Essays on Corporate Credit Risk

by

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Abstract

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"The Impact of Exchange Rate Exposure on Multinationals' Credit Risk" study explores the correlation between exchange rate exposure and multinationals credit risk. Existing credit risk models do not distinguish between domestic and multinational companies, thus excluding the unique implications of exchange rate exposure on multinationals' credit risk. Empirical results reveal that when stochastic foreign exchange exposure is incorporated into the Merton structural model, most implied default probabilities slightly increase. Parametric and nonparametric tests point to statistically significant differences between the traditional Merton model and the new multinationals model. Significant correlations are found between this rise in implied default probabilities and assets-exchange rate correlation, exchange rate volatility and physical assets' book value. The proposed multinationals model is found to have a higher predictive power, leading to economically significant improved performance over loan portfolios. A simulation experiment illustrates the superiority of the new multinationals model within observed lenient policy.

The "Homogeneous Markov Chain, Stochastic Economic, and Non-Homogeneous Models for Measuring Corporate Credit Risk" study discovers some of the factors affecting the survivability of different debt portfolios, presents quantitative and

comparative measurements for default rate and distance to default within a portfolio perspective, and suggests various approaches for simulating ratings migration. The data sample used to test the models contains the S&P long-term credit ratings of industrial companies in North America from 1985 to 2004. A comparative analysis of the alternative models reveals that the density-dependent model is the most realistic approach, outperforming the homogeneous model in describing empirically observed ratings transitions. The measurements presented in this study can be used to compare and sort different debt portfolios with respect to their credit risk components, and to price credit-sensitive securities.

"A Review of Developments in Credit Structural Models and Credit Ratings Migration Analysis" is a survey reviewing past developments in structural models of credit risk, started with the Merton's framework in 1974, and credit ratings migration analysis, as well as introduces recent schemes in these two methodologies. The advantages and disadvantages of each approach are further discussed and compared.

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Essay 1: "The Impact of Exchange Rate Exposure on Multinationals' Credit Risk"

Essay 2: "Homogeneous Markov Chain, Stochastic Economic, and Non-Homogeneous Models for Measuring Corporate Credit Risk"

The Impact of Exchange Rate Exposure on Multinationals' Credit Risk

Abstract

This article explores the correlation between exchange rate exposure and multinationals credit risk. Existing credit risk models do not distinguish between domestic and multinational companies, thus excluding the unique implications of exchange rate exposure on multinationals' credit risk. Empirical results reveal that when stochastic foreign exchange exposure is incorporated into the Merton structural model, most implied default probabilities slightly increase. Parametric and nonparametric tests point to statistically significant differences between the traditional Merton model and the new multinationals model. Significant correlations are found between this rise in implied default probabilities and assets-exchange rate correlation, exchange rate volatility and physical assets' book value. The proposed multinationals model is found to have a higher predictive power, leading to economically significant improved performance over loan portfolios. A simulation experiment illustrates the superiority of the new multinationals model within observed lenient policy.

I. Introduction

It has been more than three decades since the first appearance of the Merton structural approach for measuring corporate credit risk. Merton (1974) introduces the pioneering insight for measuring default risk as a function of assets and debt values, often referred as the option theoretic valuation of debt. The perception underlying the model states that the default threshold can be resolved by applying the Black and Scholes (1973) option theoretic formula on a deterministic debt face value, and the firm's stochastic total assets market value.

A second generation of structural models extended Merton's approach, and removed some of its problematic assumptions. Other credit risk approaches consider the credit ratings migration analysis, largely by using the Markov chain methodology, some accounting for business cycles and transition correlations. Alternative approaches for measuring corporate credit risk can also be found in credit scores, principal component analysis, feature maps, logistic regressions, hierarchical classification models, neural-networks and other reduced-form models.

Recent research, using various credit risk models, has excluded the unique implications of exchange rate exposure on multinationals. Existing models do not distinguish between domestic companies and multinationals. To explore this missing link, this study investigates whether exchange rate exposure is associated with higher credit risk for multinationals, and incorporates stochastic foreign exchange exposure within the traditional Merton model.

If multinationals perfectly hedge their currency risk, we would not expect them to carry additional credit risk. However, this is not the case in practice. Although there are numerous methods to evade foreign exchange risk, it appears that many multinationals choose not to completely hedge this type of exposure. Among the traded derivatives are currency futures, currency options and currency denominated bonds. Among the OTC instruments are currency swaps evolved from back-to-back loans or parallel loans, currency forwards and basket options.[1] Another popular method for decreasing currency risk is through timing and operating policies, such as accelerating or decelerating the timing of payments that must be made or received in foreign currencies. Other techniques include risk-sharing contracts, in which the buyer and the seller agree to split losses and gains caused by currency movements between them.

Smith and Stulz (1985) argue that foreign exchange hedging can reduce the probability of default and thereby reduce expected costs of financial distress. In contrast, Nance, Smith and Smithson (1993) claim that since designing hedging strategies and managing risks can be costly, firms have an incentive to hedge only if hedging benefits are greater than costs. They argue that firm size is related to hedging-motives. Larger firms that have access to risk management expertise are more likely to hedge than smaller

[1] A back-to-back loan is when foreign companies obtain financing in different currencies and agree to swap the principal and interest payments. A parallel loan is between a domestic firm and its foreign subsidiary.

firms, exposing bigger multinationals to less exchange rate risk. Chow, Lee and Solt (1997) describe the presumption of hedging effectiveness as reasonable only for current cash flows, but "*where the long-term effects of exchange-rates changes are difficult to ascertain, hedging effectiveness is doubtful*". Furthermore, only a relatively small portion of firms hedge exchange rate risk. Bodnar, Hayt and Marston (1998) report that only 22% of the firms in their survey hedge foreign exchange risk. Bodnar, Jong and Macrae (2003) discover that Dutch firms tend to hedge foreign exchange exposure more than U.S. firms. They state "*U.S. firms have more concerns regarding derivative usage, and focus more on accounting earnings, hence practice less foreign currency hedging relative to Dutch firms.*"

Furthermore, even multinationals that repeatedly hedge their cash-flow exposure to foreign currencies, rarely consider their global assets exposure to exchange rate fluctuations. The investigation hereafter may encourage multinationals in general, and those with higher value of physical assets in particular, to consider hedging their exchange rate exposure more frequently.

The purpose of this article is to extend the Merton structural model, to incorporate multinationals' foreign exchange exposure, and thereby to explore an additional feature of corporate credit risk exposure. This study attempts to shed light on the impact of exchange rate exposure on multinationals' credit risk. The paper proceeds as follows. Section 2 reviews the Merton model. Section 3 outlines the theoretical arguments for associating the firm's exchange rate exposure to its credit risk. Section 4 discusses the data sample and presents the empirical methodology. Section 5 summarizes the results, Section 6 points to the economic importance of the proposed multinationals model, and Section 7 concludes.

II. The Merton Model

Merton (1974) presents a model where a firm's equity can be considered as a European call option on the firm's assets. The model can be simplified by assuming that the firm has one zero-coupon bond outstanding which matures at time T. The model defines the following:

V_0: market value of the firm's assets today

V_T: market value of the firm's assets at time T

E_0: value of the firm's equity today

E_T: value of the firm's equity at time T

D: total amount of debt (interest and principal) due to be repaid at time T

σ_V: volatility of the firm's assets

σ_E: volatility of the firm's equity

r_f: risk-free interest rate (within the relevant market)

Hence, if $V_T < D$, the firm defaults at time T and the value of the equity is zero. However, if $V_T > D$, the firm pays out its debt at time T, and its equity value becomes $V_T - D$. Therefore, the value of the firm's equity at time T can be written as

$$E_T = Max\ (V_T - D,\ 0) = (V_T - D)^+ \quad \forall V_T \in \mathbb{R} \quad \forall D \in \mathbb{R} \tag{2.1}$$

The firm's equity is a European call option on the value of the assets, with the total amount of debt due to be paid at time T as the strike price. The Black-Scholes (1973) model solves the value of the firm's equity today as follows:

$$E_0 = V_0 \Phi(d_1) - De^{-r_f T} \Phi(d_2) \tag{2.2}$$

where

$$d_1 = (\ ln(V_0 / D) + (r_f + \sigma^2_V / 2)T\ ) / \sigma_V \sqrt{T}$$

$$d_2 = (\ ln(V_0 / D) + (r_f - \sigma^2_V / 2)T\ ) / \sigma_V \sqrt{T} = d_1 - \sigma_V \sqrt{T}$$

The standard Merton model is applicable purely for domestic companies, where assets and debt are denominated in the home currency. When a firm's assets and debt are denominated in different currencies, the Merton model needs to be modified.

The Black-Scholes model considers $\Phi(d_2)$ as the probability that the option will be exercised, or an in-the-money option, where Φ is the *cumulative distribution function* (CDF) of the *normal distribution*. In the Merton model, $\Phi(d_2)$ is the risk-neutral probability that the firm will not default, and $1 - \Phi(d_2) = \Phi(-d_2)$ is the risk-neutral probability that the firm will default on the debt.

More formally, default occurs if and only if, at time t, the value of the firm's assets, V_t, is below its debt value, D_t. The g*eometric Brownian motion* for the asset value of the firm is:

$$dV_t = \mu_V V_t dt + \sigma_V V_t dZ_{V,t} \tag{2.3}$$

where μ_V and σ_V are assumed to be constants. The Merton model assumes that asset value at time t will follow a *lognormal distribution*, or that the natural logarithm of the asset value at a given time t is normally distributed as follows:

$$ln(V_t) = ln(V) + (\mu_V - \sigma^2_V / 2)t + \sigma_V\sqrt{t}\varepsilon \tag{2.4}$$

with mean $ln(V) + (\mu_V - \sigma^2_V / 2)t$ and variance $\sigma^2_V t$. Therefore, the probability of default equals

$$\begin{aligned} P(V_t < D_t) &= P(ln(V_t) < ln(D_t)) \\ &= P(ln(V) + (\mu_V - \sigma^2_V / 2)t + \sigma_V\sqrt{t}\varepsilon < ln(D_t)) \\ &= P((ln(V / D_t) + (\mu_V - \sigma^2_V / 2)t) / \sigma_V\sqrt{t} < -\varepsilon) \\ &= \Phi(-((ln(V / D_t) + (\mu_V - \sigma^2_V / 2)t) / \sigma_V\sqrt{t})) \\ &= \Phi(-d_2) \end{aligned} \tag{2.5}$$

An increase in σ_V causes a decrease in d_2. As $(-d_2)$ increases, the risk-neutral probability for the firm to default on debt, $\Phi(-d_2)$, increases as well.

The Merton risk neutral default probability, $\Phi(-d_2)$, increases with the face value of debt (D), and with the volatility of the firm's assets (σ_V), but decreases with the firm's value (V_0), and with the risk free interest rate (r_f). As time to maturity T increases, the Merton risk neutral default probability converges in the limits to either 0 or 1. This depends on whether the risk free interest rate, r_f, is bigger or smaller than $\sigma^2_V / 2$.

If $r_f > \sigma^2_V / 2$ then $T \rightarrow +\infty \Rightarrow d_2 \rightarrow +\infty \Rightarrow (-d_2) \rightarrow -\infty \Rightarrow \Phi(-d_2) \rightarrow 0$.

If $r_f < \sigma^2_V / 2$ then $T \rightarrow +\infty \Rightarrow d_2 \rightarrow -\infty \Rightarrow (-d_2) \rightarrow +\infty \Rightarrow \Phi(-d_2) \rightarrow 1$.

Assuming that at any given time there is a positive probability for default, the default process is a *sub-martingale*[2]. As time passes, the debt time to maturity, T, gets smaller, therefore:

If $V_0 > D \Rightarrow ln(V_0 / D) > 0$ then $T \rightarrow 0 \Rightarrow d_2 \rightarrow +\infty \Rightarrow (-d_2) \rightarrow -\infty \Rightarrow \Phi(-d_2) \rightarrow 0$.

If $V_0 < D \Rightarrow ln(V_0 / D) < 0$ then $T \rightarrow 0 \Rightarrow d_2 \rightarrow -\infty \Rightarrow (-d_2) \rightarrow +\infty \Rightarrow \Phi(-d_2) \rightarrow 1$.

If the asset value is larger than the face value of debt, when the debt' time to maturity gets smaller, the firm's default risk decreases, and vice versa.

