) > \text{ tr deg}_G\ G(\zeta).$$

This contradiction shows that $F(\zeta)$ is algebraically disjoint from $G = F(\eta)$ over F. $\square$

Lemma 1.46 *Let $\mathfrak{F}$ be a Δ-field, and let $\mathfrak{G} = \mathfrak{F}\langle\eta\rangle_\Delta \subset \mathfrak{U}$ be a finitely Δ-generated field Δ-extension of $\mathfrak{F}$ where $\eta = (\eta_1, \ldots, \eta_r) \in \mathfrak{U}^r$ and $\mathfrak{U}$ is Δ-universal over $\mathfrak{F}$. Let $\mathfrak{P}_\eta \subset \mathfrak{F}\{y_1, \ldots, y_r\}_\Delta$ be the defining Δ-ideal of η. Let $\mathfrak{Q}$ be a prime component of the radical Δ-ideal $\mathfrak{G}\mathfrak{P}_\eta \subset \mathfrak{G}\{y_1, \ldots, y_r\}_\Delta$. And let $\zeta = (\zeta_1, \ldots, \zeta_r) \in U^r$ be a $\mathfrak{G}$-generic Δ-zero of $\mathfrak{Q}$. Then $\mathfrak{G} = \mathfrak{F}\langle\eta\rangle_\Delta$ is algebraically disjoint from $\mathfrak{F}\langle\zeta\rangle_\Delta$ over $\mathfrak{F}$.*

Proof: Since any algebraic relation between elements of $\mathfrak{G} = \mathfrak{F}\langle\eta\rangle_\Delta$ and $\mathfrak{F}\langle\zeta\rangle_\Delta$ over $\mathfrak{F}$ involves just a finite number of elements from each field, the strategy of this argument is to reduce the proof of the lemma to Lemma 1.45, which only involves finitely generated field extensions.

Define Θ be the free commutative multiplicative monoid generated by Δ. Thus an element of Θ has a unique representation in the form

$$\theta = \delta_1^{e_1} \cdots \delta_m^{e_m}$$

for natural numbers $e_1, \ldots, e_m \in \mathbb{N}$. The *order* of θ is

$$\operatorname{ord} \theta = e_1 + \cdots + e_n,$$

and the unique element of order 0 is denoted by 1. The subset of elements of order no larger that ν is denoted by $\Theta(\nu)$, i.e.

$$\Theta(\nu) = \{\theta \in \Theta \mid \operatorname{ord} \theta \leq \nu\}.$$

For each positive integer ν, let

$$A_\nu = \mathfrak{F}[(\theta y_i)_{\theta \in \Theta(\nu)}] \quad \text{and} \quad B_\nu = \mathfrak{G}[(\theta y_i)_{\theta \in \Theta(\nu)}].$$

Define the order of $f \in \mathfrak{F}\{y_1, \ldots, y_r\}_\Delta$ (*resp.* $f \in \mathfrak{G}\{y_1, \ldots, y_r\}_\Delta$) to be the least value of ν such that $f \in A_\nu$ (*resp.* $f \in B_\nu$).

Claim 1.47 *$\mathfrak{G} \cdot (\mathfrak{I} \cap A_\nu) = \mathfrak{G}\mathfrak{I} \cap B_\nu$ for any ideal $\mathfrak{I} \subset \mathfrak{F}\{y_1, \ldots, y_r\}_{\Delta}$,.*

(See [24, page 131].) The containment $\mathfrak{G} \cdot (\mathfrak{I} \cap A_\nu) \subseteq \mathfrak{G}\mathfrak{I} \cap B_\nu$ is clear. A $(u_i)_{i \in I}$ basis for $\mathfrak{G}$ over $\mathfrak{F}$ is also a basis for $\mathfrak{G}\{y_1, \ldots, y_r\}_\Delta$ over $\mathfrak{F}\{y_1, \ldots, y_r\}_\Delta$ because the Δ-derivatives of $y_1, \ldots, y_r$ are assumed to be linearly independent over $\mathfrak{G}$ and the vector spaces $\mathfrak{G}$ and $\mathfrak{F}\{y_1, \ldots, y_r\}_\Delta$ are linearly disjoint over $\mathfrak{F}$. Therefore, any $x \in \mathfrak{G}\mathfrak{I} \subset \mathfrak{G}\{y_1, \ldots, y_r\}_\Delta$ may be written uniquely in terms of this basis as $x = \Sigma_i x_i u_i$ with $x_i \in \mathfrak{F}\{y_1, \ldots, y_r\}_\Delta$. According to [24, Lemma 9, page 18], $x_i \in \mathfrak{I}$. If in addition $x \in B_\nu$, that is $x \in \mathfrak{G}\mathfrak{I} \cap B_\nu$, the order of x is equal the maximum of the orders of x_i for all i because of the uniqueness of the coefficients x_i. Therefore, $x_i \in A_\nu$ for all i, and $x = \Sigma_i x_i u_i = \Sigma_i u_i x_i \in \mathfrak{G} \cdot (\mathfrak{I} \cap A_\nu)$.

Let $\mathfrak{P}_\eta = \mathfrak{P}_{\eta,\mathfrak{F}} \subset \mathfrak{F}\{y_1, \ldots, y_r\}_\Delta$. Let $\mathfrak{P}_i$ for $i = 1, \ldots, s$ be the prime components of the radical Δ-ideal $\mathfrak{G}\mathfrak{P}_\eta = \bigcap_i \mathfrak{P}_i \subset \mathfrak{G}\{y_1, \ldots, y_r\}_\Delta$ [24, Proposition 3(b),page 131]. By the claim, $\mathfrak{G} \cdot (\mathfrak{P}_\eta \cap A_\nu) = (\mathfrak{G}\mathfrak{P}_\eta) \cap B_\nu = (\bigcap_i \mathfrak{P}_i) \cap B_\nu = \bigcap_i (\mathfrak{P}_i \cap B_\nu)$. For ν large enough, $\mathfrak{P}_i \cap B_\nu$ for $i = 1, \ldots, s$ are distinct prime components of $\mathfrak{G}(\mathfrak{P}_\eta \cap A_\nu)$ in B_ν because the components are the radicals of finitely Δ-generated Δ-ideals [24, Corollary 5, page 128].

Let $\zeta = (\zeta_1, \ldots, \zeta_r) \in \mathfrak{U}^r$ be a $\mathfrak{G}$-generic Δ-zero of one of the prime components $\mathfrak{Q}$ ($= \mathfrak{P}_i$ for some i) of $\mathfrak{G}\mathfrak{P}_\eta$. Then ζ is also a $\mathfrak{F}$-generic Δ-zero of $\mathfrak{P}_\eta$ because $\mathfrak{Q} \cap \mathfrak{F}\{y_1, \ldots, y_r\}_\Delta = \mathfrak{P}_\eta$ [24, Proposition 3(b),page 131]. Since the prime ideal $\mathfrak{P}_\eta \cap A_\nu \subset A_\nu$ is the defining ideal of $(\theta\eta_i)_{\theta \in \Theta(\nu)}$ and $(\theta\zeta_i)_{\theta \in \Theta(\nu)}$ is a $\mathfrak{G}$-generic zero of the prime component $\mathfrak{Q} \cap B_\nu$ of the radical ideal $\mathfrak{G} \cdot (\mathfrak{P}_\eta \cap A_\nu) \subset B_\nu$, Lemma 1.45 may be applied to conclude the fields $\mathfrak{F}(\theta\eta_i)_{\theta \in \Theta(\nu)}$ and $\mathfrak{F}(\theta\zeta_i)_{\theta \in \Theta(\nu)}$ are algebraically disjoint over $\mathfrak{F}$ for large ν. Since any algebraic relation over $\mathfrak{F}$ between elements of the fields $\mathfrak{G} = \mathfrak{F}\langle\eta\rangle_\Delta$ and $\mathfrak{F}\langle\zeta\rangle_\Delta$ is in fact between elements of $(\theta\eta_i)_{\theta \in \Theta(\nu)}$ and $(\theta\zeta_i)_{\theta \in \Theta(\nu)}$ for ν sufficiently large, $\mathfrak{G} = \mathfrak{F}\langle\eta\rangle_\Delta$ and $\mathfrak{F}\langle\zeta\rangle_\Delta$ are algebraically disjoint over $\mathfrak{F}$. $\square$

Corollary 1.48 *Let* $\mathcal{G} \subset \mathcal{U}$ *be a finitely* Δ*-generated* Δ*-extension of* $\mathcal{F}$ *such that* $\mathcal{G} \neq \mathcal{F}$*. If* $\sigma \in X = \mathrm{Isom}^{\Delta}_{\mathcal{F}}(\mathcal{G}, \mathcal{U})$ *is an isolated* Δ*-isomorphism, then* $\mathcal{G}$ *and* $\sigma(\mathcal{G})$ *are algebraically disjoint over* $\mathcal{F}$*.*

Proof: Let $\sigma \in X$ be isolated. By Corollary 1.44, σ is a generic point of the component $\overline{\{\sigma\}}$. For Δ-generators $\eta = (\eta_1, \ldots, \eta_r) \in \mathcal{U}^r$ of $\mathcal{G}$ over $\mathcal{F}$, since φ_η is a homeomorphism of X onto its image $\phi_\eta(X)$ (Proposition 1.42), $\varphi_\eta(\sigma)$ is also a generic point of a component of $\varphi_\eta(X)$ and, therefore, a generic point of a component of $\mathfrak{Z}_{\mathcal{G}}(\mathcal{G}\mathfrak{P})$ (Corollary 1.41). By Lemma 1.29, $\varphi_\eta(\sigma) = \sigma(\eta)$ is a $\mathcal{G}$-generic Δ-zero of a prime component of the Δ-ideal $\mathcal{G}\mathfrak{P}$. By Lemma 1.46, $\mathcal{G} = \mathcal{F}\langle\eta\rangle_\Delta$ and $\sigma(\mathcal{G}) = \mathcal{F}\langle\sigma(\eta)\rangle_\Delta$ are algebraically disjoint over $\mathcal{F}$. $\square$

Proposition 1.49 *Let* $\mathcal{G} \subset \mathcal{U}$ *be a finitely* Δ*-generated* Δ*-extension of* $\mathcal{F}$*. Then the finitely many irreducible components of* $X = \mathrm{Isom}^{\Delta}_{\mathcal{F}}(\mathcal{G}, \mathcal{U})$ *are mutually disjoint and are of the form* $C(\sigma_i)$*, where* $\sigma_1, \ldots, \sigma_r$ *are isolated* Δ*-isomorphisms of* $\mathcal{G}$ *over* $\mathcal{F}$*. If* $\mathcal{G}$ *is regular*[2] *over* $\mathcal{F}$*,* $r = 1$*.*

Because the $C(\sigma_i)$ are disjoint, they are also the connected components.

Proof: Let σ be any Δ-$\mathcal{F}$-isomorphism of $\mathcal{G}$ into $\mathcal{U}$. By Corollary 1.44, there are finitely many isolated points σ_i such that there exists an i such that $\sigma_i \longrightarrow \sigma$ (Proposition 1.21). To show σ is the Δ-$\mathcal{G}$-specialization of just one σ_i, assume σ is a Δ-$\mathcal{G}$-specialization of σ_i and σ_j for some i and j.

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[2] The field G is *regular* over F, or F is *algebraically closed* in G, if each element of G algebraic over F is in F.

Let $\varphi_i : \mathcal{G}\{\sigma_i\mathcal{G}\}_\Delta \mapsto \mathcal{G}\{\sigma\mathcal{G}\}_\Delta$ and $\varphi_j : \mathcal{G}\{\sigma_j\mathcal{G}\}_\Delta \mapsto \mathcal{G}\{\sigma\mathcal{G}\}_\Delta$ be the Δ-$\mathcal{G}$-homomorphisms of the corresponding Δ-$\mathcal{G}$-specializations. Also let $\mathcal{F}_0$ be the algebraic closure of $\mathcal{F}$ in $\mathcal{G}$. Because $\sigma_i\mathcal{F}_0$ and $\sigma_j\mathcal{F}_0$ are algebraic over $\mathcal{F}$, $\mathcal{G}\{\sigma_i\mathcal{F}_0\}_\Delta$ is the field compositum $\mathcal{G} \cdot \sigma_i\mathcal{F}_0$, and $\mathcal{G}\{\sigma_j\mathcal{F}_0\}_\Delta$ is the field compositum $\mathcal{G} \cdot \sigma_j\mathcal{F}_0$. Therefore φ_i and φ_j restricted to $\mathcal{G}\{\sigma_i\mathcal{F}_0\}_\Delta$ and $\mathcal{G}\{\sigma_j\mathcal{F}_0\}_\Delta$, respectively, are Δ-$\mathcal{G}$-isomorphisms onto the field $\mathcal{G}\{\sigma\mathcal{F}_0\}_\Delta$.

Diagram 1.50

$$\begin{tikzcd}
\mathcal{G} \arrow[r] & \mathcal{G} \cdot \sigma_i\mathcal{F}_0 \arrow[r] & \mathcal{G} \cdot \sigma_i\mathcal{G} \\
\mathcal{F} \arrow[r] \arrow[u] & \sigma_i\mathcal{F}_0 \arrow[r] \arrow[u] & \sigma_i\mathcal{G} \arrow[u]
\end{tikzcd}$$

Because σ_i is isolated, $\sigma_i\mathcal{G}$ is algebraically disjoint from $\mathcal{G}$ over $\mathcal{F}$ by Corollary 1.48 (see Diagram 1.50 above). Since $\sigma_i\mathcal{F}_0$ is a subfield of $\sigma_i\mathcal{G}$ containing $\mathcal{F}$, $\sigma_i\mathcal{G}$ is algebraically disjoint from $\mathcal{G} \cdot \sigma_i\mathcal{F}_0$ over $\sigma_i\mathcal{F}_0$. Because $\sigma_i\mathcal{G}$ over $\sigma_i\mathcal{F}_0$ is a regular extension, it follows that $\mathcal{G} \cdot \sigma_i\mathcal{F}_0$ and $\sigma_i\mathcal{G}$ are linearly disjoint over $\sigma_i\mathcal{F}_0$ [26, Theorem 3, page 57].

Let γ be the Δ-$\mathcal{G}$-isomorphism

$$\gamma : \quad \mathcal{G} \cdot \sigma\mathcal{F}_0 \to \mathcal{G} \cdot \sigma_j\mathcal{F}_0$$

that is the inverse of φ_j with its range restricted to $\mathcal{G}\{\sigma\mathcal{F}_0\}_\Delta = \mathcal{G} \cdot \sigma\mathcal{F}_0$. Then, after restricting the domain of φ_i to $\mathcal{G} \cdot \sigma_i\mathcal{F}_0$,

$$\gamma \circ \varphi_i : \quad \mathcal{G} \cdot \sigma_i\mathcal{F}_0 \to \mathcal{G} \cdot \sigma\mathcal{F}_0 \to \mathcal{G} \cdot \sigma_j\mathcal{F}_0$$

is a Δ-$\mathcal{F}$-isomorphism. And

$$\sigma_j \circ \sigma_i^{-1} : \quad \sigma_i\mathcal{G} \to \mathcal{G} \to \sigma_j\mathcal{G}$$

is a Δ-$\mathcal{F}$-isomorphism that coincides with $\gamma \circ \varphi_i$ on $\sigma_i \mathcal{F}_0$. Therefore, by the linear disjointness, $\gamma \circ \varphi_i$ and $\sigma_j \circ \sigma_i^{-1}$ extend to a Δ-$\mathcal{G}$-homomorphism of

$$(\mathcal{G} \cdot \sigma_i \mathcal{F}_0)\{\sigma_i \mathcal{G}\}_\Delta = \mathcal{G}\{\sigma_i \mathcal{G}\}_\Delta \to (\mathcal{G} \cdot \sigma_j \mathcal{F}_0)\{\sigma_j \mathcal{G}\}_\Delta = \mathcal{G}\{\sigma_j \mathcal{G}\}_\Delta.$$

By definition, σ_j is a Δ-$\mathcal{G}$-specialization of σ_i. This is impossible because the σ_i and σ_j were chosen not to be Δ-$\mathcal{G}$-specializations of each other. $\square$

For a Galois theory, it is important to know the field invariant under all Δ-$\mathcal{F}$-isomorphisms. For a treatment of this topic similar to what follows see Kolchin [24, Corollary, page 388].

Lemma 1.51 *Let $\mathcal{H}$ be a Δ-extension of $\mathcal{F}$ algebraic over $\mathcal{F}$. Any field isomorphism of $\mathcal{H}$ over $\mathcal{F}$ into $\mathcal{U}$ is in fact differential.*

Proof: Let σ be a field isomorphism (not necessarily a Δ-$\mathcal{F}$-isomorphism) of $\mathcal{H}$ over $\mathcal{F}$ into $\mathcal{U}$. Let $f(x)$ in $\mathcal{F}[x]$ be the minimal polynomial over $\mathcal{F}$ for an element α of $\mathcal{H}$. Let δ be an element of Δ. Also let $S_f(x)$ be the derivative of f with respect to x and $f^\delta(x)$ the polynomial obtained by applying δ to the coefficients of $f(x)$. Since all fields are assumed to have characteristic 0, α is separably algebraic, and, thus, $S_f(\alpha) \neq 0$. Then $\delta\alpha = -f^\delta(\alpha)/S_f(\alpha)$. Since $f(x)$ is also the minimal polynomial of $\sigma\alpha$ over $\mathcal{F}$, $\delta(\sigma\alpha) = -f^\delta(\sigma\alpha)/S_f(\sigma\alpha)$. Then, because the coefficients of $f(x)$ and $f^\delta(x)$ are fixed by σ, $\sigma\delta\alpha = \sigma(-f^\delta(\alpha)/S_f(\alpha)) = -(\sigma f^\delta)(\sigma\alpha)/S_{\sigma f}(\sigma\alpha) = -(f)^\delta(\sigma\alpha)/S_f(\sigma\alpha) = \delta\sigma\alpha$. $\square$

Proposition 1.52 *Let $\mathcal{U}$ be a Δ-universal over $\mathcal{F}$. Let $\mathcal{H}$ be a finitely Δ-generated extension of $\mathcal{F}$, not necessarily contained in $\mathcal{U}$, and let $\mathcal{I}$ be* *a finitely Δ-generated extension of $\mathcal{H}$. Then every Δ-$\mathcal{F}$-isomorphism $\sigma : \mathcal{H} \longrightarrow \mathcal{U}$ and be extended to a Δ-$\mathcal{F}$-isomorphism $\sigma' : \mathcal{I} \longrightarrow \mathcal{U}$.*

Proof: Let $\mathcal{S}$ be a set not contained in $\mathcal{U}$ or $\mathcal{I}$ that has the same cardnality as $\mathcal{I} \setminus \mathcal{H}$. And let $f : \mathcal{I} \longrightarrow \mathcal{I}^{+} = \mathcal{S} \cup \sigma(\mathcal{H})$ be a bijection such that $f(\alpha) = \sigma(\alpha)$ for all $\alpha \in \mathcal{H}$. Endow $\mathcal{I}^{+}$ with the Δ-field structure such that f is a Δ-$\mathcal{F}$-isomorphism. Because $\mathcal{I}$ is a finitely Δ-generated extension of $\mathcal{H}$, $f(\mathcal{I})$ is also a finitely Δ-generated extension of $f(\mathcal{H}) = \sigma(\mathcal{H})$. Similarly, $\mathcal{H}$ finitely Δ-generated over $\mathcal{F}$ implies $\sigma(\mathcal{H}) \subset \mathcal{U}$ finitely Δ-generated over $\sigma(\mathcal{F})$. By the definition of Δ-universal over $\mathcal{F}$, there exists a Δ-isomorphism $g : \mathcal{I}^{+} \longrightarrow \mathcal{U}$ over $\sigma(\mathcal{H})$. Then Δ-isomorphism $\sigma' = g \circ f : \mathcal{I} \rightarrow \mathcal{U}$ extends σ. $\square$

$$\begin{tikzcd}
 & & & & & & \mathcal{U} \\
\mathcal{I} \arrow[rrr, "f"] & & & \mathcal{I}^{+} \arrow[rr, dotted, "g"] & & g(\mathcal{I}^{+}) = \sigma'(\mathcal{I}) \arrow[ur] & \\
 & \mathcal{H} \arrow[ul] \arrow[rrr, "\sigma"] & & & \sigma(\mathcal{H}) \arrow[ul] \arrow[ur] & & \\
 & & \mathcal{F} \arrow[ul] \arrow[urr] & & & &
\end{tikzcd}$$

Proposition 1.53 *Let $\mathcal{G} \subset \mathcal{U}$ be a finitely Δ-generated extension of $\mathcal{F}$. The subset of $\mathcal{G}$ fixed by all elements of $X = \mathrm{Isom}^{\Delta}_{\mathcal{F}}(\mathcal{G}, \mathcal{U})$ is $\mathcal{F}$.*

Proof: Let $\alpha \in \mathcal{G}$ be fixed by all elements of X. By Corollary 1.44, there exists an isolated element $\tau \in X$. Since τ fixes α, $\alpha \in \mathcal{G} \cap \tau\mathcal{G}$. This shows that $\alpha \in \tau\mathcal{G}$ is algebraic over $\mathcal{G}$ ($\alpha \in \mathcal{G}$). Since $\mathcal{G}$ and $\tau\mathcal{G}$ are algebraically disjoint over $\mathcal{F}$ (Corollary 1.48), α is algebraic over $\mathcal{F}$.

Assume $\alpha \notin \mathcal{F}$. By the algebraic theory of isomorphisms, there exists a field isomorphism σ (not necessarily differential) of $\mathcal{F}(\alpha)$ over $\mathcal{F}$ into $\mathcal{U}$ such that $\sigma(\alpha) \neq \alpha$. By Lemma 1.51, σ is a Δ-$\mathcal{F}$-isomorphism.

There exists an extension $\sigma' \in X$ of σ to $\mathcal{G}$, by Proposition 1.52, because $\mathcal{F}(\alpha) = \mathcal{F}\langle\alpha\rangle_\Delta$ is finitely Δ-generated and $\mathcal{G}$ over $\mathcal{F}\langle\alpha\rangle_\Delta$ is also since $\mathcal{G}$ over $\mathcal{F}$ is. Since $\sigma' \in X$ does not fix α and it was assumed that all the elements of X fix α, the assumption $\alpha \notin \mathcal{F}$ must be incorrect. Therefore $\alpha \in \mathcal{F}$ $\square$

2 Pre E-Sets and Isomorphisms

This chapter reviews Kolchin's definition of E-sets (read E as epsilon) [25, Chapter 1] and presents examples in a novel manner. The E-sets have a geometric nature and are later used to define algebraic E-groups.

Fix two disjoint sets $\mathrm{E} = \{\epsilon_1, \ldots, \epsilon_r\}$ and $\Delta = \{\delta_1, \ldots, \delta_n\}$. The symbol "$(\mathrm{E}, \Delta)$" will denote the set $\mathrm{E} \cup \Delta$. However, when this symbol is used as a subscript or superscript the parenthesis are removed, e.g., $\mathfrak{F}\{y\}_{\mathrm{E},\Delta}$ or $\mathfrak{F}^{\mathrm{E},\Delta}$. If $\mathfrak{G}$ is an (E, Δ)-field, let $\mathfrak{G}^{\Delta}$ denote the field of Δ-constants of $\mathfrak{G}$. Note that $\mathfrak{G}^{\Delta}$ is a E-field. If $\mathfrak{F}$ and $\mathfrak{G}$ are fields contained in a larger field, then $\mathfrak{F} \cdot \mathfrak{G}$, or more simply $\mathfrak{F}\mathfrak{G}$, will denote their compositum.

2.1 The Category of Pre E-Sets

The objects in the category of pre E-$\mathfrak{F}$-sets [25, Chapter 1] are defined as follows.

Definition 2.54 *Let $\mathfrak{F}$ be an* E*-field and let $\mathcal{V}$ be a universal* E*-field extension of $\mathfrak{F}$. A pre* E*-$\mathfrak{F}$-set (relative to $\mathcal{V}$) is a set A for which there are given*

1. *for each element $x \in A$, an* E*-finitely generated field extension $\mathfrak{F}\langle x\rangle$ over $\mathfrak{F}$,*
2. *a pre order on A called* E-specialization over $\mathfrak{F}$ or, more simply, E-$\mathfrak{F}$-specialization (*which shall be indicated by the notation $x \longrightarrow x'$*),

3. *for each pair* (x, x') *in* A^2 *with* $x \leftrightarrow x'$*, an* E*-isomorphism* $S_{x',x} : \mathfrak{F}\langle x\rangle \approx \mathfrak{F}\langle x'\rangle$ *over* $\mathfrak{F}$,

all subject to the following axioms.

DAS1 *A has a finite subset* Φ *such that, for each* $x' \in A$*, there exists an* $x \in \Phi$ *with* $x \rightarrow x'$.

DAS2a *If* $x, x', x'' \in A$, $x \leftrightarrow x'$*, and* $x' \leftrightarrow x''$*, then* $S_{x'',x'} \circ S_{x',x} = S_{x'',x}$.

DAS2b *If* $x \in A$ *and* $S : \mathfrak{F}\langle x\rangle \approx \mathfrak{F}'$ *is a* E*-field isomorphism over* $\mathfrak{F}$*, then there exists a unique* $x' \in A$ *with* $x \leftrightarrow x'$ *such that* $\mathfrak{F}\langle x'\rangle = \mathfrak{F}'$ *and* $S_{x',x} = S$.

See Observation 6.245 for an example of a pre Δ-$\mathfrak{F}$-set.

Definition 2.55 *A subset B of the pre* E-$\mathfrak{F}$*-set A is called* $\mathfrak{F}$-irreducible (in *A*) *if there exists an* $x \in A$ *such that B is the set of all elements of A that are* E-$\mathfrak{F}$*-specializations of x. Such an x will be called an* $\mathfrak{F}$-generic element *of B. A maximal* $\mathfrak{F}$*-irreducible subset of A is called an* $\mathfrak{F}$-component *of A.*

Later in this chapter, it will be shown how to associate to any E-scheme over $\mathfrak{F}$ a pre E-$\mathfrak{F}$-set. In particular, if E is empty, to a scheme over $\mathfrak{F}$ is associated a pre E-$\mathfrak{F}$-set called a pre $\mathfrak{F}$-set.

Kolchin defines pre E-$\mathfrak{F}$-maps, as below, in a manner such that the composition of two is not necessarily a third.

Definition 2.56 *Let A and B be pre* E-$\mathfrak{F}$*-sets. A pre* E-$\mathfrak{F}$*-mapping of A to B is a mapping f of a subset A_f of A into B with the following four properties:*

1. *the $\mathfrak{F}$-generic elements of the components of A are contained in A_f;*
2. *if $x \in A_f$, then $\mathfrak{F}\langle f(x)\rangle \subset \mathfrak{F}\langle x\rangle$;*
3. *if $x \in A, x' \in A_f$, and $x \longrightarrow x'$, then $x \in A_f$ and $f(x) \longrightarrow f(x')$;*
4. *if $x, x' \in A_f$ and $x \leftrightarrow x'$, then $S_{x',x}$ extends $S_{f(x'),f(x)}$. See Diagram 2.57 below.*

Diagram 2.57

$$\begin{tikzcd}
\mathfrak{F}\langle x\rangle \arrow[r, "S_{x',x}"] & \mathfrak{F}\langle x'\rangle \\
\mathfrak{F}\langle f(x)\rangle \arrow[u, "inclusion"'] \arrow[r, "S_{f(x'),f(x)}"] & \mathfrak{F}\langle f(x')\rangle \arrow[u, "inclusion"']
\end{tikzcd}$$

To have morphisms that are composable, pre E-$\mathfrak{F}$-mappings from A to B that are everywhere defined (that is $A_f = A$) are taken to be the morphisms in the category of pre E-$\mathfrak{F}$-sets.

Definition 2.58 *The category of pre* E-$\mathfrak{F}$*-sets (relative the universal* E *- field $\mathcal{V}$) is the category with pre* E-$\mathfrak{F}$*-sets as objects and with everywhere defined* E-$\mathfrak{F}$*-mappings as morphisms.*

Later in this chapter, it will be shown how to associate to any morphism of E-schemes over $\mathfrak{F}$ an everywhere defined E-$\mathfrak{F}$-mapping.

2.2 E-Schemes

In this paragraph, the concept of E-schemes, as defined by Kovacic in a series of papers [19], [20] and [21], will be explained. Let R be an E-ring. Let $\operatorname{diffspec} R$ be the set of all E-prime ideals of R. The topology on $\operatorname{diffspec} R$ such that the closed subsets are $V(I) = \{\mathfrak{P} \in \operatorname{diffspec} R \mid \mathfrak{P} \supseteq I\}$ for $I \subset R$ is called the E-*Zariski* topology or the *Kolchin* topology [19]. Henceforth, in agreement with the prevailing custom, $\operatorname{diffspec} R$ will denote this topological space unless otherwise specified, and the notation for points of $\operatorname{diffspec} R$ will alternate between points $x, y, \ldots$ if their topological nature is to be emphasized or prime E-ideals $\mathfrak{P}, \mathfrak{Q}, \ldots$ if their algebraic properties are of interest.

Observation 2.59 *In* $\operatorname{diffspec} R$ *the closure of a point* $\mathfrak{P}$ *has the following equivalent expressions:*

1. $\overline{\{\mathfrak{P}\}}$,
2. $V(\mathfrak{P})$, *and*
3. $\{\mathfrak{Q} \in \operatorname{diffspec} R \mid \mathfrak{Q} \supseteq \mathfrak{P}\}$.

Since a set consisting of a single element is irreducible and $V(\mathfrak{P})$ is the closure of $\mathfrak{P}$ in $\operatorname{diffspec} R$ (Observation 2.59), $V(\mathfrak{P})$ is irreducible [3, Proposition 2, page 95]. Conversely,

Proposition 2.60 *Let* $\mathfrak{a} \subset R$ *be a* Δ*-ideal. If* $V(\mathfrak{a})$ *is an irreducible closed set of* $\operatorname{diffspec} R$*, the radical* Δ*-ideal* $\{\mathfrak{a}\}$ *is prime. Also, each irreducible closed set of* $\operatorname{diffspec} R$ *has a unique generic point.*

Proof: Each closed set W of $\operatorname{diffspec} R$ in the E-Zariski topology is of the form $W = V(I) = \{\mathfrak{P} \in \operatorname{diffspec} R \mid \mathfrak{P} \supseteq I\} = \{\mathfrak{P} \in \operatorname{diffspec} R \mid \mathfrak{P} \supseteq \{I\}_{\mathrm{E}}\} = V(\{I\}_{\mathrm{E}})$ for $I \subset R$ where $\{I\}_{\mathrm{E}}$ is the radical E-ideal generated by I [19, Proposition 3.2(d), page 75]. Note that for an element $f \in R$, the relation $f \in \{I\}_{\mathrm{E}}$ is equivalent to $W \subseteq V(\{f\})$.

Assume $W = V(\{I\}_{\mathrm{E}})$ is irreducible. Let f and g be elements of R such that $fg \in \{I\}_{\mathrm{E}}$. Then

$$W \subseteq V(\{fg\}) = V(\{f\}) \cup V(\{g\});$$

since W is irreducible and $V(\{f\})$ and $V(\{g\})$ are closed, $W \subseteq V(\{f\})$ or $W \subseteq V(\{g\})$. Therefore $f \in \{I\}_{\mathrm{E}}$ or $g \in \{I\}_{\mathrm{E}}$, and $\{I\}_{\mathrm{E}}$ is prime.

The prime E-ideal $\{I\}_{\mathrm{E}}$ is a generic point of $W = V(\{I\}_{\mathrm{E}})$ by Observation 2.59. If $\mathfrak{Q} \in \operatorname{diffspec} R$ is another generic point of W, then $W = V(\mathfrak{Q})$ also. From $V(\mathfrak{Q}) = V(\mathfrak{P})$, it follows that $\mathfrak{Q} \supseteq \mathfrak{P}$ and $\mathfrak{Q} \subseteq \mathfrak{P}$ by [19, Proposition 3.2(c), page 75]. Therefore $\mathfrak{Q} = \mathfrak{P}$, and the generic point is unique. □

Remark 2.61 *The topology on* $\operatorname{diffspec} R$ *generated by the closed sets of the form* $V(\mathfrak{a})$ *for* Δ*-ideals* $\mathfrak{a} \subseteq R$ *is called the E-*Zariski topology*. It is the restriction of the Zariski topology on* $\operatorname{spec} R$ *to* $\operatorname{diffspec} R$*.*

For each open set U of $\operatorname{diffspec} R$, let $\mathcal{O}_{\operatorname{diffspec} R}(U)$ be the ring of functions

$$s : U \to \coprod_{\mathfrak{P} \in U} R_{\mathfrak{P}}$$

satisfying

1. For each $\mathfrak{P} \in U$, $s(\mathfrak{P}) \in R_{\mathfrak{P}}$, and
2. there is an open cover $\{U_i\}$ of U, and $a_i, b_i \in R$ such that, for each $\mathfrak{Q} \in U_i, b_i \notin \mathfrak{Q}$ and $s(\mathfrak{Q}) = a_i/b_i \in R_{\mathfrak{Q}}$.

The pre-sheaf $\mathcal{O}_{\text{diffspec}\, R}$ is a sheaf of E-rings on diffspec R [19, page 79].

The category of *E-ringed spaces* has as objects E-ringed spaces $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf on X with values in the category of E-rings. The morphisms are pairs $(f, f^{\#}) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ where $f : X \longrightarrow Y$ is a continuous map and $f^{\#} : \mathcal{O}_Y \longrightarrow f_*(\mathcal{O}_X)$ is an E-homomorphism of sheaves of E-algebras[3]. The composition of two morphisms $(f, f^{\#}) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ and $(g, g^{\#}) : (Y, \mathcal{O}_Y) \longrightarrow (Z, \mathcal{O}_Z)$ is the morphism $(h, h^{\#}) : (X, \mathcal{O}_X) \longrightarrow (Z, \mathcal{O}_Z)$ defined by $h = g \circ f$ and the composition

$$\mathcal{O}_Z \xrightarrow{g^{\#}} g_*(\mathcal{O}_Y) \xrightarrow{g_*(f^{\#})} g_*(f_*(\mathcal{O}_X) = (g_* \circ f_*)(\mathcal{O}_X) = h_*(\mathcal{O}_X)$$

$h^{\#} = g_*(f^{\#}) \circ g^{\#}$.

For each $x \in X$, the family of E-rings $\{\mathcal{O}_X(U)\}_{x \in U}$, for all open subsets $U \subseteq X$ containing x, is a directed system in the category of E-rings. The direct limit $\mathcal{O}_x = \varinjlim_{x \in U} \mathcal{O}_X(U)$, called the *stalk of $\mathcal{O}_X$ at x*, exists in the category of E-rings because the direct limit of a directed system of algebras with a given algebraic structure again has that algebraic structure [9, Proposition 2.8, page 14]. If the underlying space X is not obvious, then $\mathcal{O}_x$ will be denoted by $\mathcal{O}_{X,x}$. For each open subset U of X containing x, the *canon-*

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[3] The sheaf $f_*(\mathcal{O}_X)$ on Y is defined by $f_*(\mathcal{O}_X)(U) = \mathcal{O}_X(f^{-1}(U))$ for all open U of Y where $\mathcal{O}_X(f^{-1}(U))$ is the ring of sections of the sheaf $\mathcal{O}_X$ over $f^{-1}(U)$.

ical E-*homomorphism* $\mathcal{J}_{x,U} : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_x$ is canonical E-homomorphism determined by the definition of $\mathcal{O}_x$ as a limit.

Given a morphism of E-ringed spaces $(f, f^{\#}) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ and a point $x \in X$, there is a well-defined E-homomorphism $f_x^{\#} : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$ of stalks, called the *induced mapping on stalks*, defined as follows. The morphism of sheaves $f^{\#} : \mathcal{O}_Y \longrightarrow f_*(\mathcal{O}_X)$ induces a homomorphism of E-rings $f^{\#}(V) : \mathcal{O}_Y(V) \longrightarrow f_*(\mathcal{O}_X)(V) = \mathcal{O}_X(f^{-1}(V))$ for every open set V. As V varies over all open subsets of Y containing $f(x)$, $f^{-1}(V)$ varies over a subset of all open subset of X containing x. Therefore, there exist an E-ring homomorphism

$$\mathcal{O}_{Y,f(x)} = \varinjlim_{f(x)\in V} \mathcal{O}_Y(V) \longrightarrow \varinjlim_{f(x)\in V} \mathcal{O}_X(f^{-1}(V)).$$

The later E-ring maps to the stalk $\mathcal{O}_{X,x}$.

The category of *local* E-*ringed spaces* has as objects E-ringed spaces $(X, \mathcal{O}_X)$ such that the stalks are local E-rings with maximal E-ideals. The morphisms $(f, f^{\#}) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ are the morphisms of E-ringed spaces such that the induced E-homomorphism on each stalk takes the maximal E-ideal of the domain into the maximal E-ideal of the range [19, page 80]. The E-ringed space $(\operatorname{diffspec} R, \mathcal{O}_{\operatorname{diffspec} R})$ is a local E-ringed space because each stalk $\mathcal{O}_x$ for $x \in \operatorname{diffspec} R$ is E-isomorphic to the local E-ring R_x, in which clearly the maximal ideal is an E-ideal. (See [19, Proposition 4.2, page 79] and Lemma 2.70.)

An *affine* E-*scheme* is a local E-ringed space $(X, \mathcal{O}_X)$ isomorphic in the category of local E-ringed spaces to $(\operatorname{diffspec} R, \mathcal{O}_{\operatorname{diffspec} R})$ for R an E-ring. The category of E-*schemes* has as objects local E-ringed spaces $(X, \mathcal{O}_X)$ with the property that every point $x \in X$ has an open neighborhood $U \subseteq X$ such that the E-ringed space $(U, \mathcal{O}_X|_U)$ is an affine E-scheme, where $\mathcal{O}_X|_U$ is the sheaf $\mathcal{O}_X$ restricted to U. The morphisms of the category of E-schemes are the morphisms of local E-ringed spaces.

An open subset $U \subseteq X$ is called an *affine open* (subset of X) if the E-ringed space $(U, \mathcal{O}_X|_U)$ is an affine E-scheme. If U is an affine open of X, there is an E-morphism of schemes $(i_U, i_U^{\#}) : (U, \mathcal{O}_X|_U) \longrightarrow (X, \mathcal{O}_X)$, called an *open immersion*, where $i_U : U \longrightarrow X$ is the inclusion and $i_U^{\#} : \mathcal{O}_X \longrightarrow (i_U)_*(\mathcal{O}_X|_U)$ is the morphism of sheaves of E-rings on X defined in the following manner. The morphism of sheaves $i_U^{\#}$ evaluated at the open subset $V \subseteq X$ is the restriction $\mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(U \cap V) = (\mathcal{O}_X|_U)(U \cap V) = (i_U)_*(\mathcal{O}_X|_U)(V)$. When no confusion is likely, $\mathcal{O}_X|_U$ will be denoted by $\mathcal{O}_U$. Some basic properties of affine opens are collected in the following observation.

Observation 2.62 *Let U be an affine open of X, and let $(i_U, i_U^{\#}) : (U, \mathcal{O}_U) \longrightarrow (X, \mathcal{O}_X)$ be the open immersion. Then*

1. *for an open subset $V \subseteq U$, $\mathcal{O}_X(V) = \mathcal{O}_U(V)$ and*

2. *for each $x \in U$, the induced local* E-$\mathfrak{F}$-*homomorphism $i_x^{\#} : \mathcal{O}_{i(x)} \longrightarrow \mathcal{O}_x$ from the open immersion $(i_U, i_U^{\#}) : (U, \mathcal{O}_X|_U) \longrightarrow (X, \mathcal{O}_X)$ is an* E-$\mathfrak{F}$-*isomorphism, .*

Proposition 2.63 *Let $(X, \mathcal{O}_X)$ be an* E*-scheme. Then each irreducible closed set of X has a unique generic point.*

Proof: Since each irreducible closed set of each open in a cover has a unique generic point (Proposition 2.60), each irreducible closed set of X has a unique generic point by Lemma 1.12. □

An E-scheme $(X, \mathcal{O}_X)$ will be called a *Noetherian* E*-scheme* if the underlying topological space X is Noetherian.

Corollary 2.64 *If $(X, \mathcal{O}_X)$ is a Noetherian* E*-scheme, then X is an NG-space.*

The next three Propositions are basic results extracted from Kovacic ([19], [21]) and are presented here for the convenience of the reader.

Proposition 2.65 *Let $\phi : R \longrightarrow S$ be an* E*-homomorphism. Then ϕ induces a morphism of topological spaces ${}^a\phi : \text{diffspec}\, S \longrightarrow \text{diffspec}\, R$ and a morphism of* E*-schemes*

$$({}^a\phi, \phi^{\#}) : (\text{diffspec}\, S, \mathcal{O}_{\text{diffspec}\, S}) \longrightarrow (\text{diffspec}\, R, \mathcal{O}_{\text{diffspec}\, R}).$$

Remark 2.66 From this proposition it easily follows that there is a contravariant functor F from E-rings to E-schemes such that $F(R) = (\text{diffspec}\, R, \mathcal{O}_{\text{diffspec}\, R})$ and $F(\phi) = ({}^a\phi, \phi^{\#})$ for an E-homomorphism $\phi : R \longrightarrow S$ between E-rings R and S.

More general results than Proposition 2.65 are true [21, Propositions 23.6, 23.7, 23.8; page 4504].

Proposition 2.67 *Let R be an* E*-ring, and let $(Z, \mathcal{O}_Z)$ be an* E*-scheme. Any* E*-homomorphism $\phi : R \longrightarrow \mathcal{O}_Z(Z)$ determines a morphism of* E*-schemes $({}^a\phi, \phi^\#) : (Z, \mathcal{O}_Z) \longrightarrow (\operatorname{diffspec} R, \mathcal{O}_{\operatorname{diffspec} R})$.*

Proposition 2.68 *Let R be an* E*-ring, and let $(Z, \mathcal{O}_Z)$ be an* E*-scheme. Put $X = \operatorname{diffspec} R$. Given a morphism of* E*-$\mathcal{F}$-schemes $(f, f^\#) : (Z, \mathcal{O}_Z) \longrightarrow (X, \mathcal{O}_X)$. Let $\varphi((f, f^\#))$ be the composition*

$$R \xrightarrow{i_R} \mathcal{O}_X(X) \xrightarrow{f^\#(X)} \mathcal{O}_Z(Z).$$

Then the map

$$\varphi : \operatorname{Mor}((Z, \mathcal{O}_Z), (X, \mathcal{O}_X)) \longrightarrow \operatorname{Hom}^{\mathrm{E}}_{\mathcal{F}}(R, \mathcal{O}_Z(Z))$$

is a bijection.

Proposition 2.69 *Let R and S be* E*-rings. Let $(f, f^\#)$ be a morphism of* E*-schemes*

$$(f, f^\#) : (\operatorname{diffspec} S, \mathcal{O}_{\operatorname{diffspec} S}) \longrightarrow (\operatorname{diffspec} R, \mathcal{O}_{\operatorname{diffspec} R}).$$

Then the sheaf morphism $f^\# : \mathcal{O}_{\operatorname{diffspec} R} \longrightarrow f_*(\mathcal{O}_{\operatorname{diffspec} S})$ induces an E*-homomorphism of global sections $f^\#(\operatorname{diffspec} R) : \mathcal{O}_{\operatorname{diffspec} R}(\operatorname{diffspec} R) \longrightarrow \mathcal{O}_{\operatorname{diffspec} S}(\operatorname{diffspec} S)$.*

Proof: The sheaf morphism $f^{\#} : \mathcal{O}_{\text{diffspec}\, R} \longrightarrow f_*(\mathcal{O}_{\text{diffspec}\, S})$ evaluated on the open $\text{diffspec}\, R$ yields the E-homomorphism

$$\begin{aligned} f^{\#}(\text{diffspec}\, R) : \mathcal{O}_{\text{diffspec}\, R}(\text{diffspec}\, R) \longrightarrow f_*(\mathcal{O}_{\text{diffspec}\, S})(\text{diffspec}\, R) \\ = (\mathcal{O}_{\text{diffspec}\, S})(f^{-1}(\text{diffspec}\, R)) \\ = \mathcal{O}_{\text{diffspec}\, S}(\text{diffspec}\, S). \end{aligned}$$

$\square$

Let $(\text{diffspec}\, R, \mathcal{O}_{\text{diffspec}\, R})$ be an affine E-scheme. For each point $x \in \text{diffspec}\, R$ and for each open subset $U \subseteq \text{diffspec}\, R$ containing x, define the E-homomorphism $\pi_{x,U} : \mathcal{O}_{\text{diffspec}\, R}(U) \longrightarrow R_x$ to be the evaluation $s(x)$ of a section $s \in \mathcal{O}_{\text{diffspec}\, R}(U)$ at the point x.

Lemma 2.70 *There exist an* E*-isomorphism* $\varphi : \mathcal{O}_x \longrightarrow R_x$ *such that* $\varphi \circ \mathcal{J}_{x,U} = \pi_{x,U}$.

$$\begin{tikzcd} \mathcal{O}_{\text{diffspec}\, R}(U) \arrow[r, "\mathcal{J}_{x,U}"] \arrow[dr, "\pi_{x,U}"'] & \mathcal{O}_x \arrow[d, "\varphi"] \\ & R_x \end{tikzcd}$$

Proof: The E-isomorphism φ is defined for each $\alpha \in \mathcal{O}_x$ by choosing a representative $s_\alpha \in \mathcal{O}_{\text{diffspec}\, R}(U_\alpha, \mathcal{O}_{U_\alpha})$ where U_α is an open subset of $\text{diffspec}\, R$ containing x and setting $\varphi(\alpha) = s_\alpha(x)$. Following the proof of [14, Proposition 2.2a, page 71], φ may be shown to be a well-defined E-isomorphism

[19, Proposition 4.2, page 79]. To show the above diagram commutes, observe that any $s \in \mathcal{O}_{\operatorname{diffspec} R}(U)$ is a representative of $\mathfrak{J}_{x,U}(s) \in \mathcal{O}_x$ and $(\varphi \circ \mathfrak{J}_{x,U})(s) = \varphi(\mathfrak{J}_{x,U}(s)) = s(x) = \pi_{x,U}(s)$ by the definition of φ and $\pi_{x,U}$.$\square$

For every E-ring R, define the E-homomorphism, called the *canonical E-homomorphism*, $\mathfrak{I}_R : R \longrightarrow \mathcal{O}_{\operatorname{diffspec} R}(\operatorname{diffspec} R)$ by $\mathfrak{I}_R(r)(\mathfrak{P}) = r/1 \in R_{\mathfrak{P}}$ for $r \in R$ and $\mathfrak{P} \in \operatorname{diffspec} R$ [19, Definition 4.4, page 79]. It is a functor from E-rings to E-rings [17, page 38] or [18, page 169]. The following lemma will be used in the proof of Theorem 2.126.

Lemma 2.71 *Let* $(\operatorname{diffspec} R, \mathcal{O}_{\operatorname{diffspec} R})$ *be an affine* E*-scheme, and let* $x \in \operatorname{diffspec} R$*. Then* $(\mathfrak{J}_{x,\operatorname{diffspec} R} \circ \mathfrak{I}_R)^{-1}(\mathfrak{P}_x) = x$ *where* $\mathfrak{P}_x$ *is the maximal* E*-ideal of* $\mathcal{O}_x$*.*

$$\begin{tikzcd}
R \arrow[r, "\mathfrak{I}_R"] & \mathcal{O}_{\operatorname{diffspec} R}(\operatorname{diffspec} R) \arrow[r, "\mathfrak{J}_{x,\operatorname{diffspec} R}"] \arrow[dr, "\pi_{x,\operatorname{diffspec} R}"'] & \mathcal{O}_x \arrow[d, "\varphi"] \\
 & & R_x
\end{tikzcd}$$

Proof: Since φ is an E-isomorphism (Lemma 2.70), $\varphi(\mathfrak{P}_x)$ is the maximal ideal of R_x. By the definition of a section $s \in \mathcal{O}_{\operatorname{diffspec} R}(\operatorname{diffspec} R)$, there exists an open subset V of $\operatorname{diffspec} R$ containing x and there exists $a, b \in R$ such that, for all $\mathfrak{Q} \in V$, $b \notin \mathfrak{Q}$ and $s(\mathfrak{Q}) = a/b \in R_{\mathfrak{Q}}$. By taking $\mathfrak{Q} = x$, the fraction a/b is a representative for $s(x) \in R_x$ and is contained in the maximal ideal $\varphi(\mathfrak{P}_x) \subset R_x$ if and only if $a \in x$ [3, Proposition 2, page 81].

By the definition of $\mathcal{I}_R(r)$ (as on the previous page), $\mathcal{I}_R(r)(\mathfrak{Q}) = r/1 \in \mathcal{O}_{X,x}$ for all $\mathfrak{Q} \in V = \text{diffspec}\, R$. In particular, for $\mathfrak{Q} = x$, $\mathcal{I}_R(r)(x)$ equals $r/1$ and is in the maximal E-ideal $\varphi(\mathfrak{P}_x)$ of R_x if and only if $r \in x$. By Lemma 2.70 for $U = \text{diffspec}\, R$, $(\varphi \circ \mathfrak{J}_{x,\text{diffspec}\, R})(\mathcal{I}_R(r)) = \pi_{x,\text{diffspec}\, R}(\mathcal{I}_R(r))$ and $\mathfrak{J}_{x,\text{diffspec}\, R}(\mathcal{I}_R(r)) = \varphi^{-1}(\pi_{x,\text{diffspec}\, R}(\mathcal{I}_R(r))) = \varphi^{-1}(\mathcal{I}_R(r)(x))$ by applying φ^{-1} and the definition of $\pi_{x,\text{diffspec}\, R}$. Because the E-isomorphism φ^{-1} maps one maximal E-ideal to another, $\mathfrak{J}_{x,\text{diffspec}\, R}(\mathcal{I}_R(r))$ is in the maximal E-ideal $\mathfrak{P}_x$ of $\mathcal{O}_x$ if and only if $r \in x$. Therefore $(\mathfrak{J}_{x,\text{diffspec}\, R} \circ \mathcal{I}_R)^{-1}(\mathfrak{P}_x) = x$. □

Let $(X, \mathcal{O}_X)$ be a E-$\mathfrak{F}$-scheme. Throughout this manuscript, the underlying spaces X will be endowed with the topological pre-order: that is $x \underset{X}{\longrightarrow} x'$ if $\overline{\{x'\}} \subseteq \overline{\{x\}}$ for $x, x' \in X$. Now assume $x \underset{X}{\longrightarrow} x'$ for $x, x' \in X$. Since any open subset $U \subseteq X$ that contains x' contains x (Lemma 1.10), there is a canonical E-homomorphism $i_{x,x'}$:

$$\mathcal{O}_{x'} = \varinjlim_{x' \in U} \mathcal{O}_X(U) \longrightarrow \varinjlim_{x \in U} \mathcal{O}_X(U) = \mathcal{O}_x.$$

Lemma 2.72 *Let $(X, \mathcal{O}_X)$ be an E-$\mathfrak{F}$-scheme, and let $x, x' \in X$ such that $x \underset{X}{\longrightarrow} x'$ for $x, x' \in X$. Let $i_{x,x'} : \mathcal{O}_{x'} \longrightarrow \mathcal{O}_x$ be the canonical E-homomorphism. Then $i^{-1}_{x,x'}(\mathfrak{P}_x) \subset \mathfrak{P}'_x$ where $\mathfrak{P}_x$ and $\mathfrak{P}'_x$ are the maximal E-ideals of $\mathcal{O}_x$ and $\mathcal{O}'_x$.*

Proof: Let $\text{diffspec}\, R \subseteq X$ be an affine open containing x'. By Lemma 1.10, $x \in \text{diffspec}\, R$. Since $U = \text{diffspec}\, R$ is one element in both of the indexing sets used in the above definitions of $\mathcal{O}_x$ and $\mathcal{O}_{x'}$, and since $\mathcal{O}_X(\text{diffspec}\, R) =$

$\mathcal{O}_{\operatorname{diffspec} R}(\operatorname{diffspec} R)$ (Observation 2.62), the diagram below commutes:

$$\begin{tikzcd}
\mathcal{O}_{\operatorname{diffspec} R}(\operatorname{diffspec} R) \arrow[r, "\mathfrak{J}_{x',\operatorname{diffspec} R}"] \arrow[dr, "\mathfrak{J}_{x,\operatorname{diffspec} R}"'] & \mathcal{O}_{x'} \arrow[d, "i_{x,x'}"] \\
& \mathcal{O}_{x}.
\end{tikzcd}$$

By inserting $\mathfrak{I}_R : R \to \mathcal{O}_{\operatorname{diffspec} R}(\operatorname{diffspec} R)$ on the left of this diagram, one obtains the commuting diagram:

$$\begin{tikzcd}
R \arrow[r, "\mathfrak{I}_R"] & \mathcal{O}_{\operatorname{diffspec} R}(\operatorname{diffspec} R) \arrow[r, "\mathfrak{J}_{x',\operatorname{diffspec} R}"] \arrow[dr, "\mathfrak{J}_{x,\operatorname{diffspec} R}"'] & \mathcal{O}_{x'} \arrow[d, "i_{x,x'}"] \\
& & \mathcal{O}_{x}.
\end{tikzcd}$$

Both $\overline{\{x\}} \cap \operatorname{diffspec} R$ and $\overline{\{x'\}} \cap \operatorname{diffspec} R$ are the irreducible closures $V(\{x\})$ and $V(\{x'\})$ of x and x' in $\operatorname{diffspec} R$ (Proposition 1.11). Because $V(\{x'\}) = \overline{\{x'\}} \cap \operatorname{diffspec} R \subset \overline{\{x\}} \cap \operatorname{diffspec} R = V(\{x\})$, it follows that $x \subset x'$ (Observation 2.59).

Let $\alpha \in \mathcal{O}_{x'}$, and assume $i_{x,x'}(\alpha) \in \mathfrak{P}_x \subseteq \mathcal{O}_x$. Since $(\mathfrak{J}_{x',\operatorname{diffspec} R} \circ \mathfrak{I}_R)^{-1}(\mathfrak{P}_x) = x'$ and $(\mathfrak{J}_{x,\operatorname{diffspec} R} \circ \mathfrak{I}_R)^{-1}(\mathfrak{P}_x) = x$ by Lemma 2.71,

$$(\mathfrak{J}_{x,\operatorname{diffspec} R} \circ \mathfrak{I}_R)^{-1}(i_{x,x'}(\alpha)) \in x.$$

Therefore,

$$(\mathfrak{J}_{x,\operatorname{diffspec} R} \circ \mathfrak{I}_R)^{-1}(i_{x,x'}(\alpha)) \in x' \tag{1}$$

since $x \subset x'$. By applying the E-homomorphism $\mathfrak{J}_{x',\operatorname{diffspec} R} \circ \mathfrak{I}_R$ to line 1,

$$(\mathfrak{J}_{x',\operatorname{diffspec} R} \circ \mathfrak{I}_R)((\mathfrak{J}_{x,\operatorname{diffspec} R} \circ \mathfrak{I}_R)^{-1}(i_{x,x'}(\alpha)))$$

$$= \mathfrak{J}_{x',\mathrm{diffspec}\,R}((\mathfrak{J}_{x,\mathrm{diffspec}\,R})^{-1}(i_{x,x'}(\alpha)))$$

$$= \mathfrak{J}_{x',\mathrm{diffspec}\,R}((\mathfrak{J}_{x',\mathrm{diffspec}\,R})^{-1}(\alpha))$$

$$= \alpha \in \mathfrak{P}'_x.$$

□

Lemma 2.73 *Let* $(X, \mathcal{O}_X)$ *and* $(Y, \mathcal{O}_Y)$ *be two* E*-schemes, and let* $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ *be a* E*-morphism between. For* $x, x' \in X$*, let* $x \underset{X}{\longrightarrow} x'$*, and let* $f(x) = f(x')$*. Then*

$$\begin{tikzcd}
\mathcal{O}_{f(x)} \arrow[r, "f^{\#}_{x'}"] \arrow[dr, "f^{\#}_{x}"'] & \mathcal{O}_{x'} \arrow[d, "i_{x,x'}"] \\
& \mathcal{O}_x .
\end{tikzcd}$$

commutes, and $i^{-1}_{x,x'}(\mathfrak{P}_x) \subset \mathfrak{P}_{x'}$ *where* $\mathfrak{P}_x$ *and* $\mathfrak{P}_{x'}$ *are the maximal* E *ideals of* $\mathcal{O}_x$ *and* $\mathcal{O}_{x'}$*.*

Proof: The last statement is the previous proposition, and the commutivity of the diagram easily follows by picking a representative $\alpha_U \in \mathcal{O}_Y(U)$ for $\alpha \in \mathcal{O}_{f(x)}$. Then $f^{\#}(U)(\alpha_U) \in f_*(\mathcal{O}_X)(U) = \mathcal{O}_X(f^{-1}(U))$ is a representative for $f^{\#}_{x'}(\alpha)$. It is also a representative for $i_{x,x'}(f^{\#}_{x'}(\alpha))$ and $f^{\#}_{x}(\alpha)$. □

Proposition 2.74 *Let* R *be a local E-ring such that the maximal prime ideal* $\mathfrak{P}$ *is an* E*-ideal. Then the canonical* E*-homomorphism* $\mathfrak{I}_R : R \longrightarrow \mathcal{O}_{\mathrm{diffspec}\,R}(\mathrm{diffspec}\,R)$ *is* E*-isomorphism. In particular, this is true if* R *is a* E*-field.*

Proof: (See [16, Proposition 3.3, page 151] for the same result.) Let $s \in \mathcal{O}_{\operatorname{diffspec} R}(\operatorname{diffspec} R)$. By definition $s(\mathfrak{P}') \in R_{\mathfrak{P}'}$ for each $\mathfrak{P}' \in \operatorname{diffspec} R$, and there is an open cover $\{U_i\}$ of $\operatorname{diffspec} R$, and $a_i, b_i \in R$, such that for each $\mathfrak{Q} \in U_i$, $b_i \notin \mathfrak{Q}$ and $s(\mathfrak{Q}) = a_i/b_i \in R_{\mathfrak{Q}}$. Since R is a local ring, the closure $\overline{\{\mathfrak{Q}\}}$ of any point $\mathfrak{Q} \in \operatorname{diffspec} R$ contains $\mathfrak{P}$ (Observation 2.59). If an open subset U of $\operatorname{diffspec} R$ contains $\mathfrak{P}$ but does not contain a point $\mathfrak{Q}$ of $\operatorname{diffspec} R$, the complement of U would be a closed subset containing $\overline{\{\mathfrak{Q}\}}$ but not $\mathfrak{P}$. Hence any open subset that contains $\mathfrak{P}$ also contains all points of $\operatorname{diffspec} R$, and $U_i = \operatorname{diffspec} R$ for some i. Therefore, there are $a, b \in R$ such that, for each $\mathfrak{Q} \in \operatorname{diffspec} R, b \notin \mathfrak{Q}$ and $s(\mathfrak{Q}) = a/b \in R_{\mathfrak{Q}}$. In particular, since $\mathfrak{P} \in \operatorname{diffspec} R$, $b \notin \mathfrak{P}$, which implies there exist $b' \in R$ such that $bb' = 1$ since R is a local ring with maximal ideal $\mathfrak{P}$. Then $\mathcal{I}_R(ab') = ab'/1 = a/b = s$, and i_R is surjective.

To prove the injectivity, let $r \in R$ such that $\mathcal{I}_R(r) = r/1 = 0$ in $\mathcal{O}_{\mathfrak{Q}}$ for each $\mathfrak{Q} \in \operatorname{diffspec} R$. In particular, $r/1 = 0$ in $\mathcal{O}_{\mathfrak{P}}$, which implies there exists $s \in R \setminus \mathfrak{P}$ such that $sr = 0$. Since s is invertible in R, $r = 0$. $\square$

2.3 E-Schemes over a Base

Let $\mathfrak{F}$ be an E-field, and let $(F, \mathcal{O}_F) = (\operatorname{diffspec} \mathfrak{F}, \mathcal{O}_{\operatorname{diffspec} \mathfrak{F}})$. The *category of* E-*schemes over* $(F, \mathcal{O}_F)$ has as objects morphisms of E-schemes

$$(f, f^{\#}) : (X, \mathcal{O}_X) \to (F, \mathcal{O}_F),$$

called E-*schemes over* $(F, \mathcal{O}_F)$ [19, Definition 13.1, page 90]. In this category, a morphism from $(f, f^{\#}) : (X, \mathcal{O}_X) \longrightarrow (F, \mathcal{O}_F)$ to $(g, g^{\#}) : (Y, \mathcal{O}_Y) \longrightarrow (F, \mathcal{O}_F)$ is a morphism $(h, h^{\#}) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ of E-schemes such that the following diagram commutes [19, Definition 13.3, page 91]:

Diagram 2.75

$$\begin{tikzcd}
(X, \mathcal{O}_X) \arrow[rr, "(h,h^{\#})"] \arrow[dr, "(f,f^{\#})"'] & & (Y, \mathcal{O}_Y) \arrow[dl, "(g,g^{\#})"] \\
& (F, \mathcal{O}_F). &
\end{tikzcd}$$

The following lemma implies that many of the E-rings and E-homomorphisms to be studied are actually E-$\mathfrak{F}$-algebras and E-$\mathfrak{F}$-homomorphisms. Let $(f, f^{\#}) : (X, \mathcal{O}_X) \longrightarrow (F, \mathcal{O}_F)$ be an E-scheme over $(F, \mathcal{O}_F)$. The definition of the sheaf morphism $f^{\#} : \mathcal{O}_F \longrightarrow f_*(\mathcal{O}_X)$ includes the existence of the E-homomorphism of global sections $f^{\#}(F) : \mathcal{O}_F(F) \longrightarrow f_*(\mathcal{O}_X)(F) = \mathcal{O}_X(X)$. Since $\mathcal{O}_F(F) = \mathfrak{F}$ (Proposition 2.74), $f^{\#}(F)$ endows $\mathcal{O}_X(X)$ with the structure of an E-$\mathfrak{F}$-algebra. For any open subset $U \subseteq X$, the restriction map $\mathcal{O}_X(X) \longrightarrow \mathcal{O}_X(U)$ composed with $f^{\#}(F)$ also endows each $\mathcal{O}_X(U)$ with the structure of an E-$\mathfrak{F}$-algebra such that the restriction maps $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V)$ are E-$\mathfrak{F}$-homomorphisms for open subsets $V \subseteq U$. Therefore, $\mathcal{O}_X$ has the structure of a sheaf of E-$\mathfrak{F}$-algebras, which is the first part of the next lemma.

Lemma 2.76 *Let* $(f, f^{\#}) : (X, \mathcal{O}_X) \longrightarrow (F, \mathcal{O}_F)$ *be an* E-*scheme over* $(F, \mathcal{O}_F)$. *Then* $\mathcal{O}_X$ *is a sheaf of* E-$\mathfrak{F}$-*algebras. Let* $(g, g^{\#}) : (Y, \mathcal{O}_Y) \longrightarrow (F, \mathcal{O}_F)$ *be an-*

other E-*scheme over* $(F, \mathcal{O}_F)$, *and let* $(h, h^\#) : (f, f^\#) \longrightarrow (g, g^\#)$ *be a morphism between them. Then* $h^\#$ *is a morphism of sheaves of* E-$\mathcal{F}$-*algebras.*

Proof: From the commutivity of Diagram 2.75, the following two morphisms of sheaves of E-rings on $(F, \mathcal{O}_F)$ are equal:

$$f^\# : \mathcal{O}_F \longrightarrow f_*(\mathcal{O}_X) = g_*(h_*(\mathcal{O}_X)) \tag{2}$$

and

$$(g \circ h)^\# : \mathcal{O}_F \xrightarrow{g^\#} g_*(\mathcal{O}_Y) \xrightarrow{g_*(h^\#)} g_*(h_*(\mathcal{O}_X)), \tag{3}$$

where the morphism (3) is the definition of the sheaf map associated to the composition $(g, g^\#) \circ (h, h^\#)$. By evaluating these two morphisms of sheaves at F and by using $\mathcal{O}_F(F) = \mathcal{F}$, the following two E-$\mathcal{F}$-homomorphisms are equal:

$$f^\#(F) : \mathcal{F} \longrightarrow g_*(h_*(\mathcal{O}_X))(F) \tag{4}$$

and $(g \circ h)^\#(F)$:

$$\mathcal{F} \xrightarrow{g^\#(F)} g_*(\mathcal{O}_Y)(F) \xrightarrow{g_*(h^\#)(F)} g_*(h_*(\mathcal{O}_X))(F). \tag{5}$$

This shows that $g_*(h^\#)(F)$ is an E-$\mathcal{F}$-homomorphism. Since $g_*(\mathcal{O}_Y)(F) = (\mathcal{O}_Y)(Y)$, $g_*(h_*(\mathcal{O}_X))(F) = h_*(\mathcal{O}_X)(Y)$ and $g_*(h^\#)(F) = (h^\#)(Y)$,

$$(h^\#)(Y) : (\mathcal{O}_Y)(Y) \longrightarrow h_*(\mathcal{O}_X)(Y)$$

is an E-$\mathcal{F}$-homomorphism. Because $h^\#$ is a natural transformation of functors, the diagram of E-homomorphisms

Diagram 2.77

$$\begin{tikzcd}
(\mathcal{O}_Y)(Y) \arrow[r, "h^{\#}(Y)"'] \arrow[d, "restriction"] & h_*(\mathcal{O}_X)(Y) \arrow[d, "restriction"] \\
(\mathcal{O}_Y)(U) \arrow[r, "h^{\#}(U)"'] & h_*(\mathcal{O}_X)(U)
\end{tikzcd}$$

commutes for open subsets $U \subset Y$. Since the restrictions and $h^{\#}(Y)$ are E-$\mathcal{F}$-homomorphisms, $h^{\#}(U)$ is also an E-$\mathcal{F}$-homomorphism for each $U \subset Y$. Therefore, $h^{\#}$ is a morphism of sheaves of E-$\mathcal{F}$-algebras. □

If R is an E-$\mathcal{F}$-algebra with structure map $i_R : \mathcal{F} \longrightarrow R$ where i_R is an E-$\mathcal{F}$-morphism, then the morphism of E-schemes associated to i_R in Proposition 2.65

$$({}^a i_R, i_R^{\#}) : (\operatorname{diffspec} R, \mathcal{O}_{\operatorname{diffspec} R}) \longrightarrow (F, \mathcal{O}_F)$$

is an E-scheme over $(F, \mathcal{O}_F)$. Similarly, an E-$\mathcal{F}$-algebra homomorphism $f : R \longrightarrow S$ yields a morphism

$$({}^a f, f^{\#}) : ({}^a i_S, i_S^{\#}) \longrightarrow ({}^a i_R, i_R^{\#})$$

of E-schemes over $(F, \mathcal{O}_F)$.

However, there are many examples of affine E-schemes $(\operatorname{diffspec} R, \mathcal{O}_{\operatorname{diffspec} R})$ over $(F, \mathcal{O}_F)$ such that R is not an E-$\mathcal{F}$-algebra [19, page 91]. For instance let $E = \{\epsilon\}$, k an E-constant field, x an algebraic indeterminate over k such that $\epsilon x = 1$, $R = k[x]$ and $\mathcal{F} = k(x)$. Then $(\operatorname{diffspec} k[x], \mathcal{O}_{\operatorname{diffspec} k[x]})$ is isomorphic to $(\operatorname{diffspec} k(x), \mathcal{O}_{\operatorname{diffspec} k(x)})$. So, a fortiori, there is a morphism of

$$(\operatorname{diffspec} k[x], \mathcal{O}_{\operatorname{diffspec} k[x]}) \longrightarrow (\operatorname{diffspec} k(x), \mathcal{O}_{\operatorname{diffspec} k(x)})$$

of E-schemes, but $k[x]$ is not an E-$k(x)$-ring [19, page 4501] or [17, page 40]. Since this phenomenon makes it difficult to prove products exist in the category of E-schemes over $(F, \mathcal{O}_F)$, Kovacic defines the following subcategory of E-schemes, called E-$\mathfrak{F}$-schemes [19, Definition 21.1, page 4501][4], in which products exist [21, Proposition 25.2, 4507].

Definition 2.78 *An* affine E-$\mathfrak{F}$-scheme *is an affine* E-*scheme* $(X, \mathcal{O}_X)$ *where* $X =$ diffspec R, R *is an* E-$\mathfrak{F}$-*algebra, and* $\mathcal{O}_X$ *is a sheaf of* E-$\mathfrak{F}$-*algebras. A* morphism of affine E-$\mathfrak{F}$-schemes *is a morphism* $(f, f^{\#})$ *of* E-*schemes in which* $f^{\#}$ *is a morphism of sheaves of* E-$\mathfrak{F}$-*algebras. An* E-$\mathfrak{F}$-scheme *is an* E-*scheme over* $(F, \mathcal{O}_F)$ *that has an open cover by affine* E-$\mathfrak{F}$-*schemes such that the sheaf morphisms induced by the inclusions are* E-$\mathfrak{F}$-*homomorphisms. The* category of E-$\mathfrak{F}$-schemes *is the full subcategory of* E-*schemes over* $(F, \mathcal{O}_F)$ *that has* E-$\mathfrak{F}$-*schemes as objects.*

If E is empty, the category of E-schemes over $(F, \mathcal{O}_F)$ is the category of E-$\mathfrak{F}$-schemes because, for any affine open diffspec R of such an E-scheme, (diffspec R, $\mathcal{O}_{\text{diffspec}\,R}$) is also an E-scheme over $(F, \mathcal{O}_F)$, and this implies R is an E-$\mathfrak{F}$-algebra [14, Proposition 2.3, page 73]. So that, when E is empty, the terms "E-schemes over spec $\mathfrak{F}$" and "E-$\mathfrak{F}$-schemes" may be used interchangeably as is done in algebraic geometry. If E is not empty, these two concepts are also equivalent for reduced E-schemes over $(F, \mathcal{O}_F)$ [21, page 4501].

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[4] The definition of E-$\mathfrak{F}$-scheme has been corrected here.

Definition 2.79 *A* E-$\mathfrak{F}$-*scheme* $(X, \mathcal{O}_X)$ *is of* E-$\mathfrak{F}$-finite type *if there exist a finite cover of* X *by affine opens of the form* $\operatorname{diffspec} R$ *for some finitely* E-*generated* E-$\mathfrak{F}$-*algebra* R.

Proposition 2.80 *If* $(X, \mathcal{O}_X)$ *is an* E-$\mathfrak{F}$-*scheme of* E-$\mathfrak{F}$-*finite type,* X *is an NG-space.*

Proof: Each finitely E-generated E-$\mathfrak{F}$-algebra R is Noetherian [24, Theorem 1, page 126]. That is every ascending sequence of radical E-ideals terminates. Since the closed sets of $\operatorname{diffspec} R$ are of the form $V(I) = \{\mathfrak{P} \in \operatorname{diffspec} R \mid \mathfrak{P} \supseteq I\}$ for $I \subset R$ and $V(I) = V(\{I\}_{\mathrm{E}})$ [19, Proposition 3.2(d), page 75], all descending sequences of closed sets of $\operatorname{diffspec} R$ terminate. Therefore, $\operatorname{diffspec} R$ is Noetherian. So is X because it is covered by finitely many Noetherian affine opens. This and Corollary 2.64 imply the proposition. ☐

Proposition 2.81 *The product of two* E-$\mathfrak{F}$-*schemes of* E-$\mathfrak{F}$-*finite type is an* E-$\mathfrak{F}$-*scheme of* E-$\mathfrak{F}$-*finite type.*

Proof: The product of two E-$\mathfrak{F}$-schemes is an E-$\mathfrak{F}$-scheme [21, Proposition 25.2, page 4507]. To show this product is an E-$\mathfrak{F}$-scheme of E-$\mathfrak{F}$-finite type, cover each of two E-$\mathfrak{F}$-schemes of E-$\mathfrak{F}$-finite type by finitely many affine opens of the form $(\operatorname{diffspec} R_i, \mathcal{O}_{\operatorname{diffspec} R_i})$ and $(\operatorname{diffspec} S_j, \mathcal{O}_{\operatorname{diffspec} S_j})$ for some finitely E-generated E-$\mathfrak{F}$-algebras R_i and S_j. Then each $R_i \otimes_{\mathfrak{F}} S_j$ is a finitely E-generated E-$\mathfrak{F}$-algebra, and the finitely many affine opens $\operatorname{diffspec} R_i \otimes_{\mathfrak{F}} S_j$ cover the product. ☐

2.4 The Functor of Points

2.4.1 Basic Definitions

Let $\mathcal{F}$ be an E-field, and let the E-field $\mathcal{V}$ be E-universal over $\mathcal{F}$. Every E-field $\mathcal{G} \subseteq \mathcal{V}$ that contains $\mathcal{F}$ has the canonical structure of an E-$\mathcal{F}$-algebra provided by the inclusion of $\mathcal{F} \subseteq \mathcal{G}$. With this structure, $(\operatorname{diffspec} \mathcal{G}, \mathcal{O}_{\operatorname{diffspec} \mathcal{G}})$ is an E-scheme over $(F, \mathcal{O}_F)$ by Proposition 2.65. Let $G = \operatorname{diffspec} \mathcal{G}$. In the remainder of this chapter, all E-schemes considered will be objects of the category of E-schemes over $(F, \mathcal{O}_F)$, and the zero $\{0\}_{\mathrm{E}}$ ideal of $\mathcal{G}$ will be denoted by "0". As a consequence of Lemma 2.76, all rings of sections and stalks are E-$\mathcal{F}$-algebras, and all restriction E-homomorphisms, all induced E-homomorphisms on stalks, etc. are E-$\mathcal{F}$-homomorphisms.

Lemma 2.82 *The stalk* $\mathcal{O}_{G,0}$ *of* G *at* 0 *is canonically* E-$\mathcal{F}$*-isomorphic to* $\mathcal{G}$ *via the* E-$\mathcal{F}$*-isomorphism* $\operatorname{res} \circ \mathcal{I}_{\mathcal{G}}$*:*

$$\mathcal{G} \xrightarrow{\mathcal{I}_{\mathcal{G}}} \mathcal{O}_G(G) \xrightarrow{\operatorname{res}} \varinjlim_{0 \in U} \mathcal{O}_G(U) = \mathcal{O}_{G,0}.$$

Proof: Since $\mathcal{G}$ is a field, $\mathcal{I}_{\mathcal{G}}$ is an E-$\mathcal{F}$-isomorphism (Proposition 2.74). Because G has only one non-empty open subset, the indexing set consists of the single element G, and res is the identity. $\square$

Let $\mathcal{G} \subseteq \mathcal{V}$ be an E-field that contains $\mathcal{F}$. Define a functor $(-)(\mathcal{G})$, called *the functor of* $\mathcal{G}$*-valued points*, from the category E-schemes over $(F, \mathcal{O}_F)$ to the category of sets as follows: for an E-scheme $(Y, \mathcal{O}_Y)$ over $(F, \mathcal{O}_F)$, let

$$Y(\mathcal{G}) = \operatorname{Mor}_{(F,\mathcal{O}_F)}((G, \mathcal{O}_G), (Y, \mathcal{O}_Y))$$

= *the set of* $\mathcal{G}$*-valued points of* Y. For a morphism of E-schemes $(f, f^{\#}) : (Y, \mathcal{O}_Y) \longrightarrow (Z, \mathcal{O}_Z)$ over $(F, \mathcal{O}_F)$, define

$$f_{\mathcal{G}} : Y(\mathcal{G}) \longrightarrow Z(\mathcal{G})$$

as the composition $f_{\mathcal{G}}((\xi, \xi^{\#})) = (f, f^{\#}) \circ (\xi, \xi^{\#})$. Then $f_{\mathcal{G}}((\xi, \xi^{\#}))$ $= (f \circ \xi, f_*(\xi^{\#}) \circ f^{\#})$ for $(\xi, \xi^{\#}) \in Y(\mathcal{G})$ by the formula for the composition of two morphisms of E-schemes.

Lemma 2.83 *Let* $(\xi, \xi^{\#})$ *and* $(\zeta, \zeta^{\#})$ *be two elements of* $Y(\mathcal{G})$. *If* $y = \xi(0) = \zeta(0)$ *and the induced* E*-homomorphisms* $\xi_y^{\#}$ *and* $\zeta_y^{\#}$ *on the stalk* $\mathcal{O}_y$ *are equal, then* $(\xi, \xi^{\#}) = (\zeta, \zeta^{\#})$.

Proof: The maps of topological spaces $\xi : G \longrightarrow Y$ and $\zeta : G \longrightarrow Y$ are equal because they are assumed to have the same value on the unique element 0 of G. It remains to show that $\xi_*(\mathcal{O}_G) = \zeta_*(\mathcal{O}_G)$ and that the morphisms of sheaves $\xi^{\#} : \mathcal{O}_{\text{diffspec}\, Y} \longrightarrow \xi_*(\mathcal{O}_G)$ and $\zeta^{\#} : \mathcal{O}_{\text{diffspec}\, Y} \longrightarrow \zeta_*(\mathcal{O}_G)$ are equal.

Let U be an open subset of Y and suppose $y \in Y$. Either $y \notin U$, or $y \in U$. If $y \notin U$, $\xi^{-1}(U)$ equals the empty set ϕ and

$$\xi_*(\mathcal{O}_G)(U) = \mathcal{O}_G(\xi^{-1}(U)) = \mathcal{O}_G(\phi) = 0.$$

For the same reasons, $\zeta_*(\mathcal{O}_G)(U) = 0$. If $y \in U$, $\xi^{-1}(U) = G$ and

$$\xi_*(\mathcal{O}_G)(U) = \mathcal{O}_G(\xi^{-1}(U)) = \mathcal{O}_G(G).$$

Also, $\zeta_*(\mathcal{O}_G)(U) = \mathcal{O}_G(G)$. Therefore the sheaves are equal.

It will now be shown that the morphisms of sheaves are equal. If $y \notin U$, since $\xi_*(\mathcal{O}_G)(U) = \zeta_*(\mathcal{O}_G)(U) = 0$, the E-$\mathfrak{F}$-homomorphisms

$$\xi^{\#}(U) : \mathcal{O}_{\mathrm{diffspec}\,Y}(U) \longrightarrow \xi_*(\mathcal{O}_G)(U) \text{ and}$$

$$\zeta^{\#}(U) : \mathcal{O}_{\mathrm{diffspec}\,Y}(U) \longrightarrow \zeta_*(\mathcal{O}_G)(U)$$

have the value 0 for each element of their domains and therefore are equal. If $y \in U$, let $s \in \mathcal{O}_Y(U)$. For any open subset W of Y containing y, let $\mathfrak{J}_{y,W} : \mathcal{O}_Y(W) \longrightarrow \mathcal{O}_{Y,y}$ be the canonical E-$\mathfrak{F}$-homomorphism of $\mathcal{O}_Y(W) \longrightarrow \varinjlim_{y\in T} \mathcal{O}_Y(T) = \mathcal{O}_{Y,y}$, and let V be an open subset of U containing y. By using the fact that sheaf map $\phi^{\#} : \mathcal{O}_Y \longrightarrow (\xi)_*(\mathcal{O}_{\mathcal{G}})$ is a natural transformation and the definition of $\xi^{\#}_y$, the following diagram commutes.

Diagram 2.84

$$\begin{tikzcd}
\mathcal{O}_Y(U) \arrow[r, "\xi^{\#}(U)"] \arrow[d, "r_{V,U}"'] & \xi_*(\mathcal{O}_G)(U) \arrow[r, "="] \arrow[d, "r_{V,U}"'] & (\mathcal{O}_G)(\xi^{-1}(U)) \arrow[d, "="'] \\
\mathcal{O}_Y(V) \arrow[r, "\xi^{\#}(V)"] \arrow[d, "\mathfrak{J}_{y,V}"'] & \xi_*(\mathcal{O}_G)(V) \arrow[r, "="] \arrow[d, "\alpha"'] & (\mathcal{O}_G)(\xi^{-1}(V)) \arrow[d, "="'] \\
\mathcal{O}_{Y,y} \arrow[r, "\xi^{\#}_y"] & \xi_*(\mathcal{O}_G)_{Y,y} \arrow[r, "="] & (\mathcal{O}_{\mathrm{diffspec}\,\mathcal{G}})_{G,0}
\end{tikzcd}$$

The maps $r_{V,U}$ are restriction maps, and α is the canonical E-$\mathfrak{F}$-homomorphism of $\xi_*(\mathcal{O}_G)(V)$ into its stalk at y. The E-$\mathfrak{F}$-homomorphism of the left column is $\mathfrak{J}_{y,U} : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_{Y,y}$, and each E-$\mathfrak{F}$-homomorphism in right column is the identity E-homomorphism because $\xi^{-1}(U) = G$ for all open subsets U of Y containing y. Therefore, $\xi^{\#}_y \circ \mathfrak{J}_{y,U} = \xi^{\#}(U)$ by the commutivity of the perimeter of Diagram 2.84. Similarly, $\zeta_y^{\#} \circ \mathcal{J}_{y,U} = \zeta^{\#}(U)$. Since $\xi_y^{\#} = \zeta_y^{\#}$, it follows that $\xi^{\#}(U) = \xi_y^{\#} \circ \mathcal{J}_{y,U} = \zeta_y^{\#} \circ \mathcal{J}_{y,U} = \zeta^{\#}(U)$, and the sheaf morphisms $\xi^{\#}$ and $\zeta^{\#}$ are equal. $\square$

In order to provide a different description of points of an E-schemes over $(F, \mathcal{O}_F)$, define the set Υ of all pairs (x, α) where $x \in X$ and $\alpha : \mathcal{O}_x \longrightarrow \mathcal{G}$ is a local E-$\mathcal{F}$-homomorphism.

Proposition 2.85 *Let* $(f, f^{\#}) : (X, \mathcal{O}_X) \longrightarrow (F, \mathcal{O}_F)$ *be an* E-*scheme over* $(F, \mathcal{O}_F)$. *The map* $\varphi : X(\mathcal{G}) \longrightarrow \Upsilon$ *that takes* $(\xi, \xi^{\#}) \in X(\mathcal{G})$ *to* $(\xi(0), \xi^{\#}_{\xi(0)})$ *is a bijection between the set of* $\mathcal{G}$*-valued points of* $(X, \mathcal{O}_X)$ *and* Υ.

Proof: Define $\varphi : X(\mathcal{G}) \longrightarrow \Upsilon$ as in the statement of the proposition. For any $(x, \alpha) \in \Upsilon$, define a continuous mapping of topological spaces $\xi : G \longrightarrow X$ by $\xi((0)) = x$. Then for an $U \subseteq X$ open,

$$\xi^{-1}(U) = \begin{cases} \{(0)\} = G & \text{if } x \in U, \\ \phi & \text{if } x \in U. \end{cases} \tag{6}$$

Composition of ξ with the structure map $X \longrightarrow F$, endows G with a structure over F. Define a sheaf morphism

$$\xi^{\#} : \mathcal{O}_X \longrightarrow \xi_* \mathcal{O}_G$$

as follows: for any open $U \subseteq X$, define

$$\xi^{\#}(U) : \mathcal{O}_X(U) \longrightarrow \xi_* \mathcal{O}_G(U) = \mathcal{O}_G(\xi^{-1}U)\xi^{-1}(U) = \begin{cases} \mathcal{O}_G(G) = \mathcal{G} & \text{if } x \in U, \\ \mathcal{O}_G(\phi) = 0 & \text{if } x \notin U, \end{cases} \tag{7}$$

to be $\xi(U) = \alpha \circ \mathrm{res}_x$ if $x \in U$, where $\mathrm{res}_x : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,x}$ is the canonical E-homomorphism, and the zero E-homomorphism if $x \notin U$. Since this definition is compatible with restriction, $\xi^{\#}$ is a morphisms of sheaves over X, and $(\xi, \xi^{\#}) \in X(\mathcal{G})$. Clearly, $\varphi((\xi, \xi^{\#})) = (x, \alpha)$. Therefore, φ is surjective. Injectivity follows from Lemma 2.83. $\square$

2.4.2 A Topology on the Set of Points

Let $(X, \mathcal{O}_X)$ be an E-$\mathcal{F}$-scheme over $(F, \mathcal{O}_F)$, and let the E-field $\mathcal{V}$ be E-universal over $\mathcal{F}$. Define a topology on $X(\mathcal{V})$ by taking the closed sets to be of the form $W(\mathcal{V}) \subset X(\mathcal{V})$ for W closed in X. The map $\pi_X : X(\mathcal{V}) \longrightarrow X$ which takes $(\xi, \xi^{\#}) \in X(\mathcal{V})$ to $\xi(0)$ is clearly continuous because, for W closed in X, $\pi_X^{-1}(W) = W(\mathcal{V})$ is closed in $X(\mathcal{V})$ by definition.

Recall that for the topological space $X(\mathcal{V})$, the topological pre-order is defined by $(\xi, \xi^{\#}) \longrightarrow (\xi', \xi'^{\#})$ if $\overline{\{(\xi, \xi^{\#})\}} \supset \overline{\{(\xi', \xi'^{\#})\}}$ (Section 1.3).

Proposition 2.86 *Let* $(f, f^{\#}) : (Y, \mathcal{O}_Y) \longrightarrow (Z, \mathcal{O}_Z)$ *be a morphism of* E-*schemes. Endow* X *and* Y *with their topological pre-orders. Then*

1. $f_{\mathcal{V}} : Y(\mathcal{V}) \longrightarrow Z(\mathcal{V})$ *is continuous, and*
2. *for* $x, x' \in X$, *if* $x \longrightarrow x'$, *then* $f_{\mathcal{V}}(x) \longrightarrow f_{\mathcal{V}}(x')$.

Proof: For $(\xi, \xi^{\#}) \in Y(\mathcal{V})$, the calculation

$$\pi_Z(f_{\mathcal{V}}((\xi, \xi^{\#}))) = \pi_Z((f \circ \xi, f_*(\xi^{\#}) \circ f^{\#})) = (f \circ \xi)(0)$$

$$= f(\xi(0)) = f(\pi_Y((\xi, \xi^{\#})))$$

shows the following Diagram 2.87 commutes.

Diagram 2.87

$$\begin{tikzcd} Y(\mathcal{V}) \arrow[r, "f_{\mathcal{V}}"'] \arrow[d, "\pi_Y"] & Z(\mathcal{V}) \arrow[d, "\pi_Z"] \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

Then $f_{\mathcal{V}}$ is continuous because any closed set of $Z(\mathcal{V})$ is of the form $W(\mathcal{V}) = \pi_Z^{-1}(W)$ for W closed in Z, and

$$f_{\mathcal{V}}^{-1}(W(\mathcal{V})) = (f_{\mathcal{V}}^{-1}(\pi_Z^{-1}(W)) = (f_{\mathcal{V}}^{-1} \circ \pi_Z^{-1})(W)$$

$$= (\pi_Y^{-1} \circ f^{-1})(W) = \pi_Y^{-1}(f^{-1}(W))$$

is closed since f and π_Y are continuous. Part 2 follows from the application of Proposition 1.17 to Part 1. ☐

It will now be shown that each irreducible closed set of $X(\mathcal{V})$ has a generic point in $X(\mathcal{V})$ (Corollary 2.96). The key to these results is the following Proposition.

Proposition 2.88 *Let* $X = \operatorname{diffspec} R$ *where* R *is a finitely* E-*generated* E-$\mathfrak{F}$-*algebra. Then* $\pi_X : X(\mathcal{V}) \to X$ *is surjective.*

Proof: Let $\mathfrak{P} \in X$. Because R is finitely E-generated over $\mathfrak{F}$, there exists a surjective E-$\mathfrak{F}$-homomorphism $\pi : \mathfrak{F}\{y_1, \ldots, y_r\}_{\mathrm{E}} \to R$ for some integer r where $y_1, \ldots, y_n$ are E-indeterminates over $\mathfrak{F}$. Let $\eta = (\eta_1, \ldots, \eta_r) \in \mathcal{V}^r$ be an

$\mathcal{F}$-generic E-zero for the prime E-ideal $\pi^{-1}(\mathfrak{P})$ of $\mathcal{F}\{y_1, \ldots, y_r\}_{\mathrm{E}}$, which exists by Proposition 2.60. Let $\phi_\eta : \mathcal{F}\{y_1, \ldots, y_r\}_{\mathrm{E}} \longrightarrow \mathcal{V}$ be the E-$\mathcal{F}$-homomorphism defined by $\phi_\eta(y_i) = \eta_i$ for $i = 1, \ldots, r$. Since $\pi^{-1}(\mathfrak{P}) \supseteq \ker \pi$, ϕ_η factors through the E-$\mathcal{F}$-homomorphism $\rho_\eta : R \longrightarrow \mathcal{V}$ defined by $\rho_\eta(\pi(y_i)) = \eta_i$ for $i = 1, \ldots, r$:

$$\phi_\eta : \mathcal{F}\{y_1, \ldots, y_r\}_{\mathrm{E}} \xrightarrow{\pi} R \xrightarrow{\rho_\eta} \mathcal{V},$$

and $\rho_\eta^{-1}(0) = \mathfrak{P}$. By Proposition 2.65, the E-$\mathcal{F}$-homomorphism ρ_η induces a morphism of E-$\mathcal{F}$-schemes $(\xi, \xi^{\#}) \in X(\mathcal{V})$. Clearly $\xi(0) = \rho_\eta^{-1}(0) = \mathfrak{P}$. Thus $\pi_X((\xi, \xi^{\#})) = \xi(0) = \mathfrak{P}$. ☐

Since, by definition, a $(X, \mathcal{O}_X)$ be an E-$\mathcal{F}$-scheme of E-$\mathcal{F}$-finite type is an E-scheme over $(F, \mathcal{O}_F)$, the functor of $\mathcal{V}$-valued points may be applied to $(X, \mathcal{O}_X)$.

Corollary 2.89 *Let* $(X, \mathcal{O}_X)$ *be an* E-$\mathcal{F}$-*scheme of* E-$\mathcal{F}$-*finite type. The map* $\pi_X : X(\mathcal{V}) \longrightarrow X$ *is surjective.*

Proof: Let $x \in X$. Choose an affine open $\operatorname{diffspec} R$ of X containing x where R is a finitely E-generated E-$\mathcal{F}$-algebra. Then $\pi_X^{-1}(\operatorname{diffspec} R) = (\operatorname{diffspec} R)(\mathcal{V})$ is a subset of $X(\mathcal{V})$. By Corollary 2.88, there exist a point (ξ, ξ_*) of $(\operatorname{diffspec} R)(\mathcal{V})$ such that $\pi_X((\xi, \xi_*)) = x$. ☐

Corollary 2.90 *Let* $(X, \mathcal{O}_X)$ *be an* E-$\mathcal{F}$-*scheme of* E-$\mathcal{F}$-*finite type. The continuous map* $\pi_X : X(\mathcal{V}) \longrightarrow X$ *induces a bijection*

$$B : \{\textit{closed subsets of } X(\mathcal{V})\} \longrightarrow \{\textit{closed subsets of } X\}$$

defined by $B(\Psi) = \pi(\Psi)$.

Proof: Since all closed sets Ψ of $X(\mathcal{V})$ are of the form $\Psi = W(\mathcal{V})$ for W closed in X and since $\pi(W(\mathcal{V})) = W$, $B(\Psi)$ is a closed set, and B is well-defined.

Let $W \subseteq X$ be a closed set. Since π is surjective (Corollary 2.89), $B(W(\mathcal{V})) = V$. Therefore, B is surjective.

Let $W_1(\mathcal{V}) \subseteq X(\mathcal{V})$ and $W_2(\mathcal{V}) \subseteq X(\mathcal{V})$ be two closed sets, for W_1 and W_2 closed in X, such that $B(W_1(\mathcal{V})) = B(W_2(\mathcal{V}))$. Then $W_1 = \pi(W_1(\mathcal{V})) = \pi(W_2(\mathcal{V})) = W_2$. Therefore, $W_1(\mathcal{V}) = W_2(\mathcal{V})$. □

Corollary 2.91 *Let* $X = \operatorname{diffspec} R$ *for some finitely* E*-generated* E-$\mathcal{F}$-*algebra* R. *The closed set* $W \subseteq X$ *is irreducible if and only if* $W(\mathcal{V}) \subseteq X(\mathcal{V})$ *is irreducible.*

Proof: The proof follows immediately from the previous corollary. □

Corollary 2.92 *Let* $(X, \mathcal{O}_X)$ *be a Noetherian* E-$\mathcal{F}$-*scheme over* $(F, \mathcal{O}_F)$. *Then* $X(\mathcal{V})$ *is Noetherian.*

Proof: Let $\{\Psi_i\}$ be a descending sequence of closed sets. Since each $\pi_X(\Psi_i)$ is closed, the sequence $\{\pi_X(\Psi_i)\}$ must stabilize. Then so does $\{\Psi_i\}$ because $\Psi_i = \pi_X^{-1}(\pi(\Psi_i))$. □

Proposition 2.93 *Let* $(X, \mathcal{O}_X)$ *be an* E-$\mathcal{F}$-*scheme of* E-$\mathcal{F}$-*finite type, and let* $(\xi, \xi^{\#}) \in X(\mathcal{V})$. *Then* $\pi_X(\overline{\{(\xi, \xi^{\#})\}}) = \overline{\{\xi(0)\}}$, *and* $\overline{\{(\xi, \xi^{\#})\}} = \pi_X^{-1}(\overline{\{\xi(0)\}}) = \overline{\{\xi(0)\}}(\mathcal{V})$.

Proof: Since π_X induces a bijection between the closed sets of $X(\mathcal{V})$ and the closed sets of X (Corollary 2.90) and since a closed set W of $X(\mathcal{V})$ contains $(\xi, \xi^{\#})$ if and only if the closed set $\pi(W)$ of X contains $\xi(0)$, $\pi_X(\overline{\{(\xi, \xi^{\#})\}}) = \overline{\{\pi_X((\xi, \xi^{\#}))\}}$. Because $\pi_X((\xi, \xi^{\#})) = \xi(0)$, also $\overline{\{\pi_X((\xi, \xi^{\#}))\}} = \overline{\{\xi(0)\}}$. Apply π_X^{-1} to the first result to obtain $\overline{\{(\xi, \xi^{\#})\}} = \pi_X^{-1}(\overline{\{\xi(0)\}})$. Since, for a closed set W of X, $\pi_X^{-1}(W) = W(\mathcal{V})$, take $W = \overline{\{\xi(0)\}}$ to conclude $\pi_X^{-1}(\overline{\{\xi(0)\}}) = \overline{\{\xi(0)\}}(\mathcal{V})$. □

From Proposition 1.17, it follows that if $(\xi, \xi^{\#}) \to (\xi', \xi'^{\#})$, then $\pi((\xi, \xi^{\#})) \to \pi((\xi', \xi'^{\#}))$. Because of Corollary 2.93, the following stronger result holds.

Proposition 2.94 *Let* $(\xi, \xi^{\#}), (\xi', \xi'^{\#}) \in X(\mathcal{V})$. *Then* $(\xi, \xi^{\#}) \to (\xi', \xi'^{\#})$ *if and only if* $\xi(0) \to \xi'(0)$.

Proof: The following statements are equivalent:

1. $(\xi, \xi^{\#}) \to (\xi', \xi'^{\#})$,
2. $\overline{\{(\xi, \xi^{\#})\}} \supset \overline{\{(\xi', \xi'^{\#})\}}$,
3. $\pi_X(\overline{\{(\xi, \xi^{\#})\}}) \supset \pi_X(\overline{\{(\xi', \xi'^{\#})\}})$,
4. $\overline{\{\xi(0)\}} \supset \overline{\{\xi'(0)\}}$ by Proposition 2.93, and
5. $\xi(0) \to \xi'(0)$.

□

Corollary 2.95 *Let* $(\xi, \xi^{\#}), (\xi', \xi'^{\#}) \in X(\mathcal{V})$. *Then* $(\xi, \xi^{\#}) \leftrightarrow (\xi', \xi'^{\#})$ *if and only if* $\xi(0) = \xi'(0)$.

Proof: By the last proposition, $(\xi, \xi^{\#}) \leftrightarrow (\xi', \xi'^{\#})$ is equivalent to $\xi(0) \leftrightarrow \xi'(0)$. This is true if and only if $\overline{\{\xi(0)\}} = \overline{\{\xi'(0)\}}$ in the topology on $X(0)$ by the definition of the topological pre-order. Since generic points are unique in E-$\mathfrak{F}$-schemes (Lemma 2.63), $\xi(0) = \xi'(0)$. Conversely, if $\xi(0) = \xi'(0)$, the argument is reversible. $\square$

Proposition 2.96 *Let* $(X, \mathcal{O}_X)$ *be an* E-$\mathfrak{F}$*-scheme of* E-$\mathfrak{F}$*-finite type. Every irreducible closed set* Ψ *of* $X(\mathcal{V})$ *has a generic point* $(\xi, \xi^{\#})$ *that is mapped by* π_X *onto the unique generic point* $\xi(0) = \pi_X((\xi, \xi^{\#}))$ *of the irreducible closed set* $\pi_X(\Psi)$.

Proof: By Corollary 2.90 and Corollary 2.91, $W = \pi_X(\Psi) \subset X$ is closed and irreducible. Let $w \in \operatorname{diffspec} R$ be the unique generic point of W, which exists by Proposition 2.60. By Corollary 2.89, there exists $(\xi, \xi^{\#}) \in X(\mathcal{V})$ such that $\pi_X((\xi, \xi^{\#})) = w$. To show $(\xi, \xi^{\#})$ is a generic point of $W(\mathcal{V})$, let $(\xi', \xi'^{\#}) \in W(\mathcal{V})$. Since $\pi_X((\xi', \xi'^{\#})) \in W$, $w \to \pi_X((\xi', \xi'^{\#}))$. By Proposition 2.94, $(\xi, \xi^{\#}) \to (\xi', \xi'^{\#})$. $\square$

Corollary 2.97 *Let* $(X, \mathcal{O}_X)$ *be an* E-$\mathfrak{F}$*-scheme of* E-$\mathfrak{F}$*-finite type over* $\mathfrak{F}$. *Then* $X(\mathcal{V})$ *is an NG-space.*

Proof: By the remarks after Definition 2.79, X is Noetherian. By Lemma 2.92, so is $X(\mathcal{V})$. Each irreducible closed set has a generic point by Corollary 2.96. ☐

2.4.3 Points on Products of E-schemes

In this section, the interrelations between products of E-$\mathcal{F}$-schemes and their $\mathcal{V}$-valued points are explored. The results will be used in the proof of Theorem 3.177 of the next chapter and are not necessary for the understanding of remaining material of this chapter.

In this section, fix two E-$\mathcal{F}$-schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ of E-$\mathcal{F}$-finite type (Definition 2.79). The product $(X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y)$ is an E-$\mathcal{F}$-scheme of E-$\mathcal{F}$-finite type (Proposition 2.81). If $\mathcal{L} \subseteq \mathcal{V}$ is an E-field extension of $\mathcal{F}$, $(\operatorname{diffspec} \mathcal{L}, \mathcal{O}_{\operatorname{diffspec} \mathcal{L}})$ is an E-$\mathcal{F}$-scheme. Let $(\xi, \xi^{\#}) \in X(\mathcal{L})$ and $(\zeta, \zeta^{\#}) \in Y(\mathcal{L})$. By the definition of product, there exists a unique $(\varphi, \varphi^{\#}) = \Phi((\xi, \xi^{\#}), (\zeta, \zeta^{\#})) \in (X \times Y)(\mathcal{L})$, called the *product of* $(\xi, \xi^{\#})$ *and* $(\zeta, \zeta^{\#})$, such that the following diagram commutes:

Diagram 2.98

$$\begin{tikzcd}
& (\operatorname{diffspec} \mathcal{L}, \mathcal{O}_{\operatorname{diffspec} \mathcal{L}}) \arrow[d, dotted, "(\varphi,\varphi^{\#})"] \arrow[ddl] \arrow[ddr] & \\
& (X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y) \arrow[dl, "(\pi_X,\pi_X^{\#})"'] \arrow[dr, "(\pi_Y,\pi_Y^{\#})"] & \\
(X, \mathcal{O}_X) & & (Y, \mathcal{O}_Y)
\end{tikzcd}$$

where the left and right long arrows represent $(\xi, \xi^{\#})$ and $(\zeta, \zeta^{\#})$; $(\pi_X, \pi_X^{\#})$ and $(\pi_Y, \pi_Y^{\#})$ are the projections.

Observation 2.99 *The map of sets* $\Phi : X(\mathcal{L}) \times Y(\mathcal{L}) \to (X \times Y)(\mathcal{L})$ *is a bijection.*

The inverse of Φ is the map that associates to $(\varphi, \varphi^{\#}) \in (X \times Y)(\mathcal{L})$ the element $((\pi_X, \pi_X^{\#}) \circ (\varphi, \varphi^{\#}), (\pi_Y, \pi_Y^{\#}) \circ (\varphi, \varphi^{\#})) \in X(\mathcal{L}) \times Y(\mathcal{L})$.

Let $(X, \mathcal{O}_X)$ be an E-scheme over $(F, \mathcal{O}_F)$, and let $(\xi, \xi^{\#}) \in X(\mathcal{V})$. The image of the induced local E-$\mathcal{F}$-homomorphism $\xi^{\#}_{\xi(0)} : \mathcal{O}_{\xi(0)} \to \mathcal{V}$ is an E-$\mathcal{F}$-field that will be denoted by $\mathcal{F}\langle(\xi, \xi^{\#})\rangle \subset \mathcal{V}$.

Lemma 2.100 *Let* $(X, \mathcal{O}_X)$ *be an* E-$\mathcal{F}$*-scheme of* E-$\mathcal{F}$*-finite type, and let* $(\xi, \xi^{\#}) \in X(\mathcal{V})$. *Then* $\mathcal{F}\langle(\xi, \xi^{\#})\rangle \subset \mathcal{V}$ *is a finitely* E-$\mathcal{F}$*-generated field.*

Proof: Because $(X, \mathcal{O}_X)$ is an E-$\mathcal{F}$-scheme of E-$\mathcal{F}$-finite type (Definition 2.79), there exist an affine open subset $\operatorname{diffspec} R \subseteq X$ for some finitely E-generated E-$\mathcal{F}$-algebra R such that $\xi(0) \in \operatorname{diffspec} R$. Because R is finitely E-generated over $\mathcal{F}$, so is $R/\xi(0)$, and therefore the quotient field $QF(R/\xi(0))$ is finitely E-generated as an E-$\mathcal{F}$-field. Since $QF(R/\xi(0)) = R_{\xi(0)}/\xi(0)R_{\xi(0)}$ and $R_{\xi(0)}$ is E-$\mathcal{F}$-isomorphic to $\mathcal{O}_{\xi(0)}$ ([19, Proposition 4.2]), $QF(R/\xi(0))$ is E-$\mathcal{F}$-isomorphic to $\mathcal{O}_{\xi(0)}/\xi(0)\mathcal{O}_{\xi(0)} = \xi^{\#}_{\xi(0)}(\mathcal{O}_{\xi(0)}) = \mathcal{F}\langle(\xi, \xi^{\#})\rangle$, which is also finitely E-$\mathcal{F}$-generated. ☐

Lemma 2.101 *Let* $\mathcal{J} \subseteq \mathcal{V}$ *be a finitely* E*-generated extension of* $\mathcal{H} \subset \mathcal{V}$ *that is not equal to* $\mathcal{H}$. *Then there exists* $\rho \in \operatorname{Isom}^E_{\mathcal{H}}(\mathcal{J}, \mathcal{V})$ *such that* ρ *is non-trivial, i.e.,* ρ *is not the identity on* $\mathcal{J}$.

Proof: From Proposition 1.49, it follows that there exists an isolated element $\rho \in \mathrm{Isom}^{E}_{\mathcal{H}}(\mathcal{J}, \mathcal{V})$. Let $\eta = (\eta_1, \ldots, \eta_m) \in \mathcal{V}^m$ be E-generators of $\mathcal{J}$ over $\mathcal{H}$, and let $\mathfrak{P} \subset \mathcal{H}\{y_1, \ldots, y_n\}_{\mathrm{E}}$ be the defining E-ideal of η. Corollary 1.41 implies that there exists a $\mathcal{J}$-generic element $\zeta \in \mathfrak{Z}_{\mathcal{J}}(\mathcal{J}\mathfrak{P})$ such that $\rho(\eta) = \zeta$. Since $\mathcal{J} \neq \mathcal{H}$, $\zeta \notin \mathcal{J}$, and ρ is not the identity. $\square$

Proposition 2.102 *Let* $(X, \mathcal{O}_X)$ *and* $(Y, \mathcal{O}_Y)$ *be two* E-$\mathcal{F}$*-schemes of* E-$\mathcal{F}$*-finite type, and let* $(Z, \mathcal{O}_Z) = (X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y)$. *Let* $(\xi, \xi^{\#}) \in X(\mathcal{V})$, $(\zeta, \zeta^{\#}) \in Y(\mathcal{V})$ *and* $(\varphi, \varphi^{\#}) = \Phi((\xi, \xi^{\#}), (\zeta, \zeta^{\#}))$. *Then* $\mathcal{F}\langle(\varphi, \varphi^{\#})\rangle$ *is equal to the compositum* $\mathcal{F}\langle(\xi, \xi^{\#})\rangle \cdot \mathcal{F}\langle(\zeta, \zeta^{\#})\rangle$. *And each element of* $\mathcal{O}_{Z,\varphi(0)}$ *is the quotient of two elements of* $(\pi_X)^{\#}_{\xi(0)}(\mathcal{O}_{X,\xi(0)})[(\pi_Y)^{\#}_{\zeta(0)}(\mathcal{O}_{Y,\zeta(0)})]$.

Proof: With $\mathcal{G} = \mathcal{V}$, the commuting triangle on the left of Diagram 2.98 induces the following commuting diagram of stalks and induced local E-$\mathcal{F}$-homomorphisms:

Diagram 2.103

$$\begin{tikzcd}
\mathcal{V} & & \mathcal{O}_{Z,\varphi(0)} \arrow[ll, "\varphi^{\#}_{\varphi(0)}"'] \\
& \mathcal{O}_{X,\xi(0)}. \arrow[ul, "\xi^{\#}_{\xi(0)}"] \arrow[ur, "(\pi_X)^{\#}_{\xi(0)}"'] &
\end{tikzcd}$$

That is $\xi^{\#}_{\xi(0)} = \varphi^{\#}_{\varphi(0)} \circ (\pi_X)^{\#}_{\xi(0)}$. Therefore, the image of $\xi^{\#}_{\xi(0)}$ is contained in the image of $\varphi^{\#}_{\varphi(0)}$, and equivalently $\mathcal{F}\langle(\xi, \xi^{\#})\rangle \subseteq \mathcal{F}\langle(\varphi, \varphi^{\#})\rangle$. Similarly, $\mathcal{F}\langle(\zeta, \zeta^{\#})\rangle \subseteq \mathcal{F}\langle(\varphi, \varphi^{\#})\rangle$. Therefore $\mathcal{H} = \mathcal{F}\langle(\xi, \xi^{\#})\rangle \cdot \mathcal{F}\langle(\zeta, \zeta^{\#})\rangle \subseteq \mathcal{F}\langle(\varphi, \varphi^{\#})\rangle = \mathcal{J}$.

To show $\mathcal{H} = \mathcal{I}$, assume $\mathcal{H}$ is properly contained in $\mathcal{I}$. Since Proposition 2.81 implies that the product $(X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y)$ is of E-F-schemes of finite E-type, Lemma 2.100 shows $\mathcal{I} = \mathcal{F}\langle(\varphi, \varphi^{\#})\rangle$ is finitely E-generated over $\mathcal{F}$ and, a fortiori, over $\mathcal{H}$. By Lemma 2.101, there exists a non-trivial E-$\mathcal{H}$-isomorphism ρ of $\mathcal{F}\langle(\varphi, \varphi^{\#})\rangle$ into $\mathcal{V}$. It will be shown that this implies $(\varphi, \varphi^{\#})$ is not unique, which is false. Therefore, the assumption that $\mathcal{H}$ is properly contained in $\mathcal{I}$ is also false.

The E-$\mathcal{F}$-homomorphism $\varphi^{\#}_{\varphi(0)} : \mathcal{O}_{\varphi(0)} \rightarrow \mathcal{V}$ factors through its image $\mathcal{I}$:

$$\varphi^{\#}_{\varphi(0)} : \quad \mathcal{O}_{Z,\varphi(0)} \xrightarrow{\varphi^{\#\#}_{\varphi(0)}} \mathcal{I} \xrightarrow{i_{\mathcal{V},\mathcal{I}}} \mathcal{V},$$

where $i_{\mathcal{V},\mathcal{I}}$ is the inclusion E-$\mathcal{F}$-homomorphism of $\mathcal{I}$ into $\mathcal{V}$ and $\varphi^{\#\#}_{\varphi(0)}$ is $\varphi^{\#}_{\varphi(0)}$ with its codomain restricted to $\mathcal{I}$. Consider also the local E-$\mathcal{F}$-homomorphism $\alpha = \rho \circ \varphi^{\#\#}_{\varphi(0)}$:

$$\alpha : \quad \mathcal{O}_{Z,\varphi(0)} \xrightarrow{\varphi^{\#\#}_{\varphi(0)}} \mathcal{I} \xrightarrow{\rho} \mathcal{V}.$$

Because ρ is non-trivial, $\varphi^{\#}_{\varphi(0)} \neq \alpha$. By Proposition 2.85, each of these local E-$\mathcal{F}$-homomorphisms correspond to different $\mathcal{V}$-valued points $(\varphi, \varphi^{\#})$ and $(\varphi', \varphi'^{\#})$ of $(X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y)$. Since $\mathcal{F}\langle(\xi, \xi^{\#})\rangle \subset \mathcal{H}$ is invariant under ρ, the compositions

$$\mathcal{O}_{X,\xi(0)} \xrightarrow{(\pi_X)^{\#}_{\xi(0)}} \mathcal{O}_{Z,\varphi(0)} \xrightarrow{\varphi^{\#\#}_{\varphi(0)}} \mathcal{I} \xrightarrow{i_{\mathcal{V},\mathcal{I}}} \mathcal{V},$$

and

$$\mathcal{O}_{X,\xi(0)} \xrightarrow{(\pi_X)^{\#}_{\xi(0)}} \mathcal{O}_{Z,\varphi(0)} \xrightarrow{\varphi^{\#\#}_{\varphi(0)}} \mathcal{I} \xrightarrow{\rho} \mathcal{V}.$$

are equal. Therefore, $\pi_X(\varphi, \varphi^{\#}) = \pi_X(\varphi', \varphi'^{\#}) = (\xi, \xi^{\#})$ again by Proposition 2.85. Similarly, $\pi_Y(\varphi, \varphi^{\#}) = \pi_Y(\varphi', \varphi'^{\#}) = (\zeta, \zeta^{\#})$. Therefore, the product of $(\xi, \xi^{\#})$ and $(\zeta, \zeta^{\#})$ is not unique.

For the proof of the last statement in the proposition, consider the following diagram

Diagram 2.104

$$\begin{tikzcd}
\mathcal{F}\langle(\xi, \xi^{\#})\rangle \otimes \mathcal{F}\langle(\zeta, \zeta^{\#})\rangle \arrow[r, "i_1 \otimes i_2"] & B \arrow[r, "m_1"] & \mathcal{F}\langle(\varphi, \varphi^{\#})\rangle \\
\mathcal{O}_{\xi(0)} \otimes \mathcal{O}_{\zeta(0)} \arrow[u, "(\xi)^{\#}_{\xi(0)} \otimes (\zeta)^{\#}_{\zeta(0)}"'] \arrow[r, "(\pi_X)^{\#}_{\xi(0)} \otimes (\pi_Y)^{\#}_{\zeta(0)}"] & \mathcal{O}_{\varphi(0)} \otimes \mathcal{O}_{\varphi(0)} \arrow[u, "\Upsilon"] \arrow[r, "m_2"] & \mathcal{O}_{\varphi(0)} \arrow[u, "\varphi^{\#}_{\varphi(0)}"]
\end{tikzcd}$$

where $B = \mathcal{F}\langle(\varphi, \varphi^{\#})\rangle \otimes \mathcal{F}\langle(\varphi, \varphi^{\#})\rangle$, m_1 and m_2 are the homomorphisms defining the ring multiplications, the middle map is $\Upsilon = \varphi^{\#}_{\varphi(0)} \otimes \varphi^{\#}_{\varphi(0)}$ and $i_1 : \mathcal{F}\langle(\xi, \xi^{\#})\rangle \to \mathcal{F}\langle(\varphi, \varphi^{\#})\rangle$ and $i_2 : \mathcal{F}\langle(\zeta, \zeta^{\#})\rangle \to \mathcal{F}\langle(\varphi, \varphi^{\#})\rangle$ are the inclusions. The image of m_1 is $\mathcal{F}\langle(\xi, \xi^{\#})\rangle[\mathcal{F}\langle(\zeta, \zeta^{\#})\rangle]$, and the image of m_2 is $(\pi_X)^{\#}_{\xi(0)}(\mathcal{O}_{X,\xi(0)})[(\pi_Y)^{\#}_{\zeta(0)}(\mathcal{O}_{Y,\zeta(0)})]$. Let α be an element of $\mathcal{O}_{Z,\varphi(0)}$ that is not the quotient of two elements of $(\pi_X)^{\#}_{\xi(0)}(\mathcal{O}_{X,\xi(0)})[(\pi_Y)^{\#}_{\zeta(0)}(\mathcal{O}_{Y,\zeta(0)})]$. Then

$$\varphi^{\#}_{\varphi(0)}(\alpha) \notin \mathcal{F}\langle(\xi, \xi^{\#})\rangle \cdot \mathcal{F}\langle(\zeta, \zeta^{\#})\rangle,$$

which contradicts the first statement of the proposition since $\varphi^{\#}_{\varphi(0)}(\alpha) \in \mathcal{F}\langle(\varphi, \varphi^{\#})\rangle$. $\square$

Thus it has been shown that a point in a product has values in the compositum of the values of its projections (Proposition 2.102). For two points of a product different conditions that ensure one specializes to the other will now be explained.

Proposition 2.105 *Consider the product* $(Z, \mathcal{O}_Z) = (X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y)$ *of two* E-$\mathcal{F}$*-schemes. Let* $(\pi_X, \pi_X^\#)$ *and* $(\pi_Y, \pi_Y^\#)$ *be the two projections onto* $(X, \mathcal{O}_X)$ *and* $(Y, \mathcal{O}_Y)$*. Consider the underlying spaces* X*,* Y *and* Z *with their topological pre-orders. Then, if* $z \longrightarrow z'$ *for* $z, z' \in Z$*,* $\pi_X(z) \longrightarrow \pi_X(z')$ *and* $\pi_Y(z) \longrightarrow \pi_Y(z')$*.*

Proof: Apply Proposition 1.17 to the continuous projection mappings of the underlying spaces. □

The following example shows the converse of this proposition is not true. Let E be empty, k be a field and t and s be two indeterminates over k. Let $(X, \mathcal{O}_X) = (\operatorname{spec} k[t], \mathcal{O}_{\operatorname{spec} k[t]})$, and let $(Y, \mathcal{O}_Y) = (\operatorname{spec} k[s], \mathcal{O}_{\operatorname{spec} k[s]})$. Then $(Z, \mathcal{O}_Z) = (X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y) = (\operatorname{spec} k[t,s], \mathcal{O}_{\operatorname{spec} k[t,s]})$. Let $z = (t-s) \subset k[t,s]$, and let $z' = (t-a, s-b) \subset k[t,s]$ for $a, b \in k$, $a \neq b$. Since z is the diagonal, $\pi_X(V(z)) = X$ and $\pi_Y(V(z)) = Y$. Also, $\pi_X(z') = (t-a)$ and $\pi_Y(z') = (s-b)$. So $\pi_X(z) \longrightarrow \pi_X(z')$, and $\pi_Y(z) \longrightarrow \pi_Y(z')$. But it is not true that $z \longrightarrow z'$. Proposition 2.108 provides a converse under a strong condition.

Lemma 2.106 *Let* $(X, \mathcal{O}_X)$ *and* $(Y, \mathcal{O}_Y)$ *be two* E-$\mathcal{F}$*-schemes. Let* $V \subseteq X$ *and* $W \subseteq Y$ *be closed subsets. Then*

$$(V \times Y) \cap (X \times W) = V \times W.$$

Proof: Assume first that $X = \operatorname{diffspec} R$ and $Y = \operatorname{diffspec} S$ are affine, where R and S are Δ-$\mathcal{F}$-algebras. Also let $V = V(I)$ and $W = V(J)$ where $I \subset R$ and $J \subset S$ are E-ideals. Then

$$V \times Y = \operatorname{diffspec}((R/I) \otimes S) = V(I \otimes S)$$

by [39, Thoerem 35, page 184] and similarly

$$X \times W = \operatorname{diffspec}(R \otimes (S/J)) = V(R \otimes J).$$

Therefore, again by [39, Thoerem 35, page 184]

$$(V \times Y) \cap (X \times W) = V(I \otimes S + R \otimes J) = \operatorname{diffspec}((R/I) \otimes (S/J)) = V \times W.$$

If X or Y is not affine, let $\{A_i\}$ and $\{B_j\}$ be an affine open covers of X and Y respectively. Also, let $V_i = V \cap A_i$ and let $W_j = W \cap B_j$. Since $\{V_i\}$ and $\{W_j\}$ are affine open covers of V and W respectively,

$$(V \times Y) \cap (X \times W) = ((\cup_l V_l) \times (\cup_m B_m)) \cap ((\cup_i A_i) \times (\cup_j W_j))$$

$$= [\cup_{l,m} (V_l \times B_m)] \cap [\cup_{i,j} (A_i \times W_j)]$$

$$= \cup_{l,m,i,j} [(V_l \times B_m) \cap (A_i \times W_j)].$$

Since $(V_l \times B_m) \cap (A_i \times W_j) \subseteq (V_i \times B_j) \cap (A_i \times W_j)$ for all l, m, i, j, this is equal to

$$\cup_{i,j} [(V_i \times B_j) \cap (A_i \times W_j)].$$

By using the result of the first paragraph of this proof, this is equal to

$$\cup_{i,j} [V_i \times W_j] = V \times W.$$

The last equality follows from the fact that $\{V_i \times W_j\}$ is a cover for $V \times W$.□

Lemma 2.107 *Consider the product* $(Z, \mathcal{O}_Z) = (X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y)$ *of two* E-$\mathfrak{F}$-*schemes. Let* $z \in Z$. *Then* $\overline{\{z\}} \subseteq \overline{\{\pi_X(z)\}} \times Y$, $\overline{\{z\}} \subseteq X \times \overline{\{\pi_Y(z)\}}$ *and* $\overline{\{z\}} \subseteq (\overline{\{\pi_X(z)\}} \times Y) \cap (X \times \overline{\{\pi_Y(z)\}}) = \overline{\{\pi_X(z)\}} \times \overline{\{\pi_Y(z)\}}$.

Proof: The point z is contained in the closed subset $\pi_X^{-1}(\overline{\pi_X(z)}) = \overline{\{\pi_X(z)\}} \times Y$. So is its closure. The last equality follows from Lemma 2.106. The proof of the second statement is the same. The third is trivial. □

Proposition 2.108 *Consider the product* $(Z, \mathcal{O}_Z) = (X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y)$ *of two* E-$\mathfrak{F}$-*schemes. Let* $z, z' \in Z$. *Let* $\pi_X(z) \longrightarrow \pi_X(z')$ *and* $\pi_Y(z) \longrightarrow \pi_Y(z')$. *If* $\overline{\{z\}} = \overline{\{\pi_X(z)\}} \times \overline{\{\pi_Y(z)\}}$, *then* $z \longrightarrow z'$.

Proof: By Lemma 2.107 and since $\overline{\pi_X(z')} \subseteq \overline{\pi_X(z)}$ and $\overline{\pi_Y(z')} \subseteq \overline{\pi_Y(z)}$, $\overline{\{z'\}} \subseteq (\overline{\{\pi_X(z')\}} \times Y) \cap (X \times \overline{\{\pi_Y(z')\}}) \subseteq (\overline{\{\pi_X(z)\}} \times Y) \cap (X \times \overline{\{\pi_Y(z)\}}) = \overline{\{\pi_X(z)\}} \times \overline{\{\pi_Y(z)\}} = \overline{\{z\}}$. □

Lemma 2.109 *Let* $(Z, \mathcal{O}_Z) = (X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y)$ *be the product of two* E-$\mathfrak{F}$-*schemes of* E-$\mathfrak{F}$-*finite type, and let* $z \in Z$. *Let* $\{\operatorname{diffspec}(R_i \otimes S_j)\}$ *be finite cover of* $X \times Y$, *where* R_i *and* S_j *are finitely* E-$\mathfrak{F}$-*generated* E-$\mathfrak{F}$-*algebras. For* $\overline{\{z\}} = \overline{\{\pi_X(z)\}} \times \overline{\{\pi_Y(z)\}} \subseteq X \times Y$, *it is necessary and sufficient that, for each* $\operatorname{diffspec}(R_i \otimes S_j)$ *in the cover,* $\overbrace{\{z\}} = \overbrace{\{\pi_X(z)\}} \times \overbrace{\{\pi_Y(z)\}} \subseteq \operatorname{diffspec} R_i \times \operatorname{diffspec} S_j = \operatorname{diffspec}(R_i \otimes S_j)$, *where* "$\overbrace{}$" *is the closure operation in* $\operatorname{diffspec} R_i$, $\operatorname{diffspec} S_j$ *and* $\operatorname{diffspec} R_i \times \operatorname{diffspec} S_j$.

Proof: For a $\operatorname{diffspec}(R_i \otimes S_j)$ in the cover, let $R = R_j$ and $S = S_j$. The closure of z in $\operatorname{diffspec} R \times \operatorname{diffspec} S$ is $\overline{\{z\}} \cap (\operatorname{diffspec} R \times \operatorname{diffspec} S)$ by

Proposition 1.11. Similarly, $\overbrace{\{\pi_X(z)\}} = \overline{\{\pi_X(z)\}} \cap \operatorname{diffspec} R$, and $\overbrace{\{\pi_Y(z)\}} = \overline{\{\pi_Y(z)\}} \cap \operatorname{diffspec} S$. Therefore, if $\overline{\{z\}} = \overline{\{\pi_X(z)\}} \times \overline{\{\pi_Y(z)\}}$, $\overbrace{\{z\}} = \overline{\{z\}} \cap (\operatorname{diffspec} R \times \operatorname{diffspec} S) = (\overline{\{\pi_X(z)\}} \times \overline{\{\pi_Y(z)\}}) \cap (\operatorname{diffspec} R \times \operatorname{diffspec} S) = (\overline{\{\pi_X(z)\}} \cap \operatorname{diffspec} R) \times (\overline{\{\pi_Y(z)\}} \cap \operatorname{diffspec} S) = \overbrace{\{\pi_X(z)\}} \times \overbrace{\{\pi_Y(z)\}}$.

To prove the sufficiency, observe $\overline{\{z\}} \subseteq \overline{\{\pi_X(z)\}} \times \overline{\{\pi_Y(z)\}}$ by Lemma 2.107. Let $\zeta \in \overline{\{\pi_X(z)\}} \times \overline{\{\pi_Y(z)\}}$, and let $\operatorname{diffspec}(R \otimes S) = \operatorname{diffspec} R \times \operatorname{diffspec} S$ be an element in the cover containing ζ. Then $\zeta \in (\overline{\{\pi_X(z)\}} \times \overline{\{\pi_Y(z)\}}) \cap (\operatorname{diffspec} R \times \operatorname{diffspec} S) = (\overline{\{\pi_X(z)\}} \cap \operatorname{diffspec} R) \times (\overline{\{\pi_Y(z)\}} \cap \operatorname{diffspec} S) = \overbrace{\{\pi_X(z)\}} \times \overbrace{\{\pi_Y(z)\}} = \overbrace{\{z\}} = \overline{\{z\}} \cap (\operatorname{diffspec} R \times \operatorname{diffspec} S)$. Therefore $\zeta \in \overline{\{z\}}$, and $\overline{\{z\}} = \overline{\{\pi_X(z)\}} \times \overline{\{\pi_Y(z)\}}$. □

Lemma 2.110 *Let* $X = \operatorname{diffspec} R$, *and let* $Y = \operatorname{diffspec} S$. *Also, let* $(\xi, \xi^{\#}) \in X(\mathcal{V})$, *and let* $(\zeta, \zeta^{\#}) \in Y(\mathcal{V})$. *As closed subspaces of* $X \times Y$,

$$(\overline{\{\xi(0)\}} \times Y) \cap (X \times \overline{\{\zeta(0)\}}) = V(\xi(0) \otimes R + S \otimes \zeta(0)).$$

Proof: By Observation 2.59, $\overline{\{\xi(0)\}} = V(\xi(0))$ and $\overline{\{\zeta(0)\}} = V(\zeta(0))$. Therefore, $\overline{\{\xi(0)\}} \times Y = V(\xi(0) \otimes S)$, and $X \times \overline{\{\zeta(0)\}} = V(R \otimes \zeta(0))$. By [19, Proposition 3.2(b), page 75], $(\overline{\{\xi(0)\}} \times Y) \cap (X \times \overline{\{\zeta(0)\}}) = V((\xi(0)) \otimes S) \cap (R \otimes \zeta(0))) = V(\xi(0) \otimes S + R \otimes \zeta(0))$. □

Consider the product $(Z, o_Z) = (X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y)$ of two E-$\mathfrak{F}$-schemes of finite E-$\mathfrak{F}$-type, and two points $\mathcal{V}$-valued points $(\xi, \xi^{\#}) \in X(\mathcal{V})$ and $(\zeta, \zeta^{\#}) \in Y(\mathcal{V})$. Let $(\phi, \phi^{\#}) = \varphi((\xi, \xi^{\#}), (\zeta, \zeta^{\#})) \in (X \times Y)(\mathcal{V})$ be the product. Recall, from the discussion at the beginning of the proof of Proposition 2.102, that the projections π_X and π_Y induce local E-$\mathcal{F}$-homomorphisms $(\pi_X)^{\#}_{\xi(0)} : \mathcal{O}_{\xi(0)} \longrightarrow \mathcal{O}_{\phi(0)}$ and $(\pi_Y)^{\#}_{\zeta(0)} : \mathcal{O}_{\zeta(0)} \longrightarrow \mathcal{O}_{\phi(0)}$. Since these are local E-$\mathcal{F}$-homomorphisms, the evaluation maps fit into the following commutative diagrams:

Diagram 2.111

$$\begin{tikzcd}
\mathcal{O}_{\xi(0)} \arrow[r, "(\pi_X)^{\#}_{\xi(0)}"] \arrow[d, "\xi^{\#}_{\xi(0)}"] & \mathcal{O}_{\phi(0)} \arrow[d, "\phi^{\#}_{\xi(0)}"] \\
\mathcal{F}\langle(\xi, \xi^{\#})\rangle \arrow[r, "i_\xi"'] & \mathcal{F}\langle(\phi, \phi^{\#})\rangle
\end{tikzcd}$$

Diagram 2.112

$$\begin{tikzcd}
\mathcal{O}_{\zeta(0)} \arrow[r, "(\pi_X)^{\#}_{\zeta(0)}"] \arrow[d, "\zeta^{\#}_{\zeta(0)}"] & \mathcal{O}_{\phi(0)} \arrow[d, "\phi^{\#}_{\xi(0)}"] \\
\mathcal{F}\langle(\zeta, \zeta^{\#})\rangle \arrow[r, "i_\zeta"'] & \mathcal{F}\langle(\phi, \phi^{\#})\rangle
\end{tikzcd}$$

where i_ξ and i_ζ are the inclusions of one field into another. Because of simple properties of tensor products and because $\phi^{\#}_{\xi(0)}$ is an E-$\mathcal{F}$-homomorphism of E-$\mathcal{F}$-algebras, the following diagram commutes:

Diagram 2.113

$$\begin{tikzcd}
\mathcal{O}_{\xi(0)} \otimes \mathcal{O}_{\zeta(0)} \arrow[r, "(\pi_X)^{\#}_{\xi(0)} \otimes (\pi_X)^{\#}_{\zeta(0)}"] \arrow[d, "\xi^{\#}_{\xi(0)} \otimes \zeta^{\#}_{\zeta(0)}"] & \mathcal{O}_{\phi(0)} \otimes \mathcal{O}_{\phi(0)} \arrow[r, "m"] \arrow[d, "\phi^{\#}_{\xi(0)} \otimes \phi^{\#}_{\xi(0)}"] & \mathcal{O}_{\phi(0)} \arrow[d, "\phi^{\#}_{\xi(0)}"] \\
\mathcal{F}\langle(\xi,\xi^{\#})\rangle \otimes \mathcal{F}\langle(\zeta,\zeta^{\#})\rangle \arrow[r, "i_\xi \otimes i_\zeta"'] & \mathcal{F}\langle(\phi,\phi^{\#})\rangle \otimes \mathcal{F}\langle(\phi,\phi^{\#})\rangle \arrow[r, "m"] & \mathcal{F}\langle(\phi,\phi^{\#})\rangle.
\end{tikzcd}$$

where m is ring multiplication. Clearly the image of $m \circ (i_\xi \otimes i_\zeta)$ is the E-$\mathfrak{F}$-algebra $\mathfrak{F}\langle(\xi,\xi^\#)\rangle[\mathfrak{F}\langle(\zeta,\zeta^\#)\rangle]$. Define

$$m_{\xi,\zeta} : \mathfrak{F}\langle(\phi,\phi^\#)\rangle \otimes \mathfrak{F}\langle(\phi,\phi^\#)\rangle \to \mathfrak{F}\langle(\xi,\xi^\#)\rangle[\mathfrak{F}\langle(\zeta,\zeta^\#)\rangle]$$

to be the E-$\mathfrak{F}$-homomorphism $m \circ (i_\xi \otimes i_\zeta)$ with its codomain restricted to $\mathfrak{F}\langle(\zeta,\zeta^\#)\rangle[\mathfrak{F}\langle(\xi,\xi^\#)\rangle]$.

Lemma 2.114 *Let X be an* E-$\mathfrak{F}$-*scheme and let $x \in X$. Let* diffspec R *be an open affine of X containing x. Consider the following sequence of* E-$\mathfrak{F}$-*homomorphisms:*

$$\vartheta_{\xi,R} : R \xrightarrow{\mathfrak{I}_R} \mathfrak{O}_{\text{diffspec}\,R}(\text{diffspec}\,R) \xrightarrow{\mathfrak{J}_{x,\text{diffspec}\,R}} \mathfrak{O}_x \xrightarrow{\xi^\#_{\xi(0)}} \mathfrak{F}\langle(\xi,\xi^\#)\rangle.$$

The kernel of the composite $\vartheta_{\xi,R} = \xi^\#_{\xi(0)} \circ \mathfrak{J}_{x,\text{diffspec}\,R} \circ \mathfrak{I}_R$ is $x \subset R$.

Proof: By Lemma 2.71, $(\mathfrak{J}_{x,\text{diffspec}\,R} \circ \mathfrak{I}_R)^{-1}(\mathfrak{P}_x) = x$ where $\mathfrak{P}_x$ is the maximal E-ideal of $\mathfrak{O}_x$. Since $\mathfrak{P}_x$ is the kernel of $\xi^\#_{\xi(0)}$, the result follows. □

Let $\{\text{diffspec}\,R_i\}_{i \in I}$ and $\{\text{diffspec}\,S_j\}_{j \in J}$ be finite covers of X and Y where R_i and S_j are E-$\mathfrak{F}$-algebras finitely E-generated over $\mathfrak{F}$. Then $\{\text{diffspec}\,(R_i \otimes S_j)\}_{i \in I, j \in J}$ is a finite cover of $X \times Y$. Let $\xi(0) \in \text{diffspec}\,R$ and $\zeta(0) \in \text{diffspec}\,S$ for $R = R_i$ and $S = S_j$ for $i \in I$ and $j \in J$. Then $\phi(0) \in \text{diffspec}\,(R \otimes S)$.

By Lemma 2.114, the following three sequences of E-$\mathfrak{F}$-homomorphisms:

$$\vartheta_{\xi,R} : R \xrightarrow{\mathfrak{I}_R} \mathfrak{O}_{\text{diffspec}\,R}(\text{diffspec}\,R) \xrightarrow{\mathfrak{J}_{\xi(0),\text{diffspec}\,R}} \mathfrak{O}_{\xi(0)} \xrightarrow{\xi^\#_{\xi(0)}} \mathfrak{F}\langle(\xi,\xi^\#)\rangle \tag{8}$$

$$\vartheta_{\zeta,s} : S \xrightarrow{\mathfrak{I}_S} \mathcal{O}_{\text{diffspec}\,S}(\text{diffspec}\,S) \xrightarrow{\mathfrak{d}_{\zeta(0),\text{diffspec}\,S}} \mathcal{O}_{\zeta(0)} \xrightarrow{\xi^{\#}_{\zeta(0)}} \mathfrak{F}\langle(\zeta,\zeta^{\#})\rangle \tag{9}$$

$$\vartheta_{\phi,R\otimes S}\colon \quad R\otimes S \xrightarrow{\mathfrak{I}_{R\otimes S}} \mathcal{O}_{\text{diffspec}\,(R\otimes S)}(\text{diffspec}\,(R\otimes S)) \xrightarrow{\mathfrak{d}_{\phi(0),\text{diffspec}\,(R\otimes S)}} \mathcal{O}_{\phi(0)} \xrightarrow{\phi^{\#}_{\phi(0)}} \mathfrak{F}\langle(\phi,\phi^{\#})\rangle \tag{10}$$

have kernels $\xi(0)$, $\zeta(0)$ and $\phi(0)$, respectively. Tensor Line 8 and Line 9 to obtain the E-$\mathfrak{F}$-homomorphism below:

$$R\otimes S \xrightarrow{\vartheta_{\xi,R}\otimes\vartheta_{\zeta,S}} \mathfrak{F}\langle(\xi,\xi^{\#})\rangle \otimes \mathfrak{F}\langle(\zeta,\zeta^{\#})\rangle. \tag{11}$$

By [39, Theorem 35, page 184], the kernel of $\vartheta_{\xi,R}\otimes\vartheta_{\zeta,S}$ is $\xi(0)\otimes S+R\otimes\zeta(0)$, which is the smallest ideal containing the images of $\xi(0)$ and $\zeta(0)$ in $S\otimes R$.

Claim 2.115 *The following two diagrams commute.*

Diagram 2.116

$$\begin{tikzcd}
R \arrow[r, "\mathfrak{I}_R"] \arrow[d, "id\otimes 1"'] & \mathcal{O}_{\text{diffspec}\,R}(\text{diffspec}\,R) \arrow[r, "\mathfrak{d}_{\xi(0),\text{diffspec}\,R}"] \arrow[d, "\mathcal{O}_{\text{diffspec}\,R}(id\otimes 1)"'] & \mathcal{O}_{\xi(0)} \arrow[r, "\xi^{\#}_{\xi(0)}"] \arrow[d, "(\pi_R)^{\#}_{\xi(0)}"'] & \mathfrak{F}\langle\xi(0)\rangle \arrow[d, "i_\xi"'] \\
R\otimes S \arrow[r, "\mathfrak{I}_{R\otimes S}"] & \mathcal{O}_{\text{diffspec}\,R\otimes S}(\text{diffspec}\,R\otimes S) \arrow[r, "\mathfrak{d}_{\phi(0),\text{diffspec}\,R\times S}"] & \mathcal{O}_{\phi(0)} \arrow[r, "\phi^{\#}_{\phi(0)}"] & \mathfrak{F}\langle\phi(0)\rangle
\end{tikzcd}$$

Diagram 2.117

$$\begin{tikzcd}
S \arrow[r, "\mathfrak{I}_S"] \arrow[d, "id\otimes 1"'] & \mathcal{O}_{\text{diffspec}\,S}(\text{diffspec}\,S) \arrow[r, "\mathfrak{d}_{\zeta(0),\text{diffspec}\,S}"] \arrow[d, "\mathcal{O}_{\text{diffspec}\,S}(id\otimes 1)"'] & \mathcal{O}_{\zeta(0)} \arrow[r, "\zeta^{\#}_{\zeta(0)}"] \arrow[d, "(\pi_S)^{\#}_{\zeta(0)}"'] & \mathfrak{F}\langle\zeta(0)\rangle \arrow[d, "i_\zeta"'] \\
R\otimes S \arrow[r, "\mathfrak{I}_{R\otimes S}"] & \mathcal{O}_{\text{diffspec}\,R\otimes S}(\text{diffspec}\,R\otimes S) \arrow[r, "\mathfrak{d}_{\phi(0),\text{diffspec}\,R\times S}"] & \mathcal{O}_{\phi(0)} \arrow[r, "\phi^{\#}_{\phi(0)}"] & \mathfrak{F}\langle\phi(0)\rangle
\end{tikzcd}$$

In Diagram 2.116, the square on the left commutes because the map taking an E-ring R to the E-ring $\mathcal{O}_{\mathrm{diffspec}\,R}(\mathrm{diffspec}\,R)$ is a functor [17, page 38] or [18, page 169]. The middle square commutes by the definition of the induced local E-homomorphism on stalks. The square on the right commutes because $(\pi_R)^{\#}_{\phi(0)}$ is a local homomorphism. The commutivity of Diagram 2.117 is proved in the same manner. By the definition of the ϑ, Claim 2.115 is equivalent to the commuting of the following two diagrams:

Diagram 2.118

$$\begin{tikzcd}
R \arrow[r, "\vartheta_{\xi,R}"] \arrow[d, "id\otimes 1"'] & \mathcal{F}\langle\xi(0)\rangle \arrow[d, "i_\xi"'] \\
R\otimes S \arrow[r, "\vartheta_{\phi,R\otimes S}"] & \mathcal{F}\langle\phi(0)\rangle
\end{tikzcd}$$

Diagram 2.119

$$\begin{tikzcd}
S \arrow[r, "\vartheta_{\zeta,S}"] \arrow[d, "1\otimes id"'] & \mathcal{F}\langle\zeta(0)\rangle \arrow[d, "i_\zeta"'] \\
R\otimes S \arrow[r, "\vartheta_{\phi,R\otimes S}"] & \mathcal{F}\langle\phi(0)\rangle
\end{tikzcd}$$

Tensor the Diagrams 2.118 and 2.119 and follow this by ring multiplication m to obtain the commuting diagram:

Diagram 2.120

$$\begin{tikzcd}
R\otimes S \arrow[r, "\vartheta_{\xi,R}\otimes\vartheta_{\zeta,S}"] \arrow[d, "(id\otimes 1)\otimes(1\otimes id)"'] & \mathcal{F}\langle\xi(0)\rangle\otimes\mathcal{F}\langle\zeta(0)\rangle \arrow[d, "i_\xi\otimes i_\zeta"'] \\
(R\otimes S)\otimes(R\otimes S) \arrow[r, "\vartheta_{\phi,R\otimes S}\otimes\vartheta_{\phi,R\otimes S}"] \arrow[d, "m"'] & \mathcal{F}\langle\phi(0)\rangle\times\mathcal{F}\langle\phi(0)\rangle \arrow[d, "m"'] \\
R\otimes S \arrow[r, "\vartheta_{\phi,R\otimes S}"] & \mathcal{F}\langle\phi(0)\rangle
\end{tikzcd}$$

Since the E-$\mathcal{F}$-homomorphism $m \circ (id \otimes 1) \otimes (1 \otimes id)$ of the left hand column is the identity, Diagram 2.120 may be simplified as

Diagram 2.121

$$\begin{tikzcd}
R \otimes S \arrow[r, "\vartheta_{\xi,R} \otimes \vartheta_{\zeta,S}"] \arrow[dr, "\vartheta_{\phi,R\otimes S}"'] & \mathcal{F}\langle(\xi,\xi^{\#})\rangle \otimes \mathcal{F}\langle(\zeta,\zeta^{\#})\rangle \arrow[d, "m_{\xi,\zeta}"] \\
 & \mathcal{F}\langle(\xi,\xi^{\#})\rangle[\mathcal{F}\langle(\zeta,\zeta^{\#})\rangle].
\end{tikzcd}$$

Claim 2.122 *The kernel of the composite* $m_{\xi,\zeta} \circ (\vartheta_{\xi,R} \otimes \vartheta_{\zeta,S})$ *is* $\phi(0)$.

Since Diagram 2.121 commutes and the kernel of $\vartheta_{\phi,R\otimes S}$ is equal to $\phi(0)$, the claim is clear.

For $(\xi,\xi^{\#}), (\xi',\xi'^{\#}) \in X(\mathcal{V})$, if $(\xi,\xi^{\#}) \leftrightarrow (\xi',\xi'^{\#})$, then $\xi(0) \leftrightarrow \xi'(0)$, and $\overline{\{\xi(0)\}} = \overline{\{\xi'(0)\}}$. Since generic points are unique in E-$\mathcal{F}$-schemes (Lemma 2.63), $\xi(0) = \xi'(0)$. Define $S_{(\xi',\xi'^{\#}),(\xi,\xi^{\#})} : \mathcal{F}\langle(\xi,\xi^{\#})\rangle \approx \mathcal{F}\langle(\xi',\xi'^{\#})\rangle$ to be the unique E-$\mathcal{F}$-isomorphism over $\mathcal{F}$ such that $S_{(\xi',\xi'^{\#}),(\xi,\xi^{\#})} \circ \xi^{\#}_{\xi(0)} = \xi'^{\#}_{\xi(0)}$.

$$\begin{tikzcd}
\mathcal{F}\langle(\xi,\xi^{\#})\rangle \arrow[r, "S_{(\xi',\xi'^{\#}),(\xi,\xi^{\#})}"] & \mathcal{F}\langle(\xi',\xi'^{\#})\rangle \\
\mathcal{O}_{\xi(0)} \arrow[u, "\xi^{\#}_{\xi(0)}"] \arrow[r, "identity"] & \mathcal{O}_{\xi'(0)} \arrow[u, "\xi'^{\#}_{\xi'(0)}"']
\end{tikzcd}$$

Lemma 2.123 *Consider the product* $(Z, o_Z) = (X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y)$ *of two* E-$\mathcal{F}$*-schemes of finite* E-$\mathcal{F}$*-type, and four* $\mathcal{V}$*-valued points*

$(\xi, \xi^{\#}), (\xi', \xi'^{\#}) \in X(\mathcal{V})$ *and* $(\zeta, \zeta^{\#}), (\zeta', \zeta'^{\#}) \in Y(\mathcal{V})$. *Let* $(\phi, \phi^{\#}) = \varphi((\xi, \xi^{\#}), (\zeta, \zeta^{\#}))$ *and* $(\phi', \phi'^{\#}) = \varphi((\xi', \xi'^{\#}), (\zeta', \zeta'^{\#}))$ *in* $(X \times Y)(\mathcal{V})$. *Assume* $(\xi, \xi^{\#}) \leftrightarrow (\xi', \xi'^{\#})$ *and* $(\zeta, \zeta^{\#}) \leftrightarrow (\zeta', \zeta'^{\#})$. *Then* $(\phi, \phi^{\#}) \to (\phi', \phi'^{\#})$ *in* $(X \times Y)(\mathcal{V})$ *if and only if* $S_{(\xi', \xi'^{\#}),(\xi, \xi^{\#})}$ *and* $S_{(\zeta', \zeta'^{\#}),(\zeta, \zeta^{\#})}$ *are compatible.*

Proof: Let $\{\operatorname{diffspec} R_i\}_{i \in I}$ and $\{\operatorname{diffspec} S_j\}_{j \in J}$ be finite covers of X and Y where R_i and S_j are E-$\mathfrak{F}$-algebras of finite E-$\mathfrak{F}$-type. Then $\{\operatorname{diffspec}(R_i \otimes S_j)\}_{i \in I, j \in J}$ is a finite cover of $X \times Y$. Then $\phi'(0) \in \operatorname{diffspec}(R \otimes S)$ for $R = R_i$ and $S = S_j$ for some $i \in I$ and $j \in J$. Then $\pi_X(\phi'(0)) = \xi'(0) \in \operatorname{diffspec} R$, and $\pi_Y(\phi'(0)) = \zeta'(0) \in \operatorname{diffspec} S$. Since $\xi(0) = \xi'(0)$ and $\zeta(0) = \zeta'(0)$ by Proposition 2.94 and Proposition 2.60, $\phi(0) \in \operatorname{diffspec}(R \otimes S)$ because $\phi(0) \in \pi_X^{-1}(\operatorname{diffspec} R) \cap \pi_Y^{-1}(\operatorname{diffspec} S) = (\operatorname{diffspec} R \times Y) \cap (X \times \operatorname{diffspec} S) = \operatorname{diffspec}(R \otimes S)$.

In the following diagram, the triangle commutes by the definitions of $S_{(\xi', \xi'^{\#}),(\xi, \xi^{\#})}$ and $S_{(\zeta', \zeta'^{\#}),(\zeta, \zeta^{\#})}$. By Claim 2.122, the kernel of $m_{\xi,\zeta} \circ (\vartheta_{\xi,R} \otimes \vartheta_{\zeta,S})$ is $\phi(0)$, and the kernel of $m_{\xi',\zeta'} \circ (\vartheta_{\xi',R} \otimes \vartheta_{\zeta',S})$ is $\phi'(0)$.

Diagram 2.124

$$\begin{tikzcd}
R \otimes S \arrow[r, "\vartheta_{\xi,R} \otimes \vartheta_{\zeta,S}"] \arrow[dr, "\vartheta_{\xi,R} \otimes \vartheta_{\zeta,S}"'] & \mathfrak{F}\langle(\xi, \xi^{\#})\rangle \otimes \mathfrak{F}\langle(\zeta, \zeta^{\#})\rangle \arrow[r, "m_{\xi,\zeta}"] \arrow[d, "S_{(\xi', \xi'^{\#}),(\xi, \xi^{\#})} \otimes S_{(\zeta', \zeta'^{\#}),(\zeta, \zeta^{\#})}"] & \mathfrak{F}\langle(\xi, \xi^{\#})\rangle[\mathfrak{F}\langle(\zeta, \zeta^{\#})\rangle] \arrow[d, "\varphi"] \\
 & \mathfrak{F}\langle(\xi', \xi'^{\#})\rangle \otimes \mathfrak{F}\langle(\zeta', \zeta'^{\#})\rangle \arrow[r, "m_{\xi',\zeta'}"] & \mathfrak{F}\langle(\xi', \xi'^{\#})\rangle[\mathfrak{F}\langle(\zeta', \zeta'^{\#})\rangle]
\end{tikzcd}$$

Then $(\phi, \phi^{\#}) \to (\phi', \phi'^{\#})$ in $(X \times Y)(\mathcal{V}) \iff \phi(0) \to \phi'(0)$ in $X \times Y$ by Proposition 2.94 $\iff \phi(0) \to \phi'(0)$ in $\operatorname{diffspec}(R \otimes S)$ by Proposition 1.11

and Observation 1.1 $\Longleftrightarrow \phi(0) \subset \phi'(0) \Longleftrightarrow$ the E-$\mathfrak{F}$-homomorphism φ exists and extends $S_{(\xi',\xi'^{\#}),(\xi,\xi^{\#})}$ and $S_{(\zeta',\zeta'^{\#}),(\zeta,\zeta^{\#})}$ making Diagram 2.124 commute $\Longleftrightarrow S_{(\xi',\xi'^{\#}),(\xi,\xi^{\#})}$ and $S_{(\zeta',\zeta'^{\#}),(\zeta,\zeta^{\#})}$ are compatible by definition. ☐

Proposition 2.125 *Consider the product* $(Z, o_Z) = (X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y)$ *of two* E-$\mathfrak{F}$-*schemes of finite* E-$\mathfrak{F}$-*type, and four* $\mathcal{V}$-*valued points* $(\xi, \xi^{\#})$, $(\xi', \xi'^{\#}) \in X(\mathcal{V})$ *and* $(\zeta, \zeta^{\#}), (\zeta', \zeta'^{\#}) \in Y(\mathcal{V})$ *such that* $(\xi, \xi^{\#}) \to (\xi', \xi'^{\#})$ *and* $(\zeta, \zeta^{\#}) \to (\zeta', \zeta'^{\#})$. *If* $\mathfrak{F}\langle(\zeta, \zeta^{\#})\rangle$ *and* $\mathfrak{F}\langle(\xi, \xi^{\#})\rangle$ *are algebraically disjoint over* $\mathfrak{F}$, *then* $(\phi, \phi^{\#}) = \varphi((\xi, \xi^{\#}) \times (\zeta, \zeta^{\#})) \to (\phi', \phi'^{\#}) = \varphi((\xi', \xi'^{\#}) \times (\zeta', \zeta'^{\#}))$.

Proof: Let $\{\text{diffspec}\,(R_i \otimes S_j)\}$ be finite cover of $X \times Y$, where R_i and S_j are finitely E-$\mathfrak{F}$-generated E-$\mathfrak{F}$-algebras. If, for some i and j, $\overline{\{\phi(0)\}} \cap \text{diffspec}\,(R_i \otimes S_j)$ is non-empty, $\phi(0) \in \text{diffspec}\,(R_i \otimes S_j)$ by Lemma 1.10. The disjointness assumption and the fact that $\mathfrak{F}\langle(\xi, \xi^{\#})\rangle \otimes 1$ and $1 \otimes \mathfrak{F}\langle(\zeta, \zeta^{\#})\rangle$ in $\mathfrak{F}\langle(\xi, \xi^{\#})\rangle \otimes \mathfrak{F}\langle(\zeta, \zeta^{\#})\rangle$ are linearly disjoint over $\mathfrak{F}$ [39, Corollary, page 185] imply that the E-$\mathfrak{F}$-homomorphism $m_{\xi,\zeta}$ in Diagram 2.121 is an E-$\mathfrak{F}$-isomorphism by [25, Proposition 1, page 2]. Because of the commutivity of that diagram, the kernel of $\vartheta_{R_i \otimes S_j}$ is the kernel of $\vartheta_{R_i} \otimes \vartheta_{S_j}$, i.e. $\phi(0) = \xi(0) \otimes S + R \otimes \zeta(0)$. By Lemma 2.110 and Lemma 2.106, $V(\phi(0)) \subseteq \text{diffspec}\,(R_i \otimes S_j)$ is a product, and therefore, $\overline{\{\phi(0)\}} \subseteq X \times Y$ is globally a product by Lemma 2.109. Apply Proposition 2.108 to conclude that $\phi(0) \to \phi'(0)$. Use Proposition 2.94 to finish the proof. ☐

2.5 The Pre E-Set Associated to a E-Scheme

In this section, it will be shown that the functor of $\mathcal{V}$-valued points when restricted from the category of E-schemes over $(\operatorname{diffspec}\mathcal{F}, \mathcal{O}_{\operatorname{diffspec}\mathcal{F}})$ to the full subcategory of E-$\mathcal{F}$-schemes of E-$\mathcal{F}$-finite type has its image in the category of pre E-$\mathcal{F}$-sets. Thus a plethora of examples of pre E-$\mathcal{F}$-sets and their morphisms are obtained. The only difficult part of this whole exercise is the existence of generic points in $\mathcal{V}$ for prime E-ideals of $\mathcal{F}\{y_1, \ldots, y_r\}_{\mathrm{E}}$ which was demonstrated in Proposition 1.24 and used in Proposition 2.96.

Theorem 2.126 *The functor of $\mathcal{V}$-valued points when restricted to the category of* E-$\mathcal{F}$*-schemes of* E-$\mathcal{F}$*-finite type has its image in the category of pre* E-$\mathcal{F}$*-sets.*

Proof: In this proof, identify the stalk $\mathcal{O}_0$ of $\mathcal{O}_{\operatorname{diffspec}\mathcal{V}}$ at the unique point $0 = \{0\}$ in $\operatorname{diffspec}\mathcal{V}$ with the E-$\mathcal{F}$-field $\mathcal{V}$ via the canonical E-$\mathcal{F}$-isomorphism (Lemma 2.82). Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be E-$\mathcal{F}$-schemes of E-$\mathcal{F}$-finite type, and let $(f, f^{\#}) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of E-$\mathcal{F}$-schemes. The proof is a step by step verification that the points of $X(\mathcal{V})$ satisfy the axioms of a pre E-set (Definition 4.2) and that the morphisms $f_{\mathcal{V}}$ are pre E-set maps (Definition 2.56). The three structures of the pre E-set $X(\mathcal{V})$ are the following:

1. for each $(\xi, \xi^{\#}) \in X(\mathcal{V})$, the E-field $\mathcal{F}\langle(\xi, \xi^{\#})\rangle = \xi^{\#}_{\xi(0)}(\mathcal{O}_{\xi(0)}) \subset \mathcal{V}$, which is a finitely E-$\mathcal{F}$-generated of $\mathcal{F}$ (Lemma 2.100),

2. for $(\xi, \xi^{\#}), (\xi', \xi'^{\#}) \in X(\mathcal{V})$, the topological pre-order $(\xi, \xi^{\#}) \longrightarrow (\xi', \xi'^{\#})$ associated to the topology on $X(\mathcal{V})$,

3. for each pair $((\xi, \xi^{\#}), (\xi', \xi'^{\#})) \in X(\mathcal{V})^2$ with $(\xi, \xi^{\#}) \leftrightarrow (\xi', \xi'^{\#})$, the unique E-$\mathfrak{F}$-isomorphism $S_{(\xi', \xi'^{\#}),(\xi, \xi^{\#})} : \mathfrak{F}\langle(\xi, \xi^{\#})\rangle \approx \mathfrak{F}\langle(\xi', \xi'^{\#})\rangle$ such that $S_{(\xi', \xi'^{\#}),(\xi, \xi^{\#})} \circ \xi^{\#}_{\xi(0)} = \xi'^{\#}_{\xi(0)}$,

$$\begin{tikzcd}
\mathfrak{F}\langle(\xi, \xi^{\#})\rangle \arrow[r, "S_{(\xi', \xi'^{\#}),(\xi, \xi^{\#})}"] & \mathfrak{F}\langle(\xi', \xi'^{\#})\rangle \\
\mathcal{O}_{\xi(0)} \arrow[u, "\xi^{\#}_{\xi(0)}"] \arrow[r, "="] & \mathcal{O}_{\xi'(0)} . \arrow[u, "\xi'^{\#}_{\xi'(0)}"']
\end{tikzcd}$$

The bottom E-$\mathfrak{F}$-homomorphism is the identity because $(\xi, \xi^{\#}) \leftrightarrow (\xi', \xi'^{\#})$ implies $\xi(0) = \xi'(0)$ by Corollary 2.95.

This structure satisfies the axioms of a pre E-$\mathfrak{F}$-set.

DAS1. Since $X(\mathcal{V})$ with the pre-order topology is an NG-space (Corollary 2.97), there exists a generic point for each of the finitely many components (Proposition 1.14). Let $\Phi = \{(\zeta_1, \zeta_1^{\#}), \dots, (\zeta_s, \zeta_s^{\#})\}$ where each $(\zeta_i, \zeta_i^{\#})$ is a generic point of a different component of $X(\mathcal{V})$. Then Φ has the property that for each $(\zeta, \zeta^{\#}) \in X(\mathcal{V})$ there exits $(\zeta_i, \zeta_i^{\#}) \in \Phi$ such that $(\zeta_i, \zeta_i^{\#}) \rightarrow (\zeta, \zeta^{\#})$ (Proposition 1.21).

DAS2a. This axiom is obviously valid.

DAS2b. Given $(\xi, \xi^{\#}) \in X(\mathcal{V})$ and an E-$\mathfrak{F}$-isomorphism $S : \mathfrak{F}\langle(\xi, \xi^{\#})\rangle \approx \mathfrak{F}'$, it must be shown that there exist a unique $(\xi', \xi'^{\#}) \in X(\mathcal{V})$ with the properties

1. $(\xi, \xi^{\#}) \leftrightarrow (\xi', \xi'^{\#})$

2. $\mathcal{F}\langle(\xi', \xi'^{\#})\rangle = \mathcal{F}'$ and

3. $S = S_{(\xi',\xi'^{\#}),(\xi,\xi^{\#})} : \mathcal{F}\langle(\xi, \xi^{\#})\rangle \approx \mathcal{F}\langle(\xi', \xi'^{\#})\rangle$.

Write the local E-$\mathcal{F}$-homomorphism $\xi^{\#}_{\xi(0)} : \mathcal{O}_{\xi(0)} \to \mathcal{V}$ as the composition of $\xi^{\#\#}_{\xi(0)} : \mathcal{O}_{\xi(0)} \to \mathcal{F}\langle(\xi, \xi^{\#})\rangle$ and the inclusion $i_1 : \mathcal{F}\langle(\xi, \xi^{\#})\rangle \to \mathcal{V}$

$$\xi^{\#}_{\xi(0)} : \quad \mathcal{O}_{X,\xi(0)} \xrightarrow{\xi^{\#\#}_{\xi(0)}} \mathcal{F}\langle(\xi, \xi^{\#})\rangle \xrightarrow{i_1} \mathcal{V},$$

where $\xi^{\#\#}_{\xi(0)}$ is $\xi^{\#}_{\xi(0)}$ with its codomain restricted to $\mathcal{F}\langle(\xi, \xi^{\#})\rangle$. Let $\xi'^{\#}_{\xi(0)}$ be the composition $i_2 \circ S \circ \xi^{\#\#}_{\xi(0)}$

$$\xi'^{\#}_{\xi(0)} : \quad \mathcal{O}_{X,\xi(0)} \xrightarrow{\xi^{\#\#}_{\xi(0)}} \mathcal{F}\langle(\xi, \xi^{\#})\rangle \xrightarrow{S} \mathcal{F}' \xrightarrow{i_2} \mathcal{V},$$

where i_2 is the inclusion $\mathcal{F}' \to \mathcal{V}$. By Proposition 2.85, $\xi'^{\#}_{\xi(0)}$ corresponds to a unique $\mathcal{V}$-valued point $(\xi', \xi'^{\#})$ of $(X, \mathcal{O}_X)$ with induced local E-$\mathcal{F}$-homomorphism $\xi'^{\#}_{\xi(0)}$. Since clearly $\xi(0) = \xi'(0)$, property (1) follows from Corollary 2.95. By definition, $\mathcal{F}\langle(\xi', \xi'^{\#})\rangle = \xi'^{\#}_{\xi'(0)}(\mathcal{O}_{\xi'(0)}) = \mathcal{F}'$, which is property (2). Since $\xi'^{\#}_{\xi'(0)} = S \circ \xi^{\#}_{\xi(0)}$ by the definition of $\xi'^{\#}_{\xi'(0)}$ and $\xi'^{\#}_{\xi'(0)} = S_{(\xi',\xi'^{\#}),(\xi,\xi^{\#})} \circ \xi^{\#}_{\xi(0)}$ by the definition of $S_{(\xi',\xi'^{\#}),(\xi,\xi^{\#})}$,

$$S \circ \xi^{\#}_{\xi(0)} = S \circ \xi^{\#}_{\xi(0)},$$

and property (3) $S_{(\xi',\xi'^{\#}),(\xi,\xi^{\#})} = S$ follows because $\xi^{\#}_{\xi(0)}$ is surjective onto $\mathcal{F}\langle(\xi', \xi'^{\#})\rangle$.

$$\begin{tikzcd}
\mathcal{F}\langle(\xi, \xi^{\#})\rangle \arrow[r, "S_{(\xi',\xi'^{\#}),(\xi,\xi^{\#})}", "S"'] & \mathcal{F}' = \mathcal{F}\langle(\xi', \xi'^{\#})\rangle \\
\mathcal{O}_{\xi(0)} \arrow[u, "\xi^{\#}_{\xi(0)}"] \arrow[r, "identity"] & \mathcal{O}_{\xi'(0)} \arrow[u, "\xi'^{\#}_{\xi'(0)}"']
\end{tikzcd}$$

For E-$\mathcal{F}$-schemes of E-$\mathcal{F}$-finite type $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, let f be a morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$. Then it will be shown $f_{\mathcal{V}}$ is an everywhere defined morphism from $X(\mathcal{V})$ to $Y(\mathcal{V})$ in the category of E-$\mathcal{F}$-sets. To demonstrate this, take $A_{f_{\mathcal{V}}} = X(\mathcal{V})$.

1. The $\mathcal{F}$-generic points of $X(\mathcal{V})$ are contained in all points $A_{f_{\mathcal{V}}}$.

2. Let $(\xi, \xi^{\#}) \in A_{f_{\mathcal{V}}}$. Then $(\xi, \xi^{\#})$ and $f_{\mathcal{V}}((\xi, \xi^{\#})) = (f \circ \xi, f_*(\xi^{\#}) \circ f^{\#})$ both induce local E-$\mathcal{F}$-homomorphisms of stalks

$$\xi^{\#}_{\xi(0)} : \mathcal{O}_{X,\xi(0)} \longrightarrow \mathcal{O}_{\mathrm{diffspec}\,\mathcal{V},(0)} = \mathcal{V}$$

and

$$(f_*(\xi^{\#}) \circ f^{\#})_{f(\xi(0))} : \mathcal{O}_{Y,f(\xi(0))} \longrightarrow \mathcal{O}_{\mathrm{diffspec}\,\mathcal{V},(0)} = \mathcal{V}$$

with images $\mathcal{F}\langle(\xi, \xi^{\#})\rangle$ and $\mathcal{F}\langle f_{\mathcal{V}}((\xi, \xi^{\#}))\rangle$, respectively. Also f induces a local E-$\mathcal{F}$-homomorphism $f^{\#}_{f(\xi(0))} : \mathcal{O}_{Y,f(\xi(0))} \longrightarrow \mathcal{O}_{X,\xi(0)}$ such that

$$(f_*(\xi^{\#}) \circ f^{\#})_{f(\xi(0))} = \xi^{\#}_{\xi(0)} \circ f^{\#}_{f(\xi(0))}$$

This implies the image of $(f_*(\xi^{\#}) \circ f^{\#})_{f(\xi(0))}$ is contained in the image of $\xi^{\#}_{\xi(0)}$. That is $\mathcal{F}\langle f_{\mathcal{V}}((\xi, \xi^{\#}))\rangle \subset \mathcal{F}\langle(\xi, \xi^{\#})\rangle$.

3. If $(\xi, \xi^{\#}), (\xi', \xi'^{\#}) \in A_{f_{\mathcal{V}}}$, and $(\xi, \xi^{\#}) \to (\xi', \xi'^{\#})$, then $(\xi, \xi^{\#}) \in A_{f_{\mathcal{V}}}$ because all points of $X(\mathcal{V})$ are in $A_{f_{\mathcal{V}}}$. By Proposition 2.86, $f_{\mathcal{V}}((\xi, \xi^{\#})) \to f_{\mathcal{V}}((\xi', \xi'^{\#}))$.

4. If $(\xi, \xi^{\#}), (\xi', \xi'^{\#}) \in A_{f_{\mathcal{V}}}$ and $(\xi, \xi^{\#}) \leftrightarrow (\xi', \xi'^{\#})$, then by (3) $f_{\mathcal{V}}((\xi, \xi^{\#})) \leftrightarrow f_{\mathcal{V}}((\xi', \xi'^{\#}))$. By Corollary 2.95, $\xi(0) = \xi'(0)$ and $f(\xi(0))$ $= f(\xi'(0))$. Thus $f^{\#}_{\xi(0)} = f^{\#}_{\xi'(0)}$ (See the commuting Diagram 2.127 below).

Diagram 2.127

$$\begin{tikzcd}
\mathcal{F}\langle(\xi, \xi^{\#})\rangle \arrow[rr, "S_{(\xi',\xi'^{\#}),(\xi,\xi^{\#})}"] & & \mathcal{F}\langle(\xi', \xi'^{\#})\rangle \\
\mathcal{O}_{\xi(0)} \arrow[u, "\xi^{\#}_{\xi(0)}"] \arrow[rr, "identity"] & & \mathcal{O}_{\xi'(0)} \arrow[u, "\xi'^{\#}_{\xi'(0)}"'] \\
\mathcal{O}_{f(\xi(0))} \arrow[u, "f^{\#}_{\xi(0)}"] \arrow[rr, "identity"] & & \mathcal{O}_{f(\xi'(0))} \arrow[u, "f^{\#}_{\xi'(0)}"']
\end{tikzcd}$$

By the definition of $S_{(\xi',\xi'^{\#}),(\xi,\xi^{\#})}$ and $S_{f_{\mathcal{V}}((\xi',\xi'^{\#})),f_{\mathcal{V}}((\xi,\xi^{\#}))}$,

$$S_{(\xi',\xi'^{\#}),(\xi,\xi^{\#})} \circ \xi^{\#}_{\xi(0)} = \xi'^{\#}_{\xi'(0)}$$

and

$$S_{f_{\mathcal{V}}((\xi',\xi'^{\#})),f_{\mathcal{V}}((\xi,\xi^{\#}))} \circ (f_*(\xi^{\#}) \circ f^{\#})_{f(\xi(0))} = (f_*(\xi'^{\#}) \circ f^{\#})_{f(\xi'(0))}. \tag{12}$$

The right hand side of Equation 12 equals $\xi'^{\#}_{\xi'(0)} \circ f^{\#}_{f(\xi'(0))}$ $= (S_{(\xi',\xi'^{\#}),(\xi,\xi^{\#})} \circ \xi^{\#}_{\xi(0)}) \circ f^{\#}_{f(\xi'(0))} = S_{(\xi',\xi'^{\#}),(\xi,\xi^{\#})} \circ (\xi^{\#}_{\xi(0)} \circ f^{\#}_{f(\xi'(0))})$. Thus,

$$S_{f_{\mathcal{V}}((\xi',\xi'^{\#})),f_{\mathcal{V}}((\xi,\xi^{\#}))} \circ (f_*(\xi^{\#}) \circ f^{\#})_{f(\xi(0))} = S_{(\xi',\xi'^{\#}),(\xi,\xi^{\#})} \circ (f_*(\xi^{\#}) \circ f^{\#})_{f(\xi(0))}.$$

Since the image of $(f_*(\xi^{\#}) \circ f^{\#})_{f(\xi(0))}$ is $\mathcal{F}\langle f_{\mathcal{V}}((\xi, \xi^{\#}))\rangle$, $S_{f_{\mathcal{V}}((\xi',\xi'^{\#})),f_{\mathcal{V}}((\xi,\xi^{\#}))}$ is the restriction of $S_{(\xi',\xi'^{\#}),(\xi,\xi^{\#})}$ to $\mathcal{F}\langle f_{\mathcal{V}}((\xi, \xi^{\#}))\rangle$. $\square$

2.6 A Pre E-Set of a Finitely Generated Extension

Let $\mathcal{G}$ be a finitely E-generated extension of $\mathcal{F}$. Let $\operatorname{Isom}^E_{\mathcal{F}}(\mathcal{G},\mathcal{V})$ be the set of E-isomorphisms of $\mathcal{G}$ over $\mathcal{F}$ into $\mathcal{V}$. Define on $\operatorname{Isom}^E_{\mathcal{F}}(\mathcal{G},\mathcal{V})$ a structure of a pre E-$\mathcal{F}$-set as follows. For $\sigma \in \operatorname{Isom}^E_{\mathcal{F}}(\mathcal{G},\mathcal{V})$ define $\mathcal{F}\langle\sigma\rangle = \sigma\mathcal{G}$. For $\sigma, \tau \in \operatorname{Isom}^E_{\mathcal{F}}(\mathcal{G},\mathcal{V})$ define $\sigma \longrightarrow \tau$ to be always true. Therefore $\sigma \leftrightarrow \tau$ is always true. Take $S_{\tau,\sigma}$ to be the composite of σ^{-1} and τ. [25, Chapter 7, page 157].

2.7 The Pre E-Set of a Strongly Normal Isomorphism

In this section, the pre E-set axioms are verified for the set of specializations of an E-strong isomorphism which is defined as follows. Recall "(E,Δ)" denotes the union of two disjoint sets $\mathrm{E} = \{\epsilon_1,\ldots,\epsilon_r\}$ and $\Delta = \{\delta_1,\ldots,\delta_n\}$.

Definition 2.128 *Let $\mathcal{G}$ be an (E,Δ)-subfield of $\mathcal{U}$. An (E,Δ)-isomorphism σ of $\mathcal{G}$ into $\mathcal{U}$ is* E-strong *if it satisfies the following two conditions.*

St1. *σ leaves invariant every element of $\mathcal{G}^\Delta$.*

St2. *$\sigma\mathcal{G} \subset \mathcal{G}\cdot\mathcal{U}^\Delta$ and $\mathcal{G} \subset \sigma\mathcal{G}\cdot\mathcal{U}^\Delta$.*

An E-strong (E,Δ)-isomorphism is the same as an E-homomorphism which is also a strong Δ-isomorphism in the sense defined by Kolchin in [24, p. 382]. Note that St2 is equivalent to $\mathcal{G}\cdot\mathcal{U}^\Delta = \sigma\mathcal{G}\cdot\mathcal{U}^\Delta$. Also it is clear that any (E,Δ)-automorphism of $\mathcal{G}$ over $\mathcal{G}^\Delta$ is an E-strong (E,Δ)-isomorphism. For any (E,Δ)-isomorphism σ of $\mathcal{G}$, let $\mathcal{G}^\Delta\langle\sigma\rangle = (\mathcal{G}\sigma\mathcal{G})^\Delta$. The first inclusion of St2 is equivalent to $\mathcal{G}\sigma\mathcal{G} \subset \mathcal{G} \cdot \mathcal{U}^{\Delta}$ which by [24, Corollary 2, p. 88] is equivalent to $\mathcal{G}\sigma\mathcal{G} = \mathcal{G} \cdot \mathcal{G}^{\Delta}\langle\sigma\rangle$. Similarly the second inclusion is equivalent to $\mathcal{G}\sigma\mathcal{G} = \sigma\mathcal{G} \cdot \mathcal{G}^{\Delta}\langle\sigma\rangle$.

If $\mathcal{G}$ is an arbitrary (E, Δ)-extension of $\mathcal{F}$, it may happen that not all elements σ of $\mathrm{Isom}_{\mathcal{F}}^{E,\Delta}(\mathcal{G}, \mathcal{U})$ are E-strong (E, Δ)-isomorphisms. See the Example 2.149. However, if there is one E-strong (E, Δ)-isomorphism, all its (E, Δ)-$\mathcal{G}$-specializations in the sense of Example 1.34 will be shown in the next propositions to have the structure of a pre E-set.

Proposition 2.129 *Every* (E, Δ)-$\mathcal{G}$-*specialization of an* E-*strong* (E, Δ)-*isomorphism of* $\mathcal{G}$ *is* E-*strong.*

Proof: Let σ' be an (E, Δ)-$\mathcal{G}$-specialization of the E-strong (E, Δ)-isomorphism σ of $\mathcal{G}$. By the definition of (E, Δ)-$\mathcal{G}$-specialization (Example 1.34), σ' is an (E, Δ)-isomorphism. Now σ is a strong Δ-isomorphism, and hence σ' is also a strong Δ-isomorphism by [24, Proposition 6, p. 390]. Since σ' is an E-homomorphism, σ' is an E-strong (E, Δ)-isomorphism. $\square$

One of the axioms for a pre E-set is that the field associated to an element is finitely E-generated. The following Proposition will be applied to obtain this result.

Proposition 2.130 *Let* $\mathcal{G}$ *be a finitely* (E, Δ)-*generated* (E, Δ)-*extension of* $\mathcal{F}$. *Then for every* E-*strong* (E, Δ)-*isomorphism* σ *of* $\mathcal{G}$ over $\mathcal{F}$, $\mathcal{G}^{\Delta}\langle\sigma\rangle = (\mathcal{G}\sigma\mathcal{G})^{\Delta}$ *is a finitely* E-*generated field extension of the* E-*field* $\mathcal{G}^{\Delta}$.