When default occurs, assuming negligible distress and bankruptcy costs, Altman, Resti and Sironi (2001) define *recovery rate* as the ratio between the asset value and the

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[2] A process with zero sloping-trend is called a *martingale*. Martingale is a "fair" process in the sense that expected gains or losses are zero. A process with an upward or downward sloping trend is called a sub-martingale.

debt face value, V_t / D_t. Since face value of debt is presumed to be constant, the *expected recovery rate* is: $E[V_t / D_t] = E[V_t] / D_t$. Since recovery rate has meaning only in case of a default, when $V_t < D_t$, the *expected recovery rate* can be written as the mean of a *truncated lognormal variable* divided by D_t as follows:

$$E[V_t / D_t \mid V_t < D_t] = E[V_t \mid V_t < D_t] / D_t \tag{2.6}$$

Liu, Lu, Kolpin and Meeker (1997) prove the following identity:[3]

$$\begin{aligned} E[V_t \mid V_t < D_t] &= e^{ln(V)+\mu vt}\Phi(-((ln(V_t / D_t) + (\mu_V + \sigma^2{}_V / 2)t\,) / \sigma_V\surd t)) \\ &/ \Phi(-((ln(V_t / D_t) + (\mu_V - \sigma^2{}_V / 2)t\,) / \sigma_V\surd t)) = Ve^{\mu vt}\Phi(-d_1) / \Phi(-d_2) \\ &= E[V_t]\ \Phi(-d_1) / \Phi(-d_2) \end{aligned} \tag{2.7}$$

where d_1 and d_2 are the parameters defined in the Black–Scholes model. Thus, the *expected recovery rate* is:

$$E[V_t / D_t \mid V_t < D_t] = E[V_t / D_t\,]\ \Phi(-d_1) / \Phi(-d_2) \tag{2.8}$$

Since all the parameters are positive, d_1 is always positive, but d_2 may be positive or negative. For every $\sigma_V > 0$ and $T > 0$, d_1 is always bigger than d_2. As a result, the relation $\Phi(d_1) > \Phi(d_2)$ is always true, and therefore

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[3] See Appendix 1.

$$1 - \Phi(d_1) < 1 - \Phi(d_2) \Rightarrow \Phi(-d_1) < \Phi(-d_2) \Rightarrow \Phi(-d_1) / \Phi(-d_2) < 1 \tag{2.9}$$

This inequality holds regardless of the sign of d_2. Therefore, the expected recovery rate always satisfies

$$E[V_t / D_t]\ \Phi(-d_1) / \Phi(-d_2) < E[V_t / D_t] \tag{2.10}$$

In the standard Merton model V_t and σ_V are not directly observable, but can be postulated. Jones, Mason and Rosenfeld (1984), among others, find the probability for a public company to default on the debt, $\Phi(-d_2)$, by solving two equations with two unknowns.[4] The first is the interpretation of Black-Scholes formula for the value of firm's equity

$$E_t = V_t\Phi(d_1) - De^{-r_fT}\Phi(d_2) \tag{2.11}$$

The Black-Scholes model suggests that the equity price is a function of asset value, time to maturity and asset volatility

$$E_t = f(V_t, t) \tag{2.12}$$

[4] Hull, Nelken and White (2004) present a different approach to find the assets log return volatility, σ_V, by using the implied volatility from options on the firm's traded stocks.

Assuming a constant equity growth *drift*, μ_E, a constant equity *diffusion*, σ_E, and a standard *Wiener process* dW_t, applying *Itô's lemma* for equity movements yields

$$df(V_t, t) = \big((\partial f(V_t, t) / \partial V_t)\mu_V V_t + (\partial f(V_t, t) / \partial t) + (\tfrac{1}{2} (\partial^2 f(V_t, t)) / \partial V^2{}_t)\sigma^2{}_V V^2{}_t \big)dt + (\partial f(V_t, t) / \partial V_t)\sigma_V V_t dW_t \tag{2.13}$$

The equity value dynamic follows a *geometric Brownian motion* of the form

$$dE_t = \mu_E E_t dt + \sigma_E E_t dW_t \tag{2.14}$$

where μ_E and σ_E are assumed to be constants. Hence, while comparing the diffusion coefficients from the equity value dynamic in equation (2.14), and the one from the Itô's lemma for equity movements in equation (2.13), and since the second and the third partial derivatives on the right hand side of Itô's lemma vanish; the second necessary equation is obtained as:

$$\sigma_E E_t = (\partial E_t / \partial V_t)\sigma_V V_t = \Phi(d_1)\sigma_V V_t \tag{2.15}$$

III. The Multinationals model

The analysis thus far describes credit risk characteristics for corporations that operate exclusively in one country. These corporations have their assets and debt denominated in their home currency. However, when the face value of debt is denominated in a different currency than the firm's assets, it can no longer be considered as constant. Thus, an extension of the Merton model is required for multinationals. Two types of multinationals can be considered. The first type has assets denominated in home currency and debt denominated in foreign currency. The second has assets denominated in foreign currency and debt denominated in home currency. These represent the two extreme cases, and combining them may capture the whole spectrum of multinationals. Attaching two stochastic dynamics, one to the asset's value and the other to the face value of debt, as oppose to attaching both stochastic processes to the assets, allows us to capturing this variety of multinationals.

An equity of a multinational corporation, with assets denominated in home currency and debt denominated in foreign currency, or with assets denominated in foreign currency and debt denominated in home currency, can be considered as a *cross-currency European option*[5]. This exotic option contains additional complexity to the pricing model, since both the underlying asset and the exchange rate are subject to stochastic processes. Even when the strike price is fixed in foreign currency, it is still changing randomly when expressed in home currency, due to exchange rate variability through time.

[5] Cross-currency options differ from *quanto options* and *currency options*. In a quanto option, both the strike price and the underlying asset are denominated in foreign currency, while the intrinsic value is repatriated at a predetermined exchange rate. A currency option is written on the currencies themselves. For further explanations, see Garman and Kohlhagen (1983), Grabbe (1983) and McDonald (2003).

Several scholars examine options involving two stochastic processes. Fischer (1978) discusses European options with uncertain exercise price. Margrabe (1978) presents a model for options to exchange one risky asset for another. Carr (1988) shows that compound options will have stochastic exercise prices, if their strike prices are denominated in foreign currencies. Rumsey (1991) introduces a cross-currency option as one involving different currencies for the exercise price, and for the underlying asset. Reiner (1992) introduces options on foreign assets, where the return has an exchange rate component. Brooks (1993) illustrates the use of multivariate cross-currency options and Benninga, Björk and Wiener (2002) use a numeraire approach to price European options, with a strike price linked to a non-domestic currency.

To incorporate a stochastic foreign exchange exposure for multinationals within the Merton model, denote S_t as the *indirect quote* spot exchange rate at time t.[6] The exchange rate stochastic process is then given by:

$$dS_t = \mu_S S_t d_t + \sigma_S S_t dZ_t \tag{3.1}$$

where μ_S and σ_S are assumed to be constants, and dZ_t is the corresponding Wiener increment. The first type of multinationals classifies corporations with assets denominated in home currency *(H)*, and debt face value, as the strike price, denominated in foreign currency *(F)*. The asset value denominated in foreign currency is:

[6] An indirect quote is foreign currency per unit of home currency. A *direct quote* is home currency per unit of foreign currency.

$$V_{F,t} = V_{H,t}S_t \tag{3.2}$$

Therefore, the first type multinational's equity value denominated in foreign currency terms, at time T, is given by:

$$E_{F,T} = Max(V_{F,T} - D_F,\ 0) = Max(V_{H,T}S_T - D_F,\ 0) \tag{3.3}$$

Since the strike price is fixed in foreign currency terms, applying Itô's lemma for equity movements yields the following dynamic in foreign currency:

$$DV_{F,,t} = V_{F,,t}(\mu_V + \mu_S + \rho_{V,S}\sigma_V\sigma_S)dt + V_{F,,t}\delta_{V,F}dY_t \tag{3.4}$$

where Y is a scalar Wiener process, $\rho_{V,\ S}$ is the assets-exchange rate correlation and the asset volatility denominated in foreign currency is:[7]

$$\delta_{V,F} = \sqrt{(\sigma^2{}_V + \sigma^2{}_S + 2\rho_{V,S}\sigma_V\sigma_S)} \tag{3.5}$$

Using the Black-Scholes model, the value of the first type multinational's equity denominated in foreign currency, at time t, is obtained as:

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[7] The *Siegel paradox* permits different percentage changes, when expressing exchange rates in direct quote and in indirect quote. However the exchange rate log return volatility, σ_S, remains the same under both notations. Since the direct quote is the reciprocal of the indirect quote, the return logarithms yield identical values, but with opposite signs, and therefore the standard deviation remains the same.

$$E_{F,t} = S_t V_{H,t} \Phi(d'_1) - D_F e^{-rF(T-t)} \Phi(d'_2) \tag{3.6}$$

where

$$d'_1 = (\ln(S_t V_{H,t} / D_F) + (r_F + \delta^2_{V,F} / 2)(T-t)) / \delta_{V,F}\sqrt{(T-t)}$$

$$d'_2 = (\ln(S_t V_{H,t} / D_F) + (r_F - \delta^2_{V,F} / 2)(T-t)) / \delta_{V,F}\sqrt{(T-t)} = d'_1 - \delta_{V,F}\sqrt{(T-t)}$$

This multinational's equity value in home currency can be obtained by a simple transformation:[8]

$$E_{H,t} = E_{F,t} / S_t \tag{3.7}$$

Thus, the first type multinational's equity value denominated in home currency, at time *t*, is:

$$E_{H,t} = V_{H,t} \Phi(d'_1) - D_F e^{-rF(T-t)} \Phi(d'_2)/S_t \tag{3.8}$$

where

$$d'_1 = (\ln(S_t V_{H,t} / D_F) + (r_F + \delta^2_{V,F} / 2)(T-t)) / \delta_{V,F}\sqrt{(T-t)}$$

$$d'_2 = (\ln(S_t V_{H,t} / D_F) + (r_F - \delta^2_{V,F} / 2)(T-t)) / \delta_{V,F}\sqrt{(T-t)} = d'_1 - \delta_{V,F}\sqrt{(T-t)}$$

The second type of multinationals has assets denominated in foreign currency *(F)*, and debt, as the strike price, denominated in home currency *(H)*. The asset value denominated in home currency is:

[8] The reader should notice that the transformation is performed only on the equity value derivation, but not on the temporary variables, d'_1 and d'_2.

$$V_{H,t} = V_{F,t} / S_t \quad (3.9)$$

Hence, the second type of multinational's equity value denominated in home currency terms, at time *T*, is given by:

$$E_{H,T} = Max(V_{H,T} - D_H, 0) = Max(V_{F,T} / S_T - D_H, 0) \quad (3.10)$$

Since the strike price is fixed in home currency terms, applying Itô's lemma for equity movements yields the following dynamic in home currency:

$$dV_{H,t} = V_{H,t}(\mu_V - \mu_S - \rho_{V,S}\sigma_V\sigma_S)dt + V_{H,t}\delta_{V,H}dY_t \quad (3.11)$$

where Y is a scalar Wiener process, $\rho_{V,\ S}$ is the assets-exchange rate correlation and the assets volatility in home currency is:

$$\delta_{V,H} = \sqrt{(\sigma^2{}_V + \sigma^2{}_S - 2\rho_{V,S}\sigma_V\sigma_S)} \quad (3.12)$$

Using the Black-Scholes model, the value of the second type of multinational's equity denominated in home currency, at time t, is obtained as:

$$E_{H,t} = V_{F,t}\Phi(d''_1) / St - D_H e^{-rH(T-t)}\Phi(d''_2) \quad (3.13)$$

where

$$d''_1 =(ln(V_{F,t} / S_t D_H) + (r_H + \delta^2{}_{V,H} / 2)(T-t)) / \delta_{V,H}\sqrt{(T-t)}$$

$d''_2 = (ln(V_{F,t} / S_tD_H) + (r_H - \delta^2_{V,H} / 2)(T-t)) / \delta_{V,H}\sqrt{(T-t)} = d''_1 - \delta_{V,H}\sqrt{(T-t)}$

The corresponding multinational's equity value in foreign currency is obtained by the simple transformation:

$$E_{F,t} = E_{H,t}S_t \quad (3.14)$$

Hence, the second type multinational's equity value denominated in foreign currency, at time *t*, is given by:

$$E_{F,t} = V_{F,t}\Phi(d''_1) - S_tD_He^{-rH(T-t)}\Phi(d''_2) \quad (3.15)$$

where

$d''_1 = (ln(V_{F,t} / S_tD_H) + (r_H + \delta^2_{V,H} / 2)(T-t)) / \delta_{V,H}\sqrt{(T-t)}$

$d''_2 = (ln(V_{F,t} / S_tD_H) + (r_H - \delta^2_{V,H} / 2)(T-t)) / \delta_{V,H}\sqrt{(T-t)} = d''_1 - \delta_{V,H}\sqrt{(T-t)}$