Proof: Let $\eta = (\eta_1, \ldots, \eta_n)$ be a finite family of (E, Δ)-generators of $\mathcal{G}$ over $\mathcal{F}$. Let σ be an E-strong (E, Δ)-isomorphism of $\mathcal{G}$ over $\mathcal{F}$. The extension $\mathcal{G}\sigma\mathcal{G} = \mathcal{G}\langle\sigma\eta\rangle_{E,\Delta}$ of $\mathcal{G}$ is a finitely (E, Δ)-generated extension by $\sigma\eta$. Let $\xi = (\xi_i)_{i \in I}$ be a family of E-generators of $\mathcal{G}^{\Delta}\langle\sigma\rangle = (\mathcal{G}\sigma\mathcal{G})^{\Delta}$ over $\mathcal{G}^{\Delta}$. Since $\mathcal{G}\sigma\mathcal{G} = \mathcal{G}\mathcal{G}^{\Delta}\langle\sigma\rangle = \mathcal{G}\langle\xi\rangle$, the family ξ also E-generates $\mathcal{G}\sigma\mathcal{G}$ over $\mathcal{G}$.

$$\begin{tikzcd}
\mathcal{G} \arrow[r] & \mathcal{G}\sigma\mathcal{G} = \mathcal{G}\langle\xi\rangle \\
\mathcal{G}^{\Delta} \arrow[u] \arrow[r] & \mathcal{G}^{\Delta}\langle\sigma\rangle = \mathcal{G}^{\Delta}\langle\xi\rangle \arrow[u]
\end{tikzcd}$$

Because this extension is (E, Δ)-finitely generated over $\mathcal{G}$ by $\sigma\eta$ and each element of $\sigma\eta$ is in an E-field generated by finitely many of the elements of the family ξ, there is a finite subfamily $(\xi_1, \ldots, \xi_m)$ of the family ξ that (E, Δ)-generates $\mathcal{G}\sigma\mathcal{G}$ over $\mathcal{G}$: that is $\mathcal{G}\mathcal{G}^{\Delta}\langle\sigma\rangle = \mathcal{G}\langle\xi_1, \ldots, \xi_m\rangle_{E,\Delta}$ $= \mathcal{G} \cdot \mathcal{G}^{\Delta}\langle\xi_1, \ldots, \xi_m\rangle_{E,\Delta}$. Since the elements of ξ are Δ-constants, $\mathcal{G}^{\Delta}\langle\sigma\rangle = \mathcal{G}^{E}\langle\xi_1, \ldots, \xi_m\rangle$ by [24, Corollary 2, p. 88]. $\square$

Proposition 2.131 *Let $\mathcal{G}$ be an (E, Δ)-field such that $\mathcal{G}$ is finitely (E, Δ)-generated over $\mathcal{F}$, and let ρ be an* E*-strong isomorphism of $\mathcal{G}$ over $\mathcal{F}$. Let $V(\rho)$ be the set of all (E, Δ)-$\mathcal{G}$-specializations of ρ. Then $V(\rho)$ is a pre* E-$\mathcal{G}^{\Delta}$*-set.*

The pre E-$\mathcal{G}^{\Delta}$-set structure on $V(\rho)$ is the following.

1. To each $\sigma \in V(\rho)$ associate $\mathcal{G}^{\Delta}\langle\sigma\rangle = (\mathcal{G}\sigma\mathcal{G})^{\Delta}$, considered as an E-field extension of $\mathcal{G}^{\Delta}$. This is finitely E-generated by Proposition 2.130.

2. The pre-order on $V(\rho)$ is that of (E, Δ)-$\mathcal{G}$-specialization in the sense of Example 1.34 with $r = 1$.

3. If σ' is a generic specialization of the E-strong (E, Δ)-isomorphism σ of $\mathcal{G}$, then there exists a unique $\mathcal{G}$-(E, Δ)-isomorphism $\mathcal{G}\sigma\mathcal{G} \approx \mathcal{G}\sigma'\mathcal{G}$ that, for each $\alpha \in \mathcal{G}$, maps α to α and $\sigma\alpha$ onto $\sigma'\alpha$. This (E, Δ)-$\mathcal{G}$-isomorphism yields on restriction to the Δ-constants an E-$\mathcal{G}^{\Delta}$-isomorphism $S_{\sigma',\sigma} : \mathcal{G}^{\Delta}\langle\sigma\rangle \approx \mathcal{G}^{\Delta}\langle\sigma'\rangle$ over $\mathcal{G}^{\Delta}$ which is called the E-$\mathcal{G}^{\Delta}$-*isomorphism induced by the generic* (E, Δ)-$\mathcal{G}$-*specialization.*

For *DAS1*, the first axiom of a pre E-set, take ϕ to the set consisting of ρ. For *DAS2* use the following proposition.

Proposition 2.132 *Let σ be an* E-*strong* (E, Δ)-*isomorphism of* $\mathcal{G}$.

1. *If σ' is a generic (E, Δ)-$\mathcal{G}$-specialization of σ, and σ'' is a generic (E, Δ)-$\mathcal{G}$-specialization of σ' (and therefore of σ), then the composite of the induced* E-$\mathcal{G}^{\Delta}$-*isomorphisms $\mathcal{G}^{\Delta}\langle\sigma\rangle \approx \mathcal{G}^{\Delta}\langle\sigma'\rangle$ and $\mathcal{G}^{\Delta}\langle\sigma'\rangle \approx \mathcal{G}^{\Delta}\langle\sigma''\rangle$ is the induced* E-$\mathcal{G}^{\Delta}$-*isomorphism $\mathcal{G}^{\Delta}\langle\sigma\rangle \approx \mathcal{G}^{\Delta}\langle\sigma''\rangle$.*

2. *If $S : \mathcal{G}^{\Delta}\langle\sigma\rangle \approx \mathcal{C}'$ is any* E-*isomorphism over $\mathcal{G}^{\Delta}$, then there exists a unique generic (E, Δ)-$\mathcal{G}$-specialization σ' of σ such that $\mathcal{G}^{\Delta}\langle\sigma'\rangle = \mathcal{C}'$, and S is the induced* E-$\mathcal{G}^{\Delta}$-*isomorphism $\mathcal{G}^{\Delta}\langle\sigma\rangle \approx \mathcal{G}^{\Delta}\langle\sigma'\rangle$.*

Proof:

1. This follows from the corresponding facts about the (E, Δ)-$\mathcal{G}$-isomorphisms $\mathcal{G}\sigma\mathcal{G} \approx \mathcal{G}\sigma'\mathcal{G}$, $\mathcal{G}\sigma'\mathcal{G} \approx \mathcal{G}\sigma''\mathcal{G}$ and $\mathcal{G}\sigma\mathcal{G} \approx \mathcal{G}\sigma''\mathcal{G}$.

2. $\mathcal{G}^{\Delta}\langle\sigma\rangle$ and $\mathcal{G}$ are linearly disjoint over $\mathcal{G}^{\Delta}$, as are $\mathcal{C}'$ and $\mathcal{G}$: therefore S can be extended to an (E, Δ)-$\mathcal{G}$-isomorphism $T : \mathcal{G}\mathcal{G}^{\Delta}\langle\sigma\rangle \approx \mathcal{G}\mathcal{C}'$. The composite mapping $\mathcal{G} \approx^{\sigma} \sigma\mathcal{G} \subset \mathcal{G}\sigma\mathcal{G} = \mathcal{G}\mathcal{G}^{\Delta}\langle\sigma\rangle \approx^{T} \mathcal{G}\mathcal{C}'$ yields an (E, Δ)-$\mathcal{G}^{\Delta}$-isomorphism $\sigma' : \mathcal{G} \approx T(\sigma\mathcal{G})$. Since $T\sigma\mathcal{G} = \sigma'\mathcal{G}$, $T : \mathcal{G}\sigma\mathcal{G} \approx \mathcal{G}\sigma'\mathcal{G}$. Therefore σ' is a generic (E, Δ)-$\mathcal{G}$-specialization of σ, $\mathcal{C}' = \mathcal{G}^{\Delta}\langle\sigma'\rangle$ [24, Corollary 2, p. 88], and S is the induced E-$\mathcal{G}^{\Delta}$-isomorphism $S_{\sigma',\sigma}$. The uniqueness is clear.

□

2.8 Two Examples and an Application

Throughout this section, let $\Delta = \{\delta\}$, and write δw as w' for some Δ-ring element w. Proposition 2.140 of this section will be applied to prove Proposition 6.276 in the chapter of examples.

2.8.1 Example 1

This section presents an example, which Kolchin suggested in a private conversation, of an (E, Δ)-extension $\mathcal{G}$ over $\mathcal{F}$ with no new Δ-constants such that some (E, Δ)-isomorphisms are E-strong but others are not (Example 2.149). To such an (E, Δ)-isomorphism that is E-strong, one may apply Proposition 2.131 to conclude that all its E-specializations form the underlying set of a pre

E-set. To exhibit such an example, it suffices to take E to be empty, and consider the functions $\eta_{a,b} = a + \log(x+b)$ of x which are Δ-$\mathbb{Q}$-zeros of the prime Δ-ideal generated by the Δ-polynomial $y'' + (y')^2$, where $x' = 1$ and a and b are Δ-constants. The Δ-$\mathbb{Q}$-isomorphism $\sigma_{c,d} : \mathbb{Q}\langle\eta_{a,b}\rangle_\Delta \longrightarrow \mathbb{Q}\langle\eta_{a+c,b+d}\rangle_\Delta$, where c and d are Δ-constants and $\sigma_{c,d}(\eta_{a,b}) = \eta_{a+c,b+d}$, is not strong if $d \neq 0$. But, if $d = 0$, $\sigma_{c,d}$ is strong.

Lemma 2.133 *Let R be a Δ-ring that is a factorial domain of characteristic zero. Extend δ to a derivation of the quotient field Q. For any $\alpha \in Q$, write the reduced fraction $\alpha = \Pi p_i^{n_i} / \Pi q_j^{m_j}$ where the p_i and q_j are non-associate irreducible elements of R and the n_i and m_j are positive integers. If $q_j' \notin (q_j)$, then q_j is in the denominator of the reduced fraction of α' with an exponent of $m_j + 1$.*

Proof: Examine the numerator of α':

$$(\Pi p_i^{n_i})'\Pi q_j^{m_j} - \Pi p_i^{n_i}(\Pi q_j^{m_j})' = (\Pi p_i^{n_i})'\Pi q_j^{m_j} - \Sigma_j(\Pi p_i^{n_i}) m_j q_j' q_j^{m_j-1} \Pi_{k \neq j} q_k^{m_k}.$$

The only term not divisible by $q_j^{m_j}$ is $(\Pi p_i^{n_i}) m_j q_j' q_j^{m_j-1} \Pi_{k \neq j} q_k^{m_k}$. So the power of q_i in the factorization of the numerator is $m_i - 1$. Since $q_i^{2m_i}$ is present in the denominator of the derivative formula for α', in the reduced fraction of α' the irreducible element q_j is present in the denominator with an exponent of $m_j + 1$. □

Proposition 2.134 *Let k be a field of characteristic zero, and let $k(x)$ be the rational function field in one indeterminate x such that $x' = 1$ and*

$a' = 0$ for every $a \in k$. For any $\alpha \in k(x)$, write the reduced fraction $\alpha = \Pi p_i^{n_i}/\Pi q_j^{m_j}$ where the p_i and q_j are different irreducible elements of $k[x]$ and the n_i and m_j are positive integers. If one $m_j = 1$, α is not a derivative of any element of $k(x)$.

Corollary 2.135 *Let $\mathcal{U}$ be Δ-universal extension of the constant field $\mathcal{C}$. Let $x \in \mathcal{U}$ be a Δ-zero of $y' - 1 \in \mathcal{C}\{y\}_\Delta$. For $i = 1, \ldots, n$, let $p_i(x) \in \mathcal{C}[x]$ be non-associate and irreducible. Then the reciprocals of the p_i are linearly independent over $\mathcal{C}$ modulo $(\mathcal{C}(x))'$.*

Proof: Express any linear combination $\Sigma_i c_i/p_i(x)$ ($c_i \in \mathcal{C}$ and all $c_i \neq 0$) of the reciprocals of the $p_i(x)$ over $\mathcal{C}$ as a rational fraction α in reduced form. Since the numerator is not divisible by any $p_i(x)$, the denominator of α has each $p_i(x)$ as a factor with exponent exactly 1. Then apply Proposition 2.134.

$\square$

Lemma 2.136 *Let $\mathcal{F}$ be a Δ-field. Endow $\mathcal{F}\{y\}_\Delta$ with an orderly ranking. Let $A \in \mathcal{F}\{y\}_\Delta$ be a Δ-polynomial with separant 1, i.e., A is of the form $\delta^r y +$ terms of order less than r. Then $[A]_\Delta$ is a prime Δ-ideal.*

Proof: By [24, Theorem 3(b), page 155], the prime general component of A is $[A]_\Delta : 1^\infty = [A]_\Delta$. $\square$

Lemma 2.137 *Let $\mathcal{F}$ be a Δ-field, and let $\mathcal{C} = \mathcal{F}^\Delta$. If $\alpha \in \mathcal{F}$ and $\alpha' = 1$, then α is transcendental over $\mathcal{C}$.*

Proof: If α were algebraic over $\mathcal{C}$, it would be the zero of an irreducible polynomial $p(z) \in \mathcal{C}[z]$ of minimal degree. Let $p(z) = z^r + c_{r-1}z^{r-1} + \ldots + c_0$ with $c_i \in \mathcal{C}$. By evaluating $p(z)$ at α and then applying the derivation, $\alpha'(r\alpha^{r-1} + (r-1)c_{r-1}\alpha^{r-2} + \ldots + c_1) = 0$. By the minimality, $r\alpha^{r-1} + (r-1)c_{r-1}\alpha^{r-2} + \ldots + c_1 \neq 0$: so that $\alpha' = 0$, which is a contradiction. Therefore α is transcendental over $\mathcal{C}$. $\square$

In the proof of the next proposition, the following order on polynomials will be utilized. (See [15, Lemma 3, page 58] for a similar argument.) Let $z_1, \ldots, z_n$ be algebraic indeterminates over $\mathcal{F}$. Let $g \in \mathcal{F}[z_1, \ldots, z_n]$, and let d be the degree of g in the indeteminates $z_1, \ldots, z_n$, with the convention $\deg 0 = -1$. Write $g = \Sigma_M \alpha_M M$ where the M are monomials in $z_1, \ldots, z_n$ and $\alpha_M \in \mathcal{F}$. Let $c(g)$ denote the number of terms $\alpha_M M$ ($\alpha_M \neq 0$) of degree d in g. Define the level(g) to be $(\deg g, c(g))$ in the lexicographical order on $\mathbb{N} \times \mathbb{N}$.

Let $a_i \in \mathcal{F}$ for $i = 1, \ldots, n$, and define a Δ-ring structure $\mathcal{F}[z_1, \ldots, z_n]_\Delta$ on $\mathcal{F}[z_1, \ldots, z_n]$ by $z_i' = a_i$ for $i = 1, \ldots, n$.

Lemma 2.138 *Assume that $a_1, \ldots, a_n$ are linearly independent over $\mathcal{C}$ modulo $\delta\mathcal{F}$. For each $g \in \mathcal{F}[z_1, \ldots, z_n]_\Delta$ of degree d,* $\deg g' \geq d - 1$. *If $g \neq 0$ and at least one of the non-zero coefficients of a term of degree d is in $\mathcal{C}$, then* level$(g') <$ level(g).

Proof: Write g in the form

$$g = \Sigma_{\deg M = d}\ \alpha_M M + \Sigma_{\deg N = d-1}\ \ \alpha_N N + P$$

where $\alpha_M, \alpha_N \in \mathfrak{F}$, $P \in \mathfrak{F}[z_1, \ldots, z_n]$ and deg $P < d - 1$. Then, since

$$\delta(\alpha_M M) = \delta\alpha_M M + \Sigma_i\ \Sigma_{z_i | M}\ \ n_i \alpha_M a_i \frac{M}{z_i}$$

for a monomial M of positive degree and integers n_i,

$$g' = \Sigma_{\deg M = d}\ \alpha'_M M + \Sigma_{\deg\ N = d-1}\ \ (\alpha'_N + \Sigma_{\deg\ L = d,\ L = N z_i}\ \ n_{L,N} \alpha_L a_i) N + Q$$

where $n_{L,N}$ are positive integers, $Q \in \mathfrak{F}[z_1, \ldots, z_n]$ and deg $Q < d - 1$.

Assume that $\alpha'_M \neq 0$ for at least one monomial M of degree d in g. Then deg $g' =$ deg $g > d - 1$. If also $\alpha'_{M'} = 0$ for at least one monomial M' of degree d in g, then $c(g') < c(g)$. Therefore, level$(g') <$ level(g).

Assume the negative of the assumption of the last paragraph: $\alpha'_M = 0$ for all monomials M of degree d in g. If $g \neq 0$, then deg $g' <$ deg g. Therefore, level$(g') <$ level(g). To show deg $g' \geq d - 1$, first assume deg $g \leq 0$. Then $g \in \mathfrak{C}$, and deg $g' = -1 \geq d - 1$. On the other hand, if deg $g > 0$, choose a monomial N of degree $d - 1$ such that, for some i, $N z_i$ is present in g, i.e., $a_{N z_i} \neq 0$. In g', the coefficient of N, $a'_N + \Sigma_{L = N z_i} \alpha_L a_i$, is not equal to 0 because $a_1, \ldots, a_n$ are assumed in 1 to be linearly independent over $\mathfrak{C}$ modulo $\mathfrak{F}'$. This proves deg $g' = d - 1$. ☐

Lemma 2.139 *Assume that $a_1, \ldots, a_n$ are linearly independent over $\mathfrak{C}$ modulo $\delta\mathfrak{F}$. The Δ-$\mathfrak{F}$-ring $\mathfrak{F}[z_1, \ldots, z_n]_\Delta$ is Δ-simple, i.e., has no proper non-trivial Δ-ideal.*

Proof: Let $\mathfrak{P} \subset \mathcal{F}[z_1, \ldots, z_n]_\Delta$ be a proper Δ-ideal. Assume there exists a nonzero element of $\mathfrak{P}$. Let $g \in \mathfrak{P}$ have the lowest level of all nonzero elements of $\mathfrak{P}$. Since $\mathfrak{P}$ is proper and, therefore, has no non-zero elements of degree 0, $d = \deg\, g > 0$. Multiply g by a non-zero element of $\mathcal{F}$ to ensure that one of the terms of degree d has 1 for a coefficient. This new non-zero element, which again is denoted by g, is also in $\mathfrak{P}$ and has level less than or equal to all of the non-zero elements of $\mathfrak{P}$. By Lemma 2.138, $\text{level}(g') < \text{level}(g)$. Since $g' \in \mathfrak{P}$, $g' = 0$. However, by the first part of the same lemma, $-1 = \deg\, g' \geq d - 1 \geq 0$ since $d > 0$. This contradiction shows $\mathfrak{P}$ is the zero Δ-ideal. $\square$

Proposition 2.140 *Let $\mathcal{U}$ be Δ-universal extension of the Δ-field $\mathcal{F}$, and let $\mathcal{C} = \mathcal{F}^\Delta$. For $i = 1, \ldots, n$, let $a_i \in \mathcal{F}$, and let $b_i \in \mathcal{U}$ be such that $b_i' = a_i$. The following four conditions are equivalent:*

1. *$a_1, \ldots, a_n$ are linearly independent over $\mathcal{C}$ modulo $\delta\mathcal{F}$,*
2. *$b_1, \ldots, b_n$ are algebraically independent over $\mathcal{F}$, and $\mathcal{F}\{b_1, \ldots, b_n\}_\Delta$ is Δ-simple,*
3. *$1, b_1, \ldots, b_n$ are linearly independent over $\mathcal{F}$, and $(\mathcal{F}\{b_1, \ldots, b_n\}_\Delta)^\Delta = \mathcal{C}$,*
4. *$1, b_1, \ldots, b_n$ are linearly independent over $\mathcal{F}$, and $(\mathcal{F}\langle b_1, \ldots, b_n\rangle_\Delta)^\Delta = \mathcal{C}$.*

Proof: $1 \Longrightarrow 2$. Define a Δ-ring structure $\mathcal{F}[z_1, \ldots, z_n]_\Delta$ on $\mathcal{F}[z_1, \ldots, z_n]$ by $z_i' = a_i$ for $i = 1, \ldots, n$. Clearly, $\mathcal{F}\{z_1, \ldots, z_n\}_\Delta = \mathcal{F}[z_1, \ldots, z_n]_\Delta$. To show $\mathfrak{F}\{b_1, \ldots, b_n\}_\Delta = \mathfrak{F}[b_1, \ldots, b_n]_\Delta$ is Δ-simple, define a surjective Δ-$\mathfrak{F}$-homomorphism $\rho : \mathfrak{F}[z_1, \ldots, z_n]_\Delta \to \mathfrak{F}[b_1, \ldots, b_n]_\Delta$ over $\mathfrak{F}$ by $\rho(z_i) = b_i$ for $i = 1, \ldots, n$. Then ρ is a Δ-$\mathfrak{F}$-isomorphism because the kernel of ρ, which is a Δ-ideal, must be the zero ideal by Lemma 2.139. Therefore, $\mathfrak{F}\{b_1, \ldots, b_n\}_\Delta$, the codomain of ρ, also has no non-trivial Δ-ideal, and $b_1, \ldots, b_n$ are algebraically independent over $\mathfrak{F}$ because $z_1, \ldots, z_n$ are algebraically independent over $\mathfrak{F}$ and $\rho(z_i) = b_i$ for every i.

$2 \implies 3$. Let g be a non-zero element of $(\mathfrak{F}\{b_1, \ldots, b_n\}_\Delta)^\Delta$. Because g is a Δ-constant, $(g) \subset \mathfrak{F}\{b_1, \ldots, b_n\}_\Delta$ is a Δ-ideal and must be the unit Δ-ideal by 2. Because, by assumption, $\mathfrak{F}\{b_1, \ldots, b_n\}_\Delta$ is a polynomial ring in the algebraically independent indeterminates $b_1, \ldots, b_n$, $g \in \mathfrak{F}$ and $g \in \mathfrak{C} = \mathfrak{F}^\Delta$. That $b_1, \ldots, b_n$ are algebraically independent over $\mathfrak{F}$ clearly implies that $1, b_1, \ldots, b_n$ are linearly independent over $\mathfrak{F}$.

$3 \implies 1$. Assume $a_1, \ldots, a_n$ are linearly dependent over $\mathfrak{C}$ modulo $\delta\mathfrak{F}$, i.e., $\Sigma_i\ \alpha_i a_i = \delta f$ for $\alpha_i \in \mathfrak{C}$ and $f \in \mathfrak{F}$. Since $1, b_1, \ldots, b_n$ are assumed to be linearly independent over $\mathfrak{F}$, the element $\Sigma_i\ \alpha_i b_i - f$ $(\in \mathfrak{F}\{b_1, \ldots, b_n\}_\Delta)$ is not in $\mathfrak{F}$ and is a Δ-constant of $\mathfrak{F}\{b_1, \ldots, b_n\}_\Delta$. This contradicts 3. Therefore, the $a_1, \ldots, a_n$ are linearly independent over $\mathfrak{C}$ modulo $\delta\mathfrak{F}$. This proves 1.

$3 \iff 4$. For the non-obvious implication, assume 3. Assume $g \in \mathfrak{F}\langle b_1, \ldots, b_n\rangle^\Delta$. Then

$$\mathfrak{a} = \{a \in \mathfrak{F}\{b_1, \ldots, b_n\}_\Delta \mid ag \in \mathfrak{F}\{b_1, \ldots, b_n\}_\Delta\}$$

is a Δ-ideal because g is a Δ-constant. Since it is non-zero and $\mathfrak{F}\{b_1, \ldots, b_n\}_\Delta$ is Δ-simple, $1 \in \mathfrak{a}$, which implies $g \in \mathfrak{F}\{b_1, \ldots, b_n\}^\Delta$. $\square$

Remark 2.141 *If* $a_1, \ldots, a_n$ *are linearly independent over* $\mathfrak{C}$ *modulo* $\delta\mathfrak{F}$ (Part 1. of the proposition)*, then, for each* i*, no* Δ*-zero* c_i *of* $y' - a_i$ *is in* $\mathfrak{F}$*. In particular,* $b_i \notin \mathfrak{F}$ *for all* i*.*

Proof: If some $c_i \in \mathfrak{F}$, then $a_1, \ldots, a_n$ would not be linearly independent over $\mathfrak{C}$ modulo $\mathfrak{F}'$ because $c_i' = a_i$. $\square$

Remark 2.142 *If* $n = 1$*, the converse of the previous* remark *is true: if no* Δ*-zero of* $y' - a_1$ *is in* $\mathfrak{F}$*, then* a_1 *is linearly independent over* $\mathfrak{C}$ *modulo* $\mathfrak{F}'$ (Part 1 of the proposition).

Proof: If a_1 is linearly dependent over $\mathfrak{C}$ modulo $\mathfrak{F}'$, there is a Δ-zero of $y' - a_1$ in $\mathfrak{F}$. $\square$

Corollary 2.143 *If no* Δ*-zero of* $y' - a_1$ *is in* $\mathfrak{F}$*, then* b_1 *is transcendental over* $\mathfrak{F}$*.*

Proof: By the last remark, a_1 is linearly independent over $\mathfrak{C}$ modulo $\mathfrak{F}'$. Part 1. $\Longrightarrow$ Part 2. of the proposition implies b_1 is transcendental over $\mathfrak{F}$.$\square$

Corollary 2.144 (The Ostrowski Theorem) *If* $b_1, \ldots, b_n$ *are algebraically dependent over* $\mathfrak{F}$ *and* $(\mathfrak{F}\langle b_1, \ldots, b_n\rangle_\Delta)^\Delta = \mathfrak{C}$*, then* $1, b_1, \ldots, b_n$ *are linearly dependent over* $\mathfrak{F}$*.*

Proof: (See [24, Exercise 4, page 407] or [23, page 1155].) The contrapositive of $4 \Longrightarrow 2$ is that, if $\mathcal{F}\{b_1, \dots, b_n\}_\Delta$ has a non-trivial Δ-ideal or $b_1, \dots, b_n$ are algebraically dependent over $\mathcal{F}$, then $(\mathcal{F}\langle b_1, \dots, b_n\rangle_\Delta)^\Delta \neq \mathcal{C}$ or $1, b_1, \dots, b_n$ are linearly dependent over $\mathcal{F}$. Therefore, if $b_1, \dots, b_n$ are algebraically dependent over $\mathcal{F}$ and $(\mathcal{F}\langle b_1, \dots, b_n\rangle_\Delta)^\Delta = \mathcal{C}$, then $1, b_1, \dots, b_n$ are linearly dependent over $\mathcal{F}$. □

Corollary 2.145 *Let $\mathcal{U}$ be Δ-universal extension of the constant field $\mathcal{C}$. Let $x \in \mathcal{U}$ be a Δ-zero of $y' - 1 \in \mathcal{C}\{y\}_\Delta$. For $i = 1, \dots, n$, let $c_i \in \mathcal{C}$ such that $c_i \neq c_j$ if $i \neq j$, and let $b_i \in \mathcal{U}$ be a Δ-zero of the Δ-polynomial $y' - \frac{1}{x+c_i} \in \mathcal{C}(x)\{y\}_\Delta$. Then $b_1, \dots, b_n$ are algebraically independent over $\mathcal{C}(x)$, and $(\mathcal{C}(x)\langle b_1, \dots, b_n\rangle_\Delta)^\Delta = \mathcal{C}$.*

Proof: By Corollary 2.135 with irreducible $p_i(x) = x + c_i$ for $i = 1, \dots, n$, $\frac{1}{x+c_1}, \dots, \frac{1}{x+c_n}$ are linearly independent over $\mathcal{C}$ modulo $\delta(\mathcal{C}(x))$. Then apply Proposition 2.140. □

Corollary 2.146 *Let $\mathcal{U}$ be Δ-universal extension of Δ-field $\mathcal{F}$, and let $\mathcal{C} = \mathcal{F}^\Delta$.*

1. *The Δ-ideal $\mathfrak{P} = [y'' + (y')^2]_\Delta \subset \mathcal{F}\{y\}_\Delta$ is prime.*

   If $\eta \in \mathcal{U}$ is an $\mathcal{F}$-generic Δ-zero of $\mathfrak{P}$, then $\eta' \neq 0$.

2. *Let $\zeta \in \mathcal{U}$ be any Δ-zero of $\mathfrak{P}$ such that $\zeta' \neq 0$. Let $x = 1/\zeta'$. Then $x' = 1$, $\zeta = 1/x$, x is transcendental over $\mathcal{C}$, and the field $\mathcal{F}\langle\zeta\rangle_\Delta = \mathcal{F}(\zeta, x)$.*

(a) If no Δ-zero of the Δ-polynomial $y'-1/x \in \mathcal{F}(x)\{y\}_\Delta$ is in $\mathcal{F}(x)$, then ζ and x are algebraically independent over $\mathcal{C}$.

(b) If $\mathcal{F} = \mathcal{C}$, then no Δ-zero of the Δ-polynomial $y' - 1/x \in \mathcal{F}(x)\{y\}_\Delta$ is in $\mathcal{F}(x)$, ζ is $\mathcal{F}$-generic Δ-zero of the prime Δ-ideal $[y'' + (y')^2]_\Delta \subset \mathcal{F}\{y\}_\Delta$, and $(\mathcal{F}\langle\eta\rangle_\Delta)^\Delta = \mathcal{F}$.

Proof: 1. For the ranking [24, page 75] defined by order, the separant of $y'' + (y')^2$ is 1. Therefore, the Δ-ideal $\mathfrak{P}$ is prime by Lemma 2.136. Since the order of every element of $\mathfrak{P}$ is greater than 1 and the order of y' is one, $y' \notin \mathfrak{P}$, and $\eta' \neq 0$ since η is $\mathcal{F}$-generic.

2. Let $\zeta \in \mathfrak{U}$ be any Δ-zero of $\mathfrak{P}$ such that $\zeta' \neq 0$. Then $(1/\zeta')' = -\zeta''/(\zeta')^2 = (\zeta')^2/(\zeta')^2 = 1$. So that, $x' = (1/\zeta')' = 1$, and $\zeta' = 1/x$. Since $\zeta' = 1/x$, all higher derivatives of ζ are rational functions of x, and the field $\mathcal{C}\langle\zeta\rangle_\Delta = \mathcal{C}(\zeta, x)$.

2(a). If no Δ-zero of $y - 1/x \in \mathcal{C}(x)\{y\}_\Delta$ is in $\mathcal{F}(x)$, ζ is transcendental over $\mathcal{F}(x)$ by Corollary 2.8.1. Since x is transcendental over $\mathcal{C}$ by Lemma 2.137, x and η are algebraically independent over $\mathcal{C}$.

2(b). If $\mathcal{F}^\Delta = \mathcal{F}$, by Corollary 2.135 for $n = 1$ and $p_1(x) = x$, $1/x \notin \delta\mathcal{F}(x)$, and $1/x$ is linearly independent over $\mathcal{F}$ modulo $\delta\mathcal{F}(x)$. By Remark 2.141, no Δ-zero of the Δ-polynomial $y' - 1/x \in \mathcal{F}(x)\{y\}_\Delta$ is in $\mathcal{F}(x)$.

To show ζ is a $\mathcal{F}$-generic Δ-zero of $\mathfrak{P}$, observe that any $f(y) \in \mathcal{F}\{y\}_\Delta$ may be written in the form

$$f(y) = \Sigma_{i,j}\, c_{i,j} y^i (y')^j + p(y)$$

with $c_{i,j} \in \mathcal{F}$ and $p(y) \in \mathfrak{P}$. If $f(\zeta) = 0$, then $\Sigma_{i,j}\ c_{i,j}\zeta^i/x^j = 0$. Since x and ζ are algebraically independent over $\mathcal{F} = \mathcal{C}$ (part 2(a)), all $c_{i,j} = 0$, and $f(y) = p(y) \in \mathfrak{P}$. The last statement follows from the last statement of Corollary 2.145. $\square$

Definition 2.147 *Let $\mathcal{W}$ be a Δ-vector space over a Δ-field $\mathcal{F}$. Any set $\Sigma \subseteq \mathcal{W}$ is* Δ-linearly independent over $\mathcal{F}$ *if the family $(\theta\alpha)_{\theta\in\Theta,\alpha\in\Sigma}$ is linearly independent over $\mathcal{F}$. Let $\mathcal{R}$ be a Δ-ring. A family $(\alpha_i)_{i\in I}$ of elements of a Δ-overring of $\mathcal{R}$ is* Δ-algebraically independent over $\mathcal{R}$ *or, more simply,* Δ-$\mathcal{R}$-algebraically independent, *or* Δ-$\mathcal{R}$-independent, *if the family $(\theta\alpha)_{\theta\in\Theta,\alpha\in\Sigma}$ is algebraically independent over $\mathcal{R}$.*

Corollary 2.148 *Let $\mathcal{U}$ be (E,Δ)-universal extension of the (E,Δ)-field $\mathcal{F}$, and let $\mathcal{C} = \mathcal{F}^{\Delta}$. For $i = 1, \dots, n$, let $a_i \in \mathcal{F}$, and let $b_i \in \mathcal{U}$ be such that $b_i' = a_i$. The following four conditions are equivalent:*

1. *$a_1, \dots, a_n$ are* E-*linearly independent over $\mathcal{C}$ modulo $\delta\mathcal{F}$,*
2. *$b_1, \dots, b_n$ are* E-*algebraically independent if the family over $\mathcal{F}$, and $\mathcal{F}\{b_1, \dots, b_n\}_{\mathrm{E},\Delta}$ is Δ-simple,*
3. *$1, b_1, \dots, b_n$ are* E-*linearly independent over $\mathcal{F}$, and $(\mathcal{F}\{b_1, \dots, b_n\}_{\mathrm{E},\Delta})^{\Delta} = \mathcal{C}$,*
4. *$1, b_1, \dots, b_n$ are* E-*linearly independent over $\mathcal{F}$, and $(\mathcal{F}\langle b_1, \dots, b_n\rangle_{\mathrm{E},\Delta})^{\Delta} = \mathcal{C}$.*

Proof: For each positive integer ν, let $\Psi(\nu)$ be the set of monomials in E of order less than or equal to ν. Then for each ν, i and $\psi \in \Psi(\nu)$, ψb_i is a Δ-zero of the Δ-polynomial $y' - \psi a_i$. Since $\mathcal{U}$ is clearly also a Δ-universal extension of the Δ-field $\mathcal{F}$, Proposition 2.140 may be applied to the families $(\psi b_i)_{\psi \in \Psi(\nu), i=1,\ldots,n}$ and $(\psi a_i)_{\psi \in \Psi(\nu), i=1,\ldots,n}$ for each ν.

The equivalence of the four parts of the proposition may be verified by the following four observations which are true because each E-algebraic relation only has a finite number of E-derivatives:

1. $(\Psi_i a_i)_{\psi \in \Psi(\nu), 1 \leq i \leq n}$ are linearly independent over $\mathcal{C}$ modulo $\delta\mathcal{F}$ for all ν if and only if the $a_1, \ldots, a_n$ are E-linearly independent over $\mathcal{C}$ modulo $\delta\mathcal{F}$,
2. $(\Psi_i a_i)_{\psi \in \Psi(\nu), 1 \leq i \leq n}$ are algebraically independent over $\mathcal{C}$ modulo $\delta\mathcal{F}$ for all ν if and only if the $b_1, \ldots, b_n$ are E-algebraically independent over $\mathcal{C}$ modulo $\delta\mathcal{F}$,
3. $\mathcal{F}\{(\psi b_i)_{\psi \in \Psi(\nu), 1 \leq i \leq n}\}_{\mathrm{E},\Delta}$ is Δ-simple for all ν if and only if $\mathcal{F}\{b_i\}_{\mathrm{E},\Delta}$ is Δ-simple,
4. $(\mathcal{F}\{(\psi b_i)_{\psi \in \Psi(\nu), 1 \leq i \leq n}\}_{\mathrm{E},\Delta})^{\Delta} = \mathcal{C}$ for all ν if and only if $(\mathcal{F}\{b_i\}_{\mathrm{E},\Delta})^{\Delta} = \mathcal{C}$.

□

In Section 2.8.3, Corollary 2.148 will be used to construct an example similar to one of Johnson, Reinhart and Rubel [15].

Example 2.149 *Let $\mathcal{U}$ be Δ-universal extension of a field of Δ-constants $\mathcal{C}$. Let η be a Δ-zero in $\mathcal{U}$ of $y'' + (y')^2 \in \mathcal{C}\{y\}_{\Delta}$ such that $\eta' \neq 0$. (Such*

Δ-zeros are of the form η is of the form $a+log(x+b)$ for Δ-constants a and b.) Then the Δ-constants of $\mathcal{C}\langle\eta\rangle_\Delta$ are $\mathcal{C}$, and $\mathcal{C}\langle\eta\rangle_\Delta$ over $\mathcal{C}$ is not strongly normal (in the sense of Kolchin). But some non-trivial Δ-$\mathcal{C}$-isomorphisms are strong.

Corollary 2.146 will be applied with $\mathcal{F} = \mathcal{C}$ and $\xi = \eta$. By Part 2 of the corollary, $x' = 1$ where $x = 1/\eta'$. By Part 2(b), no Δ-zero of $y' - 1/x$ is in $\mathcal{C}(x)$ since $\mathcal{C} = \mathcal{C}^\Delta$, and the Δ-constants of $\mathcal{C}\langle\eta\rangle_\Delta$ equals $\mathcal{C}$. By Part 2(a) of the corollary, η and x are algebraically independent over $\mathcal{C}$.

Each Δ-$\mathcal{C}$-isomorphism σ of $\mathcal{C}(x)$ is strong and is of the form $\sigma(x) = x+k$ for $k \in \mathcal{K} = \mathcal{U}^\Delta$ [24, Example A, page 404]. Since $\mathcal{C}(x)$ over $\mathcal{C}$ is finitely Δ-generated and since $\mathcal{C}\langle\eta\rangle_\Delta$ over $\mathcal{C}(x)$ is finitely Δ-generated (Part 2 of Corollary 2.146), σ extends to a non-unique Δ-$\mathcal{C}$-isomorphism $\overline{\sigma} : \mathcal{C}\langle\eta\rangle_\Delta \to \mathcal{U}$ by Proposition 1.52. Note that $\eta' = 1/x \in \mathcal{C}(x)$ and $(\overline{\sigma}\eta)' = \overline{\sigma}\eta' = \overline{\sigma}(1/x) = \sigma(1/x) = 1/\sigma x = 1/(x+k) \in \mathcal{C}(x+k) \subset \mathcal{K}(x)$.

If $k \neq 0$, Corollary 2.145 implies $(\mathcal{C}\langle\eta, \overline{\sigma}\eta\rangle_\Delta)^\Delta = (\mathcal{C}(x)\langle\eta, \overline{\sigma}\eta\rangle_\Delta)^\Delta = \mathcal{C}$ and η and $\overline{\sigma}\eta$ are algebraically independent over $\mathcal{C}(x)$ since $\eta' = 1/x$ is not equal to $(\overline{\sigma}\eta)' = 1/(x+k) \in \mathcal{K}(x)$. Consequently, $\overline{\sigma}\eta \notin \mathcal{K}\mathcal{C}\langle\eta\rangle_\Delta$, and $\overline{\sigma}$ is not strong.

If $k = 0$, $(\overline{\sigma}\eta)' = \eta' \in \mathcal{C}(x)$. Therefore, $\overline{\sigma}$ is a strong Δ-isomorphism of $\mathcal{C}\langle\eta\rangle_\Delta$ over $\mathcal{C}(x)$ of the form $\overline{\sigma}\eta = \eta + d$ for $d \in \mathcal{K}$ and, also, a strong Δ-isomorphism of $\mathcal{C}\langle\eta\rangle_\Delta$ over $\mathcal{C}$. The set of all specializations of $\overline{\sigma}$ is a pre $\mathcal{C}$-set by Proposition 2.131.

2.8.2 Example 2

The second example of this section is of a Δ-field extension $\mathfrak{G}$ over $\mathfrak{F}$ such that the image of $\phi_\eta(X)$ is not open in $\mathfrak{Z}_{\mathfrak{G}}(\mathfrak{G}\mathfrak{P}_{\eta,\mathfrak{F}})$ (See the paragraph before Definition 1.34 for the definition of X and the paragraph before Corollary 1.36 for the definition of $\phi_\eta(X)$).

Example 2.150 *Let* $\mathfrak{F} = \mathfrak{C}\langle x\rangle_\Delta = \mathfrak{C}(x)$ *where* $x' = 1$, *and let* $\mathfrak{C}$ *be a* Δ-*field of* Δ-*constants. Let* $\eta \in \mathfrak{U}$ *be an* $\mathfrak{F}$-*generic* Δ-*zero of* $\mathfrak{P} = [y'' + (y')^2]_\Delta \subset \mathfrak{F}\{y\}_\Delta$ *such that* $\eta' \neq 0$. *Let* $\mathfrak{G} = \mathfrak{F}\langle\eta\rangle_\Delta$, *and let* $X = \mathrm{Isom}^\Delta_{\mathfrak{F}}(\mathfrak{G}, \mathfrak{U})$. *Then* $\phi_\eta(X)$ *is not open in* $\mathfrak{Z}_{\mathfrak{G}}(\mathfrak{G}\mathfrak{P})$.

By Corollary 2.146, $\mathfrak{P}$ is a prime Δ-ideal. For each $c \in \mathfrak{C}$, the Δ-ideal $\mathfrak{P}_c = [y' - \frac{1}{x+c}]_\Delta \subset \mathfrak{F}\{y\}$ is prime by Lemma 2.136. It contains $\mathfrak{P}$ because

$$y'' + (y')^2 = (y' - \frac{1}{x+c})(y' + \frac{1}{x+c}) + (y' - \frac{1}{x+c})'$$

and is proper because $\mathfrak{P}$ has no element of order 1.

Let ζ_c be an $\mathfrak{F}$-generic Δ-zero of $\mathfrak{P}_c$ for $c \in \mathfrak{C}$: that is $\mathfrak{A}_{\mathfrak{F}}(\{\zeta_c\}) = \mathfrak{P}_c$. Clearly, all the ζ_c for $c \in \mathfrak{C}$ are distinct. Because $\mathfrak{P}_c \supset \mathfrak{P}$ and, therefore, $\mathfrak{G}\mathfrak{P}_c \supset \mathfrak{G}\mathfrak{P}$, it follows that $\zeta_c \in \mathfrak{Z}_{\mathfrak{G}}(\mathfrak{G}\mathfrak{P})$. Since ζ_c is an $\mathfrak{F}$-generic Δ-zero of $\mathfrak{P}_c$, the Δ-ideal $\mathfrak{A}_{\mathfrak{G}}(\{\zeta_c\}) \cap \mathfrak{F}\{y\}_\Delta = \mathfrak{P}_c$. Because this properly contains $\mathfrak{P}$, as shown above, the element $\zeta_c \notin \phi_\eta(X)$ by Lemma 1.38.

Because $\phi_\eta(\mathrm{id}) = \eta \in \phi_\eta(X)$, $\phi_\eta(X)$ is not empty. If $\phi_\eta(X)$ were open in $\mathfrak{Z}_{\mathfrak{G}}(\mathfrak{G}\mathfrak{P})$, by the definition of the $\mathfrak{G}$-Δ-Zariski topology on $\mathfrak{Z}_{\mathfrak{G}}(\mathfrak{G}\mathfrak{P})$, $\phi_\eta(X)$ would equal the complement of $\mathfrak{Z}_{\mathfrak{G}}(\{f_1, \ldots, f_d\}_\Delta) \cap \mathfrak{Z}_{\mathfrak{G}}(\mathfrak{G}\mathfrak{P})$ in $\mathfrak{Z}_{\mathfrak{G}}(\mathfrak{G}\mathfrak{P})$ for

$f_i \in \mathfrak{G}\{y\}_\Delta$ and $f_i \notin \mathfrak{G}\mathfrak{P}$, $i = 1, \ldots, d$. It will be shown that each such f_i has only a finite number of Δ-zeros of the form ζ_c, for $c \in \mathfrak{C}$. Since $\mathfrak{C}$ is infinite, there is one, in fact an infinite number, $\zeta_c \in \mathfrak{Z}_\mathfrak{G}(\mathfrak{G}\mathfrak{P})$ such that $\zeta_c \notin \mathfrak{Z}_\mathfrak{G}(\{f_1, \ldots, f_d\}_\Delta)$. But, because $\zeta_c \notin \phi_\eta(X)$, this will prove $\phi_\eta(X)$ is not open in $\mathfrak{Z}_\mathfrak{G}(\mathfrak{G}\mathfrak{P})$.

Each $f_i = f_i(y) \in \mathfrak{G}\{y\}_\Delta$ may be written as $f_i(y) = \Sigma_{j,k} a_{j,k} y^k (y')^j + p(y)$ where $a_{j,k} \in \mathfrak{G}$ and $p(y) \in \mathfrak{G}\mathfrak{P}$. By Corollary 2.146, $\mathfrak{G} = \mathfrak{C}\langle x, \eta\rangle_\Delta = \mathfrak{C}(x, \eta)$ with $x = 1/\eta'$, $x' = 1$, $\eta' = 1/x$ and x and η are algebraically independent over $\mathfrak{C}$. Therefore, each $a_{j,k} = g_{j,k}/h_{j,k}$ for polynomials $g_{j,k}, h_{j,k} \in \mathfrak{C}[x, \eta]$, $h_{j,k} \neq 0$. Then

$$f_i(y) = \Sigma_{j,k}\, \frac{g_{j,k}}{h_{j,k}} y^k (y')^j + p(y) = \frac{1}{q}\, (\Sigma_{j,k}\, q_{j,k} y^k (y')^j) + p(y).$$

such that $q \neq 0$ is the product of the denominators $h_{j,k}$ and $q_{j,k} \in \mathfrak{C}[x, \eta]$. Because $x, \eta, y, y' \in \mathfrak{G}\{y\}_\Delta$ are algebraically independent over $\mathfrak{C}$, $\Sigma_{j,k}\, q_{j,k} y^k (y')^j$ may be written in the form $\Sigma_{l,m}\, r_{l,m}(y')\eta^l y^m$ for $r_{l,m}(y') \in \mathfrak{C}[x][y']$. As a result of these considerations, $f_i(y)$ may be written in the form

$$f_i(y) = \frac{1}{q}\, (\Sigma_{l,m}\, r_{l,m}(y')\eta^l y^m) + p(y). \tag{13}$$

Therefore,

$$q(f_i(y) - p(y)) = \Sigma_{l,m}\, r_{l,m}(y')\eta^l y^m. \tag{14}$$

Assume $c \neq 0$ and ζ_c is a Δ-zero of $f_i(y)$. Since $\eta' = \frac{1}{x}$, $\zeta_c' = \frac{1}{x+c}$ and $c \neq 0$, η and ζ_c are algebraically independent over $\mathfrak{C}(x)$ by Corollary 2.145. By evaluating both sides of Equation (14) at $y = \zeta_c$,

$$0 = \Sigma_{l,m}\, r_{l,m}(\zeta_c')\eta^l (\zeta_c)^m = \Sigma_{l,m}\, r_{l,m}(\frac{1}{x+c})\eta^l (\zeta_c)^m, \tag{15}$$

this implies $0 = r_{l,m}(\frac{1}{x+c}) \in \mathcal{C}(x)$ for all l and m.

Let $r_{l,m}(y') = p_r(x)(y')^r + \ldots + p_0(x) \in \mathcal{C}[x][y']$ for $p_r(x), \ldots, p_0(x) \in \mathcal{C}[x]$ such that $p_r(x) \neq 0$. If $r \geq 1$, $0 = r_{l,m}(\frac{1}{x+c}) = p_r(x)(\frac{1}{x+c})^r + \ldots + p_0(x)$ implies $x + c$ divides $p_r(x)$.

Therefore, if $f_i(y)$ has an infinite number of such Δ-zeros ζ_c ($c \in \mathcal{C}$), $p_r(x)$, which is divisible by $x + c$ for an infinite number of $c \in \mathcal{C}$, must be equal to 0. This implies that the degree r of $r_{l,m}(y')$ in y' is zero for all l and m, i.e., $r_{l,m}(y') = p_0(x) \in \mathcal{C}[x]$. The equation (15) and the algebraic independence of η and ζ over $\mathcal{C}(x)$ imply $0 = r_{l,m}(y')$ for all l and m. Since $q \in \mathcal{G}$ is not zero, Equation (13) implies $f_i(y) = p(y) \in \mathfrak{P}$ by , which contradicts the definition of f_i.

2.8.3 An Application of the Techniques Developed

The main objective of [15] by Johnson, Reinhart and Rubel is to construct a prime (E, Δ)-ideal $\mathfrak{P} \subset \mathcal{F}\{y\}_{E,\Delta}$ such that all (E, Δ)-zeros $\zeta \in \mathcal{U}$ of $\mathfrak{P}$ generate (E, Δ)-field extensions $\mathcal{F}\langle\zeta\rangle_{E,\Delta}$ over $\mathcal{F}$ that have infinite transcendence degree over $\mathcal{F}$. Since examples of such prime ideals follow simply from the techniques just developed, it is worth making a short digression to present because it is of general interest. Recall $\Delta = \{\delta\}$ is this section.

Lemma 2.151 *Let z be an* (E, Δ)*-indeterminate over the* (E, Δ)*-field* $\mathcal{H}$. *Let $a \in \mathcal{H}\langle z\rangle_{\mathrm{E}}$ and $a \notin \mathcal{H}$. Then*

1. 1 *and a are* E*-linearly independent over* $\mathcal{H}$,

2. $a \notin \delta\mathcal{H}\langle z\rangle_{\mathrm{E},\Delta}$, i.e., a has no primitive in $\mathcal{H}\langle z\rangle_{\mathrm{E},\Delta}$,

3. $(\mathcal{H}\langle z\rangle_{\mathrm{E},\Delta})^{\Delta} = \mathcal{H}^{\Delta}$ and

4. a is E*-linearly independent over $\mathcal{H}^{\Delta}$ modulo $\delta\mathcal{H}\langle z\rangle_{\mathrm{E},\Delta}$.*

Proof:

1. If a and 1 were E-linearly dependent over $\mathcal{H}$, then a would satisfy a linear E-polynomial with coefficients in $\mathcal{H}$. However, each element of $\mathcal{H}\langle z\rangle_{\mathrm{E}}$ not in $\mathcal{H}$ is E-algebraically independent over the E-field $\mathcal{H}$ ([24, Exercise 8, page 159] and for a proof see Exercise 5.226 of this paper).

2. Let $\mathrm{E} = \{\epsilon_1, \ldots, \epsilon_n\}$ and choose the ranking on the (E, Δ)-indeterminate z such that the rank of $\delta^r\epsilon_1^{r_1}, \cdots, \epsilon_n^{r_n} z$ is $(r, r_1, \ldots, r_n)$ in the lexicographical order on $\mathbb{N}^{n+1}$. Extend this to a ranking of $\mathcal{H}\{z\}_{\mathrm{E},\Delta}$. For an element $f \in \mathcal{H}\{z\}_{\mathrm{E},\Delta}$, let S_f denote the separant of f.

   Let $b \in \mathcal{H}\langle z\rangle_{E,\Delta}$ be represented as the quotient c/d with $c, d \in \mathcal{H}\{z\}_{\mathrm{E},\Delta}$ and $d \neq 0$ such that the maximum of the rank of c and the rank of d is the least possible among all such representations. Let w be the highest ranking derivative of z present in c or d.

   Suppose $a = b'$ where $b = c/d$ as above. If the rank of $w = (0, \ldots, 0)$, then $c \in \mathcal{H}$, $d \in \mathcal{H}$, $c/d \in \mathcal{H}$, and $a = (c/d)' \in \mathcal{H}$. Assume the rank of w is greater than $(0, \ldots, 0)$ and write $(c/d)' =$

$$\frac{c' \cdot d - c \cdot d'}{d^2} = \frac{S_c d - c S_d}{d^2} w' + \frac{\text{terms of rank } < \text{ rank } w'}{d^2}.$$

If $(S_c d - cS_d) \neq 0$, then $(c/d)' \notin \mathcal{H}\langle z\rangle_{\mathrm{E}}$ because $w' \notin \mathcal{H}\langle z\rangle_{\mathrm{E}}$. If $(S_c d - cS_d) = 0$, then, since $c \neq 0$ and $d \neq 0$, $S_d \neq 0$ because otherwise $S_c = 0$. But $S_c/S_d = c/d = b$ is a representation of b such that S_c and S_d have lower rank than c and d, which is contrary to the assumptions on c and d.

3. For each positive integer ν, let $\Psi(\nu)$ be the monomials in E of order less than or equal to ν. Since $(\psi z)_{\psi\in\Psi(\nu)}$ is a finite set of Δ-indeterminates over $\mathcal{H}$ and each Δ-constant of $\mathcal{H}\langle z\rangle_{E,\Delta}$ is in $\mathcal{H}\langle(\psi z)_{\psi\in\Psi(\nu)}\rangle_{\Delta}$ for some ν, Corollary 5.228 implies that $(\mathcal{H}\langle z\rangle_{E,\Delta})^{\Delta} = \mathcal{H}^{\Delta}$.

4. Every E-linear combination of a over $\mathcal{H}$ is not in $\mathcal{H}$ because a and 1 are E-linearly independent over $\mathcal{H}$ (part 1), is in $\mathcal{H}\langle z\rangle_{\mathrm{E}}$ by assumption, and not in $\delta\mathcal{H}\langle z\rangle_{E,\Delta}$ by part 2. Therefore a is E-independent over $\mathcal{H}$ modulo $\delta\mathcal{H}\langle z\rangle_{E,\Delta}$. A fortiori, a is E-linearly independent over $\mathcal{H}^{\Delta}$ modulo $\delta\mathcal{H}\langle z\rangle_{E,\Delta}$, since $\mathcal{H}^{\Delta} \subseteq \mathcal{H}$.

□

Proposition 2.152 *Let* E *be non-empty, and* $\Delta = \{\delta\}$. *Let* z *be an* (E,Δ)*-indeterminate over the* (E,Δ)*-field* $\mathcal{H}$. *And, let* y *be an* (E,Δ)*-indeterminate over the* (E,Δ)*-field* $\mathcal{F} = \mathcal{H}\langle z\rangle_{E,\Delta}$. *Let* $a \in \mathcal{H}\langle z\rangle_{\mathrm{E}}$, *and* $a \notin \mathcal{H}$. *Then for all* (E,Δ)*-zeros* b *of the prime* (E,Δ)*-ideal* $[\delta y - a]_{E,\Delta} \subset \mathcal{F}\{y\}_{E,\Delta}$, b *is* E*-algebraically independent over* $\mathcal{F}$, *and the algebraic transcendence degree of* $\mathcal{F}\langle b\rangle_{E,\Delta}$ *over* $\mathcal{F}$ *is infinite.*

Proof: By part 4 of Lemma 2.151, a is E-linearly independent over $\mathfrak{F}^{\Delta} = \mathfrak{H}^{\Delta}$ modulo $\delta(\mathfrak{F})$ (See [15, Theorem 5, page 59]). This is the condition 1 of Corollary 2.148. For any $b \in \mathfrak{U}$ that is an (E, Δ)-zero of the prime (E, Δ)-ideal $[\delta y - a]_{E,\Delta} \subset \mathfrak{F}\{y\}_{E,\Delta}$, condition 2 of Corollary 2.148 implies b is E-algebraically independent over $\mathfrak{F}$. Since E is non-empty, $\mathfrak{G} = \mathfrak{F}\langle b\rangle_{E,\Delta}$ has E-transcendence degree one over $\mathfrak{F}$ and has infinite algebraic transcendence degree over $\mathfrak{F}$. $\square$

3 Differential Algebraic Groups and E-Strongly Normal Extensions.

3.1 Differential Algebraic Groups

Definition 3.153 [25, page 33] *An* E-$\mathfrak{F}$*-group* (*relative to the universal* E-*field* $\mathcal{V}$) *is a set* G *which has both a group structure* (*usually written multiplicatively*) *and a pre* E-$\mathfrak{F}$*-set structure relative to the universal* E-*field* $\mathcal{V}$, *subject to the following axioms.*

DAG1a *If* $x_1, x_2 \in G$, *then* $\mathfrak{F}\langle x_1 x_2\rangle \subset \mathfrak{F}\langle x_1\rangle\mathfrak{F}\langle x_2\rangle$.

DAG1b *If* $x_1, x_2 \in G$, *then* $\mathfrak{F}\langle {x_1}^{-1} x_2\rangle \subset \mathfrak{F}\langle x_1\rangle\mathfrak{F}\langle x_2\rangle$.

DAG2a *If* $x_1, x_2, x_1', x_2' \in G$ *and* $x_1 \leftrightarrow x_1'$, $x_2 \leftrightarrow x_2'$, *and* $S_{x_1', x_1}, S_{x_2', x_2}$ *are compatible*[5], *then* $x_1 x_2 \longrightarrow x_1' x_2'$. *If moreover* $x_1 x_2 \leftrightarrow x_1' x_2'$, *and* h *is an* E-$\mathfrak{F}$*-homomorphism of finitely* E-*generated* E-*overrings of* $\mathfrak{F}$ *in*

---

[5] Let R_1 and R_2 be rings over the ring R, all in a common field, and let ϕ_1 and ϕ_2 be ring homomorphisms of R_1 and R_2, respectively, into another field over R. Then ϕ_1 and ϕ_2 are *compatible*, if there exists an extension of ϕ_1 and ϕ_2 to the ring $R[R_1, R_2]$.

$\mathfrak{U}$ such that $h, S_{x'_1,x_1}, S_{x'_2,x_2}$ are compatible, then h and $S_{x'_1x'_2,x_1x_2}$ are compatible.

DAG2b If $x_1, x_2, x'_1, x'_2 \in G$ and $x_1 \to x'_1$, $x_2 \to x'_2$, then there exist elements $x_1^*, x_2^* \in G$ with $x_1 \leftrightarrow x_1^*$, $x_2 \leftrightarrow x_2^*$ such that x_1^*, x_2^* are algebraically disjoint over $\mathfrak{F}$ and $x_1^*x_2^* \to x'_1x'_2$ (i.e., $\mathfrak{F}\langle x_1^*\rangle$ and $\mathfrak{F}\langle x_2^*\rangle$ are algebraically disjoint over $\mathfrak{F}$), and such that, if $x_1^*x_2^* \leftrightarrow x'_1x'_2$ and $x_2^* \leftrightarrow x'_2$, then $S_{x'_1x'_2,x_1^*x_2^*}$, $S_{x'_2,x_2^*}$ are compatible.

DAG2c If $x_1, x_2, x'_1, x'_2 \in G$ and $x_1 \leftrightarrow x'_1$, $x_2 \leftrightarrow x'_2$, and $S_{x'_1,x_1}$, $S_{x'_2,x_2}$ are compatible, then $x_1^{-1}x_2 \to x_1'^{-1}x'_2$. If moreover $x_1^{-1}x_2 \leftrightarrow x_1'^{-1}x'_2$, and h is an E-*$\mathfrak{F}$-homomorphism of finitely* E-*generated* E-*overrings of $\mathfrak{F}$ in $\mathfrak{U}$ such that $h, S_{x'_1,x_1}, S_{x'_2,x_2}$ are compatible, then h, $S_{x_1'^{-1}x'_2,x_1^{-1}x_2}$ are compatible.*

DAG2d If $x_1, x_2, x'_1, x'_2 \in G$ and $x_1 \to x'_1$, $x_2 \to x'_2$, then there exist elements $x_1^*, x_2^* \in G$ with $x_1 \leftrightarrow x_2^*$, $x_2 \leftrightarrow x_2^*$ such that x_1^* and x_2^* are algebraically disjoint over $\mathfrak{F}$ and $x_1^{*-1}x_2^* \to x_1'^{-1}x'_2$.

DAG3 The unity element 1 of G is contained in an $\mathfrak{F}$-component (Definition 2.55) of G having an $\mathfrak{F}$-generic element x that is regular over $\mathfrak{F}$, i.e. $\mathfrak{F}$ is algebraically closed in $\mathfrak{F}\langle x\rangle$.

3.2 E-$\mathfrak{F}$-Group Schemes

Definition 3.154 *An* E-$\mathfrak{F}$-group scheme *is a group object $(Z, \mathcal{O}_Z)$ in the category of* E-*$\mathfrak{F}$-schemes: that is $(Z, \mathcal{O}_Z)$ is an* E-*$\mathfrak{F}$-scheme, and there are* *three morphisms* mult, inv and id *from* $(Z, \mathcal{O}_Z) \times (Z, \mathcal{O}_Z)$ *to* $(Z, \mathcal{O}_Z)$, *from* $(Z, \mathcal{O}_Z)$ *to* $(Z, \mathcal{O}_Z)$, *and from* $(\operatorname{diffspec} \mathcal{F}, \mathcal{O}_{\operatorname{diffspec} \mathcal{F}})$ *to* $(Z, \mathcal{O}_Z)$, *respectively, which satisfy the group axioms.*

Let $(Y, \mathcal{O}_Y)$ and $(X, \mathcal{O}_X)$ be two E-$\mathcal{F}$-schemes of E-$\mathcal{F}$-finite type, and let $(f, f^{\#}) : (Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$ be a morphism of E-$\mathcal{F}$-schemes. Let $\mathcal{G}$ be an E-field extension of $\mathcal{F}$. Denote by $(Y_{\mathcal{G}}, \mathcal{O}_{Y_{\mathcal{G}}})$, $(X_{\mathcal{G}}, \mathcal{O}_{X_{\mathcal{G}}})$ and $(f_{\mathcal{G}}, f^{\#}_{\mathcal{G}})$ the base extensions of $(Y, \mathcal{O}_Y)$, $(X, \mathcal{O}_X)$ and $(f, f^{\#})$ by $(i, i^{\#}) : (\operatorname{diffspec} \mathcal{G}, \mathcal{O}_{\operatorname{diffspec} \mathcal{G}}) \longrightarrow (\operatorname{diffspec} \mathcal{F}, \mathcal{O}_{\operatorname{diffspec} \mathcal{F}})$, which is induced by the inclusion $i : \mathcal{F} \longrightarrow \mathcal{G}$ (Proposition 2.65). This means

$$(Y_{\mathcal{G}}, \mathcal{O}_{Y_{\mathcal{G}}}) = (\operatorname{diffspec} \mathcal{G}, \mathcal{O}_{\operatorname{diffspec} \mathcal{G}}) \times_{(\operatorname{diffspec} \mathcal{F}, \mathcal{O}_{\operatorname{diffspec} \mathcal{F}})} (Y, \mathcal{O}_Y),$$

a similar definition for $(X_{\mathcal{G}}, \mathcal{O}_{X_{\mathcal{G}}})$, and $(f_{\mathcal{G}}, f^{\#}_{\mathcal{G}}) : (Y_{\mathcal{G}}, \mathcal{O}_{Y_{\mathcal{G}}}) \longrightarrow (X_{\mathcal{G}}, \mathcal{O}_{X_{\mathcal{G}}})$ is the projection from the product below:

$$\begin{tikzcd}
(Y_{\mathcal{G}}, \mathcal{O}_{Y_{\mathcal{G}}}) \arrow[r, "(f_{\mathcal{G}}, f^{\#}_{\mathcal{G}})"'] \arrow[d] & (X_{\mathcal{G}}, \mathcal{O}_{X_{\mathcal{G}}}) \arrow[r] \arrow[d] & (\operatorname{diffspec} \mathcal{G}, \mathcal{O}_{\operatorname{diffspec} \mathcal{G}}) \arrow[d, "(i, i^{\#})"'] \\
(Y, \mathcal{O}_Y) \arrow[r, "(f, f^{\#})"'] & (X, \mathcal{O}_X) \arrow[r] & (\operatorname{diffspec} \mathcal{F}, \mathcal{O}_{\operatorname{diffspec} \mathcal{F}})
\end{tikzcd}$$

This paragraph consists of straight forward generalizations of standard results for group schemes [2, Section 4.1, pages 94-99]. Let $(Z, \mathcal{O}_Z)$ be an E-$\mathcal{F}$-group scheme with defining morphism mult, inv and id. Then, the E-$\mathcal{G}$-scheme $(Z_{\mathcal{G}}, \mathcal{O}_{Z_{\mathcal{G}}})$ is an E-$\mathcal{G}$-group scheme with defining maps $\text{mult}_{\mathcal{G}}$, $\text{inv}_{\mathcal{G}}$ and $\text{id}_{\mathcal{G}}$. Also, the $\mathcal{G}$-valued points $Z(\mathcal{G})$ have a canonical structure of a group. For each element $\xi = (\xi, \xi^\#) \in Z(\mathcal{G})$, *left multiplication* L_ξ is defined as the composition

$$\begin{tikzcd}
(Z_{\mathcal{G}}, \mathcal{O}_{Z_{\mathcal{G}}}) \arrow[r, "(s,s^\#)\times(\mathrm{id}_{Z_{\mathcal{G}}},\mathrm{id}_{Z_{\mathcal{G}}^\#})"] & (\mathrm{diffspec}\,\mathcal{G}, \mathcal{O}_{\mathrm{diffspec}\,\mathcal{G}}) \times (Z_{\mathcal{G}}, \mathcal{O}_{Z_{\mathcal{G}}}) \arrow[d, "(\xi,\xi^\#)\times(\mathrm{id}_Z,\mathrm{id}_Z^\#)"'] \\
 & (Z_{\mathcal{G}}, \mathcal{O}_{Z_{\mathcal{G}}}) \times (Z_{\mathcal{G}}, \mathcal{O}_{Z_{\mathcal{G}}}) \arrow[d, "(\mathrm{mult}_{\mathcal{G}},\mathrm{mult}_{\mathcal{G}}^\#)"'] \\
 & (Z_{\mathcal{G}}, \mathcal{O}_{Z_{\mathcal{G}}}),
\end{tikzcd}$$

where $(\mathrm{id}_{Z_{\mathcal{G}}}, \mathrm{id}_{Z_{\mathcal{G}}^\#})$ is the identity morphism of $(Z_{\mathcal{G}}, \mathcal{O}_{Z_{\mathcal{G}}})$ and $(s, s^\#) : (Z_{\mathcal{G}}, \mathcal{O}_{Z_{\mathcal{G}}}) \to (\mathrm{diffspec}\,\mathcal{G}, \mathcal{O}_{\mathrm{diffspec}\,\mathcal{G}})$ is the $\mathcal{G}$-structure morphism. Because L_ξ has inverse $L_{\xi^{-1}}$, it is an isomorphism of E-$\mathcal{G}$-schemes.

Remark 3.155 *As a consequence of this, for each* $(\zeta, \zeta^\#) \in Z(\mathcal{G})$, *the induced* E-$\mathcal{G}$-*homomorphism on stalks* $L_\xi^\# : \mathcal{O}_{\mathrm{mult}_{\mathcal{G}}(\xi(0),\zeta(0))} \to \mathcal{O}_{\zeta(0)}$ *is an* E-$\mathcal{G}$-*isomorphism.*

Examples of E-$\mathcal{F}$-group schemes are constructed by a slight modification of examples of algebraic groups schemes. Let $R = \mathbb{Q}\{(y_{ij})_{i=1,\ldots,m,j=1,\ldots,m}\}_{\mathrm{E}}$ for a positive integer m and E-indeterminates $(y_{ij})_{i=1,\ldots,m,j=1,\ldots,m}$ over $\mathbb{Q}$, let $d = \det(y_{ij})$, and let $Z = \mathrm{diffspec}\, R_d$. The E-$\mathbb{Q}$-group scheme $\mathbf{GL}_{\mathbb{Q}}^{\mathrm{E}}(n)$ or $\mathbf{GL}^{\mathrm{E}}(n)$ is defined to be $(Z, \mathcal{O}_Z)$, where mult, inv and id are the E-$\mathbb{Q}$-scheme morphisms induced via Proposition 2.65 from $R_d \to R_d \otimes R_d$ such that $y_{ij} \to \sum_{k=1,\ldots,m} y_{ik} \otimes y_{kj}$, from $R_d \to R_d$ such that $y_{ij} \to z_{ij}$, where (z_{ij}) is the inverse of the matrix of (y_{ij}), and from $R_d \to \mathbb{Q}$ such that $y_{ij} \to \delta_{ij}$, where δ_{ij} is the Kronecker Delta. The E-$\mathbb{Q}$-group scheme $\mathbf{G}_m^{\mathrm{E}}$ is defined to be $\mathbf{GL}^{\mathrm{E}}(1)$. Similarly, let $S = \mathbb{Q}\{y\}_{\mathrm{E}}$, and let $Z = \operatorname{diffspec} S$. The E-$\mathbb{Q}$-group scheme $\mathbf{G}_a^{\mathrm{E}}$ is $(Z, \mathcal{O}_Z)$, where mult, inv and id are the E-$\mathbb{Q}$-scheme morphisms induced via Proposition 2.65 from $S \longrightarrow S \otimes S$ such that $y \longrightarrow y \otimes 1 + 1 \otimes y$, from $S \longrightarrow S$ such that $y \longrightarrow -y$, and from $S \longrightarrow \mathbb{Q}$ such that $y \longrightarrow 0$. See Definition **??** for a generalization $\mathbb{W}^{\Delta}(g_2, g_3)$ of an elliptic curve.

Definition 3.156 *An* E-*scheme* $(X, \mathcal{O}_X)$ *is* connected *if the topological space* X *is connected. An* E-*scheme* $(X, \mathcal{O}_X)$ *is* irreducible *if* X *is irreducible. Let* $\mathcal{F}$ *be an* E-*field. An* E-*scheme* $(X, \mathcal{O}_X)$ *over* $(\operatorname{diffspec}\mathcal{F}, \mathcal{O}_{\operatorname{diffspec}\mathcal{F}})$ *is* geometrically connected (resp. geometrically irreducible) *if the product*

$$(X_{\mathcal{G}}, \mathcal{O}_{X_{\mathcal{G}}}) = (\operatorname{diffspec}\mathcal{G}, \mathcal{O}_{\operatorname{diffspec}\mathcal{G}}) \times_{(\operatorname{diffspec}\mathcal{F}, \mathcal{O}_{\operatorname{diffspec}\mathcal{F}})} (X, \mathcal{O}_X)$$

is connected (resp. *irreducible*) *for all* E-*fields* $\mathcal{G}$ *algebraic over* $\mathcal{F}$.

Definition 3.157 *Let* $(Z, \mathcal{O}_Z)$ *be an* E-$\mathcal{F}$-*group scheme. Define* the connected component of the identity Z° *to be the connected component of* Z *containing the image of the identity* $\operatorname{id}(0)$.

Lemma 3.158 *Let* $(X, \mathcal{O}_X)$ *be an* E-*scheme. Every irreducible closed subset* $W \subseteq X$ *is connected.*

Proof: By Proposition 2.63, W has a unique generic point $w \in W$, i.e. $\overline{\{w\}} = W$. Then W is connected because the closure of a connected set is connected. $\square$

In general the converse of this lemma is false. However, the connected component of the identity of an E-$\mathcal{F}$-group scheme will be shown to be not only irreducible but also geometrically irreducible. The following outline may help the reader to understand the proof of this result, which constitute most of the remainder of this section. Let the E-field $\mathcal{V}$ be E-universal over $\mathcal{F}$, and let $\mathcal{F}^\circ \subset \mathcal{V}$ be the algebraic closure of $\mathcal{F}$ in $\mathcal{V}$. There is a natural structure of an E-field on $\mathcal{F}^\circ$. And let $(Z, \mathcal{O}_Z)$ be an E-$\mathcal{F}$-group scheme. Since the connected component Z° has an $\mathcal{F}$-valued point (the identity), Corollary 3.171 implies $(Z^\circ)_{\mathcal{F}^\circ}$ is connected. Because $\mathcal{V}$ is regular over $\mathcal{F}^\circ$, the fiber of the projection $(Z^\circ)_\mathcal{V} \longrightarrow (Z^\circ)_{\mathcal{F}^\circ}$ is connected, and, therefore, $(Z^\circ)_\mathcal{V}$ is also connected (Proposition 3.174). The image of the connected set $(Z_\mathcal{V})^\circ$ under the continuous projection $p : Z_\mathcal{V} \longrightarrow Z$ is connected and is contained in Z° since it is connected and contains the image of the identity id(0). Therefore, $(Z_\mathcal{V})^\circ \subseteq p^{-1}(Z^\circ) = (Z^\circ)_\mathcal{V}$. Since $(Z_\mathcal{V})^\circ$ is the largest connected set of $Z_\mathcal{V}$ containing the identity and since $(Z^\circ)_\mathcal{V}$ is connected, $(Z_\mathcal{V})^\circ = (Z^\circ)_\mathcal{V}$. If two different components intersect, there exist a $\mathcal{V}$-valued point with image in the intersection because $\mathcal{V}$ is E-universal over $\mathcal{F}$. Some $\mathcal{V}$-valued points are just in one component. But the homogeneity of the E-$\mathcal{V}$-group scheme $(Z_\mathcal{V}, \mathcal{O}_{Z_\mathcal{V}})$ implies that the image each $\mathcal{V}$-valued point is contained in the same number of irreducible components. Therefore, $(Z^\circ)_\mathcal{V}$, $(Z^\circ)_{\mathcal{F}^\circ}$ and Z° are irreducible by Proposition 3.175.

Then Proposition 3.166 is used to conclude, the main result of this section Proposition 3.176: the residue field of a $\mathcal{F}$-generic point of the connected component of the identity of an E-$\mathfrak{F}$-group scheme is regular over $\mathfrak{F}$. Several of the techniques of this section are similar to those of S.G.A. 3 [13, Expose 6, Section 2, pages 296-300] and E.G.A 4 [12].

Statements equivalent to regularity are listed in the following proposition from [6, Theorem 2, Lecture 14, page 6].

Proposition 3.159 *Let the field L be an extension of the field K. Then the following conditions are equivalent.*

1. *L is a regular extension of K.*
2. *If L' is a field extension of K, the algebra $L' \otimes_F L$ has no zero divisors.*
3. *If $\overline{K}$ is an algebraic closure of K, the algebra $\overline{K} \otimes_F L$ has no zero divisors.*

A similar result holds if the fields are replaced by E-fields.

Proposition 3.160 *Let the* E*-field L be an extension of the* E*-field K. Then the following conditions are equivalent.*

1. *L is a regular extension of K.*
2. *If L' is an* E*-field extension of K, the* E*-algebra $L' \otimes_K L$ has no zero divisors.*
3. *If the* E*-field $\overline{K}$ is an algebraic closure of K, the* E*-algebra $\overline{K} \otimes_K L$ has no zero divisors.*

Proof: Assume 1. Parts 2 and 3 follow from Proposition 3.159. Assume 3, and let M be any algebraic closure of K. The field M has a canonical structure of an E-field. By 3, the E-ring $M \otimes_K L$ has no zero divisors. Since the ring $M \otimes_K L$, without its E-structure, also has no zero divisors, Proposition 3.159 implies part 1. $\square$

The following lemma is a reformulation of well-known facts about radical E-ideals in Notherian E-$\mathfrak{F}$-algebras in terms of tensor products [24, Corollary 4, page 128] and [24, Proposition 3, page 131]. In it, $\otimes$ will mean $\otimes_F$.

Lemma 3.161 *Let the* E-$\mathfrak{F}$*-field* $\mathfrak{G}$ *be an extension of the* E*-field* $\mathfrak{F}$*, let* R *be an* E-$\mathfrak{F}$*-algebra finitely* E*-generated over* $\mathfrak{F}$*, and let* $\mathfrak{P} \subset R$ *be a prime* E*-ideal. Then the* E*-ideal* $\mathfrak{G} \otimes \mathfrak{P} \subset \mathfrak{G} \otimes R$ *is radical and is the intersection of a finite number of components (minimal prime* E*-ideals containing* $\mathfrak{G} \otimes \mathfrak{P}$*)* $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_s$ *of* $\mathfrak{G} \otimes R$*. If* $\mathfrak{G}$ *is regular over* $\mathfrak{F}$*,* $s = 1$*. Moreover, for each* $\mathfrak{Q}_j$*,* $(1 \otimes R) \cap \mathfrak{Q}_j = 1 \otimes \mathfrak{P}$ *and* $i^{-1}(\mathfrak{Q}_j) = \mathfrak{P}$*, where* $i : R \to \mathfrak{F} \otimes R$ *is the canonical inclusion. Finally, each radical* E*-ideal* $\mathfrak{R} \subseteq \mathfrak{G} \otimes R$ *is* E*-generated by a finite number of* $f_1, \ldots, f_r \in \mathfrak{R} \subseteq \mathfrak{G} \otimes R$*, i.e.* $\mathfrak{R} = \{f_1, \ldots, f_r\}_{\mathrm{E}}$*.*

Proof: Let $(\eta_1, \ldots, \eta_n)$ E-generate R over $\mathfrak{F}$, and let $y_1, \ldots, y_n$ be E-indeterminates over $\mathfrak{G}$. The substitution E-$\mathfrak{F}$-homomorphism $\phi_{\mathfrak{F}} : \mathfrak{F}\{y_1, \ldots, y_n\}_{\mathrm{E}} \to R$, defined by $\phi_{\mathfrak{F}}(y_i) = \eta_i$, is surjective and extends to a surjective E-$\mathfrak{F}$-homomorphism $\phi_{\mathfrak{G}} : \mathfrak{G}\{y_1, \ldots, y_n\}_{\mathrm{E}} \to \mathfrak{G} \otimes R$. The E-ideal $\phi_{\mathfrak{F}}^{-1}(\mathfrak{P})$ is prime, and the E-ideal $\mathfrak{G} \cdot \phi_{\mathfrak{F}}^{-1}(\mathfrak{P})$ is radical [24, Proposition 7(b), page 25]. There exist a finite number of components $\mathfrak{Q}'_1, \ldots, \mathfrak{Q}'_s \subset \mathfrak{G}\{y_1, \ldots, y_n\}_{\mathrm{E}}$ of

$\mathcal{G} \cdot \phi_{\mathcal{F}}^{-1}(\mathfrak{P})$ such that $\mathfrak{Q}'_1 \cap \ldots \cap \mathfrak{Q}'_s = \mathcal{G} \cdot \phi_{\mathcal{F}}^{-1}(\mathfrak{P}) \subset \mathcal{G}\{y_1, \ldots, y_n\}_{\mathrm{E}}$ and $\mathfrak{Q}'_i \cap \mathcal{F}\{y_1, \ldots, y_n\}_{\mathrm{E}} = \phi_{\mathcal{F}}^{-1}(\mathfrak{P})$ for each $\mathfrak{Q}'_i$ and, if $\mathcal{G}$ is regular over $\mathcal{F}$, $s = 1$ [24, Proposition 3, page 131].

Sinc $\varphi_{\mathcal{G}}$ is surjective, $\mathfrak{Q}_j = \phi_{\mathcal{G}}(\mathfrak{Q}'_j)$ is a prime E-ideal of $\mathcal{G} \otimes R$ for each j, and $\mathcal{G} \otimes \mathfrak{P} = \phi_{\mathcal{G}}(\mathcal{G} \cdot \phi_{\mathcal{F}}^{-1}(\mathfrak{P}))$ is radical E-ideal of $\mathcal{G} \otimes R$. Clearly, each $\mathfrak{Q}_j$ is minimal, and

$$\mathcal{G} \otimes \mathfrak{P} = \phi_{\mathcal{G}}(\mathcal{G} \cdot \phi_{\mathcal{F}}^{-1}(\mathfrak{P})) = \phi_{\mathcal{G}}(\mathfrak{Q}'_1 \cap \ldots \cap \mathfrak{Q}'_s) = \mathfrak{Q}_1 \cap \ldots \cap \mathfrak{Q}_s.$$

For each $\mathfrak{Q}'_j$, since $\mathfrak{Q}'_j \cap \mathcal{F}\{y_1, \ldots, y_n\}_{\mathrm{E}} = \phi_{\mathcal{F}}^{-1}(\mathfrak{P})$, it follows that

$$\mathfrak{Q}_j \cap (1 \otimes R) = \phi_{\mathcal{G}}(\mathfrak{Q}'_j \cap \mathcal{F}\{y_1, \ldots, y_n\}_{\mathrm{E}}) = \phi_{\mathcal{G}}(\phi_{\mathcal{F}}^{-1}(\mathfrak{P})) = 1 \otimes \mathfrak{P}$$

and $i^{-1}(\mathfrak{Q}_j) = \mathfrak{P}$ since $1 \otimes R$ is the image of i.

The last sentence of the lemma follows because there exist a finite number of generators of radical E-ideal $\mathcal{G} \cdot \phi_{\mathcal{F}}^{-1}(\mathfrak{P})$ by [24, Corollary 4, page 128], and the image of this finite set under $\phi_{\mathcal{G}}$ E-generates $\mathcal{G} \otimes \mathfrak{P}$. $\square$

If $\mathcal{G}$ is a Galois extension of $\mathcal{F}$ with Galois group G, define an action of G on $\mathcal{G} \otimes R$ by the formula

$$\gamma(\alpha \otimes r) = \gamma\alpha \otimes r,$$

for $\gamma \in G$, $\alpha \in \mathcal{G}$ and $r \in R$.

Proposition 3.162 *Let the* E-$\mathcal{F}$-*field* $\mathcal{G}$ *be a finite Galois extension of* $\mathcal{F}$ *with Galois group* G, *let* R *be an* E-$\mathcal{F}$-*algebra finitely* E-*generated over* $\mathcal{F}$, *and let* $\mathfrak{P} \subseteq R$ *be a prime* E-*ideal.*

1. *The fixed field of* $\mathfrak{G} \otimes R$ *under* G *is* $1 \otimes R$.

2. *If* $\mathfrak{Q} \subseteq \mathfrak{G} \otimes R$ *is a prime* E-*ideal such that* $\mathfrak{Q} \cap (1 \otimes R) = 1 \otimes \mathfrak{P}$, *then* $\mathfrak{Q}$ *is a component of* $1 \otimes \mathfrak{P}$ *in* $\mathfrak{G} \otimes R$. *If* $\mathfrak{Q}_1$ *is any component of* $\mathfrak{P} \otimes R$, *there exists a* $\gamma \in G$ *such that* $\gamma\mathfrak{Q}_1 = \mathfrak{Q}$.

3. *The action of* G *on the prime components of* $\mathfrak{G} \otimes \mathfrak{P}$ *is transitive.*

Proof: Let $f = \Sigma_i \; \alpha_i \otimes r_i \in \mathfrak{G} \otimes R$ with $\alpha_i \in \mathfrak{G}$ and $r_i \in R$ such that the r_i are linearly independent over $\mathfrak{F}$. Assume $\gamma f = f$ for all $\gamma \in G$. Then $\Sigma_i \; (\gamma\alpha_i - \alpha_i) \otimes r_i = \Sigma_i \; \gamma\alpha_i \otimes r_i - \Sigma_i \; \alpha_i \otimes r_i = \gamma f - f = 0$. Since the $1 \otimes r_i \in \mathfrak{G} \otimes R$ are also linear independent over $\mathfrak{G} \otimes 1$, $\gamma\alpha_i - \alpha_i = 0$ and $\gamma\alpha_i = \alpha_i$ for all $\gamma \in G$. Since $\mathfrak{G}$ is a Galois extension of $\mathfrak{F}$, $\alpha_i \in \mathfrak{F}$ for all i. Therefore, $f = \Sigma_i \; \alpha_i \otimes r_i \in \mathfrak{F} \otimes R = 1 \otimes R$.

Let the prime E-ideal $\mathfrak{Q}$ have the property specified in part 2. Let $\mathfrak{Q}_1, \mathfrak{Q}_2, \ldots, \mathfrak{Q}_s$ be the prime components of $\mathfrak{G} \otimes \mathfrak{P}$ (Lemma 3.161). Clearly the image of a component under an element of G is another component. The prime E-ideal $\mathfrak{Q}$ is contained in $\bigcup_{\gamma \in G} \gamma\mathfrak{Q}_1$ because, if not, there exist $a \in \mathfrak{Q}$ such that $a \notin \bigcup_{\gamma \in G} \gamma\mathfrak{Q}_1$. And for each $\tau \in G$, $\tau a \notin \mathfrak{Q}_1$: otherwise, a would be in $\bigcup_{\gamma \in G} \gamma\mathfrak{Q}_1$ by applying τ^{-1}. Therefore, $x = \prod_{\gamma \in G} \gamma a \in \mathfrak{Q}$ is not in $\mathfrak{Q}_1$ since $\mathfrak{Q}_1$ is a prime ideal. Since x is clearly invariant under the action of G, $x \in \mathfrak{F} \otimes R$ by the part 1 of this lemma. Therefore, $x \in (\mathfrak{F} \otimes R) \cap \mathfrak{Q} = (1 \otimes R) \cap \mathfrak{Q} = 1 \otimes \mathfrak{P}$ by assumption. However, since $1 \otimes \mathfrak{P} = (\mathfrak{F} \otimes R) \cap \mathfrak{Q}_1$, $x \in \mathfrak{Q}_1$ is a contradiction. Therefore, $\mathfrak{Q} \subset \bigcup_{\gamma \in G} \gamma\mathfrak{Q}_1$, and $\mathfrak{Q} \subseteq \gamma\mathfrak{Q}_1$ for some $\gamma \in G$ by [1, Proposition 1.11, page 8]. Because $\gamma\mathfrak{Q}_1$ is minimal, $\mathfrak{Q} = \gamma\mathfrak{Q}_1$ is a component.

To prove part 3, let $\mathfrak{Q}$ and $\mathfrak{Q}_1$ be any two components of $\mathfrak{G} \otimes \mathfrak{P}$. Since $\mathfrak{Q} \cap (1 \otimes R) = 1 \otimes \mathfrak{P}$ (Lemma 3.161), there exists $\gamma \in G$ such that $\mathfrak{Q} = \gamma\mathfrak{Q}_1$ by part 2. □

Corollary 3.163 *Proposition 3.162 remains true if, instead of assuming* $\mathfrak{G}$ *is a finite Galois Extension of* $\mathfrak{F}$*, it is assumed* $\mathfrak{G}$ *is the algebraic closure* $\mathfrak{F}^\circ$ *of* $\mathfrak{F}$ *is* $\mathfrak{V}$*. The Galois group of the proposition is replaced by the pro-finite Galois Group of* $\mathfrak{F}^\circ$ *over* $\mathfrak{F}$*.*

Proof: The proof of part 1 is the same. Let the prime E-ideal $\mathfrak{Q}_0$ have the property specified in part 2 of the proposition with $\mathfrak{G} = \mathfrak{F}^\circ$: that is $\mathfrak{Q}_0 \subseteq \mathfrak{F}^\circ \otimes R$ is a prime E-ideal such that $\mathfrak{Q}_0 \cap (1 \otimes R) = 1 \otimes \mathfrak{P}$. Let $\mathfrak{Q}_1$ be any prime component of $\mathfrak{F}^\circ \otimes \mathfrak{P}$ and let $\mathfrak{Q}_2, \ldots, \mathfrak{Q}_s$ be the others (Lemma 3.161).

For $i = 0, 1, \ldots, s$, by Lemma 3.161, $\mathfrak{Q}_i = \{f_{i,1}, \ldots, f_{i,r_i}\}_{\mathrm{E}}$ for $f_{i,1}, \ldots, f_{i,r_i} \in \mathfrak{F}^\circ \otimes R$. Let $f_{i,k} = \Sigma_l\, \alpha_{i,k,l} \otimes r_{i,k,l}$ for $\alpha_{i,k,l} \in \mathfrak{F}^\circ$ and $r_{i,k,l} \in R$. Let G be the Galois group of the smallest Galois extension $\mathfrak{G} \subset \mathfrak{F}^\circ$ of $\mathfrak{F}$ containing all the $\alpha_{i,k,l}$. For each i, let $\mathfrak{Q}'_i = \mathfrak{Q}_i \cap (\mathfrak{G} \otimes R)$. Then clearly $\mathfrak{Q}'_i = \{f_{i,1}, \ldots, f_{i,r_i}\}_{\mathrm{E}} \subset \mathfrak{G} \otimes R$, and $\mathfrak{F}^\circ \otimes_{\mathfrak{G}} \mathfrak{Q}'_i = \mathfrak{F}^\circ \otimes_{\mathfrak{G}} \{f_{i,1}, \ldots, f_{i,r_i}\}_{\mathrm{E}} = \{f_{i,1}, \ldots, f_{i,r_i}\}_{\mathrm{E}} = \mathfrak{Q}_i$. Also, $\mathfrak{Q}'_i \cap (1 \otimes R) = (\mathfrak{Q}_i \cap (\mathfrak{G} \otimes R)) \cap (1 \otimes R) = \mathfrak{Q}'_i \cap (1 \otimes R) = 1 \otimes \mathfrak{P}$. Therefore, by part 2 of the proposition, all the $\mathfrak{Q}'_i \subset \mathfrak{G} \otimes R$ are components of $\mathfrak{G} \otimes \mathfrak{P}$.

It will now be shown that $\mathfrak{Q}'_1, \ldots \mathfrak{Q}'_s$ are all the components. Assume $\mathfrak{Q}'$ is another component of $\mathcal{G} \otimes \mathfrak{P}$. Let $\mathfrak{T}$ be a component of $\mathcal{F}^\circ \otimes_{\mathcal{G}} \mathfrak{Q}'$ in $\mathcal{F}^\circ \otimes_{\mathcal{G}} (\mathcal{G} \otimes R) = \mathcal{F}^\circ \otimes_{\mathcal{F}} R$. Since $\mathfrak{T} \cap (1 \otimes R) = (\mathfrak{T} \cap (\mathcal{G} \otimes R) \cap (1 \otimes R) = \mathfrak{Q}' \cap (1 \otimes R) = \mathfrak{P}$, some component $\mathfrak{Q}_i \subseteq \mathfrak{T}$. Then $\mathfrak{Q}'_i = \mathfrak{Q}_i \cap (\mathcal{G} \otimes R) \subseteq \mathfrak{T} \cap (\mathcal{G} \otimes R) = \mathfrak{Q}'$. Since $\mathfrak{Q}'$ is a component and, therefore, minimal, $\mathfrak{Q}'_i = \mathfrak{Q}'$.

Consequently, the component $\mathfrak{Q}'_0$ is one of the $\mathfrak{Q}'_1, \ldots \mathfrak{Q}'_s$. Say $\mathfrak{Q}'_0 = \mathfrak{Q}'_1$. Then, $\mathfrak{Q}_0 = \mathcal{F}^\circ \otimes_{\mathcal{G}} \mathfrak{Q}'_0 = \mathcal{F}^\circ \otimes_{\mathcal{G}} \mathfrak{Q}'_1 = \mathfrak{Q}_1$ is one of the components of $\mathcal{F}^\circ \otimes \mathfrak{P}$. Similarly, the action of the pro-finite Galois group is transitive because, by part 3 of the proposition, it is transitive on the $\mathfrak{Q}'_1, \ldots \mathfrak{Q}'_s$ and because if, for an element γ of that group, $\gamma \mathfrak{Q}'_1 = \mathfrak{Q}'_i$, then $\gamma \mathfrak{Q}_1 = \mathcal{F}^\circ \otimes_{\mathcal{G}} \gamma \mathfrak{Q}'_1 = \mathcal{F}^\circ \otimes_{\mathcal{G}} \mathfrak{Q}'_i = \mathfrak{Q}_i$. □

Lemma 3.164 *Let $\mathcal{G}$ be an* E*-field extension of $\mathcal{F}$, and let $(X, \mathcal{O}_X)$ be an* E*-$\mathcal{F}$-scheme of finite* E*-type. The projection $(\pi, \pi^\#) : (X_{\mathcal{G}}, \mathcal{O}_{X_{\mathcal{G}}}) \longrightarrow (X, \mathcal{O}_X)$ is surjective. Moreover, if $\mathcal{G}$ is a regular extension of $\mathcal{F}$ and if $x \in X$, the fiber $\pi^{-1}(x)$ consists of one element.*

Proof: Let $\mathfrak{P} \in X$, and let diffspec $A \subseteq X$ be an affine open containing $\mathfrak{P}$ where A is a finitely E-$\mathcal{F}$-generated E-$\mathcal{F}$-algebra. Then diffspec $\mathcal{G} \times$ diffspec $A =$ diffspec $(\mathcal{G} \otimes A)$ is an affine open of $X_{\mathcal{G}}$. The projection $(p, p^\#)$: diffspec $(\mathcal{G} \otimes A) \longrightarrow$ diffspec A is the restriction of $(\pi, \pi^\#)$ to diffspec $(\mathcal{G} \otimes A)$ and is induced by the inclusion $i : A \longrightarrow (\mathcal{G} \otimes A)$. By Lemma 3.161, $\mathcal{G} \otimes \mathfrak{P} \subseteq \mathcal{G} \otimes A$ is a radical E-ideal with prime components $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_s$ such that $i^{-1}(\mathfrak{Q}_j) = \mathfrak{P}$ for all j. Therefore, $p(\mathfrak{Q}_j) = \mathfrak{P}$ for all j. Since p is the restriction of π to diffspec $(\mathcal{G} \otimes A)$, π is surjective.

If $\mathcal{G}$ is a regular extension of $\mathcal{F}$, let $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_s$ be the prime components of $\mathcal{G} \otimes \mathfrak{P}$. Since

$$0 \longrightarrow \mathfrak{P} \longrightarrow A \xrightarrow{\phi} A/\mathfrak{P} \longrightarrow 0$$

is exact,

$$0 \longrightarrow \mathcal{G} \otimes \mathfrak{P} \longrightarrow \mathcal{G} \otimes A \xrightarrow{\mathrm{id} \otimes \phi} \mathcal{G} \otimes (A/\mathfrak{P}) \longrightarrow 0$$

is exact. Therefore,

$$(\mathcal{G} \otimes A)/(\mathcal{G} \otimes \mathfrak{P}) = \mathcal{G} \otimes (A/\mathfrak{P}) \subset \mathcal{G} \otimes \mathrm{QF}(A/\mathfrak{P}),$$

where $\mathrm{QF}(A/\mathfrak{P})$ is the quotient field of $A/\mathfrak{P}$, and $(\mathrm{id} \otimes \phi)(\mathfrak{Q}_1), \ldots,$ $(\mathrm{id} \otimes \phi)(\mathfrak{Q}_s)$ are the components of the radical E-ideal generated by $(\mathcal{G} \otimes 0) \subset (\mathcal{G} \otimes A)$. If s were greater than 1, one may choose, for each $1 \leq j \leq s$, a non-zero $x_j \in (\mathrm{id} \otimes \phi)(\mathfrak{Q}_j)$ not contained in any $(\mathrm{id} \otimes \phi)(\mathfrak{Q}_k)$ for $k \neq j$ ([1, Proposition 1.11, page 8]). The product of all of the x_j is in the intersection of all the $(\mathrm{id} \otimes \phi)(\mathfrak{Q}_j)$ and, therefore, some power of the product is equal to zero. This means each x_j is a zero divisor. Since $\mathcal{G} \otimes (A/\mathfrak{P}) \subset \mathcal{G} \otimes \mathrm{QF}(A/\mathfrak{P})$ and $\mathcal{G} \otimes \mathrm{QF}(A/\mathfrak{P})$ has no zero divisors by Proposition 3.160, $s = 1$. Thus, there is only one prime component of the radical E-ideal generated by $0 \otimes \mathcal{G}$, and $\pi^{-1}(x)$ consists of one element. □

Lemma 3.165 *Let $(X, \mathcal{O}_X)$ be an* E-$\mathcal{F}$*-scheme, and let $(\xi, \xi^{\#}) \in X(\mathcal{V})$ be a $\mathcal{V}$-valued point of $(X, \mathcal{O}_X)$. For each finite set $\alpha_1, \ldots, \alpha_r$ of elements of the* *residue field* $\mathcal{F}\langle(\xi,\xi^{\#})\rangle = \xi^{\#}_{\xi(0)}(\mathcal{O}_{\xi(0)})$, *there exists an open affine* diffspec C *of* $(X,\mathcal{O}_X)$ *containing* $\xi(0)$ *and a finite set* $\gamma_1,\ldots,\gamma_r \in C$ *such that* $i_{\xi,C}(\gamma_i) = \alpha_i$ *for* $i = 1,\ldots,r$ *where* $i_{\xi,C} = \xi^{\#}_{\xi(0)} \circ \mathcal{J}_{\xi(0),\mathrm{diffspec}\,C} \circ \mathcal{I}_C$ *is the composite of canonical* E-$\mathcal{F}$-*homomorphisms* :

$$C \xrightarrow{\mathcal{I}_C} \mathcal{O}_{\mathrm{diffspec}\,C}(\mathrm{diffspec}\,C) \xrightarrow{\mathcal{J}_{\xi(0),\mathrm{diffspec}\,C}} \mathcal{O}_{\xi(0)} \xrightarrow{\xi^{\#}_{\xi(0)}} \mathcal{F}\langle(\xi,\xi^{\#})\rangle. \tag{16}$$

Proof: Since $\xi^{\#}_{\xi(0)}$ surjective, there exist $\alpha'_1,\ldots,\alpha'_r \in \mathcal{O}_{\xi(0)}$ such that $\xi^{\#}_{\xi(0)}(\alpha'_i) = \alpha_i$ for all i. Because $\mathcal{O}_{\xi(0)}$ is the direct limit of $\mathcal{O}_{\mathbf{diffspec}\,B}(\mathrm{diffspec}\,B)$ over all open affines diffspec B containing $\xi(0)$, each α'_i is in the image of a $\beta_i \in \mathcal{O}_{\mathbf{diffspec}\,B_i}(\mathrm{diffspec}\,B_i)$ under $\mathcal{J}_{\xi(0),\mathbf{diffspec}\,B_i}$ for some open affines diffspec B_i. Since the open affines are a basis for X ([19, Proposition 3.5, page 76]), there is an open affine diffspec B contained in each diffspec B_i and containing $\xi(0)$ such that each β_i restricts to an element of $\mathcal{O}_{\mathrm{diffspec}\,B}(\mathrm{diffspec}\,B)$, which will also be denoted by β_i, and $(\xi^{\#}_{\xi(0)} \circ \mathcal{J}_{\xi(0),\mathrm{diffspec}\,B})(\beta_i) = \alpha_i$. By the definition, each section β_i of $\mathcal{O}_{\mathbf{diffspec}\,B}(\mathrm{diffspec}\,B)$ is defined by an element $\gamma_i \in C_i$ for some open affine diffspec C_i containing $\xi(0)$. Again, since the open affines are a basis for X, there is an open affine diffspec C contained in each diffspec C_i and containing $\xi(0)$ such that each γ_i restricts to an element of $\mathcal{O}_{\mathbf{diffspec}\,C}(\mathrm{diffspec}\,C)$, which will also be denoted by γ_i, and $(\xi^{\#}_{\xi(0)} \circ \mathcal{J}_{\xi(0),\mathbf{diffspec}\,B_i} \circ \mathcal{I}_C)(\gamma_i) = \alpha_i$. ☐

Proposition 3.166 *Let* $(X,\mathcal{O}_X)$ *be an irreducible* E-$\mathcal{F}$-*scheme of* E-$\mathcal{F}$-*finite type, and let* $\mathcal{F}\langle(\eta,\eta^{\#})\rangle = \eta^{\#}_{\eta(0)}(\mathcal{O}_{\eta(0)})$ *be residue field of an* $\mathcal{F}$-*generic element*

$(\eta, \eta^{\#})$ of the pre E-$\mathcal{F}$*-set $X(\mathcal{V})$ (Theorem 2.126). If $(X, \mathcal{O}_X)$ is geometrically irreducible, then $\mathcal{F}\langle(\eta, \eta^{\#})\rangle$ is a regular extension of $\mathcal{F}$.*

Proof: Let $\mathcal{F}^\circ \subset \mathcal{V}$ be an algebraically closed E-field extension of $\mathcal{F}$. By the definition of geometrically irreducible,

$$(Y, \mathcal{O}_Y) = (\operatorname{diffspec} \mathcal{F}^\circ, \mathcal{O}_{\operatorname{diffspec} \mathcal{F}^\circ}) \times_{(\operatorname{diffspec} \mathcal{F}, \mathcal{O}_{\operatorname{diffspec} \mathcal{F}})} (X, \mathcal{O}_X)$$

is irreducible. Let $(\eta, \eta^{\#}) \in X(\mathcal{V})$ be an $\mathcal{F}$-generic element. Let $\alpha_1, \ldots, \alpha_r \in \mathcal{F}\langle(\eta, \eta^{\#})\rangle$ be E-generators of the field $\mathcal{F}\langle(\eta, \eta^{\#})\rangle$ over $\mathcal{F}$. By Lemma 3.165, the exists an an open affine diffspec C of $(X, \mathcal{O}_X)$ containing $\eta(0)$ and finite set $\gamma_1, \ldots, \gamma_r \in C$ such that $i_{\eta,C}(\gamma_i) = \alpha_i$ for $i = 1, \ldots, r$, where $i_{\eta,C} : C \to \mathcal{F}\langle(\eta, \eta^{\#})\rangle$ is the canonical map defined in the previous lemma. Since an open subset of an irreducible topological space is also irreducible, diffspec C and diffspec $\mathcal{F}^\circ \times$ diffspec $C =$ diffspec $(\mathcal{F}^\circ \otimes C)$ are irreducible. By Proposition 2.94 and Lemma 1.10, $\eta(0)$ is a generic point of diffspec C. By definition, $\eta(0)$ is the minimal prime E-ideal of C.

By Lemma 3.161, $\mathcal{F}^\circ \otimes \eta(0) \subset \mathcal{F}^\circ \otimes C$ is a radical E-ideal and is the intersection of a finite number of components $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_s$ of $\mathcal{F}^\circ \otimes C$ such that $\mathfrak{Q}_i \cap (1 \otimes R) = 1 \otimes \eta(0)$ for all i. Each $\mathfrak{Q}_i$ must be a minimal prime E-ideal of $\mathcal{F}^\circ \otimes C$ because, if $\mathfrak{Q} \subseteq \mathfrak{Q}_i$ is an E-prime ideal, $\mathfrak{Q} \cap (1 \otimes C) \subseteq \mathfrak{Q}_i \cap (1 \otimes C) = 1 \otimes \eta(0)$. But, since $1 \otimes \eta(0) \subset 1 \otimes C$ is minimal, $\mathfrak{Q} \cap (1 \otimes C)) = 1 \otimes \eta(0)$. By Corollary 3.163, $\mathfrak{Q}$ is a component of $\mathcal{F}^\circ \otimes \eta(0)$. Therefore, $\mathfrak{Q} = \mathfrak{Q}_i$. Since diffspec $(\mathcal{F}^\circ \otimes C)$ is irreducible, there is only one minimal prime E-ideal of $\mathcal{G} \otimes R$, and the radical E-ideal $\mathcal{F}^\circ \otimes \eta(0)$ is a prime E-ideal.

In diagram 16 in Lemma 3.165, the image of $\eta(0) \subset C$ under $\mathfrak{J}_{\eta(0),\mathrm{diffspec}\, C} \circ \mathfrak{I}_C$ is the maximal ideal of $\mathcal{O}_{\eta(0)}$ (Lemma 2.71) and is the kernel of $\eta^{\#}_{\eta(0)}$. Therefore $\eta(0)$ is the kernel of $i_{\eta,C} = \eta^{\#}_{\eta(0)} \circ \mathfrak{J}_{\eta(0),\mathrm{diffspec}\, C} \circ \mathfrak{I}_C$, and the following sequence is exact:

$$\begin{tikzcd} 0 \arrow[r] & \eta(0) \arrow[r, "inclusion"] & C \arrow[r, "i_{\eta,C}"] & i_{\eta,C}(C) \arrow[r] & 0. \end{tikzcd}$$

Since

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{F}^{\circ} \otimes \eta(0) \arrow[r, "\mathrm{id}\otimes inclusion"] & \mathfrak{F}^{\circ} \otimes C \arrow[r, "\mathrm{id}\otimes i_{\eta,C}"] & \mathfrak{F}^{\circ} \otimes i_{\eta,C}(C) \arrow[r] & 0 \end{tikzcd}$$

is exact and $\mathfrak{F}^{\circ} \otimes \eta(0)$ is a prime E-ideal, $\mathfrak{F}^{\circ} \otimes i_{\eta,C}(C)$ is an integral domain. Since the $\alpha_i \in i_{\eta,C}(C)$ were chosen to be E-$\mathfrak{F}$-generators of $\mathfrak{F}\langle(\eta, \eta^{\#})\rangle$, the quotient field of $i_{\eta,C}(C)$ is $\mathfrak{F}\langle(\eta, \eta^{\#})\rangle$. Because $\mathfrak{F}^{\circ} \otimes \mathfrak{F}\langle(\eta, \eta^{\#})\rangle$ is contained in the quotient field of the integral domain $\mathfrak{F}^{\circ} \otimes i_{\eta,C}(C)$, the ring $\mathfrak{F}^{\circ} \otimes \mathfrak{F}\langle(\eta, \eta^{\#})\rangle$ is an integral domain and thus does not have zero divisors. Proposition 3.160 implies $\mathfrak{F}\langle(\eta. \eta^{\#})\rangle$ is a regular extension of $\mathfrak{F}$. $\square$

Lemma 3.167 *Let* $(X, \mathcal{O}_X)$ *and* $(Y, \mathcal{O}_Y)$ *be two* E-$\mathfrak{F}$-*schemes of* E-$\mathfrak{F}$-*finite type, and let their product* $(Z, \mathcal{O}_Z)$ *be their product* (*Proposition 2.81*). *Then the projection* $\pi_Y : Z \to Y$ *is an open map.*

Proof: To show π_Y is open it is enough to show it is open when restricted to each element of a cover of Z. Since $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are E-$\mathfrak{F}$-schemes of E-$\mathfrak{F}$-finite type, X and Y have covers of the form $\{\mathrm{diffspec}\, R_i\}_{i\in I}$ and $\{\mathrm{diffspec}\, S_j\}_{j\in J}$ where R_i and S_j are E-$\mathfrak{F}$-algebras finitely E-generated over

$\mathfrak{F}$. Therefore $\{\operatorname{diffspec} R_i \times \operatorname{diffspec} S_j\}_{i\in I, j\in J}$ is a cover of $X \times Y$, and it is sufficient to prove that for each (i, j) the projection $\operatorname{diffspec} R_i \times \operatorname{diffspec} S_j \longrightarrow \operatorname{diffspec} S_j$ is open.

For a fixed (i, j), let $R = R_i$, $S = S_j$ and $\pi : \operatorname{diffspec} R \times \operatorname{diffspec} S \longrightarrow \operatorname{diffspec} S$ be the projection. Since $\{\operatorname{diffspec}(R \otimes S)_f\}_{f\in R\otimes S}$ an open basis of $\operatorname{diffspec} R \times \operatorname{diffspec} S = \operatorname{diffspec}(R \otimes S)$, the proof will be completed by showing that $\pi(\operatorname{diffspec}(R \otimes S)_f) \subseteq \operatorname{diffspec} S$ is open for all $f \in R \otimes S$ ([19, Proposition 3.5, page 76]).

Let $f \in R \otimes S$, and write $f = \Sigma_k \ \alpha_k \otimes f_k$ with $\alpha_k \in R$ and $f_k \in S$ such that the α_k are linearly independent over $\mathfrak{F}$. It will now be shown that the projection

$$\pi(\operatorname{diffspec}(R \otimes S)_f) = \bigcup_{k=1,\ldots,t} \operatorname{diffspec} S_{f_k}$$

and, therefore, $\pi(\operatorname{diffspec}(R \otimes S)_f)$ is open. Suppose $\mathfrak{P} \in \pi(\operatorname{diffspec}(R \otimes S)_f) \subseteq \operatorname{diffspec} S$, and let $\mathfrak{Q} \in \operatorname{diffspec}(R \otimes S)_f$ such that $\pi(\mathfrak{Q}) = \mathfrak{P}$. Then $i_S^{-1}(\mathfrak{Q}) = \mathfrak{P}$ where $i_S : S \longrightarrow R \otimes S$ is the injective E-$\mathfrak{F}$-homomorphism that defines π. Then, since $f \notin \mathfrak{Q} \supseteq i_S(\mathfrak{P}) = R \otimes \mathfrak{P}$, at least one $f_k \notin \mathfrak{P} = \pi(\mathfrak{Q})$. Therefore, $\mathfrak{P} \in \operatorname{diffspec} S_{f_k}$ for some f_k.

On the other hand, suppose $\mathfrak{P} \in \bigcup_{k=1,\ldots,t} \operatorname{diffspec} S_{f_k}$. One may assume $f_1, \ldots, f_r \notin \mathfrak{P}$ for some $r \geq 1$ and $f_k \in \mathfrak{P}$ for $k > r$. By Lemma 3.161, there exist prime components $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_s$ of $\mathfrak{G} \otimes \mathfrak{P}$ in $\mathfrak{G} \otimes R$ such that $\mathfrak{G} \otimes \mathfrak{P} = \mathfrak{Q}_1 \cap \ldots \cap \mathfrak{Q}_s$. Let $\phi : R \longrightarrow R/\mathfrak{P}$ be the canonical E-$\mathfrak{F}$-homomorphism. Since

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{P} \arrow[r] & R \arrow[r, "\phi"] & R/\mathfrak{P} \arrow[r] & 0 \end{tikzcd}$$

is exact,

$$\mathcal{G}\otimes\mathfrak{P} \longrightarrow \mathcal{G}\otimes R \xrightarrow{\mathrm{id}\otimes\phi} \mathcal{G}\otimes(R/\mathfrak{P}) \longrightarrow 0$$

is exact. Assume $f\in\mathfrak{Q}_i$ for each $i=1,\ldots,s$. Then $f\in\mathcal{G}\otimes\mathfrak{P}$. Therefore

$$0=(\mathrm{id}\otimes\phi)(f)=\Sigma_{k=1,\ldots,r}\quad \alpha_k\otimes\phi(f_k)$$

since $\phi(f_k)=0$ for $k>r$. This relation is impossible because $\phi(f_k)\neq 0$ for $1\leq k\leq r$ and the $\alpha_i\otimes 1$ are linearly independent over $1\otimes(R/\mathfrak{P})$ since the α_i are linearly independent over $\mathcal{F}$. Thus f is not contained in one of the $\mathfrak{Q}_i$. If $\mathfrak{Q}_1$ is a component such that $f\notin\mathfrak{Q}_1$, then $\mathfrak{Q}_1\in\operatorname{diffspec}(\mathcal{G}\otimes R)_f$, and $\mathfrak{P}=\pi(\mathfrak{Q}_1)\in\pi(\operatorname{diffspec}(\mathcal{G}\otimes R)_f)$. $\square$

Proposition 3.168 *Let* $(X,\mathcal{O}_X)$ *an* E-$\mathcal{F}$*-scheme of* E-$\mathcal{F}$*-finite type, and let the* E-$\mathcal{F}$*-field* $\mathcal{F}^\circ$ *be an algebraic closure of* $\mathcal{F}$*. Then the projection*

$$(X_{\mathcal{F}^\circ},\mathcal{O}_{X_{\mathcal{F}^\circ}})=(\operatorname{diffspec}\mathcal{F}^\circ,\mathcal{O}_{\operatorname{diffspec}\mathcal{F}^\circ})\times_{(\operatorname{diffspec}\mathcal{F},\mathcal{O}_{\operatorname{diffspec}\mathcal{F}})}(X,\mathcal{O}_X)\longrightarrow(X,\mathcal{O}_X)$$

is closed.

Proof: It must be shown that the projection $X_{\mathcal{F}^\circ}\longrightarrow X$ on the underlying topological space is closed. Since $(X,\mathcal{O}_X)$ an E-$\mathcal{F}$-scheme of E-$\mathcal{F}$-finite type, there is a finite cover of the form $\{\operatorname{diffspec}R_i\}_{i\in I}$ where each R_i is an E-$\mathcal{F}$-algebra finitely E-generated over $\mathcal{F}$. Therefore $\{\operatorname{diffspec}(\mathcal{F}^\circ\otimes R_i)\}_{i\in I}$ covers the product. Thus it is enough to show that the projection $(p,p^{\#})$: $\operatorname{diffspec}(\mathcal{F}^\circ\otimes R)\longrightarrow\operatorname{diffspec}R$ is closed for any E-$\mathcal{F}$-algebra R that is finitely

E-generated over $\mathfrak{F}$. Note that $(p, p^{\#})$ is induced by the inclusion $i : R \to \mathfrak{F}^{\circ} \otimes R$ (Proposition 2.65).

Each closed subset of diffspec$(\mathfrak{F}^{\circ} \otimes R)$ is of the form $V(\mathfrak{R})$ for some radical E-ideal $\mathfrak{R} \subseteq \mathfrak{F}^{\circ} \otimes R$ ([19, Corollary 2.8, page 74]). Lemma 3.161 implies $\mathfrak{R}$ is the intersection of a finite number of prime E-ideals $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_s$ of $\mathfrak{G} \otimes R$. For $j = 1, \ldots, s$, let $\mathfrak{P}_j = i^{-1}(\mathfrak{Q}_j)$. Each $\mathfrak{P}_j$ is a prime E-ideal of R, and $i^{-1}(\mathfrak{R}) = i^{-1}(\bigcap_j \mathfrak{Q}_j) = \bigcap_j i^{-1}(\mathfrak{Q}_j) = \bigcap_j \mathfrak{P}_j$.

That $p(V(\mathfrak{R}))$ is closed is implied by the following:

Claim 3.169 $p(V(\mathfrak{R})) = V(i^{-1}(\mathfrak{R}))$.

To prove the claim, let $\mathfrak{Q} \in V(\mathfrak{R})$. Then $\mathfrak{Q} \supseteq \mathfrak{R}$, and $p(\mathfrak{Q}) = i^{-1}(\mathfrak{Q}) \supseteq i^{-1}(\mathfrak{R})$. Therefore, $p(\mathfrak{Q}) \in V(i^{-1}(\mathfrak{R}))$.

Let $\mathfrak{P} \in V(i^{-1}(\mathfrak{R}))$. Since $\mathfrak{P} \supseteq i^{-1}(\mathfrak{R}) = \bigcap_j \mathfrak{P}_j$ and $\mathfrak{P}$ is a prime ideal, $\mathfrak{P} \supseteq \mathfrak{P}_d$ for some d, $1 \leq d \leq s$ ([1, Proposition 1.11, page 8]). Then $\mathfrak{F}^{\circ} \otimes \mathfrak{P} \supseteq \mathfrak{F}^{\circ} \otimes \mathfrak{P}_d$. Let $\mathfrak{P}' \subset \mathfrak{F}^{\circ} \otimes R$ be a prime component of $\mathfrak{F}^{\circ} \otimes \mathfrak{P}$ so that $\mathfrak{P}' \cap (1 \otimes R) = \mathfrak{P}$ by Lemma 3.161. Then

$$\mathfrak{P}' \supseteq \mathfrak{K} \otimes \mathfrak{P} \supseteq \mathfrak{K} \otimes \mathfrak{P}_d = \bigcap_{j=1,\ldots,r} \mathfrak{Q}'_j$$

where the $\mathfrak{Q}'_j$ are the prime components of $\mathfrak{F}^{\circ} \otimes \mathfrak{P}_d$ in $\mathfrak{F}^{\circ} \otimes R$. Again by ([1, Proposition 1.11, page 8]), $\mathfrak{P}' \supseteq \mathfrak{Q}'_c$ for some c, $1 \leq c \leq r$.

Since $\mathfrak{Q}_d \cap (1 \otimes R) = 1 \otimes \mathfrak{P}_d$, by Corollary 3.163 part 2, $\mathfrak{Q}_d$ is a prime component of $\mathfrak{F}^{\circ} \otimes \mathfrak{P}_d$ and, thus, is one of the $\mathfrak{Q}'_j$. By the same corollary, there exists γ an element of the pro-finite Galois group of $\mathfrak{F}^{\circ}$ over $\mathfrak{F}$ such that $\gamma\mathfrak{Q}'_c = \mathfrak{Q}_d$. Let $\mathfrak{P}'' = \gamma\mathfrak{P}'$. Then

$$\mathfrak{P}'' = \gamma\mathfrak{P}' \supseteq \gamma\mathfrak{Q}'_c = \mathfrak{Q}_d \supset \bigcap_j \mathfrak{Q}_j = \mathfrak{R}.$$

Therefore, $\mathfrak{P}'' \in V(\mathfrak{R})$ and $p(\mathfrak{P}'') = \mathfrak{P}$. This establishes the claim. $\square$

Proposition 3.170 *Let* $(X, \mathcal{O}_X)$ *and* $(Y, \mathcal{O}_Y)$ *be two* E-$\mathcal{F}$*-schemes of* E-$\mathcal{F}$*-finite type, and let* $(f, f^\#) : (Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$ *be a morphism of* E-$\mathcal{F}$*-schemes. Suppose* $(Y, \mathcal{O}_Y)$ *is geometrically connected and non-empty, and* $(X, \mathcal{O}_X)$ *is connected. Then* $(X, \mathcal{O}_X)$ *is geometrically connected.*

Proof: Let $\mathcal{F}^\circ$ be an algebraic closure of $\mathcal{F}$ with its induced E-structure. Let $(f_{\mathcal{F}^\circ}, f^\#_{\mathcal{F}^\circ}) : (Y_{\mathcal{F}^\circ}, \mathcal{O}_{Y_{\mathcal{F}^\circ}}) \longrightarrow (X_{\mathcal{F}^\circ}, \mathcal{O}_{X_{\mathcal{F}^\circ}})$ be the morphism of E-$\mathcal{F}^\circ$-schemes induced from $(f, f^\#)$ by extending the base by $(\operatorname{diffspec}\mathcal{F}^\circ, \mathcal{O}_{\operatorname{diffspec}\mathcal{F}^\circ}) \longrightarrow (\operatorname{diffspec}\mathcal{F}, \mathcal{O}_{\operatorname{diffspec}\mathcal{F}})$. Let $(q, q^\#) : (Y_{\mathcal{F}^\circ}, \mathcal{O}_{Y_{\mathcal{F}^\circ}}) \longrightarrow (Y, \mathcal{O}_Y)$ and $(p, p^\#) : (X_{\mathcal{F}^\circ}, \mathcal{O}_{X_{\mathcal{F}^\circ}}) \longrightarrow (X, \mathcal{O}_X)$ be the projections.

$$\begin{tikzcd}
(Y_{\mathcal{F}^\circ}, \mathcal{O}_{Y_{\mathcal{F}^\circ}}) \arrow[r, "(f_{\mathcal{F}^\circ}, f^\#_{\mathcal{F}^\circ})"] \arrow[d, "(q, q^\#)"'] & (X_{\mathcal{F}^\circ}, \mathcal{O}_{X_{\mathcal{F}^\circ}}) \arrow[r] \arrow[d, "(p, p^\#)"'] & (\operatorname{diffspec}\mathcal{F}^\circ, \mathcal{O}_{\operatorname{diffspec}\mathcal{F}^\circ}) \arrow[d] \\
(Y, \mathcal{O}_Y) \arrow[r, "(f, f^\#)"] & (X, \mathcal{O}_X) \arrow[r] & (\operatorname{diffspec}\mathcal{F}, \mathcal{O}_{\operatorname{diffspec}\mathcal{F}}).
\end{tikzcd}$$

The continuous maps on the underlying spaces $p : X_{\mathcal{F}^\circ} \longrightarrow X$ and $q : Y_{\mathcal{F}^\circ} \longrightarrow Y$ are open by Lemma 3.167 and closed by Proposition 3.168. To prove $(X, \mathcal{O}_X)$ is geometrically connected is suffices to show that $(X_{\mathcal{F}^\circ}, \mathcal{O}_{X_{\mathcal{F}^\circ}})$ is connected, i.e. $X_{\mathcal{F}^\circ}$ is connected. Assume $X_{\mathcal{F}^\circ} = U_1 \cup U_2$ where $U_1, U_2 \subseteq X_{\mathcal{F}^\circ}$ are non-empty disjoint open and closed subsets. For $i = 1, 2$, $p(U_i) \subset X$ is non-empty, open and closed, and $p(U_i) = X$ because X is connected. Since Y is non-empty, so are $V_i = f_{\mathcal{F}^\circ}^{-1}(U_i) \subseteq Y_{\mathcal{F}^\circ}$ for $i = 1, 2$ because p is surjective (Lemma 3.164). Since $Y_{\mathcal{F}^\circ}$ is connected and V_1 and V_2 are non-empty, open and closed subsets, $Y_{\mathcal{F}^\circ} = V_1 = V_2$. But then $U_1 = f_{\mathcal{F}^\circ}(V_1) = f_{\mathcal{F}^\circ}(V_2) = U_2$, which is contrary to the assumption. □

Corollary 3.171 *Let* $(X, \mathcal{O}_X)$ *be a connected* E-$\mathcal{F}$*-scheme of* E-$\mathcal{F}$*-finite type with an* $\mathcal{F}$*-rational point* $(\zeta, \zeta^{\#}) : (\operatorname{diffspec}\mathcal{F}, \mathcal{O}_{\operatorname{diffspec}\mathcal{F}}) \longrightarrow (X, \mathcal{O}_X)$. *Then* $(X, \mathcal{O}_X)$ *is geometrically connected.*

Proof: In the proposition, take $(Y, \mathcal{O}_Y) = (\operatorname{diffspec}\mathcal{F}, \mathcal{O}_{\operatorname{diffspec}\mathcal{F}})$ and $(f, f^{\#}) = (\zeta, \zeta^{\#})$. Clearly $\operatorname{diffspec}\mathcal{F}$ is non-empty. Observe that $\operatorname{diffspec}\mathcal{F}$ is geometrically connected because $(\operatorname{diffspec}\mathcal{G}, \mathcal{O}_{\operatorname{diffspec}\mathcal{G}}) =$

$$(\operatorname{diffspec}\mathcal{F}, \mathcal{O}_{\operatorname{diffspec}\mathcal{F}}) \times_{(\operatorname{diffspec}\mathcal{F}, \mathcal{O}_{\operatorname{diffspec}\mathcal{F}})} (\operatorname{diffspec}\mathcal{G}, \mathcal{O}_{\operatorname{diffspec}\mathcal{G}})$$

for all E-fields $\mathcal{G}$ containing $\mathcal{F}$ and $\operatorname{diffspec}\mathcal{G}$ is connected since it has only one point. □

Corollary 3.172 *The connected component of an* E-$\mathcal{F}$*-group scheme is geometrically connected.*

Proof: In the last corollary, take the $\mathcal{F}$-rational point to be the E-$\mathcal{F}$-morphism $(\mathrm{id}, \mathrm{id}^{\#})$. □

Lemma 3.173 *Let X and X' be two topological spaces, and let $f : X' \to X$ be an open surjective continuous map such that $f^{-1}(x)$ is connected* (resp. *irreducible*) *for all $x \in X$. Then for X' to be connected* (resp. *irreducible*) *it is necessary and sufficient for X to be connected* (resp. *irreducible*).

Proof: ([12, Lemma 4.4.2, page 59]) Assume X' is connected. Let $X = U_1 \cup U_2$ where U_1 and U_2 are two disjoint open subsets of X. Because f is continuous, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are two disjoint open subsets of X', and $X' = f^{-1}(U_1) \cup f^{-1}(U_2)$. Since X' is connected, either $f^{-1}(U_1)$ or $f^{-1}(U_2)$ is empty. Since f is surjective, either U_1 or U_2 is empty. Therefore X is connected.

Assume X is connected. Let $X' = U_1 \cup U_2$ where U_1 and U_2 are two disjoint open subsets of X'. For each $x \in X$, the subset $f^{-1}(x)$ of X' with the induced topology is the disjoint union of the two open subsets $f^{-1}(x) \cap U_1$ and $f^{-1}(x) \cap U_2$. One of these must be empty because $f^{-1}(x)$ is assumed to be connected. Therefore either $f^{-1}(x) \subset U_1$ or $f^{-1}(x) \subset U_2$, and $f(U_1)$ and $f(U_2)$ are disjoint. Because f is a surjective open map, $X = f(U_1) \cup f(U_2)$ is the disjoint union of two open sets. Since X is connected, either $f(U_1)$ or $f(U_2)$ is empty. Again by the surjectivity of f, either U_1 or U_2 is empty, and X' is connected.

Assume X' is irreducible. Since f is surjective, $X = f(X')$. Then X is irreducible because the closure of image of an irreducible closed set is irreducible (Lemma 1.7).

Assume X is irreducible. Let V_1' and V_2' be two closed sets of X' such that $V_1' \cup V_2' = X'$. Let $V_i = \{x \in X \mid f^{-1}(x) \subseteq V_i'\}$ for $i = 1, 2$. For $i = 1, 2$, $V_i = X - f(X' - V_i')$ is closed because $X' - V_i'$ and $f(X' - V_i')$ are open. For all $x \in X$, $f^{-1}(x)$ is irreducible by hypothesis and is the union of the closed sets $f^{-1}(x) \cap V_1'$ and $f^{-1}(x) \cap V_2'$. Since one of these closed sets must equal $f^{-1}(x)$, $x \in V_1$ or $x \in V_2$: that is $X = V_1 \cup V_2$. As X was assumed to be irreducible, $X = V_1$ or $X = V_2$, and $X' = V_1'$ or $X' = V_2'$. □

Proposition 3.174 *Let* $\mathcal{F}$ *be an algebraically closed* E*-field, and let* $\mathcal{G}$ *be an* E*-field extension of* $\mathcal{F}$*. Let* $(X, \mathcal{O}_X)$ *be an* E-$\mathcal{F}$*-scheme of finite* E*-type. Then,* $(X, \mathcal{O}_X)$ *is connected* (resp. *irreducible*) *if and only if* $(X_{\mathcal{G}}, \mathcal{O}_{X_{\mathcal{G}}})$ *is connected* (resp. *irreducible*)*.*

Proof: The proof is an application of Lemma 3.173 to the map π where $(\pi, \pi^{\#}) : (X_{\mathcal{G}}, \mathcal{O}_{X_{\mathcal{G}}}) \longrightarrow (X, \mathcal{O}_X)$ is the projection. The map π is surjective by Lemma 3.164 and open by Lemma 3.167. Since $\pi^{-1}(x)$ irreducible implies that it is also connected (Lemma 3.158), it is sufficient to show that the fiber $\pi^{-1}(x)$ is irreducible for all $x \in X$. Since clearly $\mathcal{G}$ is regular over the algebraically closed field $\mathcal{F}$, Lemma 3.164 implies $\pi^{-1}(x)$ consists of one element, and, therefore, is irreducible. □

Proposition 3.175 *Let* $(Z, \mathcal{O}_Z)$ *be an* E-$\mathcal{F}$*-group scheme. Then* Z° *is geometrically irreducible.*

Proof: Let $\mathcal{V}$ be an E-universal extension of $\mathcal{F}$, and let $\mathcal{F}^\circ$ be the algebraic closure of $\mathcal{F}$ in $\mathcal{V}$. Since, by definition, $(Z^\circ, \mathcal{O}_{Z^\circ})$ is connected and the identity is an $\mathcal{F}$-rational point of $(Z^\circ, \mathcal{O}_{Z^\circ})$, Corollary 3.171 implies $((Z^\circ)_{\mathcal{F}^\circ}, \mathcal{O}_{(Z^\circ)_{\mathcal{F}^\circ}})$ is connected. By Proposition 3.174, $((Z^\circ)_{\mathcal{V}}, \mathcal{O}_{(Z^\circ)_{\mathcal{V}}})$ is connected.

Let $(\pi, \pi^{\#}) : (Z_{\mathcal{V}}, \mathcal{O}_{Z_{\mathcal{V}}}) \longrightarrow (Z, \mathcal{O}_Z)$ be the projection. Since the image of a connected set under a continuous map is connected, $\pi((Z_{\mathcal{V}})^\circ)$ is connected. Because $\pi((Z_{\mathcal{V}})^\circ)$ contains $\mathrm{id}(0)$, $\pi((Z_{\mathcal{V}})^\circ) \subseteq Z^\circ$. Since $\pi^{-1}(Z^\circ) = (Z^\circ)_{\mathcal{V}}$, $(Z_{\mathcal{V}})^\circ \subseteq \pi^{-1}(Z^\circ) = (Z^\circ)_{\mathcal{V}}$. Because $(Z_{\mathcal{V}})^\circ$ is the largest connected subset of $Z_{\mathcal{V}}$ containing $\mathrm{id}_{\mathcal{V}}(0)$, $(Z_{\mathcal{V}})^\circ = (Z^\circ)_{\mathcal{V}}$.

Since the E-$\mathcal{V}$-scheme $(Z_{\mathcal{V}}, \mathcal{O}_{Z_{\mathcal{V}}})$ is clearly of finite E-$\mathcal{V}$-type, $Z_{\mathcal{V}}$ is an NG-space (Proposition 2.80). Therefore, $Z_{\mathcal{V}}$ is the union of a finite number of closed irreducible closed sets $V_1, \ldots, V_r$ (Proposition 1.14). Assume there exist $i \neq j$ such that $V_i \cap V_j$ is non-empty. Let $z \in V_i \cap V_j$, and let $y \in V_i$ such that $y \notin V_k$ for $k \neq i$. Since $\mathcal{V}$ is E-universal over $\mathcal{F}$, Corollary 2.89 may be used to conclude that there exist two $\mathcal{V}$-valued points $\psi_z = (\psi_z, \psi_z^{\#}), \psi_y = (\psi_y, \psi_y^{\#}) \in Z(\mathcal{V})$ such that $\psi_z(0) = z$ and $\psi_y(0) = y$. Let $\zeta = (\zeta, \zeta^{\#}) = (\psi_z, \psi_z^{\#}) \cdot (\psi_y, \psi_y^{\#})^{-1}$. The morphism $L_\zeta : (Z, \mathcal{O}_Z) \longrightarrow (Z, \mathcal{O}_Z)$ maps y to z and induces an E-$\mathcal{F}$-isomorphism of stalks $\mathcal{O}_{z(0)} \approx \mathcal{O}_{y(0)}$.

Since the minimal prime E-ideals of $\mathcal{O}_{z(0)}$ and $\mathcal{O}_{y(0)}$ correspond bijectively to the maximal closed irreducible subsets of $Z_{\mathcal{V}}$ containing z and y, $\mathcal{O}_{z(0)}$ has at least two minimal prime E-ideals, and $\mathcal{O}_{y(0)}$ has only one. This is impossible because $\mathcal{O}_{z(0)} \approx \mathcal{O}_{y(0)}$ by Remark 3.155. Therefore, all the irreducible closed components of $Z_{\mathcal{V}}$ are disjoint and are the connected components (Lemma 3.158). In particular, the connected component $(Z_{\mathcal{V}})^\circ$ containing the image of the identity is also irreducible. Since $(Z_{\mathcal{V}})^\circ = (Z^\circ)_{\mathcal{V}} = ((Z^\circ)_{\mathcal{F}^\circ})_{\mathcal{V}}$, Proposition 3.174 implies $(Z^\circ)_{\mathcal{F}^\circ}$ is irreducible. ☐

Proposition 3.176 *Let* $(Z, \mathcal{O}_Z)$ *be an* E-$\mathcal{F}$*-group scheme. Then* Z° *is irreducible. Let* $(\phi, \phi^\#)$ *be an* $\mathcal{F}$*-generic* $\mathcal{V}$*-valued point of* $(Z^\circ, \mathcal{O}_{Z^\circ})$*, where* $\mathcal{O}_{Z^\circ}$ *is the restriction of* $\mathcal{O}_Z$ *to* Z°*. Then* $\mathcal{F}\langle(\phi, \phi^\#)\rangle$ *is a regular extension of* $\mathcal{F}$*.*

Proof: By Proposition 3.175, Z° is geometrically irreducible, i.e. $(Z^\circ)_{\mathcal{F}^\circ}$ is irreducible. By Lemma 3.164, the projection of $(Z^\circ)_{\mathcal{F}^\circ}$ to Z° is surjective. Since the image of an irreducible set is irreducible, Z° is also irreducible. The result then follows from Proposition 3.166. ☐

This result will be used in the next section to verify *DAG3* in the definition of an E-$\mathcal{F}$-group.

3.3 The Differential Algebraic Group Associated to an E-$\mathcal{F}$-Group Scheme

Since, by Definition 2.78, every E-$\mathcal{F}$-scheme is an E-scheme over $(\operatorname{diffspec}\mathcal{F}, \mathcal{O}_{\operatorname{diffspec}\mathcal{F}})$, one may apply the functor of $\mathcal{V}$-valued points to E-$\mathcal{F}$-schemes to obtain a pre E-$\mathcal{F}$-set (Theorem 2.126). If the E-$\mathcal{F}$-scheme is a group object in the category of E-$\mathcal{F}$-schemes, the pre E-$\mathcal{F}$-set thus obtained is in fact an E-$\mathcal{F}$-group, as the following theorem shows. Examples of E-groups are obtained as corollaries. The remainder of this section constitutes a proof of Theorem 3.177.

Theorem 3.177 *If* $(Z, \mathcal{O}_Z)$ *is an* E-$\mathcal{F}$-*group scheme of* E-$\mathcal{F}$-*finite type,* $G = Z(\mathcal{V})$ *is an* E-$\mathcal{F}$-*group.*

Corollary 3.178 *If* $(Z, \mathcal{O}_Z)$ *is an* $\mathcal{F}$-*group scheme of* $\mathcal{F}$-*finite type,* $G = Z(\mathcal{V})$ *is a* $\mathcal{F}$-*group (an* E-$\mathcal{F}$-*group with* E *empty).*

Definition 3.179 *Let* $(Z, \mathcal{O}_Z)$ *be the* $\mathbb{Q}$-*group scheme* $\boldsymbol{GL}^{\mathrm{E}}_{\mathbb{Q}}(n)$ *(See the paragraph before Definition 3.156 for the definition of types of group schemes.). The* E-$\mathbb{Q}$-*group* $Z(\mathcal{V})$ *is denoted by* $\mathrm{GL}^{\mathrm{E}}_{\mathbb{Q}}(\mathrm{n})$ *or* $\mathrm{GL}^{\mathrm{E}}(\mathrm{n})$. *If* $n = 1$, $\mathrm{GL}^{\mathrm{E}}(\mathrm{n})$ *is called* $\mathrm{G}^{\mathrm{E}}_{\mathrm{m}}$. *The* E-$\mathbb{Q}$-*groups* $\boldsymbol{G}^{\mathrm{E}}_{a}(\mathcal{V})$ *and* $\boldsymbol{W}^{\mathrm{E}}(\mathcal{V})$ *are denoted by* $\mathrm{G}^{\mathrm{E}}_{\mathrm{a}}$ *and* W^{E}, *respectively.*

Kolchin in [25] uses the E-groups $\mathrm{GL}^{\mathrm{E}}(\mathrm{n})$, $\mathrm{G}^{\mathrm{E}}_{\mathrm{m}}$ and $\mathrm{G}^{\mathrm{E}}_{\mathrm{a}}$ without defining them precisely [25, page 28-29]. However, it is clear that he means the E-groups of the above definition.

The set $G = \mathrm{Hom}((\mathrm{diffspec}\,\mathcal{V}, \mathcal{O}_{\mathrm{diffspec}\,\mathcal{V}}), (Z, \mathcal{O}_Z))$ is clearly a group because $(Z, \mathcal{O}_Z)$ is a group object in the category of E-$\mathcal{F}$-schemes ([27, Proposition 1, page 75]). By Theorem 2.126, G is a pre E-$\mathcal{F}$-set, and the application of the functor of $\mathcal{V}$-valued points to the maps defining group law and the inverse on $(Z, \mathcal{O}_Z)$ yields pre E-$\mathcal{F}$-maps. Because the functor of $\mathcal{V}$-valued points preserves limits ([27, Theorem 1, page 112]), $(Z \times Z)(\mathcal{V})$ is the product $Z(\mathcal{V}) \times Z(\mathcal{V}) = G \times G$ in pre E-$\mathcal{F}$-sets. That $G(\mathcal{V})$ is an E-$\mathcal{F}$-group will be verified by checking that each axiom of an E-$\mathcal{F}$-group (Definition 4.2) is satisfied.

Abbreviate the $\mathcal{V}$-valued point $(\xi, \xi^{\#}) \in Z(\mathcal{V})$, the stalk $\mathcal{O}_{Z,\xi(0)}$, the induced local E-$\mathcal{F}$-homomorphism $\xi^{\#}_{\xi(0)}$ and the residue field $\mathcal{F}\langle(\xi, \xi^{\#})\rangle$ by ξ, $\mathcal{O}_{\xi}$, $\xi^{\#}_{\xi}$ and $\mathcal{F}\langle\xi\rangle$, respectively.

DAG1a. If $x_1, x_2 \in G$, $(x_1, x_2) \in G \times G$ with $\mathcal{F}\langle(x_1, x_2)\rangle = \mathcal{F}\langle x_1\rangle\mathcal{F}\langle x_2\rangle$ by Proposition 2.102 or [25, Theorem 2, page 99]. Denote $\mathrm{mult}(x_1, x_2)$ by x_1x_2. Because $\mathrm{mult}(\mathcal{V})$ is a pre E-$\mathcal{F}$-mapping (Definition 2.57), property (b) of the definition of pre E-$\mathcal{F}$-mapping implies $\mathcal{F}\langle x_1x_2\rangle \subset \mathcal{F}\langle(x_1, x_2)\rangle = \mathcal{F}\langle x_1\rangle\mathcal{F}\langle x_2\rangle$.

DAG1b. The pre E-$\mathcal{F}$-morphism $(\mathrm{mult} \circ (\mathrm{inv} \times \mathrm{id}_G))(\mathcal{V})$ (where id_G is the identity E-$\mathcal{F}$-scheme morphism of G to itself) maps (x_1, x_2) to $x_1^{-1}x_2$. By the same reasoning as in the proof of *DAG1a.*, $\mathcal{F}\langle {x_1}^{-1}x_2\rangle \subset \mathcal{F}\langle x_1\rangle\mathcal{F}\langle x_2\rangle$.

DAG2a. If $x_1, x_2, x_1', x_2' \in G$, $x_1 \leftrightarrow x_1'$, $x_2 \leftrightarrow x_2'$, and $S_{x_1',x_1}, S_{x_2',x_2}$ are compatible, $(x_1, x_2) \rightarrow (x_1', x_2')$ in $G \times G$ by Lemma 2.123 or [25, Proposition 6, page 100]. Since $\mathrm{mult}(\mathcal{V})$ is a pre E-$\mathcal{F}$-mapping, $x_1x_2 \rightarrow x_1'x_2'$ by part 3 of Definition 2.56 of a pre E-$\mathcal{F}$-mapping.

If in addition $x_1x_2 \leftrightarrow x_1'x_2'$ and if h is an E-$\mathcal{F}$-homomorphism of finitely E-generated E-overrings $\mathcal{D}$ and $\mathcal{D}'$ of $\mathcal{F}$ in $\mathcal{V}$ such that $h, S_{x_1',x_1}, S_{x_2',x_2}$ are compatible, it will now be shown h and $S_{x_1'x_2',x_1^*x_2^*}$ are also compatible. By definition of compatibility, there exist an E-$\mathcal{F}$-homomorphism

$$\varphi : \mathcal{F}\{\mathcal{D} \cup \mathcal{F}\langle x_1\rangle \cup \mathcal{F}\langle x_2\rangle\}_{\mathrm{E}} \rightarrow \mathcal{F}\{\mathcal{D}' \cup \mathcal{F}\langle x_1'\rangle \cup \mathcal{F}\langle x_2'\rangle\}_{\mathrm{E}}$$

extending h, S_{x_1',x_1} and S_{x_2',x_2}. Let K be the kernel of φ, and let

$$\overline{\varphi} : (\mathcal{F}\{\mathcal{D} \cup \mathcal{F}\langle x_1\rangle \cup \mathcal{F}\langle x_2\rangle\}_{\mathrm{E}})_K \rightarrow QF(\mathcal{F}\{\mathcal{D}' \cup \mathcal{F}\langle x_1'\rangle \cup \mathcal{F}\langle x_2'\rangle\}_{\mathrm{E}})$$

be the extension of φ to the localization of $\mathcal{F}\{\mathcal{D} \cup \mathcal{F}\langle x_1\rangle \cup \mathcal{F}\langle x_2\rangle\}_{\mathrm{E}}$ at the prime E-ideal K onto the quotient field $\mathrm{QF}(\mathcal{F}\{\mathcal{D}' \cup \mathcal{F}\langle x_1'\rangle \cup \mathcal{F}\langle x_2'\rangle\}_{\mathrm{E}})$ of $\mathcal{F}\{\mathcal{D}' \cup \mathcal{F}\langle x_1'\rangle \cup \mathcal{F}\langle x_2'\rangle\}_{\mathrm{E}}$.

Since $\overline{\varphi}$ restricted to both $\mathcal{F}\langle x_1\rangle$ and $\mathcal{F}\langle x_2\rangle$ are E-$\mathcal{F}$-isomorphisms, no non-zero element of either $\mathcal{F}\langle x_1\rangle$ and $\mathcal{F}\langle x_2\rangle$ is in the kernel of $\overline{\varphi}$. Therefore, $\mathcal{F}\langle x_1\rangle$ and $\mathcal{F}\langle x_2\rangle$ are contained in $(\mathcal{F}\{\mathcal{D} \cup \mathcal{F}\langle x_1\rangle \cup \mathcal{F}\langle x_2\rangle\}_{\mathrm{E}})_K$. Also note that $\mathcal{F}\{\mathcal{F}\langle x_1\rangle \cup \mathcal{F}\langle x_2\rangle\}_{\mathrm{E}} = \mathcal{F}\langle x_1\rangle[\mathcal{F}\langle x_2\rangle]$: thus, $\mathcal{F}\langle x_1\rangle[\mathcal{F}\langle x_2\rangle] \subseteq (\mathcal{F}\{\mathcal{D} \cup \mathcal{F}\langle x_1\rangle \cup \mathcal{F}\langle x_2\rangle\}_{\mathrm{E}})_K$.

The verification of *DAG2a* will be completed by first showing that $\mathcal{F}\langle x_1x_2\rangle \subset (\mathcal{F}\{\mathcal{D} \cup \mathcal{F}\langle x_1\rangle \cup \mathcal{F}\langle x_2\rangle\}_{\mathrm{E}})_K$ and then by showing $\overline{\varphi}$ restricted to $\mathcal{F}\langle x_1x_2\rangle$ equals $S_{x_1'x_2',x_1x_2}$. Since $\overline{\varphi}$ is also an extension of h, $\overline{\varphi}$ restricts to $\mathcal{F}\{D \cup \mathcal{F}\langle x_1x_2\rangle\}_{\mathrm{E}}$, which implies h and $S_{x_1'x_2',x_1x_2}$ are compatible and completes the proof of axiom *DAG2a*.

The following commuting diagram will be shown to be commutative.

Diagram 3.180

$$\begin{tikzcd}
\mathcal{F}\langle x_1\rangle[\mathcal{F}\langle x_2\rangle] \arrow[r, "\subset"] & \mathcal{F}\langle x_1\rangle \cdot \mathcal{F}\langle x_2\rangle & \mathcal{F}\langle x_1x_2\rangle \arrow[l, "\supset"'] \\
\mathcal{O}_{x_1} \otimes \mathcal{O}_{x_2} \arrow[u, "m_1\circ((x_1)^{\#}_{x_1}\otimes(x_2)^{\#}_{x_2})"'] \arrow[r, "m_2\circ(p^{\#}_{x_1}\otimes q^{\#}_{x_2})"] & \mathcal{O}_{(x_1,x_2)} \arrow[u, "(x_1,x_2)^{\#}_{(x_1,x_2)}"'] & \mathcal{O}_{x_1x_2} \arrow[u, "(x_1x_2)^{\#}_{(x_1x_2)}"'] \arrow[l, "\mathrm{mult}^{\#}_{(x_1x_2)}"'] \\
\mathcal{O}_{x_1'} \otimes \mathcal{O}_{x_2'} \arrow[u, "\|"] \arrow[r, "m_2'\circ(p^{\#}_{x_1'}\otimes q^{\#}_{x_2'})"] \arrow[d, "m_1'\circ((x_1')^{\#}_{x_1'}\otimes(x_2')^{\#}_{x_2'})"] & \mathcal{O}_{(x_1',x_2')} \arrow[u, "i_{(x_1,x_2),(x_1',x_2')}"'] \arrow[d, "(x_1',x_2')^{\#}_{(x_1',x_2')}"] & \mathcal{O}_{x_1'x_2'} \arrow[u, "\|"'] \arrow[l, "\mathrm{mult}^{\#}_{(x_1'x_2')}"'] \arrow[d, "(x_1'x_2')^{\#}_{(x_1'x_2')}"] \\
\mathcal{F}\langle x_1'\rangle[\mathcal{F}\langle x_2'\rangle] \arrow[r, "\subset"] & \mathcal{F}\langle x_1'\rangle \cdot \mathcal{F}\langle x_2'\rangle & \mathcal{F}\langle x_1'x_2'\rangle \arrow[l, "\supset"']
\end{tikzcd}$$

Since $x_1 \leftrightarrow x_1'$, $x_2 \leftrightarrow x_2'$ and $x_1x_2 \leftrightarrow x_1'x_2'$ in G, then $x_1(0) = x_1'(0)$, $x_2(0) = x_2'(0)$ and $(x_1x_2)(0) = (x_1'x_2')(0)$ by Corollary 2.95. Consequently, $\mathcal{O}_{x_1} = \mathcal{O}_{x_1'}$, $\mathcal{O}_{x_2} = \mathcal{O}_{x_2'}$ and $\mathcal{O}_{x_1x_2} = \mathcal{O}_{x_1'x_2'}$. This establishes the two equalities in the diagram.

Since $(x_1, x_2) \longrightarrow (x_1', x_2')$ in $G \times G$, by applying Lemma 2.73 to mult and the two projections $p, q : Z \times Z \longrightarrow Z$, the following three diagrams commute.

Diagram 3.181

$$\begin{tikzcd}
\mathcal{O}_{(x_1,x_2)} & \mathcal{O}_{x_1x_2} \arrow[l, "\mathrm{mult}^{\#}_{(x_1x_2)}"'] \\
\mathcal{O}_{(x_1',x_2')} \arrow[u, "i_{(x_1,x_2),(x_1',x_2')}"'] & \mathcal{O}_{x_1'x_2'} \arrow[l, "\mathrm{mult}^{\#}_{(x_1'x_2')}"'] \arrow[u, "\|"]
\end{tikzcd}$$

This is the square in the middle right of Diagram 3.184.

Diagram 3.182

$$\begin{tikzcd}
\mathcal{O}_{x_1} \arrow[r, "p^{\#}_{x_1}"] & \mathcal{O}_{(x_1,x_2)} \\
\mathcal{O}_{x_1'} \arrow[r, "p^{\#}_{x_1'}"] \arrow[u, "\|"] & \mathcal{O}_{(x_1',x_2')} \arrow[u, "i_{(x_1,x_2),(x_1',x_2')}"']
\end{tikzcd}$$

Diagram 3.183

$$\begin{tikzcd}
\mathcal{O}_{x_2} \arrow[r, "q^{\#}_{x_2}"] & \mathcal{O}_{(x_1,x_2)} \\
\mathcal{O}_{x_2'} \arrow[r, "q^{\#}_{x_2'}"] \arrow[u, "\|"] & \mathcal{O}_{(x_1',x_2')} \arrow[u, "i_{(x_1,x_2),(x_1',x_2')}"']
\end{tikzcd}$$

The square in the middle left of Diagram 3.184 commutes because both the horizontal homomorphisms in the last two diagrams extend to $\mathcal{O}_{x_1} \otimes \mathcal{O}_{x_2} = \mathcal{O}_{x'_1} \otimes \mathcal{O}_{x'_2}$. The top right square commutes because local homomorphisms pass to the residue field as described before Lemma 2.114 and $\mathcal{F}\langle(x_1, x_2)\rangle = \mathcal{F}\langle x_1\rangle \cdot \mathcal{F}\langle x_2\rangle$ (Proposition 2.102). The proof of the commutivity of the bottom right square is similar.

The top left square commutes because the following diagram commutes

Diagram 3.184

$$\begin{tikzcd}
\mathcal{F}\langle x_1\rangle \otimes \mathcal{F}\langle x_2\rangle \arrow[r, "i_1 \otimes i_2"] & \mathcal{F}\langle(x_1, x_2)\rangle \otimes \mathcal{F}\langle(x_1, x_2)\rangle \arrow[r, "m_1"] & \mathcal{F}\langle(x_1, x_2)\rangle \\
\mathcal{O}_{x_1} \otimes \mathcal{O}_{x_2} \arrow[u, "(x_1)^{\#}_{x_1} \otimes (x_2)^{\#}_{x_2}"'] \arrow[r, "p^{\#}_{x_1} \otimes q^{\#}_{x_2}"] & \mathcal{O}_{(x_1,x_2)} \otimes \mathcal{O}_{(x_1,x_2)} \arrow[u, "\Upsilon"] \arrow[r, "m_2"] & \mathcal{O}_{(x_1,x_2)} \arrow[u, "(x_1,x_2)^{\#}_{(x_1,x_2)}"]
\end{tikzcd}$$

and the image of m_1 is $\mathcal{F}\langle x_1\rangle[\mathcal{F}\langle x_2\rangle]$, where m_1 and m_2 are the homomorphisms defining the ring multiplication, the middle map is $\Upsilon = (x_1, x_2)^{\#}_{(x_1,x_2)} \otimes (x_1, x_2)^{\#}_{(x_1,x_2)}$, and $i_1 : \mathcal{F}\langle x_1\rangle \to \mathcal{F}\langle(x_1, x_2)\rangle$ and $i_2 : \mathcal{F}\langle x_2\rangle \to \mathcal{F}\langle(x_1, x_2)\rangle$ are the inclusions. The bottom left square commutes for the same reason.

Let $\alpha \in \mathcal{F}\langle(x_1 x_2)\rangle$. To show $\alpha \in (\mathcal{F}\{\mathcal{D} \cup \mathcal{F}\langle x_1\rangle \cup \mathcal{F}\langle x_2\rangle\}_{\mathrm{E}})_K$, refer to Diagram 3.184 and choose $\alpha' \in \mathcal{O}_{x'_1 x'_2} = \mathcal{O}_{x_1 x_2}$ such that $(x_1 x_2)^{\#}_{(x_1 x_2)}(\alpha') = \alpha$. Let $\alpha'' = \mathrm{mult}^{\#}_{(x'_1 x'_2)}(\alpha') \in \mathcal{O}_{(x'_1, x'_2)}$. By the second statement in Proposition 2.102, there exist $\beta', \gamma' \in \mathcal{O}_{x'_1} \otimes \mathcal{O}_{x'_2}$ such that

$$(m_2 \circ (p^{\#}_{x'_1} \otimes q^{\#}_{x'_2}))(\beta') = \beta'' \ , \ (m_2 \circ (p^{\#}_{x'_1} \otimes q^{\#}_{x'_2}))(\gamma') = \gamma''$$

are elements of $\mathcal{O}_{(x_1',x_2')}$, $\gamma'' \notin \mathfrak{P}_{(x_1',x_2')}$ (the maximal prime E-ideal of $\mathcal{O}_{(x_1',x_2')}$), and $\alpha'' = \beta''/\gamma'' \in \mathcal{O}_{(x_1',x_2')}$. By Lemma 2.72, $i_{(x_1,x_2),(x_1',x_2')}(\gamma'') \notin \mathfrak{P}_{(x_1,x_2)}$ (the maximal prime E-ideal of $\mathcal{O}_{(x_1,x_2)}$).

The map $(m_1' \circ ((x_1')^{\#}_{x_1'} \otimes (x_2')^{\#}_{x_2'})) \circ (m_1 \circ ((x_1)^{\#}_{x_1} \otimes (x_2)^{\#}_{x_2}))^{-1}$ down the right hand side of Diagram 3.184 is clearly φ because they are equal when restricted to $\mathcal{F}\langle x_1 \rangle$ and $\mathcal{F}\langle x_2 \rangle$. Let $K' \subset \mathcal{O}_{x_1'} \otimes \mathcal{O}_{x_2'} = \mathcal{O}_{x_1} \otimes \mathcal{O}_{x_2}$ be the inverse image of $K \subset \mathcal{F}\langle x_1 \rangle[\mathcal{F}\langle x_2 \rangle]$ under the E-homomorphism $m_1 \circ ((x_1)^{\#}_{x_1} \otimes (x_2)^{\#}_{x_2})$ in the upper left square of Diagram 3.184. Then $(m_1' \circ ((x_1')^{\#}_{x_1'} \otimes (x_2')^{\#}_{x_2'}))(\mathcal{K}') = 0$ since $\varphi(\mathcal{K}) = 0$. Because the lower left square of Diagram 3.184 commutes $(m_2 \circ (p^{\#}_{x_1'} \otimes q^{\#}_{x_2'}))(K') \subset \mathfrak{P}_{(x_1',x_2')}$. Therefore, since $\gamma'' \notin \mathfrak{P}_{(x_1',x_2')}$,

$$\gamma'' \notin (m_2 \circ (p^{\#}_{x_1'} \otimes q^{\#}_{x_2'}))(K').$$

By the commutivity of the middle left square and the upper left squares, $\gamma = (m_1 \circ ((x_1)^{\#}_{x_1} \otimes (x_2)^{\#}_{x_2}))(\gamma') \notin K \subset \mathcal{F}\langle x_1 \rangle[\mathcal{F}\langle x_2 \rangle]$. Let $\beta = (m_1 \circ ((x_1)^{\#}_{x_1} \otimes (x_2)^{\#}_{x_2}))(\beta')$. Then $\alpha = \beta/\gamma$, and this implies $\alpha \in (\mathcal{F}\{\mathcal{D} \cup \mathcal{F}\langle x_1 \rangle \cup \mathcal{F}\langle x_2 \rangle\}_{\mathrm{E}})_K$.

It remains to show that $\overline{\varphi}$ restricted to $\mathcal{F}\langle x_1 x_2 \rangle_{\mathrm{E}}$ is $S_{x_1'x_2',x_1x_2}$. By the last result, any $\alpha \in \mathcal{F}\langle x_1 x_2 \rangle \subseteq \mathcal{F}\langle x_1 \rangle_{\mathrm{E}} \cdot \mathcal{F}\langle x_2 \rangle_{\mathrm{E}}$ may be written as the quotient β/γ with $\beta, \gamma \in \mathcal{F}\langle x_1 \rangle_{\mathrm{E}}[\mathcal{F}\langle x_2 \rangle_{\mathrm{E}}]$ such that $\gamma \notin K$. Referring to Diagram 3.184, since $m_1 \circ ((x_1)^{\#}_{x_1} \otimes (x_2)^{\#}_{x_2})$:

$$\mathcal{O}_{x_1'} \otimes \mathcal{O}_{x_2'} = \mathcal{O}_{x_1} \otimes \mathcal{O}_{x_2} \longrightarrow \mathcal{F}\langle x_1 \rangle \otimes \mathcal{F}\langle x_2 \rangle \longrightarrow \mathcal{F}\langle x_1 \rangle[\mathcal{F}\langle x_2 \rangle]$$

and $(x_1x_2)^{\#}_{(x_1x_2)} : \mathcal{O}_{x'_1x'_2} = \mathcal{O}_{x_1x_2} \longrightarrow \mathcal{F}\langle x_1x_2\rangle$ are surjective, one may choose $\beta', \gamma' \in \mathcal{O}_{x'_1} \otimes \mathcal{O}_{x'_2}$ and $\alpha' \in \mathcal{O}_{(x'_1x'_2)}$ such that

$$(m_1 \circ ((x_1)^{\#}_{x_1} \otimes (x_2)^{\#}_{x_2}))(\beta') = \beta,$$

$$(m_1 \circ ((x_1)^{\#}_{x_1} \otimes (x_2)^{\#}_{x_2}))(\gamma') = \gamma \quad \text{and}$$

$$((x_1x_2)^{\#}_{(x_1x_2)})(\alpha') = \alpha = \beta/\gamma.$$

Let $\beta'' = (m'_2 \circ ((p)^{\#}_{q} \otimes (x'_2)^{\#}_{x'_2}))(\beta') \in \mathcal{O}_{(x'_1,x'_2)}$, $\gamma'' = (m'_2 \circ ((p)^{\#}_{x'_1} \otimes (q)^{\#}_{x'_2}))(\gamma') \in \mathcal{O}_{(x'_1,x'_2)}$ and $\alpha'' = (\mathrm{mult}^{\#}_{(x'_1x'_2)})(\alpha') \in \mathcal{O}_{(x'_1,x'_1)}$. By the commutitivity of Diagram 3.184,

$$(x_1,x_2)^{\#}_{(x_1,x_2)}(i_{(x_1,x_2),(x'_1,x'_2)}(\beta'' - \alpha''\gamma'')) = \beta - \alpha\gamma = 0.$$

Since the maximal E-ideal $\mathfrak{P}_{(x_1,x_2)} \subset \mathcal{O}_{(x_1,x_2)}$ is the kernel of $i_{(x_1,x_2),(x'_1,x'_2)}$, $i_{(x_1,x_2),(x'_1,x'_2)}(\beta'' - \alpha''\gamma'') \in \mathfrak{P}_{(x_1,x_2)}$. Because, by Lemma 2.72, $\mathfrak{P}_{(x'_1,x'_2)} \supseteq (i_{(x_1,x_2),(x'_1,x'_2)})^{-1}(\mathfrak{P}_{(x_1,x_2)})$, it follows that $\beta'' - \alpha''\gamma'' \in \mathfrak{P}_{(x'_1,x'_2)}$. Then $(x'_1,x'_2)^{\#}_{(x'_1,x'_2)}(\beta'' - \alpha''\gamma'') = 0$ since $\mathfrak{P}_{(x'_1,x'_2)}$ is the kernel of $(x'_1,x'_2)^{\#}_{(x'_1,x'_2)}$.

Let $\alpha''' = (x'_1,x'_2)^{\#}_{(x'_1,x'_2)}(\alpha'')$, $\beta''' = (x'_1,x'_2)^{\#}_{(x'_1,x'_2)}(\beta'')$ and $\gamma''' = (x'_1,x'_2)^{\#}_{(x'_1,x'_2)}(\gamma'')$. Then, $\beta'''-\alpha'''\gamma''' = 0$. That the three elements α''', β''' and γ''' independent of the choices above of α', β' and γ' maybe deduced from the commutitivity of Diagram 3.184 and the fact that $\mathfrak{P}_{(x'_1,x'_2)} \supseteq (i_{(x_1,x_2),(x'_1,x'_2)})^{-1}(\mathfrak{P}_{(x_1,x_2)})$ (Lemma 2.72).

As previously explained,

$$\varphi = (m'_1 \circ ((x'_1)^{\#}_{x'_1} \otimes (x'_2)^{\#}_{x'_2})) \circ (m_1 \circ ((x_1)^{\#}_{x_1} \otimes (x_2)^{\#}_{x_2}))^{-1}$$

is the left hand column of Diagram 3.184, and, similarly, $S_{x'{}_1x'{}_2,x_1x_2} = (x'_1x'_2)^{\#}_{(x'_1x'_2)} \circ ((x_1x_2)^{\#}_{(x_1x_2)})^{-1}$ is the right hand column. Again by the commutivity of this diagram, $\beta''' = \varphi(\beta)$, $\gamma''' = \varphi(\gamma)$ and $\alpha''' = S_{x'{}_1x'{}_2,x_1x_2}(\alpha)$. Since $\gamma \notin K$, $\gamma''' \neq 0$, and

$$S_{x'{}_1x'{}_2,x_1x_2}(\alpha) = \alpha''' = \beta'''/\gamma''' = \varphi(\beta)/\varphi(\gamma) = \overline{\varphi}(\beta/\gamma) = \overline{\varphi}(\alpha).$$

Therefore, $\overline{\varphi}$ restricted to $\mathfrak{F}\langle x_1x_2\rangle$ is $S_{x'_1x'_2,x_1x_2}$.

DAG2b. Let x_1, x_2, x'_1, x'_2 be elements of G such that $x_1 \rightarrow x'_1$ and $x_2 \rightarrow x'_2$. By [25, Lemma 1(b), page 32], there exist elements $x^*_1, x^*_2 \in G$ such that $x^*_1 \leftrightarrow x_1, x^*_2 \leftrightarrow x_2$, x^*_1 and x^*_2 are algebraically disjoint, and $S_{x'_1,x^*_1}$ and $S_{x'_2,x^*_2}$ are compatible. Then $x^*_1 \rightarrow x'_1$ and $x^*_2 \rightarrow x'_2$ by transitivity, and $(x^*_1, x^*_2) \rightarrow (x'_1, x'_2)$ in $G \times G$ by Proposition 2.125. Because $\mathrm{mult}(\mathcal{V})$ is a pre E-$\mathfrak{F}$-mapping, $x^*_1x^*_2 \rightarrow x'_1x'_2$. When in addition $x^*_1x^*_2 \leftrightarrow x'_1x'_2$ and $x^*_2 \leftrightarrow x'_2$, because $(\mathrm{mult} \times \mathrm{id}_G) \circ (\mathrm{id}_G \times \Delta)$ ($\Delta : G \times G \rightarrow G$ is the diagonal pre E-$\mathfrak{F}$-mapping)

$$\begin{tikzcd} G \times G \arrow[r, "\mathrm{id}_G \times \Delta"] & G \times G \times G \arrow[r, "\mathrm{mult} \times \mathrm{id}_G"] & G \times G \end{tikzcd}$$

is a pre E-$\mathfrak{F}$-mapping, $(x^*_1, x^*_2) \rightarrow (x'_1, x'_2)$ implies $(x^*_1x^*_2, x^*_2) \rightarrow (x'_1x'_2, x'_2)$ because of part 3 in definition of a pre E-$\mathfrak{F}$ mapping (Definition 2.56 3). Lemma 2.123 implies $S_{x'_1x'_2,x^*_1x^*_2}$ and $S_{x'_2,x^*_2}$ are compatible.

DAG2c. Since $\mathrm{inv}_{\mathcal{V}}$ is a pre E-$\mathfrak{F}$-map, if $x_1 \leftrightarrow x'_1$, then , by [25, part c of Proposition 1, page 33] $x_1^{-1} \leftrightarrow x_1'^{-1}$, $\mathfrak{F}\langle x_1\rangle = \mathfrak{F}\langle x_1^{-1}\rangle$ and $S_{x'_1,x_1} = S_{x_1'^{-1},x_1^{-1}}$. Then apply *DAG2a.*

DAG2d. By first part of the proof *DAG2b*, there exist elements $x_1^*, x_2^* \in G$ such that $x_1^* \leftrightarrow x_1, x_2^* \leftrightarrow x_2$, x_1^* and x_2^* are algebraically disjoint, and $(x_1^*, x_2^*) \to (x_1', x_2')$. Then $x_1^{*-1}x_2^* \to x_1'^{-1}x_2'$ because $\mathrm{mult} \circ (\mathrm{inv} \times \mathrm{id}_G)$ is a pre E-map (Definition 2.56 3).

DAG3. This is Proposition 3.176.

3.4 The Differential Group Associated to an E-Strongly Normal Extension

In this section and the next, $\mathcal{F}$ will denote an (E, Δ)-field, $\mathcal{U}$ an (E, Δ)-field that is (E, Δ)-universal over $\mathcal{F}$, and $\mathcal{C}$ the Δ-constants of $\mathcal{F}$. Then $\mathcal{K} = \mathcal{U}^{\Delta}$ may be considered as an E-field. As such, it is E-universal over $\mathcal{C}$, considered as an E-field. The (E, Δ)-field $\mathcal{G} \subset \mathcal{U}$ will contain $\mathcal{F}$.

Definition 3.185 *An* E*-strongly normal extension* $\mathcal{G}$ *of the* (E, Δ)*-field* $\mathcal{F}$ *is a finitely* (E, Δ)*-generated extension* $\mathcal{G}$ *of* $\mathcal{F}$ *such that every* (E, Δ)*-*$\mathcal{F}$*-isomorphism of* $\mathcal{G}$ *is* E*-strong (Definition 2.128).*

Remark 3.186 *If* $\mathcal{G}$ *over* $\mathcal{F}$ *is* E*-strongly normal, it is not necessarily a strongly normal extension for* Δ *because* $\mathcal{G}$ *over* $\mathcal{F}$ *might not be finitely* Δ*-generated. A strongly normal extension for* (E, Δ) *is an* E*-strongly normal extension if each* (E, Δ)*-isomorphism leaves invariant not only every element of* $\mathcal{G}^{E,\Delta}$ *but also those of* $\mathcal{G}^{\Delta}$*.*

Let $G = \mathrm{Isom}_{\mathcal{F}}^{\mathrm{E},\Delta}(\mathcal{G}, \mathcal{U})$. In this section, it will be shown that, if $\mathcal{G}$ is an E-strongly normal extension of $\mathcal{F}$, then G has the structure of an E-$\mathcal{C}$-group

(Theorem 3.197). Also, in this paper, a strongly normal extension (in the sense of Kolchin [24, p.393]) for a finite set Γ of commuting derivations will refer to a finitely Γ-generated Γ-field extension $\mathfrak{G}$ over $\mathfrak{F}$ such that every Γ-isomorphism of $\mathfrak{G}$ over $\mathfrak{F}$ is strong (in the sense of Kolchin) for Γ.

Proposition 3.187 *If $\mathfrak{G}$ is an* E*-strongly normal extension of $\mathfrak{F}$, then $\mathfrak{F}$ and $\mathfrak{G}$ have the same field of Δ-constants.*

Proof: By Definition 2.128 St1, the Δ-constants in $\mathfrak{G}$ are invariant under every isomorphism of $\mathfrak{G}$ over $\mathfrak{F}$. Since any element of $\mathfrak{G}$ fixed by all E-$\mathfrak{F}$-isomorphisms of $\mathfrak{G}$ is in $\mathfrak{F}$ (Proposition 1.53), the Δ-constants of $\mathfrak{G}$ are contained in $\mathfrak{F}$. □

Proposition 3.188 *Let $\mathfrak{G}$ be a finitely (E, Δ)-generated extension of $\mathfrak{F}$ having the same field of Δ-constants as $\mathfrak{F}$. Let $\sigma_1 \dots \sigma_r$ be (E, Δ)-$\mathfrak{F}$-isomorphisms of $\mathfrak{G}$ such that every (E, Δ)-$\mathfrak{F}$-isomorphism of $\mathfrak{G}$ is an (E, Δ)-$\mathfrak{G}$-specialization of one of these. If $\sigma_k\mathfrak{G} \subset \mathfrak{G}\mathfrak{U}^\Delta$ for all k, $(1 \leq k \leq r)$, then $\mathfrak{G}$ is* E*-strongly normal over $\mathfrak{F}$.*

Proof: Let σ be any (E, Δ)-$\mathfrak{F}$-isomorphism of $\mathfrak{G}$. Since $\mathfrak{G}^\Delta = \mathfrak{F}^\Delta$, σ fixes $\mathfrak{G}^\Delta$. By considering σ as a Δ-homomorphism and the remark after [24, Proposition 6, page 390], $\sigma\mathfrak{G} \subset \mathfrak{G}\mathfrak{U}^\Delta$ since $\sigma_i\mathfrak{G} \subset \mathfrak{G}\mathfrak{U}^\Delta$.

To prove that σ is E-strong, it remains to show $\mathfrak{G} \subset \sigma\mathfrak{G}\mathfrak{U}^\Delta$. Following the technique of the proof in [24, Proposition 10, page 393], one may show that the (E, Δ)-$\mathfrak{F}$-isomorphism $\sigma^{-1} : \sigma\mathfrak{G} \approx \mathfrak{G}$ can be extended to an (E, Δ)-$\mathfrak{F}$-isomorphism φ of $\mathfrak{G}\sigma\mathfrak{G}$ by Proposition 1.52. The restriction of φ to $\mathfrak{G}$ is an

(E, Δ)-$\mathfrak{F}$-isomorphism τ of $\mathfrak{G}$ over $\mathfrak{F}$. Thus, $\varphi : \mathfrak{G}\sigma\mathfrak{G} \approx \tau\mathfrak{G} \cdot \mathfrak{G}$ is an (E, Δ)-$\mathfrak{F}$-isomorphism, $\varphi\mathfrak{G} = \tau\mathfrak{G}$, $\varphi(\sigma\mathfrak{G}) = \mathfrak{G}$, and $\varphi(\mathfrak{G}^{\Delta}\langle\sigma\rangle) = \mathfrak{G}^{\Delta}\langle\tau\rangle$. By the final result of the last paragraph, $\tau\mathfrak{G} \subseteq \mathfrak{G}\mathfrak{G}^{\Delta}\langle\tau\rangle$. Therefore $\mathfrak{G} = \varphi^{-1}(\tau\mathfrak{G}) \subseteq \varphi^{-1}(\mathfrak{G}\mathfrak{G}^{\Delta}\langle\tau\rangle) = \varphi^{-1}\mathfrak{G} \cdot \varphi^{-1}(\mathfrak{G}^{\Delta}\langle\tau\rangle) = \sigma\mathfrak{G} \cdot \mathfrak{G}^{\Delta}\langle\sigma\rangle \subset \sigma\mathfrak{G}\mathfrak{U}^{\Delta}$. □

Corollary 3.189 *Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be extensions of $\mathfrak{F}$ such that $\mathfrak{G}_1\mathfrak{G}_2$ has the same field of Δ-constants as $\mathfrak{F}$. If $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are* E*-strongly normal over $\mathfrak{F}$, then so is $\mathfrak{G}_1\mathfrak{G}_2$.*

Proof: Obviously $\mathfrak{G}_1\mathfrak{G}_2$ is a finitely (E, Δ)-generated extension of $\mathfrak{F}$. If σ is any isomorphism of $\mathfrak{G}_1\mathfrak{G}_2$ over $\mathfrak{F}$, then the restriction σ_i of σ to $\mathfrak{G}_i$ is an E-strong (E, Δ)-isomorphism of $\mathfrak{G}_i$ so that $\sigma(\mathfrak{G}_1\mathfrak{G}_2) = \sigma_1\mathfrak{G}_1 \cdot \sigma_2\mathfrak{G}_2 \subset \mathfrak{G}_1\mathfrak{U}^{\Delta} \cdot \mathfrak{G}_2\mathfrak{U}^{\Delta} = (\mathfrak{G}_1\mathfrak{G}_2) \cdot \mathfrak{U}^{\Delta}$. It follows by Proposition 3.188 that $\mathfrak{G}_1\mathfrak{G}_2$ is an E-strongly normal extension of $\mathfrak{F}$. □

The following proposition shows why G is a group.

Proposition 3.190 *Let $\mathfrak{G}$ be any Δ-field in $\mathfrak{U}$. Each* E*-strong* (E, Δ)*-isomorphism of $\mathfrak{G}$ can be extended to a unique* (E, Δ)*–automorphism of $\mathfrak{G}\mathfrak{U}^{\Delta}$ over $\mathfrak{U}^{\Delta}$. Conversely, the restriction to $\mathfrak{G}$ of each* (E, Δ)*–automorphism of $\mathfrak{G}\mathfrak{U}^{\Delta}$ over $\mathfrak{U}^{\Delta}$ is a* E*-strong* (E, Δ)*-isomorphism of $\mathfrak{G}$.*

Proof: By [24, Corollary 1, page 87], $\mathfrak{G}$ and $\mathfrak{U}^{\Delta}$ are linearly disjoint over $\mathfrak{G}^{\Delta}$. Also, if σ is any E-strong (E, Δ)-isomorphism of $\mathfrak{G}$ (Definition 2.128), then $\sigma\mathfrak{G}$ and $\mathfrak{U}^{\Delta}$ are also linearly disjoint over $\mathfrak{G}^{\Delta}$. Therefore σ can be extended to a unique (E, Δ)-isomorphism $s : \mathcal{G}\mathcal{U}^{\Delta} \approx \sigma\mathcal{G} \cdot \mathcal{U}^{\Delta}$ over $\mathcal{U}^{\Delta}$. Because σ is E-strong (E, Δ)-isomorphism, $\sigma\mathcal{G}\mathcal{U}^{\Delta} = \mathcal{G}\mathcal{U}^{\Delta}$, and s is an (E, Δ)-automorphism of $\mathcal{G}\mathcal{U}^{\Delta}$ over $\mathcal{U}^{\Delta}$. The converse is clear. □

This proposition canonically identifies the set of all E-strong (E, Δ)-isomorphisms of $\mathcal{G}$ with the set of all (E, Δ)-automorphisms of $\mathcal{G}\mathcal{U}^{\Delta}$ over $\mathcal{U}^{\Delta}$. Because the set of all (E, Δ)-automorphisms of $\mathcal{G}\mathcal{U}^{\Delta}$ over $\mathcal{U}^{\Delta}$ has a natural group structure, this identification induces a group structure on the set of all E-strong (E, Δ)-isomorphisms of $\mathcal{G}$. If $\mathcal{F}$ is an (E, Δ)-subfield of $\mathcal{G}$, the set of all E-strong (E, Δ)-isomorphisms of $\mathcal{G}$ over $\mathcal{F}$ can be canonically identified with the group G of all (E, Δ)-automorphisms of $\mathcal{G}\mathcal{U}^{\Delta}$ over $\mathcal{F}\mathcal{U}^{\Delta}$, which is a subgroup of the group of all (E, Δ)-automorphisms of $\mathcal{G}\mathcal{U}^{\Delta}$ over $\mathcal{U}^{\Delta}$.

Recall the definitions of the E-type, E-dimension and typical E-dimension of a pre E-set in [25, page 31]. If $\mathcal{H}$ over $\mathcal{F}$ (considered as an E-field) is E-extension that is finitely E-generated by $\rho = (\rho_1, \ldots, \rho_n)$, $\omega_{\rho/\mathcal{F}}$ will denote the E-transcendence polynomial of ρ over $\mathcal{F}$ ([24, page 117]).

Proposition 3.191 *Let $\mathcal{G}$ be an* E-*strongly normal extension of $\mathcal{F}$, and let $\mathcal{C}$ denote the field of Δ-constants of $\mathcal{F}$. For every isomorphism σ of $\mathcal{G}$ over $\mathcal{F}$, define $\mathcal{C}\langle\sigma\rangle = (\mathcal{G}\sigma\mathcal{G})^{\Delta}$. Then $\mathcal{C}\langle\sigma\rangle$, as an* E-*field extension of $\mathcal{C}$, is finitely* E-*generated over $\mathcal{C}$. Moreover, $\mathcal{G}$ is finitely* E-*generated over $\mathcal{F}$, and, for every isolated isomorphism σ of $\mathcal{G}$ over $\mathcal{F}$, the* E-*type* (*resp.* E-*dimension, typical* E-*dimension*) *of $\mathcal{C}\langle\sigma\rangle$ over $\mathcal{C}$ is equal to the* E-*type* (*resp.*

E-*dimension, typical* E-*dimension) of* $\mathcal{G}$ *over* $\mathcal{F}$.