Similar to Jones, Mason and Rosenfeld (1984), a second equation for multinationals can be postulated while considering that the equity price is a function of two stochastic variables: market value of assets and exchange rate, as follows:

$$E_t = f(V_t, S_t, t) \quad (3.16)$$

Applying Itô's lemma in two dimensions for equity movements yields:[9]

$$df(V_t, S_t, t) = \big((\partial f(V_t, S_t, t) / \partial V_t)\mu_V V_t + (\partial f(V_t, S_t, t) / \partial S_t)\mu_S S_t \\ + (\partial f(V_t, S_t, t) / \partial t)\big)dt + ½ \big((\partial^2 f(V_t, S_t, t) / \partial V^2_t)\sigma^2_V V^2_t \\ + (\partial^2 f(V_t, S_t, t) / \partial S^2_t)\sigma^2_S S^2_t + 2V_t S_t \rho_{V,S}\sigma_V\sigma_S \partial^2 f(V_t, S_t, t) / \partial V_t \partial S_t\big)dt \\ + (\partial f(V_t, S_t, t) / \partial V_t)\sigma_V V_t dY_t + (\partial f(V_t, S_t, t) / \partial S_t)\sigma_S S_t dZ_t \quad (3.17)$$

Comparing the diffusion coefficients from the equity value single-dimension dynamic in equation (2.14), to the one from the Itô's lemma in two dimensions for equity movements in equation (3.17) requires to accommodate also the assets-exchange rate correlation thus, the second necessary equation is obtained as:

$$\sigma^2_E E^2_t = ((\partial E_t / \partial V_t)\sigma_V V_t)^2 + ((\partial E_t / \partial S_t)\sigma_S S_t)^2 \\ + 2\rho_{V,S}((\partial E_t / \partial V_t)\sigma_V V_t)((\partial E_t / \partial S_t)\sigma_S S_t) \quad (3.18)$$

The second necessary equation for the first multinational type is:

$$\sigma^2_E E^2_{F,t} = (S_t \Phi(d'_1)\sigma_{VH} V_{H,t})^2 + (V_{H,t}\Phi(d'_1)\sigma_S S_t)^2 \\ + 2\rho_{V,S}(S_t\Phi(d'_1)\sigma_{VH}V_{H,t})(V_{H,t}\Phi(d'_1)\sigma_S S_t) \quad (3.19)$$

Expressing the equity in the same home currency as the assets yields:

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[9] For further discussion on multi-dimensional Itô's lemma see Kwok (1998), Hull (2003), or Baz and Chacko (2004).

$$\sigma_E E_{H,t} = V_{H,t}\Phi(d'_1)\sqrt{(|\sigma^2_{VH} + \sigma^2_S + 2\rho_{V,S}\,\sigma_{VH}\,\sigma_S|)} \tag{3.20}$$

The second necessary equation for the second multinational type is:

$$\sigma^2_E E^2_{H,t} = ((\Phi(d''_1) / S_t)\sigma_{VF}V_{F,t})^2 - ((V_{F,t}\Phi(d''_1) / S^2_t)\sigma_S S_t)^2 + 2\rho_{V,S}(\Phi(d''_1) / S_t)\sigma_{VF}V_{F,t})(V_{F,t}\Phi(d''_1) / S^2_t)\sigma_S S_t) \tag{3.21}$$

Similarly, expressing the equity in the same foreign currency as the assets yields:

$$\sigma_E E_{F,t} = V_{F,t}\Phi(d''_1)\sqrt{(|\sigma^2_{VF} - \sigma^2_S + 2\rho_{V,S}\,\sigma_{VF}\,\sigma_S|)} \tag{3.22}$$

These two equations alone cannot solve the two unobserved variables V_t and σ_V. As opposed to the two-equation system within the Merton model for domestic companies, multinational firms have another unknown. The asset-exchange rate correlation, $\rho_{V,\ S}$ is also unobservable, and hence a third equation is required. This correlation is neither quoted nor actively traded. The absence of a liquid market for implied correlations requires traders of multi-asset options to estimate correlation matrices from historical data, using *Efron's bootstrap technique*[10]. However, in the Merton model, no historical data are available, and hence combining two types of risk; the *Vega risk* and the *correlation risk* derives the third necessary equation.

[10] Efron (1979) presents the bootstrap procedure to resample observed data repeatedly and simulate the sampling distribution. Since the technique assumes independency among observations, time series data requires additional step of dividing the sample into random cohorts.

Vega of a European option, *v*, is the rate of change in the option value with respect to changes in the volatility of the underlying asset[11]. If Vega is high in absolute terms, the option's value is more sensitive to small changes in volatility, and vice versa. For a European call (or put) option on a non-dividend-paying stock, *Vega risk* is measured as:

$$\upsilon = \partial\pi / \partial\sigma = S_0(\sqrt{T})\Psi(d_1) \tag{3.23}$$

where π is the option market value, σ is the underlying asset volatility, S_0 is the underlying stock price, T is the time to maturity and Ψ is the *probability density function* (PDF) of the standard normal distribution, and is given by

$$\Psi(x) = (1 / \sqrt{(2\pi)})exp(-x^2 / 2) \tag{3.24}$$

In the context of the Merton model, Vega of a *domestic company* is the rate of change of the firm's equity value with respect to the volatility of total assets, and is given by

$$\upsilon = V_t(\sqrt{(T-t)})\Psi(d_1) \tag{3.25}$$

[11] Hull and White (1987) show that even though the Black-Scholes model assumes constant volatility, there is no contradiction in calculating Vega risk as sensitivity to volatility changes. It is a good approximation under the assumption that asset price and its volatility are uncorrelated.

Expressing the equity in foreign currency, Vega of the first multinational type can be written as:

$$\upsilon_1 = \partial E_{F,t} / \partial \delta_{V,F} = S_t V_{H,t}(\sqrt{(T-t)})\Psi(d'_1) \tag{3.26}$$

With equity in home currency, Vega of the second multinational type can be written as:

$$\upsilon_2 = \partial E_{H,t} / \partial \delta_{V,H} = (V_{F,t} / S_t)(\sqrt{(T-t)})\Psi(d''_1) \tag{3.27}$$

Correlation risk is the risk exposure due to miscalculation of correlation or, within the Merton model context, the risk of sudden changes in equity value with respect to changes in the asset-exchange rate correlation. These changes may occur due to dynamic foreign exchange hedging, or the level of *pass-through* the firm is exposed to. The *correlation risk* is defined as

$$\zeta = \partial E / \partial \rho_{V,S} \tag{3.28}$$

To find the correlation risk of multinationals, the *Plackett's identity* can be used. Plackett (1954) describes density in terms of its characteristics function, and proves the following theorem: for multivariate normal density function f and vector X of n random variables, with means zero and unit variances

$$\partial f / \partial \rho_{i,j} = \partial^2 f / \partial X_i \partial X_j \tag{3.29}$$

Reiss and Wystup (2001) use the Plackett's identity to derive a correlation risk for European claims in the two-dimensional Black-Scholes model. Following their analysis, the equity value can be written as a *bivariate normal density function* of two stochastic variables: $E_t = f(V_t, S_t, t)$, where μ_V, σ_V, μ_S and σ_S are assumed to be constants. Thus the correlation risk is

$$\zeta = \partial E / \partial \rho_{V,S} = V_t S_t \sigma_V \sigma_S (T-t) \partial^2 f(V_t, S_t) / \partial V_t \partial S_t \tag{3.30}$$

Expressing equity value in foreign currency, correlation risk of the first multinational type can be written as:

$$\zeta_1 = \partial E_{F,t} / \partial \rho_{V,S} = V_{H,t} S_t \sigma_V \sigma_S (T-t) \partial^2 E_{F,t} / \partial V_t \partial S_t = V_{H,t} S_t \sigma_V \sigma_S (T-t) \Phi(d'_1) \tag{3.31}$$

Similarly, expressing equity value in home currency, correlation risk of the second multinational type can be written as:

$$\zeta_2 = \partial E_{H,t} / \partial \rho_{V,S} = V_{F,t} S_t \sigma_V \sigma_S (T-t) \partial^2 E_{H,t} / \partial V_t \partial S_t = -V_{F,t} \sigma_V \sigma_S (T-t) \Phi(d''_1) / S_t \tag{3.32}$$

To establish the third necessary equation, the *chain rule* decomposes the *correlation risk* into *Vega risk* and an additional component. For the first multinational type

$$\zeta_1 = V_{H,t}S_t\sigma_V\sigma_S(T\text{-}t)\Phi(d'_1) = \partial E_{F,t} / \partial\rho_{V,S} \tag{3.33}$$

$$= (\partial E_{F,t} / \partial\delta_{V,F})(\partial\delta_{V,F} / \partial\rho_{V,S})$$

$$= \upsilon_1(\sigma_V\sigma_S) / \sqrt{(\sigma^2_V + \sigma^2_S + 2\rho_{V,S}\, \sigma_V\, \sigma_S)}$$

$$= S_tV_{H,t}(\sqrt{(T\text{-}t)})\Psi(d'_1)\ (\sigma_V\sigma_S) / \sqrt{(\sigma^2_V + \sigma^2_S + 2\rho_{V,S}\, \sigma_V\, \sigma_S)}$$

and for the second multinational type

$$\zeta_2 = -V_{F,t}\sigma_V\sigma_S(T\text{-}t)\Phi(d'_1) / S_t = \partial E_{H,t} / \partial\rho_{V,S} \tag{3.34}$$

$$= (\partial E_{H,t} / \partial\delta_{V,H})(\partial\delta_{V,H} / \partial\rho_{V,S})$$

$$= \upsilon_2(-\sigma_V\sigma_S) / \sqrt{(\sigma^2_V + \sigma^2_S - 2\rho_{V,S}\, \sigma_V\, \sigma_S)}$$

$$= (V_{F,t} / S_t)(\sqrt{(T\text{-}t)})\Psi(d''_1)\ (-\sigma_V\sigma_S) / \sqrt{(\sigma^2_V + \sigma^2_S - 2\rho_{V,S}\, \sigma_V\, \sigma_S)}$$

Therefore, the third equation for the first multinational type is obtained as:

$$\sqrt{(\sigma^2_V + \sigma^2_S + 2\rho_{V,S}\, \sigma_V\, \sigma_S)} = \Psi(d'_1) / (\Phi(d'_1)\sqrt{(T\text{-}t)}) \tag{3.35}$$

and for the second multinational type as:

$$\sqrt{(\sigma^2_V + \sigma^2_S - 2\rho_{V,S}\, \sigma_V\, \sigma_S)} = \Psi(d''_1) / (\Phi(d''_1)\sqrt{(T\text{-}t)}) \tag{3.36}$$

Multinationals often have a more complex structure of assets and debt, with respect to their geographic location. Since multinationals may issue debt in more than one geographic region simultaneously, the total volatility of a portfolio of exchange rates can be estimated by using a weighted combination of exchange rates, their log return volatility, and their correlations. In this case, the total volatility of the portfolio of exchange rates is expressed as:

$$\sigma^2_{P(S)} = \Sigma_{i=1}{}^{n} w^2_{i(S)} \sigma^2_{i(S)} + 2\Sigma_{i=1}{}^{n-1} \Sigma_{j=i+1}{}^{n} w_{i(S)} w_{j(S)} \rho_{i(S)j(S)} \sigma_{i(S)} \sigma_{j(S)} \tag{3.37}$$

where $w_{i(S)}$ represents the regional weight, $\sigma_{i(S)}$ is the individual exchange rate volatility, and $\rho_{i(S),j(S)}$ denotes the correlation between a pair of exchange rates, i and j. Multinationals may also have both assets and debt exposed to foreign exchange risk, at the same time. When a multinational can be identified as having both its assets and debt diversified over several markets, domestic and foreign, its equity value can be computed as a portfolio of the first and the second types of multinationals. In this case, equations (3.6) and (3.15) can be batched together to estimate the equity value denominated in foreign currency. Similarly, equations (3.8) and (3.13) can be batched together to estimate the equity value denominated in home currency as:

$$\begin{aligned} E_{H,t} = {} & w_{VH,t} V_{H,t} \Phi(d'_1) + w_{VF,t} V_{F,t} \Phi(d''_1) / S_t \\ & - w_{DH} D_H e^{-rH(T-t)} \Phi(d''_2) - w_{DF} D_F e^{-rF(T-t)} \Phi(d'_2) / S_t \end{aligned} \tag{3.38}$$

where $w_{VH,t} + w_{VF,t} = 1$ as the weights of the assets, and $w_{DH} + w_{DF} = 1$ as the weights of debt, both at home and foreign markets correspondingly.