Proof: That $\mathcal{C}\langle\sigma\rangle$ is a finitely E-generated field extension of $\mathcal{C}$ for every isomorphism σ of $\mathcal{G}$ over $\mathcal{F}$ is Proposition 2.130. To show $\mathcal{G}$ is finitely E-generated over $\mathcal{F}$, let σ be an isolated (E, Δ)-isomorphism of $\mathcal{G}$ over $\mathcal{F}$ that specializes to the identity isomorphism. By [24, Corollary (b), page 388], σ leaves fixed the algebraic closure $\mathcal{F}^\circ$ of $\mathcal{F}$ in $\mathcal{G}$. Let $\eta = (\eta_1, \ldots, \eta_n)$ be a family of (E, Δ)-generators of $\mathcal{G}$ over $\mathcal{F}$, i.e. $\mathcal{G} = \mathcal{F}\langle\eta\rangle_{\mathrm{E},\Delta}$, and let $\xi = (\xi_1, \ldots, \xi_r)$ be a family of E-generators of $\mathcal{C}\langle\sigma\rangle$ over $\mathcal{C}$, i.e. $\mathcal{C}\langle\sigma\rangle = \mathcal{C}\langle\xi\rangle_{\mathrm{E}}$. Since $\mathcal{G}\langle\sigma\eta\rangle_{\mathrm{E},\Delta} = \mathcal{G}\sigma\mathcal{G} = \mathcal{G}\mathcal{C}\langle\sigma\rangle = \mathcal{G}\langle\xi\rangle_{\mathrm{E},\Delta}$, each coordinate of ξ is in the E-field generated over $\mathcal{G}$ by a finite number of Δ-derivatives of $\sigma\eta$. Denote the set Δ-derivatives of $\sigma\eta$ by $\vartheta = (\vartheta_1, \ldots, \vartheta_s)$. Then ϑ E-generates $\mathcal{G}\sigma\mathcal{G}$ over $\mathcal{G}$.

Claim 3.192 $\mathcal{F}^\circ\langle\vartheta\rangle_{\mathrm{E}} = \sigma\mathcal{G}$

Proof: By the definition of ϑ, $\mathcal{F}^\circ\langle\vartheta\rangle_{\mathrm{E}} \subset \sigma\mathcal{G}$. Let $\alpha \in \sigma\mathcal{G}$. Then $\alpha \in \mathcal{G}\sigma\mathcal{G} = \mathcal{G}\langle\sigma\eta\rangle_{\mathrm{E},\Delta} = \mathcal{G} \cdot \mathcal{F}^\circ\langle\vartheta\rangle_{\mathrm{E}}$. If $(\gamma_i)_{i\in I}$ is a basis for $\mathcal{F}^\circ\langle\vartheta\rangle_{\mathrm{E}}$ over $\mathcal{F}^\circ$, $\alpha = (\Sigma\, g_i\gamma_i)/(\Sigma\, g'_j\gamma_j)$, with g_i and g'_j in $\mathcal{G}$ and not all the g'_j are 0. Therefore, $\Sigma\, g'_j(\gamma_j\alpha) - \Sigma\, g_i\gamma_i = 0$, and the family $(\gamma_j\alpha, \gamma_i)$ of elements of $\sigma\mathcal{G}$ is linearly dependent over $\mathcal{G}$. Since σ is isolated, $\sigma\mathcal{G}$ and $\mathcal{G}$ are algebraically disjoint over $\mathcal{F}$ (Corollary 1.48). *A fortiori*, they are also algebraically disjoint over $\mathcal{F}^\circ$. Since $\mathcal{G}$ is regular over $\mathcal{F}^\circ$, $\sigma\mathcal{G}$ and $\mathcal{G}$ are linearly disjoint over $\mathcal{F}^\circ$ ([26, Theorem 3, page 57]). By this disjointness, the family $(\gamma_j\alpha, \gamma_i)$ is linearly dependent over $\mathcal{F}^\circ$. So there exists f_i and f'_i elements of $\mathcal{F}^\circ$, not all 0, such that $\Sigma\ f'_j(\gamma_j\alpha) - \Sigma\ f_i\gamma_i = 0$. Because the γ_j are linearly independent over $\mathfrak{F}^\circ$, $\Sigma\ f'_j\gamma_j \neq 0$. Therefore $\alpha = (\Sigma\ f_i\gamma_i)/(\Sigma\ f'_j\gamma_j) \in \mathfrak{F}^\circ\langle\vartheta\rangle$. ☐

Since the E-field $\sigma\mathfrak{G} = \mathfrak{F}^\circ\langle\vartheta\rangle_{\mathrm{E}}$ is finitely E-generated over $\mathfrak{F}^\circ$, $\mathfrak{G} = \mathfrak{F}^\circ\langle\sigma^{-1}\vartheta\rangle_{\mathrm{E}}$ (as E-fields) is also finitely E-generated over $\mathfrak{F}^\circ$. Because any intermediate extension of a finitely E-generated extension is finitely E-generated [24, Chapter 2, Proposition 14, p. 112], it follows that $\mathfrak{F}^\circ$ is finitely (E, Δ)-generated over $\mathfrak{F}$, and, hence, also finitely E-generated over $\mathfrak{F}$ (because $\mathfrak{F}^\circ$ is algebraic). Thus $\mathfrak{G}$ is finitely E-generated over $\mathfrak{F}$.

Then

$$\omega_{\sigma^{-1}\vartheta/\mathfrak{F}} = \omega_{\vartheta/\mathfrak{F}} = \omega_{\vartheta/\mathfrak{G}}$$

since the first equality would be true for any E-$\mathfrak{F}$-isomorphism σ ([24, page 387]) and the second equality holds by [24, Comment on page 117] because $\sigma\mathfrak{G}$ and $\mathfrak{G}$ are algebraically disjoint over $\mathfrak{F}$ (Corollary 1.48). Also,

$$\omega_{\xi/\mathfrak{G}} = \omega_{\xi/\mathfrak{C}}$$

because $\mathfrak{G}$ and $\mathfrak{C}\langle\sigma\rangle$ are linearly disjoint over $\mathfrak{C}$ [24, Corollary 1,page 87] and [24, Comment on page 117]. Because ϑ and ξ both E-generate $\mathfrak{G}\sigma\mathfrak{G}$ over $\mathfrak{G}$, the E-birational invariants (E-type, E-dimension, typical E-dimension) of $\omega_{\vartheta/\mathfrak{G}}$ and $\omega_{\zeta/\mathfrak{G}}$ are equal ([24, page 118] or [25, page 7]). By utilizing the above equalities, the E-birational invariants of $\omega_{\sigma^{-1}\vartheta/\mathfrak{G}}$ and $\omega_{\zeta/\mathfrak{C}}$ are also the same. Thus the E-type (*resp.* E-dimension, typical E-dimension) of $\mathfrak{C}\langle\sigma\rangle$ over $\mathfrak{C}$ is equal to the E-type (*resp.* E-dimension, typical E-dimension) of $\mathfrak{G}$ over $\mathfrak{F}$. ☐

The following two propositions and corollary will be used to show that G satisfies the axioms of an E-$\mathfrak{F}$-group in Theorem 3.197.

Proposition 3.193 *Let σ and τ be two* E*-strong* (E,Δ)*-isomorphisms of* $\mathcal{G}$*. Then* $\mathcal{G}^\Delta\langle\sigma\rangle\mathcal{G}^\Delta\langle\sigma\tau\rangle = \mathcal{G}^\Delta\langle\sigma\rangle\mathcal{G}^\Delta\langle\tau\rangle = \mathcal{G}^\Delta\langle\sigma\tau\rangle\mathcal{G}^\Delta\langle\tau\rangle$*, and* $\mathcal{G}^\Delta\langle\sigma^{-1}\rangle = \mathcal{G}^\Delta\langle\sigma\rangle$ *as* E*-fields.*

Proof: By considering the fields in the statement of the proposition as just Δ-fields, and σ and τ as just strong Δ-isomorphisms, Kolchin's result [24, Proposition 5, p. 390] may be applied to obtain these equalities as fields in $\mathcal{U}$. Because they are also E-fields, they are equal as E-fields. □

Proposition 3.194 *Let $\sigma, \sigma', \tau, \tau'$ be* E*-strong* (E,Δ)*-isomorphisms of* $\mathcal{G}$*.*

1. *If (σ',τ') is a specialization of (σ,τ) then $(\sigma'^{-1},\sigma'^{-1}\tau')$ is a specialization of $(\sigma^{-1},\sigma^{-1}\tau)$.*

2. *Suppose that σ' and τ' are generic specializations of σ and τ, respectively. If (σ',τ') is a specialization of (σ,τ), then the induced* E*-isomorphisms* $\mathcal{G}^\Delta\langle\sigma\rangle \approx \mathcal{G}^\Delta\langle\sigma'\rangle$ *and* $\mathcal{G}^\Delta\langle\tau\rangle \approx \mathcal{G}^\Delta\langle\tau'\rangle$ *are compatible, and conversely.*

3. *Suppose that σ' and τ' are generic specializations of σ and τ, respectively, let $h : \mathcal{D} \longrightarrow \mathcal{D}'$ be an* E*-homomorphism between subrings of* $\mathcal{U}^\Delta$*. If h and the induced* E*-isomorphisms* $\mathcal{G}^\Delta\langle\sigma\rangle \approx \mathcal{G}^\Delta\langle\sigma'\rangle$ *and* $\mathcal{G}^\Delta\langle\tau\rangle \approx \mathcal{G}^\Delta\langle\tau'\rangle$ *are compatible, then σ'^{-1} is a generic specialization of*

σ^{-1} and $\sigma'^{-1}\tau'$ is a specialization of $\sigma^{-1}\tau$; when the latter specialization is generic, then h and the induced E*-isomorphisms $\mathcal{G}^{\Delta}\langle\sigma^{-1}\rangle \approx \mathcal{G}^{\Delta}\langle\sigma'^{-1}\rangle$ and $\mathcal{G}^{\Delta}\langle\sigma^{-1}\tau\rangle \approx \mathcal{G}^{\Delta}\langle\sigma'^{-1}\tau'\rangle$ are compatible.*

Proof: Since σ, σ', τ and τ' are E-strong (E, Δ)-isomorphisms, it follows from Kolchin's corresponding result [24, Proposition 8(a), page 391] for strong Δ-isomorphisms, that $(\sigma'^{-1}, \sigma'^{-1}\tau')$ is a specialization of $(\sigma^{-1}, \sigma^{-1}\tau)$ over $\mathcal{G}$. This remains a specialization over $\mathcal{G}$ when σ, σ', τ and τ' are considered as (E, Δ)-isomorphisms by [24, Lemma 1, page 385].

Part 2 is proved a manner similar to that of part 1: (σ', τ') is a specialization of (σ, τ), when σ, σ', τ and τ' are considered as strong Δ-isomorphisms if and only if the induced isomorphisms considered as non-differential isomorphisms are compatible. The result then follows when σ, σ', τ and τ' are again considered as (E, Δ)-isomorphisms.

Part 3 follows from the same considerations as in the previous part. $\square$

Corollary 3.195 *1. If σ' is a specialization of σ, then σ'^{-1} is a specialization of σ^{-1}. When the former specialization is generic, then so is the latter, and the induced isomorphisms $\mathcal{G}^{\Delta}\langle\sigma\rangle \approx \mathcal{G}^{\Delta}\langle\sigma'\rangle$ and $\mathcal{G}^{\Delta}\langle\sigma^{-1}\rangle \approx \mathcal{G}^{\Delta}\langle\sigma'^{-1}\rangle$ coincide.*

2. Suppose that σ' and τ' are generic specializations of σ and τ, respectively, such that the induced E*-isomorphisms $\mathcal{G}^{\Delta}\langle\sigma\rangle \approx \mathcal{G}^{\Delta}\langle\sigma'\rangle$ and $\mathcal{G}^{\Delta}\langle\tau\rangle \approx \mathcal{G}^{\Delta}\langle\tau'\rangle$ are compatible, then $\sigma'\tau'$ is a specialization of $\sigma\tau$.*

When the last specialization is generic, and $h : \mathcal{D} \longrightarrow \mathcal{D}'$ is an E-*homomorphism between subrings of $\mathcal{U}^{\Delta}$ such that h and the induced* E-*isomorphisms $\mathcal{G}^{\Delta}\langle\sigma\rangle \approx \mathcal{G}^{\Delta}\langle\sigma'\rangle$ and $\mathcal{G}^{\Delta}\langle\tau\rangle \approx \mathcal{G}^{\Delta}\langle\tau'\rangle$ are compatible, then h and the induced* E-*isomorphism $\mathcal{G}^{\Delta}\langle\sigma\tau\rangle \approx \mathcal{G}^{\Delta}\langle\sigma'\tau'\rangle$ are compatible.*

Proof: The first assertion follows from part 1 of the proposition, in the special case in which $\tau = \sigma, \tau' = \sigma'$. Since $\mathcal{G}^{\Delta}\langle\sigma^{-1}\rangle = \mathcal{G}^{\Delta}\langle\sigma\rangle$ (Proposition 3.193), the second assertion follows from part 3 of the proposition, in the special case in which $\tau = \sigma, \tau' = \sigma'$, and h is the induced E-isomorphism $\mathcal{G}^{\Delta}\langle\sigma\rangle \approx \mathcal{G}^{\Delta}\langle\sigma'\rangle$.

Because of part 1, one may replace σ, σ' by $\sigma^{-1}, \sigma'^{-1}$. Part 2 then follows from part 3 of the proposition. ☐

Endow G with the following pre E-C-set structure.

1. For each $\sigma \in G$, let $\mathcal{C}\langle\sigma\rangle = (\mathcal{G}\sigma\mathcal{G})^{\Delta}$.
2. For each $(\sigma, \sigma') \in G^2$, let $\sigma \longrightarrow \sigma'$ mean that σ' is an (E, Δ)-$\mathcal{F}$-specialization of σ (Definition 1.34).
3. For each $(\sigma, \sigma') \in G^2$ with $\sigma \leftrightarrow \sigma'$ (that is, with σ' a generic E-$\mathcal{F}$-specialization of σ), let $S_{\sigma',\sigma}$ denote the induced E-$\mathcal{F}$-isomorphism $\mathcal{C}\langle\sigma\rangle \approx \mathcal{C}\langle\sigma'\rangle$ (Definition in Proposition 2.131).

Proposition 3.196 *With the above pre* E-$\mathcal{C}$-*set structure, G is a pre* E-$\mathcal{C}$-*set.*

Proof: By Proposition 1.49, G has a finite number of isolated is elements $\sigma_1, \ldots, \sigma_2$, and the closure of each one is a pre E-$\mathcal{C}$-set (Proposition 2.131). Since the union of a finite number of pre E-$\mathcal{C}$-sets is again a pre E-$\mathcal{C}$-set, G is a pre E-$\mathcal{C}$-set. □

Recall the three different types of specialization, each of which will be used in the proof of the main theorem of this section. For (E, Δ)-$\mathcal{F}$-isomorphisms σ and σ' of $\mathcal{G}$ into $\mathcal{U}$, σ (E, Δ)-$\mathcal{G}$-specializes to σ' if there exists an (E, Δ)-$\mathcal{G}$-homomorphism $\phi : \mathcal{G}\{\sigma\mathcal{G}\}_{(\mathrm{E},\Delta)} \longrightarrow \mathcal{G}\{\sigma'\mathcal{G}\}_{(\mathrm{E},\Delta)}$ such that $\phi(\alpha) = \alpha$ and $\phi(\sigma\alpha) = \sigma'\alpha$ for all α in $\mathcal{G}$ (Definition 1.34). If $\mathcal{G}$ is finitely (E, Δ)-generated over $\mathcal{F}$ by $\eta = \{\eta_1, \ldots, \eta_n\}$, this is equivalent to the family $\sigma\eta$ (E, Δ)-$\mathcal{G}$-specializes to the family $\sigma'\eta$ (Example 1.4) by Lemma 1.35. The third kind of specialization is the pre order associated to a pre E-set (Definition 2.54). In the statement of the theorem, the elements of G are defined to be (E, Δ)-$\mathcal{F}$-isomorphisms of $\mathcal{G}$, which in the course of the proof will be shown to be a pre E-$\mathcal{F}^{\Delta}$-set. As such, the pre order $\sigma \longrightarrow \sigma'$ is called an E-$\mathcal{F}^{\Delta}$-specialization.

Theorem 3.197 *Let* $\mathcal{G}$ *be an* E*-strongly normal extension of the* Δ*-field* $\mathcal{F}$ *with field of* Δ*-constants* $\mathcal{C}$*, and let* $G = \mathrm{Isom}^{\mathrm{E},\Delta}_{\mathcal{F}}(\mathcal{G}, \mathcal{V})$*. With the pre* E-$\mathcal{C}$*-set structure defined above,* G *is an* E-$\mathcal{C}$*-group. Furthermore, the field* $\mathcal{G}$ *is finitely* E*-generated, and, as such, the* E*-type* (resp. E*-dimension, typical* E*-dimension) of* $\mathcal{G}$ *over* $\mathcal{F}$ *equals the* E*-type* (resp. E*-dimension, typical* E*-dimension) of the* E-$\mathcal{C}$*-group* G*.*

Proof: It has been established that G is an E-$\mathcal{C}$-set G (Observation 3.196) and is a group (Proposition 3.190). Axiom *DAG 1* follows from Proposition 3.193. Axiom *DAG 2a* follows from part 2 of Corollary 3.195. Part 3 of Proposition 3.194 implies Axiom *DAG 2c*.

To prove parts *DAG2b* and *DAG2d*, let $\sigma, \sigma', \tau, \tau'$ be E-strong (E, Δ)-isomorphisms of $\mathcal{G}$ over $\mathcal{F}$ with $\sigma \longrightarrow \sigma'$ and $\tau \longrightarrow \tau'$. Fix a family $\eta = (\eta_1 \ldots \eta_n)$ of (E, Δ)-generators of $\mathcal{G}$ over $\mathcal{F}$, and let p (*resp.* q) denote the defining (E, Δ)-ideal of $\sigma^{-1}\eta$ (*resp.* $\tau\eta$) in the (E, Δ)–polynomial algebra $\mathcal{G}\{y_1, \ldots, y_n\}_{\mathrm{E},\Delta}$ (*resp.* $\mathcal{G}\{z_1, \ldots, z_n\}_{\mathrm{E},\Delta}$). Let $\mathcal{G}_a$ denote the algebraic closure of $\mathcal{G}$ in $\mathcal{U}$. Then $\mathcal{G}_a p$ and $\mathcal{G}_a q$ have components $p_1, \ldots, p_r$ and $q_1, \ldots, q_s$ such that the quotient fields $QF(\mathcal{G}_a\{y_1, \ldots, y_n\}_{\mathrm{E},\Delta}/p_i)$ for $i = 1, \ldots, r$ and $QF(\mathcal{G}_a\{z_1, \ldots, z_n\}_{\mathrm{E},\Delta}/q_j)$ for $j = 1, \ldots, s$ are regular over $\mathcal{G}_a$ [24, Proposition 3, page 131]. By [25, Corollary, page 132], each (E, Δ)–ideal $r_{kl} = \{p_k \cup q_l\}_{(\mathrm{E},\Delta)}$ of $\mathcal{G}_a\{y_1, \ldots, y_n, z_1, \ldots, z_n\}_{\mathrm{E},\Delta}$ is prime. Therefore, r_{kl} has a $\mathcal{G}_a$-generic (E, Δ)-zero $(\eta^{(k,l)}, \xi^{(k,l)})$ where $\eta^{(k,l)}$ is a generic zero of $r_{kl} \cap \mathcal{G}_a\{y_1, \ldots, y_n\}_{\mathrm{E},\Delta} = p_k$ and therefore of $p_k \cap \mathcal{G}\{y_1, \ldots, y_n\}_{\mathrm{E},\Delta} = p$, so that $\eta^{(k,l)}$ is a $\mathcal{G}$-generic (E, Δ)-specialization of $\sigma^{-1}\eta$ over $\mathcal{G}$ and hence over $\mathcal{F}$. Therefore $\eta^{(k,l)}$ is the image of η by an E-strong (E, Δ)-isomorphism of $\mathcal{G}$ over $\mathcal{F}$, which is denoted by σ_{kl}^{-1} and is defined by $\eta^{(k,l)} = \sigma_{kl}^{-1}\eta$. By [24, Lemma 2, page 386], $\sigma^{-1} \leftrightarrow \sigma_{kl}^{-1}$. Similarly $\xi^{(k,l)} = \tau_{kl}\eta$ for some E-strong (E, Δ)-isomorphism τ_{kl} of $\mathcal{G}$ over $\mathcal{F}$ with $\tau \leftrightarrow \tau_{kl}$. By hypothesis $\sigma \longrightarrow \sigma'$, whence $\sigma^{-1} \longrightarrow \sigma'^{-1}$ (part 1 of the Corollary 3.195) so that $\sigma'^{-1}\eta$ is an (E, Δ)-zero of p and hence of some p_k. Similarly, $\tau'\eta$ is an

(E, Δ)-zero of some q_l. Thus, $(\sigma'^{-1}\eta, \tau'\eta)$ is an (E, Δ)-zero of r_{kl}, thus $(\sigma_{kl}^{-1}\eta, \tau_{kl}\eta) \to_{\mathcal{G}_a} (\sigma'^{-1}\eta, \tau'\eta)$ and hence over $\mathcal{G}$. It follows [24, Lemma 2, page 386], that $(\tau_{kl}, \sigma_{kl}^{-1}) \to_{\mathcal{G}} (\tau', \sigma'^{-1})$, and hence by Proposition 3.194 1 and 2, that $(\tau_{kl}^{-1}, \tau_{kl}^{-1}\sigma_{kl}^{-1}) \to_{\mathcal{G}} (\tau'^{-1}, \tau'^{-1}\sigma'^{-1})$ and that if $\tau_{kl}^{-1}\sigma_{kl}^{-1} \leftrightarrow \tau'^{-1}\sigma'^{-1}$ and $\tau_{kl}^{-1} \leftrightarrow \tau'^{-1}$, then the induced E-$\mathcal{C}$-isomorphisms $\mathcal{C}\langle\tau_{kl}^{-1}\sigma_{kl}^{-1}\rangle \approx \mathcal{C}\langle\tau'^{-1}\sigma'^{-1}\rangle$ and $\mathcal{C}\langle\tau_{kl}^{-1}\rangle \approx \mathcal{C}\langle\tau'^{-1}\rangle$ are compatible. By part 1 of the Corollary 3.195, then $\sigma_{kl}\tau_{kl} \to \sigma'\tau'$ and if $\sigma_{kl}\tau_{kl} \leftrightarrow \sigma'\tau'$ and $\tau_{kl} \leftrightarrow \tau'$, then the induced E-$\mathcal{C}$-isomorphisms $\mathcal{C}\langle\sigma_{kl}\tau_{kl}\rangle \approx \mathcal{C}\langle\sigma'\tau'\rangle$ and $\mathcal{C}\langle\tau_{kl}\rangle \approx \mathcal{C}\langle\tau'\rangle$ are compatible. This proves *DAG2b*, and (because $\sigma^{-1} \to \sigma'^{-1}$ whenever $\sigma \to \sigma'$) also part *DAG2d*.

To prove axiom *DAG3*, one must show that if σ is an isolated isomorphism of $\mathcal{G}$ over $\mathcal{F}$ with $\sigma \to id_{\mathcal{G}}$, then $\mathcal{C}\langle\sigma\rangle$ is regular over $\mathcal{C}$. Since $\sigma \to id_{\mathcal{G}}$, $\sigma\mathcal{F}^\circ = \mathcal{F}^\circ$ ([24, Proposition 2(b), page 388]). Since $\mathcal{G}$ is regular over $\mathcal{F}^\circ$, clearly $\sigma\mathcal{G}$ is regular over $\sigma\mathcal{F}^\circ = \mathcal{F}^\circ$. By Corollary 1.48, $\sigma\mathcal{G}$ is algebraically disjoint from $\mathcal{G}$ over $\mathcal{F}$, and, *a fortiori*, they are algebraically disjoint over $\mathcal{F}^\circ$. Because $\mathcal{G}$ is regular over $\mathcal{F}^\circ$, $\sigma\mathcal{G}$ is linearly disjoint from $\mathcal{G}$ over $\mathcal{F}^\circ$ ([26, Theorem 3, page 57]).

$$\begin{tikzcd}
\mathcal{G} \arrow[r] & \mathcal{G}\sigma\mathcal{G} \\
\mathcal{F}^\circ \arrow[u] \arrow[r] & \sigma\mathcal{G}, \arrow[u]
\end{tikzcd}$$

Recall ([26, Corollary 6, page 58]), that if K and L are field extensions of field k in a larger field and if they linearly disjoint over k, then K is regular over k if and only if KL is regular over L. Therefore, $\mathcal{G}\sigma\mathcal{G}$ is regular over $\mathcal{G}$.

Since $\mathcal{G}$ and $\mathcal{C}\langle\sigma\rangle$ are linearly disjoint over $\mathcal{C}$ ([24, Corollary 2, page 88])

$$\begin{tikzcd}
\mathcal{G} \arrow[r] & \mathcal{G}\mathcal{C}\langle\sigma\rangle = \mathcal{G}\sigma\mathcal{G} \\
\mathcal{C} \arrow[u] \arrow[r] & \mathcal{C}\langle\sigma\rangle . \arrow[u]
\end{tikzcd}$$

and $\mathcal{G}\mathcal{C}\langle\sigma\rangle = \mathcal{G}\sigma\mathcal{G}$, that $\mathcal{G}\sigma\mathcal{G}$ is regular over $\mathcal{G}$ implies $\mathcal{C}\langle\sigma\rangle$ is regular over $\mathcal{C}$, which is *DAG3*. This establishes G as a E-$\mathcal{C}$-group. ☐

Definition 3.198 *By virtue of Theorem 3.197, the set of* E*-strong* (E, Δ)*-isomorphisms of the* E*-strongly normal extension* $\mathcal{G}$ *over* $\mathcal{F}$ *has a natural structure of an* E-$\mathcal{C}$*-group relative to the* E*-universal field* $\mathcal{U}^{\Delta}$*. This* E-$\mathcal{C}$*-group is called the Galois group of* $\mathcal{G}$ *over* $\mathcal{F}$*, and it is denoted by* $G_{\mathrm{E}}(\mathcal{G}/\mathcal{F})$ *or* $G(\mathcal{G}/\mathcal{F})$*. The* $\mathcal{C}$*-component of the identity of* $G_{\mathrm{E}}(\mathcal{G}/\mathcal{F})$ *is denoted by* $G^{\circ}_{\mathrm{E}}(\mathcal{G}/\mathcal{F})$ *or* $G^{\circ}(\mathcal{G}/\mathcal{F})$*.*

It is desirable to consider E-$\mathcal{C}$-groups and, more generally, E-$\mathcal{F}$-sets relative to the E-field $\mathcal{U}$, which, as an E-field, is clearly E-universal over E-field $\mathcal{F}$. Therefore, when referring to an E-$\mathcal{C}$-group that is not the Galois group of an E-strongly normal extension (or an E-$\mathcal{F}$-set) it is meant, unless the contrary is indicated, an E-$\mathcal{C}$-group (or an E-$\mathcal{F}$-set) relative to the universal field $\mathcal{U}$ considered as an E-field. When G is such an E-$\mathcal{C}$-group, then $\mathrm{G}_{\mathcal{K}}$ is an E-$\mathcal{C}$-group relative to the E-universal E-field $\mathcal{K} = \mathcal{U}^{\Delta}$. Furthermore by [25, page 144] every E-$\mathcal{C}$-group relative to $\mathcal{K}$ is obtainable in this way from an E-$\mathcal{C}$-group relative to $\mathcal{U}$. When $\mathcal{X}$ is an E-$\mathcal{F}$-set and $\alpha \in \mathcal{X}(\mathcal{U})$, $\mathcal{F}\langle\alpha\rangle_{\mathrm{E}}$ or just $\mathcal{F}\langle\alpha\rangle$ will denote the E-field generated by α over $\mathcal{F}$.

Definition 3.199 *If G is any* E-$\mathcal{C}$-*group, a G*-extension of $\mathcal{F}$ *is any* E-*strongly normal extension $\mathcal{G}$ of $\mathcal{F}$ such that $G(\mathcal{G}/\mathcal{F})$ is* E-$\mathcal{C}$-*isomorphic to an* E-$\mathcal{C}$-*subgroup of $G_{\mathcal{K}}$. When $G(\mathcal{G}/\mathcal{F})$ is* E-$\mathcal{C}$-*isomorphic to $G_{\mathcal{K}}$ itself, the* (E, Δ)-*extension $\mathcal{G}$ over $\mathcal{F}$ is called* full. *A* linear extension *of $\mathcal{F}$ is an* E-GL(n)-extension of $\mathcal{F}$ *for some natural number n.* (*See Definition 3.179.*)

3.5 Extending the Constants.

Definition 3.200 [25, page 48] *Let $\mathcal{C}$ be an* E-*field, let $\mathcal{V}$ be another* E-*field that is* E-*universal over $\mathcal{F}$, and let $\mathcal{D} \subset \mathcal{V}$ be an* E-*field containing $\mathcal{F}$ over which $\mathcal{V}$ is* E-*universal. Let G be an* E-$\mathcal{C}$-*group relative to $\mathcal{V}$, and let H be an* E-$\mathcal{D}$-*group relative to $\mathcal{V}$. An* E-($\mathcal{D}, \mathcal{C}$)-homomorphism of H into G *is a group homomorphism $f : H \longrightarrow G$ that satisfies the following three conditions:*

1. *if $y \in H$, then $\mathcal{D}\langle y\rangle \supset \mathcal{C}\langle f(y)\rangle$,*
2. *if $y, y' \in H$ and $y \longrightarrow y'$ over $\mathcal{D}$, then $f(y) \longrightarrow f(y')$ over $\mathcal{C}$,*
3. *if $y, y' \in H$ and $y \leftrightarrow y'$ over $\mathcal{D}$, then $S_{\mathcal{D},y',y}$ extends $S_{\mathcal{C},f(y'),f(y)}$.*

Definition 3.201 [25, page 49] *An* E-$\mathcal{D}$-*group structure on G is said to be* induced (by the given E-$\mathcal{C}$-group structure on G) *if the following two conditions are satisfied:*

1. *the identity map* id_G *on the set* G *is an* E-$(\mathcal{D}, \mathcal{C})$-*homomorphism from* G *with the structure of an* E-$\mathcal{D}$-*group to* G *with the structure of the* E-$\mathcal{C}$-*group* G*;*
2. *every* E-$(\mathcal{D}, \mathcal{C})$-*homomorphism of an* E-$\mathcal{D}$-*group into* G *is an* E-$\mathcal{D}$-*homomorphism.*

The following generalization of [24, Theorem 2, page 396] interprets, for an E-extension $\mathcal{C}'$ of $\mathcal{C}$ in $\mathcal{K}$ $(= \mathcal{U}^{\Delta})$, the induced E-$\mathcal{C}'$-group of the E-$\mathcal{C}$-group $G(\mathcal{G}/\mathcal{F})$.

Theorem 3.202 *Let* $\mathcal{G} \subset \mathcal{U}$ *be an* E-*strongly normal extension of* $\mathcal{F}$*. Denote the field of* Δ-*constants of* $\mathcal{F}$ *by* $\mathcal{C}$*, and let* $\mathcal{C}' \subset \mathcal{K}$ *be an* (E, Δ)-*extension of* $\mathcal{C}$ *such that* $\mathcal{U}$ *is* (E, Δ)-*universal over* $\mathcal{F}\mathcal{C}'$*. Then* $\mathcal{U}$ *is* (E, Δ)-*universal over* $\mathcal{G}\mathcal{C}'$*, and* $\mathcal{G}\mathcal{C}'$ *is an* E-*strongly normal extension of* $\mathcal{F}\mathcal{C}'$ *with field of* Δ-*constants* $\mathcal{C}'$*. Furthermore, the* E-$\mathcal{C}'$-*group* $G(\mathcal{G}\mathcal{C}'/\mathcal{F}\mathcal{C}')$ *is the induced* E-$\mathcal{C}'$-*group of the* E-$\mathcal{C}$-*group* $G(\mathcal{G}/\mathcal{F})$*, both these groups being identified with each other by means of their canonical identifications with the group of* (E, Δ)-*automorphisms of* $\mathcal{G}\mathcal{K}$ *over* $\mathcal{F}\mathcal{K}$ *(See Proposition 3.190).*

$$\begin{tikzcd}
\mathcal{G} \arrow[r] & \mathcal{G}\mathcal{C}' \arrow[r] & \mathcal{G}\mathcal{K} \\
\mathcal{F} \arrow[r] \arrow[u] & \mathcal{F}\mathcal{C}' \arrow[r] \arrow[u] & \mathcal{F}\mathcal{K}' \arrow[u]
\end{tikzcd}$$

Remark 3.203 *The similar result of Kolchin's* [24, Theorem 2, page 396] *cannot be used directly to prove this theorem because a* E-*strongly normal extension* $\mathcal{G}$ over $\mathcal{F}$ *is not necessarily finitely* Δ-*generated, and, thus,* $\mathcal{G}$ over $\mathcal{F}$ *is not necessarily a strongly normal* Δ-*extension in the sense of Kolchin. By definition,* $\mathcal{G}$ *is only finitely* (E, Δ)-$\mathcal{F}$-*generated.*

Proof: Since $\mathcal{G}\mathcal{C}'$ is finitely (E, Δ)-generated over $\mathcal{F}\mathcal{C}'$, [24, Proposition 4(b), page 133] shows that $\mathcal{U}$ is (E, Δ)-universal over $\mathcal{G}\mathcal{C}'$. That $\mathcal{C}'$ is the field of Δ-constants of $\mathcal{F}\mathcal{C}'$ and $\mathcal{G}\mathcal{C}'$ follows from [24, Corollary 2, page 88]. If σ is any (E, Δ)-isomorphism of $\mathcal{G}\mathcal{C}'$ over $\mathcal{F}\mathcal{C}'$, then the restriction of σ to $\mathcal{G}$ is an (E, Δ)-isomorphism of $\mathcal{G}$ over $\mathcal{F}$ and as such is E-strong. Hence, $\sigma(\mathcal{G}\mathcal{C}') = \sigma\mathcal{G}\sigma\mathcal{C}' \subset \mathcal{G} \cdot \mathcal{K} \cdot \mathcal{C}' = \mathcal{G}\mathcal{C}' \cdot \mathcal{K}$, and similarly $\mathcal{G}\mathcal{C}' \subset \sigma(\mathcal{G}\mathcal{C}') \cdot \mathcal{K}$: that is σ is E-strong. Therefore $\mathcal{G}\mathcal{C}'$ is E-strongly normal over $\mathcal{F}\mathcal{C}'$, and $G(\mathcal{G}\mathcal{C}'/\mathcal{F}\mathcal{C}')$ is a E-$\mathcal{C}'$-group (Theorem 3.197). Denote by $\mathcal{C}'\langle\sigma\rangle$ the E-$\mathcal{C}'$-field associated to any $\sigma \in G(\mathcal{G}\mathcal{C}'/\mathcal{F}\mathcal{C}')$.

Define $\mathrm{id}_G : G(\mathcal{G}\mathcal{C}'/\mathcal{F}\mathcal{C}') \longrightarrow G(\mathcal{G}/\mathcal{F})$ by identifying $\sigma \in G(\mathcal{G}\mathcal{C}'/\mathcal{F}\mathcal{C}')$ with the (E, Δ)-automorphism of $\mathcal{G}\mathcal{C}' \cdot \mathcal{K} = \mathcal{G}\mathcal{K}$ over $\mathcal{F}\mathcal{C}' \cdot \mathcal{K} = \mathcal{F}\mathcal{K}$ that extends σ (Proposition 3.190), and then with the E-strong (E, Δ)-isomorphism of $\mathcal{G}$ over $\mathcal{F}$ to which σ restricts. Then

$$\mathcal{G}\mathcal{C}'\langle\sigma\rangle = \mathcal{G}\mathcal{C}' \cdot \mathcal{C}'\langle\sigma\rangle = \mathcal{G}\mathcal{C}'\sigma(\mathcal{G}\mathcal{C}') = \mathcal{G}\mathrm{id}_{\mathrm{G}}\sigma\mathcal{G} \cdot \mathcal{C}' = \mathcal{G}\mathcal{C}\langle\mathrm{id}_{\mathrm{G}}\sigma\rangle\mathcal{C}',$$

and, by [24, Corollary 2, p. 88],

$$\mathcal{C}'\langle\sigma\rangle = \mathcal{C}\langle\mathrm{id}_{\mathrm{G}}\sigma\rangle\mathcal{C}'. \tag{17}$$

If σ' is an E-$\mathcal{C}'$-specialization of σ in $G(\mathcal{G}\mathcal{C}'/\mathcal{F}\mathcal{C}')$, then $(\sigma'\alpha)_{\alpha\in\mathcal{G}}$ is an (E,Δ)-$\mathcal{G}\mathcal{C}'$-specialization of $(\sigma\alpha)_{\alpha\in\mathcal{G}}$, and hence over $\mathcal{G}$, so that $\mathrm{id}_G\sigma'$ is an E-$\mathcal{C}'$-specialization of $\mathrm{id}_G\sigma$ in $G(\mathcal{G}/\mathcal{F})$. When the E-$\mathcal{C}'$-specialization in $G(\mathcal{G}\mathcal{C}'/\mathcal{F}\mathcal{C}')$ is $\mathcal{C}'$-generic, then there exists an (E,Δ)-isomorphism

$$\mathcal{G}\mathcal{C}'\sigma(\mathcal{G}\mathcal{C}') \approx \mathcal{G}\mathcal{C}'\sigma'(\mathcal{G}\mathcal{C}') \tag{18}$$

over $\mathcal{G}\mathcal{C}'$ mapping $\sigma\alpha$ onto $\sigma'\alpha$ for every $\alpha\in\mathcal{G}$, and this restricts to an (E,Δ)-isomorphism

$$\mathcal{G}\cdot\mathrm{id}_{\mathrm{G}}\sigma\mathcal{G} \approx \mathcal{G}\cdot\mathrm{id}_{\mathrm{G}}\sigma'\mathcal{G} \tag{19}$$

over $\mathcal{G}$, so that the E-$\mathcal{C}$-specialization in $G(\mathcal{G}/\mathcal{F})$ is $\mathcal{C}$-generic. This restricts to the induced E-$\mathcal{C}$-isomorphism

$$S^{\mathcal{C}}_{\mathrm{id}_{\mathrm{G}}\sigma',\mathrm{id}_{\mathrm{G}}\sigma} : \mathcal{C}\langle\mathrm{id}_{\mathrm{G}}\sigma\rangle \approx \mathcal{C}\langle\mathrm{id}_{\mathrm{G}}\sigma'\rangle, \tag{20}$$

which is also a restriction of the (E,Δ)-isomorphism 18. Moreover, the (E,Δ)-isomorphism 18 also restricts to the induced E-$\mathcal{C}$-isomorphism

$$S^{\mathcal{C}'}_{\sigma',\sigma} : \mathcal{C}'\langle\sigma\rangle \approx \mathcal{C}'\langle\sigma'\rangle. \tag{21}$$

Therefore, the restriction of $S^{\mathcal{C}'}_{\sigma',\sigma}$ to $\mathcal{C}\langle\mathrm{id}_{\mathrm{G}}\sigma\rangle$ is the induced E-$\mathcal{C}$-isomorphism $S^{\mathcal{C}}_{\mathrm{id}_{\mathrm{G}}\sigma',\mathrm{id}_{\mathrm{G}}\sigma} : \mathcal{C}\langle\mathrm{id}_{\mathrm{G}}\sigma\rangle \approx \mathcal{C}\langle\mathrm{id}_{\mathrm{G}}\sigma'\rangle$. Therefore, $S^{\mathcal{C}'}_{\sigma',\sigma}$ is an extension of $S^{\mathcal{C}}_{\mathrm{id}_{\mathrm{G}}\sigma',\mathrm{id}_{\mathrm{G}}\sigma}$. This shows that id_G is an E-$(\mathcal{C}',\mathcal{C})$-homomorphism.

Now let H be any E-$\mathcal{C}'$-group relative to the universal E-field $\mathcal{K}$, and let $f : H \longrightarrow G(\mathcal{G}/\mathcal{F})$ be any E-$(\mathcal{C}',\mathcal{C})$-homomorphism. To complete the proof of the theorem, it must be shown that $f' = \mathrm{id}_G{}^{-1}f$ from H to $G(\mathcal{G}\mathcal{C}'/\mathcal{F}\mathcal{C}')$ is an

$$\begin{tikzcd}
H \arrow[r, "f'"] \arrow[dr, "f"'] & G(\mathcal{G}\mathcal{C}'/\mathcal{F}\mathcal{C}') \arrow[d, "\mathrm{id}_G"] \\
 & G(\mathcal{G}/\mathcal{F})
\end{tikzcd}$$

E-$\mathcal{C}'$-homomorphism [25, Chapter 1, Section 2, p. 37]; that is f' is a homomorphism of groups and an everywhere defined pre E-$\mathcal{C}'$-mapping (Definition 2.56). Clearly f' is a homomorphism of groups. By [25, Corollary 1, p. 90], it suffices to show that the restriction, also denoted by f', of f' to the $\mathcal{C}'$-generic elements of H is a pre E-$\mathcal{C}'$-mapping.

Property 1 of the definition of pre E-$\mathcal{C}'$-mapping is clear, i.e. the domain of definition of f contains the $\mathcal{C}'$-generic elements of H. For any $y \in H$, $\mathcal{C}'\langle y\rangle \supset \mathcal{C}\langle f(y)\rangle$ because f is an E-$(\mathcal{C}', \mathcal{C})$-homomorphism. From this and the equation 17, the following containment may be deduced:

$$\mathcal{C}'\langle y\rangle \supset \mathcal{C}\langle f(y)\rangle\mathcal{C}' = C\langle \mathrm{id}_G f'(y)\rangle \cdot \mathcal{C}' = \mathcal{C}'\langle f'(y)\rangle.$$

This is property 2 of the definition of a pre E-$\mathcal{C}'$-mapping.

For properties 3 and 4, if $y \leftrightarrow y'$ in H, then $f(y) \leftrightarrow f(y')$ in $\mathrm{G}(\mathcal{G}/\mathcal{F})$ because f is an E-$(\mathcal{C}', \mathcal{C})$-homomorphism. For the same reason, $S^{\mathcal{C}'}_{y',y}$ extends the induced E-$\mathcal{C}$-isomorphism $S^{\mathcal{C}}_{f(y'),f(y)}$

$$\begin{tikzcd}
\mathcal{C}'\langle y\rangle \arrow[r, "S^{\mathcal{C}'}_{y',y}"] & \mathcal{C}'\langle y'\rangle \\
\mathcal{C}\langle f(y)\rangle \arrow[u, "\cup"] \arrow[r, "S^{\mathcal{C}}_{f(y'),f(y)}"] & \mathcal{C}\langle f(y')\rangle, \arrow[u, "\cup"]
\end{tikzcd}$$

and hence $S^{\mathcal{C}}_{f(y'),f(y)}$ and $id_{\mathcal{C}'}$ are bicompatible. Therefore, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C}'\langle y\rangle \arrow[r, "S^{\mathcal{C}'}_{y',y}"] & \mathcal{C}'\langle y'\rangle \\
\mathcal{C}\langle f(y)\rangle\cdot\mathcal{C}' \arrow[u, "\cup"] \arrow[r, "\varphi"] & \mathcal{C}\langle f(y')\rangle\cdot\mathcal{C}', \arrow[u, "\cup"]
\end{tikzcd}$$

where the E-isomorphism φ extends $\mathrm{id}_{\mathcal{C}'}$ and $S^{\mathcal{C}}_{f(y'),f(y)}$.

Since $\mathcal{G}$ and $\mathcal{C}[\mathcal{C}'\langle y\rangle]$ are linearly disjoint over $\mathcal{C}$, as are $\mathcal{G}$ and $\mathcal{C}[\mathcal{C}'\langle y'\rangle]$, it follows that $id_{\mathcal{G}}$ and $S^{\mathcal{C}'}_{y',y}$ are bicompatible. Therefore, the top square of the following diagram commutes:

$$\begin{tikzcd}
\mathcal{G}\mathcal{C}'\langle y\rangle \arrow[r, "\alpha"] & \mathcal{G}\mathcal{C}'\langle y'\rangle \\
\mathcal{G}\mathcal{C}\langle f(y)\rangle\cdot\mathcal{C}' \arrow[u, "\cup"] \arrow[r, "\beta"] & \mathcal{G}\mathcal{C}\langle f(y')\rangle\cdot\mathcal{C}' \arrow[u, "\cup"] \\
\mathcal{G}\mathcal{C}\langle \mathrm{id}_{\mathrm{G}}f'(y)\rangle\mathcal{C}' \arrow[u, "="] \arrow[r, "\beta"] & \mathcal{G}\mathcal{C}\langle \mathrm{id}_{\mathrm{G}}f'(y')\rangle\mathcal{C}' \arrow[u, "="] \\
\mathcal{G}\mathcal{C}'\langle f'(y)\rangle \arrow[u, "="] \arrow[r, "\gamma"] & \mathcal{G}\mathcal{C}'\langle f'(y')\rangle \arrow[u, "="] \\
\mathcal{G}\mathcal{C}'\cdot\mathcal{C}'\langle f'(y)\rangle \arrow[u, "="] \arrow[r, "\lambda"] & \mathcal{G}\mathcal{C}'\cdot\mathcal{C}'\langle f'(y')\rangle \arrow[u, "="] \\
\mathcal{G}\mathcal{C}'\cdot f'(y)(\mathcal{G}\mathcal{C}') \arrow[u, "="] \arrow[r, "\mu"] & \mathcal{G}\mathcal{C}'\cdot f'(y')(\mathcal{G}\mathcal{C}') \arrow[u, "="]
\end{tikzcd}$$

where the (E,Δ)-isomorphism α extends $\mathrm{id}_{\mathcal{G}}$ and $S^{\mathcal{C}'}_{y',y}$ and the (E,Δ)-isomorphism β extends φ, $\mathrm{id}_{\mathcal{G}}$, $S^{\mathcal{C}}_{f(y'),f(y)}$ and $\mathrm{id}_{\mathcal{C}'}$. Since $f = \mathrm{id}_G f'$, the third line of this diagram is also

$$\beta:\ \mathcal{G}\mathcal{C}\langle \mathrm{id}_G f'(y)\rangle \mathcal{C}' \longrightarrow \mathcal{G}\mathcal{C}\langle \mathrm{id}_G f'(y')\rangle \mathcal{C}'$$

extending $\mathrm{id}_{\mathcal{G}}$, $S_{\mathrm{id}_G f'(y'),\mathrm{id}_G f'(y)}$ and $\mathrm{id}_{\mathcal{C}'}$. By equation 17, $\mathcal{C}\langle \mathrm{id}_G f'(y)\rangle \mathcal{C}' = \mathcal{C}'\langle f'(y)\rangle$, and $\mathcal{C}\langle \mathrm{id}_G f'(y')\rangle \mathcal{C}' = \mathcal{C}'\langle f'(y')\rangle$. Because id_G is an E-$(\mathcal{C}', \mathcal{C})$-homomorphism, as was shown in the first part of this proof, the forth line

$$\gamma:\ \mathcal{G} \cdot \mathcal{C}'\langle f'(y)\rangle \longrightarrow \mathcal{G} \cdot \mathcal{C}'\langle f'(y')\rangle$$

is an (E, Δ)-isomorphism extending $\mathrm{id}_{\mathcal{G}}$ and $S^{\mathcal{C}'}_{f'(y'),f'(y)}$. The fifth line of the diagram is the (E, Δ)-isomorphism

$$\lambda:\ \mathcal{G}\mathcal{C}' \cdot \mathcal{C}'\langle f'(y)\rangle \longrightarrow \mathcal{G}\mathcal{C}' \cdot \mathcal{C}'\langle f'(y')\rangle,$$

which is obtained by writing '$\mathcal{G}\mathcal{C}'$' instead of '$\mathcal{G}$'. Clearly, the (E, Δ)-isomorphism λ extends $\mathrm{id}_{\mathcal{G}\mathcal{C}'}$ and $S^{\mathcal{C}'}_{f'(y'),f'(y)}$. By the E-strong normality of $\mathcal{G}\mathcal{C}'$ over $\mathcal{F}\mathcal{C}'$, λ is the same as the (E, Δ)-isomorphism, in the sixth line,

$$\mu:\ \mathcal{G}\mathcal{C}' \cdot f'(y)(\mathcal{G}\mathcal{C}') \longrightarrow \mathcal{G}\mathcal{C}' \cdot f'(y')(\mathcal{G}\mathcal{C}')$$

that extends $\mathrm{id}_{\mathcal{G}\mathcal{C}'}$ and that maps $f'(y)\alpha$ onto $f'(y')\alpha$ for every $\alpha \in \mathcal{G}\mathcal{C}'$. Therefore $f'(y) \leftrightarrow f'(y')$ in $G(\mathcal{G}\mathcal{C}'/\mathcal{F}\mathcal{C}')$, which is property 3. Property 4 is obtained by restricting the top and bottom lines in the above diagram to the Δ-constants, i.e. $S^{\mathcal{C}'}_{y',y}$ extends $S^{\mathcal{C}'}_{f'(y'),f'(y)}$. $\square$

Proposition 3.204 *Let $\mathcal{G}$ be an* E*-strongly normal extension of $\mathcal{F}$, with $\mathcal{C}$ the subfield of Δ-constants, and let φ be an* (E, Δ)*-isomorphism of $\mathcal{G}$ over*

$\mathcal{C}$ *such that* $\mathcal{U}$ *is universal over* $\varphi\mathcal{G}$. *Then* $\varphi\mathcal{G}$ *is an* E-*strongly normal extension of* $\varphi\mathcal{F}$. *There is a unique* (E,Δ)-*isomorphism* $\mathcal{G}\cdot\mathcal{K}\approx\varphi\mathcal{G}\cdot\mathcal{K}$ *over* $\mathcal{K}$ *that extends* φ (*that also shall be denoted by* φ). *When* $G(\mathcal{G}/\mathcal{F})$, *respectively* $G(\varphi\mathcal{G}/\varphi\mathcal{F})$, *is canonically identified with the group of* (E,Δ)-*automorphisms of* $\mathcal{G}\cdot\mathcal{K}$ *over* $\mathcal{F}\cdot\mathcal{K}$, *respectively* $\varphi\mathcal{G}\cdot\mathcal{K}$ *over* $\varphi\mathcal{F}\cdot\mathcal{K}$. *the formula* $T_\varphi(\sigma)=\varphi\cdot\sigma\cdot\varphi^{-1}$ *defines an* E-$\mathcal{C}$-*isomorphism* $T_\varphi : G(\mathcal{G}/\mathcal{F})\approx G(\varphi\mathcal{G}/\varphi\mathcal{F})$.

Remark 3.205 *When* φ *is an* (E,Δ)-*isomorphism of* $\mathcal{G}$ *over* $\mathcal{F}$, *then* $\varphi\in G(\mathcal{G}/\mathcal{F})$. *After* $G(\mathcal{G}/\mathcal{F})$ *and* $G(\varphi\mathcal{G}/\varphi\mathcal{F})$ *are canonically identified with the group of automorphisms of the differential field* $\mathcal{G}\mathcal{K}=\varphi\mathcal{G}\cdot\mathcal{K}$ *over* $\mathcal{F}\mathcal{K}$, *then they coincide as groups* (*but not necessarily as* $\mathcal{C}$-*groups*), *and* T_φ *is the inner* E-*automorphism determined by* φ.

Proof: Let τ be any (E,Δ)-isomorphism of $\varphi\mathcal{G}$ over $\varphi\mathcal{F}$. The (E,Δ)-isomorphism $\varphi^{-1} : \varphi\mathcal{G}\approx\mathcal{G}$ can be extended to some (E,Δ)-isomorphism ψ : $\varphi\mathcal{G}\cdot\tau\varphi\mathcal{G}\approx\mathcal{G}\cdot\psi\tau\varphi\mathcal{G}$, and evidently the formula $\alpha\mapsto\psi\tau\varphi\alpha$ defines an isomorphism of $\mathcal{G}$ over $\mathcal{F}$. Therefore, since $\psi\tau\varphi$ is E-strong, the field of constants $\mathcal{C}'$ of $\mathcal{G}\cdot\psi\tau\varphi\mathcal{G}$ has the property that

$$\mathcal{G}\mathcal{C}'=\mathcal{G}\cdot\psi\tau\varphi\mathcal{G}=\psi\tau\varphi\mathcal{G}\cdot\mathcal{C}'. \tag{22}$$

By applying ψ^{-1} to equation 22,

$$\varphi\mathcal{G}\cdot\mathcal{C}\langle\tau\rangle=\varphi\mathcal{G}\cdot\tau\varphi\mathcal{G}=\tau\varphi\mathcal{G}\cdot\mathcal{C}\langle\tau\rangle$$

since ψ^{-1} maps $\mathcal{G}$ onto $\varphi\mathcal{G}$ and $\mathcal{C}'$ onto the field of Δ-constants $\mathcal{C}\langle\tau\rangle$ of $\varphi\mathcal{G}\cdot\tau\varphi\mathcal{G}$. Therefore, τ is E-strong, and, hence, $\varphi\mathcal{G}$ is an E-strongly normal extension of $\varphi\mathcal{F}$.

Since $\mathcal{G}$ and $\mathcal{K}$ are linearly disjoint over $\mathcal{C}$, as are $\varphi\mathcal{G}$ and $\mathcal{K}$, φ can be extended to a unique (E, Δ)-isomorphism $\mathcal{G}\mathcal{K} \approx \varphi\mathcal{G} \cdot \mathcal{K}$ over $\mathcal{K}$, and denote it, too, by φ. Making the canonical identifications, one can see that for each $\sigma \in G(\mathcal{G}/\mathcal{F})$, $\varphi \cdot \sigma \cdot \varphi^{-1} \in G(\varphi\mathcal{G}/\varphi\mathcal{F})$. Therefore one can define a mapping $T_\varphi : G(\mathcal{G}/\mathcal{F}) \approx G(\varphi\mathcal{G}/\varphi\mathcal{F})$ by the formula $T_\varphi(\sigma) = \varphi \cdot \sigma \cdot \varphi^{-1}$, and it is clear that T_φ is a group isomorphism. Since $\varphi\mathcal{G} \cdot \mathcal{C}\langle T_\varphi(\sigma)\rangle = \varphi\mathcal{G} \cdot (\varphi \cdot \sigma \cdot \varphi^{-1})\varphi\mathcal{G} = \varphi(\mathcal{G}\sigma\mathcal{G}) = \varphi(\mathcal{G}\mathcal{C}\langle\sigma\rangle) = \varphi\mathcal{G} \cdot \mathcal{C}\langle\sigma\rangle$, one may infer that $\mathcal{C}\langle T_\varphi(\sigma)\rangle = \mathcal{C}\langle\sigma\rangle$. Furthermore, if $\sigma \leftrightarrow \sigma'$, then there exists an (E, Δ)-isomorphism $\mathcal{G}\sigma\mathcal{G} \approx \mathcal{G}\sigma'\mathcal{G}$ over $\mathcal{G}$ mapping $\sigma\alpha$ onto $\sigma\alpha$ $(\alpha \in G)$ and inducing the E-$\mathcal{C}$-isomorphism $S_{\sigma',\sigma} : \mathcal{C}\langle\sigma\rangle \approx \mathcal{C}\langle\sigma'\rangle$. Since φ maps $\mathcal{G}\sigma\mathcal{G}$, respectively $\mathcal{G}\sigma'\mathcal{G}$, onto $\varphi\mathcal{G}\cdot T_\varphi(\sigma)\varphi\mathcal{G}$, respectively $\varphi\mathcal{G} \cdot T_\varphi(\sigma')\varphi\mathcal{G}$, and leaves Δ-constants fixed, one obtains an (E, Δ)-isomorphism $\varphi\mathcal{G}\cdot T_\varphi(\sigma)\varphi\mathcal{G} \approx \varphi\mathcal{G}\cdot T_\varphi(\sigma')\varphi\mathcal{G}$ over $\varphi\mathcal{G}$ mapping $T_\varphi(\sigma)\varphi\alpha$ onto $T_\varphi(\sigma')\varphi\alpha$ $(\alpha \in G)$, so that $T_\varphi(\sigma) \leftrightarrow T_\varphi(\sigma')$ and $S_{T_\varphi(\sigma'),T_\varphi(\sigma)} = S_{\sigma',\sigma}$. Thus, T_φ restricted to the $\mathcal{C}$-generic elements of G is a pre E-$\mathcal{C}$-map, and T_φ is an E-$\mathcal{C}$-isomorphism by [25, Corollary 1, p. 90]. $\square$

4 The Fundamental Theorems

4.1 The Topology on E-Sets

In this section, let $\mathcal{F}$ be an E-field, and let $\mathcal{V}$ be an E-extension of $\mathcal{F}$ that is E-universal over $\mathcal{F}$. And consider $\mathcal{H}$ an E-extension of $\mathcal{F}$ over which $\mathcal{V}$ need not be E-universal. Also, let A be a pre E-$\mathcal{F}$-set relative to $\mathcal{V}$ (Section 2.1 or [24, page 29]). Then $x \in A$ is defined to be *rational* over $\mathcal{H}$ if $\mathcal{F}\langle x\rangle \subset \mathcal{H}$ [25, page 29]. In a similar manner, define x to be *algebraic* (*resp.*, E-*algebraic* or *regular*) if $\mathcal{H}\mathcal{F}\langle x\rangle$ is an algebraic (*resp.*, E-algebraic or regular) extension of $\mathcal{H}$. Denote by $A_{\mathcal{H}}$ the set of elements of A rational over $\mathcal{H}$. In particular, $A_{\mathcal{V}}$ is the set A.

Let G be an E-$\mathcal{F}$-group (Section 3.153 or [25, page 33]). A *homogeneous* E-$\mathcal{F}$-*space for* G is a set M on which is given a structure of a homogeneous space for the group G and a structure of a pre E-$\mathcal{F}$-set subject to axioms, which are similar to those for an Δ-$\mathcal{F}$-group [25, page 34]. The homogeneous E-$\mathcal{F}$-space M for G is *principal* if it is principal as a homogeneous for G and satisfies additional axioms [25, page 35].

A subset V of the pre E-$\mathcal{F}$-subset A is $\mathcal{F}$-*irreducible* (*in* A) if there exists $x \in V$ such that V is the set of all E-specializations of x over $\mathcal{F}$ [25, page 30]. Such an x is called an E-$\mathcal{F}$-generic element of V. If the set B of A is the union of finitely many $\mathcal{F}$-irreducible subsets of A, then B has the structure of a pre E-$\mathcal{F}$-set that is induced by the restriction of the pre E-$\mathcal{F}$-set structure on A. Such a B is called a *pre* E-$\mathcal{F}$-*subset* (*of* A). A maximal $\mathcal{F}$-irreducible subset of A is called an $\mathcal{F}$-*component* (*of* A)

An E-$\mathcal{F}$-set is a pre E-$\mathcal{F}$-subset of a homogeneous E-$\mathcal{F}$-space for an E-$\mathcal{F}$-group [25, page 37]. Then the E-$\mathcal{H}$-subsets of M are the closed subsets of a Noetherian topology on M [25, Theorem 1 page 72], which is called the E-*Zariski topology relative to* $\mathcal{H}$ or more simply the E-$\mathcal{H}$-*topology.* If $\mathcal{H} = \mathcal{V}$, the reference to $\mathcal{V}$ is usually omitted, and it is called the E-*Zariski topology* or more simply the E-*topology.* Each E-$\mathcal{F}$-set will be considered to have the topology induced from the E-$\mathcal{H}$-topology on its the ambient homogeneous E-$\mathcal{F}$-space for an E-$\mathcal{F}$-group. For an E-$\mathcal{F}$-set A, the subset $A_{\mathcal{H}} = \{v \in A \mid \mathcal{F}\langle v \rangle \subseteq \mathcal{H}\}$ will be called E-*dense in* A if, for each closed E-closed subset C of A with $A \neq C$, $A_{\mathcal{H}}$ is not contained in C. Kolchin shows that, if $\mathcal{H}$ is constrainedly closed [25, page 79], then $A_{\mathcal{H}}$ is E-dense in A [25, Proposition 3, page 84].

Any E-$\mathcal{F}$-group G has a natural structure of a principal homogeneous E-$\mathcal{F}$-space for G, which is called the *regular* E-$\mathcal{F}$-*space for* G. Consequently, any pre E-$\mathcal{F}$-set contained in the E-$\mathcal{F}$-group G is an E-$\mathcal{F}$-subset. An E-$\mathcal{F}$-*subgroup* is a subgroup of G that is an E-$\mathcal{F}$-subset and satisfies all the E-$\mathcal{F}$-group axioms [25, page 37]. By [25, Proposition 1, page 87], a subgroup that is also an E-$\mathcal{F}$-subset is an E-$\mathcal{F}$-subgroup.

Definition 4.206 *The* $\mathcal{F}$-*component of the identity of an* E-*group* G *is denoted by* G°.

4.2 Fundamental Theorems

In this the rest of this chapter, let $\mathcal{F}$ be an (E,Δ)-field, and let $\mathcal{U}$ be an (E,Δ)-extension of $\mathcal{F}$ which is (E,Δ)-universal over $\mathcal{F}$. Then $\mathcal{K} = \mathcal{U}^{\Delta}$ considered as an E-field is clearly E-universal over $\mathcal{C} = \mathcal{F}^{\Delta}$ considered as an E-field and, thus, constrainedly closed. Also, $\mathcal{G}$ will denote an E-strongly normal extension of $\mathcal{F}$.

The following theorem establishes a Galois correspondence between the set of intermediate differential fields of a E-strongly normal extension and the set of E-$\mathcal{C}$-subgroups of its Galois group when the field of Δ-constants is constrainedly closed. The proofs are very similar to [25, Chapter 6, Section 4].

Theorem 4.207 (First Fundamental Theorem) *Let $\mathcal{G}$ be an* E*-strongly normal extension of $\mathcal{F}$ with field of Δ-constants $\mathcal{C}$.*

1. *If $\mathcal{F}_1$ is an* (E,Δ)*-field with $\mathcal{F} \subset \mathcal{F}_1 \subset \mathcal{G}$, then $\mathcal{G}$ is* E*-strongly normal over $\mathcal{F}_1$, $G(\mathcal{G}/\mathcal{F}_1)$ is an* E-$\mathcal{C}$*-subgroup of $G(\mathcal{G}/\mathcal{F})$, and the set of invariants of $G(\mathcal{G}/\mathcal{F}_1)$ in $\mathcal{G}$ is $\mathcal{F}_1$.*

2. *If G_1 is an* E-$\mathcal{C}$*-subgroup of $G(\mathcal{G}/\mathcal{F})$ and $\mathcal{F}_1$ denotes the set of invariants of G_1 in $\mathcal{G}$, then $\mathcal{F}_1$ is an* (E,Δ)*-field with $\mathcal{F} \subset \mathcal{F}_1 \subset \mathcal{G}$, and, if the elements of G_1 rational over $\mathcal{C}$ are* E*-dense in G_1, then $G(\mathcal{G}/\mathcal{F}_1) = G_1$.*

3. If $\mathcal{C}$ is constrainedly closed [25, page 76] *as an* E*-field, parts 1 and 2 establish a bijective correspondence between* (E, Δ)*-subfields* $\mathcal{F}_1$ *with* $\mathcal{F} \subset \mathcal{F}_1 \subset \mathcal{G}$ *and* E*-subgroups* $G_1 \subseteq G(\mathcal{G}/\mathcal{F})$.

Remark 4.208 *It would be preferable to remove the hypothesis of constrainedly closed from part 3. In a special case, this is accomplished in Corollary 4.212. Also, if* $\mathcal{C}$ *is* E*-universal over some* E*-field, then* $\mathcal{C}$ *is constrainedly closed* [25, page 76].

Proof: To prove part 1, let $\mathcal{F}_1$ be an (E, Δ)-field with $\mathcal{F} \subset \mathcal{F}_1 \subset \mathcal{G}$. Every (E, Δ)-isomorphism of $\mathcal{G}$ over $\mathcal{F}_1$ is over $\mathcal{F}$, too, and hence is E-strong. Therefore $\mathcal{G}$ is E-strongly normal over $\mathcal{F}_1$, and the Galois group $G(\mathcal{G}/\mathcal{F}_1)$ is an E-$\mathcal{C}$-group by Theorem 3.197. It is obviously a subgroup and an E-$\mathcal{C}$-subset [25, page 30 and 37] of $G(\mathcal{G}/\mathcal{F})$. Thus, $G(\mathcal{G}/\mathcal{F}_1)$ is an E-$\mathcal{C}$-subgroup of $G(\mathcal{G}/\mathcal{F})$. By definition, every element of $\mathcal{F}_1$ is an invariant of $G(\mathcal{G}/\mathcal{F}_1)$ in $\mathcal{G}$, and, by Proposition 1.53, every such invariant is in $\mathcal{F}_1$.

For part 2, let $\mathcal{F}_1$ be the set of invariants of G_1 in G. It is obvious that $\mathcal{F}_1$ is a (E, Δ)-field with $\mathcal{F} \subset \mathcal{F}_1 \subset \mathcal{G}$, and therefore, by part 1, $G(\mathcal{G}/\mathcal{F}_1)$ is E-$\mathcal{C}$-subgroup of $G(\mathcal{G}/\mathcal{F})$. Of course $G_1 \subset G(\mathcal{G}/\mathcal{F}_1)$. It must be shown that $G_1 = G(\mathcal{G}/\mathcal{F}_1)$ under the hypothesis that the elements of G_1 rational over $\mathcal{C}$ are E-dense in G_1.

Assume that $G_1 \neq G(\mathcal{G}/\mathcal{F}_1)$. Fix E-$\mathcal{C}$-generic elements $\sigma_1 \ldots \sigma_r$ of the E-$\mathcal{C}$-components of G_1. By assumption, there exists an element $\tau \in G(\mathcal{G}/\mathcal{F}_1)$ that is not a E-specialization of any σ_k. Fixing elements $\eta_1, \ldots, \eta_n \in \mathcal{G}$ with

$\mathcal{F}\langle\eta_1, \ldots, \eta_n\rangle_{E,\Delta} = \mathcal{G}$, by Lemma 1.35, for each index k there exists a differential polynomial $F_k \in \mathcal{G}\{y_1, \ldots, y_n\}_{(\mathrm{E},\Delta)}$ that vanishes at $(\sigma_k\eta_1, \ldots, \sigma_k\eta_n)$ but not at $(\tau\eta_1, \ldots, \tau\eta_n)$. Then F_k vanishes at $(\sigma\eta_1, \ldots, \sigma\eta_n)$ for all σ in the component of σ_k. The product $\Pi_i F_i$ is a differential polynomial in $\mathcal{G}\{y_1, \ldots, y_n\}_{(\mathrm{E},\Delta)}$ that vanishes at $(\sigma\eta_1, \ldots, \sigma\eta_n)$ for every $\sigma \in G_1$ but not for every $\sigma \in G(\mathcal{G}/\mathcal{F}_1)$. Let F be such a differential polynomial with as few non-zero terms as possible. Also suppose that one of the coefficients in F is 1. Consider any $\sigma' \in (G_1)_{\mathcal{C}}$. Then σ' is an (E, Δ)-automorphism of $\mathcal{G}$ over $\mathcal{F}$. Since $F^{\sigma'}(\sigma\eta_1, \ldots, \sigma\eta_n) = \sigma'(F(\sigma'^{-1}\sigma\eta_1, \ldots, \sigma'^{-1}\sigma\eta_n)), F^{\sigma'}$ vanishes at $(\sigma\eta_1, \ldots, \sigma\eta_n)$ for every $\sigma \in G_1$, because $\sigma'^{-1}\sigma \in G_1$. And therefore $F - F^{\sigma'}$ does too. Since $F - F^{\sigma'}$ has fewer terms than F, $F - F^{\sigma'}$ must vanish at $(\sigma\eta_1, \ldots, \sigma\eta_n)$ for every $\sigma \in G(\mathcal{G}/\mathcal{F}_1)$. Hence for any $\alpha \in \mathcal{G}$, $F - \alpha(F - F^{\sigma'})$ vanishes at $(\sigma\eta_1, \ldots, \sigma\eta_n)$ for every $\sigma \in \mathcal{G}_1$ but not for every $\sigma \in G(\mathcal{G}/\mathcal{F}_1)$. If $F - F^{\sigma'}$ were not zero, one could choose α so that $F - \alpha(F - F^{\sigma'})$ has fewer terms than F and is nonzero. Therefore $F - F^{\sigma'} = 0$ for $\sigma' \in (G_1)_{\mathcal{C}}$.

By part 1, the set $\{\sigma \in G(\mathcal{G}/\mathcal{F}) \mid F = F^{\sigma}\}$ is the E-$\mathcal{C}$-group leaving invariant the (E, Δ)-$\mathcal{F}$-field generated by the coefficients of F. In particular, it is an E-$\mathcal{C}$-subset of $G(\mathcal{G}/\mathcal{F})$ and a closed subset of the E-$\mathcal{C}$-topology on $G(\mathcal{G}/\mathcal{F})$. If the closed set $\{\sigma \in G(\mathcal{G}/\mathcal{F}) \mid F = F^{\sigma}\} \cap G_1$ were not all of G_1, there would be an element of $(G_1)_{\mathcal{C}}$ not in $\{\sigma \in G(\mathcal{G}/\mathcal{F}) \mid F = F^{\sigma}\} \cap G_1$ since (by hypothesis) $(G_1)_{\mathcal{C}}$ is E-dense in G$_1$. Therefore, $G_1 \subseteq \{\sigma \in G(\mathcal{G}/\mathcal{F}) \mid F = F^{\sigma}\}$, or $F = F^{\sigma}$ for all $\sigma \in G_1$. Since $\mathcal{F}_1$ is the (E, Δ)-field invariant under the action of G_1, $F \in \mathcal{F}_1\{y_1, \ldots, y_n\}$. However, then $F^{\sigma} = F$ for every

$\sigma \in G(\mathcal{G}/\mathcal{F}_1)$, so that $F(\sigma\eta_1 \ldots \sigma\eta_n) = \sigma F(e\eta_1 \ldots e\eta_n) = 0$, since the identity e of G_1 is contained in G_1. This contradiction shows that $G_1 = G(\mathcal{G}/\mathcal{F}_1)$ under the hypothesis that the elements of G_1 rational over $\mathcal{C}$ are E-dense in G_1.

For part 3, the hypothesis that $\mathcal{C}$ is constrainedly closed implies that the elements of G_1 rational over $\mathcal{C}$ are E-dense in G_1 ([25, Proposition 3, page 84]). ☐

It would be desirable to strengthen part 3 this theorem by removing the restriction on $\mathcal{C}$. After two preliminary lemmas,the next theorem does this for a certain type of small E-$\mathcal{C}$-subgroup.

Lemma 4.209 *Let G and K be field extensions of F. Let H' be a subfield of GK containing K. Put $H = G \cap H'$. If H' and G are linearly disjoint over H and if K and H are linearly disjoint over F, then $H' = HK$.*

$$\begin{tikzcd}
G \arrow[r] & GK \\
H = G \cap H' \arrow[u] \arrow[r] & H' \arrow[u] \\
F \arrow[u] \arrow[r] & K \arrow[u]
\end{tikzcd}$$

Proof: Evidently $HK \subset H'$. Consider any element $\varphi \in H'$. Fix a basis (c_i) of K over F. By considering φ as an element of GK, one may write $\varphi = (\Sigma\beta_i c_i)/(\Sigma\gamma_j c_j)$, where the β_i and γ_j are elements of G, and therefore $\Sigma\gamma_j(c_j\varphi) - \Sigma\beta_i c_i = 0$. Thus the elements $c_j\varphi$ and c_i of H' are linearly dependent over G. By the first hypothesis, they must be linearly dependent over H, that is there exist elements β_i' and γ_j' of H, not all 0, such that $\Sigma\gamma_j'(c_j\varphi) - \Sigma\beta_i'c_i = 0$. By the second hypothesis, the elements c_j of K are linearly independent over H, and therefore $\Sigma\gamma_j'c_j \neq 0$, so that $\varphi = (\Sigma\beta_i'c_i)/(\Sigma\gamma_j'c_j) \in HK$. This shows that $HK = H'$. □

Lemma 4.210 *Let* $\mathcal{G}$ *over* $\mathcal{F}$ *be an* E-*strongly normal extension of* (E,Δ)-*fields, and let* $G = G(\mathcal{G}/\mathcal{F})$, *the associated* E-$\mathcal{C}$-*group of* (E,Δ)-*isomorphisms. Let* H *be an* E-$\mathcal{C}$-*subgroup of* G *and* $\mathcal{H}$ *be the* (E,Δ)-*field of invariants of* H *in* $\mathcal{G}$. *If* $\mathcal{C}\langle\sigma\rangle \subset \mathcal{C}\mathcal{U}^{\mathrm{E},\Delta}$ *for all* $\sigma \in H$, *then* $\mathcal{G}$ *over* $\mathcal{H}$ *as an* (E,Δ)-*extension is strongly normal in the sense of Kolchin.*

Proof: For all $\sigma \in H$, $\mathcal{C}\langle\sigma\rangle \subset \mathcal{C}\mathcal{U}^{\mathrm{E},\Delta}$ implies $\sigma\mathcal{G} \subset \mathcal{G}\sigma\mathcal{G} = \mathcal{G}\mathcal{C}\langle\sigma\rangle \subset \mathcal{G}(\mathcal{C}\mathcal{U}^{\mathrm{E},\Delta}) = \mathcal{G}\mathcal{U}^{\mathrm{E},\Delta}$. Since σ leaves invariant Δ-constants, it also leaves invariant (E,Δ)-constants. By [24, Propostion 10, page 393], $\mathcal{G}$ over $\mathcal{H}$ as an (E,Δ)-extension is strongly normal as an (E,Δ)-extension in the sense of Kolchin. □

Lemma 4.211 *Let* F *be an* E-*field, and let* $\mathcal{V} \supset F$ *be an* E-*field that is* E-*universal over* F. *Let* B *be an* E-$\mathcal{F}$-*set. Let* $C{=}F^{\mathrm{E}}$, *and let* C_a *be the algebraic closure of* C *in* $\mathcal{V}^{\mathrm{E}}$. *If* $B_{\mathcal{V}} \subset B_{F\mathcal{V}^{\mathrm{E}}}$, *then* B_{FC_a} *is* E-*dense in* B.

Proof: Since $\mathcal{V}$ is a constrainedly closed extension of F ([25, Proposition 3, page 84]), $B_{\mathcal{V}}$ is dense in B [25, Proposition 3, page 84]. However, each point of B rational over $\mathcal{V}$ is rational over $F\mathcal{V}^{\mathrm{E}}$ by assumption. But an element constrained over F rational over $F\mathcal{V}^{\mathrm{E}}$ is, in fact, rational over FC_a because an E-extension constrained over $\mathcal{C}$ has E-constants algebraic over $\mathcal{C}$ [24, Proposition 7(d), page 142]. Therefore, the set B_{FC_a} is E-dense in B. □

The formulation of the following corollary was influenced by Chapter 3 of Sit's thesis [34], in which he considers Δ-subfields of $\mathcal{F}\langle t\rangle_\Delta$ over which $\mathcal{F}\langle t\rangle_\Delta$ is strongly normal in the sense of Kolchin, where t is a Δ-indeterminant over the Δ-field $\mathcal{F}$. For instance, the previous lemma is a generalization of [34, Lemma 2.1, page 652] from an affine E-Zariski closed subset of $\mathcal{V}^n$ to an E-$\mathcal{F}$-subset that is not necessarily affine. In this corollary, these ideas have been combined with those of Kolchin in the second part of his proof of the fundamental theorem for strongly normal extensions ([24, Theorem 3, page 398]). It will be applied in the last chapter of this paper to improve slightly upon a theorem of Sit.

Corollary 4.212 *Let* $\mathcal{L} = \mathcal{U}^{\mathrm{E},\Delta}$ *and let* $G = G(\mathcal{G}/\mathcal{F})$. *Let* $\mathcal{I}$ *be the set of* (E, Δ)*-subfields* $\mathcal{H}$ *of* $\mathcal{G}$ *containing* $\mathcal{F}$ *such that* $\mathcal{G}$ *over* $\mathcal{H}$ *is strongly normal as an* (E, Δ)*-field extension (in the sense of Kolchin), and let* $\mathcal{S}$ *be the set of* E-$\mathcal{C}$*-subgroups* H *of* G *such that* $H_{\mathcal{U}^\Delta} \subset H_{\mathcal{C}\mathcal{L}}$. *Then there is a Galois correspondence between* $\mathcal{I}$ *and* $\mathcal{S}$.

Proof: Let $\mathcal{H} \in \mathcal{I}$. Then by part 1 of the First Fundamental Theorem 4.207, there exists an E-$\mathcal{C}$-subgroup $H = G(\mathcal{G}/\mathcal{H})$ of G such that the (E, Δ)-field of invariants of H is $\mathcal{H}$. Let $\sigma \in H_{\mathcal{U}^\Delta}$. Because $\mathcal{G}$ over $\mathcal{H}$ is strongly normal as an (E, Δ)-extension (in the sense of Kolchin), σ is a strong (in the sense of

Kolchin) (E,Δ)-isomorphism of $\mathcal{G}$ over $\mathcal{H}$, and $\mathcal{G}\sigma\mathcal{G} \subset \mathcal{G}\cdot\mathcal{L}$. Apply Lemma 4.209 to the case $G = \mathcal{G}$, $K = \mathcal{L}$, $F = \mathcal{G}^{\mathrm{E},\Delta}$, $H' = (\mathcal{G}\mathcal{L})^{\Delta}$ and $H = \mathcal{G}^{\Delta}$.

$$\begin{tikzcd}
\mathcal{G} \arrow[r] & \mathcal{G}\mathcal{L} \\
(\mathcal{G}\mathcal{L})^{\Delta} \cap \mathcal{G} = \mathcal{G}^{\Delta} \arrow[u] \arrow[r] & (\mathcal{G}\mathcal{L})^{\Delta} \arrow[u] \\
\mathcal{G}^{\mathrm{E},\Delta} \arrow[u] \arrow[r] & \mathcal{L} \arrow[u]
\end{tikzcd}$$

By [24, Corollary 1, page 87], $\mathcal{G}^{\Delta}$ and $\mathcal{L}$ are linearly disjoint over $\mathcal{G}^{\mathrm{E},\Delta}$ by the linear disjointness of E-constants, and $\mathcal{G}$ and $(\mathcal{G}\mathcal{L})^{\Delta}$ are linearly disjoint over $\mathcal{G}^{\Delta}$ by the linear disjointness of Δ-constants. This Lemma then implies $(\mathcal{G}\mathcal{L})^{\Delta} = \mathcal{G}^{\Delta}\mathcal{L}$. Then, $\mathcal{C}\langle\sigma\rangle = (\mathcal{G}\sigma\mathcal{G})^{\Delta} \subset (\mathcal{G}\cdot\mathcal{L})^{\Delta} = \mathcal{G}^{\Delta}\cdot\mathcal{L} = \mathcal{C}\mathcal{L}$. Therefore, $\sigma \in H_{\mathcal{C}\mathcal{L}}$, and $H \in \mathcal{S}$.

Let $H \in \mathcal{S}$, and let $\mathcal{H}$ be the corresponding subfield of invariants of H in $\mathcal{G}$. If the elements of H rational over $\mathcal{C}$ are E-dense in H, by part 1 of the First Fundamental Theorem 4.207, $\mathcal{H}$ is a differential field with $\mathcal{F} \subset \mathcal{H} \subset \mathcal{G}$ and $H = G(\mathcal{G}/\mathcal{H})$. By Lemma 4.210, $\mathcal{G}$ over $\mathcal{H}$ is strongly normal as an (E,Δ)-extension (in the sense of Kolchin), and $\mathcal{H} \in \mathcal{I}$.

Let $\mathcal{D} = \mathcal{C}^{E} = \mathcal{G}^{(E,\Delta)}$, and let $\mathcal{D}_a$ be the algebraic closure of $\mathcal{D}$ in $\mathcal{L}$. For all $H \in \mathcal{S}$, the set of elements of H rational over $\mathcal{C}\mathcal{D}_a$ is E-dense in H by Lemma 4.211. (For the affine case, see a lemma of Sit [34, Chapter 2, Section 2].) By results in the two paragraphs above, if $\mathcal{D}_a \subseteq \mathcal{C}$ or equivalently if $\mathcal{D}_a\mathcal{C} = \mathcal{C}$, then there is a Galois correspondence between $\mathcal{I}$ and $\mathcal{S}$.

To prove the corollary without assuming $\mathcal{D}_a \subseteq \mathcal{C}$, let $H \in \mathcal{S}$, and let $\mathcal{H}$ be the set of invariants of H in $\mathcal{G}$. It will be shown that $H = G(\mathcal{G}/\mathcal{H})$. Let

$\mathcal{H}'$ denote the set of invariants of H in $\mathcal{G}\mathcal{D}_a$. Then $\mathcal{H}'$ is an (E, Δ)-field with $\mathcal{F}\mathcal{D}_a \subset \mathcal{H}' \subset \mathcal{G}\mathcal{D}_a$ and $\mathcal{G} \cap \mathcal{H}' = \mathcal{H}$.

$$\begin{tikzcd}
\mathcal{G} \arrow[r] & \mathcal{G}\mathcal{D}_a \\
\mathcal{G} \cap \mathcal{H}' = \mathcal{H} \arrow[u] \arrow[r] & \mathcal{H}' \arrow[u] \\
\mathcal{F} \arrow[u] \arrow[r] & \mathcal{F}\mathcal{D}_a \arrow[u] \\
\mathcal{D} \arrow[u] \arrow[r] & \mathcal{D}_a \arrow[u]
\end{tikzcd}$$

Claim 4.213 *The fields $\mathcal{G}$ and $\mathcal{H}'$ are linearly disjoint over $\mathcal{H}$.*

To prove this, consider elements $\varphi_1, \ldots \varphi_s \in \mathcal{H}'$ that are linearly dependent over $\mathcal{G}$. It must be shown that they are linearly dependent over $\mathcal{H}$. It may be assumed that $s > 1$ and no $s-1$ of them are linearly dependent over $\mathcal{G}$. Then there exist nonzero elements $\alpha_1, \ldots, \alpha_s \in \mathcal{G}$ with $\sum\limits_{1 \leq j \leq s} \alpha_j \varphi_j = 0$. Dividing by α_s one may suppose that $\alpha_s = 1$. For any $\sigma \in H$, since $\mathcal{H}'$ is invariant under H, $\sum\limits_{1 \leq j \leq s} (\sigma\alpha_j)\varphi_j = 0$, and therefore $\sum\limits_{1 \leq j \leq s-1} (\sigma\alpha_j - \alpha_j)\varphi_j = 0$.

Take $\sigma \in H_{\mathcal{C}\mathcal{D}_a}$ so that, by definition, $\mathcal{C}\langle\sigma\rangle$ is algebraic over $\mathcal{C}$. By part 1 of the Definition of a pre E-set, $\mathcal{C}\langle\sigma\rangle$ is finitely E-generated over $\mathcal{C}$. Therefore, the degree of $\mathcal{G}\mathcal{C}\langle\sigma\rangle$ over $\mathcal{G}$ is finite. Let $f_i : \mathcal{G}\mathcal{C}\langle\sigma\rangle \longrightarrow \mathcal{G}\mathcal{D}_a$ for $i = 1, \ldots, t$ denote the finite set of isomorphisms (not necessarily differential) over $\mathcal{G}$. Suppose $f_1 = id$. Lemma 1.51 shows that each f_i is, in fact, an (E, Δ)-isomorphism. By Lemma 4.210, σ is a strong isomorphism of $\mathcal{G}$ over $\mathcal{F}$ (in the sense of Kolchin) such that $(\mathcal{G}\sigma\mathcal{G})^{\Delta} = \mathcal{C}\langle\sigma\rangle \subset \mathcal{C}\mathcal{D}_a$ and $\mathcal{G}\sigma\mathcal{G} = \mathcal{G}\mathcal{C}\langle\sigma\rangle \subset \mathcal{G}\mathcal{D}_a$. Consider the (E, Δ)-isomorphisms of $\mathcal{G}$ defined as $\sigma_i = f_i\sigma$ for $i = 1, \ldots, t$. So $\sigma = \sigma_1$. For each i, by the definitions of f_i and σ_i, there exist (E, Δ)-$\mathcal{G}$-isomorphisms $\psi_i : \mathcal{G}\sigma\mathcal{G} \to \mathcal{G}\sigma_i\mathcal{G}$ such that $\psi_i(\alpha) = \alpha$ and $\psi_i(\sigma\alpha) = \sigma_i\alpha$ for all $\alpha \in \mathcal{G}$. That is $\sigma \leftrightarrow \sigma_i$ in G. Since $\sigma \longrightarrow \sigma_i$ and since H is E-closed in $G(\mathcal{G}/\mathcal{F})$, $\sigma_i \in H$.

So that $\sum_{1\le j\le s-1}(\sigma_k\alpha_j - \alpha_j)\varphi_j = 0$ (for $1 \le k \le t$). If $\sigma\alpha_1 - \alpha_1 \neq 0$, then, because f_k is an isomorphism over $\mathcal{G}$, $0 \neq f_k(\sigma\alpha_1 - \alpha_1) = f_k\sigma\alpha_1 - f_k\alpha_1 = \sigma_k\alpha_1 - \alpha_1$. So, one may divide by $\sigma_k\alpha_1 - \alpha_1$ for each k to obtain

$$\sum_{1\le j\le s-1}(\sigma_k\alpha_1 - \alpha_1)^{-1}(\sigma_k\alpha_j - \alpha_j)\varphi_j = 0 \quad (\text{for } 1 \le k \le t). \tag{23}$$

Set $\alpha'_j = \sum_{1\le k\le t}(\sigma_k\alpha_1 - \alpha_1)^{-1}(\sigma_k\alpha_j - \alpha_j) = \mathrm{Tr}\ (\sigma\alpha_1 - \alpha_1)^{-1}(\sigma\alpha_j - \alpha_j)$ (Tr is the trace of $\mathcal{G}\mathcal{D}_a$ over $\mathcal{G}$). By summing the equations 23, one would have $\sum_{1\le j\le s-1}\alpha'_j\varphi_j = 0$, $\alpha'_j \in \mathcal{G}(1 \le j \le s-1)$, $\alpha'_1 = \mathrm{Tr}\ 1 \neq 0$. This contradicts the linear independence of $\varphi_1, \ldots, \varphi_{s-1}$ over $\mathcal{G}$. Therefore, $\sigma\alpha_1 = \alpha_1$ for every $\sigma \in H_{\mathcal{C}\mathcal{D}_a}$. Since $H_{\mathcal{C}\mathcal{D}_a}$ is E-dense in H, $\sigma\alpha_1 = \alpha_1$ for every $\sigma \in H_{\mathcal{U}^{\Delta}}$. Therefore, $\alpha_1 \in \mathcal{H}$. Similarly, $\alpha_k \in \mathcal{H}$ for every index k, so that $\varphi_1, \ldots, \varphi_s$ are linearly dependent over $\mathcal{H}$. This establishes the claim.

By the claim and Lemma 4.209, $\mathcal{H}' = \mathcal{H}\mathcal{D}_a$. It follows from Theorem 3.202 that $G(\mathcal{G}/\mathcal{H}) = G(\mathcal{G}\mathcal{D}_a/\mathcal{H}\mathcal{D}_a) = G(\mathcal{G}\mathcal{D}_a/\mathcal{H}')$. Because it has been shown that $H_{\mathcal{C}\mathcal{D}_a}$ is E-dense in H and $\mathcal{H}'$ is the (E, Δ)-subfield of invariants of H in $\mathcal{G}\mathcal{D}_a$, the Galois correspondence (part 2 of the First Fundamental Theorem 4.207) implies $G(\mathcal{G}\mathcal{D}_a/\mathcal{H}') = H$ and, thus, $G(\mathcal{G}/\mathcal{H}) = H$. Since

$\mathcal{H} \in \mathcal{I}$, this establishes the Galois correspondence of the theorem. ☐

Corollary 4.214 *Assume that* $\mathcal{C}$ *is constrainedly closed over* $\mathcal{F}$ *as an* E-*field. Let* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *be* (E, Δ)-*differential fields contained in* $\mathcal{G}$ *and containing* $\mathcal{F}$. *Then* $G(\mathcal{G}/\mathcal{F}_1\mathcal{F}_2) = G(\mathcal{G}/\mathcal{F}_1) \cap G(\mathcal{G}/\mathcal{F}_2)$, *and* $G(\mathcal{G}/\mathcal{F}_1 \cap \mathcal{F}_2)$ *is the smallest* E-$\mathcal{C}$-*subgroup of* $G(\mathcal{G}/\mathcal{F})$ *containing* $G(\mathcal{G}/\mathcal{F}_1)G(\mathcal{G}/\mathcal{F}_2)$.

Proof: Observe that an (E, Δ)-isomorphism of $\mathcal{G}$ leaves invariant every element of $\mathcal{F}_1\mathcal{F}_2$ if and only if it leaves invariant every element of $\mathcal{F}_1$ and every element of $\mathcal{F}_2$. Thus the first assertion is true, because, under the assumptions of this corollary, the Galois correspondences of the First Fundamental Theorem and Corollary 4.212 imply the subgroups Galois groups are uniquely determined by the invariant subfields.

For the second assertion, the smallest E-$\mathcal{C}$-subgroup of $G(\mathcal{G}/\mathcal{F})$ containing $G(\mathcal{G}/\mathcal{F}_1)G(\mathcal{G}/\mathcal{F}_2)$ is of the form $G(\mathcal{G}/\mathcal{F}')$, where $\mathcal{F}' \subset \mathcal{F}_1 \cap \mathcal{F}_2$, so that $G(\mathcal{G}/\mathcal{F}') \supset G(\mathcal{G}/\mathcal{F}_1 \cap \mathcal{F}_2)$. On the other hand, $G(\mathcal{G}/\mathcal{F}_1 \cap \mathcal{F}_2)$ is a E-$\mathcal{C}$-subgroup of $G(\mathcal{G}/\mathcal{F})$ containing $G(\mathcal{G}/\mathcal{F}_1)$ and $G(\mathcal{G}/\mathcal{F}_2)$, so that $G(\mathcal{G}/\mathcal{F}') \subset G(\mathcal{G}/\mathcal{F}_1 \cap \mathcal{F}_2)$. ☐

Theorem 4.215 *Assume that* $\mathcal{C}$ *is constrainedly closed over* $\mathcal{F}$ *as an* E-*field. Let* $\mathcal{F}_1 \subseteq \mathcal{G}$ *be an* (E, Δ)-*field containing* $\mathcal{F}$. *Then the following four conditions are equivalent.*

1. $\mathcal{F}_1$ *is an* E-*strongly normal extension of* $\mathcal{F}$.
2. *For each element* $\alpha \in \mathcal{F}_1$ *with* $\alpha \notin \mathcal{F}$, *there exists an* E-*strong isomorphism* σ_1 *of* $\mathcal{F}_1$ *over* $\mathcal{F}$ *such that* $\sigma_1\alpha \neq \alpha$.

3. $G(\mathcal{G}/\mathcal{F}_1)$ *is a normal* E*-subgroup of* $G(\mathcal{G}/\mathcal{F})$*.*

4. $\sigma\mathcal{F}_1 \subset \mathcal{F}_1\mathcal{U}^\Delta$ *for every* $\sigma \in G(\mathcal{G}/\mathcal{F})$*.*

When these conditions are satisfied, then, for each $\sigma \in G(\mathcal{G}/\mathcal{F})$*, the restriction* σ_1 *of* σ *to* $\mathcal{F}_1$ *is an element of* $G(\mathcal{F}_1/\mathcal{F})$*, and the formula* $\sigma \mapsto \sigma_1$ *defines a surjective* E-$\mathcal{C}$*-homomorphism* $G(\mathcal{G}/\mathcal{F}) \to G(\mathcal{F}_1/\mathcal{F})$ *with kernel* $G(\mathcal{G}/\mathcal{F}_1)$*.*

Remark 4.216 *In the proof below, only the implication condition 4 implies condition 3 uses part 3 of the First Fundamental Theorem.*

Proof: If condition 1 is satisfied, then, by part 1 of the First Fundamental Theorem 4.207, the set of invariants of $G(\mathcal{F}_1/\mathcal{F})$ in $\mathcal{F}_1$ is $\mathcal{F}$, so that part 2 is satisfied. Let condition 2 be satisfied. The normalizer N of $G(\mathcal{F}_1/\mathcal{F})$ in $G(\mathcal{G}/\mathcal{F})$ is a E-$\mathcal{C}$-subgroup of $G(\mathcal{G}/\mathcal{F})$ containing $G(\mathcal{F}_1/\mathcal{F})$ [25, Corollary 2, page 103]. By the First and Corollary 4.212, there exists a differential field $\mathcal{F}_2$ with $\mathcal{F} \subset \mathcal{F}_2 \subset \mathcal{F}_1$ such that $G(\mathcal{G}/\mathcal{F}_2) = N$. If σ_1 is any E-strong isomorphism of $\mathcal{F}_1$ over $\mathcal{F}$, σ_1 can be extended to an E-$\mathcal{F}$-isomorphism of $\mathcal{G}$, that is, to an element $\sigma \in G(\mathcal{G}/\mathcal{F})$ (Proposition 1.52). Then for any $\tau \in G(\mathcal{G}/\mathcal{F}_1)$ and any $\beta \in \mathcal{F}_1$, $\sigma\beta = \sigma_1\beta \in \mathcal{F}_1\mathcal{U}^\Delta$, hence $\tau\sigma\beta = \sigma\beta$ and $\sigma^{-1}\tau\sigma\beta = \beta$, so that $\sigma^{-1}\tau\sigma \in G(\mathcal{G}/\mathcal{F}_1)$. Thus, $\sigma \in N$, so that σ_1 leaves invariant every element of $\mathcal{F}_2$. Since σ is an extension of an arbitrary element of $G(\mathcal{F}_1/\mathcal{F})$, it follows by condition 2 that $\mathcal{F}_2 = \mathcal{F}$, that is, $N = G(\mathcal{G}/\mathcal{F})$. Therefore, condition 3 is proved from condition 2. Next, let condition 3 be satisfied. Consider any $\sigma \in G(\mathcal{G}/\mathcal{F})$ and any $\beta \in \mathcal{F}_1$. For every $\tau \in G(\mathcal{G}/\mathcal{F}_1)$, $\sigma^{-1}\tau\sigma \in G(\mathcal{G}/\mathcal{F}_1)$, so that $\sigma^{-1}\tau\sigma\beta = \beta$ and $\tau\sigma\beta = \sigma\beta$. Since by Theorem 4.30, $G(\mathcal{G}/\mathcal{F}_1) = G(\mathcal{G}\mathcal{C}\langle\sigma\rangle/\mathcal{F}_1\mathcal{C}\langle\sigma\rangle)$, and since $\sigma\beta \in \mathcal{G}\sigma\mathcal{G} = \mathcal{G}\mathcal{C}\langle\sigma\rangle$, $\sigma\beta$ is an invariant of $G(\mathcal{G}\mathcal{C}\langle\sigma\rangle/\mathcal{F}_1\mathcal{C}\langle\sigma\rangle)$ in $\mathcal{G}\mathcal{C}\langle\sigma\rangle$, and hence, by the first part of the First Fundamental Theorem and Corollary 4.212, $\sigma\beta \in \mathcal{F}_1\mathcal{C}\langle\sigma\rangle$. Therefore condition 4 is satisfied. Let condition 4 be satisfied. If σ' is any isomorphism of $\mathcal{F}_1$ over $\mathcal{F}$, then σ' can be extended to an element $\sigma \in G(\mathcal{G}/\mathcal{F})$. Then because of condition 4 $\sigma'\mathcal{F}_1 = \sigma\mathcal{F}_1 \subset \mathcal{F}_1\mathcal{K}$. It follows by [24, Proposition 10, page 393], that condition 1 is satisfied. Therefore all four conditions are equivalent.

Let the conditions be satisfied. It is obvious that the restriction mapping defined by the formula $\sigma \mapsto \sigma_1$ is a group homomorphism $G(\mathcal{G}/\mathcal{F}) \to G(\mathcal{G}/\mathcal{F}_1)$ with kernel $G(\mathcal{G}/\mathcal{F}_1)$. It has already been observed that every isomorphism of $\mathcal{F}_1$ over $\mathcal{F}$ can be extended to an isomorphism of $\mathcal{G}$, and this shows that the homomorphism is surjective. It remains to prove that it is a E-$\mathcal{C}$-homomorphism. First of all, $\mathcal{C}\langle\sigma\rangle = (\mathcal{G}\sigma\mathcal{G}) \cap \mathcal{U}^{\Delta} \supset (\mathcal{F}_1\sigma_1\mathcal{F}_1) \cap \mathcal{U}^{\Delta} = \mathcal{C}\langle\sigma_1\rangle$. Next, if σ' is an E-specialization of σ in $G(\mathcal{G}/\mathcal{F})$, then, by definition, $\sigma \underset{\mathcal{G}}{\to} \sigma'$ (Definition 1.34). By Lemma 1.35, this is equivalent to $(\sigma'\alpha)_{\alpha\in\mathcal{G}}$ is an (E, Δ)-$\mathcal{G}$-specialization of $(\sigma\alpha)_{\alpha\in\mathcal{G}}$ over $\mathcal{G}$, so that a fortiori $(\sigma_1'\alpha)_{\alpha\in\mathcal{G}}$ is an (E, Δ)-$\mathcal{G}$-specialization of $(\sigma_1\alpha)_{\alpha\in\mathcal{G}}$ over $\mathcal{F}_1$, that is, σ_1' is a differential specialization of σ_1 by Lemma 1.35. Finally, if σ' is a generic specialization of σ, then by the above, σ_1' is a generic specialization of σ_1. Since the induced E-$\mathcal{C}$-isomorphism $S_{\sigma',\sigma}$: $\mathcal{C}\langle\sigma\rangle \approx \mathcal{C}\langle\sigma'\rangle$ is a restriction of the (E, Δ)-$\mathcal{G}$-

isomorphism $\mathcal{G}\sigma\mathcal{G} \approx \mathcal{G}\sigma'\mathcal{G}$ mapping $\sigma\alpha$ onto $\sigma'\alpha$ ($\alpha \in \mathcal{G}$), and the induced E-$\mathcal{C}$-isomorphism $S_{\sigma_1',\sigma_1} : \mathcal{C}\langle\sigma_1\rangle \approx \mathcal{C}\langle\sigma_1'\rangle$ is a restriction of the (E, Δ)-$\mathcal{G}$-isomorphism $\mathcal{F}_1\sigma_1\mathcal{F}_1 \approx \mathcal{F}_1\sigma_1'\mathcal{F}_1$ over $\mathcal{F}_1$ mapping $\sigma\alpha$ onto $\sigma'\alpha$ ($\alpha \in \mathcal{F}_1$), it is evident that $S_{\sigma',\sigma}$ is an extension of S_{σ_1',σ_1}. This shows that the restriction mapping is a E-$\mathcal{C}$-homomorphism and completes the proof of the theorem.□

Corollary 4.217 *Assume that* $\mathcal{C}$ *is constrainedly closed over* $\mathcal{F}$ *as an* E-*field. Let* $\mathcal{F}^\circ$ *denote the algebraic closure of* $\mathcal{F}$ *in* $\mathcal{G}$. *Then* $G(\mathcal{G}/\mathcal{F}^\circ) = G^o(\mathcal{G}/\mathcal{F})$ *(Definition 4.206),* $\mathcal{F}^\circ$ *is an* E-*strongly normal extension of* $\mathcal{F}$*, and* $G(\mathcal{F}^\circ/\mathcal{F}) \approx G(\mathcal{G}/\mathcal{F})/G^o(\mathcal{G}/\mathcal{F})$. *In particular, the degree of* $\mathcal{F}^\circ$ *over* $\mathcal{F}$ *equals the index of* $G^o(\mathcal{G}/\mathcal{F})$ *in* $G(\mathcal{G}/\mathcal{F})$*, so that* $\mathcal{F}$ *is algebraically closed in* $\mathcal{G}$ *if and only if* $G(\mathcal{G}/\mathcal{F})$ *is connected, and* $\mathcal{G}$ *is algebraic over* $\mathcal{F}$ *if and only if* $G(\mathcal{G}/\mathcal{F})$ *is finite.*

Proof: By Proposition 1.49, there exists an isolated (E, Δ)-$\mathcal{F}$-isomorphism $\sigma_0 \in G^o(\mathcal{G}/\mathcal{F})$ such that $\sigma_0 \underset{\mathcal{G}}{\rightarrow}$ id. By part b of [25, Corollary to Proposition 2, page 388], the set of invariants of $G^o(\mathcal{G}/\mathcal{F})$ is $\mathcal{F}^\circ$. Therefore, by the First and Second theorems, $G^o(\mathcal{G}/\mathcal{F}) = G(\mathcal{G}/\mathcal{F}^\circ)$. As $G^o(\mathcal{G}/\mathcal{F})$ is a normal E-$\mathcal{C}$-subgroup of $G(\mathcal{G}/\mathcal{F})$ [25, Theorem 1, page 39], the previous theorem shows that $\mathcal{F}^\circ$ is E-strongly normal over $\mathcal{F}$ and $G(\mathcal{F}^\circ/\mathcal{F}) \approx G(\mathcal{G}/\mathcal{F})/G^o\mathcal{G}/\mathcal{F})$. □

Corollary 4.218 *Assume that* $\mathcal{C}$ *is constrainedly closed over* $\mathcal{F}$ *as an* E-*field. Assume that* $\mathcal{G}\mathcal{H}$ *and* $\mathcal{F}$ *have the same field of* Δ-*constants. Then* $\mathcal{G} \cap \mathcal{H}$ *is an* E-*strongly normal extension of* $\mathcal{F}$.

Proof: By Corollary 3.189, $\mathcal{G}\mathcal{H}$ is E-strongly normal over $\mathcal{F}$. By Theorem 5.43, $G(\mathcal{G}\mathcal{H}/\mathcal{G})$ and $G(\mathcal{G}\mathcal{H}/\mathcal{H})$ are normal E-$\mathcal{C}$-subgroups of $G(\mathcal{G}\mathcal{H}/\mathcal{F})$, so that their product is also [25, Corollary 2, page 109]. By Corollary 5.42, the product is $G(\mathcal{G}\mathcal{H}/\mathcal{G} \cap \mathcal{H})$. Since it is normal in $G(\mathcal{G}\mathcal{H}/\mathcal{F})$, it follows by Theorem 5.43 that $\mathcal{G} \cap \mathcal{H}$ is E-strongly normal over $\mathcal{F}$. □

Theorem 4.219 *Assume that $\mathcal{C}$ is constrainedly closed over $\mathcal{F}$ as an* E-*field. Let $\mathcal{E}$ be an extension of $\mathcal{F}$ such that $\mathcal{U}$ is universal over $\mathcal{E}$ as an* E-*strongly normal extension and the field of Δ-constants of $\mathcal{G}\mathcal{E}$ is $\mathcal{C}$. Then $\mathcal{G}\mathcal{E}$ is an* E-*strongly normal extension of $\mathcal{E}$, for each element $\tau \in G(\mathcal{G}\mathcal{E}/\mathcal{E})$ the restriction τ_1 of τ to $\mathcal{G}$ is an element of $G(\mathcal{G}/\mathcal{G} \cap \mathcal{E})$, and the formula $\tau \mapsto \tau_1$ defines an* E-$\mathcal{C}$-*isomorphism $G(\mathcal{G}\mathcal{E}/\mathcal{E}) \approx G(\mathcal{G}/\mathcal{G} \cap \mathcal{E})$.*

Proof: For any (E, Δ)-$\mathcal{E}$-isomorphism τ of $\mathcal{G}\mathcal{E}$, τ_1 is obviously an (E, Δ)-isomorphism of $\mathcal{G}$ over $\mathcal{G} \cap \mathcal{E}$ and hence is E-strong. Therefore,

$$\tau(\mathcal{G}\mathcal{E}) \subseteq \mathcal{G}\mathcal{E} \cdot \tau(\mathcal{G}\mathcal{E}) = \mathcal{G}\mathcal{E}\tau\mathcal{G} \cdot \mathcal{E} = \mathcal{G}\tau_1\mathcal{G} \cdot \mathcal{E} = \mathcal{G}\mathcal{C}\langle\tau_1\rangle \cdot \mathcal{E} = \mathcal{G}\mathcal{E}\mathcal{C}\langle\tau_1\rangle \subset \mathcal{G}\mathcal{E}\mathcal{U}^{\Delta}.$$

It follows from Proposition 3.188, $\mathcal{G}\mathcal{E}$ is E-strongly normal over $\mathcal{E}$.

Clearly the formula $\tau \mapsto \tau_1$ defines an injective group homomorphism $G(\mathcal{G}\mathcal{E}/\mathcal{E}) \to G(\mathcal{G}/\mathcal{G} \cap \mathcal{E})$. It also follows from the above sequence of equalities that $\mathcal{G}\mathcal{E}\mathcal{C}\langle\tau\rangle = \mathcal{G}\mathcal{E}\mathcal{C}\langle\tau_1\rangle$ and by [24, Corollary 2, page 88] $\mathcal{C}\langle\tau\rangle = \mathcal{C}\langle\tau_1\rangle$. If τ and τ' are elements of $G(\mathcal{G}\mathcal{E}/\mathcal{E})$ and $\tau \to \tau'$, then $(\tau'\beta)_{\beta \in \mathcal{G}\mathcal{E}}$ is an (E, Δ)-specialization of $(\tau\beta)_{\beta \in \mathcal{G}\mathcal{E}}$ over $\mathcal{G}\mathcal{E}$, so that $(\tau_1'\beta)_{\beta \in \mathcal{G}}$ is an (E, Δ)-specialization of $(\tau_1\beta)_{\beta \in \mathcal{G}}$ over $\mathcal{G}$, whence $\tau_1 \to \tau_1'$. If moreover $\tau \leftrightarrow \tau'$, then $\tau_1 \leftrightarrow \tau_1'$, and the (E, Δ)-isomorphism $\mathcal{G}\mathcal{E}\cdot\tau(\mathcal{G}\mathcal{E}) \approx \mathcal{G}\mathcal{E}\cdot\tau'(\mathcal{G}\mathcal{E})$ over $\mathcal{G}\mathcal{E}$ mapping $\tau\beta$ onto $\tau\beta'(\beta \in \mathcal{G}\mathcal{E})$ is an extension of the (E, Δ)-isomorphism $\mathcal{G}\tau_1\mathcal{G} \approx \mathcal{G}\tau_1'\mathcal{G}$ over $\mathcal{G}$ mapping $\tau_1\beta$ onto $\tau_1'\beta$ $(\beta \in \mathcal{G})$. Since these two (E, Δ)-isomorphisms are extensions of the induced E-isomorphisms $S_{\tau',\tau} : \mathcal{C}\langle\tau\rangle \approx \mathcal{C}\langle\tau'\rangle$ and $S_{\tau_1',\tau_1} : \mathcal{C}\langle\tau_1\rangle \approx \mathcal{C}\langle\tau_1'\rangle$, and since $\mathcal{C}\langle\tau\rangle = \mathcal{C}\langle\tau_1\rangle$ and $\mathcal{C}\langle\tau'\rangle = \mathcal{C}\langle\tau_1'\rangle$, $S_{\tau',\tau} = S_{\tau_1',\tau_1}$. It follows that the injective group homomorphism is an E-$\mathcal{C}$-homomorphism. Its image is an E-$\mathcal{C}$-subgroup G_1 of $G(\mathcal{G}/\mathcal{G}\cap\mathcal{E})$. If an element $\alpha \in \mathcal{G}$ is an invariant of G_1, then it is an invariant of $G(\mathcal{G}\mathcal{E}/\mathcal{E})$, whence $\alpha \in \mathcal{E}$. Thus, the set of invariants of G_1 in $\mathcal{G}$ is $\mathcal{G}\cap\mathcal{E}$, so that $G_1 = G(\mathcal{G}/\mathcal{G}\cap\mathcal{E})$ by the First Fundamental Theorem and Corollary 4.212. This completes the proof of the theorem. ☐

5 Disjointness from Derivatives

5.1 Introduction

In this chapter, a concept of disjointness introduced by Kolchin is defined and analyzed. It is used in two ways to construct E-strongly normal extensions.

In this introduction, one should consider the commuting derivations Δ as the union of two disjoint subsets Δ' and Δ''. Let $\mathcal{F}$ be a Δ-field, and let $\mathcal{A}'$ a Δ'-extension of $\mathcal{F}$ generated by η' a family of elements of some Δ-extension of $\mathcal{F}$. A family η of elements of some Δ-extension of $\mathcal{F}$ will be chosen that satisfy the same Δ'-equations η' satisfies but no Δ''-equation other than those derived from the Δ'-equations satisfied by η'. It will be shown that $\mathcal{F}\langle\eta\rangle_\Delta$ is free from several types Δ''-relations. In particular, if $\mathcal{F}\langle\eta\rangle_\Delta$ over $\mathcal{F}$ is regular,

$\mathfrak{F}\langle\eta\rangle_\Delta$ is free from Δ''-constants not in $\mathfrak{F}$.

This control of constants is essential for creating examples of E-strongly normal extensions. In fact, an E-strongly normal extension will be constructed with Galois group E-isomorphic to any given connected E-group. This construction uses the special extensions without new constants described in the last paragraph and the technique of the logarithmic derivative. The method of proof of this result is new even for algebraic groups in Kolchin's setting [22, Theorem 2, page 880] and does not require the field of constants to be algebraically closed as does the result of Kolchin.

A second use of these extensions will be to define a functor from pre Δ'-sets to pre Δ-sets. This takes a Δ'-group to a Δ-group and is compatible with the Galois theory (Section 5.6).

5.2 Definition of Δ''-Free

In this section, $\mathfrak{F}$ will always denote a Δ-field.

Definition 5.220 *Let $\mathcal{A}$ be a Δ-$\mathfrak{F}$-algebra. Let Δ' be a finite commuting subset of the vector space of derivations of $\mathfrak{U}$ spanned by Δ over $\mathfrak{F}$. Let $\mathcal{A}'$ be a Δ'-$\mathfrak{F}$-subalgebra of $\mathcal{A}$ such that $\mathcal{A}'$ generates $\mathcal{A}$ as a Δ-$\mathfrak{F}$-algebra. Define $\mathcal{A}$ to be* Δ/Δ'-$\mathfrak{F}$-free over *$\mathcal{A}'$ if any Δ'-$\mathfrak{F}$-homomorphism of $\mathcal{A}'$ into a Δ-field extension of $\mathfrak{F}$ can be extended to a Δ-$\mathfrak{F}$-homomorphism of $\mathcal{A}$. If Δ is the disjoint union of two subsets Δ' and Δ'', define $\mathcal{A}$ to be* Δ''-$\mathfrak{F}$-free over *$\mathcal{A}'$ if $\mathcal{A}$ is Δ/Δ'-$\mathfrak{F}$-free over $\mathcal{A}'$.*

Kolchin [25, Section 7, page 19] uses the terminology "$\mathcal{A}'$ and Δ are Δ'-disjoint over $\mathfrak{F}$" instead of $\mathcal{A}$ is Δ/Δ'-$\mathfrak{F}$-free over $\mathcal{A}'$. It might seem strange that Kolchin's terminology does not refer to the ring $\mathcal{A}$ that $\mathcal{A}'$ Δ-generates. But $\mathcal{A}$ is implicit in Kolchin's definition because the Δ'-algebra $\mathcal{A}'$ is assumed to be contained in some larger unspecified Δ-algebra, so that $\mathcal{A}$ is uniquely determined by $\mathcal{A}'$ and the Δ-algebra containing it.

The following proposition shows that if $\mathcal{A}$ is Δ/Δ'-$\mathfrak{F}$-free over $\mathcal{A}'$ the Δ'-$\mathfrak{F}$-isomorphism class of $\mathcal{A}'$ determines the Δ-$\mathfrak{F}$-isomorphism class of $\mathcal{A}'_\Delta$.

Proposition 5.221 *Let $\mathcal{A}$ and $\mathcal{B}$ be Δ-$\mathfrak{F}$-algebras that are integral domains. Let $\mathcal{A}'$ and $\mathcal{B}'$ be Δ'-$\mathfrak{F}$-subalgebras of $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A}$ is Δ/Δ'-$\mathfrak{F}$-free over $\mathcal{A}'$ and $\mathcal{B}$ is Δ/Δ'-$\mathfrak{F}$-free over $\mathcal{B}'$. If $\mathcal{A}'$ and $\mathcal{B}'$ are Δ'-$\mathfrak{F}$-isomorphic, then $\mathcal{A} = \mathcal{A}'_\Delta$ and $\mathcal{B} = \mathcal{B}'_\Delta$ are Δ-$\mathfrak{F}$-isomorphic.*

Proof: In the definition of Δ/Δ''-free, the extension Δ-homomorphism is clearly unique because it is determined by the action of the Δ'-homomorphism on Δ'-ring generators. Let $\varphi' : \mathcal{A}' \longrightarrow \mathcal{B}'$ be a given Δ'-$\mathfrak{F}$-isomorphism, and let $\chi' : \mathcal{B}' \longrightarrow \mathcal{A}'$ be inverse Δ'-$\mathfrak{F}$-isomorphism. Then φ' and χ' extend to unique Δ-$\mathfrak{F}$-homomorphisms $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ and $\chi : \mathcal{B} \longrightarrow \mathcal{A}$. The composite Δ-$\mathfrak{F}$-homomorphism $\chi\varphi : \mathcal{A} \longrightarrow \mathcal{A}$ is the unique Δ-$\mathfrak{F}$-homomorphism extending the identity Δ'-$\mathfrak{F}$-isomorphism of $\mathcal{A}'$ and, therefore, is the identity Δ-$\mathfrak{F}$-isomorphism of $\mathcal{A}$. Similarly, $\varphi\chi$ is the identity, and, therefore, φ is a Δ-$\mathfrak{F}$-isomorphism. □

Corollary 5.222 *Let $\mathcal{A}$ be an integral domain and Δ/Δ'-$\mathcal{F}$-free over $\mathcal{A}'$. Then each Δ'-automorphism of $\mathcal{A}'$ extends uniquely to a Δ-automorphism of $\mathcal{A} = \mathcal{A}'_{\Delta}$.*

The following is the first basic proposition of Kolchin about this concept of Δ/Δ'-$\mathcal{F}$-free extensions ([25, Proposition 9, page 20]).

Proposition 5.223 *Let $\eta = (\eta_j)_{j\in J}$ be a family of elements of a Δ-field extension $\mathcal{U}$ that is Δ-universal over $\mathcal{F}$, let Δ' be a commutative linearly independent subset of $\mathcal{F}\Delta$, and let $\mathfrak{P}'$ and $\mathfrak{P}$ denote, respectively, the defining Δ'-ideal of η in $\mathcal{F}\{(y_j)_{j\in J}\}_{\Delta'}$ and the defining Δ-ideal of η in $\mathcal{F}\{(y_j)_{j\in J}\}_{\Delta}$. Then the following three conditions are equivalent.*

1. *$\mathcal{F}\{\eta\}_{\Delta}$ is Δ''-free over $\mathcal{F}\{\eta\}_{\Delta'}$.*

2. *$\mathcal{F}\{\eta\}_{\Delta}$ is Δ''-free over $\mathcal{F}\langle\eta\rangle_{\Delta'}$.*

3. *$\mathfrak{P} = \{\mathfrak{P}'\}_{\Delta}$.*

The equivalence of condition 1 and condition 3 in this proposition shows that $\mathcal{F}\{\eta\}_{\Delta}$ is Δ''-free over $\mathcal{F}\{\eta\}_{\Delta'}$ if and only if $\{\mathfrak{P}'\}_{\Delta}$ is the defining Δ-ideal of η in $\mathcal{F}\{(y_j)_{j\in J}\}_{\Delta}$. This observation enables one to construct a Δ'-$\mathcal{F}$-algebra $\mathcal{B}' \subset \mathcal{U}$ which is Δ'-isomorphic over $\mathcal{F}$ to $\mathcal{A}'$ and such that $\mathcal{B}'_{\Delta}$ is Δ/Δ'-$\mathcal{F}$-free over $\mathcal{B}'$. [25, Proposition 11, page 22]. Just take $\xi = (\xi_j)_{j\in J}$ to be an $\mathcal{F}$-generic Δ-zero of $\{\mathfrak{P}'\}_{\Delta}$. Then, by the equivalence stated, $\mathcal{F}\{\xi\}_{\Delta}$ is

Δ/Δ'-$\mathfrak{F}$-free over $\mathcal{B}'$, and $\mathcal{B}'$ is Δ'-isomorphic over $\mathfrak{F}$ to $\mathcal{A}'$ because ξ, as is η, is an $\mathfrak{F}$-generic Δ'-zero of $\mathfrak{P}'$.

The proof that condition 3 implies condition 1 is straight forward application of the definition of Δ''-freeness. For simplicity, assume that the indexing set J is finite, i.e. $\eta = (\eta_1, \ldots, \eta_n)$. Let $\varphi' : \mathcal{A}' \to \mathfrak{U}$ be a Δ'-$\mathfrak{F}$-homomorphism. Then, $\varphi'(\eta)$ is a Δ'-zero of $\mathfrak{P}'$ and, thus, a Δ-zero of $\mathfrak{P} = \{\mathfrak{P}'\}_\Delta$. Since η is an $\mathfrak{F}$-generic Δ-zero of its defining ideal $\mathfrak{P}$, $\eta \to \varphi'(\eta)$, and, thus, φ' extends to a Δ-$\mathfrak{F}$-homomorphism $\varphi : \mathcal{A}_\Delta = \mathfrak{F}\{\eta\}_\Delta \to \mathfrak{F}\{\varphi'\eta\}_\Delta$.

The following proof, which is slightly different than that of Kolchin, that condition 1 implies condition 3 will serve to motivate the next proposition. Clearly, $\mathfrak{P} \supset \{\mathfrak{P}'\}_\Delta$. Assume that there exists $F \in \mathfrak{P} \subset \mathfrak{F}\{y_1, \ldots, y_n\}_\Delta$ with $F \notin \{\mathfrak{P}'\}_\Delta$. Since $\mathfrak{P} \cap \mathfrak{F}\{y_1, \ldots, y_n\}_{\Delta'} = \mathfrak{P}'$ [25, Proposition 8, page 16], the Δ-polynomial $F \notin \mathfrak{F}\{y_1, \ldots, y_n\}_{\Delta'}$ and, thus, must involve some Δ''-derivatives of some y_i. Since $\mathfrak{U}$ is a Δ-universal over $\mathfrak{F}$, one may choose Δ-zero $\xi = (\xi_1, \ldots, \xi_n) \in \mathfrak{U}^n$ of $\{\mathfrak{P}'\}_\Delta \subset \mathfrak{F}\{y_1, \ldots, y_n\}_\Delta$ such that $F(\xi) \neq 0$. Because ξ is a zero of $\mathfrak{P}'$, there is a Δ'-homomorphism of $\mathfrak{F}\{\eta\}_{\Delta'}$ onto $\mathfrak{F}\{\xi\}_{\Delta'}$ sending η to ξ. This Δ'-homomorphism cannot extend to a Δ-homomorphism from $\mathfrak{F}\{\eta\}_\Delta$ to $\mathfrak{F}\{\xi\}_\Delta$ because $F(\eta) = 0$ and $F(\xi) \neq 0$. Therefore, $\mathcal{A}$ is not Δ/Δ'-$\mathfrak{F}$-free over $\mathcal{A}'$.

The existence of the Δ-polynomial F 'prevents' $\mathcal{A}$ from being Δ/Δ'-$\mathfrak{F}$-free over $\mathcal{A}'$. Since $F \notin \mathfrak{F}\{y_1, \ldots, y_n\}_{\Delta'}$, proper Δ''-derivatives of Δ'-derivatives of $(y_1, \ldots, y_n)$ are present in F. Since η is a Δ-zero of F, some Δ''-derivatives of Δ' derivatives of η are algebraically dependent over $\mathcal{A}'$. Thus, the algebraic independence of certain of the ring generators of $\mathcal{A}$ over $\mathcal{A}'$ is a necessary condition for freeness. This is made precise in the following proposition, which is a generalization of the results of Sit (with Δ' empty) [35, Corollaries 1 and 2, page 25].

Proposition 5.224 *Let $\xi = (\xi_1, \ldots, \xi_n)$ be Δ-generators of a Δ-field over $\mathcal{F}$. Assume that Δ is the union of two disjoint subsets Δ' and Δ''. Then the following statements are equivalent.*

1. *The Δ-$\mathcal{F}$-algebra $\mathcal{F}\{\mathcal{F}\langle\xi\rangle_{\Delta'}\}_{\Delta}$ is Δ''-$\mathcal{F}$-free over $\mathcal{F}\langle\xi\rangle_{\Delta'}$.*
2. *Every transcendence basis for the field $\mathcal{F}\langle\xi\rangle_{\Delta'}$ over $\mathcal{F}$ is Δ''-algebraically independent over $\mathcal{F}$.*
3. *There exists one transcendence basis for the field $\mathcal{F}\langle\xi\rangle_{\Delta'}$ over $\mathcal{F}$ that is Δ''-algebraically independent over $\mathcal{F}$.*

Proof: Because there always exists a transcendence basis for the field $\mathcal{F}\langle\xi\rangle_{\Delta'}$ over $\mathcal{F}$, condition 2 implies condition 3. Assuming condition 3, let the transcendence basis $(t_i)_{i\in I}$ for the field $\mathcal{F}\langle\xi\rangle_{\Delta'}$ over $\mathcal{F}$ be Δ''-algebraically independent over $\mathcal{F}$.

Claim 5.225 $\mathcal{F}\{\mathcal{F}\langle\xi\rangle_{\Delta'}\}_{\Delta} = \mathcal{F}\langle\xi\rangle_{\Delta'}[(\theta'' t_i)_{i\in I, \theta''\in\,\Theta_{\Delta''}}]$

Proof: The right hand side is clearly contained in the left. To prove the claim, it must be shown that all the Δ-derivatives of $\alpha \in \mathcal{F}\langle\xi\rangle_{\Delta'}$ are in the right hand side. If $\alpha \in \mathcal{F}((t_i)_{i\in I})$, this is clear by the formula for the derivative of a quotient. If α is algebraic over $\mathcal{F}((t_i)_{i\in I})$ and not in $\mathcal{F}((t_i)_{i\in I})$, let $f(x) \in \mathcal{F}((t_i)_{i\in I})[x]$ be the minimal polynomial for α. For $\delta'' \in \Delta''$, let $S_f(x) = df/dx$, and let $f^{\delta''}(x)$ be the polynomial obtained from $f(x)$ by differentiating the coefficients of $f(x)$ with respect to δ''. Then, $S_f(\alpha)\delta''\alpha + f^{\delta''}(\alpha) = 0$. Since the degree of $S_f(x)$ in x is less than than the degree of the minimal polynomial, $S_f(\alpha) \neq 0$. Then, $\delta''\alpha = -f^{\delta''}(\alpha)/S_f(\alpha)$ is an element of the right hand side because the coefficients of $f^{\delta''}(x)$, $f^{\delta''}(\alpha)$ and $1/S_f(\alpha)$ are in the right hand side. □

To show condition 1, let $\rho : \mathcal{F}\langle\xi\rangle_{\Delta'} \to \mathcal{H}$ be a Δ'-$\mathcal{F}$-homomorphism to an Δ-$\mathcal{F}$-field $\mathcal{H}$. The $\theta''t_i$ for all $i \in I$ and all $\theta'' \in \Theta_{\Delta''}$ of positive order, in addition to being algebraically independent over $\mathcal{F}$, are algebraically independent over $\mathcal{F}\langle\xi\rangle_{\Delta'}$ because an algebraic relation over $\mathcal{F}\langle\xi\rangle_{\Delta'}$ would contradict the algebraic independence of the family $(\theta''t_i)_{i\in I,\theta''\in\Theta''}$ over $\mathcal{F}$. Therefore, one may extend ρ to an $\mathcal{F}$-homomorphism of $\mathcal{F}\langle\xi\rangle_{\Delta'}[(\theta''t_i)_{i\in I,\theta''\in\,\Theta_{\Delta''}}]$ by defining $\rho(\theta''t_i) = \theta''\rho(t_i)$ for all $i \in I$ and all $\theta'' \in \Theta_{\Delta''}$. To complete the proof of condition 1, it will be shown that ρ is a Δ-$\mathcal{F}$-isomorphism.

To show ρ is an Δ''-$\mathcal{F}$-homomorphism, since ρ restricted to $\mathcal{F}[(\theta''t_i)_{i\in I,\theta''\in\,\Theta_{\Delta''}}]$ clearly is, it must be shown that $\rho\delta''\alpha = \delta''\rho\alpha$ for all δ'' in $\Theta_{\Delta''}$ and for $\alpha \in \mathcal{F}\langle\xi\rangle_{\Delta'}$ algebraic over $\mathcal{F}((t_i)_{i\in I})$. If α is not in $\mathcal{F}((t_i)_{i\in I})$, as before, let $f(x) \in \mathcal{F}((t_i)_{i\in I})[x]$ be the minimal polynomial for α. Then, for $\delta \in \Delta''$, $S_f(\alpha)\delta''\alpha + f^{\delta''}(\alpha) = 0$, $S_f(\alpha) \neq 0$, and $\delta''\alpha = -f^{\delta''}(\alpha)/S_f(\alpha)$ is an element of $\mathcal{F}\langle\xi\rangle_{\Delta'}[(\theta''t_i)_{i\in I,\theta''\in\,\Theta_{\Delta''}}]$, the domain of ρ. Since ρ restricted to $\mathcal{F}\langle\xi\rangle_{\Delta'}$ is an isomorphism, $\rho\alpha$ satisfies $(\rho f)(x)$ and $S_{\rho f}(\rho\alpha) = \rho(S_f(\alpha)) \neq 0$.

Apply δ'' to $(\rho f)(\alpha) = 0$ to obtain $S_{\rho f}(\rho\alpha)\delta''\rho\alpha + (\rho f)^{\delta''}(\rho\alpha) = 0$ and $\delta''\rho\alpha = -(\rho f)^{\delta''}(\rho\alpha)/S_{\rho f}(\rho\alpha)$. Since the coefficients of f are in $\mathcal{F}((t_i)_{i\in I})$ where ρ and δ'' commute, $(\rho f)^{\delta''}(x) = \rho(f^{\delta''})(x)$. Therefore,

$$\delta''\rho\alpha = -(\rho f)^{\delta''}(\rho\alpha)/S_{\rho f}(\rho\alpha) = -\rho(f^{\delta''})(\rho\alpha)/S_{\rho f}(\rho\alpha)$$

$$= -\rho((f^{\delta''})(\alpha))/\rho(S_f(\alpha)) = -\rho((f^{\delta''})(\alpha)/S_f(\alpha)) = \rho\delta''\alpha.$$

This Δ''-$\mathcal{F}$-homomorphism ρ is also a Δ'-$\mathcal{F}$-homomorphism because ρ restricted to $\mathcal{F}\langle\xi\rangle_{\Delta'}$ was assumed to be a Δ'-$\mathcal{F}$-isomorphism and because, for all θ'' in $\Theta_{\Delta''}$ and all δ' in Δ',

$$\rho(\delta'\theta''t_i) = \rho(\theta''\delta't_i) = \theta''\rho(\delta't_i) = \theta''\delta'\rho(t_i) = \delta'\theta''\rho(t_i) = \delta'\rho(\theta''t_i).$$

Therefore, ρ is a Δ-homomorphism of $\mathcal{F}\{\mathcal{F}\langle\xi\rangle_{\Delta'}\}_{\Delta}$. This shows $\mathcal{F}\{\mathcal{F}\langle\xi\rangle_{\Delta'}\}_{\Delta}$ is Δ''-$\mathcal{F}$-free over $\mathcal{F}\langle\xi\rangle_{\Delta'}$.

Assume condition 1. Let $(t_i)_{i\in I}$ be a transcendence basis of $\mathcal{F}\langle\xi\rangle_{\Delta'}$ over $\mathcal{F}$. Let $(y_i)_{i\in I}$ be a family of Δ''-indeterminates over $\mathcal{F}\langle\xi\rangle_{\Delta'}$. Define an isomorphism over $\mathcal{F}$ of fields $\varphi : \mathcal{F}((t_i)_{i\in I}) \mapsto \mathcal{F}((y_i)_{i\in I})$ such that $\varphi(t_i) = y_i$ for each $i \in I$. Then because each element of $\mathcal{F}\langle\xi\rangle_{\Delta'}$ is algebraic over $\mathcal{F}((t_i)_{i\in I})$, φ extends to an isomorphism of $\mathcal{F}\langle\xi\rangle_{\Delta'}$ into an algebraically closed field containing $\mathcal{F}\langle(y_i)_{i\in I}\rangle_{\Delta''}$. Endow the image $\mathcal{H}$ of φ with the unique Δ'-structure such that φ is a Δ'- $\mathcal{F}$-isomorphism mapping each t_i to y_i for i in I. Then $\mathcal{H}\{(y_i)_{i\in I}\}_{\Delta''}$ has a structure of a Δ''-$\mathcal{F}$-algebra because the elements in $\mathcal{H}$ not in $\mathcal{F}((y_i)_{i\in I})$ are algebraic over $\mathcal{F}((y_i)_{i\in I})$ and, as shown in the proof of the claim, have uniquely determined Δ''-derivatives in $\mathcal{H}\{(y_i)_{i\in I}\}_{\Delta''}$.

The Δ'-structure on $\mathcal{H}$ may be extended to all of $\mathcal{H}\{(y_i)_{i\in I}\}_{\Delta''}$ by defining $\delta'(\theta'' y_i) = \theta''\delta' y_i$ for each θ'' in $\Theta_{\Delta''}$, each $\delta' \in \Delta'$ and $i \in I$. Because $\delta'\delta'' y_i = \delta''\delta' y_i$, the derivation $\delta'\delta'' - \delta''\delta'$ on $\mathcal{F}((y_i)_{i\in I})$ is the zero derivation. Since it extends uniquely to the zero derivation on $\mathcal{H}$, $\delta'\delta''\beta = \delta''\delta'\beta$ for β in $\mathcal{H}$ not in $\mathcal{F}(y_i)_{i\in I}$. This shows that there is a well-defined Δ-structure on $\mathcal{H}_{\Delta''}$.

Because condition 3 implies condition 1, $\mathcal{H}_{\Delta''}$ is Δ''-$\mathcal{F}$-free over $\mathcal{H}$. By Lemma 8.59, since φ from $\mathcal{F}\langle\xi\rangle_{\Delta'}$ to $\mathcal{H}$ is an Δ'-isomorphism, $(\mathcal{F}\langle\xi\rangle_{\Delta'})_{\Delta}$ and $\mathcal{H}\{(y_i)_{i\in I}\}_{\Delta''}$ are Δ-isomorphic over $\mathcal{F}$ by an isomorphism that sends t_i to y_i. Because the $(y_i)_{i\in I}$ are Δ''-algebraically independent over $\mathcal{F}$, the $(t_i)_{i\in I}$ are Δ''-algebraically independent over $\mathcal{F}$ also. □

The goal of the rest of this section is to analyze the constants of free extensions. First recall the following exercise of Kolchin [24, Exercise 8, page 159].

Exercise 5.226 *Let $\mathcal{U}$ a Δ-field universal over $\mathcal{F}$. Let $t_1, \ldots, t_n \in \mathcal{U}$ be Δ-algebraically independent over $\mathcal{F}$. Let $P, Q \in \mathcal{F}\{y_1, \ldots, y_n\}_\Delta$ such that $PQ \notin \mathcal{F}$, and* $\gcd(P, Q) = 1$. *Prove that $u = P(t_1, \ldots, t_n)/Q(t_1, \ldots, t_n)$ is Δ-transcendental over $\mathcal{F}$.*

To do this exercise, consider $t_1, \ldots, t_n$ as Δ-indeterminates over $\mathcal{F}$. Choose orderly rankings for $\mathcal{F}\{y_1, \ldots, y_n\}_\Delta$ and $\mathcal{F}\{z\}_\Delta$. Assume $g \in \mathcal{F}\{z\}_\Delta$ is of lowest rank among the non-zero Δ-polynomials satisfied by u. Let $g = I_d v_g^d + I_{d-1} v_g^{d-1} + \ldots + I_0$ where d is a positive integer, v_g is the leader of g, and the I_k are Δ-polynomials in $\mathcal{F}\{z\}_\Delta$ of lower rank than v_g. Because I_0 and I_d are of lower rank than g, $I_0(u) \neq 0$ and $I_d(u) \neq 0$. If ord $v_g = 0$, substitute P/Q for z, clear denominators and observe P divides Q. But $\gcd(P, Q) = 1$, so it may be assumed that ord $v > 0$. Let v_g, v_P and v_Q be the leaders of g, P and Q, respectively, and S_g, S_P and S_Q the separants. Write $v_g = \theta z$, where θ is the non-empty product of r derivations from Δ.

Claim 5.227 $v_g(P/Q) = \theta(P/Q) = [Q^{r-1}(S_P\theta v_P Q - PS_Q\theta v_Q) + W]/Q^{r+1}$ *such that* W *is the sum of terms of rank lower than the maximum rank of* θv_P *and* θv_Q.

Proof: The claim is clearly true for $r = 1$. Assume the claim is true for r. By differentiating $v_g(P/Q) = \theta(P/Q) = (S_P\theta v_P Q - PS_Q\theta v_Q)/Q^2 + W/Q^{r+1}$ with respect to one of the $\delta \in \Delta$, $\delta v_g(P/Q) =$

$$\delta\theta(P/Q) = (S_P\delta\theta v_P Q - PS_Q\delta\theta v_Q)/Q^2 + V/Q^3 + (\delta W Q - (r+1)W\delta Q)/Q^{r+2}$$

such that the rank of V is lower than the maximum rank of $\delta\theta v_P$ and $\delta\theta v_Q$. Since $\delta W Q$ and $(r+1)W\delta Q$ also have lower than the maximum rank of $\delta\theta v_P$ and $\delta\theta v_Q$, after adding the three fractions, the claim is true for $r+1$. $\square$

Let t be a positive integer such that $Q^t \cdot I_j(P/Q)v_g^j(P/Q)$ is a Δ-polynomial, in $\mathcal{F}\{y_1, \ldots, y_n\}_\Delta$, for each $j = 0, \ldots, d$. By substituting u into $Q^t g(z)$, one obtains the zero Δ-polynomial

$$Q^t g(u) = Q^t(I_d(P/Q)v_g^d(P/Q) + I_{d-1}(P/Q)v_g^{d-1}(P/Q) + \ldots + I_0(P/Q)).$$

If rank$P >$ rank$Q$, then, by the claim, the sum of the highest ranking terms of $Q^t g(P/Q)$ is the Δ-polynomial $Q^t I_n(P/Q)(Q^{r-1}S_P\theta v_P Q)^d$ which is equal to zero because $Q^t g(u) = 0$. So that, since $I_n(P/Q) \neq 0$ and $Q \neq 0$, it follows that $S_P = 0$ and $P \in \mathfrak{F}$. Thus, $Q \in \mathfrak{F}$ because rank$P >$ rank$Q$. This is contrary to the assumption $PQ \notin \mathfrak{F}$. If rank$Q >$ rank$P$, the same type of contradiction results.

If ord $P =$ ord Q, by the claim,

$$Q^t I_n(P/Q)Q^{(r-1)d}(S_P\theta v_P Q - PS_Q\theta v_Q)^d$$

$$= Q^t I_n(P/Q)Q^{(r-1)d}(S_P Q - PS_Q)^d(\theta v_P)$$

is the sum of the highest ranking terms of $Q^t g$ and is equal to 0. Therefore, $(S_P Q - PS_Q) = 0$. Then, P divides S_P because $\gcd(P, Q) = 1$. But, this is impossible because S_P has lower rank than P. Thus the exercise is complete.

Corollary 5.228 *Let $\mathfrak{U}$ a Δ-field Δ-universal over $\mathfrak{F}$. Let $t_1, \ldots, t_n \in \mathfrak{U}$ be Δ-algebraically independent over $\mathfrak{F}$. Then $\mathfrak{F}\langle t_1, \ldots, t_n\rangle^\Delta = \mathfrak{F}^\Delta$.*

Proof: The condition that an element be a Δ-constant is a Δ-relation on that element. This is impossible by the previous exercise. ☐

The next lemma is well-known.

Lemma 5.229 (*The Algebraic Constant Lemma*) *Let $\mathfrak{G}$ over $\mathfrak{F}$ be an extension of Δ-fields. A Δ-constant of $\mathfrak{G}$ algebraic over $\mathfrak{F}$ is algebraic over the Δ-constants of $\mathfrak{F}$.*

Proof: Let α be a Δ-constant of $\mathcal{G}$ algebraic over $\mathcal{F}$. Let $f(x) \in \mathcal{F}[x]$ be the minimal polynomial of α over $\mathcal{F}$. Write $f(x) = \Sigma_{i=1,\ldots,d}\, a_i x^i$ for $a_i \in \mathcal{F}$. Then, for each $\delta \in \Delta$, $S_f(\alpha)\delta\alpha + f^{\delta}(\alpha) = 0$, where $S_f(x)$ is the derivative of f with respect to x and $f^{\delta}(x)$ is the polynomial obtained by applying δ to the coefficients of $f(x)$. Since $\delta\alpha = 0$, $f^{\delta}(\alpha) = 0$. Because the leading coefficient of $f(x)$ is 1, the degree of $f^{\delta}(x)$ is less than that of $f(x)$. Since $f(x)$ is the minimal polynomial of α, $f^{\delta}(x) = 0$. Consequently, $\delta a_i = 0$ for $i = 1, \ldots, d$ and all $\delta \in \Delta$. Therefore, the coefficients of $f(x)$ are δ-constants in $\mathcal{F}$, and α is algebraic over $\mathcal{F}^{\Delta}$. □

Lemma 5.230 (*No New Δ''-Constant Lemma*) *Assume that Δ is the union of two disjoint subsets Δ' and Δ''. Let $\xi = (\xi_1, \ldots, \xi_n)$ be a finite family of elements of $\mathcal{U}$. If the Δ-ring $\mathcal{F}\{\mathcal{F}\langle\xi\rangle_{\Delta'}\}_{\Delta}$ is Δ''-$\mathcal{F}$-free over $\mathcal{F}\langle\xi\rangle_{\Delta'}$, then the Δ''-constants of $\mathcal{F}\langle\xi\rangle_{\Delta}$ are contained in the algebraic closure of $\mathcal{F}^{\Delta''}$ in $\mathcal{F}\langle\xi\rangle_{\Delta'}$. If $\mathcal{F}\langle\xi\rangle_{\Delta'}$ is a regular extension of $\mathcal{F}$, $\mathcal{F}\langle\xi\rangle_{\Delta}$ and $\mathcal{F}$ have the same Δ''-constants.*

Proof: By Proposition 5.224, there is a transcendence basis $(t_i)_{i\in I}$ for the field $\mathcal{F}\langle\xi\rangle_{\Delta'}$ over $\mathcal{F}$ that is Δ''-algebraically independent over $\mathcal{F}$. By Corollary 5.228, the Δ''-constants of $\mathcal{F}\langle(t_i)_{i\in I}\rangle_{\Delta''}$ are in $\mathcal{F}$. Let γ in $\mathcal{F}\langle\xi\rangle_{\Delta}$ be algebraic over $\mathcal{F}\langle(t_i)_{i\in I}\rangle_{\Delta''}$ and a Δ''-constant. Then there is a finite subset $t_1, \ldots, t_n$ of the family $(t_i)_{i\in I}$ such that $\gamma \in \mathcal{F}\langle t_1, \ldots, t_n\rangle_{\Delta''}$X. The Algebraic Constant Lemma 5.229 can then be applied to show γ is algebraic over the Δ''-constants of $\mathcal{F}\langle(t_i)_{i\in I}\rangle_{\Delta''}$. Thus, γ is algebraic over $\mathcal{F}^{\Delta''}$ by Corollary 5.228. If $\mathcal{F}\langle\xi\rangle_{\Delta'}$

is regular over $\mathfrak{F}$, then $\mathfrak{F}\langle\xi\rangle_\Delta$ is regular over $\mathfrak{F}$ ([25, Proposition 10(c), page 21]) and, therefore, $\gamma \in \mathfrak{F}$. □

If the Δ-ring $\mathfrak{F}\{\mathfrak{F}\langle\xi\rangle_{\Delta'}\}_\Delta$ is Δ''-$\mathfrak{F}$-free over $\mathfrak{F}\langle\xi\rangle_{\Delta'}$ and if $\mathfrak{F}\langle\xi\rangle_\Delta$ is a not regular extension of $\mathfrak{F}$, there may be some Δ''-constants in $\mathfrak{F}\langle\xi\rangle_{\Delta'}$ algebraic over $\mathfrak{F}$. For example, take $\mathfrak{F} = \mathbb{Q}$, $\Delta' = \emptyset$ and $\mathfrak{P}_{\Delta'} = (y^2+1) \subset \mathbb{Q}[y]$ a prime ideal. Then $\{\mathfrak{P}\}_{\Delta',\Delta''} = \{y^2+1\}_{\Delta''} \subset \mathbb{Q}\{y\}_{\Delta''}$ is a Δ-prime ideal ([25, Proposition 8, page 16]). Let ξ be a $\mathbb{Q}$-generic zero of $\{y^2+1\}_{\Delta''}$ in $\mathfrak{U}$. Then, by Propostion 5.223, $\mathbb{Q}(\xi)_\Delta$ is Δ-$\mathbb{Q}$-free over $\mathbb{Q}(\xi)$. And, since $\delta''y \in \{y^2+1\}_\Delta$ for $\delta'' \in \Delta''$, ξ is a Δ''-constant of $\mathbb{Q}(\xi)$. In fact, the same technique shows that, if $\mathfrak{P}_{\Delta'} = (f)$ where $f \in \mathbb{Q}[y]$ is an irreducible polynomial, $\delta''\xi = 0$ for a $\mathbb{Q}$-generic zero ξ of $\{f\}_{\Delta''}$ in $\mathfrak{U}$. A proof of a similar fact will be given in the first part of Proposition 5.232.

The next two propositions consider Δ'-constants of $\mathfrak{F}\langle\xi\rangle_\Delta$ instead of Δ''-constants. Sometimes $\mathfrak{F}\langle\xi\rangle_\Delta$ will contain no Δ'-constants not in $\mathfrak{F}\langle\xi\rangle_{\Delta'}$. Sometimes it will.

Proposition 5.231 *Let $\xi = (\xi_1, \ldots, \xi_n)$ be a finite family of elements of $\mathfrak{U}$. If the Δ-ring $\mathfrak{F}\{\mathfrak{F}\langle\xi\rangle_{\Delta'}\}_\Delta$ is Δ''-$\mathfrak{F}$-free over $\mathfrak{F}\langle\xi\rangle_{\Delta'}$ and if ξ are Δ'-independent over $\mathfrak{F}$, then $(\mathfrak{F}\langle\xi\rangle_{\Delta'})^{\Delta'} = ((\mathfrak{F}\langle\xi\rangle_\Delta))^{\Delta'} = \mathfrak{F}^{\Delta'}$.*

Proof: The set of all the Δ'-derivatives of ξ is a transcendence basis for $\mathfrak{F}\langle\xi\rangle_{\Delta'}$ over $\mathfrak{F}$. By Proposition 5.224, they are Δ''-algebraically independent over $\mathfrak{F}$, and all the Δ''-derivatives of ξ are Δ'-independent. By Corollary 5.228, there are no new Δ'-constants, and the conclusion follows. □

Proposition 5.232 *Let* card $\Delta' =$ card $\Delta'' = 1$ *and* $\xi = (\xi_1)$, *and let* $\xi \in \mathfrak{U}$. *Let* $f(y) \in \mathcal{F}^{\Delta}\{y\}_{\Delta'}$ *such that* $f(y) = \Sigma a_{ij} y^i (\delta' y)^j$ *with* $a_{ij} \in \mathcal{F}^{\Delta}$. *Assume* $f(\xi) = 0$, *and the* Δ*-ring* $\mathcal{F}\{\mathcal{F}\langle\xi\rangle_{\Delta'}\}_{\Delta}$ *is* Δ''*-*$\mathcal{F}$*-free over* $\mathcal{F}\langle\xi\rangle_{\Delta'}$. *If* $f(y)$ *is of order zero, i.e.* $a_{ij} = 0$ *for* $j > 0$, *then* $\delta'\xi = 0$ *and* $\delta''\xi = 0$. *If not, then* $\delta'\xi_1/\delta''\xi_1$ *is a* Δ'*-constant of* $\mathcal{F}\langle\xi\rangle_{\Delta}$ *not in* $\mathcal{F}\langle\xi\rangle_{\mathrm{E}}$.

Proof: Let $\mathfrak{P}'$ and $\mathfrak{P}$ denote, respectively, the prime defining Δ'-ideals of ξ in $\mathcal{F}\{(y_j)_{j\in J}\}_{\Delta'}$ and the defining Δ-ideal of ξ in $\mathcal{F}\{(y_j)_{j\in J}\}_{\Delta}$. By Proposition 5.223, $\mathfrak{P} = \{\mathfrak{P}'\}_{\Delta}$.

If $\delta' y$ is not present in f, then ξ is algebraic over $\mathcal{F}^{\Delta}$. Let $g \in \mathcal{F}^{\Delta}[y]$ be the minimal polynomial for ξ. Clearly, $g \in \mathfrak{P}'$, and $\delta' g \in \mathfrak{P}'$. Let S be dg/dy. Because g is the minimal polynomial, $S \notin \mathfrak{P}'$. Since $\delta' g = S\delta' y$ and since $\mathfrak{P}'$ is prime, $\delta' y \in \mathfrak{P}' \subset \mathfrak{P}$, and $\delta'\xi = 0$.

If $\delta' y$ is present in f, $\delta' f = S\delta'^2 y + (\partial f/\partial y)\delta' y$ and $\delta'' f = S\delta''\delta' y + (\partial f/\partial y)\delta'' y$ are elements of $\mathfrak{P} = \{\mathfrak{P}'\}_{\Delta}$, where $S = \partial f/\partial \delta' y$ and $S \notin \mathfrak{P}$. Then,

$$\delta'' y \cdot \delta' f - \delta' y \cdot \delta'' f = S(\delta'' y \delta'^2 y - \delta' y \delta'' \delta' y)$$

is also an element of $\mathfrak{P}$. Since $S \notin \mathfrak{P}$, $\delta'' y \delta'^2 y - \delta' y \delta'' \delta' y$ is. Because ξ is a Δ-zero of $\mathfrak{P}$, $\delta''\xi\delta'^2\xi - \delta'\xi\delta''\delta'\xi = 0$, and $\delta'(\delta'\xi_1/\delta''\xi_1) = 0$. ☐

The last proposition applies to familiar equations. For instance, if $f(y) = (\delta' y)^2 - y^3 - ay - b$, the Weiestrass $\wp$-function results in new Δ'-constants. If $f(y) = y - \delta' y$, the exponential function results in new Δ'-constants. In this case, the constant of the proposition is $\delta'\xi_1/\delta''\xi_1 = \xi_1/\delta''\xi_1$.

5.3 The E-Group Induced from an Algebraic Group.

In this section, let $\mathfrak{F}$ be a Δ-field and let Δ' be a commutative linearly independent of the vector space spanned by Δ over $\mathfrak{F}$. Let $\mathfrak{U}$ be a Δ-extension of $\mathfrak{F}$ that is Δ-universal over $\mathfrak{F}$. In [25, Chapter 2, Section 3, page 56], Kolchin develops a procedure for associating to each Δ'-$\mathfrak{F}$-group (relative to the Δ'-field $\mathfrak{U}$) a Δ-$\mathfrak{F}$-group G (relative to the Δ-field $\mathfrak{U}$) which is called the induced Δ-$\mathfrak{F}$-group and will be denoted by G_Δ. The elements of G_Δ are defined to be the same as those of G. If the Δ'-subfield of $\mathfrak{U}$ associated to x in G is $\mathfrak{F}\langle x\rangle_{\Delta'}$, the Δ-subfield of $\mathfrak{U}$ associated to x in G_Δ is $\mathfrak{F}\langle \mathfrak{F}\langle x\rangle_{\Delta'}\rangle_\Delta$.

Heuristically, to each open affine B of G defined by a Δ'-ideal $\mathfrak{P}'$ of $\mathfrak{F}\{y_1, \ldots, y_n\}_{\Delta'}$, one may associate the open affine B_Δ of G_Δ defined by the Δ-ideal $\{\mathfrak{P}'\}_\Delta$ of $\mathfrak{F}\{y_1, \ldots, y_n\}_\Delta$. To the element x of G, thought of as a Δ'-zero in $\mathfrak{U}^n$ of $\mathfrak{P}'$, corresponds the element x of G_Δ, thought of as a Δ-zero of $\{\mathfrak{P}'\}_\Delta$. The Δ'-rational functions giving the group law on G also give the group law on G_Δ. An $\mathfrak{F}$-generic element v of G, which is a generic zero of some $\mathfrak{P}'$ as above, will be an $\mathfrak{F}$-generic element of G_Δ if and only if it is a generic zero of $\{\mathfrak{P}'\}_\Delta$ [25, Theorem 3(2c), page 58]. The discussion in the last section implies v will be an $\mathfrak{F}$-generic element of G_Δ if v is a $\mathfrak{F}$-generic element of G and $(\mathfrak{F}\langle x\rangle_{\Delta'})_\Delta$ is Δ/Δ'-$\mathfrak{F}$-free over $\mathfrak{F}\langle x\rangle_{\Delta'}$.

Definition 5.233 [25, page 56]*Let Δ' be a commutative linearly independent subset of $\mathfrak{F}\Delta$. Let G be an Δ'-$\mathfrak{F}$-group (relative to the Δ'-field $\mathfrak{U}$) and H be an Δ-$\mathfrak{F}$-group (relative to the Δ-field $\mathfrak{U}$). A (Δ, Δ')-$\mathfrak{F}$-homomorphism of*

H into G is a group homomorphism $f : H \longrightarrow G$ that satisfies the following three conditions:

1. *if $y \in H$, then $\mathfrak{F}\langle f(y)\rangle_{\Delta'} \subset \mathfrak{F}\langle y\rangle_{\Delta}$,*
2. *if $y, y' \in H$ and $y \rightarrow^{\Delta} y'$, then $f(y) \rightarrow^{\Delta'} f(y')$,*
3. *if $y, y' \in H$ and $y \leftrightarrow^{\Delta} y'$, then $S_{\Delta,y',y}$ extends $S_{\Delta',f(y'),f(y)}$.*

Definition 5.234 [25, page 57] *Let G be an Δ'-$\mathfrak{F}$-group relative to the universe $\mathfrak{U}$. A Δ-$\mathfrak{F}$-group structure on G, denoted by G_{Δ}, is said to be induced (by the given Δ'-$\mathfrak{F}$-group structure on G) if the following two conditions are satisfied:*

1. id_G *is a (Δ, Δ')-$\mathfrak{F}$-homomorphism;*
2. *every (Δ, Δ')-$\mathfrak{F}$-homomorphism of a Δ-$\mathfrak{F}$-group into G is a Δ-$\mathfrak{F}$-homomorphism.*

5.4 Varying the Universal field

For $\mathfrak{F}$ a Δ-field, the functor "extending the universal field of $\mathfrak{F}$", has been developed by Kolchin. (See [25, Chapter 2, Section 1, Varying the universal differential field, page 45] and [25, Chapter 8, Section 10, The Lie-Cassidy-Kovacic method, page 247]). Let $\mathcal{V}$ and $\mathfrak{U}$ be Δ-extensions of $\mathfrak{F}$ that are Δ-universal over $\mathfrak{F}$ and such that $\mathfrak{U} \subseteq \mathcal{V}$. The functor "extending the universal field of $\mathfrak{F}$" takes the category of Δ-$\mathfrak{F}$-groups (relative to $\mathfrak{U}$) and Δ-$\mathfrak{F}$-group homomorphisms to the category of Δ-$\mathfrak{F}$-groups (relative to $\mathcal{V}$) and Δ-$\mathfrak{F}$-groups homomorphisms. Heuristically, a set defined as the Δ-zeros in $\mathcal{U}$ of a system of Δ-equations is taken to the set of Δ-zeros in $\mathcal{V}$ of the same system of Δ-equations.

5.5 The Logarithmic Derivative

The purpose of this section is to prove every connected E-group is isomorphic to the Galois group of an E-strongly normal extension. The technique of proof was suggested by Kovacic, who after looking at the examples of E-strongly normal extensions for subgroups of G_a and G_m, realized that the No New Δ''-Constant Lemma 5.230, in the generality presented in the last section, together with standard reasoning with the logarithmic derivative would achieve the desired result.

For the definition of the logarithmic derivative see [25, page 236]. In this paragraph, an heuristic description of the logarithmic derivative will be given. Let $\mathfrak{F}$ be an E-field, and let $\mathcal{V}$ be an E-extension of $\mathfrak{F}$ that is E-universal over $\mathfrak{F}$. Let G be a connected E-$\mathfrak{F}$-group (relative to the E-field $\mathcal{V}$). Let $\mathcal{H} \subset \mathcal{V}$ an E-extension of $\mathfrak{F}$, with $\mathcal{U}$ not necessarily universal over $\mathcal{H}$. Let χ be an E-derivation (χ commutes with the action of E) of $\mathcal{H}$ into $\mathcal{V}$ over $\mathfrak{F}$. For each element g of G rational over $\mathcal{H}$, evaluation at g of E-$\mathfrak{F}$-functions on G defined at g composed with χ is local E-derivation at g. If g is E-$\mathcal{H}^{\chi}$-affine, this local derivation can be extended to a unique tangent vector to G at g [25, Section 8, Chapter 8]. By right translating this tangent vector to all of G, one obtains an element $l\chi(g)$ of the Lie algebra $\mathcal{L}_\Delta(G)$ of invariant E-derivations of G which is called the logarithmic derivative of g relative to χ. Thus, for any local derivation χ and $g \in G$, there exists a unique element $l\chi(g)$ of the Lie algebra $\mathcal{L}_\Delta(G)$ with the property that

$$l\chi(g)(f)(g) = \delta(f(g))$$

for every E-$\mathcal{F}$-function f.

In the remainder of this section, let $\mathcal{F}$ be an (E, Δ)-field, let $\mathcal{C} = \mathcal{F}^\Delta$, and let $\mathcal{U}$ be an (E, Δ)-extension of $\mathcal{F}$ that is (E, Δ)-universal over $\mathcal{F}$. Let G be a connected E-$\mathcal{C}$-group (relative to the E-field $\mathcal{U}^\Delta$). By extending the universal E-field from $\mathcal{U}^\Delta$ to $\mathcal{U}$, considered as an E-field (Section 5.4 or [25, Chapter 2, Section 1, page 44]), G may be considered as E-$\mathcal{C}$-group (relative to the E-field $\mathcal{U}$). For each δ in Δ and any g in $G_\mathcal{U}$, the conditions described in the last paragraph for the existence of the logarithmic derivative are satisfied, and $l\delta(g)$ is in $\mathcal{L}_\mathrm{E}(G)$.

The following lemma is one of the well known properties of the logarithmic derivative [25, Proposition 8, page 236] and will be used a few times.

Lemma 5.235 *Let $x, y \in G_\mathcal{U}$. If $l\delta x = l\delta y$ for all $\delta \in \Delta$, there exist an element $c \in G_{\mathcal{U}^\Delta}$ such that $c = x^{-1}y$.*

Proof: Assume $l\delta x = l\delta y$ for all $\delta \in \Delta$. By [25, Remark after Theorem 3, page 237], for $w, z \in G$, $l\delta(wz) = l\delta(w) + \tau_w^\#(l\delta(z))$ where $\tau_w^\#$ is the isomorphism of the Lie algebra induced by conjugation with w. By letting

$w = x$ and $z = x^{-1}y$, $l\delta(y) = l\delta(x) + \tau_x^{\#}(l\delta(x^{-1}y))$. So $0 = \tau_x^{\#}(l\delta(x^{-1}y))$, and $0 = l\delta(x^{-1}y)$. Then $c = x^{-1}y \in G_{\mathfrak{U}^\Delta}$ [25, Proposition 8(c), page 236]. □

Definition 5.236 *The element* $\alpha \in G_{\mathfrak{U}}$ *is a* G-primitive over $\mathfrak{F}$ *if the logarithmic derivative* $l\delta(\alpha) \in \mathfrak{L}_{\mathrm{E},\mathfrak{F}}(G)$ *for each* $\delta \in \Delta$*. A* G-primitive extension *is an extension of* $\mathfrak{F}$ *of the form* $\mathfrak{F}\langle\alpha\rangle$ *where* α *is a* G*-primitive over* $\mathfrak{F}$*.*

Proposition 5.237 *Let* α *be a* G*-primitive over* $\mathfrak{F}$ *such that the field of* Δ*-constants of* $\mathfrak{F}\langle\alpha\rangle_{\mathrm{E},\Delta}$ *is* $\mathfrak{C}$*. Then* $\mathfrak{F}\langle\alpha\rangle_{\mathrm{E},\Delta}$ *is an* E*-strongly normal extension of* $\mathfrak{F}$ *(relative to* (E,Δ)*-field* $\mathfrak{U}$*), and the map* $c : G(\mathfrak{F}\langle\alpha\rangle_{\mathrm{E},\Delta}/\mathfrak{F}) \mapsto G$ *defined by* $c(\sigma) = \alpha^{-1}\sigma\alpha$ *defines an injective* E-$\mathfrak{C}$*-homomorphism of* E-$\mathfrak{C}$*-groups (relative to the* E*-field* $\mathfrak{U}^\Delta$*).*

Proof: Since α is a G-primitive over $\mathfrak{F}$, $l\delta(\alpha) \in \mathfrak{L}_{\mathrm{E},\mathfrak{F}}(G)$ for each $\delta \in \Delta$. So that, for any (E,Δ)-isomorphism σ of $\mathfrak{F}\langle\alpha\rangle_{\mathrm{E},\Delta}$ over $\mathfrak{F}$, $\sigma(l\delta(\alpha)) = l\delta(\alpha)$ for $\delta \in \Delta$. Also, $l\delta(\sigma\alpha) = \sigma(l\delta(\alpha))$ for all $\delta \in \Delta$ by [25, Proposition 8(b), page 236]. Therefore, $l\delta(\sigma\alpha) = l\delta(\alpha)$, and, by Lemma 5.235, $c(\sigma) = \alpha^{-1}\sigma\alpha$ is an element of $G_{\mathfrak{U}^\Delta}$. Since

$$\sigma(\mathfrak{F}\langle\alpha\rangle_{\mathrm{E},\Delta}) \subset \mathfrak{F}\langle\alpha\rangle_{\mathrm{E},\Delta}\sigma(\mathfrak{F}\langle\alpha\rangle_{\mathrm{E},\Delta}) = \mathfrak{F}\langle\alpha,\sigma\alpha\rangle_{\mathrm{E},\Delta}$$

$$= \mathfrak{F}\langle\alpha, c(\sigma))\rangle_{\mathrm{E},\Delta} = \mathfrak{F}\langle\alpha\rangle_{\mathrm{E},\Delta}\mathfrak{C}\langle c(\sigma)\rangle_{\mathrm{E}},$$

$\mathfrak{F}\langle\alpha\rangle$ is E-strongly normal over $\mathfrak{F}$ by Proposition 3.188. By definition, $\mathfrak{C}\langle\sigma\rangle = (\mathfrak{F}\langle\alpha\rangle_{\mathrm{E},\Delta}\sigma(\mathfrak{F}\langle\alpha\rangle_{\mathrm{E},\Delta}))^\Delta$. Therefore, $\mathfrak{C}\langle\sigma\rangle = \mathfrak{C}\langle c(\sigma)\rangle_{\mathrm{E}}$ by [24, Corollary 2 to Theorem 1, page 88]. For any $\sigma, \tau \in G(\mathfrak{F}\langle\alpha\rangle_{\mathrm{E},\Delta}/\mathfrak{F})$, c is a group homomorphism since $\alpha c(\sigma\tau) = \sigma\tau\alpha = \sigma(\alpha c(\tau)) = \sigma\alpha \circ c(\tau) = \alpha c(\sigma)c(\tau)$. If σ

is in the kernel of c, $\sigma\alpha = \alpha c(\sigma) = \alpha$ and, hence, $\sigma = id_{\mathcal{F}\langle\alpha\rangle}$ because α (E,Δ)-generates $\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}$. Therefore c is injective.

To prove that c is an E-$\mathcal{C}$-homomorphism, it will be shown to be pre E-$\mathcal{C}$-mapping (Definition 2.56). Then, since c is a homomorphism, [25, Corollary 1, page 90] implies that c is an E-$\mathcal{C}$-homomorphism. Parts 1,2 and 3 of the Definition 2.56 follow by taking the domain to consist only of $\mathcal{C}$-generic elements and from the fact that $\mathcal{C}\langle\sigma\rangle = \mathcal{C}\langle c(\sigma)\rangle_{\mathrm{E}}$. To show part 4 of the definition, take $\sigma \leftrightarrow \sigma'$ two $\mathcal{C}$-generic elements. By the definition of $\mathcal{C}$-generic E-specialization in $G(\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}/\mathcal{F})$, there exists an (E,Δ)-$\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}$-isomorphism $\varphi : \mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}\sigma(\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}) \approx \mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}\sigma'(\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta})$ that maps $\sigma\beta$ onto $\sigma'\beta$ for each $\beta \in \mathcal{F}\langle\alpha\rangle$. Therefore, $\varphi(c(\sigma)) = \varphi(\alpha^{-1}\sigma\alpha) = \alpha^{-1}\sigma'\alpha = c(\sigma')$. Thus, the induced E-$\mathcal{C}$-isomorphism $S_{c(\sigma'),c(\sigma)}$ obtained by restricting φ to $\mathcal{C}\langle\sigma\rangle = \mathcal{C}\langle c(\sigma)\rangle_{\mathrm{E}}$, is exactly the induced E-$\mathcal{C}$-isomorphism $S_{\sigma',\sigma}$, and $c(\sigma) \leftrightarrow c(\sigma')$. □

The following Lemma has a pivotal role in the next theorem.

Lemma 5.238 *Let G be a connected* E-$\mathcal{C}$-*group (relative to the* $\mathcal{U}$). *Let η and ξ be elements of $G_{\mathcal{U}}$. Assume η is $\mathcal{C}$-generic and $\mathcal{C}\langle\eta\rangle^{\Delta} = \mathcal{C}$. If $l\delta(\eta) = l\delta(\xi)$ for all $\delta \in \Delta$, then ξ is $\mathcal{C}$-generic, and $\eta \leftrightarrow \xi$ in G.*

Proof: By Lemma 5.235, there exists $\gamma \in G_{\mathcal{U}^{\Delta}}$ such that $\eta\gamma = \xi$. By the Theorem on the linear disjointness of Δ-constants [24, Corollary 1, page 87], $\mathcal{C}\langle\eta\rangle$ and $\mathcal{C}\langle\gamma\rangle$ are linearly disjoint over $\mathcal{C}$. By [25, Theorem 1(d), page 39],

$\eta\gamma$ is a $\mathcal{C}$-generic element of $G_{E,\Delta}$. Since $\eta\gamma = \xi$, ξ is $\mathcal{C}$-generic. Because G is connected, $\eta \leftrightarrow \xi$ in G. $\square$

For the proof the next Theorem, one uses the fact that the elements of G (relative to the E-field $\mathcal{U}^{\Delta}$) are contained in $(G_{\mathrm{E},\Delta})_{\mathcal{U}^{\Delta}}$, as the following discussion indicates. An E-$\mathcal{C}$-group G (relative to the E-field $\mathcal{U}^{\Delta}$) is given. Let $G_{\mathcal{U}}$ (relative to $\mathcal{U}$) be the E-$\mathcal{C}$-group obtained from G (relative to $\mathcal{U}^{\Delta}$) by extending the universal differential field from $\mathcal{U}^{\Delta}$ to $\mathcal{U}$. The elements of G (relative to $\mathcal{U}^{\Delta}$) are the elements $(G_{\mathcal{U}})_{\mathcal{U}^{\Delta}}$ of the E-group $G_{\mathcal{U}}$ (relative to $\mathcal{U}$) rational over $\mathcal{U}^{\Delta}$. Let $G_{\mathrm{E},\Delta}$ (relative to the (E,Δ)-field $\mathcal{U}$) be the (E,Δ)-$\mathcal{C}$-group obtained from the E-$\mathcal{C}$-group $G_{\mathcal{U}}$ (relative to $\mathcal{U}$) by extending the derivations from E to (E,Δ). From the discussion in the preceding section on the (E,Δ)–$\mathcal{C}$-group $G_{\mathrm{E},\Delta}$, the elements of the E-$\mathcal{C}$-group $G_{\mathcal{U}}$ are included in the elements of the (E,Δ)-$\mathcal{C}$-group $G_{\mathrm{E},\Delta}$. Therefore the elements of the E-$\mathcal{C}$-group G (relative to $\mathcal{U}^{\Delta}$) are elements $(G_{\mathrm{E},\Delta})_{\mathcal{U}^{\Delta}}$ of (E,Δ) $\mathcal{C}$-group $G_{\mathrm{E},\Delta}$ (relative to the (E,Δ)-field $\mathcal{U}$).

Theorem 5.239 *Let G be a connected* E-$\mathcal{C}$-*group* (*relative to the* E-*field* $\mathcal{U}^{\Delta}$). *Let η be a $\mathcal{C}$-generic element of $G_{\mathrm{E},\Delta}$. Then, $\mathcal{G} = \mathcal{C}\langle\eta\rangle_{\mathrm{E},\Delta}$ is* E-*strongly normal over $\mathcal{F} = \mathcal{C}\langle l\delta_1\eta\rangle_{\mathrm{E},\Delta}\cdots\mathcal{C}\langle l\delta_m\eta\rangle_{\mathrm{E},\Delta}$ (relative to the* (E,Δ)-*field* $\mathcal{U}$) *with Galois group $G(\mathcal{G}/\mathcal{F})$ (relative to the* E-*field* $\mathcal{U}^{\Delta}$) E-$\mathcal{C}$-*isomorphic to G.*

Proof: Since the E-$\mathcal{C}$-group G (relative to the E-field $\mathcal{U}^{\Delta}$) is connected, the E-$\mathcal{C}$-group G (relative to the E-field $\mathcal{U}$) is connected [25, Section 1, page 44].

This implies that the (E, Δ)-$\mathcal{C}$-group $G_{E,\Delta}$ (relative to the (E, Δ)-field $\mathcal{U}$) is connected [25, Theorem 3, page 58].

By Proposition 5.223, $\mathcal{C}\{\mathcal{C}\langle\eta\rangle_{\mathrm{E}}\}_\Delta$ is Δ-free over $\mathcal{C}\langle\eta\rangle_{\mathrm{E}}$. Because $G_{\mathrm{E},\Delta}$ is connected, $\mathcal{G} = \mathcal{C}\langle\eta\rangle_{\mathrm{E},\Delta}$ is a regular extension of $\mathcal{C}$ by the third axiom for E-groups. The No New Δ''-Constant Lemma 5.230 then implies that the Δ-constants of $\mathcal{G} = \mathcal{C}\langle\eta\rangle_{\mathrm{E},\Delta}$ are in $\mathcal{C}$.

Set $\mathcal{G} = \mathcal{C}\langle\eta\rangle_{\mathrm{E},\Delta}$ and $\mathcal{F} = \mathcal{C}\langle l\delta_1\eta\rangle_{\mathrm{E},\Delta} \cdots \mathcal{C}\langle l\delta_m\eta\rangle_{\mathrm{E},\Delta}$. Since for each $\delta \in \Delta$, $l\delta : G_{\mathrm{E},\Delta} \longrightarrow (\mathcal{L}_{\mathrm{E},\mathcal{F}}(G))_{\mathrm{E},\Delta}$ is a pre (E, Δ)-mapping [25, Corollary, page 243], $\mathcal{C}\langle l\delta\eta\rangle_{\mathrm{E},\Delta} \subseteq \mathcal{C}\langle\eta\rangle_{\mathrm{E},\Delta}$ for each $\delta\Delta$. Therefore, $\mathcal{F} \subset \mathcal{G}$, and $\mathcal{G}^\Delta = \mathcal{F}^\Delta = \mathcal{C}$. By construction, η is a G-primitive over $\mathcal{F}$. By Proposition 5.237, $\mathcal{G}$ is strongly E-normal over $\mathcal{F}$, and the map $c : G(\mathcal{G}/\mathcal{F}) \mapsto G$ defined by $c(\sigma) = \eta^{-1}\sigma\eta$ is an injective E-$\mathcal{C}$-homomorphism.

To show that c is surjective, let β be any element of the connected E-$\mathcal{C}$-group G (relative to the universal E-field $\mathcal{U}^\Delta$). Using the identification of the elements of the E-$\mathcal{C}$-group G (relative to the E-field $\mathcal{U}^\Delta$) with the subset $(G_{\mathrm{E},\Delta})_{\mathcal{U}^\Delta}$ of the elements of the (E, Δ)–$\mathcal{C}$-group $G_{\mathrm{E},\Delta}$ (relative to the (E, Δ)-field $\mathcal{U}$), consider β as an element of $G_{\mathrm{E},\Delta}$. Because $l\delta(\eta\beta) = l\delta(\eta) + \tau_\eta^* l\delta(\beta) = l\delta(\eta)$, Lemma 5.238 implies $\eta \leftrightarrow \eta\beta$. Then, by part 3 in the definition of a pre set, there is an (E, Δ)–isomorphism $S_{(\mathrm{E},\Delta),\eta\beta,\eta} : \mathcal{C}\langle\eta\rangle_{\mathrm{E},\Delta} \approx \mathcal{C}\langle\eta\beta\rangle_{\mathrm{E},\Delta}$ over $\mathcal{C}$. Let $\sigma = S_{(\mathrm{E},\Delta),\eta\beta,\eta}$. By *DAS 2b* in the definition of a pre set, there exist a unique element x of $G_{\mathrm{E},\Delta}$ such that $\eta \leftrightarrow x$, $S_{(\mathrm{E},\Delta),x,\eta} = \sigma$ and $\sigma(\mathcal{C}\langle\eta\rangle_{\mathrm{E},\Delta}) = \mathcal{C}\langle x\rangle_{\mathrm{E},\Delta}$. This element x is the definition of $\sigma\eta$ [25, page 30]. Therefore, $\sigma\eta = \eta\beta$. For all $\delta \in \Delta$, the computation $\sigma l\delta(\eta) = l\delta(\sigma\eta) =$

$l\delta(\eta\beta) = l\delta(\eta) + \tau_\eta^* l\delta(\beta) = l\delta(\eta)$ shows that $\mathcal{F}$ is invariant under σ, and, hence, $\sigma \in G(\mathcal{G}/\mathcal{F})$. Then, c is surjective since $c(\sigma) = \eta^{-1}\sigma\eta = \beta$. Because a bijective E-$\mathcal{C}$-homomorphism of E-$\mathcal{C}$-groups is an E-$\mathcal{C}$-isomorphism [25, Corollary 4, page 97], c is an E-$\mathcal{C}$-isomorphism. □

For given E-group, the procedure in the next corollary constructs an E-strongly normal extension in two stages.

Corollary 5.240 *Assume* $\Delta = \{\delta\}$. *Let* G *be a connected* E-$\mathcal{C}$-*group* (*relative to the* E-*field* $\mathcal{U}^\Delta$). *Let* $G_{\mathrm{E},\Delta}$ *be the* (E,Δ)-$\mathcal{C}$-*group* (*relative to the* (E,Δ)-*field* $\mathcal{U}$) *obtained by first extending the universal* E-*field from* $\mathcal{U}^\Delta$ *to* $\mathcal{U}$ *and then by extending the the derivations from* E *to* (E,Δ). *One can always chose a* $\mathcal{C}$-*generic element* a *of* $\mathcal{L}_{\mathrm{E},\mathcal{C}}(G)_{\mathrm{E},\Delta}$ *and then an element* b *of* $G_{\mathrm{E},\Delta}$ *satisfying the equation* $l\delta(X) = a$. *Then* b *is a* C-*generic element of* $G_{\mathrm{E},\Delta}$, *and* $\mathcal{C}\langle b\rangle_{(\mathrm{E},\Delta)}$ *over* $\mathcal{C}\langle a\rangle_{(\mathrm{E},\Delta)}$ *is* E-*strongly normal* (*relative to the* (E,Δ)-*field* $\mathcal{U}$) *with Galois group* E-$\mathcal{C}$-*isomorphic to* G.