The multinationals model incorporates stochastic foreign exchange exposure in the traditional Merton model. This way, it can predict several relationships between implied default probabilities and assets-exchange rate correlation, as well as exchange rate volatility. **Figure 1** illustrates a sensitivity analysis between assets-exchange rate correlation and default probability. In this example, the following values are selected: equity value equals \$4B, face value of debt equals \$3B, risk free rate is 5%, time to maturity of debt is 5 years, assets' log return volatility equals 0.5, exchange rate log return volatility equals 0.1, and the assets-exchange rate correlation varies between −1 and 1. Simulation results show that for these selected values, the asset market values decrease from \$6.1078B to \$5.6784B, suggesting that the implied default probability is a monotonic increasing concave function of the assets-exchange rate correlation.

Figure 2 demonstrates a sensitivity analysis between exchange rate volatility and default probability. In this example, the following values are selected: equity value equals \$5B, face value of debt equals \$2B, risk free rate is 5%, time to maturity of debt is 4 years, assets' log return volatility equals 0.5, assets-exchange rate correlation equals 0.1, and exchange rate log return volatility varies from 0 to 0.4. The simulation reveals that for these selected values, the asset market values decrease from \$6.5233B to \$6.3298B, proposing that the implied default probability is a monotonic increasing convex function of the exchange rate volatility.

The theoretical explanation for these sensitivities can be found in the first type of multinationals model, described in equations (3.5) and (3.6). Volatility of the first type multinational assets, $\delta_{V,F}$, is an increasing function of assets-exchange rate correlation, $\rho_{V,S}$, as well as exchange rate volatility, σ_S. Since d'_2 is a decreasing function of $\delta_{V,F}$, a rise in either $\rho_{V,S}$, or σ_S, increases the default probability, $\Phi(-d'_2)$. The sensitivity between the default probability, $\Phi(-d'_2)$, and the exchange rate volatility, σ_S, takes a convex form due to the quadratic exponent of the later, in equation (3.5). Thus, when either $\rho_{V,S}$, or σ_S increase, the impact of exchange rate exposure on multinationals' credit risk, measured by the difference between the implied default probabilities of the two models, is expected to increase.

IV. Data and Methodologies

We obtain our data from the following sources: Compustat North America, which contains the Industrial Quarterly, Industrial Annual and Segments, and Global Insight (previously called DRI - Data Resource Incorporated). After merging all data fields, the final dataset contains 5,431 observations for 412 multinationals, where 12 of them defaulted, from 1996 to 2003.

The Compustat North America Industrial Quarterly database provides data on multinationals, their country of origin and their equity market value. These companies having sales in more than one geographic region, excluding those within the financial service industries, assemble the multinationals sample. Multinationals' stock prices at the end of the first, second and third months of each quarter are collected and combined with the number of shares outstanding, to assess the equity market value at the end of each month. This helps us to obtain the multinationals' log return volatility of market value of equity. Property, plant and equipment and industry classification codes are also collected for each multinational, to examine whether the impact of foreign exchange exposure on multinationals' credit risk is more pronounced among companies holding large value of physical assets, or within specific industries.

The Compustat North America Industrial Annual database contains the firms' liabilities and their time to maturity. Delianedis and Geske (2003) propose the following method to construct the debt structure and its time to maturity. We divide total liabilities into current liabilities, debt due in one, two, three, four and five years, long-term debt, deferred taxes, minority interest and other long-term liabilities. Current liabilities are considered to mature in six months, while all other debt components, except those with explicitly stated time frame, are assumed to mature in ten years. In order not to overstate short-term debt, account receivables, cash, short-term investment, and marketable securities are deducted from account payables. Since the models consider only one debt time to maturity, the *Macaulay duration* is used to combine all debt components into a single maturity.[12]

The Global Insight International Monthly database provides data on the term structure of interest rates among several countries, as well as exchange rates. Different risk-free interest rates are estimated for each country from government bills, notes and

[12] Macaulay duration calculates the weighted average term to maturity of cash flows for bonds: Macaulay Duration = (Σ(Present Value of Cash Flow * Time to Cash flow)) / Price of the bond.

bonds, bank lending rates for businesses and certificates of deposits corresponding to the Compustat debt maturity, for each month. Spot exchange rates between various currencies are collected, from 1996 to 2003, and used to estimate all combinations of exchange rates, their log return volatility, as well as their correlations.

Since no information is available on where liabilities are issued, it is assumed that multinationals issue their debt within the geographical areas they operate in, along with the corresponding weights. Since only the first type of multinationals is considered, equity and assets are presumed to be located in the home country.[13] The dataset contains multinationals that operate in 38 different countries.[14] For simplicity, these countries are batched into twelve geographic indices: Africa, Asia, Canada, Central America, Central Europe, Eastern Europe, Middle East, Oceania, Scandinavia, South America, U.K. and U.S. For each geographic index, the appropriate risk-free interest rate is computed as an equally weighted average index of the rates from the countries in that region. For example, the Scandinavian risk-free rate index is computed as an equally weighted average of the rates in Denmark, Finland, Norway and Sweden. We construct twelve such indices for the 38 countries. Next, the log return volatility of the cross-currency exchange

[13] The second multinational type requires a switch in the assumptions regarding the asset's stochastic process, while the first multinational type maintains the same asset's dynamic as the traditional Merton model, and hence is more natural for comparison.

[14] These are: Argentina, Australia, Austria, Belgium, Brazil, Canada, Chile, Colombia, Denmark, England, Finland, France, Germany, Greece, Hungary, India, Ireland, Israel, Italy, Japan, Mexico, Netherlands, New Zealand, Norway, Peru, Philippines, Poland, Portugal, Russia, Singapore, South Africa, South Korea, Spain, Sweden, Switzerland, Thailand, U.S. and Venezuela.

rates among all indices, and the foreign exchange correlations among all the twelve indices are calculated.

The Compustat North America Segments database includes multinationals' sales per geographical region or specific country. Using the segment type and name, net-sales are classified within the twelve geographic indices. Then the corresponding weights, out of the total net sales, are assigned to these geographic indices, to assess the appropriate exchange rate and risk-free interest rate at all time. For example, a multinational corporation located in Australia, is classified as having its assets and equity in Oceania. If the multinational' net-sales are identified as 50% in Canada, 30% in Asia and 20% in Central Europe, debt is assumed to be located with respect to these geographic indices, along with the corresponding weights. In this case, the risk free interest rate, r_F, in equation (3.8) is constructed from 50% of the Canadian, 30% of the Asian, and 20% of the Central European risk-free regional interest rates. Volatility of the exchange rate, σ_S, between the home market (i.e. Oceania) and the foreign markets (i.e. Canada, Asia and Central Europe), in equation (3.5) is composed as a standard deviation of a portfolio with the appropriate exchange rates and the corresponding weights as illustrated in equation (3.37).

Crosbie and Bohn (2003) and Bharath and Shumway (2004) discuss an alternative and preferable method to calculate the relevant variables without solving the Merton model simultaneous equations, (2.11) and (2.15).[15] This is done through sequential iterations. The additional complexity of the multinationals model proposed here, relative to the standard Merton model, requires additional iterations before values converge, as described hereafter.

Following KMV methodology, for the traditional Merton model, the iterative procedure starts by considering $\Phi(d_1) = 1$ in equation (2.15), and by assigning an initial guess of $\sigma_V = \sigma_E[E/(E+D)]$. Then, using these values of σ_V within equation (2.11) to infer the market value of each multinational's assets, throughout the whole period. From this point on, further iterations can compute from past observations the implied log return of market value of assets, generating each time a new series of σ_V. All iterations disregard exchange rates, exchange rates' volatility and correlations between assets and exchange rates. The risk-free interest rate is obtained as a single rate observed in the home country of the multinational, and not as a portfolio of risk-free interest rates, derived from different geographic regions.

The iterative procedure for the multinationals model contains two stages. The first stage starts by considering $\rho_{V,S} = 0$ in equation (3.35). Then taking $\Phi(d'_1) = 1$ in equation (3.20), and assigning an initial guess of $\sigma_V = \sqrt{((\sigma_E(E/(E+D)))-\sigma^2_S)}$ for every observation, with the same market value of equity, equity volatility and face value of debt as in the Merton iterative procedure. Then, using these values of σ_V within equation (3.8) to infer the market value of each multinational's assets, throughout the whole period. Subsequent

[15] Moody's KMV avoids solving the simultaneous equations, since in practice "*the model linking equity and asset volatility holds only instantaneously*".

iterations compute the implied log return of market value of assets, generating each time a new series of σ_V. After several such iterations, the assets exchange rate correlation, $\rho_{V,S}$, is estimated with a higher accuracy from past observations of the same multinational. The second stage implements several more iterations, where each time a new assets-exchange rate correlation and log return volatility of market value of assets are computed, and then incorporated into equation (3.8), to infer new series of market value of assets. Iterations continue until both models converge, when absolute difference between successive σ_V's drops below one percent. In most cases, successive σ_V's converge with even higher accuracy after few such iterations.

Log return volatilities of the market value of assets, the market value of equity, and the portfolio of exchange rates are converted into annual volatilities. Thus, the risk-neutral implied default probabilities, derived from both models, represent one-year expected default probabilities. Both the traditional Merton model and the multinationals model proposed here are examined over the same final dataset. This is meant to reveal whether foreign exchange exposure has any consequences over multinationals' credit risk. The implied default probabilities derived from these two models are compared and analyzed using several statistical tests; an independent two-sample t-test, Wilcoxon-Mann-Whitney test, ROC curves, intra-cohort analysis and Probit models. These tests are described in the following section.

V. Empirical Results

This section reports the empirical findings. **Table 1** presents a summary statistics on the multinationals in our sample, their origin geographic region and the initial year of which data are available for them. The numbers of defaulted multinationals are indicated in brackets. Altogether, there are 412 multinationals, of which 12 have defaulted, from 1996 to 2003. **Figure 3** illustrates the distribution for the S&P credit ratings, and **Figure 4** presents the distribution of the industries within the final dataset. Out of the 5,431 observations, 1,221 observations, or 22.5% are at or below 'BB', the highest category classified as non-investment-grade or "junk bonds". This relatively high percentage of low credit ratings explains the distribution of the implied default probabilities evolved from the Merton model and the proposed multinationals model in **Figure 5**.

Table 2 presents a descriptive statistics for the relevant variables in the multinationals model. On average, the multinationals in the selected sample are relatively high leveraged with mean of face value of debt equals to $3.986B, and mean of market value of equity equals to $6.725B. The medians are $1.552B and $2.169B, respectively. The relatively high mean of debt-to-equity ratio, estimated to be 0.593, arises from our definition of total outstanding debt.[16] This includes not only short-term and long-term liabilities, but also deferred taxes, minority interest, account payables and other long-term liabilities, causing the risk-neutral measure of default probabilities to exceed the physical measure.

[16] As a prudent benchmark, one may consider the S&P 500 average debt-to-equity ratio, estimated as 0.33 by Peter Lynch (a former manager of the Fidelity Magellan fund) on several web sites throughout the recent years.

To authenticate the impact of exchange rate exposure on multinational's credit risk, the implied default probabilities are compared and analyzed using an independent *two-sample Student's t-test*. Assuming that implied default probabilities in these two models are normally distributed, with equal variances, their means are normally distributed and hence the difference between the means is also normally distributed as a linear combination of normal variables. Under the null hypothesis, both models yield the same mean default probability, while the alternative hypothesis suggests higher mean default probability within the multinationals model:

$H_0: 1/n\Sigma\Phi(-d'_2) = 1/n\Sigma\Phi(-d_2)$

$H_1: 1/n\Sigma\Phi(-d'_2) > 1/n\Sigma\Phi(-d_2)$

The *t-statistic* is obtained as:

$$T = (1/n\Sigma\Phi(-d'_2) - 1/n\Sigma\Phi(-d_2)) / \sqrt{(S^2(1/m + 1/n))} \tag{5.1}$$

where

$$S^2 = (1/(m+n-2))(\Sigma_{i=1}^{m}(\Phi_i(-d_2) - 1/n\Sigma\,\Phi(-d_2))^2 + \Sigma_{j=1}^{n}(\Phi_j(-d'_2) - 1/n\Sigma\Phi(-d'_2))^2)$$

and where m and n are the sample sizes (in this case $m = n = 5{,}431$), $\Phi_i(-d_2)$ denotes the implied default probability for multinational i under the Merton model, and $\Phi_j(-d'_2)$ describes the implied default probability for multinational j under the proposed multinationals model.