Proof: There exist a $\mathcal{C}$-generic element a of $\mathcal{L}_{\mathrm{E},\mathcal{C}}(G)_{\mathrm{E},\Delta}$ because of the definition of pre (E,Δ)-sets. That b exists follows from the surjectivity of the logarithmic derivative [25, Proposition 11, page 240].

Let η be a generic element of $G_{\mathrm{E},\Delta}$. Set $\mathcal{G} = \mathcal{C}\langle\eta\rangle_{\mathrm{E},\Delta}$ and $\mathcal{F} = \mathcal{C}\langle l\delta\eta\rangle_{\mathrm{E},\Delta}$. By the previous theorem, $\mathcal{G}$ over $\mathcal{F}$ is a E-strongly normal extension with Galois group $G(\mathcal{G}/\mathcal{F})$ which is E-$\mathcal{C}$-isomorphic to G (relative to the universal E-field $\mathcal{U}^\Delta$). The proof of this corollary will be accomplished by showing that $\mathcal{C}\langle b\rangle_{\mathrm{E},\Delta}$ is (E,Δ)-isomorphic to $\mathcal{C}\langle\eta\rangle_{\mathrm{E},\Delta}$ over $\mathcal{C}$.

Because η is a $\mathcal{C}$-generic element of $G_{\mathrm{E},\Delta}$ and the logarithmic derivative $l\delta$ is a surjective (E,Δ)-$\mathcal{C}$-mapping, $l\delta\eta$ is a $\mathcal{C}$-generic element of $\mathcal{L}_{\mathrm{E},\mathcal{C}}(G)_{\mathrm{E},\Delta}$ because, if t is any element of $\mathcal{L}_{\mathrm{E},\mathcal{C}}(G)_{\mathrm{E},\Delta}$ and ξ is an element of $G_{\mathrm{E},\Delta}$ such that $l\delta\xi = t$, then $\eta \longrightarrow \xi$ implies $l\delta\eta \longrightarrow l\delta\xi = t$ since $l\delta$ is pre (E,Δ)-mapping [25, Corollary, page 242]. Because a and $l\delta\eta$ are both $\mathcal{C}$-generic elements of $\mathcal{L}_{\mathrm{E},\mathcal{C}}(G)_{\mathrm{E},\Delta}$, there exists an (E,Δ)–isomorphism φ over $\mathcal{C}$ from $\mathcal{C}\langle a\rangle_{\mathrm{E},\Delta}$ to $\mathcal{C}\langle l\delta\eta\rangle_{\mathrm{E},\Delta}$. By Proposition 1.52 on extensions of (E,Δ)-isomorphisms, φ extends to an (E,Δ)-$\mathcal{C}$-isomorphism, also called φ, from $\mathcal{C}\langle b\rangle_{\mathrm{E},\Delta}$ to $\mathcal{U}$.

Since b is an element of $G_{\mathrm{E},\Delta}$, by *DAS 2b* in the definition of pre sets, there exist a unique x in $G_{\mathrm{E},\Delta}$ with $b \leftrightarrow x$ such that $\mathcal{C}\langle x\rangle_{\mathrm{E},\Delta} = \varphi(\mathcal{C}\langle b\rangle_{\mathrm{E},\Delta})$ and $S_{(\mathrm{E},\Delta),\mathcal{C},b,x} = \varphi$. Since isomorphisms over $\mathcal{C}$ commute with the logarithmic derivative [25, Proposition 8, page 236], $l\delta(x) = l\delta(\varphi b) = \varphi(l\delta(b)) = \varphi a = l\delta(\eta)$. By Lemma 5.238, x is a $\mathcal{C}$-generic element of $G_{\mathrm{E},\Delta}$, and $x \leftrightarrow \eta$. Therefore, $b \leftrightarrow \eta$, and b is a $\mathcal{C}$-generic element of $G_{\mathrm{E},\Delta}$. Because $S_{(\mathrm{E},\Delta),\eta,b} : \mathcal{C}\langle b\rangle_{\mathrm{E},\Delta} \approx \mathcal{C}\langle \eta\rangle_{\mathrm{E},\Delta}$ is an (E,Δ)-$\mathcal{C}$-isomorphism and $S_{(\mathrm{E},\Delta),\eta,b}(\mathcal{C}\langle a\rangle)_{\mathrm{E},\Delta} = \mathcal{C}\langle l\delta(\eta)\rangle_{\mathrm{E},\Delta}$, by Proposition 3.204, $\mathcal{C}\langle b\rangle_{(\mathrm{E},\Delta)}$ over $\mathcal{C}\langle a\rangle_{(\mathrm{E},\Delta)}$ is E-strongly normal with Galois group E-$\mathcal{C}$-isomorphic to G. $\square$

5.6 The E-Strongly Normal Extension Corresponding to the E-Group Induced from an Algebraic Group.

One often studies classical linear ordinary differential equations with coefficients depending on parameters. For each value of the parameter, the differential Galois group might change. However, generically, the Galois group does not change with the parameters. Therefore the monodromy matrices, which of course do depend on the parameters, can be viewed as functions of the parameters with values in the generic linear Galois group. These monodromy matrices are elements of a Galois group, which is a differential group contained in the generic linear Galois group, and satisfy differential relations with respect to the parameters. This relationship is made precise in this chapter.

If A is an Δ-ring which is a subset of an (E,Δ)-ring, A_{E} will denote the (E,Δ)-ring generated by A. If A is an Δ-ring which is a subset of an (E,Δ)-field, $A_{(\mathrm{E})}$ will denote the (E,Δ)-field generated by A. Always $(A^{\Delta})_{(\mathrm{E})} \subset (A_{(\mathrm{E})})^{\Delta}$. Also, please note that, if A and B are two Δ-rings which are subsets of an (E,Δ)-field, $(A[B])_{(\mathrm{E})} = A_{(\mathrm{E})} \cdot B_{(\mathrm{E})}$.

In this section, the following notations will be used. Let $\mathcal{U}$ an (E,Δ)-field that is (E,Δ)-universal over some (E,Δ)-field. Consider E as the union of two disjoint subsets E' and E''. Let $\mathcal{F}'$ be an (E',Δ)-subfield of $\mathcal{U}$ such that $\mathcal{U}$ is universal over $\mathcal{F}_{(\mathrm{E}'')}$ as (E,Δ)-fields. This implies that $\mathcal{U}$ considered as an (E',Δ)-field is also (E',Δ)-universal over $\mathcal{F}'$. Let $\mathcal{G}'$ be an (E',Δ)-subfield of $\mathcal{U}$ which is an E'-strongly normal extension of $\mathcal{F}'$ relative to the universal (E',Δ)-field $\mathcal{U}$. Also, let $\mathcal{G} = (\mathcal{G}')_{(\mathrm{E}'')}$, $\mathcal{F} = (\mathcal{F}')_{(\mathrm{E}'')}$, $\mathcal{C}' = \mathcal{G}'^{\Delta} = \mathcal{F}'^{\Delta}$ and $\mathcal{C} = \mathcal{G}^{\Delta}$. This definition of $\mathcal{C}$ is a change in notation from the usual $\mathcal{C} = \mathcal{F}^{\Delta}$. (See Remark 5.242.)

$$\begin{tikzcd}
\mathcal{G}' \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{G} = \mathcal{G}\mathcal{C} \\
\mathcal{F}' \arrow[r] \arrow[u] & \mathcal{F} \arrow[r] \arrow[u] & \mathcal{F}\mathcal{C} \arrow[u] \\
\mathcal{C}' \arrow[r] \arrow[u] & \mathcal{F}^{\Delta} \arrow[r] \arrow[u] & \mathcal{C} \arrow[u]
\end{tikzcd}$$

All the results in this section relate the Galois groups of the (E', Δ)-fields $\mathcal{G}'$ over $\mathcal{F}'$ to the Galois group of the (E, Δ)-fields $\mathcal{G}$ over $\mathcal{F}\mathcal{C}$ and constitute a straight forward application of basic definitions. In one's first reading of this material, the reader may assume that E' is empty. The theorems are presented in the increased generality, with E' not empty, because no extra work is involved and they might be useful.

Lemma 5.241 *Let* $\mathcal{G}'$ *be an* (E', Δ)*-subfield of* $\mathcal{U}$ *which is an* E'*-strongly normal extension of the* E'*-field* $\mathcal{F}'$ *relative to the* (E', Δ)*-universal* (E', Δ)*-field* $\mathcal{U}$*. Assume* $\mathcal{U}$ *is* (E, Δ)*-universal over* $\mathcal{G} = \mathcal{G}'_{(\mathrm{E}'')}$*. Then any* (E, Δ)*-isomorphism* σ *of* $\mathcal{G} = \mathcal{G}\mathcal{C}$ *into* $\mathcal{U}$ *over* $\mathcal{F}\mathcal{C}$ *is* E*-strong. Furthermore,*
$(\mathcal{G}\sigma\mathcal{G})^{\Delta}$
$= ((\mathcal{G}'\sigma\mathcal{G}')_{(\mathrm{E}'')})^{\Delta} = \mathcal{C}((\mathcal{G}'\sigma\mathcal{G}')^{\Delta})_{(\mathrm{E}'')}$*, and* $\mathcal{C}\langle\sigma\rangle = \mathcal{C} \cdot \mathcal{C}'\langle\sigma\rangle_{(\mathrm{E}'')}$*.*

Remark 5.242 *The field generated by the* E''*-derivatives of* $\mathcal{G}'$ *may contain new* Δ*-constants not in the field generated by the* E''*-derivatives of* $\mathcal{F}'$*. An example of a strongly normal extension of* Δ*-fields* $\mathcal{G}'$ *over* $\mathcal{F}'$ *with this property is any* $\mathcal{G}'$ *generated by a Weierstrassian over a field of* Δ*-constants*

$\mathcal{F}'$. *(See* [24, Examples, page 405] *and Corollary 5.232.) This means that, in the lemma, for* σ E*-strong it must leave fixed a field* $\mathcal{C}$ *of* Δ*-constants that might include* Δ*-constants not in* $\mathcal{C}'$.

Proof: Because σ is an (E, Δ)-isomorphism of $\mathcal{G}$ over $\mathcal{F}\mathcal{C}$ and $\mathcal{C} = \mathcal{G}^{\Delta}$, σ leaves the Δ-constants $\mathcal{C}$ of $\mathcal{G}$ invariant. Since σ restricted to $\mathcal{G}'$ is E$'$-strong, $\sigma\mathcal{G}' \subset \mathcal{G}'\mathcal{U}^{\Delta}$ and $\mathcal{G}' \subset \sigma\mathcal{G}'\mathcal{U}^{\Delta}$. Then,

$$\sigma\mathcal{G} = \sigma(\mathcal{G}'_{(\mathrm{E}'')}) = (\sigma\mathcal{G}')_{(\mathrm{E}'')} \subset (\mathcal{G}'\mathcal{U}^{\Delta})_{(\mathrm{E}'')} = \mathcal{G}'_{(\mathrm{E}'')}(\mathcal{U}^{\Delta})_{(\mathrm{E}'')} = \mathcal{G}\mathcal{U}^{\Delta},$$

and

$$\mathcal{G} = \mathcal{G}'_{(\mathrm{E}'')} \subset (\sigma\mathcal{G}'\mathcal{U}^{\Delta})_{(\mathrm{E}'')} = (\sigma\mathcal{G}')_{(\mathrm{E}'')}(\mathcal{U}^{\Delta})_{(\mathrm{E}'')} = \sigma(\mathcal{G}'_{(\mathrm{E}'')})\mathcal{U}^{\Delta} = \sigma(\mathcal{G})\mathcal{U}^{\Delta}.$$

Therefore, σ is E-strong.

For the first equality,

$$(\mathcal{G}\sigma\mathcal{G})^{\Delta} = (\mathcal{G}'_{(\mathrm{E}'')}\sigma(\mathcal{G}'_{(\mathrm{E}'')}))^{\Delta} = (\mathcal{G}'_{(\mathrm{E}'')}(\sigma\mathcal{G}')_{(\mathrm{E}'')})^{\Delta} = ((\mathcal{G}'\sigma\mathcal{G}')_{(\mathrm{E}'')})^{\Delta}.$$

Since the E$'$-strong normality of σ implies $\mathcal{G}'\sigma\mathcal{G}' = \mathcal{G}'(\mathcal{G}'\sigma\mathcal{G}')^{\Delta}$, above sequence of equalities is equal to

$$((\mathcal{G}'(\mathcal{G}'\sigma\mathcal{G}')^{\Delta})_{(\mathrm{E}'')})^{\Delta} = (\mathcal{G}'_{(\mathrm{E}'')}((\mathcal{G}'\sigma\mathcal{G}')^{\Delta})_{(\mathrm{E}'')})^{\Delta} = (\mathcal{G} \cdot ((\mathcal{G}'\sigma\mathcal{G}')^{\Delta})_{(\mathrm{E}'')})^{\Delta}$$

$$= (\mathcal{G} \cdot \mathcal{C}((\mathcal{G}'\sigma\mathcal{G}')^{\Delta})_{(\mathrm{E}'')})^{\Delta} = \mathcal{C}((\mathcal{G}'\sigma\mathcal{G}')^{\Delta})_{(\mathrm{E}'')},$$

where the last equality follows from [24, Corollary 2, page 88] because $\mathcal{G}$ and the Δ-constants $\mathcal{C}((\mathcal{G}\sigma\mathcal{G})^{\Delta})_{(E'')}$ are linearly disjoint over $\mathcal{C}$. From this, the last equality of the proposition follows from the definitions of $\mathcal{C}\langle\sigma\rangle$ and $\mathcal{C}'\langle\sigma\rangle$ as $(\mathcal{G}\sigma\mathcal{G})^{\Delta}$ and $(\mathcal{G}'\sigma\mathcal{G}')^{\Delta}$. □

Proposition 5.243 *Let* $\mathcal{G}'$ *be an* (E', Δ)*-subfield of* $\mathcal{U}$ *which is an* E'*-strongly normal extension of* $\mathcal{F}'$ *relative to the universal* (E', Δ)*-field* $\mathcal{U}$*. Then* $\mathcal{G}$ *is an* E*-strongly normal extension of* $\mathcal{F}\mathcal{C}$ *relative to the universal* (E, Δ)*-field* $\mathcal{U}$*. The map* ρ *from the* E-$\mathcal{C}$*-group* $G(\mathcal{G}/\mathcal{F}\mathcal{C})$ *to the* E'-$\mathcal{C}$*-group* $G(\mathcal{G}'\mathcal{C}/\mathcal{F}'\mathcal{C})$ *that associates an* (E, Δ)-$\mathcal{F}\mathcal{C}$*-isomorphism of* $\mathcal{G}$ *its restriction to an* (E', Δ)-$\mathcal{F}'\mathcal{C}'$*-isomorphism of* $\mathcal{G}'\mathcal{C}$ *is an injective* $(\mathrm{E}, \mathrm{E}')$-$\mathcal{C}$*-homomorphism* (*Definition 5.233*)*. Furthermore,* $\mathcal{C}\langle\sigma\rangle = \mathcal{C} \cdot \mathcal{C}'\langle\rho(\sigma)\rangle_{(\mathrm{E}'')}$.

Proof: Because $\mathcal{G}'$ over $\mathcal{F}'$ is finitely (E', Δ)-generated, $\mathcal{G}$ over $\mathcal{F}$ and, therefore, $\mathcal{G}$ over $\mathcal{F}\mathcal{C}$ is finitely (E, Δ)-generated. By Lemma 5.241, any (E, Δ)-isomorphism of $\mathcal{G}$ over $\mathcal{F}\mathcal{C}$ is E-strong. Consequently, $\mathcal{G}$ over $\mathcal{F}\mathcal{C}$ is E-strongly normal.

By Theorem 3.202, $G(\mathcal{G}'\mathcal{C}/\mathcal{F}'\mathcal{C})$ is the induced E'-$\mathcal{C}$-group of the E'-$\mathcal{C}$-group $G(\mathcal{G}'/\mathcal{F}')$, both being identified with each other by means of their canonical identifications with the group of (E', Δ)-automorphisms of $\mathcal{G}'\mathcal{U}^{\Delta}$ over $\mathcal{F}'\mathcal{U}^{\Delta}$. The injectivity follows immediately from the definition of ρ. That ρ is a group homomorphism is clear by identifying the E-group $G(\mathcal{G}/\mathcal{F}\mathcal{C})$ with (E, Δ)-automorphisms of $\mathcal{G}\mathcal{U}^{\Delta}$ over $\mathcal{F}\mathcal{C}\mathcal{U}^{\Delta} = \mathcal{F}\mathcal{U}^{\Delta}$ and the E'-group $G(\mathcal{G}'\mathcal{C}/\mathcal{F}'\mathcal{C})$ with (E', Δ)-automorphisms of $\mathcal{G}'\mathcal{U}^{\Delta}$ over $\mathcal{F}'\mathcal{U}^{\Delta}$ and observing that the restriction ρ preserves composition in these groups.

To show ρ is an $(\mathrm{E}, \mathrm{E}')$-$\mathcal{C}$-homomorphism each part of Definition 5.233 will be verified. For $\sigma \in G(\mathcal{G}/\mathcal{F}\mathcal{C})$, $\mathcal{C}\langle\sigma\rangle_{\mathrm{E}} = \mathcal{C} \cdot \mathcal{C}'\langle\rho(\sigma)\rangle_{(\mathrm{E}'')}$ by Lemma 5.241. Since $\mathcal{C}\langle\rho(\sigma)\rangle_{\mathrm{E}'} = \mathcal{C} \cdot \mathcal{C}'\langle\rho(\sigma)\rangle_{\mathrm{E}'}$, it follows that $\mathcal{C}\langle\sigma\rangle_{\mathrm{E}} \supset \mathcal{C}\langle\rho(\sigma)\rangle_{\mathrm{E}'}$. If $\sigma \rightarrow \tau$ for $\sigma, \tau \in G(\mathcal{G}/\mathcal{F}\mathcal{C}$, then, by the definition of specialization, there is an

(E, Δ)-homomorphism $\varphi : \mathcal{G}[\sigma\mathcal{G}] \to \mathcal{G}[\tau\mathcal{G}]$ over $\mathcal{G}$ such that $\varphi(\sigma\alpha) = \tau\alpha$ for all $\alpha \in \mathcal{G}$. Since $\mathcal{G}'\mathcal{C} \subset \mathcal{G}$, the restriction of φ to $\mathcal{G}'\mathcal{C}[\rho(\sigma)(\mathcal{G}'\mathcal{C})]$ is an (E', Δ)-homomorphism $\mathcal{G}\mathcal{H}[\rho(\sigma)(\mathcal{G}\mathcal{H})] \to \mathcal{G}\mathcal{H}[\rho(\sigma)(\mathcal{G}\mathcal{H})]$ over $\mathcal{G}\mathcal{H}$ which takes $\sigma\alpha$ to $\tau\alpha$ for all $\alpha \in \mathcal{G}\mathcal{H}$. Therefore, by definition, $\rho(\sigma) \to \rho(\tau)$. If $\sigma \leftrightarrow \tau$, then the (E, Δ)-homomorphism φ, defined above, is an (E, Δ)-isomorphism and, therefore, extends to an (E, Δ)-isomorphism, also denoted by φ, of the E-field $\mathcal{G}\sigma\mathcal{G}$ to the field $\mathcal{G}\tau\mathcal{G}$. The restriction of this (E, Δ)-isomorphism to the Δ-constants $\mathcal{G}\sigma\mathcal{G} = \mathcal{C}\langle\sigma\rangle_{\mathrm{E}}$ is the induced E-$\mathcal{C}$-isomorphism $S_{\mathrm{E};\tau,\sigma} : \mathcal{C}\langle\sigma\rangle_{\mathrm{E}} \to \mathcal{C}\langle\tau\rangle_{\mathrm{E}}$. The (E, Δ)-isomorphism φ also restricts to an (E', Δ)-$\mathcal{C}$-isomorphism from the (E', Δ)-field $\mathcal{G}'\rho(\sigma)\mathcal{G}'$ to the (E', Δ)-field $\mathcal{G}'\rho(\tau)\mathcal{G}'$, which in turn restricts to the induced E$'$-$\mathcal{C}$-isomorphism $S_{\mathrm{E}';\rho(\tau),\rho(\sigma)} : \mathcal{C}\langle\rho(\sigma)\rangle_{\mathrm{E}'} \to \mathcal{C}\langle\rho(\tau)\rangle_{\mathrm{E}'}$. Since $\mathcal{C}\langle\rho(\sigma)\rangle_{\mathrm{E}'} \subset \mathcal{C}\langle\sigma\rangle_{E}$, $S_{\mathrm{E};\tau,\sigma}$ extends $S_{\mathrm{E}';\rho(\tau),\rho(\sigma)}$. $\square$

This Proposition, in the case E' is empty, can be used to produce examples of E-strongly normal extensions. Start with a Δ-extension $\mathcal{G}'$ over $\mathcal{F}'$ which is strongly normal (in the sense of Kolchin) such that the coefficients of the differential equations defining $\mathcal{G}'$ over $\mathcal{F}'$ depend on parameter t. Assume that the Δ-field $\mathcal{F}$ is closed with respect to differentiation by t. Differentiate the elements of $\mathcal{G}$ with respect to t to generate a $\{d/dt, \Delta\}$-field extension $\mathcal{G}$. Then if $(\mathcal{G})^{\Delta} \subset \mathcal{F}$, $\mathcal{G}$ over $\mathcal{F}$ is $\{d/dt\}$-strongly normal over $\mathcal{F}$.

Corollary 5.244 *In the above proposition, assume* $\mathcal{C} = \mathcal{G}^{\Delta} \subset \mathcal{F}^{\Delta} = \mathcal{C}'$. *Then the injective* $(\mathrm{E}, \mathrm{E}')$-$\mathcal{C}$-*homomorphism* $\rho : G(\mathcal{G}/\mathcal{F}) \to G(\mathcal{G}'/\mathcal{F}')$ *identifies the* E-$\mathcal{C}$-*group* $G(\mathcal{G}/\mathcal{F})$ *with an* E-$\mathcal{C}$-*subgroup of the* E-$\mathcal{C}$-*group* $G(\mathcal{G}'/\mathcal{F}')_{\mathrm{E}}$

induced from the E'-$\mathcal{C}$*-group* $G(\mathcal{G}'/\mathcal{F}')$ *by extending the derivations to* E *(Definition 5.234).*

Proof: Kolchin proved that the induced E-$\mathcal{C}$-group $G(\mathcal{G}'/\mathcal{F}')_{\mathrm{E}}$ always exists [25, Theorem 3, page 58]. By Definition 5.234 of the induced E-group, the $(\mathrm{E}, \mathrm{E}')$-$\mathcal{C}$-homomorphism ρ of the last proposition extends to a unique E-$\mathcal{C}$-homomorphism $\overline{\rho} : G(\mathcal{G}/\mathcal{F}) \longrightarrow G(\mathcal{G}'/\mathcal{F}')_{\mathrm{E}}$. It is also injective because ρ and $\overline{\rho}$ are equal on the elements of $G(\mathcal{G}/\mathcal{F})$. The image of an E-$\mathcal{C}$-group under an E-$\mathcal{C}$ homomorphism is a E-$\mathcal{C}$-subgroup [25, Proposition 4, page 92]. Because ρ is a bijective E-$\mathcal{C}$-homomorphism of $G(\mathcal{G}/\mathcal{F})$ to its image, the E-$\mathcal{C}$-group $G(\mathcal{G}/\mathcal{F})$ and its image in $G(\mathcal{G}'/\mathcal{F}')_{\mathrm{E}}$ are E-$\mathcal{C}$-isomorphic [25, Corollary 4, page 97]. $\square$

6 Examples

In Kolchin's *Differential Algebraic Groups* [25] several results necessary for the presentation of examples of the Galois are not carefully explained. An especially critical one is that $\mathcal{V}^n$ with coordinate addition is an E-group. Additionally, it is not shown that the Zariski closed subsets of the E-set $\mathcal{V}^n$ (Section 4.1) are the closed subsets of the E-Zariski topology on $\mathcal{V}^n$ (Section 1.4). Without this result, [25, Proposition 12, page 24] cannot be applied to classify the E-subgroups of the E-group $\mathcal{V}^n$. This classification is essential for the classification of all E-subgroups of an a given E-group. These discrepancies are rectified in the first section.

In the remaining sections, examples of the Galois theory are presented. The classification of the E-$\mathfrak{F}$-subgroups of $\mathcal{V}^n$ (6.254) is important because it is used to determine the E-$\mathfrak{F}$-subgroups of an arbitrary E-$\mathfrak{F}$-group, and this necessary in order to exhibit examples of the bijective correspondence of the Fundamental Theorem of Galois Theory.

The first aspect of the examples entails the presentation of an E-strongly normal extension $\mathcal{G}$ over $\mathcal{F}$ and exhibits its Galois group $G(\mathcal{G}/\mathcal{F})$. This is possible even if $\mathcal{C} = \mathcal{F}^\Delta$ is not constrainedly closed due to Theorem 3.197. The second and more difficult aspect of the examples exhibits the bijective correspondence between the E-$\mathcal{C}$-subgroups of G and the (E, Δ)-subfields of $\mathcal{G}$ containing $\mathcal{F}$. If $\mathcal{C}$ is constrainedly closed, Theorem 4.207 asserts that this correspondence exists.

6.1 Subgroups of G_a^{E}

In this section, $\mathfrak{F}$ will denote an E-field over which $\mathcal{V}$ is E-universal.

Two topologies on the set $\mathcal{V}^n$ will be considered. The first is the E-$\mathfrak{F}$-Zariski topology on $\mathcal{V}^n$ defined in Section 1.4, for which the closed sets are the E-zeros of E-ideals $\mathfrak{A} \subseteq \mathfrak{F}\{y_1, \dots, y_2\}_{\mathrm{E}}$. The second is E-$\mathfrak{F}$-Zariski topology on the E-$\mathfrak{F}$-set $\mathcal{V}^n$ defined in Section 4.1, for which the closed sets are the E-$\mathfrak{F}$-subsets of $\mathcal{V}^n$. The two topologies will be shown to be identical.

An outline of this section is as follows. First it will be observed that every $\mathfrak{F}$-closed subset of E-Zariski topology on $\mathcal{V}^n$ is a pre E-$\mathfrak{F}$-set. Then it will be shown that $\mathcal{V}^n$ with coordinate addition is an E-$\mathfrak{F}$-group. Since $\mathcal{V}^n$ is a homogeneous space over itself, every pre E-$\mathfrak{F}$-subset of $\mathcal{V}^n$ is an E-$\mathfrak{F}$-set by definition. The E-$\mathfrak{F}$-subsets of the E-$\mathfrak{F}$-set $\mathcal{V}^n$ are then proved to coincide with the $\mathfrak{F}$-closed subsets of the E-Zariski topology on $\mathcal{V}^n$. This makes it possible to demonstrate that the E-$\mathfrak{F}$-subgroups of $\mathcal{V}^n$ are defined by linear E-ideals.

Observation 6.245 *The set of* E*-zeros* $\mathfrak{Z}_{\mathfrak{F}}(\mathfrak{A}) \subset \mathcal{V}^n$ *of an* E*-ideal* $\mathfrak{A} \subset \mathfrak{F}\{y_1, \dots, y_n\}_{\mathrm{E}}$ *is a pre* E-$\mathfrak{F}$*-set (Definition 2.54).*

Proof: Define a pre E-$\mathfrak{F}$-structure on the elements of $\mathfrak{Z}_{\mathfrak{F}}(\mathfrak{A})$ by

1. for each element $x = (x_1, \dots, x_n) \in \mathfrak{Z}_{\mathfrak{F}}(\mathfrak{A})$ let $\mathfrak{F}\langle x \rangle$ equal the finitely E-generated extension $\mathfrak{F}\langle x \rangle_{\mathrm{E}}$ of $\mathfrak{F}$,

2. a pre order on $\mathfrak{Z}_{\mathfrak{F}}(\mathfrak{A})$ defined in Example 1.4, i.e. for $x, x' \in \mathfrak{Z}_{\mathfrak{F}}(\mathfrak{A})$, $x \underset{\mathfrak{F}}{\rightarrow} x'$ if there exists an E-$\mathfrak{F}$-homomorphism of $\varphi : \mathfrak{F}\{x_1, \ldots, x_n\}_{\mathrm{E}} \rightarrow \mathfrak{F}\{x'_1, \ldots, x'_n\}_{\mathrm{E}}$ such that $\varphi(x) = x'$, and
3. for each pair (x, x'), with $x, x' \in \mathfrak{Z}_{\mathfrak{F}}(\mathfrak{A})$, such that $x \underset{\mathfrak{F}}{\rightarrow} x'$ and $x' \underset{\mathfrak{F}}{\rightarrow} x$ define the E-$\mathfrak{F}$-isomorphism $S_{x,x'} : \mathfrak{F}\langle x \rangle \approx \mathfrak{F}\langle x' \rangle$ to be the extension of $\varphi : \mathfrak{F}\{x_1, \ldots, x_n\}_{\mathrm{E}} \rightarrow \mathfrak{F}\{x'_1, \ldots, x'_n\}_{\mathrm{E}}$ to $\mathfrak{F}\langle x \rangle$.

Since $\mathfrak{Z}_{\mathfrak{F}}(\mathfrak{A})$ is an NG-space (Proposition 1.31) with the pre order of Structure 2 (Observation 1.26), Proposition 1.14 implies that there exist a finite number of generic points, which is Axiom *DAS 1*. Axiom *DAS 2a* is clear. For Axiom *DAS 2b*, if $x \in \mathfrak{Z}_{\mathfrak{F}}(\mathfrak{A})$ and $S : \mathfrak{F}\langle x \rangle \rightarrow \mathfrak{F}'$ is an E-$\mathfrak{F}$-isomorphism, let $x' = S(x)$. Then, from the definition of specialization in Structure 2, $x \leftrightarrow x'$ and $S_{x',x} = S$. □

Let A and B be two pre E-$\mathfrak{F}$-sets relative to $\mathcal{V}$. A pre E-$\mathfrak{F}$-mapping φ of A into B is a *complete pre* E-$\mathfrak{F}$-*equivalence of A with B* [25, page 31] if φ is everywhere defined, bijective, and its inverse is an everywhere defined pre E-$\mathfrak{F}$-mapping of B into A. The following lemma shows that for any E-ideal $\mathfrak{A} \subset \mathfrak{F}\{y_1, \ldots, y_n\}_{\mathrm{E}}$ there exists a complete pre equivalence between the pre E-$\mathfrak{F}$-set $\mathfrak{Z}_{\mathfrak{F}}(\mathfrak{A}) \subset \mathcal{V}^n$ of the previous observation and the pre E-$\mathfrak{F}$-set $(\operatorname{diffspec} \mathfrak{F}\{y_1, \ldots, y_n\}_{\mathrm{E}}/\mathfrak{A})(\mathcal{V})$ (Theorem 2.126).

Proposition 6.246 *Let R be an* E-$\mathcal{V}$-*algebra that is* E-$\mathfrak{F}$-*generated by $\eta = (\eta_1, \ldots, \eta_n)$, and let $\mathfrak{A} \subset \mathfrak{F}\{y_1, \ldots, y_n\}_{\mathrm{E}}$ be the kernel of the canonical* E-$\mathfrak{F}$-*homomorphism $\mathfrak{F}\{y_1, \ldots, y_n\}_{\mathrm{E}} \rightarrow R$ that maps each y_i to η_i. Put $X =$*

diffspec R, *and put* $V = \operatorname{diffspec} \mathcal{V}$. *Let* $(\xi, \xi^{\#}) \in X(\mathcal{V})$, *and let* $\varphi((\xi, \xi^{\#}))$ *be the composition*

$$R \xrightarrow{\mathcal{I}_R} \mathcal{O}_X(X) \xrightarrow{\xi^{\#}(X)} \mathcal{O}_V(V) = \mathcal{V}.$$

The function $\psi : X(\mathcal{V}) \to \mathfrak{Z}_{\mathcal{F}}(\mathfrak{A}) \subset \mathcal{V}^n$ *defined by* $\psi((f, f^{\#})) = \varphi((f, f^{\#}))(\eta)$ *is a complete pre* E-$\mathcal{F}$*-equivalence of pre* E-$\mathcal{F}$*-sets such that* $\mathcal{F}\langle(\xi, \xi^{\#})\rangle = \mathcal{F}\langle\psi(\xi, \xi^{\#})\rangle$ *and, if* $(\xi, \xi^{\#}) \leftrightarrow (\xi, \xi^{\#})$, $S_{(\xi', \xi'^{\#}),(\xi, \xi^{\#})} = S_{\psi((\xi', \xi'^{\#})), \psi((\xi, \xi^{\#}))}$.

Proof: Let $(Z, \mathcal{O}_Z)$ be an E-$\mathcal{F}$-scheme, and let $(f, f^{\#}) : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a morphism of E-$\mathcal{F}$-schemes. Let $\varphi((f, f^{\#}))$ be the composition

$$R \xrightarrow{\mathcal{I}_R} \mathcal{O}_X(X) \xrightarrow{\xi^{\#}(X)} \mathcal{O}_Z(Z).$$

Then the map

$$\varphi : \operatorname{Mor}((Z, \mathcal{O}_Z), (X, \mathcal{O}_X)) \longrightarrow \operatorname{Hom}^{\mathrm{E}}_{\mathcal{F}}(R, \mathcal{O}_Z(Z))$$

is a bijection by Proposition 2.68. When $Z = V = \operatorname{diffspec} \mathcal{V}$, it follows from Proposition 2.74 that $\mathcal{O}_Z(Z) = \mathcal{O}_V(V) = \mathcal{V}$ and that the function

$$\varphi : X(\mathcal{V}) = \operatorname{Mor}((V, \mathcal{O}_V), (X, \mathcal{O}_X)) \longrightarrow \operatorname{Hom}^{\mathrm{E}}_{\mathcal{F}}(R, \mathcal{V})$$

is a bijection. Since $\mathfrak{Z}_{\mathcal{F}}(\mathfrak{A})$ is the set of E-zeros of $\mathfrak{A}$, the map

$$ev_\eta : \operatorname{Hom}^{\mathrm{E}}_{\mathcal{F}}(R, \mathcal{V}) \to \mathfrak{Z}_{\mathcal{F}}(\mathfrak{A})$$

defined by $ev_\eta(\rho) = \rho(\eta)$ for $\rho \in \operatorname{Hom}^{\mathrm{E}}_{\mathcal{F}}(R, \mathcal{V})$ is also a bijection. As a consequence of this, the composition

$$\psi = ev_\eta \circ \varphi : X(\mathcal{V}) \to \mathfrak{Z}_{\mathcal{F}}(\mathfrak{A})$$

is also a bijection.

That ψ is an everywhere defined pre E-$\mathcal{F}$-map will be verified by checking each of the conditions of the definition (Definition 2.56).

1. $X(\mathcal{V}) = X(\mathcal{V})_\psi$ because ψ is defined at each element of $X(\mathcal{V})$.

2. For all $(\xi, \xi^\#) \in X(\mathcal{V})$, since $\varphi((\xi, \xi^\#))$ is defined as the composition

$$R \xrightarrow{\mathcal{I}_R} \mathcal{O}_X(X) \xrightarrow{\xi^\#(X)} \mathcal{V},$$

   and $\varphi((\xi, \xi^\#))(\xi(0)) = 0$, the stalk $\mathcal{O}_{\xi(0)}$ may be inserted as follows:

$$R \xrightarrow{\mathcal{I}_R} \mathcal{O}_X(X) \xrightarrow{\partial_{\xi(0), X}} \mathcal{O}_{\xi(0)} \xrightarrow{\xi^\#_{\xi(0)}} \mathcal{V},$$

   where the definition of the last two homomorphisms appears before Lemma 2.71. Because $\mathcal{O}_{\xi(0)}$ is E-$\mathcal{F}$-isomorphic to the local ring $R_{\xi(0)}$ (Lemma 2.70), $\xi^\#_{\xi(0)}(\mathcal{O}_{\xi(0)}) = \mathcal{F}\langle \varphi((\xi, \xi^\#))(\eta)\rangle_{\mathrm{E}}$. Because

$$\mathcal{F}\langle(\xi, \xi^\#)\rangle = \xi^\#_{\xi(0)}(\mathcal{O}_{\xi(0)}) = \mathcal{F}\langle \varphi((\xi, \xi^\#))(\eta)\rangle_{\mathrm{E}} = \mathcal{F}\langle \psi((\xi, \xi^\#))\rangle,$$

   $\mathcal{F}\langle(\xi, \xi^\#)\rangle \supseteq \mathcal{F}\langle \psi((\xi, \xi^\#))\rangle$, for all $(\xi, \xi^\#) \in X(\mathcal{V})$.

3. If $(\xi, \xi^\#) \to (\xi', \xi'^\#)$, then $\psi((\xi, \xi^\#)) \to \psi((\xi', \xi'^\#))$ for $(\xi, \xi^\#), (\xi', \xi'^\#) \in X(\mathcal{V})$, because the following are equivalent.

   (a) $(\xi, \xi^\#) \to (\xi', \xi'^\#)$.

   (b) $\xi(0) \subset \xi'(0) \subset R$ by Proposition 2.94.

(c) $\psi((\xi,\xi^{\#})) \to \psi((\xi',\xi'^{\#}))$. Since the kernels of $\varphi((\xi,\xi^{\#}))$ and $\varphi((\xi',\xi'^{\#}))$ are $\xi(0)$ and $\xi'(0)$, respectively (Lemma 2.71), (b) is equivalent to the existence of the commutative diagram below

$$\begin{tikzcd}
\eta \in R \arrow[r, "\varphi((\xi,\xi^{\#}))"'] \arrow[dr, "\varphi((\xi',\xi'^{\#}))"'] & \mathfrak{F}\{\varphi((\xi,\xi^{\#}))(\eta)\}_{\mathrm{E}} \arrow[d, "\rho"] \\
 & \mathfrak{F}\{\varphi((\xi',\xi'^{\#}))(\eta)\}_{\mathrm{E}},
\end{tikzcd}$$

where ρ is an E-$\mathfrak{F}$-homomorphism such that $\rho(\varphi((\xi,\xi^{\#}))(\eta)) = \varphi((\xi',\xi'^{\#}))(\eta)$.

4. For $(\xi,\xi^{\#}),(\xi',\xi'^{\#}) \in X(\mathcal{V})$, if $(\xi,\xi^{\#}) \leftrightarrow (\xi,\xi^{\#})$, then $\xi(0) = \xi'(0)$ by Corollary 2.95 and the following diagram is commutative

$$\begin{tikzcd}
R \arrow[r, "\mathfrak{I}_R"] & \mathcal{O}_X(X) \arrow[r, "\mathfrak{J}_{\xi(0),X}"] & \mathcal{O}_{\xi(0)} \arrow[r, "\xi^{\#}_{\xi(0)}"] \arrow[dr, "\xi^{\#}_{\xi(0)}"'] & \mathfrak{F}\langle(\xi,\xi^{\#})\rangle \arrow[d, "S_{(\xi',\xi'^{\#}),(\xi,\xi^{\#})}"] \\
 & & & \mathfrak{F}\langle(\xi',\xi'^{\#})\rangle.
\end{tikzcd}$$

by the definition of $S_{(\xi',\xi'^{\#}),(\xi,\xi^{\#})}$. Therefore,

$$S_{(\xi',\xi'^{\#}),(\xi,\xi^{\#})}(\varphi((\xi,\xi^{\#}))(\eta)) = \varphi((\xi',\xi'^{\#}))(\eta).$$

Since the domains of $S_{(\xi',\xi'^{\#}),(\xi,\xi^{\#})}$ and $S_{\psi((\xi',\xi'^{\#}),\psi(\xi,\xi^{\#}))}$ are the same (Property 2), $S_{(\xi',\xi'^{\#}),(\xi,\xi^{\#})} = S_{\psi((\xi',\xi'^{\#}),\psi(\xi,\xi^{\#}))}$, and $S_{(\xi',\xi'^{\#}),(\xi,\xi^{\#})}$ extends $S_{\psi((\xi',\xi'^{\#}),\psi(\xi,\xi^{\#}))}$.

Since ψ is bijective, its inverse function exist and is clearly an everywhere defined pre E-$\mathfrak{F}$-mapping. Therefore, ψ is a complete pre E-$\mathfrak{F}$-equivalence.□

Let $(Z, \mathcal{O}_Z)$ be an affine E-$\mathfrak{F}$-group scheme (Definition 3.154). By Theorem 3.177, $Z(\mathcal{V})$ is an E-$\mathfrak{F}$-group. Since Z is affine, Proposition 6.246 implies that $\psi(Z(\mathcal{V})) \subseteq \mathcal{V}^n$ is an pre E-$\mathfrak{F}$-set. Endow the set $\psi(Z(\mathcal{V}))$ with the group structure which renders ψ an isomorphism of groups. The next proposition shows that with this group structure $\psi(Z(\mathcal{V}))$ is an E-$\mathfrak{F}$-group.

Proposition 6.247 *Let* $(Z, \mathcal{O}_Z)$ *be an affine* E-$\mathfrak{F}$*-group scheme with* $Z =$ diffspec R*. Then,* $\psi(Z(\mathcal{V}))$ *is an* E-$\mathfrak{F}$*-group, and* ψ *is an* E-$\mathfrak{F}$*-isomorphism of* E-$\mathfrak{F}$*-groups.*

Proof: Since $\mathfrak{F}\langle(\xi, \xi^\#)\rangle = \mathfrak{F}\langle\psi(\xi, \xi^\#)\rangle$ and, if $(\xi, \xi^\#) \leftrightarrow (\xi, \xi^\#)$, $S_{(\xi', \xi'^\#),(\xi,\xi^\#)}$ $= S_{\psi((\xi',\xi'^\#)),\psi((\xi,\xi^\#))}$ (Proposition 6.246), ψ merely renames the elements of $Z(\mathcal{V})$ without altering the structure elements of the E-$\mathfrak{F}$-group $Z(\mathcal{V})$. Therefore, the structure elements of $\psi(Z(\mathcal{V}))$ satisfy the axioms of an E-$\mathfrak{F}$-group because they satisfy the same axioms for $Z(\mathcal{V})$ (Theorem 3.177). Also, ψ is E-$\mathfrak{F}$-isomorphism because an everywhere defined pre E-$\mathfrak{F}$-map that is a homomorphism is an E-$\mathfrak{F}$-homomorphism [25, Proposition 3,89] and because a bijective E-$\mathfrak{F}$-homomorphism is an E-$\mathfrak{F}$-isomorphism [25, Corollary 4, page 97]. ☐

Corollary 6.248 *The pre* E-$\mathbb{Q}$*-set* $\mathcal{V}^n$ *with coordinate addition is an* E-$\mathbb{Q}$*-group and* ψ *is* E-$\mathbb{Q}$*-isomorphism from* $(G_a^{\mathrm{E}})^n$ *onto* $\mathcal{V}^n$*.*

Proof: Recall from Section 3.2 that the E-$\mathbb{Q}$-group scheme $\mathbf{G}_a^{\mathrm{E}}$ is defined to be $(Z, \mathcal{O}_Z)$, where $S = \mathbb{Q}\{y\}_{\mathrm{E}}$ with E-$\mathbb{Q}$-indeteminate y, $Z =$ diffspec S

and mult, inv and id are the E-$\mathbb{Q}$-scheme morphisms induced via Proposition 2.65 from $S \longrightarrow S \otimes S$ such that $y \longrightarrow y \otimes 1 + 1 \otimes y$, from $S \longrightarrow S$ such that $y \longrightarrow -y$, and from $S \longrightarrow \mathbb{Q}$ such that $y \longrightarrow 0$. Let $(Y, \mathcal{O}_Y)$ be the product E-group scheme $(\mathbf{G}_a^{\mathrm{E}})^n$. Then $Y = \operatorname{diffspec} R$ where $R = \mathbb{Q}\{y_1, \dots, y_n\}_{\mathrm{E}}$ and $y_1, \dots, y_n$ are E-$\mathbb{Q}$-indeteminates. The E-$\mathbb{Q}$-scheme morphisms mult, inv and id are induced via Proposition 2.65 from $\varphi : R \longrightarrow R \otimes R$ such that $\varphi(y_i) = y_i \otimes 1 + 1 \otimes y_i$, from $\iota : R \longrightarrow R$ such that $\iota : (y_i) = -y_i$, and from $\varsigma : R \longrightarrow \mathbb{Q}$ such that $\varsigma(y_i) = 0$.

Because the functor of $\mathcal{V}$-valued points preserves limits ([27, Theorem 1, page 112]),

$$\begin{aligned}(G_a^{\mathrm{E}})^n &= (\operatorname{Hom}^{\mathrm{E}}((\operatorname{diffspec} \mathcal{V}, \mathcal{O}_{\operatorname{diffspec} \mathcal{V}}), (Z, \mathcal{O}_Z))^n \\ &= \operatorname{Hom}^{\mathrm{E}}((\operatorname{diffspec} \mathcal{V}, \mathcal{O}_{\operatorname{diffspec} \mathcal{V}}), (Z, \mathcal{O}_Z)^n) \\ &= \operatorname{Hom}^{\mathrm{E}}((\operatorname{diffspec} \mathcal{V}, \mathcal{O}_{\operatorname{diffspec} \mathcal{V}}), (Y, \mathcal{O}_Y)).\end{aligned}$$

For $({}^a\rho, \rho^{\#}), ({}^a\tau, \tau^{\#}) \in (G_a^{\mathrm{E}})^n$ induced by E-$\mathbb{Q}$-homomorphisms $\rho, \tau : R \rightarrow \mathcal{V}$, the sum $({}^a\rho, \rho^{\#}) + ({}^a\tau, \tau^{\#})$ is induced by the composition

$$R \xrightarrow{\varphi} R \otimes R \xrightarrow{\rho \otimes \tau} \mathcal{V} \otimes \mathcal{V} \xrightarrow{m} \mathcal{V},$$

where m is multiplication in the field $\mathcal{V}$. Therefore, $(\rho + \tau)(y_i) = \rho(y_i) + \tau(y_i)$, $\psi(({}^a\rho, \rho^{\#}) + ({}^a\tau, \tau^{\#})) = \psi(({}^a\rho, \rho^{\#})) + \psi(({}^a\tau, \tau^{\#}))$ and the group law on $\mathcal{V}^n$ is coordinate addition. $\square$

Corollary 6.249 *The pre* E-$\mathbb{Q}$-*set* $\mathcal{V}^* = \mathcal{V} \backslash \{0\}$ *with the group operation of multiplication is an* E-$\mathbb{Q}$-*group, and* ψ *is* E-$\mathbb{Q}$-*isomorphism from* G_m^{E} *onto* $\mathcal{V}^*$.

Proof: The proof is similar to that of the previous corollary. $\square$

Let $\mathfrak{F}$ be an E-$\mathbb{Q}$-field over which $\mathcal{V}$ is E-universal. To simplify notation, $\mathcal{V}^n$ will sometimes denote the E-$\mathfrak{F}$-group induced from the E-$\mathbb{Q}$-group by extending the basic differential field from $\mathbb{Q}$ to $\mathfrak{F}$ (Definition 3.201 or [25, Theorem 2, page 49]). Similarly, $\mathcal{V}^*$ will sometimes denote an E-$\mathfrak{F}$-group. The meaning will be clear from the context.

Recall from Section 4.1 or [25, page 37] that an E-$\mathfrak{F}$-set is defined to be a pre E-$\mathfrak{F}$-subset of a homogeneous E-$\mathfrak{F}$-space for an E-$\mathfrak{F}$-group. Since $\mathcal{V}^n$ is an E-$\mathfrak{F}$-group (Corollary 6.248), it is a homogenous E-$\mathfrak{F}$-space for itself. Therefore, the pre E-$\mathfrak{F}$-set $\mathfrak{Z}_{\mathfrak{F}}(\mathfrak{A}) \subseteq \mathcal{V}^n$(Observation 6.245) is an E-$\mathfrak{F}$-set. Conversely, it will be shown (Proposition 6.251) that any E-$\mathfrak{F}$-set contained in $\mathcal{V}^n$ is closed set of the E-$\mathfrak{F}$-Zariski topology on $\mathcal{V}^n$ of the form $\mathfrak{Z}_{\mathfrak{F}}(\mathfrak{A})$ for an E-ideal $\mathfrak{A} \subseteq \mathfrak{F}\{y_1, \ldots, y_n\}_{\mathrm{E}}$. To prepare for the proof of this result, E-$\mathfrak{F}$-mappings and E-$\mathfrak{F}$-functions will be reviewed in the next few paragraphs.

Let W be an irreducible E-$\mathfrak{F}$-set, and let $v_W \in W$ be an $\mathfrak{F}$-generic element. Denote by $\mathfrak{F}_a$ the algebraic closure of $\mathfrak{F}$ in $\mathcal{V}$ and by $\underset{\mathfrak{F}_a}{\longrightarrow}$ the E-specialization over $\mathfrak{F}_a$ in the induced E-$\mathfrak{F}_a$-subset of W [25, Theorem 2, page 49]. Let A and B be E-sets (Section 4.1). An E-$\mathfrak{F}$-*mapping* [25, page 121] is a pre E-$\mathfrak{F}$-mapping f of A into B (Definition 2.56) such that, for each element $v_0 \in A_f$ (the domain of f) and for each $\mathfrak{F}_a$-generic element v_W of an $\mathfrak{F}_a$-component W of A containing v_0, $f(v_W)$ is holomorphic at $v_W \underset{\mathfrak{F}_a}{\longrightarrow} v_0$ and its value is $f(v_0)$ (See [25, page 114] for the definitions of 'holomorphic' and 'value'). Denote by $\mathfrak{M}_{\mathfrak{F}}(A, B)$ or $\mathfrak{M}_{\mathrm{E},\mathfrak{F}}(A, B)$ the set of E-$\mathfrak{F}$-mappings of A into B and by

$\mathfrak{M}_{\mathcal{F},v}(A,B)$ or $\mathfrak{M}_{\mathrm{E}.\mathcal{F},v}(A,B)$ the set of E-$\mathcal{F}$-mappings of A into B that are defined at a given element $v \in A$. If $\mathcal{F} = \mathcal{V}$, put $\mathfrak{M}(A,B) = \mathfrak{M}_{\mathcal{F}}(A,B)$ and $\mathfrak{M}_v(A,B) = \mathfrak{M}_{\mathcal{F},v}(A,B)$.

If B is the E-$\mathcal{F}$-group $\mathcal{V}$, an E-$\mathcal{F}$-mapping of A into B is called an *E-$\mathcal{F}$-function.* Put $\mathfrak{F}_{\mathcal{F}}(A) = \mathfrak{M}_{\mathcal{F}}(A,\mathcal{V})$ and $\mathfrak{F}_{\mathcal{F},v}(A) = \mathfrak{M}_{\mathcal{F},v}(A,\mathcal{V})$ for $v \in A$. If $\mathcal{F} = \mathcal{V}$, put $\mathfrak{F}(A) = \mathfrak{F}_{\mathcal{F}}(A)$ and $\mathfrak{F}_v(A) = \mathfrak{F}_{\mathcal{F},v}(A)$. All these sets of E-$\mathcal{F}$-functions are E-$\mathcal{F}$-algebras [25, page 136].

The set of E-$\mathcal{F}$-functions $\mathfrak{F}_{\mathcal{F},v}(A)$ defined at $v \in V$ is a local E-$\mathcal{F}$-ring with maximal E-ideal $\mathfrak{m}_{\mathcal{F},v}(V)$ consisting of the elements $\varphi \in \mathfrak{F}_{\mathcal{F},v}(A)$ such that $\varphi(v) = 0$ [25, page 195]. The E-$\mathcal{F}$-homomorphism $\mathrm{ev}_{\mathcal{F},v(V)} : \mathfrak{F}_{\mathcal{F},v} \longrightarrow \mathcal{V}$ defined by $\mathrm{ev}_{\mathcal{F},v}(V)(\varphi) = \varphi(v)$ for $\varphi \in \mathfrak{F}_{\mathcal{F},v}(A)$ is called the *evaluation* E-$\mathcal{F}$-*homomorphism on V at v* [25, page 195].

For any E-subfield $\mathcal{F} \subseteq \mathcal{V}$, the projections $\mathrm{pr}_j : (v_1,\ldots,v_n) \to v_i$ defined by $\mathrm{pr}_j((v_1,\ldots,v_n) = v_i$ for $(v_1,\ldots,v_n) \in \mathcal{V}^n$ are examples of E-$\mathcal{F}$-functions [25, Proposition 6(f), page 129]. Moreover, when W is an irreducible E-$\mathcal{F}$-subset of $\mathcal{V}^n$, the inclusion $\mathrm{in}_{\mathcal{V}^n,W} : W \longrightarrow \mathcal{V}^n$ is an E-$\mathcal{F}$-mapping [25, Proposition 6(e), page 129]. Therefore, the composition $\kappa_j = \kappa_{W,j} = \mathrm{pr}_j \circ \mathrm{in}_{\mathcal{V}^n,W}$ is also an E-$\mathcal{F}$-function on W, i.e. $\kappa_j \in \mathfrak{F}_{\mathcal{F}}(W)$. Also, $\mathfrak{F}_{\mathcal{F}}(W) = \mathcal{F}\langle\kappa_1,\ldots,\kappa_n\rangle_{\mathrm{E}}$ [25, page 137]. The E-$\mathcal{F}$-functions $\kappa_1,\ldots,\kappa_n$ are called the *canonical coordinate functions on W*. Since $\kappa_i(w) = w_i$ for $w = (w_1,\ldots,w_n) \in W \subseteq \mathcal{V}^n$, the κ_i are everywhere defined on $\mathcal{V}^n$, i.e. $\kappa_i \in \mathfrak{F}_{\mathcal{F},w}(W)$ for $i = 1,\ldots,n$ and all $w \in W$.

Lemma 6.250 *The canonical coordinate* E-$\mathcal{F}$-*functions* $\kappa_1,\ldots,\kappa_n \in \mathfrak{F}(\mathcal{V}^n)$

are E-$\mathfrak{F}$-*independent, and the* E-$\mathfrak{F}$-*ring of everywhere defined* E-$\mathfrak{F}$-*functions on* $\mathcal{V}^n$ *equals* $\mathfrak{F}\{\kappa_1, \ldots, \kappa_n\}_{\mathrm{E}}$.

Proof: Let $\eta = (\eta_1, \ldots, \eta_n) \in \mathcal{V}^n$ be an $\mathfrak{F}$-generic element of the E-$\mathfrak{F}$-group $\mathcal{V}^n$. By the definition of E-specialization in $\mathcal{V}^n$, the family $(\eta_1, \ldots, \eta_n)$ is E-$\mathfrak{F}$-algebraically independent (Definition 2.147). Because, as noted above, the κ_i are everywhere defined on $\mathcal{V}^n$, $\kappa_i \in \mathfrak{F}_{\mathfrak{F},\eta}(\mathcal{V}^n)$ for all i. Assume the family $(\kappa_1, \ldots, \kappa_n)$ is E-$\mathfrak{F}$-dependent over some finitely generated E-extension $\mathfrak{F}$ of $\mathbb{Q}$. Then, since $\mathrm{ev}_{\mathfrak{F},\eta}$ is an E-$\mathfrak{F}$-homomorphism, the family

$$(\mathrm{ev}_{\mathfrak{F},\eta}(\mathcal{V}^n)(\kappa_1), \ldots, \mathrm{ev}_{\mathfrak{F},\eta}(\mathcal{V}^n)(\kappa_n)) = (\kappa_1(\eta), \ldots, \kappa_n(\eta)) = (\eta_1, \ldots, \eta_n)$$

would be E-$\mathfrak{F}$-dependent, which contradicts the E-$\mathfrak{F}$-independence of $(\eta_1, \ldots, \eta_n)$.

Let $\varphi \in \mathfrak{F}_{\mathfrak{F}}(\mathcal{V}^n)$ be an E-$\mathfrak{F}$-function that is defined for all $\eta \in \mathcal{V}^n$. Since $\mathfrak{F}_{\mathfrak{F}}(\mathcal{V}^n) = \mathfrak{F}\langle \kappa_1, \ldots, \kappa_n \rangle$, there exist relatively prime E-polynomials $P, Q \in \mathfrak{F}\{y_1, \ldots, y_n\}$ such that $\varphi = P(\kappa_1, \ldots, \kappa_n)/Q(\kappa_1, \ldots, \kappa_n)$. Since the family $(\kappa_1, \ldots, \kappa_n)$ is E-$\mathfrak{F}$-independent, P and Q are unique up to multiplication by an element of $\mathfrak{F}$. Assume $Q \notin \mathfrak{F}$. Then there exists an E-zero $\eta = (\eta_1, \ldots, \eta_n) \in \mathcal{V}^n$ of Q. By [25, Proposition 9, page 137], φ is not defined at η, which is a contradiction. Therefore, $Q \in \mathfrak{F}$. □

To prove the next proposition, the following correspondence between E-$\mathfrak{F}$-subsets and radical E-ideals of a certain Noetherian E-$\mathfrak{F}$-algebra [25, Proposition 1, page 69] will be used. Let M be a homogeneous E-$\mathfrak{F}$-space for an

E-$\mathfrak{F}$-group G, and let V be an $\mathfrak{F}$-irreducible E-$\mathfrak{F}$-subset of M. Let v be an $\mathfrak{F}$-generic element of V, and let t be an $\mathfrak{F}\langle v\rangle$-generic element of the connected component of the identity G^0. For an $\alpha \in \mathfrak{F}[\langle vt\rangle \cup \mathfrak{F}\langle t\rangle]_{\mathrm{E}}$ and an element $v' \in V$, define v' to be a *zero of* α, if, when t' is a $\mathfrak{F}\langle v'\rangle$-generic element of G^0, then α is in the kernel the E-$\mathfrak{F}$-homomorphism

$$S : \mathfrak{F}[\langle vt\rangle \cup \mathfrak{F}\langle t\rangle]_{\mathrm{E}} \to \mathfrak{F}[\langle v't'\rangle \cup \mathfrak{F}\langle t'\rangle]_{\mathrm{E}}$$

that extends $S_{v't',vt}$ and $S_{t',t'}$. The subset A of V is a *set of zeros* of α if each element of A is a zero of α. The element $v' \in V$ is a *zero of a subset* $\mathfrak{A}$ of $\mathfrak{F}[\langle vt\rangle \cup \mathfrak{F}\langle t\rangle]_{\mathrm{E}}$ if v' is a zero of every element of $\mathfrak{A}$ [25, pages 68-69]. If A is an E-$\mathfrak{F}$-subset of V, the set $\mathfrak{A}$ of elements of $\mathfrak{F}[\langle vt\rangle \cup \mathfrak{F}\langle t\rangle]_{\mathrm{E}}$ for which A is a set of zeros is a radical E-ideal, and A is $\mathfrak{F}$-irreducible if and only if $\mathfrak{A}$ is prime [25, Proposition 1(a), page 69]. And, if $\mathfrak{A}$ is a subset of $\mathfrak{F}[\langle vt\rangle \cup \mathfrak{F}\langle t\rangle]_{\mathrm{E}}$, then the set of zeros of $\mathfrak{A}$ is an E-$\mathfrak{F}$-subset of V [25, Proposition 1(b), page 69]. For $V = \mathcal{V}^n$, this characterization of the E-$\mathfrak{F}$-subsets will be used in the following proposition.

Proposition 6.251 *Let $\mathfrak{F}$ be an* E*-field over which $\mathcal{V}$ is* E*-universal. Then all* E*-$\mathfrak{F}$-subsets of $\mathcal{V}^n$ are of the form $\mathfrak{Z}_{\mathfrak{F}}(\mathfrak{A})$, where $\mathfrak{A} \subset \mathfrak{F}\{y_1, \ldots, y_n\}_{\mathrm{E}}$ is an* E*-ideal and $y_1, \ldots, y_n$ are* E*-$\mathfrak{F}$-indeterminates.*

Proof: Consider the E-$\mathbb{Q}$-group $\mathcal{V}^n$ as an E-$\mathfrak{F}$-group by extending the basic field from $\mathbb{Q}$ to $\mathfrak{F}$ [25, Theorem 2, page 49]. Let $v = (v_1, \ldots, v_n) \in \mathcal{V}^n$ be an $\mathfrak{F}$-generic element. By the definition of specialization in the pre E-$\mathfrak{F}$-set $\mathcal{V}^n$

(See Observation 6.245), v is $\mathfrak{F}$-generic if and only if the family $(v_1, \ldots, v_n)$ is E-$\mathfrak{F}$-independent. Therefore, an $\mathfrak{F}$-generic element of $\mathcal{V}^n$ exists, and, by definition, $\mathcal{V}^n$ is irreducible [25, page 30]. The E-$\mathfrak{F}$-independent elements $v_1, \ldots, v_n$ will be taken to be the indeterminates $y_1, \ldots, y_n$ in the statement of the proposition.

Let W be an E-$\mathfrak{F}$-subset of $\mathcal{V}^n$, and let t be an $\mathfrak{F}\langle v\rangle$-generic element of G°. Since the action of the E-$\mathfrak{F}$-group $\mathcal{V}^n$ on the E-$\mathfrak{F}$-set $\mathcal{V}^n$ is coordinate addition, this action will be written additively $v+t$. The set $\mathfrak{A}$ of elements of $\mathfrak{F}[\langle v+t\rangle_{\mathrm{E}} \cup \mathfrak{F}\langle t\rangle_{\mathrm{E}}]$ for which W is a set of zeros of a radical E-ideal that is the intersection of finitely many prime components, because $\mathfrak{F}[\langle v+t\rangle_{\mathrm{E}} \cup \mathfrak{F}\langle t\rangle_{\mathrm{E}}]$ is the ring of quotients of of a finitely generated E-$\mathfrak{F}$-algebra [25, page 70]. And W is the union of the zeros of the components of the radical Δ-ideal $\mathfrak{A}$. Therefore, to prove the proposition, it is sufficient to show that each $\mathfrak{F}$-component of W is the set of E-zeros of some E-ideal of $\mathfrak{F}\{v_1, \ldots, v_n\}_{\mathrm{E}}$. Hence W may be assumed to be $\mathfrak{F}$-irreducible, so that $\mathfrak{A}$ is prime [25, Proposition 1(a), page 69].

Let w be an $\mathfrak{F}$-generic element of W, and let s be an $\mathfrak{F}\langle w\rangle$-generic element of $\mathcal{V}^n$. Then, by definition, $\mathfrak{A}$ is the kernel of the E-$\mathfrak{F}$-homomorphism

$$S : \mathfrak{F}[\langle v+t\rangle_{\mathrm{E}} \cup \mathfrak{F}\langle t\rangle_{\mathrm{E}}] \longrightarrow \mathfrak{F}[\langle w+s\rangle_{\mathrm{E}} \cup \mathfrak{F}\langle s\rangle_{\mathrm{E}}]$$

extending $S_{w+s,v+t}$ and $S_{s,t}$. Let $w' \in \mathcal{V}^n$. Then $w \longrightarrow w'$ if and only if there exists an E-$\mathfrak{F}$-homomorphism

$$T : \mathfrak{F}[\langle w+s\rangle_{\mathrm{E}} \cup \mathfrak{F}\langle s\rangle_{\mathrm{E}}] \longrightarrow \mathfrak{F}[\langle w'+s'\rangle_{\mathrm{E}} \cup \mathfrak{F}\langle s'\rangle_{\mathrm{E}}]$$

extending $S_{w'+s'.,w+s}$ and $S_{s',s}$, where s' is an $\mathfrak{F}\langle w'\rangle$-generic element of $\mathcal{V}^n$ [25, page 68-69].

Let $\varphi_1, \ldots, \varphi_s$ be E-generators of $\mathfrak{A}$, i.e. $\mathfrak{A} = \{\varphi_1, \ldots, \varphi_s\}_{\mathrm{E}} \subseteq \mathfrak{F}[\langle v + t\rangle_{\mathrm{E}} \cup \mathfrak{F}\langle t\rangle_{\mathrm{E}}]$. Let $\{\gamma_j\}_{\in I}$ be a basis for $\mathfrak{F}\langle t\rangle_{\mathrm{E}}$ over $\mathfrak{F}$. Since S is defined at each φ_k, there exists finitely many $G_{k,j}, G'_{k,j} \in \mathfrak{F}\{v_1, \ldots, v_n\}_{\mathrm{E}}$ such that

$$\varphi_k = \frac{\Sigma\ G_{k,j}(v)\gamma_j}{\Sigma\ G'_{k,j}(v)\gamma_j}$$

and $\Sigma\ G'_{k,j}(v)\gamma_j \notin \mathfrak{A}$ for $k = 1, \ldots, s$. This last condition is equivalent to $\Sigma\ G'_{k,j}(w)S(\gamma_j) = S(\Sigma\ G'_{k,j}(v))S(\gamma_j) = S(\Sigma\ G'_{k,j}(v)\gamma_j) \neq 0$.

Let

$$S' = T \circ S : \mathfrak{F}[\langle v + t\rangle_{\mathrm{E}} \cup \mathfrak{F}\langle t\rangle_{\mathrm{E}}] \to \mathfrak{F}[\langle w' + s'\rangle_{\mathrm{E}} \cup \mathfrak{F}\langle s'\rangle_{\mathrm{E}}].$$

This E-$\mathfrak{F}$-homomorphism extends $S_{w'+s',v+t}$ and $S_{s',t}$. Again, since S' is defined at each φ_k, it again follows that $\Sigma\ G'_{k,j}(w')S(\gamma_j) = S'(\Sigma\ G'_{k,j}(v))S'(\gamma_j) = S'(\Sigma\ G'_{k,j}(v)\gamma_j) \neq 0$.

Let $\mathfrak{B} = \{(G_{k,j})\}_{\mathrm{E}}$. It will be shown that $\mathfrak{Z}_{\mathfrak{F}}(\mathfrak{B}) = W$. If $w' \in W$, the $S'(\varphi_k) = \frac{S'(\Sigma\ G_{k,j}(v)\gamma_j)}{S'(\Sigma\ G'_{k,j}(v)\gamma_j)} = 0$. and, since the denominator is not zero, $\Sigma\ G_{k,j}(w')S'(\gamma_j) = S'(\Sigma\ G_{k,j}(v)\gamma_j) = 0$. Since S' restricted to $\mathfrak{F}\langle t\rangle_{\mathrm{E}}$ is an isomorphism, $S'(\gamma_j)$ is also a basis for $\mathfrak{F}\langle s'\rangle_{\mathrm{E}}$ over $\mathfrak{F}$. As $s' \in \mathcal{V}^n$ is $\mathfrak{F}\langle w'\rangle$-generic if and only if the family $s' = (s'_1, \ldots, s'_n)$ is E-$\mathfrak{F}\langle w'\rangle_{\mathrm{E}}$-independent, $\mathfrak{F}\langle w'\rangle_{\mathrm{E}}$ and $\mathfrak{F}\langle s'\rangle_{\mathrm{E}}$ are linearly disjoint over $\mathfrak{F}$. Therefore, $S'(\gamma_j)$ is a basis for $\mathfrak{F}\langle w'\rangle_{\mathrm{E}} \cdot \mathfrak{F}\langle s'\rangle_{\mathrm{E}}$ over $\mathfrak{F}\langle w'\rangle_{\mathrm{E}}$. Because $\Sigma\ G_{k,j}(w')S'(\gamma_j) = S'(\Sigma\ G_{k,j}(v)\gamma_j) = 0$, $G_{k,j}(w') = 0$ for all k and all j, and $w' \in \mathfrak{Z}_{\mathfrak{F}}(\mathfrak{B})$.

To show $\mathfrak{Z}_{\mathfrak{F}}(\mathfrak{B}) \supseteq W$, let $w' \in \mathfrak{Z}_{\mathfrak{F}}(\mathfrak{B}) \subseteq \mathcal{V}^n$. Then, there exists an $\mathfrak{F}\langle w'\rangle$-generic element $s' \in \mathcal{V}^n$ and an E-$\mathfrak{F}$-homomorphism

$$T' : \mathfrak{F}[\langle v+t\rangle_{\mathrm{E}} \cup \mathfrak{F}\langle t\rangle_{\mathrm{E}}] \to \mathfrak{F}[\langle w'+s'\rangle_{\mathrm{E}} \cup \mathfrak{F}\langle s'\rangle_{\mathrm{E}}]$$

extending $S_{w'+s',v+s}$ and $S_{s',s}$. Since $T'(\varphi_k) = \frac{T'(\Sigma\ G_{k,j}(v)\gamma_j)}{T'(\Sigma\ G'_{k,j}(v)\gamma_j)}$, the denominator is not zero and $T'(G_{k,j}(v)) = G_{k,j}(w') = 0$, $\mathfrak{B} = \{(G_{k,j})\}_{\mathrm{E}}$ is in the kernel of T'. Therefore, T' factors through an E-homomorphism extending $S_{w'+s',w+s}$ and $S_{s',s}$, i.e. the following diagram commutes.

$$\begin{tikzcd}
\mathfrak{F}[\langle v+t\rangle_{\mathrm{E}} \cup \mathfrak{F}\langle t\rangle_{\mathrm{E}}] \arrow[r, "S"] \arrow[dr, "T'"'] & \mathfrak{F}[\langle w+s\rangle_{\mathrm{E}} \cup \mathfrak{F}\langle s\rangle_{\mathrm{E}}] \arrow[d, "T"] \\
 & \mathfrak{F}[\langle w'+s'\rangle_{\mathrm{E}} \cup \mathfrak{F}\langle s'\rangle_{\mathrm{E}}].
\end{tikzcd}$$

As noted above, this implies $w' \in W$. □

Corollary 6.252 *The* E-$\mathfrak{F}$-*Zariski topology on the* E-*group* $\mathcal{V}^n$ *relative to* $\mathfrak{F}$ ([25, page 73] or Section 4.1) *is the same as the* E-$\mathfrak{F}$-*Zariski topology on* $\mathcal{V}^n$ *relative to* $\mathfrak{F}$ *in the sense of Section 1.4.*

Proof: By the proposition, every closed set of the E-Zariski topology on the E-$\mathfrak{F}$-group $\mathcal{V}^n$ relative to $\mathfrak{F}$ is of the form $\mathfrak{Z}_{\mathfrak{F}}(\mathfrak{A})$ for some E-ideal $\mathfrak{A} \subset \mathfrak{F}\{y_1, \ldots, y_n\}_{\mathrm{E}}$ and, thus, is a closed set of E-$\mathfrak{F}$-Zariski topology relative to $\mathfrak{F}$ in the sense of Section 1.4. On the other hand, Observation 6.245 shows that every subset of $\mathcal{V}^n$ of the form $\mathfrak{Z}_{\mathfrak{F}}(\mathfrak{A})$ for some E-ideal $\mathfrak{A} \subset \mathfrak{F}\{y_1, \ldots, y_n\}_{\mathrm{E}}$ is a pre E-$\mathfrak{F}$-set and, thus, an E-$\mathfrak{F}$-subset. □

Corollary 6.253 *The* E-$\mathfrak{F}$-*subsets of* $\mathcal{V}^n$ *correspond bijectively to the radical* E-*ideals of* $\mathfrak{F}\{\kappa_1, \ldots, \kappa_n\}_{\mathrm{E}}$*, where* $\kappa_1, \ldots, \kappa_n \in \mathfrak{F}_{\mathfrak{F}}(\mathcal{V}^n)$ *are the canonical coordinate* E-$\mathfrak{F}$-*functions. This correspondence associates to each* E-$\mathfrak{F}$-*subset* $W \subseteq \mathcal{V}^n$ *the radical* E-*ideal*

$$\mathfrak{A}_W = \{\varphi \in \mathfrak{F}\{\kappa_1, \ldots, \kappa_n\}_{\mathrm{E}} \mid ev_{\mathfrak{F},w}(\mathcal{V}^n)(\varphi) = 0 \text{ for all } w \in W\}$$

and to each radical E-*ideal* $\mathfrak{A} \subseteq \mathfrak{F}\{\kappa_1, \ldots, \kappa_n\}_{\mathrm{E}}$ *the* E-$\mathfrak{F}$-*subset of zeros*

$$\mathfrak{Z}(\mathfrak{A}) = \{w \in \mathcal{V}^n \mid ev_{\mathfrak{F},w}(\mathcal{V}^n)(\varphi) = 0 \text{ for all } \varphi \in \mathfrak{A}\}.$$

Proof: Since the elements $\kappa_1, \ldots, \kappa_n$ of $\mathfrak{F}_{\mathfrak{F}}(\mathcal{V}^n)$ are E-$\mathfrak{F}$-independent (Lemma 6.250), they may be taken to be the indeterminates $y_1, \ldots, y_n$ in the statement of the proposition.

It is well-known that the E-Zariski closed sets of $\mathcal{V}^n$ over $\mathfrak{F}$ (in the sense of Section 1.4) correspond bijectively to the radical E-ideals of $\mathfrak{F}\{\kappa_1, \ldots, \kappa_n\}_{\mathrm{E}}$ [24, page 150]. This correspondence associates to each E-$\mathfrak{F}$-subset $W \subseteq \mathcal{V}^n$ the radical E-ideal

$$\mathfrak{A}_W = \{\varphi \in \mathfrak{F}\{\kappa_1, \ldots, \kappa_n\}_{\mathrm{E}} \mid \varphi(w) = 0 \text{ for all } w \in W\}$$

and to each radical E-ideal $\mathfrak{A} \subseteq \mathfrak{F}\{\kappa_1, \ldots, \kappa_n\}_{\mathrm{E}}$ the E-$\mathfrak{F}$-subset

$$\mathfrak{Z}(\mathfrak{A}) = \{w \in \mathcal{V}^n \mid \varphi(w) = 0 \text{ for all } \varphi \in \mathfrak{A}\}.$$

Since $\varphi(w) = ev_{\mathfrak{F},w}(\mathcal{V}^n)(\varphi)$, the corollary follows. □

Let $\mathfrak{F}\{y_1, \ldots, y_n\}_{\mathrm{E},1}$ denote the elements of degree one in $\mathfrak{F}\{y_1, \ldots, y_n\}_{\mathrm{E}}$. An element of $\mathfrak{F}\{y_1, \ldots, y_n\}_{\mathrm{E},1}$ is called a *linear* E-*polynomial with coefficients in* $\mathfrak{F}$ or, more simply, a *linear* E-*polynomial*, if the reference to $\mathfrak{F}$ is clear. A Δ-ideal of $\mathcal{F}\{y_1, \ldots, y_n\}_{\mathrm{E}}$ is *linear*, if it is E-generated by linear E-polynomials.

Corollary 6.254 [25, page 151] *Every* E-$\mathcal{F}$-*subgroup of the* E-$\mathcal{F}$-*group* $\mathcal{V}^n$ *is of the form* $\mathfrak{Z}_{\mathcal{F}}(\mathfrak{A})$ *for some linear* E-*ideal* $\mathfrak{A} \subseteq \mathcal{F}\{\kappa_1, \ldots, \kappa_n\}_{\mathrm{E}}$.

Proof: Since an E-$\mathcal{F}$-subgroup H is a subset that is both a group and an E-$\mathcal{F}$-subset [25, page 87 and Proposition 1, page 87], Corollary 6.253 shows that it is the set of zeros of an E-ideal $\mathfrak{A} \subset \mathcal{F}\{\kappa_1, \ldots, \kappa_n\}_{\mathrm{E}}$. And, since the E-zeros of $\mathfrak{A}$ in $\mathcal{V}^n$ are closed under addition (the group law), [25, Proposition 12, page 24] implies that $\mathfrak{A}$ is linear. □

For every positive integer k, the E-$\mathcal{F}$-homomorphism $\mu_k : G_m^{\mathrm{E}} \to G_m^{\mathrm{E}}$ defined by $\mu_k(v) = v^k$ for $v \in G_m^{\mathrm{E}}$ is surjective with finite kernel [25, Corollary 2, page 238]. The kernel $\ker \mu_k$ of μ_k is an E-$\mathcal{F}$-subgroup of G_m^{E} [25, Corollary 2, page 96].

Let $\mathrm{E} = \{\epsilon_1, \ldots, \epsilon_m\}$. Since $\mathbb{Q}\{y, 1/y\}_{\mathrm{E}} \supset \mathbb{Q}\{y\}_{\mathrm{E}}$, as sets

$$G_m^{\mathrm{E}} = (\operatorname{diffspec} \mathbb{Q}\{y, 1/y\}_{\mathrm{E}})(\mathcal{V}) \subset (\operatorname{diffspec} \mathbb{Q}\{y\}_{\mathrm{E}})(\mathcal{V}) = G_a^{\mathrm{E}}.$$

The canonical coordinate function $\kappa \in \mathfrak{F}(G_a^{\mathrm{E}})$ restricts to an E-$\mathcal{F}$-function, also denoted by κ, on $G_m^{\mathrm{E}} \subset G_a^{\mathrm{E}}$. With this notation,

$$G_m^{\mathrm{E}} = \{\eta \in G_a^{\mathrm{E}} \mid \mathrm{ev}_{\mathbb{Q},\eta}(\kappa) \neq 0\}.$$

Then $1/\kappa \in \mathfrak{F}(G_m^{\mathrm{E}})$, and $\epsilon_i\kappa/\kappa \in \mathfrak{F}(\mathfrak{U}^*)$ for all i. For $v \in G_m^{\mathrm{E}}$, the E-$\mathcal{F}$-mapping $l\mathrm{E}(v) = (\epsilon_1 v/v, \ldots, \epsilon_m v/v) : G_m^{\mathrm{E}} \to (G_a^{\mathrm{E}})^m$ [25, Proposition 6(f), page 129] is the logarithmic derivation on G_m^{E} relative to E [24, Example 1, page 352]. By [25, Proposition 3, page 89], it is an E-$\mathfrak{F}$-homomorphism. The kernel of $l\mathrm{E}$ is the E-$\mathfrak{F}$-subgroup $(G_m^{\mathrm{E}})_{\mathcal{V}^{\mathrm{E}}} \subset G_m^{\mathrm{E}}$ consisting of elements rational over $\mathcal{V}^{\mathrm{E}}$. Since the topology on G_m^{E} is the topology induced by the E-Zariski topology of the E-set G_a^{E}, the E-$\mathfrak{F}$-subgroup $(G_m^{\mathrm{E}})_{\mathcal{V}^{\mathrm{E}}}$ is the zero set of the E-ideal $[\epsilon_1\kappa, \ldots, \epsilon_m\kappa]_{\mathrm{E}} \subset \mathfrak{F}\{\kappa\}_{\mathrm{E}}$ in G_m^{E}. By forgetting the E-structure, $(G_m^{\mathrm{E}})_{\mathcal{V}^{\mathrm{E}}}$ may be considered to be an $\mathfrak{F}$-group $\mathfrak{F}$-isomorphic to G_m relative to the algebraically closed extension $\mathcal{V}^{\mathrm{E}}$ of $\mathbb{Q}$.

Corollary 6.255 *Every* E-$\mathfrak{F}$-*subgroup* G *of* G_m^{E} *is either* $\ker \mu_k$ *for a positive integer* k *or* $G_{\mathfrak{L}} = \{v \in G_m^{\mathrm{E}} \mid \varphi(l\mathrm{E}v) = 0 \text{ for } \varphi \in \mathfrak{L}\}$ *where* $\mathfrak{L} \subset \mathfrak{F}\{\kappa_1, \ldots, \kappa_m\}_{\mathrm{E}}$ *is a linear* E-*ideal containing the* E-*ideal* $[(\epsilon_i\kappa_j - \epsilon_j\kappa_i)_{i=1,\ldots,m, j=1,\ldots,m}]_E$*, and the* κ_i *are the canonical coordinate functions on* $(G_a^{\mathrm{E}})^m$*. Also,* $\ker \mu_k \subseteq \ker \mu'_k$ *if and only if* k *divides* k'*, and* $G_{\mathfrak{L}} \subseteq G_{\mathfrak{L}'}$ *if and only if* $\mathfrak{L}' \subseteq \mathfrak{L}$.

Proof: This proposition is proved in [7, Section 4]. Consider the short exact sequence of E-$\mathfrak{F}$-homomorphisms

$$\begin{tikzcd} 0 \arrow[r] & (G_m^{\mathrm{E}})_{\mathcal{V}^{\mathrm{E}}} \arrow[r, "\mathrm{incl}"] & G_m^{\mathrm{E}} \arrow[r, "l\mathrm{E}"] & (G_a^{\mathrm{E}})^m \arrow[r] & 0. \end{tikzcd}$$

Let G be an E-$\mathfrak{F}$-subgroup of G_m^{E}. Then $l\mathrm{E}(G)$ is an E-$\mathfrak{F}$-subgroup of $(G_a^{\mathrm{E}})^m$ [25, Proposition 4(a), page 92]. As such, there exists a linear E-ideal $\mathfrak{L}_G \subset \mathfrak{F}\{\kappa_1, \ldots, \kappa_m\}_{\mathrm{E}}$ containing the E-ideal $[(\epsilon_i\kappa_j - \epsilon_j\kappa_i)_{i=1,\ldots,m, j=1,\ldots,m}]_E$ such that $w \in l\mathrm{E}(G)$ if and only if $\varphi(w) = 0$ for all $\varphi \in \mathfrak{L}_G$. Since $l\mathrm{E}$ is surjective, there is a bijection between E-$\mathcal{F}$-subgroups G such that $(G_m^{\mathrm{E}})_{\mathcal{V}^{\mathrm{E}}} \subset G$ and linear E-ideals $\mathfrak{L}_G$ containing $[(\epsilon_i\kappa_j - \epsilon_j\kappa_i)_{i=1,\ldots,m,j=1,\ldots,m}]_E$ [7, Proposition 31, page 937; Corollary 1, page 937]. Therefore, if $(G_m^{\mathrm{E}})_{\mathcal{V}^{\mathrm{E}}} \subset G$, $G = \{v \in G_m^{\mathrm{E}} \mid \varphi(l\mathrm{E}(v)) = 0 \text{ for } \varphi \in \mathfrak{L}_G\} = G_{\mathfrak{L}}$. Clearly, $G_{\mathfrak{L}} \subseteq G_{\mathfrak{L}'}$ if and only if $\mathfrak{L}' \subseteq \mathfrak{L}$.

If $G \cap (G_m^{\mathrm{E}})_{\mathcal{V}^{\mathrm{E}}}$ is not equal to $(G_m^{\mathrm{E}})_{\mathcal{V}^{\mathrm{E}}}$, $G \cap (G_m^{\mathrm{E}})_{\mathcal{V}^{\mathrm{E}}}$ is a proper E-$\mathcal{F}$-subgroup of $(G_m^{\mathrm{E}})_{\mathcal{V}^{\mathrm{E}}}$. By forgetting the E-structure, $(G_m^{\mathrm{E}})_{\mathcal{V}^{\mathrm{E}}}$ may be considered to be an $\mathcal{F}$-group $\mathcal{F}$-isomorphic to G_m relative to the algebraically closed extension $\mathcal{V}^{\mathrm{E}}$ of $\mathbb{Q}$. By [24, Proposition 26(b), page 362], the only $\mathbb{Q}$-subgroups of $(G_m^{\mathrm{E}})_{\mathcal{V}^{\mathrm{E}}}$ are the finite E-$\mathbb{Q}$-subgroups $G(k)$ that are defined by the ideal $\{\kappa^k - 1\} \subset \mathbb{Q}[\kappa, 1/\kappa]$. Let $\mathfrak{A} \subset \mathbb{Q}\{\kappa\}_{\mathrm{E}}$ be the E-ideal defining the Δ-$\mathcal{F}$-subset of G_a^{E} that induces G when restricted to G_m^{E}. Then $\{\kappa^k - 1\} = \mathbb{Q}[\kappa] \cap \mathfrak{A}$. Therefore, $\kappa^k - 1 \in \mathfrak{A}$. This implies that $\epsilon_i\kappa \in \mathfrak{A}$ since κ is invertible on all of G_m^{E}, and $\ker \mu_k = (G_m^{\mathrm{E}})_{\mathcal{V}^{\mathrm{E}}} \cap G = G$. $\square$

Since for each E-$\mathcal{F}$-group G, there exists an E-extension $\mathcal{G}$ of $\mathcal{F}$ such that G is covered by a finite number of open E-$\mathcal{G}$-affine E-$\mathcal{G}$-subsets [25, Corollary, page 140] and, since an E-$\mathcal{G}$-affine E-$\mathcal{G}$-subset is a subset of G that is E-$\mathcal{G}$-isomorphic to an E-$\mathcal{G}$-subset of $\mathcal{V}^n$ [25, page 138], the topology of G is constructed by gluing together a finite number of topological spaces with the E-$\mathcal{G}$-Zariski topology in the sense of Section 1.4.

If an E-$\mathcal{F}$-group G is in the image of the functor of $\mathcal{V}$ valued points restricted to E-$\mathcal{F}$-group schemes, the cover by a finite number of open E-$\mathcal{G}$-affine E-$\mathcal{G}$-subsets of the last paragraph may, in fact be defined over $\mathcal{F}$. Assume $G = Z(\mathcal{V})$ where $(Z, \mathcal{O}_Z)$ is an E-$\mathcal{F}$-group scheme, and let $\{U_i\}$ be a finite open cover of Z, where $U_i = \operatorname{diffspec} R_i$ and each R_i is an E-$\mathcal{F}$-algebra of finitely E-generated over $\mathcal{F}$. Then $\{U_i(\mathcal{V})\}$ is a finite open cover of $Z(\mathcal{V})$. Moreover, each $U_i(\mathcal{V})$ is an open E-$\mathcal{F}$-subset of $Z(\mathcal{V})$. The function ψ in Proposition 6.246 is a complete pre E-$\mathcal{F}$-equivalence between each $U_i(\mathcal{V})$ and $V_i = \operatorname{Hom}^{\mathrm{E}}_{\mathcal{F}}(R_i, \mathcal{V}) \subseteq \mathcal{V}^n$ for some integer n.

6.2 G_a^{E}-extensions

In the remainder of this chapter, $\mathcal{F}$ will denote an (E, Δ)-field, and $\mathcal{U}$ will denote an (E, Δ)-field universal over $\mathcal{F}$. The field $\mathcal{K}$ of Δ-constants of $\mathcal{U}$ is, as an E-field, E-universal over the Δ-constants $\mathcal{C}$ of $\mathcal{F}$.

The symbol '$\mathcal{U}$' will also denote the (E, Δ)-$\mathcal{F}$-group (Corollary 6.248) that is (E, Δ)-$\mathcal{F}$-isomorphic to $G_a^{\mathrm{E},\Delta}$. Let $\kappa \in \mathfrak{F}(\mathcal{U})$ be the canonical coordinate function on $\mathcal{U}$. Then, $\delta_i \kappa \in \mathfrak{F}(\mathcal{U})$, and the E-$\mathcal{F}$-mapping $l\Delta = (\delta_1 \kappa, \dots, \delta_m \kappa) : \mathcal{U} \to \mathcal{U}^n$ [25, Proposition 6, page 129] is the logarithmic derivation on $\mathcal{U}$ relative to Δ [24, Example 1, page 352]. By [25, Proposition 3, page 89], it is an (E, Δ)-$\mathcal{F}$-homomorphism. The kernel of $l\Delta$ is the (E, Δ)-$\mathcal{F}$-subgroup $\mathcal{K}$ is the zero set of the (E, Δ)-ideal $[\delta_1 y, \dots, \delta_m y]_{\mathrm{E},\Delta} \subset \mathcal{F}\{y\}_{\mathrm{E},\Delta}$. By forgetting the Δ-structure, $\mathcal{K}$ may be considered to be an E-$\mathcal{F}$-group E-$\mathcal{F}$-isomorphic to G_a^{E} (Corollary 6.248).

Definition 6.256 *An element $\alpha \in \mathcal{U}$ is Δ-primitive over $\mathcal{F}$ if $l\Delta\alpha \in \mathcal{F}^m$; that is, for suitable elements $a_1, \dots, a_m \in \mathcal{F}$, α satisfies the system of differential equations*

$$\delta_i \alpha = a_i \ (1 \le i \le m).$$

Let α be Δ-primitive over $\mathcal{F}$, and suppose that the field of Δ-constants of $\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}$ is $\mathcal{C} = \mathcal{F}^{\Delta}$. For any (E,Δ)-isomorphism σ of $\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}$ over $\mathcal{F}$, $(\delta_1(\sigma\alpha),\ldots,\delta_m(\sigma\alpha)) = (\sigma(\delta_1\alpha),\ldots,\sigma(\delta_m\alpha)) = (\delta_1\alpha,\ldots,\delta_m\alpha)$; hence the difference $c(\sigma)$=$\sigma\alpha - \alpha$ is in the kernel of the above homomorphism $l\delta$. As

$$\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}\sigma(\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}) = \mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}\mathcal{F}\langle\sigma\alpha\rangle_{\mathrm{E},\Delta}$$

$$= \mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}\mathcal{F}\langle\alpha + c(\sigma)\rangle_{\mathrm{E},\Delta} = \mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}\mathcal{C}\langle c(\sigma)\rangle_{\mathrm{E},\Delta},$$

it follows that $\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}$ is E-strongly normal over $\mathcal{F}$, and $\mathcal{C}\langle\sigma\rangle_{\mathrm{E}} = (\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}\sigma(\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}))^{\Delta} = \mathcal{C}\langle c(\sigma)\rangle_{\mathrm{E}}$. For any two elements $\sigma,\sigma' \in G(\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}/\mathcal{F})$ (regarded as elements of $\mathrm{Aut}_{E,\Delta}(\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}\mathcal{K}/\mathcal{F}\mathcal{K})$ by means of Proposition 3.190),

$$\alpha + c(\sigma\sigma') = \sigma\sigma'\alpha = \sigma(\alpha + c(\sigma')) = \sigma\alpha + c(\sigma') = \alpha + c(\sigma) + c(\sigma')$$

since $c(\sigma') \in \mathcal{K}$ and, thus, $\sigma(c(\sigma')) = c(\sigma')$. Therefore, $c(\sigma\sigma') = c(\sigma) + c(\sigma')$, and, evidently, $c(\sigma) = 0$ only when $\sigma = id_{\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}}$. This proves the first part of the following proposition.

Proposition 6.257 *Let α be a Δ-primitive over $\mathcal{F}$, and suppose that the field of Δ-constants of $\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}$ is $\mathcal{C} = \mathcal{F}^{\Delta}$. Then, each (E,Δ)-$\mathcal{F}$-isomorphism σ of $\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}$ into $\mathcal{U}$ is of the form $\sigma\alpha = \alpha + c(\sigma)$ for $c(\sigma) \in \mathcal{K}$. In addition, $\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}$ is* E-*strongly normal over $\mathcal{F}$, and the mapping $c : G(\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}/\mathcal{F}) \rightarrow \mathcal{K}$ defined by $c(\sigma) = \sigma\alpha - \alpha$ for $\sigma \in G(\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}/\mathcal{F})$ is an injective* E-$\mathcal{C}$-*homomorphism of* E-*groups relative to the* E-*universal field $\mathcal{K}$. Consequently, $\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}$ is a $\mathrm{G}_{\mathrm{a}}^{\mathrm{E}}$-extension of $\mathcal{F}$.*

Proof: Although this proof is the same as that of Proposition 5.237, reading it in the simplest case can be beneficial. It will first be shown that c restricted to the $\mathcal{C}$-generic elements of $G(\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}/\mathcal{F})$ is a pre E-$\mathcal{C}$-mapping. Then by [25, Corollary 1, p. 90], c is an E-$\mathcal{C}$-homomorphism. Parts 1 and 2 of the definition of a pre E-$\mathcal{C}$-mapping (Definition 2.56) are obvious. For Part 3, if σ' is a generic E-$\mathcal{C}$-specialization of σ, then there exists an (E,Δ)-$\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}$-isomorphism $\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}\mathcal{F}\langle\sigma\alpha\rangle_{\mathrm{E},\Delta} \approx \mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}\mathcal{F}\langle\sigma'\alpha\rangle_{\mathrm{E},\Delta}$ mapping $\sigma\alpha$ onto $\sigma'\alpha$ and, therefore, mapping $c(\sigma)$ to $c(\sigma')$. Hence, $c(\sigma')$ is a generic specialization of $c(\sigma)$ over $\mathcal{C}$, which is Part 3. By restricting the above isomorphism to the Δ-constants, $S_{\sigma',\sigma} : \mathcal{C}\langle\sigma\rangle_{\mathrm{E}} \approx \mathcal{C}\langle\sigma'\rangle_{\mathrm{E}}$. Since $\mathcal{C}\langle\sigma\rangle_{\mathrm{E}} = \mathcal{C}\langle c(\sigma)\rangle_{\mathrm{E}}$ and $\mathcal{C}\langle\sigma'\rangle_{\mathrm{E}} = \mathcal{C}\langle c(\sigma')\rangle_{\mathrm{E}}$, the isomorphisms $S_{\sigma',\sigma}$ and $\mathrm{S}_{c(\sigma'),c(\sigma)}$ coincide, which is part 4. □

Proposition 6.258 *Let G be an* E-$\mathcal{C}$-*subgroup of* G_a^{E}. *Let $\mathcal{L} \subseteq \mathcal{C}\{y\}_{\mathrm{E}}$ be the linear* E-*ideal defining G* (Corollary 6.254). *Let $b \in \mathcal{U}$ be a $\mathcal{C}$-generic* (E,Δ)-*zero of* $\mathcal{L}_{\Delta,E} \subset \mathcal{C}\{y\}_{\mathrm{E},\Delta}$. *Let $a = (a_1, \ldots, a_m) = l\Delta b$. Put $\mathcal{F} = \mathcal{C}\langle a\rangle_{E,\Delta}$, and $\mathcal{G} = \mathcal{F}\langle b\rangle_{E,\Delta}$. Then $\mathcal{G}$ over $\mathcal{F}$ is an* E-*strongly normal extension with Galois group* E-$\mathcal{C}$-*isomorphic to G.*

Proof: Although this proof is the same as Theorem 5.239, reading it in the simplest case can be beneficial. Observe that $\mathcal{G}$ is generated by a Δ-primitive element b over $\mathcal{F}$ and has no new Δ-constants because $\mathcal{G}^{\Delta} = \mathcal{F}^{\Delta} = \mathcal{K}$ by the No New Δ''-Constant Lemma 5.230. By Proposition 6.257, c, as defined above, is an injective E-$\mathcal{C}$-homomorphism from $G(\mathcal{G}/\mathcal{F})$ to G_a^{E}. It is surjective because, for any $v \in \mathcal{K}$, the (E,Δ)-$\mathcal{F}$-isomorphism σ_v of $\mathcal{G}$ over $\mathcal{G}$ defined by $\sigma_v(b) = b + v$ is an E-strongly normal isomorphism of $\mathcal{G}$ over $\mathcal{F}$ such that $c(\sigma_v) = \sigma_v(b) - b = (b + v) - b = v$. Therefore, the Galois group is E-$\mathcal{C}$-isomorphic to G because a bijective E-$\mathcal{C}$-homomorphism is an E-$\mathcal{C}$-isomorphism of E-$\mathcal{C}$-groups [25, Corollary 4, p. 97]. $\square$

Let $\mathcal{G}$ be Δ-strongly normal over $\mathcal{F}$ with Galois group $G \subset \mathbf{G}_a^{\mathrm{E}}$. Theorem 3.197 shows that G is an E-$\mathcal{C}$-group where $\mathcal{C} = \mathcal{F}^{\Delta}$. By Corollary 6.254, G is set of E-zeros of a linear E-ideal $\mathcal{L}_G \subseteq \mathcal{C}\{\kappa\}_{\mathrm{E}}$, where κ is the canonical coordinate function on $\mathbf{G}_a^{\mathrm{E}}$. Each E-$\mathcal{C}$-subgroup $H \subseteq G$ is also the E-zeros of a linear E-ideal $\mathcal{L}_H \subseteq \mathcal{C}\{y\}_{\mathrm{E}}$ such that $\mathcal{L}_G \subseteq \mathcal{L}_H$. Recall, for a linear E-ideal $\mathcal{L}$, let $\mathcal{L}_1$ be the set of linear elements of $\mathcal{L}$ having degree one. Then $\mathcal{L} = [\mathcal{L}_1]_{\mathrm{E}}$. For each H, the following proposition exhibits the subfield of $\mathcal{G}$ invariant under the action of H and, thus, specifies the Galois correspondence.

Proposition 6.259 *Let $\mathcal{F}$ be an (E, Δ)-field such that $\mathcal{C} = \mathcal{F}^{\Delta}$ is constrainedly closed. Let $\mathcal{G}$ be an* E*-strongly normal extension of $\mathcal{F}$ with Galois group $G \subseteq G_a^{\mathrm{E}}$. Assume that $\mathcal{G} = \mathcal{F}\langle b\rangle_{\mathrm{E},\Delta}$ where $b \in \mathcal{U}$ is a Δ-primitive over $\mathcal{F}$. Then, the Galois correspondence associates to each* E-$\mathcal{C}$-*subgroup H of G the (E, Δ)-subfield $\mathcal{H} \subseteq \mathcal{G}$ equal to $\mathcal{F}\langle (L(b))_{L\in\mathcal{L}_{H,1}}\rangle_{\mathrm{E},\Delta}$, where $\mathcal{L}_H \subseteq \mathcal{C}\{\kappa\}_{\mathrm{E}}$ is the linear* E*-ideal defining H and κ is the canonical coordinate function on G_a^{E} .*

$$\begin{tikzcd}
G \arrow[r] & \mathcal{F} \arrow[d] \\
H \arrow[u] \arrow[r] & \mathcal{H} = \mathcal{G}^H = \mathcal{F}\langle (Lb)_{L\in\mathcal{L}_{H,1}}\rangle_{\mathrm{E},\Delta} \arrow[d] \\
1 \arrow[u] \arrow[r] & \mathcal{G}
\end{tikzcd}$$

Remark 6.260 *It should be possible to prove that every* E*-strongly normal extension of* $\mathcal{F}$ *with Galois group* $G \subseteq G_a^{\mathrm{E}}$ *is* (E, Δ)*-generated by a* Δ*-primitive over* $\mathcal{F}$ *by the method of* [24, Corollary 2, page 427].

Proof: It may be assumed that $G \subseteq \mathcal{K}$ because $G \subseteq G_a^{\mathrm{E}}$ and G_a^{E} and $\mathcal{K}$ are E-$\mathcal{F}$-isomorphic. Thus, $\mathcal{L}_H$ may be considered to be a subset of $\mathcal{C}\{y\}_{\mathrm{E}}$ where y is an E-indeterminate over $\mathcal{C}$. Each $\sigma \in H$ is of the form $\sigma_h(b) = b + h$ for $h = \sigma \in H$ (Proposition 6.257).

Since $\mathcal{C}$ is constrainedly closed, the Fundamental Theorem 4.207 implies that there is a Galois correspondence that associates to each E-$\mathcal{C}$-subgroup H the (E, Δ)-subfield $\mathcal{H} = \mathcal{G}^H$. Therefore, to prove the proposition it suffices to show $\mathcal{G}^H = \mathcal{F}\langle (L(b))_{L\in\mathcal{L}_H}\rangle_{\mathrm{E},\Delta}$.

For each $L \in \mathcal{L}_{H,1}$, the element $Lb \in \mathcal{G}$ is invariant under H because

$$\sigma_h(L(b)) = L(\sigma_h(b)) = L(b + h) = L(b) + L(h) = L(b)$$

for $h \in \mathfrak{Z}_{\mathcal{C}}(\mathcal{L}_H) \subseteq \mathcal{K}$. Therefore, $\mathcal{G}^H \supseteq \mathcal{F}\langle (L(b))_{L\in\mathcal{L}_{H,1}}\rangle_{\mathrm{E},\Delta}$. To show these fields are equal, assume they are not. Then, by the Fundamental Theorem

4.207, there exist an E-$\mathcal{C}$-subgroup $I \subseteq G$ with fixed field $\mathcal{F}\langle (L(b))_{L \in \mathcal{L}_{H,1}} \rangle_{\mathrm{E},\Delta}$ that properly contains H.

$$\begin{tikzcd}
G \arrow[rr] & & \mathcal{F} \arrow[d, "\cap"'] \\
I \arrow[u, "\cup"] \arrow[r] & & \mathcal{F}\langle (L(b))_{L \in \mathcal{L}_{H,1}} \rangle_{\mathrm{E},\Delta} \arrow[d, "\cap"'] \\
H \arrow[u, "\cup"] \arrow[rr] & & \mathcal{G}^H
\end{tikzcd}$$

Let $\tau \in I$ such that $\tau \notin H$. By Proposition 6.257, $\tau(b) = b + c(\tau)$ for $c(\tau) \in \mathcal{K}$. Since $\tau \in I$, $\tau(L(b)) = L(b)$ for all $L \in \mathcal{L}$. Thus, for all $L \in \mathcal{L}_{H,1}$,

$$L(b) + L(c(\tau)) = L(b + c(\tau)) = L(\tau(b)) = \tau(L(b)) = L(b),$$

$L(c(\tau)) = 0$, $c(\tau) \in \mathfrak{Z}(\mathcal{L})$, and $\tau \in H$, which is a contradiction. ☐

For simplicity, assume $\Delta = \{\delta\}$ throughout the remainder of this section.

The following is a simple example of an E-strongly normal extension $\mathcal{G}$ over $\mathcal{F}$ such that the transcendence degree of $\mathcal{G}$ over $\mathcal{F}$ is infinite in the usual algebraic sense. Let $a \in \mathcal{F}$ be linearly E-$\mathcal{C}$-independent modulo $\delta\mathcal{F}$ (Definition 2.147). For instance, if $a \in \mathcal{U}$ is (E, Δ)-independent over $\mathcal{C}$, the element $a \in \mathcal{F} = \mathcal{C}\langle a \rangle_{\mathrm{E},\Delta}$ satisfies this condition by Exercise 5.226. Let $b \in \mathcal{U}$ be an (E, Δ)-zero of the (E, Δ)-ideal $\{\delta y - a\}_{\mathrm{E},\Delta} \subset \mathcal{F}\{y\}_{\mathrm{E},\Delta}$. By Corollary 2.148, $\mathcal{F}\langle b \rangle^{\Delta}_{\mathrm{E},\Delta} = \mathcal{F}^{\Delta}$, and b is E-independent over $\mathcal{F}$, which implies $\mathcal{G} = \mathcal{F}\langle b \rangle_{\mathrm{E},\Delta}$ has infinite transcendence degree over $\mathcal{F}$. Since b is a Δ-primitive over $\mathcal{F}$ (Definition 6.256), $\mathcal{G}$ is E-strongly normal over $\mathcal{F}$ by Proposition 6.257, and the mapping $c : G(\mathcal{F}\langle b\rangle_{\mathrm{E},\Delta}/\mathcal{F}) \to \mathcal{K}$ such that $c(\sigma) = \sigma b - b$ for $\sigma \in G(\mathcal{F}\langle a\rangle_{\mathrm{E},\Delta}/\mathcal{F})$ is an injective E-$\mathcal{C}$-homomorphism of E-groups relative to the E-field $\mathcal{K}$. The mapping c is surjective because the E-independence of b over $\mathcal{F}$ implies, for each $\gamma \in \mathcal{K}$, there exists an (E, Δ)-isomorphism $\sigma_\gamma \in G(\mathcal{F}\langle b\rangle_{\mathrm{E},\Delta}/\mathcal{F})$ such that $\sigma_\gamma(b) = b+\gamma$, and $c(\sigma_\gamma) = \sigma b - b = \gamma$. Therefore, the Galois group is E-$\mathcal{C}$-isomorphic to $\mathcal{K}$ because a bijective E-$\mathcal{C}$-homomorphism is an E-$\mathcal{C}$-isomorphism of E-$\mathcal{C}$-groups [25, Corollary 4, p. 97].

Let $\mathcal{C}\{y\}_{\mathrm{E},1}$ denote the elements of degree one in $\mathcal{C}\{y\}_{\mathrm{E}}$. Another proof that $G(\mathcal{F}\langle \alpha\rangle_{\mathrm{E},\Delta}/\mathcal{F}) = \mathcal{K}$ follows from the next proposition by observing that there is no $L(y) \in \mathcal{C}\{y\}_{\mathrm{E},1}$ such that $L(a) \in \delta\mathcal{F}$. If this where so, there would exist an $f \in \mathcal{F}$ such that $L(a) = \delta f$. Then $L(b) - f$ would be a Δ-constant, and $L(b) \in \mathcal{F}$. However, $1, b, \epsilon b, \epsilon^2 b, \ldots$ are linearly independent over $\mathcal{F}$ by Proposition 2.140.

In the next proposition, if b is Δ-primitive over $\mathcal{F}$, the Galois group of $\mathcal{G} = \mathcal{F}\langle b\rangle_{\mathrm{E},\Delta}$ over $\mathcal{F}$ is completely determined by $a = \delta b \in \mathcal{F}$.

Proposition 6.261 *Let b be a Δ-primitive over $\mathcal{F}$, and let $\mathcal{G} = \mathcal{F}\langle b\rangle_{\mathrm{E},\Delta}$. Assume that $\mathcal{G}^\Delta = \mathcal{F}^\Delta$. Let $a = \delta b$, let $\mathcal{L}_1 = \{L(y) \in \mathcal{C}\{y\}_{\mathrm{E},1} \mid L(a) \in \delta\mathcal{F}\}$, and let $\mathcal{L} = [\mathcal{L}_1]_{\mathrm{E}}$. Let $G = Gal(\mathcal{G}/\mathcal{F})$, and let $c : G \to \mathcal{V}$ be the* E-*$\mathcal{F}$-homomorphism defined by $c(\sigma) = \sigma(b) - b$. Then, the defining* E-*ideal $\mathfrak{A}_{c(G)} \subset \mathcal{C}\{y\}_{\mathrm{E}}$ of $c(G)$ is $\mathcal{L}$.*

Proof: Since $\mathcal{G}$ over $\mathcal{F}$ is E-strongly normal (Proposition 6.257), G is an E-$\mathcal{C}$-group. Therefore, $c(G) \subseteq \mathcal{V}$ is also an E-$\mathcal{C}$-group . By Corollary 6.254,

$\mathfrak{A}_{c(G)} \subset \mathcal{C}\{y\}_{\mathrm{E}}$ is a linear E-ideal, i.e. there exist $M_1, \dots, M_r \in \mathcal{C}\{y\}_{\mathrm{E},1}$ such that $[M_1, \dots, M_r]_{\mathrm{E}} = \mathfrak{A}_{c(G)}$. Also, Proposition 6.257 shows that each $\sigma \in G$ is of the form $\sigma_v(b) = b + v$ for $v \in \mathfrak{Z}(\mathfrak{A}_{c(G)})$, and $c(\sigma_v) = v$.

Since, for each i, $M_i(v) = 0$ for all $v \in c(G)$,

$$\sigma_v(M_i(b)) - M_i(b) = M_i(\sigma_v(b)) - M_i(b) = M_i(\sigma_v(b) - b) = M_i(v) = 0.$$

Therefore, $\sigma_v(M_i(b)) = M_i(b)$, and $M_i(b)$ is invariant under all elements of G. Thus, $M_i(b) \in \mathcal{F}$, and $M_i(b) = f$ for some $f \in \mathcal{F}$. Hence,

$$M_i(a) = M_i(\delta b) = \delta(M_i(b)) = \delta f.$$

Therefore, $\mathfrak{A}_{c(G)} \subseteq \mathcal{L}$.

On the other hand, let $L(y) \in \mathcal{L}_1$. Then $L(a) = \delta f$ for $f \in \mathcal{F}$, and $L(b) - f$ is a Δ-constant because $\delta(L(b) - f) = L(\delta b) - \delta f = L(a) - \delta f = 0$. Therefore, $L(b) - f \in \mathcal{C} \subseteq \mathcal{F}$, and $L(b) \in \mathcal{F}$. Hence, $\sigma_v(L(b)) = L(b)$, and, for all $\sigma_v \in G$, the computation

$$L(v) = L(\sigma_v(b) - b) = L(\sigma_v(b)) - L(b) = \sigma_v(L(b)) - L(b) = 0$$

shows that $\mathfrak{A}_{c(G)} \supseteq \mathcal{L}$. □

By Corollary 6.254, each E-subgroup G of G_a^{E} is defined by a linear E-ideal $\mathcal{L} \subset \mathcal{F}\{y\}_{\mathrm{E}}$. If $\mathrm{E} = \{\epsilon\}$, $\mathcal{L}$ is generated by one element $L \in \mathcal{F}\{y\}_{\mathrm{E}}$ [24, Exercise 1, page 155].

Corollary 6.262 *Assume* $\mathrm{E} = \{\epsilon\}$ *and* $\Delta = \{\delta\}$. *Let* $\mathcal{H}$ *be an algebraically closed* (E, Δ)*-field such that* $\mathcal{H}^{\Delta} = \mathcal{H}$, *and* $\mathcal{F} = \mathcal{H}\langle x\rangle_{\mathrm{E},\Delta}$, *where* $x \in \mathcal{U}$,

$\epsilon x = 0$ and $\delta x = 1$. Then, there is no Δ-primitive E*-strongly normal extension of $\mathcal{F}$ with Galois group* E*-$\mathcal{C}$-isomorphic to G_a^{E}.*

Remark 6.263 *This remains true if the hypothesis that $\mathcal{H}$ be an algebraically closed is omitted; the following proof must be modified to take the structure of irreducibles into account in the partial fraction decomposition.*

Proof: Assume that there exist an Δ-primitive E-strongly normal extension $\mathcal{G}$ of $\mathcal{F}$ with Galois group G that is E-$\mathcal{H}$-isomorphic to $G^{\mathrm{E}}_{a,\mathcal{H}}$. Let $b \in \mathcal{U}$ be a Δ-primitive over $\mathcal{F}$ such that $\delta b = a \in \mathcal{F}$ and $\mathcal{G} = \mathcal{F}\langle b\rangle_{\mathrm{E},\Delta}$. Let $a = p(x) + \Sigma_{i,j} \dfrac{h_{i,j}}{(x-h_i)^j}$ for $p(x) \in \mathcal{H}[x]$ and $h_i, h_{i,j} \in \mathcal{H}$, be the partial fraction decomposition of a. If $h_{i,1} = 0$ for all i, $a = \delta f$ for $f \in \mathcal{F}$, and $b - f \in \mathcal{G}$ is a Δ-constant not in $\mathcal{F}$, which contracts the assumption that $\mathcal{G}$ over $\mathcal{F}$ is E-strongly normal (Proposition 3.187). Therefore, $h_{i,1} \neq 0$ for at least one i, and there exists a non-zero $L(y) =$

$$\begin{vmatrix} h_{1,1} & h_{2,1} & \dots & h_{r,1} & y \\ \epsilon h_{1,1} & \epsilon h_{2,1} & \dots & \epsilon h_{r,1} & \epsilon y \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \epsilon^r h_{1,1} & \epsilon^r h_{2,1} & \dots & \epsilon^r h_{r,1} & \epsilon^r y \end{vmatrix}$$

$\in \mathcal{H}\{y\}_{\mathrm{E},1}$ such that the finitely many $h_{i,1}$ span over $\mathcal{H}^{\mathrm{E},\Delta}$ the linear space of E-zeros of $L(y)$. By Lemma 6.264 below, since $L(h_{i,1}) = 0$ for all i, $L(a) \in \delta\mathcal{F}$. By Proposition 6.261, $L(y) \in \mathcal{L}_G$, and G is not $\mathcal{V}$. ☐

Lemma 6.264 *Assume* $\mathrm{E} = \{\epsilon\}$ *and* $\Delta = \{\delta\}$*. Let $\mathcal{H}$ be an algebraically closed* (E, Δ)*-field such that $\mathcal{H}^\Delta = \mathcal{H}$, and let $\mathcal{F} = \mathcal{H}\langle x\rangle_{\mathrm{E},\Delta}$, where $x \in \mathcal{U}$, $\epsilon x = 0$ and $\delta x = 1$. Let $M(y) \in \mathcal{H}^\Delta\{y\}_{\mathrm{E},1}$. For $\alpha \in \mathcal{F}$, let $\alpha =$*

$p(x) + \Sigma_{i,j} \dfrac{h_{i,j}}{(x-h_i)^j}$ for $p(x) \in \mathcal{H}[x]$ and $h_i, h_{i,j} \in \mathcal{H}$, be the partial fraction decomposition of α. Then, $M(\alpha) \in \delta\mathcal{F}$ if and only if $M(h_{i,1}) = 0$ for all i.

Proof: The only terms in the above representation of α not in $\delta\mathcal{F}$ are those with $j = 1$. Since $\delta M(y) = M(\delta y)$, if $j > 1$, $M(\frac{h_{i,j}}{(x-h_i)^j}) \in \delta\mathcal{F}$ because $\frac{h_{i,j}}{(x-h_i)^j} \in \delta\mathcal{F}$. Therefore, the condition $M(a) \in \delta\mathcal{F}$ is equivalent to $M(\Sigma_i \dfrac{h_{i,1}}{(x-h_i)}) \in \delta\mathcal{F}$. Since

$$\epsilon(\frac{h_{i,1}}{(x-h_i)}) = \frac{\epsilon h_{i,1}}{(x-h_i)} - \frac{h_{i,1}\epsilon h_i}{(x-h_i)^2},$$

by induction $\frac{\epsilon^k h_{i,1}}{(x-h_i)} = \frac{\epsilon h_{i,1}}{(x-h_i)}$ + an element of $\delta\mathcal{F}$. By the linearity of M, $M(\alpha) \in \delta\mathcal{F}$ is equivalent to $\Sigma_i \dfrac{M(h_{i,1})}{(x-h_i)} \in \delta\mathcal{F}$. This is true if and only if $M(h_{i,1}) = 0$ for all i since all the $\frac{1}{x-h_i}$ are linearly independent over $\mathcal{H}$ modulo $\delta\mathcal{F}$ (Corollary 2.135). $\square$

The next two results establish procedures for the construction of all G_a^{E}-extensions under the condition $\mathrm{E} = \{\epsilon\}$.

Proposition 6.265 *Assume $E = \{\epsilon\}$ and $\Delta = \{\delta\}$. Let $\mathcal{H}$ be an (E, Δ)-field, and let $h \in \mathcal{H}$. Let $\mathcal{U}$ be (E, Δ)-universal over $\mathcal{H}$. Let $L(y) \in \mathcal{H}^{\Delta}\{y\}_{\mathrm{E},1}$ of positive order n with the coefficient of the highest order term equal to 1.*

1. *Let $a \in \mathcal{U}$ be an (E, Δ)-zero of $[L(y) - \delta h]_{\mathrm{E},\Delta} \subset \mathcal{H}\{y\}_{\mathrm{E},\Delta}$ such that $a, \epsilon a, \ldots, \epsilon^{n-1}a$ are linearly independent over $(\mathcal{H}\langle a\rangle_{\mathrm{E},\Delta})^{\Delta}$ modulo $\delta((\mathcal{H}\langle a\rangle_{\mathrm{E},\Delta})$*

2. *Let* $b \in \mathcal{U}$ *be an* (E, Δ)*-zero of* $M = [\delta y - a, L(y) - h]_{E,\Delta} \subset \mathcal{H}\langle a\rangle_{\mathrm{E},\Delta}\{y\}_{E,\Delta}$.

Put $\mathcal{F} = \mathcal{H}\langle a\rangle_{E,\Delta}$ *and* $\mathcal{G} = \mathcal{F}\langle b\rangle_{E,\Delta}$. *Then,* $\mathcal{G}$ *is an* E*-strongly normal extension of* $\mathcal{F}$ *with Galois group* $G \subseteq \mathcal{V}$ *defined by the* E*-ideal* $\mathcal{L} = [L]_{\mathrm{E}}$.

Proof: First [24, Lemma 5, page 137] will be applied to show that M is a proper (E, Δ)-ideal. Then, because M is linear, it is prime [24, Section 5, page 151]. Since $\mathcal{U}$ is (E, Δ)-universal over $\mathcal{H}$, there exists an (E, Δ)-zero $b \in \mathcal{U}$ as required.

To apply the proposition, $\{\delta y - a, L(y) - h\}$ must be an coherent autoreduced set of M relative to some fixed ranking. It is clearly autoreduced. The coherence of the follows by letting $L'(y) = L(y) - \epsilon^n y$ and computing

$$\delta(L(y) - h) - \epsilon^n(\delta y - a) = \delta L'(y) + \epsilon^n a - \delta h$$

$$= \delta L'(y) + \epsilon^n a - \delta h - (L(a) - \delta h) = \delta L'(y) - L'(a) = L'(\delta y - a).$$

To show M is a proper (E, Δ)-ideal, assume that it is not. Then $1 \in M$. Since 1 is partially reduced with respect to $\{\delta y - a, L(y) - h\}$, [24, Lemma 5, page 137] implies that $1 \in (\delta y - a, L(y) - h) \subset \mathcal{H}\langle a\rangle_{\mathrm{E},\Delta}\{y\}_{E,\Delta}$, which is impossible because 1 is reduced with respect to $\{\delta y - a, L(y) - h\}$ and is not zero.

Next, $\mathcal{G}^{\Delta} = \mathcal{F}^{\Delta}$ follows from assumption 1 and Proposition 2.140 since $\delta(\epsilon^k b) = \epsilon^k a$ for $k = 0, \ldots, n-1$ and the set $b, \epsilon b, \ldots, \epsilon^{n-1} b$ generate $\mathcal{G}$ as a field extension of $\mathcal{F}$. Because $\mathcal{G}$ is (E, Δ)-generated by a Δ-primitive element over $\mathcal{F}$, Proposition 6.257 implies the $\mathcal{G}$ is E-strongly normal over $\mathcal{F}$. By Proposition 6.261, since $L(a) = \delta h$, $L(y) \in \mathcal{L}$. Suppose there where a linear $M \in \mathcal{L}$ of lower order than L. Then, again by Proposition 6.261, there exist $f \in \mathcal{F}$ such that $M(a) = \delta f$. Then, $M(b) - f$ is a Δ-constant in $\mathcal{F}$. Therefore, $M(b) = f_1 \in \mathcal{F}$. However, by Proposition 2.140, $1, b, \epsilon b, \ldots, \epsilon^{n-1} b$ are linearly independent over $\mathcal{F}$. This contradiction shows that L is linear of minimal order in $\mathcal{L}$. Therefore, $\mathcal{L} = [L]_{\mathrm{E}}$. □

One may apply this proposition to the example of the introduction. Let $\mathcal{H} = \mathbb{C}(t, x, \cos t, \sin t)$ $(\epsilon t = 1, \epsilon x = 0, \delta t = 0, \delta x = 0)$, $h = 0$, $\gamma = \cos t / \sin t$, $a = \sin t / x \in \mathcal{H}$, and let $L(y) = \epsilon y - \gamma y \in \mathcal{H}^{\mathrm{E}}\{y\}_{\mathrm{E},1}$. Then a is an (E, Δ)-zero of $[L(y)]_{\mathrm{E},\Delta}$, and a is linearly independent over $(\mathcal{H}\langle a\rangle_{\mathrm{E},\Delta})^{\Delta} = (\mathcal{H})^{\Delta} = \mathbb{C}(t, \cos t, \sin t)$ modulo $\delta(\mathcal{H}\langle a\rangle_{\mathrm{E},\Delta}) = \delta(\mathcal{H})$ by Corollary 2.135. Let $b = \log x \sin t$. Then b is an (E, Δ)-zero of $[\delta y - a, L(y)]_{\mathrm{E},\Delta}$. By Proposition 6.265, $\mathcal{G} = \mathcal{H}\langle b\rangle_{\mathrm{E},\Delta}$ is an E-strongly normal extension of $\mathcal{F} = \mathcal{H}$ with Galois group $\mathfrak{Z}_{[L(y)]_{\mathrm{E}}} \subseteq \mathcal{V}$.

The following corollary reformulates the previous proposition so that it is directly applicable to the construction of other examples.

Corollary 6.266 *Let F be an (E, Δ)-field, let $f \in F$, and let $d_1, \ldots, d_n \in F^{\mathrm{E}} \subset \mathcal{U}$. Let $M(y) \in F^{\Delta}\{y\}_{\mathrm{E},1}$ of positive order n with the coefficient of the highest order term equal to 1, and, for $i = 1, \ldots n$, let $e_i \in \mathcal{U}^{\mathrm{E}}$ be an Δ-zero of $\delta y - d_i \in F\{y\}_{\Delta}$. Assume*

1. *$d_1, \ldots, d_n$ are linearly independent over F^{Δ} modulo δF,*

2. *there exist* $\eta_1, \dots, \eta_n \in F^{\Delta}$ *such that* $\eta_1, \dots, \eta_n$ *are* E*-zeros of* $M(y)$ *linearly independent over* $F^{\mathrm{E},\Delta}$*, and*

3. *there exist an* (E, Δ)*-zero* $\eta \in F$ *of* $M(y) - f \in F\{y\}_{\mathrm{E},\Delta}$.

Let $c = \eta + \Sigma_i \eta_i e_i$. *Then* $F\langle c\rangle_{\mathrm{E},\Delta}$ *is* E*-strongly normal over* F *with Galois group* $G \subseteq \mathcal{V}$ *defined by the* E*-ideal* $\mathcal{L} = [M(y)]_{\mathrm{E}}$.

Proof: The hypothesises of the proposition will be verified by taking $\mathcal{H} = F$, $\mathcal{F} = F$, $h = f$, $L = M$, $a = \delta\eta + \Sigma\ \eta_i d_i \in F$ and $b = c$. Clearly

$$\delta c = \delta(\eta + \Sigma_i \eta_i e_i) = \delta\eta + \Sigma_i \eta_i \delta e_i = \delta\eta + \Sigma_i \eta_i d_i = a.$$

The computation

$$M(a) = M(\delta c) = \delta(M(c)) = \delta(M(\eta + \Sigma_i \eta_i e_i)) = \delta(M(\eta) + \Sigma_i M(\eta_i e_i))$$

$$= \delta(M(\eta) + \Sigma_i M(\eta_i) e_i) = \delta(M(\eta)) = \delta f$$

shows that a is an (E, Δ)-zero of $[M(y) - \delta f]_{\mathrm{E},\Delta}$. Also, the computation $M(c) = M(\eta + \Sigma_i \eta_i e_i) = M(\eta) + \Sigma_i e_i M(\eta_i) = M(\eta) = f$ demonstrates the c is an (E, Δ)-zero of $[\delta y - a, M(y) - f]_{E,\Delta} \subset F\{y\}_{E,\Delta}$.

It remains to be shown that $a, \epsilon a, \dots, \epsilon^{n-1} a$ are linearly independent over F^{Δ} modulo δF. Since $d_1, \dots, d_n$ are linearly independent over F^{Δ} modulo δF, Proposition 2.140 implies that $(F\langle e_1, \dots, e_n\rangle_{\mathrm{E},\Delta})^{\Delta} = F^{\Delta}$ and $e_1, \dots, e_n$ are algebraically independent over F. Because $\eta_1, \dots, \eta_n$ is are linearly independent over $F^{\mathrm{E},\Delta}$, the matrix $(\epsilon^{i-1}\eta_j)_{i=1,\dots,n; j=1,\dots,n}$ is invertible

[24, Theorem 1, page 86], and, therefore, the map φ of $F\langle e_1, \ldots, e_n\rangle_{\mathrm{E},\Delta}$ defined by $\varphi(e_i) = \Sigma_j\ \epsilon^{i-1}\eta_j e_j$ is an automorphism of $F\langle e_1, \ldots, e_n\rangle_{\mathrm{E},\Delta}$ over F. The composite ρ of this with the translation automorphism that sends $\varphi(e_i)$ to $\epsilon^{i-1}\eta + \varphi(e_i)$ is an automorphism of $F\langle e_1, \ldots, e_n\rangle_{\mathrm{E},\Delta}$ such that $\rho(e_i) = \epsilon^{i-1}\eta + \Sigma_j\ \epsilon^{i-1}\eta_j e_j$. Therefore, $\rho(e_1), \ldots, \rho(e_n)$ are algebraically independent over F, and $(F\langle\rho(e_1), \ldots, \rho(e_n)\rangle_{\mathrm{E},\Delta})^{\Delta} = F^{\Delta}$. Proposition 2.140 implies that $\delta(\rho(e_1)), \ldots, \delta(\rho(e_n))$ are linearly independent over F^{Δ} modulo δF. The computation

$$\delta(\rho(e_1)) = \delta(\eta + \Sigma_j\ \eta_j e_j) = \delta\eta + \Sigma_j\ \eta_j\delta(e_j) = \delta\eta + \Sigma_j\ \eta_j d_j = a,$$

$$\delta(\rho(e_2)) = \delta(\epsilon\eta + \Sigma_j\ \epsilon\eta_j e_j) = \delta\epsilon\eta + \Sigma_j\ \epsilon\eta_j\delta(e_j) = \delta\epsilon\eta + \Sigma_j\ \epsilon\eta_j d_j = \epsilon a,$$

$$\delta(\rho(e_3)) = \epsilon^2 a, \ldots,$$

$$\delta(\varphi(e_i)) = \epsilon^{i-1}a, \ldots,$$

$$\delta(\varphi(e_n)) = \epsilon^{n-1}a$$

completes the proof. □

A particularly simple example may be obtained by taking, in this last corollary, $F = \mathbb{C}(t, x)$ $(\epsilon t = 1, \epsilon x = 0, \delta t = 0, \delta x = 1)$, $d_i = 1/(x - i)$ for $i = 0, \ldots, n-1$, $e_i = \ln(x - i)$, $M = \epsilon^n y$, $f = 0$ and $\eta = 0$. By Corollary 2.135, $1/(x - i)$ for $i = 1, \ldots, n$ are linearly independent over $\mathbb{C}(t)$ modulo δF. Let $\eta_1 = 1, \ldots, \eta_n = t^{n-1}$ be a fundamental system for $\epsilon^n y$, and let

$$c = \ln x + t\ln(x - 1) + \cdots + t^{n-1}\ln(x - (n - 1)).$$

Then, $F\langle c\rangle_{\mathrm{E},\Delta} = F(c, \ln x, \ldots, \ln(x-(n-1)))$, and $F\langle c\rangle_{\mathrm{E},\Delta}$ is E-strongly normal over $\mathbb{C}(t,x)$. The operation of an element $g = \alpha_0 + t\alpha_1 + \ldots + t^{n-1}\alpha_{n-1}$ of Galois group $\mathfrak{Z}_{[\epsilon^n y]_{\mathrm{E}}} = \{v \in \mathcal{V} \mid \epsilon^n v = 0\} = \{\alpha_0 + t\alpha_1 + \ldots + t^{n-1}\alpha_{n-1} \mid \alpha_i \in \mathbb{C}\}$ is defined by $gc = (\alpha_0 + \ln x) + t(\alpha_1 + \ln(x-1)) + \cdots + t^{n-1}(\alpha_{n-1} + \ln(x-(n-1)))$. If $f = x$, η may be taken to be $t^n x/(n)!$. Then

$$c = t^n x/(n)! + \ln x + t\ln(x-1) + \cdots + t^{n-1}\ln(x-(n-1)),$$

and the Galois group is the same.

6.3 G_m^{E}-extensions

First it is shown that certain (E,Δ)-extensions with no new Δ-constants are G_m^{E}-extensions. Let $\mathcal{K} = \mathcal{U}^{\Delta}$, and let $\Delta = \{\delta_1, \ldots, \delta_m\}$. Then, for each E-$\mathcal{K}$-subgroup G of G_m, an E-strongly normal extension is constructed with G as its Galois group.

The symbol '$\mathcal{U}^*$' will also denote the (E,Δ)-$\mathcal{F}$-group (Corollary 6.249) that is (E,Δ)-$\mathcal{F}$-isomorphic to G_m^{E}. Let $\kappa \in \mathfrak{F}(\mathcal{U}^*)$ be the canonical coordinate function on $\mathcal{U}^*$. Then, $\delta_i\kappa/\kappa \in \mathfrak{F}(\mathcal{U}^*)$, and the E-$\mathcal{F}$-mapping $l\Delta = (\delta_1\kappa/\kappa, \ldots, \delta_m\kappa/\kappa) : \mathcal{U}^* \longrightarrow \mathcal{U}^m$ [25, Proposition 6, page 129] is the logarithmic derivative on $\mathcal{U}^*$ relative to Δ [24, Example 1, page 352]. By [25, Proposition 3, page 89], it is an (E,Δ)-$\mathcal{F}$-homomorphism. The kernel of $l\Delta$ is the (E,Δ)-$\mathcal{F}$-subgroup $\mathcal{K}^*$, which is the set of (E,Δ)-zeros of the (E,Δ)-ideal $[\delta_1 y, \ldots, \delta_m y] \subset \mathcal{F}\{y\}_{\mathrm{E},\Delta}$ contained in $\mathcal{U}^*$. By forgetting the E-structure, $\mathcal{K}^*$ may be considered to be an E-$\mathcal{F}$-group E-$\mathcal{F}$-isomorphic to G_m^{E} (Corollary 6.248).

Definition 6.267 *An element* $\alpha \in \mathcal{U}^*$ *is* Δ-exponential *over* $\mathcal{F}$ *if* $(\alpha^{-1}\delta_1\alpha, \ldots, \alpha^{-1}\delta_m\alpha) \in \mathcal{F}^m$*; that is, if for suitable elements* $a_1, \ldots, a_m \in \mathcal{F}$*,* α *satisfies the system of differential equations*

$$\delta_i\alpha = a_i\alpha \quad (1 \leq i \leq m).$$

Let α be Δ-exponential over $\mathcal{F}$, and suppose that the field of Δ-constants of $\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}$ is $\mathcal{C}$ $(= \mathcal{F}^{\Delta})$. For any isomorphism σ of $\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}$ over $\mathcal{F}$ and $\delta \in \Delta$,

$$(\alpha^{-1}\sigma\alpha)^{-1}\delta(\alpha^{-1}\sigma\alpha) = (\alpha^{-1}\sigma\alpha)^{-1}[\delta(\alpha^{-1})\sigma\alpha + \alpha^{-1}\delta(\sigma\alpha)]$$

$$= (\sigma\alpha)^{-1}\alpha[-\alpha^{-1}\delta\alpha\ \alpha^{-1}\sigma\alpha + \alpha^{-1}\delta(\sigma\alpha)] = -\alpha^{-1}\delta\alpha + (\sigma\alpha)^{-1}\delta(\sigma\alpha)$$

$$= -\alpha^{-1}\delta\alpha + \sigma(\alpha^{-1}\delta\alpha) = -\alpha^{-1}\delta\alpha + \alpha^{-1}\delta\alpha = 0.$$

Therefore,

$$l\Delta(c(\alpha^{-1}\sigma\alpha)) = ((\alpha^{-1}\sigma\alpha)^{-1}\delta_1(\alpha^{-1}\sigma\alpha), \ldots, (\alpha^{-1}\sigma\alpha)^{-1}\delta_m(\alpha^{-1}\sigma\alpha)) = 0.$$

Hence the element $c(\sigma) = \alpha^{-1}\sigma\alpha$ is in the kernel of $l\Delta$ and is a Δ-constant. Just as in the case of an element Δ-primitive over $\mathcal{F}$, $\mathcal{F}\langle\alpha\rangle$ is E-strongly normal over $\mathcal{F}$ because

$$\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}\sigma(\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}) = \mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}\mathcal{F}\langle\sigma\alpha\rangle_{\mathrm{E},\Delta}$$

$$= \mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}\mathcal{F}\langle\alpha \cdot c(\sigma)\rangle_{\mathrm{E},\Delta} = \mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}\mathcal{C}\langle c(\sigma)\rangle_{\mathrm{E},\Delta}.$$

The mapping $c : G(\mathcal{F}\langle\alpha\rangle/\mathcal{F}) \to G_m^{\mathrm{E}}$ is clearly a group homomorphism. It is injective because $1 = c(\sigma) = \alpha^{-1}\sigma\alpha$ implies $\alpha = \sigma\alpha$ and $\sigma = \mathrm{id}_{\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}}$.

Proposition 6.268 *Let α be a Δ-exponential over $\mathcal{F}$, and suppose that $\mathcal{C} = (\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta})^{\Delta}$. Then, each (E,Δ)–$\mathcal{F}$-isomorphism σ of $\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}$ into $\mathcal{U}$ is of the form $\sigma\alpha = \alpha \cdot c(\sigma)$ for $c(\sigma) \in \mathcal{K}^*$. In addition, $\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}$ is* E*-strongly normal over $\mathcal{F}$, and the mapping $c : G(\mathcal{F}\langle\alpha\rangle/\mathcal{F}) \longrightarrow G_m^{\mathrm{E}}$ is an injective* E-$\mathcal{C}$*-homomorphism of* E*-groups relative to the* E*-field $\mathcal{K}$. Consequently, $\mathcal{F}\langle\alpha\rangle_{\mathrm{E},\Delta}$ is a G_m^{E}-extension of $\mathcal{F}$.*

Proof: The proof is the same as Proposition 5.237. □

Proposition 6.269 *Assume $\Delta = \{\delta\}$, and let $\mathcal{C}$ be an* E*-subfield of $\mathcal{K}$ over which $\mathcal{K} = \mathcal{U}^{\Delta}$ is* E*-universal. Let G be a connected* E-$\mathcal{C}$*-subgroup of G_m^{E}. Let $\mathfrak{P}$ the prime* E*-ideal in $\mathcal{C}\{y\}_{\mathrm{E}}$ defining G. Let b be a generic zero in $\mathcal{U}$ of $\mathfrak{P}_{\Delta,E} \subset \mathcal{C}\{y\}_{\mathrm{E},\Delta}$. Let $a = \delta b/b$. Put $\mathcal{F} = \mathcal{C}\langle a\rangle_{E,\Delta}$, and $\mathcal{G} = \mathcal{F}\langle b\rangle_{E,\Delta}$. Then $\mathcal{G}$ over $\mathcal{F}$ is an* E*-strongly normal extension with Galois group G.*

Proof: Although the proof is the same as Proposition 5.239, reading it in the simplest cases can be beneficial. Observe that $\mathcal{G}$ is generated by a Δ-exponential element b over $\mathcal{F}$ and has no new Δ-constants because $\mathcal{G}^{\Delta} = \mathcal{F}^{\Delta} = \mathcal{K}$ by the No New Δ''-Constant Lemma 5.230. By the previous proposition, c, defined above, is an injective E-$\mathcal{K}$-homomorphism from $G(\mathcal{G}/\mathcal{F})$ to G_a. It is obviously onto G because, for any $v \in G(\mathcal{K})$, the (E,Δ)-$\mathcal{F}$-isomorphism of $\mathcal{G}$ over $\mathcal{F}$ defined by $b \longrightarrow b \cdot v$ defines an E-strongly normal isomorphism of $\mathcal{G}$ over $\mathcal{F}$ with image under c equal to v. Therefore the Galois group is E-$\mathcal{C}$-isomorphic to G because a bijective E-homomorphism is an isomorphism of E-groups [25, Corollary 4, p. 97]. □

Now, for simplicity, assume $\mathrm{E} = \{\epsilon\}$ and $\Delta = \{\delta\}$ in the remainder of this section. The following proposition exhibits a Galois correspondence under certain assumptions.

Recall that for every positive integer k, the E-$\mathfrak{F}$-homomorphism $\mu_k : G_m^{\mathrm{E}} \to G_m^{\mathrm{E}}$ defined by $\mu_k(v) = v^k$ for $v \in G_m^{\mathrm{E}}$ is surjective with finite kernel [25, Corollary 2, page 238]. The kernel $\ker \mu_k$ of μ_k is an E-$\mathfrak{F}$-subgroup of G_m^{E} [25, Corollary 2, page 96].

Proposition 6.270 *Assume that* $\mathfrak{C} = \mathfrak{F}^{\Delta}$ *is constrainedly closed, and assume that* $\mathfrak{G}$ *is an* E*-strongly normal extension of* $\mathfrak{F}$ *that is* (E, Δ)*-generated over* $\mathfrak{F}$ *by a* Δ*-exponential* b *over* $\mathfrak{F}$*. Let* $G = \mathrm{Gal}(\mathfrak{G}/\mathfrak{F}) \subseteq G_m^{\mathrm{E}}$ *be the Galois group. Then,* $G = \ker \mu_r$ *for a positive integer* r*, or* $G_{\mathfrak{L}} = \{v \in G_m^{\mathrm{E}} \mid L(l\mathrm{E}(v)) = 0 \text{ for } L(\kappa) \in \mathfrak{L}\}$ *where* $\mathfrak{L} \subset \mathfrak{F}\{\kappa\}_{\mathrm{E}}$ *is a linear* E*-ideal and* κ *is the canonical coordinate function on* G_m^{E}*.*

1. *If* $G = \ker \mu_r$*, then each* E-$\mathfrak{C}$*-subgroup* H *is* E-$\mathfrak{C}$*-isomorphic to* $\ker \mu_s$ *for some divisor* s *of* r*, and* $\mathfrak{G}^H = \mathfrak{F}\langle b^s\rangle_{\mathrm{E},\Delta}$*.*

2. *If* $G = G_{\mathfrak{L}}$*, then each* E-$\mathfrak{C}$*-subgroup* H *is* E-$\mathfrak{C}$*-isomorphic to either* $\ker \mu_s$ *for some positive integer* s *or* $G_{\mathfrak{L}'}$ *where* $\mathfrak{L}' \subset \mathfrak{F}\{\kappa\}_{\mathrm{E}}$ *is a linear* E*-ideal such that* $\mathfrak{L} \subseteq \mathfrak{L}'$*. If* $H = \ker \mu_s$*,* $\mathfrak{G}^H = \mathfrak{F}\langle b^s\rangle_{\mathrm{E},\Delta}$*. If* $H = G_{\mathfrak{L}'}$*,* $\mathfrak{G}^H = \mathfrak{F}\langle (L(\epsilon b/b))_{L \in \mathfrak{L}'}\rangle_{\mathrm{E},\Delta}$*.*

Remark 6.271 *It should be possible to prove that every* E*-strongly normal extension of* $\mathfrak{F}$ *with Galois group* $G \subseteq G_m^{\mathrm{E}}$ *is* (E, Δ)*-generated by a* Δ*-primitive element of* $\mathfrak{U}$ *by the method of* [24, Corollary 2, page 427].

Proof: The assertions about the E-$\mathcal{C}$-subgroups of G_m^{E} were proven in Corollary 6.255. Assume $G = \ker\ \mu_r$. For each $\sigma \in \ker\ \mu_r$, the action of σ on $\mathcal{G}$ is by an (E, Δ)-isomorphism σ_ζ defined by $\sigma_\zeta(b) = \zeta b$ for some $\zeta \in \mathcal{K}$ (Proposition 6.268). Since $\sigma^r = 1$, $\zeta^r = 1$. By Corollary 6.255, each E-$\mathcal{C}$-subgroup H of $\ker\ \mu_r$ is of the form $\ker\ \mu_s$ for s dividing r. Since $\mathcal{C}$ is constrainedly closed, the Fundamental Theorem 4.207 implies that there is a Galois correspondence that associates to each such E-$\mathcal{C}$-subgroup $\ker\ \mu_s$ of G the (E, Δ)-subfield $\mathcal{H} = \mathcal{G}^{\ker\ \mu_s}$. Therefore, to prove the proposition it suffices to show $\mathcal{G}^{\ker\ \mu_s} = \mathcal{F}\langle b^s\rangle_{\mathrm{E},\Delta}$. Clearly, $\mathcal{F}\langle b^s\rangle_{\mathrm{E},\Delta}$ is invariant under $\ker\ \mu_s$. Thus, $\mathcal{G}^{\ker\ \mu_s} \supseteq \mathcal{F}\langle b^s\rangle_{\mathrm{E},\Delta}$. Assume $\mathcal{G}^{\ker\ \mu_s}$ properly contains $\mathcal{F}\langle b^s\rangle_{\mathrm{E},\Delta}$. Then, by the Fundamental Theorem 4.207, there exist an E-$\mathcal{C}$-subgroup $\ker\ \mu_{s'} \subseteq \ker\ \mu_r$ with fixed field $\mathcal{F}\langle b^s\rangle_{\mathrm{E},\Delta}$ that properly contains $\ker\ \mu_s$.

$$\begin{tikzcd}
G = \ker\ \mu_r \arrow[r] & \mathcal{F} \arrow[d, "\cap"'] \\
\sigma_\xi \in \ker\ \mu_{s'} \arrow[u, "\cup"] \arrow[r] & \mathcal{F}\langle b^s\rangle_{\mathrm{E},\Delta} \arrow[d, "\cap"'] \\
\ker\ \mu_s \arrow[u, "\cup"] \arrow[r] & \mathcal{G}^{G(s)}
\end{tikzcd}$$

Let $\sigma_\xi \in \ker\ \mu_{s'}$ such that $\sigma_\xi \notin \ker\ \mu_s$. Since $\mathcal{F}\langle b^s\rangle_{\mathrm{E},\Delta}$ is invariant under σ_ξ,

$$b^s = \sigma_\xi(b^s) = (\sigma_\xi(b))^s = (\xi \cdot b)^s = \xi^s \cdot b^s.$$

That is $\xi^s = 1$, and $\sigma_\xi \in \ker\ \mu_s$. This contradiction shows that $\mathcal{G}^{\ker\ \mu_s} = \mathcal{F}\langle b^s\rangle_{\mathrm{E},\Delta}$.

Assume $G = G_{\mathcal{L}}$. If $H = \ker\ \mu_s$, $\mathcal{G}^H = \mathcal{G}^H = \mathcal{F}\langle b^s\rangle_{\mathrm{E},\Delta}$ by reasoning similar to the first part of this proof. Otherwise, $H = G_{\mathfrak{L}'}$ where $\mathfrak{L}' \subset \mathcal{F}\{\kappa\}_{\mathrm{E}}$ is a linear E-ideal such that $\mathfrak{L} \subseteq \mathfrak{L}'$ (Corollary 6.255). Furthermore, $\sigma_\zeta \in H$ if and only if $l\mathrm{E}(\zeta) \in \mathfrak{Z}_{\mathfrak{L}'}$. Because, for all linear $L' \in \mathfrak{L}'$ and all $\sigma_\zeta \in H$,

$$\sigma_\zeta(L'(\epsilon b/b)) = L'(\epsilon(\sigma_\zeta(b))/\sigma_\zeta(b)) = L'(\epsilon(\zeta b))/\zeta b)$$

$$= L'(\epsilon b/b + \epsilon\zeta/\zeta) = L'(\epsilon b/b) + L'(\epsilon\zeta/\zeta) = L'(\epsilon b/b),$$

the (E, Δ)-field $\mathcal{F}\langle(L'(\epsilon b/b))_{L' \in \mathfrak{L}'}\rangle_{\mathrm{E},\Delta}$ is invariant under $G_{\mathfrak{L}'}$. Therefore, $\mathcal{G}^H \supseteq \mathcal{F}\langle(L'(\epsilon b/b))_{L' \in \mathfrak{L}'}\rangle_{\mathrm{E},\Delta}$.

To complete the proof of the proposition, assume $\mathcal{G}^H$ properly contains $\mathcal{F}\langle(L'(\epsilon b/b))_{L' \in \mathfrak{L}'}\rangle_{\mathrm{E},\Delta}$. Since $\mathfrak{C}$ is constrainedly closed, the Fundamental Theorem 4.207 implies that there exist an E-$\mathfrak{C}$-subgroup $I \subseteq G$ with fixed field $\mathcal{F}\langle(L'(\epsilon b/b))_{L' \in \mathfrak{L}'}\rangle_{\mathrm{E},\Delta}$ that properly contains G^H.

$$\begin{tikzcd}
G \arrow[rr] & & \mathcal{F} \arrow[d, "\cap"'] \\
\sigma_\xi \in I \arrow[u, "\cup"] \arrow[rr] & & \mathcal{F}\langle(L'(\epsilon b/b))_{L' \in \mathfrak{L}'}\rangle_{\mathrm{E},\Delta} \arrow[d, "\cap"'] \\
H \arrow[u, "\cup"] \arrow[rr] & & \mathcal{G}^H
\end{tikzcd}$$

Let $\sigma_\xi \in I$ such that $\sigma_\xi \notin H$. Since $\mathcal{F}\langle(L'(\epsilon b/b))_{L' \in \mathfrak{L}'}\rangle_{\mathrm{E},\Delta}$ is invariant under σ_ξ, for each $L' \in \mathfrak{L}'$,

$$L'(\epsilon b/b) = \sigma_\xi(L'(\epsilon b/b)) = L'(\sigma_\xi(\epsilon b/b)) = L'(\epsilon(\sigma_\xi b)/\sigma_\xi(b)) = L'(\epsilon(\xi b)/\xi b)$$

$$= L'(\epsilon\xi/\xi + \epsilon b/b) = L'(\epsilon\xi/\xi) + L'(\epsilon b/b).$$

Therefore, $L'(\epsilon\xi/\xi) = 0$, and $\sigma_\xi \in H$. This contradiction shows that $\mathcal{G}^H = \mathcal{F}\langle (L'(\epsilon b/b))_{L' \in \mathcal{L}'} \rangle_{\mathrm{E},\Delta}$. □

The following proposition characterizes certain E-exponential G_m^{E}-extensions by the structure of $\mathcal{F}$.

Proposition 6.272 *Let $\mathcal{G}$ be an* E*-strongly normal extension of $\mathcal{F}$ that is* (E, Δ)*-$\mathcal{F}$-generated by an Δ-exponential c over $\mathcal{F}$. Let $a = \delta c/c$, let $\mathcal{L}_1^* = \{L(y) \in \mathcal{C}\{y\}_{\mathrm{E},1} \mid L(\epsilon a) \in \delta\mathcal{F}\}$, and let $\mathcal{L}^* = [\mathcal{L}_1^*]_{\mathrm{E}}$. Then* $\mathrm{Gal}(\mathcal{G}/\mathcal{F}) = G_{\mathcal{L}^*}$

Proof: By Proposition 6.269, $\mathrm{Gal}(\mathcal{G}/\mathcal{F}) \subset G_m^{\mathrm{E}}$. Corollary 6.255 implies $\mathrm{Gal}(\mathcal{G}/\mathcal{F}) = G_{\mathcal{L}}$ for some E-ideal $\mathcal{L} \subset \mathcal{C}\{y\}_{\mathrm{E}}$. Then $v \in G_{\mathcal{L}}$ if and only if $L(\epsilon v/v) = 0$ for every $L(y) \in \mathcal{L}$.

Let $b = \epsilon c/c$. Clearly, $\delta b = \epsilon a$. Let $L(y) \in \mathcal{L}$ be of degree one. Then, $L(\epsilon v/v) = 0$ for every $v \in G$. And

$$\sigma_v(L(\epsilon c/c)) - L(\epsilon c/c) = L(\sigma_v(\epsilon c/c)) - L(\epsilon c/c)$$

$$= L((\epsilon\sigma_v(c))/\sigma_v(c)) - L(\epsilon c/c) = L(\epsilon(vc)/vc) - L(\epsilon c/c)$$

$$= L(\epsilon c/c + \epsilon v/v) - L(\epsilon c/c) = L(\epsilon c/c) + L(\epsilon v/v) - L(\epsilon c/c) = L(\epsilon v/v) = 0.$$

Therefore, $\sigma_v(L(\epsilon c/c)) = L(\epsilon c/c)$, and $L(\epsilon c/c)$ is invariant under all elements of G. Thus $L(\epsilon c/c) \in \mathcal{F}$, and $L(\epsilon c/c) = f$ for some $f \in \mathcal{G}$. The computation

$$L(\epsilon a) = L(\delta b) = \delta(L(b)) = \delta(L(\epsilon c/c)) = \delta f$$

shows $L \in \mathcal{L}^*$, and $\mathcal{L} \subseteq \mathcal{L}^*$ since $\mathcal{L}$ is generated by elements of degree 1.

On the other hand, let $L(y) \in \mathcal{L}_1^*$. Then $L(\epsilon a) = \delta f$ for $f \in \mathcal{F}$, and $L(b) - f$ is a Δ-constant because $\delta(L(b) - f) = L(\delta b) - \delta f = L(\epsilon a) - \delta f = 0$. Therefore, $L(b) - f \in \mathcal{C} \subseteq \mathcal{F}$, and $L(b) \in \mathcal{F}$. Hence, $\sigma_v(L(b)) = L(b)$, and, for all $\sigma_v \in G$, the computation

$$L(\epsilon v/v) = L(\sigma_v(\epsilon c/c) - \epsilon c/c) = L(\sigma_v(\epsilon c/c)) - L(\epsilon c/c)$$

$$= \sigma_v(L(\epsilon c/c)) - L(\epsilon c/c) = \sigma_v(L(b)) - L(b) = 0$$

shows $L(y) \in \mathcal{L}$ and $\mathcal{L} \supseteq \mathcal{L}^*$ since $\mathcal{L}^*$ is generated by elements of degree 1. □

Corollary 6.273 *Let $\mathcal{H}$ be an algebraically closed (E, Δ)-field such that $\mathcal{H}^\Delta = \mathcal{H}$, and let $\mathcal{F} = \mathcal{H}\langle x\rangle_{\mathrm{E},\Delta}$, where $x \in \mathcal{U}$, $\epsilon x = 0$ and $\delta x = 1$. Then, there is no E-strongly normal extension of $\mathcal{F}$ that is (E, Δ)-generated by a Δ-exponential over $\mathcal{F}$ and has Galois group E-$\mathcal{H}$-isomorphic to G_m^{E}.*

Remark 6.274 *This remains true if the hypothesis that $\mathcal{H}$ be an algebraically closed is omitted; the following proof must be modified to take the structure of irreducibles into account in the partial fraction decomposition.*

Proof: Assume that such an E-strongly normal extension $\mathcal{G}$ of $\mathcal{F}$ exists. Let $b \in \mathcal{U}$ be a Δ-exponential over $\mathcal{F}$ such that $\delta b = ab$ for $a \in \mathcal{F}$, and $\mathcal{G} = \mathcal{F}\langle b\rangle_{\mathrm{E},\Delta}$. Let $a = p(x) + \Sigma_{i,j} \dfrac{h_{i,j}}{(x - h_i)^j}$ for $p(x) \in \mathcal{H}[x]$ and $h_i, h_{i,j} \in \mathcal{H}$, be the partial fraction decomposition of a.

By Proposition 6.272, since the Galois group is G_m^{E}, there does not exist a non-zero $L(y) \in \mathcal{H}\{y\}_{\mathrm{E},1}$ such that $L(\epsilon a) \in \delta\mathcal{F}$. If all of the $\epsilon h_{i,1} = 0$, then

$\epsilon a \in \delta\mathcal{F}$, and $L(\epsilon a) \in \delta\mathcal{F}$ for $L(y) = y$. However, if there exists a non-zero $h_{i,1}$, there exists a non-zero $L(y) =$

$$\begin{vmatrix} \epsilon h_{1,1} & \epsilon h_{2,1} & \ldots & \epsilon h_{r,1} & y \\ \epsilon^2 h_{1,1} & \epsilon^2 h_{2,1} & \ldots & \epsilon^2 h_{r,1} & \epsilon y \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \epsilon^r h_{1,1} & \epsilon^{r+1} h_{2,1} & \ldots & \epsilon^{r+1} h_{r,1} & \epsilon^r y \end{vmatrix}$$

$\in \mathcal{H}\{y\}_{\mathrm{E},1}$ such that the finitely many $\epsilon h_{i,1}$ span over $\mathcal{H}^{\mathrm{E},\Delta}$ the linear space of E-zeros of $L(y)$. By Lemma 6.264, since $L(\epsilon h_{i,1}) = 0$ for all i, $L(\epsilon a) \in \delta\mathcal{F}$. $\square$

Let F be a Δ-field, and let $z_1, \ldots, z_n$ be algebraic indeterminates over F. For an element $f \in F(z_1, \ldots, z_n)$, choose $g, h \in F[z_1, \ldots, z_n]$ such that $f = g/h$. If $f \neq 0$, the *generalized degree* $\mathrm{Deg}(f) = \deg(g) - \deg(h)$ is independent of the possible choices for g and h used to represent f. By considering the formulas for addition and differentiation of fraction, $\mathrm{Deg}(\frac{df}{dz_i}) < \mathrm{Deg}(f)$ for all i, and, for $a, b \in F(z_1, \ldots, z_n)$, $\mathrm{Deg}(ab) = \mathrm{Deg}(a) + \mathrm{Deg}(b)$, $\mathrm{Deg}(a + b) \leq \max(\mathrm{Deg}(a), \mathrm{Deg}(b))$, and $\mathrm{Deg}\ \delta f \leq \mathrm{Deg}\ f$ for $\delta \in \Delta$.

Lemma 6.275 *Let $z_1, \ldots, z_n$ be algebraic indeterminates over F, and let $a_1, \ldots, a_n \in F$. Define a Δ-structure on $F(z_1, \ldots, z_n)$ by $\delta z_i = a_i$ for $i = 1, \ldots, n$. Let $a \in F(z_1, \ldots, z_n)$ such that $\mathrm{Deg}(a) > 0$. If $f \in F(z_1, \ldots, z_n)$ is a Δ-zero of $\delta y - ay \in F(z_1, \ldots, z_n)\{y\}_\Delta$ for a Δ-indeterminate y over $F(z_1, \ldots, z_n)$, then $f = 0$.*

Proof: If $f \neq 0$,

$$\mathrm{Deg}(af) = \mathrm{Deg}(a) + \mathrm{Deg}(f) > \mathrm{Deg}(f) \geq \mathrm{Deg}(\delta f),$$

which can not be true because $af = \delta f$. $\square$

The following proposition shows how to construct an E-strongly normal extension for a given connected E-subgroup of G_m^{E}.

Proposition 6.276 *Assume* $E = \{\epsilon\}$ *and* $\Delta = \{\delta\}$. *Let the* (E, Δ)*-field* $\mathcal{U}$ *be* (E, Δ)*-universal over the* (E, Δ)*-field* $\mathcal{D}$ *of* Δ*-constants.*

1. *Let* $G_{\mathcal{L}} \subset G_m^{\mathrm{E}}$ *be an* E*-subgroup of* G_m^{E} *defined over an* (E, Δ)*-subfield* $\mathcal{D} \subset \mathcal{U}^{\Delta}$ *by the* E*-ideal* $\mathcal{L} = \{L(y)\}_{\mathrm{E}} \subset \mathcal{D}\{y\}_{\mathrm{E}}$ *where* $L(y) \in \mathcal{D}\{y\}_{\mathrm{E},1}$ *of positive order* n *with the coefficient of the highest order term equal to* 1.

2. *Let the* (E, Δ)*-field* $\mathcal{C} \subset \mathcal{U}^{\Delta}$ *be a strongly normal extension of* $\mathcal{D}$*, considered as an* E*-field, that is* E*-generated over* $\mathcal{D}$ *by a fundamental system* $1, \eta_1, \ldots, \eta_n$ *of* E*-zeros of* $L(\epsilon y)$.

3. *Let* $\mathcal{E}$ *be a subfield of* $\mathcal{C}$ *such that* η_k*, for some* k*, is transcendental over* $\mathcal{E}$*,* $\mathcal{C} = \mathcal{E}(\eta_k)$ *and the generalized degree in the indeterminate* x Deg $\eta_i \neq 1$ *for* $i \neq k$.

4. *Let the* (E, Δ)*-field* $\mathcal{B} \subset \mathcal{U}^{\mathrm{E}}$ *be finitely* Δ*-generated over* $\mathcal{C}^{\mathrm{E}}$*, satisfy the condition* $\mathcal{B}^{\Delta} = \mathcal{C}^{\mathrm{E}}$*, and contain the elements* $f_1, \ldots, f_n$ *that are assumed to be linearly independent over* $\mathcal{B}^{\Delta}$ *modulo* $\delta\mathcal{B}$.

5. *Let* $\mathcal{F} = \mathcal{C} \cdot \mathcal{B}$*, and let* $f \in \mathcal{F}$*. Let* $\eta \in \mathcal{F}$ *be an* (E, Δ)*-zero* η *of* $L(\epsilon y) - f \in \mathcal{F}\{y\}_{\mathrm{E},\Delta}$.

6. *For each* $i = 1, \ldots, n$, *let* $g_i \in \mathcal{U}^{\mathrm{E}}$ *be a* δ*-primitive of* f_i, *i.e.* $\delta g_i = f_i$.

7. *Let* $\mathcal{H} = \mathcal{F}\langle g_1, \ldots, g_n\rangle_{\mathrm{E},\Delta}$, *and let* c *be an* $\mathcal{H}$*-generic* (E, Δ)*-zero of*

$$N = [\delta y - (\delta\eta + \Sigma\ \eta_i f_i)y,\ \epsilon y - (\epsilon\eta + \Sigma\ \epsilon\eta_i g_i)y]_{\mathrm{E},\Delta} \subset \mathcal{H}\{y\}_{\mathrm{E},\Delta}.$$

$$\begin{tikzcd}
\mathcal{C} \arrow[r] & \mathcal{F} = \mathcal{C} \cdot \mathcal{B} \arrow[r] & \mathcal{H} = \mathcal{F}\langle g_1, \ldots, g_n\rangle_{\mathrm{E},\Delta} \\
\mathcal{D} \arrow[u] \arrow[r] & \mathcal{D} \cdot \mathcal{B} \arrow[u] \arrow[r] & \mathcal{D}\mathcal{B}\langle g_1, \ldots, g_n\rangle_{\mathrm{E},\Delta} \arrow[u] \\
\mathcal{C}^{\mathrm{E}} = \mathcal{B}^{\Delta} \arrow[u] \arrow[r] & \mathcal{B} \arrow[u] \arrow[r] & \mathcal{B}\langle g_1, \ldots, g_n\rangle_{\mathrm{E},\Delta} \arrow[u]
\end{tikzcd}$$

Then $\mathcal{H}\langle c\rangle_{\mathrm{E},\Delta} = \mathcal{F}\langle c\rangle_{\mathrm{E},\Delta}$ *is* E*-strongly normal over* $\mathcal{F}$ *with Galois group* $G_{\mathcal{L}}$.

Remark 6.277 *If the elements of the* (E, Δ)*-fields in the proposition are of analytic functions of two variables,* c *may be taken to be* $\exp(\eta + \Sigma\ \eta_i g_i)$.

Remark 6.278 *Condition 2 of the proposition holds for the example following the proof of this proposition. However, is not true in general, e.g.* $\eta_1 = \sin t$, $\eta_2 = \cos t$ *and* $L(y) = \delta^2 y + y$.

Proof: Since that $f_1, \ldots, f_n$ are linearly independent over $\mathcal{B}^{\Delta}$ modulo $\delta\mathcal{B}$ is condition 1 of Propositon 2.140, this proposition implies $(\mathcal{B}\langle g_1, \ldots, g_n\rangle_{\mathrm{E},\Delta})^{\Delta} = \mathcal{B}^{\Delta}$, $g_1, \ldots, g_n$ are algebraically independent over $\mathcal{B}$, and $1, g_1, \ldots, g_n$ are linearly independent over $\mathcal{B}$. Since $\mathcal{B}\langle g_1, \ldots, g_n\rangle_{\mathrm{E},\Delta}$ and $\mathcal{C}$ are linearly disjoint over $\mathcal{C}^{\mathrm{E}} = \mathcal{B}^{\Delta}$ [24, Corollary 1, page 87], $\mathcal{H}^{\Delta} = \mathcal{F}^{\Delta}$ [24, Corollary 2, page 88].

Since $\mathcal{B}\langle g_1, \ldots, g_n\rangle_{\mathrm{E},\Delta}$ and $\mathcal{F}$ are linearly disjoint over $\mathcal{B}$ [26, Proposition 1, page 50], $1, g_1, \ldots, g_n$ are linearly independent over $\mathcal{F}$. Then condition 4 of Propositon 2.140 implies $f_1, \ldots, f_n$ are linearly independent over $\mathcal{F}^{\Delta}$ modulo $\delta\mathcal{F}$ and $g_1, \ldots, g_n$ are algebraically independent over $\mathcal{F}$.

In this paragraph, the following diagram will be used to show that for all positive integers j the only Δ-zero of $\delta y - jay$ in $\mathcal{H}$ is 0.

$$\begin{tikzcd}
\mathcal{C}\mathcal{E}(\eta_k) \arrow[r] & \mathcal{F} = \mathcal{E} \cdot \mathcal{B}(\eta_k) \arrow[r] & \mathcal{H} = \mathcal{E}\mathcal{B}(g_1, \ldots, g_n)(\eta_k) \\
\mathcal{E} \arrow[u] \arrow[r] & \mathcal{E} \cdot \mathcal{B} \arrow[u] \arrow[r] & \mathcal{E}\mathcal{B}(g_1, \ldots, g_n) \arrow[u] \\
\mathcal{C}^{\mathrm{E}} = \mathcal{B}^{\Delta} \arrow[u] \arrow[r] & \mathcal{B} \arrow[u] \arrow[r] & \mathcal{B}(g_1, \ldots, g_n) \arrow[u]
\end{tikzcd}$$

The subfield $\mathcal{E} \subset \mathcal{C}$ such that η_k is transcendental over $\mathcal{E}$ and $\mathcal{C}$ is algebraic over $\mathcal{E}(\eta_k)$ was assumed to exist in Condition 3 of the proposition. Since $\mathcal{E}(\eta_k)$ and $\mathcal{E}\mathcal{B}(g_1, \ldots, g_n)$ are linearly disjoint over $\mathcal{E}$ [26, Proposition 1, page 50], η_k is transcendental over $\mathcal{E}\mathcal{B}(g_1, \ldots, g_n)$. Since $g_1, \ldots, g_n$ are algebraically independent over $\mathcal{F}$ and, therefore, over $\mathcal{E} \cdot \mathcal{B}(\eta_k) \subset \mathcal{F}$, the elements $\eta_k, g_1, \ldots, g_n$ are algebraically independent over $\mathcal{E} \cdot \mathcal{B}$. For x contained in the Δ-field $\mathcal{E}\mathcal{B}(g_1, \ldots, g_n)(\eta_k)$, let Deg x be the generalized degree of x with respect to the indeterminates $\eta_k, g_1, \ldots, g_n$. For all positive integers j, since $\mathrm{Deg}(a + b) \leq \max(\mathrm{Deg}(a), \mathrm{Deg}(b))$, condition 3 implies Deg $ja > 0$. Lemma 6.275 implies the only Δ-zero of $\delta y - jay$ in $\mathcal{E}\mathcal{B}(g_1, \ldots, g_n)(\eta_k)$ is 0.

Since $y = 0$ is a (E, Δ)-zero of N, it follows that N is a proper (E, Δ)-

ideal, and an $\mathcal{H}$-generic (E, Δ)-zero $c \in \mathcal{U}$ exists.

Let $a = \delta\eta + \Sigma\ \eta_i f_i\ (\in \mathcal{F})$ and $b = \epsilon\eta + \Sigma\ \epsilon\eta_i g_i\ (\in \mathcal{H})$. Clearly, $\epsilon a = \delta b$. For any orderly ranking, the set $\{\delta y - ay, \epsilon y - by\}$ is coherent and autoreduced because $\epsilon(\delta y - ay) - \delta(\epsilon y - by) = 0$. No polynomial non-zero $p(y) \in \mathcal{H}[y] \subset \mathcal{H}\{y\}_{\mathrm{E},\Delta}$ is contained in N because if $p(y) \in N$ then because $p(y)$ is partially reduced with respect to $\{\delta y - ay, \epsilon y - by\}$ [25, Proposition 14, page 26] implies $p(y) \in (\delta y - ay, \epsilon y - by)$. This is impossible since $p(y)$ is reduced and non-zero. Therefore, c is not algebraic over $\mathcal{H}$. This and the fact that $\mathcal{H}[c] = \mathcal{H}\{c\}_{\mathrm{E},\Delta}$ imply that c is transcendental over $\mathcal{H}$.

To show that $\mathcal{H}\langle c\rangle_{\mathrm{E},\Delta} = \mathcal{F}\langle g_1, \ldots, g_n\rangle_{\mathrm{E},\Delta}\langle c\rangle_{\mathrm{E},\Delta} = \mathcal{F}\langle c\rangle_{\mathrm{E},\Delta}$, it suffices to demonstrate that $g_i \in \mathcal{F}\langle c\rangle_{\mathrm{E},\Delta}$ for $i = 1, \ldots, n$. First observe that because $\epsilon\eta_1, \ldots, \epsilon\eta_n$ is a fundamental system of zeros for $L(y)$, the Wronskian matrix $(\epsilon^j\eta_i)_{i=1,\ldots,n; j=1,n}$ is invertible. Also, $b = \epsilon c/c \in \mathcal{F}\langle c\rangle_{\mathrm{E},\Delta}$. Therefore, the following system of linear equations obtained by repeatedly differentiating $b = \epsilon\eta + \Sigma_i \epsilon\eta_i g_i$ by ϵ may be solved for $g_1, \ldots, g_n$:

$$b = \epsilon\eta + \Sigma_i \epsilon\eta_i g_i$$

$$\epsilon b = \epsilon^2\eta + \Sigma_i \epsilon^2\eta_i g_i$$

$$\cdots$$

$$\epsilon^{n-1} b = \epsilon^n\eta + \Sigma_i \epsilon^n\eta_i g_i.$$

From this it also follows that $\mathcal{F}(b, \epsilon b, \ldots, \epsilon^{n-1}b) = \mathcal{F}(g_1, \ldots, g_n)$ because $\eta \in \mathcal{F}$. Since $g_1, \ldots, g_n$ are algebraically independent over $\mathcal{F}$, so are $b, \epsilon b, \ldots, \epsilon^{n-1}b$.

For later use, observe that since $\mathcal{F}^{\Delta} = (\mathcal{F}(g_1, \ldots, g_n))^{\Delta} = (\mathcal{F}(b, \epsilon b, \ldots, \epsilon^{n-1} b))^{\Delta}$ and $\delta(\epsilon^i b) = \epsilon^i(\delta b) = \epsilon^i(\epsilon a) = \epsilon^{i+1} a$, Propositon 2.140 implies $\epsilon a, \ldots, \epsilon^n a$ are linearly independent over $\mathcal{B}^{\Delta}$ modulo $\delta\mathcal{B}$ and $1, b, \epsilon b, \ldots, \epsilon^{n-1}$ are linearly independent over $\mathcal{F}$.

It has been demonstrated that $\mathcal{H}^{\Delta} = \mathcal{F}^{\Delta}$. To show $(\mathcal{F}\langle c\rangle_{\mathrm{E},\Delta})^{\Delta} = \mathcal{F}^{\Delta}$, it will now be proved that $(\mathcal{H}\langle c\rangle_{\mathrm{E},\Delta})^{\Delta} = \mathcal{H}^{\Delta}$. Let $\alpha \in \mathcal{G} = \mathcal{H}\langle c\rangle_{\mathrm{E},\Delta}$ be a non-zero Δ-constant. First assume $\alpha \in \mathcal{H}\{c\}_{\Delta,E} = \mathcal{H}[c]$ and is of minimal degree in c. It has been shown that c is transcendental over $\mathcal{H}$. Write $\alpha = a_r c^r + a_{r-1} c^{r-1} + \ldots + a_0$ where $r > 0$ and $a_i \in \mathcal{H}$ for $i = 0, \ldots, r$. Then $\delta\alpha = A_r c^r + A_{r-1} c^{r-1} + \ldots + A_0$ where $A_i = \delta a_i + i a a_i$ for $i = 0$ to r. Since $\delta\alpha = 0$ and the powers of c are linearly independent over $\mathcal{H}$, it follows that $A_i = 0$ for $i = 0, \ldots, r$. Because it has been shown that the only Δ-zero of $\delta y - iay$ in $\mathcal{H}$ is 0, $a_i = 0$ for $i = 1, \ldots, n$. Since $A_0 = 0$, $\delta a_0 = 0$, and $\alpha = a_0 \in \mathcal{F}^{\Delta}$. Similarly, if $1/\alpha \in \mathcal{F}\langle b\rangle_{\Delta,E}[c]$, then $1/\alpha \in \mathcal{F}^{\Delta}$.

Second, if neither α nor $1/\alpha$ is in $\mathcal{H}[c]$, let $\alpha = A/B$ where A and B are in $\mathcal{H}[c]$ of positive degree such that A has the minimal degree among all such choices of A and B. It may be assumed that $\delta B \neq 0$ because otherwise $\delta A = 0$ and $A \in \mathcal{H}^{\Delta}$. Since $\delta\alpha = 0$, $A/B = \delta A/\delta B$. Write $A = a_r c^r +, \ldots, + a_0$, $a_i \in \mathcal{H}$ for $i = 0$ to r, and $B = b_s c^s +, \ldots, + b_0$, $b_i \in \mathcal{H}$ for $i = 0$ to s. Both a_0 and b_0 may not be 0 because then the numerator and the denominator of α may be divided by c resulting in a fraction representing α with a lower degree numerator. If $b_0 = 0$ and $a_0 \neq 0$, divide the numerator and the denominator by a_0, then the derivatives of both have no constant terms and may be divided by c again to produce an equivalent fraction with lower degree numerator. If $b_0 \neq 0$ and $a_0 = 0$, apply the same reasoning. If $b_0 \neq 0$ and $a_0 \neq 0$, from $\delta g f = g \delta f$, by comparing zeroth degree terms in c, it follows that $\delta b_0 a_0 = b_0 \delta a_0$. Therefore $\delta(a_0/b_0) = 0$. Divide the numerator and the denominator both by g_0. The zeroth degree terms in c of both the numerator and the denominator are Δ-constants. Differentiate them and divide both by c to produce an equivalent fraction with lower degree numerator. So, $\alpha \in \mathcal{F}^{\Delta}$.

Since c is a Δ-exponential over $\mathcal{F}$ and $(\mathcal{F}\langle c\rangle_{\mathrm{E},\Delta})^{\Delta} = \mathcal{F}^{\Delta}$, Proposition 6.268 implies $\mathcal{F}\langle c\rangle_{\mathrm{E},\Delta}$ over $\mathcal{F}$ is E-strongly normal. It remains to show that the Galois group G of $\mathcal{F}\langle c\rangle_{\mathrm{E},\Delta}$ over $\mathcal{F}$ is $G_{\mathcal{L}}$. Because c is transcendental over $\mathcal{F}$, G is not finite. By Corollary 6.255, $G = G_{\mathcal{M}}$ for some linear E-ideal $\mathcal{M} \subset \mathcal{F}^{\Delta}\{y\}_{\mathrm{E}}$. Since it may be verified that $L(\epsilon a) = \delta f$, Proposition 6.272 implies $L(y) \in \mathcal{M}$. Let $M(y) \in \mathcal{M}$. Again by Proposition 6.272, $M(\epsilon a) = \delta h$ for some $h \in \mathcal{F}$. Then $M(b) - h \in \mathcal{F}^{\Delta} \subseteq \mathcal{F}$, and $M(b) = h'$ for $h' \in \mathcal{F}$. Since $1, b, \epsilon b, \ldots, \epsilon^{n-1} b$ are linearly independent over $\mathcal{F}$, $M(y)$ has order greater than or equal to the order of $L(Y)$. Hence $\{L(y)\}_{\mathrm{E}} = \mathcal{M} = \mathcal{L}$, and $G = G_{\mathcal{L}}$. □

A particularly simple example may be obtained by taking, in Proposition 6.276, $\mathcal{D} = \mathbb{C}$, $L = \epsilon^n y$, $\mathcal{C} = \mathcal{D}(t)$ with $\epsilon t = 1$ and $\delta t = 0$, $\mathcal{B} = \mathbb{C}(x)$, $\mathcal{F} = \mathbb{C}(t, x)$, $\eta_i = t^i$ for $i = 1, \ldots, n$, $f_i = 1/(x + i - 1)$ for $i = 1, \ldots, n$, $g_i = \ln(x+i-1)$ for $i = 1, \ldots, n$ and $\eta = 0$. A fundamental system of E-zeros of $\epsilon^{n+1} y$ is $1, t, t^2, \ldots, t^n$. By Corollary 2.135, $1/(x), 1/(x+1), \ldots, 1/(x+n-1)$

are linearly independent over $\mathbb{C} = \mathcal{B}^{\Delta}$ modulo $\delta\mathcal{B}$. Then, $a = t/(x) + t^2/(x+1) + \cdots + t^n/(x+n-1)$, and $b = \ln(x) + 2t\ln(x+1) + \cdots + nt^{n-1}\ln(x+n-1)$. One may take

$$c = \exp(t\ln x + t^2\ln(x+1) + \cdots + t^n\ln(x+n-1)).$$

Then, $\mathcal{F}\langle c\rangle_{\mathrm{E},\Delta} = \mathcal{F}(c, \ln x, \ldots, \ln(x+n-1)))$, and $\mathcal{F}\langle c\rangle_{\mathrm{E},\Delta}$ is E-strongly normal over $\mathcal{F}$. The operation of the Galois group $G_{[\epsilon^n y]_{\mathrm{E}}} = \{v \in \mathcal{V}^* \mid \epsilon^n(\epsilon v/v) = 0\} = \{\exp(\alpha_0 + t\alpha_1 + \ldots + t^n\alpha_n) \mid \alpha_i \in \mathbb{C}\}$ on c is induced by addition in the exponents. If $f = x$, η may be taken to be $t^{n+1}x/(n+1)!$. Then,

$$c = \exp(t^{n+1}x/(n+1)! + t\ln x + t^2\ln(x+1) + \cdots + t^n\ln(x+n-1)),$$

and the Galois group is the same.

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