A comparison of the two models yields a mean default probability of 0.1833 and 0.2293 within the Merton and the multinationals models, respectively. Similar variances of implied default probabilities, 0.0671 and 0.0716 within the Merton and the multinationals models, respectively, permits the two-sample Student's t-test. The t-value for the difference is 9.1049, leading to a strong rejection of the null hypothesis with a significance level of $\alpha = 0.01$.[17] To ascertain that these results are not driven by few extreme values, the number of higher, lower and equal default probabilities are compared between the two models. The results show that 4,033 observations (74.3%) yield higher default probabilities within the multinationals model, 1,281 observations (23.6%) yield lower default probabilities within the multinationals model, and 117 observations (2.2%) yield equal default probabilities for both models.[18] Not only are the implied default probabilities larger in their qualitative measure within the multinationals model, but also they are larger in their quantitative measure. The medians evolved from the multinationals model and the Merton model, 0.1169 and 0.0518, respectively, provide additional support for the higher implied default probabilities often found when incorporating foreign exchange exposure into a multinationals' credit risk model.

For purpose of robustness, we conduct the nonparametric *Wilcoxon-Mann-Whitney* test, often used to detect differences in central tendency between any two

[17] The t-distribution with $\alpha = 0.01$ and ∞ degrees of freedom admits a critical value of 2.6259.

[18] These results essentially imply that the variability of the face value of debt damages multinationals in 74.3% of the cases, but facilitates in 23.6% of the time.

distributions.[19] This allows us to compare independent distribution-free samples, without assuming that observations are normally distributed. The Wilcoxon-Mann-Whitney test first merges the two batches of data, the Merton and the multinationals implied default probabilities, while observations are tagged, so that the batch to which observations belong is recorded. The combined dataset is sorted in ascending order and ranks are assigned to each observation. These ranks are integers in consecutive order. The statistic SR_1 is defined as the sum of ranks held by observations from the first batch (i.e. the Merton model), and the statistic SR_2 as the sum of ranks held by observations from the second batch (i.e. the multinationals model). The *Mann-Whitney U-statistics* are computed as:

$$U_1 = SR_1 - n_1(n_1 + 1) / 2 \text{ and } U_2 = SR_2 - n_2(n_2 + 1) / 2 \tag{5.2}$$

where in this case, $n_1 = n_2$ is the number of observations in each batch, and $n(n+1)/2$ is the sum of all ranks in a batch. The null distribution of the *U-statistic* is approximately Gaussian with

$$\mu_U = n_1 n_2 / 2 \text{ and } \sigma_U = \sqrt{(n_1 n_2 (n_1 + n_2 + 1) / 12)} \tag{5.3}$$

[19] The technique was first developed by Wilcoxon (1945), and expanded by Mann and Whitney (1947).

The standardized departure from the null distribution is $(U_2 - \mu_U) / \sigma_U$.[20] The test yields: $SR_1 = 27{,}322{,}106$; $SR_2 = 31{,}674{,}847$; $n_1 = n_2 = 5{,}431$; $U_1 = 12{,}571{,}510$; $U_2 = 16{,}694{,}251$; $\mu_U = 14{,}747{,}881$; $\sigma_U = 163{,}405$ and the standardized departure is *13.32*. The *Cumulative Distribution Function* (CDF) of the normal distribution admits a *P-value* of *1.000* associated with this departure, suggesting that the difference between the Merton and the multinationals implied default probabilities is highly significant.

Next we examine the determinants triggering the rise in implied default probabilities once incorporating stochastic foreign exchange exposure. The gap between the multinationals and the Merton models points to the impact of exchange rate exposure on multinationals' credit risk. It may be derived from several plausible sources, such as: debt-ratio, assets-exchange rate correlation, exchange rate volatility, physical assets' book value, and foreign debt-ratio. Debt-ratio, calculated as total debt divided by total assets is a viable parameter in measuring companies' total risk and thus, may affect variables associated with credit risk. Assets-exchange rate correlation and exchange rate volatility are expected to have a direct effect on assets volatility, $\delta_{V,F}$, as derived in the theoretical model, hence on total multinationals' credit risk. Foreign debt-ratio, computed as the fraction of non-domestic debt out of one's total debt may proxy the debt exposure to foreign currency movements. Large physical assets' value, collected here as property, plant and equipment, may intensify any exchange rate exposure effect on multinationals' credit risk as well.

[20] Exact probabilities of either of these statistics, for small samples, can be found in special tables. For large number of observations, the Gaussian distribution is often used as a good approximation.

The final dataset contains multinationals from 198 different industries. These can be batched into several categories, including metals, oil and gas, real estate, electronics, food and beverages, textile and clothes, wood products, books and publishing, chemicals, machineries, other manufactures, agricultural services, healthcare services, transportation services, communication services, energy services, stores services, entertainment services, and other services. These 19 industry types are further grouped into two industry classes: service-providers and goods-manufactures. To examine whether the impact of exchange rate exposure on multinationals' credit risk is more pronounced among specific industries, these two *categorical predictors* are added to the analysis.[21] The difference between the implied default probabilities is regressed, as a dependent variable, against debt-ratio, assets-exchange rate correlation, exchange rate volatility, physical assets' book value, and foreign debt-ratio as independent variables, and industry-class and industry-type as categorical predictors. The cross-sectional analysis is therefore

$$\begin{aligned} \Phi(-d'_2) - \Phi(-d_2) = \alpha &+ \beta_1 Debt\_Ratio + \beta_2 Assets\_ER\_Corr \\ &+ \beta_3 ER\_Volatility + \beta_4 Physical\_Assets\_Book\_Value \\ &+ \beta_5 Foreign\_Debt\_Ratio + \beta_6 Industry\_Class + B_7 Industry\_Type + \varepsilon \end{aligned} \tag{5.4}$$

[21] Categorical predictors, also called *indicator variables* or *dummy variables*, can be used to find correlation between variables in the same way continuous variables are used.

Table 3 summarizes the regression analysis results. The computed F-statistic suggests that the model itself is statistically significant.[22] The regression equation admits a $R^2 = 0.3694$, and slightly below, an adjusted $R^2 = 0.3686$.[23] About 37 percent of the model can be explained by the independent variables, largely due to the assets-exchange rate correlation, the exchange rate log-return volatility and the physical assets' book value, found to be highly statistically significant. The realized positive and significant relations between assets-exchange rate correlation, exchange rate volatility, and the impact of exchange rate exposure on multinationals' credit risk are as predicted by the multinationals theoretical model, and as demonstrated in **Figures 1 – 2**.

The third statistically significant coefficient points to a positive and strong correlation between the physical assets' book value, and the impact of exchange rate exposure on multinationals' credit risk. The regression analysis reveals that multinationals with higher book value of property, plant and equipment, also bear a greater effect of exposure to foreign exchange, on their credit risk. This phenomenon can be attributed to the fact that multinationals often possess their physical assets in several geographic regions. Thus, when larger value of physical assets is exposed to foreign exchange variability, this exposure's negative effect on multinationals' credit risk,

[22] The power analysis of *F-test* reflects the proportion of variance accounted for by some source in the population (*PVs*), relative to the residual variance proportion *(PVe)*: $F^2 = PVs / PVe$. For further explanations, see Cohen (1977).

[23] The adjusted R-square, generalizing for the number of variables included in the regression equation, only slightly deviates from the R-square. This can be used as another indication for a well-fitted solution to the dataset.

worsen.[24] A word of caution is needed here. One may not conclude that larger book value of assets triggers higher default risk. The discussion above merely points to a strong empirical link between book value of assets and the impact of exchange rate exposure on multinationals' credit risk.

Debt-ratio, does not contribute much to the predictive power of the regression equation. No clear relation is found between the impact of exchange rate exposure on multinationals' credit risk and their debt-ratio, measured as face value of debt over assets' market value. Foreign debt-ratio captured as the weight of debt denominated in foreign currency out of total debt admits a negative coefficient, but lack statistically significance. This may suggest that, to some extent, a firm diversifying debt issuances over several geographical regions can form a limited natural hedge against foreign currency fluctuations. The coefficients of both industry-class and industry-type are not statistically significant as well. This suggests that there is no correlation between the impact of exchange rate exposure on multinationals' credit risk and a specific or a group of industries. Thus, we further conclude that the impact of exchange rate exposure on multinationals' credit risk emerges across a wide range of industries and sectors, as well as various levels of leverage.

One may suspect that there are linear relationships among the independent variables causing the regression model coefficients to be unstable. In particular, such

[24] The fact that physical assets' book value is highly associated with the impact of exchange rate exposure on multinationals' credit risk proves that the traditional Merton model does not fully capture assets exposure to foreign currency movements.

relationship may occur between debt-ratio and industry type or industry class. To refute any *multicollinearity* among the predictors, the *variance inflation factor* is computed. However, none of these values exceed 10, and therefore the standard errors are not inflated.

The multinationals model is found to often assign slightly higher default probabilities to multinationals, relative to the standard Merton model. This is due to the foreign exchange exposure. An appealing investigation will be to examine, whether any of the models outperforms the other. A *Receiver Operating Characteristics* (ROC) of a classifier shows its performance as a trade off between type-I and type-II errors.[25] This test has been discussed and used in the finance literature, among others by Kealhofer (2003), Stein (2003) and Jarrow and Van Deventer (2004). A type-I error relates to failing to identify a default in advance, or to assigning a lower default probability than a fixed cutoff point. A type-II error relates to misidentifying an eventually non-default firm as a default candidate, or to assigning a higher default probability than the fixed cutoff point. Minimizing one type of error often comes at the expense of increasing the other. The following *contingency table* illustrates this trade off.

	Actual Default	**Actual Non-Default**
Model Predicts Default	TP	FP (Type-II Error)
Model Predicts Non-Default	FN (Type-I Error)	TN

[25] Moody's uses the term: *Cumulative Accuracy Profiles* (CAP), rather than ROC. Other common names in the literature are *lift-curves*, *power curves* and *dubbed-curves*. All convey the same information. For further explanation, see Sobehart, Keenan and Stein (2000) and Stein (2002).

A default is been classified as a positive (P) event while a non-default as a negative (N) one. If the model correctly predicts the outcome within a one-year interval, it is a true (T) classifier, where if the model is wrong, it is a false (F) classifier. Entries within columns are complements to 1, thus the probabilities of TP and FN, as well as FP and TN sum to 1.

In order not to contrast the models with only one predetermined cutoff point, typically, a ROC curve of false positive rate on the horizontal axis versus true positive rate along the vertical axis is plotted, while allowing the cutoff point to vary. In this way, the probabilities of type-II errors are plotted against one minus the probabilities of type-I errors. The ROC curve always goes through two points, (0, 0) and (1, 1). The point (0, 0) classifies all cases as non-default, hence the classifier discovers all the negative cases, but finds no positives. The point (1, 1) classifies all cases as positive, hence all default cases are correctly predicted, but all non-default cases are mistakenly identified. A classifier with random guesses will have a ROC curve along the diagonal between these two points while a better predictive power will result in a ROC curve closer to the north-west corner. When the locus of points generating the ROC curve is closer to the (0, 1) corner, the model has less type-I error and / or less type-II error hence a higher predictive strength.

Figures 6 – 8 present the ROC curves for the standard Merton model, for the multinationals model and when both models are aligned together for a comparative analysis. The curves show that there is a little difference between the models with respect to their predictive powers. In several occasions the multinationals model slightly deviates below the traditional Merton model, in others it rises above. Since the multinationals model considers also foreign exchange risk, and hence it is somewhat stricter than the Merton model with respect to the multinationals' dataset, it tends to assign higher default probabilities, thus it often ends with less type-I error probabilities, but occasionally contains more type-II error probabilities.

An *intra-cohort analysis* yields similar yet not entirely identical patterns when first sorting the firms into cohorts according to the Merton model and then resorting within each of the cohorts by the multinationals model, or while reversing the order of the two sorts. The intra-cohort analysis is a nonparametric statistical procedure presented by Miller (1998), and is later used by Kealhofer (2003). Its goal is to evaluate whether one model has additional information not included in a second classifier. The technique takes a population of firms one year prior to default and sorts these firms into cohorts according to a first classifier, where each cohort contains observations with about the same default probabilities. Then a second classifier is used to sort the same firms again, within each of the predetermined cohorts. If the second model adds predictive power, there should be a relatively higher default rate for the riskier firms within each cohort, as determined by the second classifier. Otherwise, default rates are expected to follow a uniform distribution. Reversing the order of the two sorts can be used to evaluate whether the first classifier contains information not captured by the second model.

Tables 4 – 5 describe the direct intra-cohort analysis as well as the reversed test. A comparison of these two tables points to a slight difference between the predictive powers of the two models. When first sorting the implied default probabilities within the Merton model and then by the multinationals model, few more observations are batched around higher percentiles, than while reversing the sorting order. **Figures 9 – 10** generate the histograms derived from these two tables, suggesting that the multinationals model contains a slight superior predictive strength.

The *Probit regression analysis* can also be used as a statistical test to compare the predictive power of both models. Since the default event is a *binary outcome*, it can be regressed, as a dependent variable, against the continuous independent variable of implied default probabilities, derived from each model. In this case, the Probit models take the form:

$$Default\_Event_i = \alpha + \beta Default\_Probability_i + \varepsilon_i \tag{5.5}$$

where

$Default\_Event_i = 0$ *if no default and 1 if default*

Table 6 presents the Probit models analyses for the Merton model and for the multinationals model. Both panels yield similar coefficients with similar significance level, suggesting that no further predictive power exists in any of the models.

VI. The Economic Importance of the Multinationals model

To estimate the economic value of the proposed multinationals model, we perform simulation experiments using the final dataset, the implied default probabilities derived from both the Merton model and the multinationals model, and the power analysis. This analysis illustrates the economic consequences of making lending decisions based on each model.

Our setting describes a realistic environment where lenders measure the borrowers' creditworthiness, and charge monotonic increasing interest rate payments with respect to default risk. All lenders compute implied default probabilities and make independent lending decisions based on the Merton model and the multinationals model. In case of the cutoff-based lending point falls below the implied default probability, the borrower is not granted credit, and the lender takes neither profits nor losses. If however, the cutoff-based lending point is higher than the implied default probability, the loan is accredited and further profits or losses are expected.

We make simplifying assumptions that all multinationals seek loans with the same amount of one million dollar and for the same time-horizon of one year. If a loan transaction matures prior to default, the lender receives its predetermined interest rate charges as a profit. However, if default event occurs before the end of the lending year, the lender bears a loss of one million dollar minus the recovery rate (RR). Underwriting fees are presumed to be constant and equal for all transactions, and hence can be ignored.

We also consider two types of lenders. The first lender allows interest rate charges to vary within a narrower interval. It assigns the borrower a cost function of *(3+10* Default Probability)%*, thus interest rate payments are near 3% for the high credit quality multinationals, and may reach up to 13% for the riskier ones. The second lender charges lower payments from high credit quality borrowers, but higher interest rate payments from riskier multinationals. It assigns the borrower a cost function of *(2+20* Default Probability)%*, thus interest rate payments are near 2% for the high credit quality multinationals, and may approach 22% for the riskier borrowers.

A profitable lending transaction occurs when the cutoff-based lending point falls above the implied default probability and when the borrower does not default prior to loan maturity. Therefore, the probability for a profitable lending transaction is the complement of a type-II error (TN in the contingency table). A costly lending transaction occurs when the cutoff-based lending point falls above the implied default probability and when the borrower defaults prior to loan expiration date. Thus, the likelihood for a costly lending transaction is the type-I error probability (FN in the contingency table).

The total lender's expected profit / loss profiles are estimated first with implied default probabilities derived from the Merton model, and then from the multinationals model. These profiles are compared over 21 different portfolios constructed from 21 different cutoff-points on the ROC curves. Considering the cutoff-based lending point to be 1.0 composes the first portfolio, thus loans are given to all possible borrowers. Considering the cutoff-based lending point to be 0.0 composes the last portfolio, thus loans are not granted to any of the multinationals, and therefore no profits or losses are expected.

Figures 11 – 12 illustrate that the slight difference in the models' predictive power, leads to economically significant differences in portfolio performance. For example, at the most profitable cutoff-based lending points, with the first cost function and a recovery rate equals to 0.2 million dollars, the multinationals model creates an excess return of $27.15M to the lender. At the most profitable cutoff-based lending points, with the second cost function and a recovery rate equals to 0.4 million dollars, the multinationals model creates an excess return of $51.01M to the lender.

In practice worldwide financial institutions tend to set their cutoff-based lending point based on risk management practices. A cutoff point closer to 100% is considered lenient since most applicants are granted loan. In contrast, a cutoff point closer to 0% is considered strict since most borrowers are denied loan. Different banks may choose various decision criteria, but common leniency is noticed. The Canadian Bankers Association (CBA) survey, from 1997, shows that 87% of small-business loan requests are approved at the seven largest banks. At the same year, the Canadian Federation of Independent Business (CFIB) reports on 89% banking average loan approval rate for small-businesses. Other U.S. lenders report on their web sites on even higher lending approval rates over the past years. Under the lenient policy, only firms with exceptionally high credit-risk are not granted credit. These relatively high lending approval rates correspond to cutoff-based lending interval between 0.8 and 0.9. Within this region, the supremacy of the multinationals model indisputably leads to economically significant advantages for lenders.

VII. End Notes

This article aims to investigate the correlation between exchange rate exposure and multinationals' credit risk. A unique credit risk model for multinationals is created, by incorporating stochastic foreign exchange exposure into the traditional Merton model. A comparison between the implied default probabilities of the standard Merton model and the proposed multinationals model reveals that most implied default probabilities rise when foreign exchange exposure is entitled. Parametric and nonparametric methodologies find this difference to be statistically significant. The two-sample Student's t-test suggests that the difference in implied default probabilities derived from the models is highly statistically significant. The distribution-free nonparametric Wilcoxon-Mann-Whitney test assesses this finding. The proposed multinationals model captures a 0.046 increase in the average default probability among multinationals in the data sample, relative to the standard Merton model. We therefore conclude that foreign exchange exposure has a small but robust negative impact on multinationals credit risk.

The small negative effect, a foreign exchange exposure has on multinationals' credit risk is exceedingly associated with assets-exchange rate correlation, exchange rate volatility, and physical assets' book value. However, no clear relation has discovered between the impact of exchange rate exposure and debt-ratio, foreign debt-ratio, industry type and industry class. According to the parametric ROC curves, the nonparametric intra-cohort analyses, and the Probit models, the proposed multinationals model slightly outperforms the traditional Merton model with regards to their predictive powers. This small difference, however, leads to economically significant differences in portfolio performances.

Since a theoretical reasoning is established, and supported by an empirical significant correlation between exchange rate volatility and the impact of exchange rate exposure on multinationals' credit risk, it is probable that more rapid movements in foreign exchange rates might prompt a stronger such effect. Further lines of investigation could examine how multinationals' credit risk varies around historic events, relatively to a control group of domestic firms. The European market crisis from 1992 to 1993, the Mexican crisis from 1994 to 1995, and the Asian financial crisis from 1997 to 1998, all followed by exchange rates turmoil, can be used as event studies for this purpose. This article may also lead to more appealing inquiries, such as whether an optimal strategy for global asset and debt allocation can be found, or whether agent problems arise from the impact of exchange rate exposure on multinationals' credit risk.[26]

Furthermore, the methods presented in this article can be used in much broader applications than merely the impact of exchange rate exposure on multinationals' credit risk. Debt covenants and renegotiation over its constraints are some of the factors causing

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[26] Endogenous selection of home and foreign portfolio weights for equity value maximization on the expense of debt in equation (3.38) may cause conflicts between shareholders and creditors.

the face value of debt to become stochastic. The framework used here can be modified for these variants as well.

Appendix 1

If $V \sim Lognormal$ then: $ln(V) \sim N(\mu, \sigma^2)$. In addition: $Z = (ln(V) – \mu)/\sigma \sim N(0, 1)$. The conditional mean of V given $V < D$ (within the truncated log normal distribution), can be expressed as

$$E[V|V < D] = E[e^{\sigma Z+\mu}|\ e^{\sigma Z+\mu} < D] = E[e^{\sigma Z+\mu}|Z <(ln(D)-\mu)/\sigma]$$

This can be simplified but using the following notations

$$h_1(\mu, \sigma) = (ln(D)-\mu)/\sigma \sim N(0, 1)$$

$$h_2(\mu, \sigma) = \Phi(h_1(\mu, \sigma))$$

Where Φ is the Cumulative Distribution Function (CDF) of the standard normal distribution. Using the general theorems (Greene 2000)

$$E[x|x < a] = \int_{-\infty}^{a} xf(x|x < a)dx$$

$$f(x|x < a) = f(x) / Prob(x < a)$$

$$x \sim N(\mu, \sigma) \Rightarrow Prob(x < a) = \Phi((a-\mu)/\sigma)$$

The conditional mean becomes

$$E[V|V < D] = (\int_{-\infty}^{h2(\mu,\sigma)} exp(\sigma Z + \mu)exp(-z^2 / 2)dz) / (\sqrt{(2\pi)})h_2(\mu,\sigma)$$

$$= (exp(\mu + \sigma^2 / 2)\int_{-\infty}^{h1(\mu,\sigma)} exp(-(z-\sigma)^2 / 2)dz) / (\surd(2\pi))h_2(\mu,\sigma)$$

$$= (exp(\mu + \sigma^2 / 2)\Phi((ln(D)-\mu) / \sigma - \sigma) / \Phi((ln(D)-\mu) / \sigma)$$

By substituting the mean $ln(V) + (\mu_V - \sigma^2{}_V / 2)t$ instead of μ, and the standard deviation $\sigma_V\surd t$ instead of σ, after a simple algebra the required identity is obtained■

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Figure 1

Sensitivity Analysis between Assets-Exchange Rate Correlation and Default Probability

The implied default probability of the first type multinationals model, $\Phi(-d'_2)$, is plotted as a function of assets-exchange rate correlation, $-1 \leq \rho_{V,S} \leq 1$, holding everything else constant.

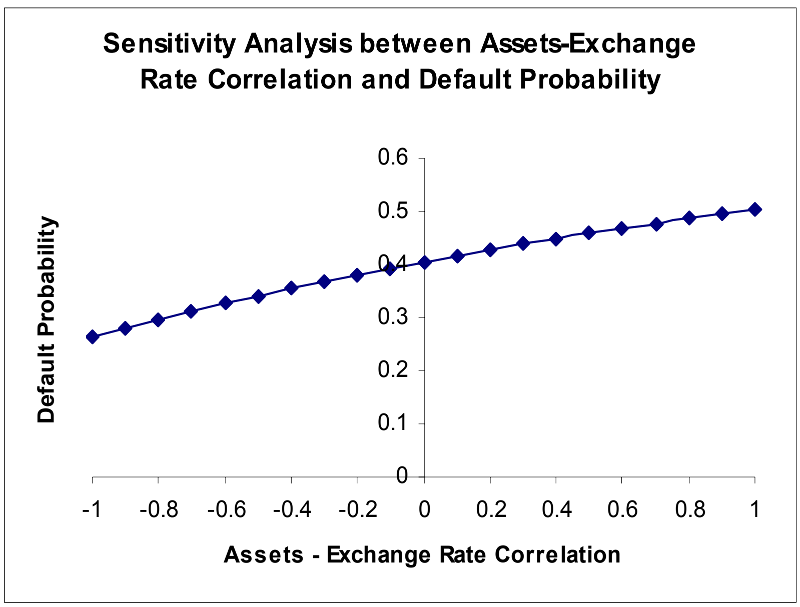


Figure 2

Sensitivity Analysis between Exchange Rate Volatility and Default Probability

The implied default probability of the first type multinationals model, $\Phi(-d'_2)$, is plotted as a function of exchange rate volatility, σ_S, holding everything else constant.

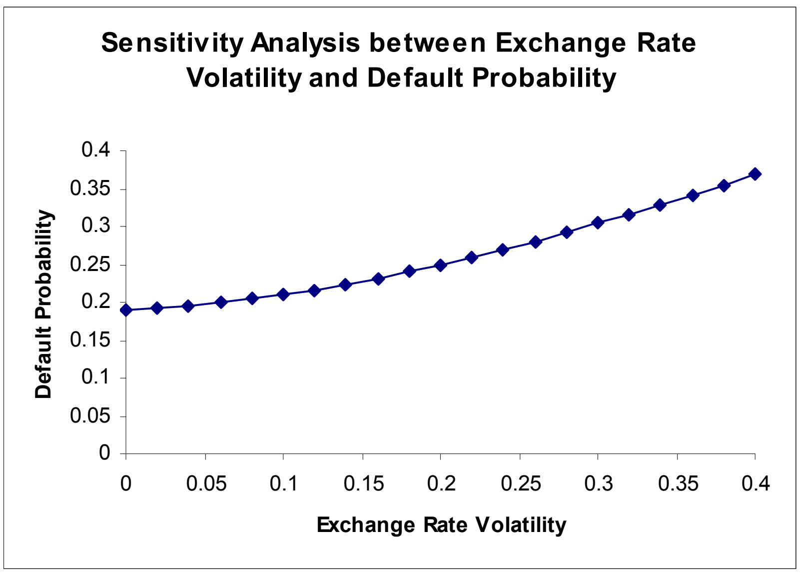
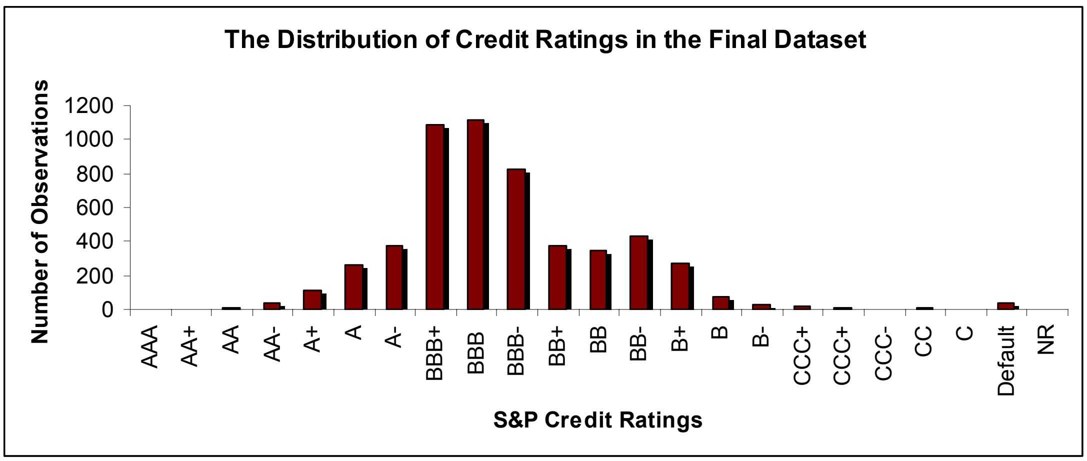


Figure 3

The Distribution of Credit Ratings in the Final Dataset

The distribution of credit ratings in the final dataset shows that out of the 5,431 observations, 1,221 observations, or 22.5% are at or below ‘BB’, the highest category classified as non-investment-grade, or “junk bonds”.



NR = Not Rated

Figure 4

The Distribution of Industries in the Final Dataset

The distribution of industries in the final dataset is presented as a pie chart.

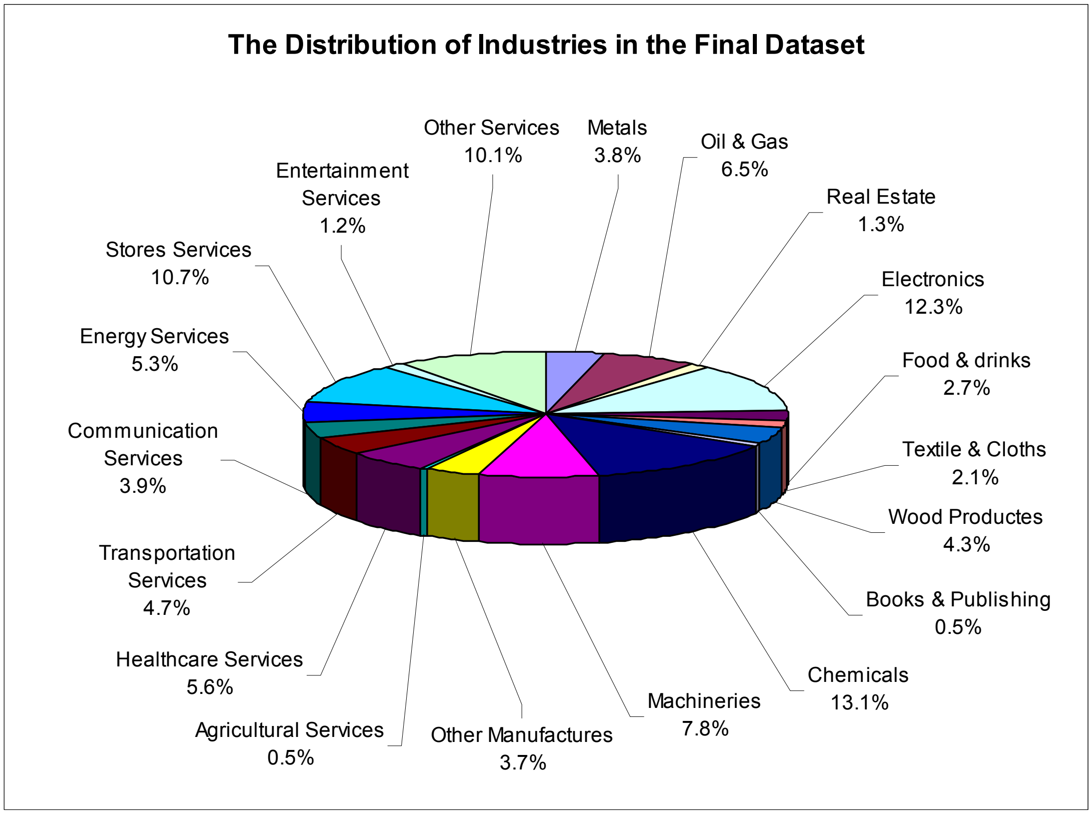


Figure 5

Distributions of Default Probabilities for both Models

The histograms of default probabilities for both the standard Merton model and the multinationals model are plotted, with bins width equal to 0.1. Both distributions may be approximated by truncated normal distributions and thus can be compared through a *two-sample Student's t-test.*

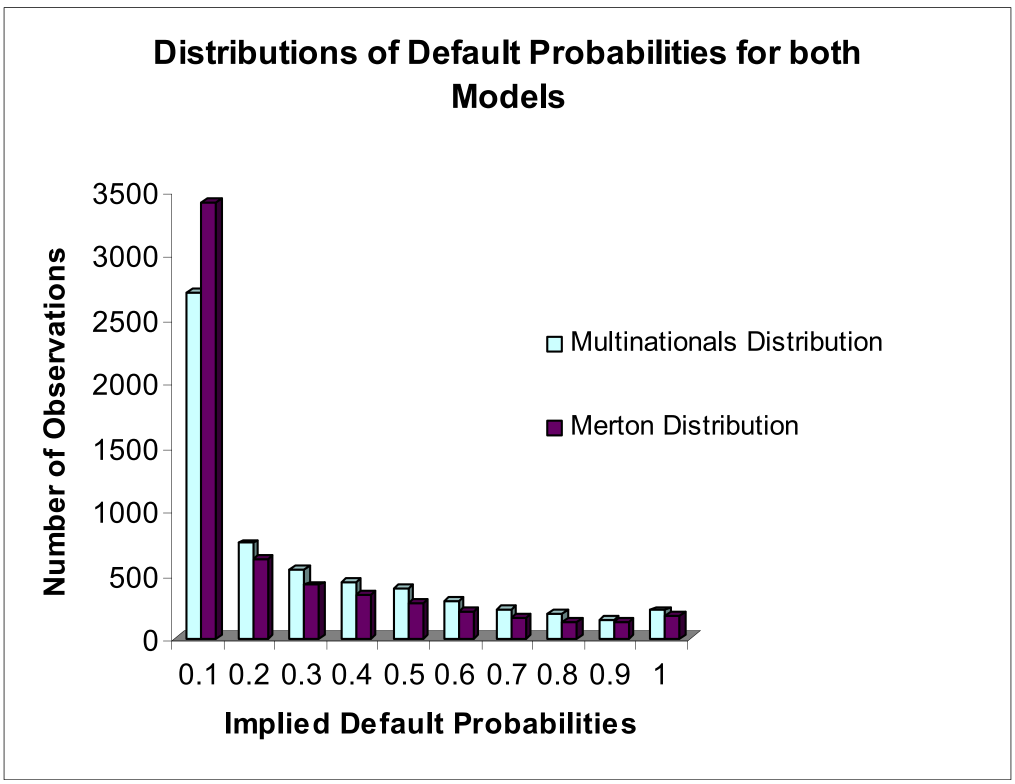


Figure 6

ROC Curve for the Merton Model

The Receiver Operating Characteristics (ROC) curve for the standard Merton model is presented below.

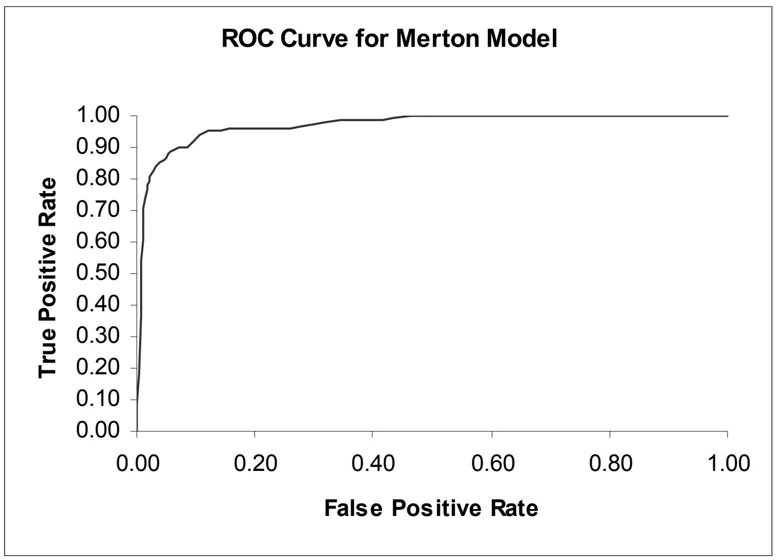


Figure 7

ROC Curve for the Multinationals Model

The Receiver Operating Characteristics (ROC) curve for the multinationals model is presented below.

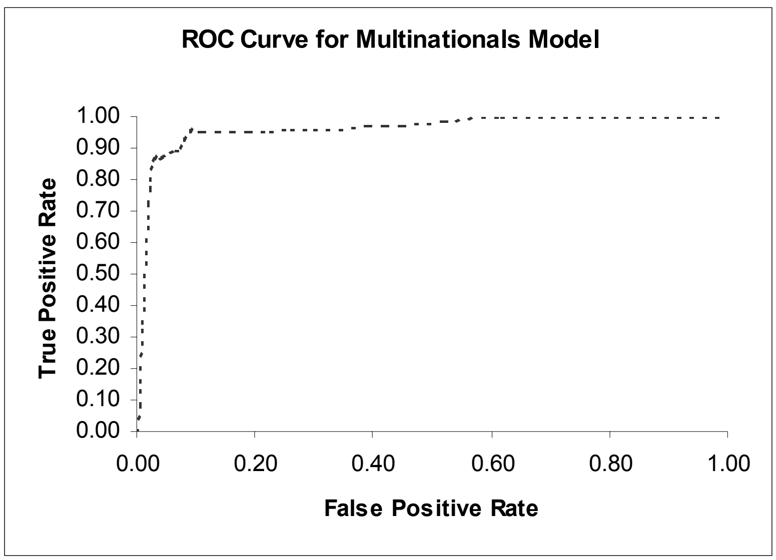


Figure 8

ROC Curves for the Merton and the Multinationals Models Aligned Together

Aligning the two ROC curves of both models together for a comparison.

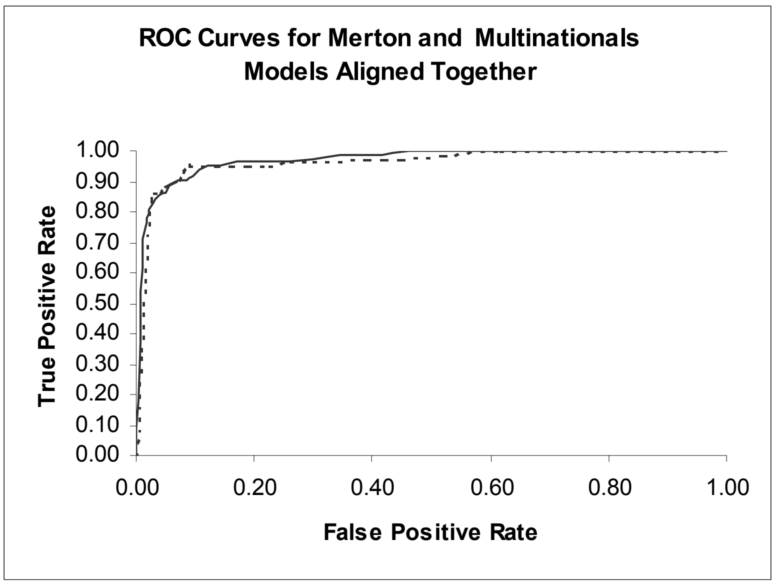


Figure 9

Intra-Cohort Analysis with the Merton Model as the First Measure and the Multinationals Model as the Second Measure

A histogram of the intra-cohort analysis twelve months prior to default as derived from Table 4 is presented below. The columns represent the number of defaults as sorted by the second measure within each cohort after the first sort based on the first classifier. The positively sloped dashed lines illustrate non-uniform distributions with a slightly stronger predictive power to the second classifier, the multinationals model.

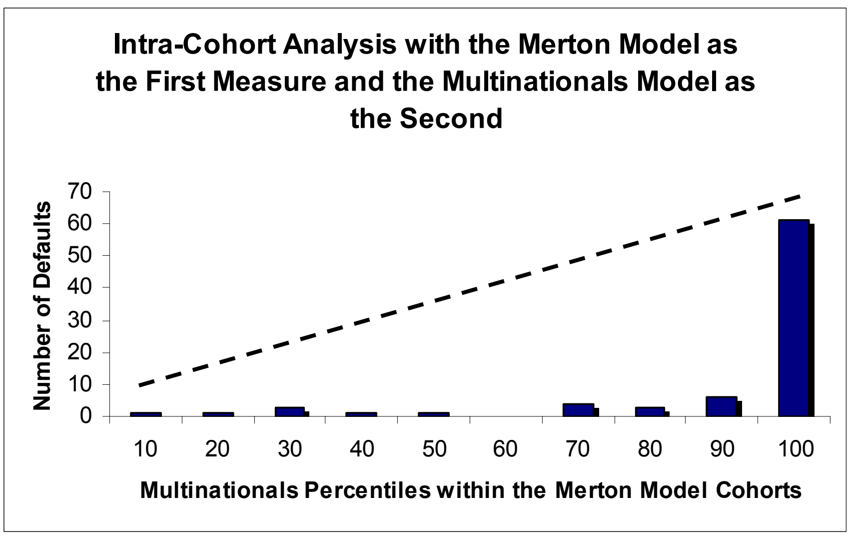


Figure 10

Reversed Intra-Cohort Analysis with the Multinationals Model as the First Measure and the Merton Model as the Second Measure

A histogram of the intra-cohort analysis twelve months prior to default as derived from Table 5 is presented below. The columns represent the number of defaults as sorted by the second measure within each cohort after the first sort based on the first classifier. The positively sloped dashed lines illustrate non-uniform distributions with a slightly stronger predictive power to the second classifier, the Merton model.

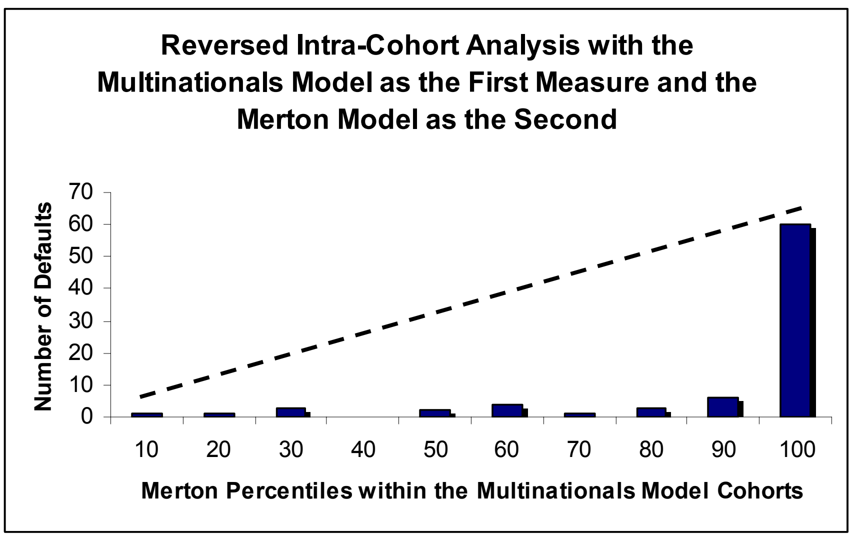


Figure 11

Profit / Loss Profile, Example I

The economic importance simulation is conducted over several different monotonic increasing cost functions with respect to implied default probabilities, as well as a wide spectrum of recovery rates. We measure the profitability of a lender holding 21 different portfolios constructed from 21 different cutoff-points on the ROC curves. Below is an example of profit / loss profiles for both models illustrating the economic consequence of differences in power.

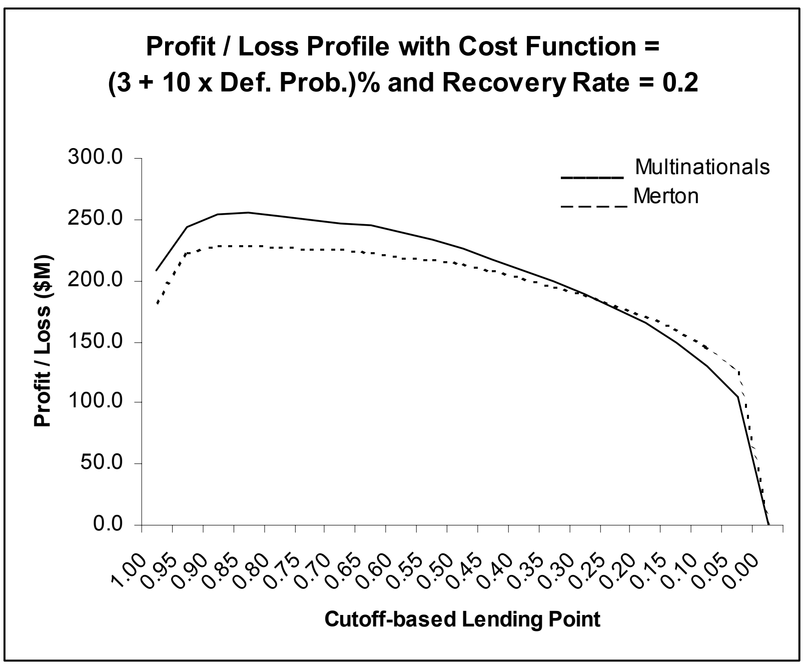


Figure 12

Profit / Loss Profile, Example II

The economic importance simulation is conducted over several different monotonic increasing cost functions with respect to implied default probabilities, as well as a wide spectrum of recovery rates. We measure the profitability of a lender holding 21 different portfolios constructed from 21 different cutoff-points on the ROC curves. Below is another example of profit / loss profiles for both models illustrating the economic consequence of differences in power.

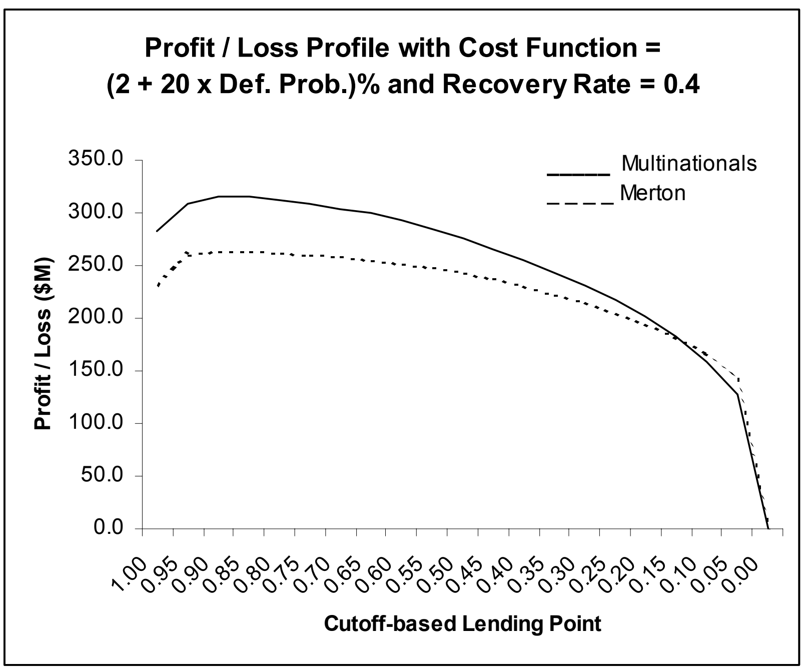


Table 1

Summary Statistics

Below is a summary statistics on the multinationals in the final dataset, along with their origin geographic region (home market) and the starting year data are available for them. Altogether there are 5,431 observations derived from 412 multinationals from 1996 to 2003. The numbers in the brackets represent the defaulted multinationals, eleven from the U.S. and one from Central Europe.

	U.S.	Africa	Middle East	Oceania	Asia	Central Europe	U.K.	Scandinavia	Eastern Europe	Canada	Central America	South America
1996	3						1					
1997	43		1				4	1		3	1	1
1998	153					2	4			6		1
1999	54 (1)			1		3		1		5		1
2000	45 (3)					2		2		3		
2001	36 (3)							1		3		
2002	22 (4)					2 (1)				1		
2003	4						1			1		

Table 2

Descriptive Statistics

A descriptive statistics for the relevant variables in the proposed multinationals model is presented below.

	Mean	St. Dev.	Median	Skewness	Kurtosis
Market Value of Equity ($B)	6.725	16.361	2.169	6.674	56.940
Equity' Log Return Volatility	0.349	0.380	0.248	4.615	51.381
Face Value of Debt ($B)	3.986	6.618	1.552	4.647	36.709
Time to Maturity (Years)	4.987	1.792	5.071	-0.076	-0.185
Risk Free Interest Rate	0.055	0.016	0.053	4.459	34.607
Combined Exchange Rate	2.771	18.965	0.940	11.474	141.829
Exchange Rate' Log Return Volatility	0.109	0.128	0.076	1.843	4.763
Market Value of Assets ($B)	9.321	19.331	3.290	5.689	41.695
Book Value of Physical Assets ($B)	2.173	3.912	0.704	4.500	30.181
Assets' Log Return Volatility	0.353	0.360	0.266	5.065	52.110
COV(Exchange Rate, Assets)	0.003	0.030	0.000	18.025	392.492
Implied Default Probability	0.229	0.267	0.117	1.201	0.471

Table 3

Cross-Sectional Analysis

A cross-sectional analysis is used to investigate the sources of the impact of exchange rate exposure on multinationals' credit risk. The linear relation, presented below, describes the difference between the implied default probabilities from the multinationals and the Merton models, as a dependent variable representing the impact of foreign exchange exposure on multinationals' credit risk, regressed against debt-ratio, assets-exchange rate correlation, exchange rate volatility, physical assets' book value, and foreign debt ratio, as independent variables, and industry class and industry type as categorical predictors. The variance inflation factor is used to refute any multicollinearity among the independent variables.

$$\Phi(-d'_2) - \Phi(-d_2) = \alpha + \beta_1 Debt\_Ratio + \beta_2 Assets\_ER\_Corr + \beta_3 ER\_Volatility + \beta_4 Physical\_Assets\_Book\_Value$$
$$+ \beta_5 Foreign\_Debt\_Ratio + \beta_6 Industry\_Class + B_7 Industry\_Type + \varepsilon$$

	Coefficient	Standard Error	t-value	Prob. > t	Variance Inflation Factor
Intercept	- 0.01924	0.00353			0
Debt Ratio	- 0.00000898	0.00001009	- 0.89	0.3733	1.00110
Assets-Exchange Rate Correlation	0.88725	0.03993	22.22	< 0.0001	1.00151
Exchange Rate Volatility	0.48711	0.00984	49.48	< 0.0001	1.10241
Physical Assets' Book Value	0.00163	0.00030969	5.25	< 0.0001	1.01593
Foreign Debt Ratio	- 0.01459	0.00817	- 1.79	0.0740	1.07939
Industry Class	- 0.00338	0.00498	- 0.68	0.4976	4.14824
Industry Type	0.00081175	0.00043236	1.88	0.0605	4.11048

F-value = 449.95 [(Prob. > F) < 0.0001]; R^2 = 0.3694; Adjusted R^2 = 0.3686

Table 4

Intra-Cohort Analysis, Part I

The results of the intra-cohort analysis for the Merton and the multinationals models are presented here.

		Second Measure - Multinationals Model									
		0-10%	**10-20%**	**20-30%**	**30-40%**	**40-50%**	**50-60%**	**60-70%**	**70-80%**	**80-90%**	**90-100%**
First Measure - Merton Model	**0-10%**	1									
	10-20%		1								
	20-30%			1							
	30-40%										
	40-50%				1						
	50-60%							2	1		1
	60-70%							1			
	70-80%							1	2		
	80-90%									3	3
	90-100%			2		1				3	57
Total:		1	1	3	1	1	0	4	3	6	61

Table 5

Intra-Cohort Analysis, Part II

The results of the reversed intra-cohort analysis for the Merton and the multinationals models are presented here.

First Measure - Multinationals Model	Second Measure - Merton Model									
	0-10%	**10-20%**	**20-30%**	**30-40%**	**40-50%**	**50-60%**	**60-70%**	**70-80%**	**80-90%**	**90-100%**
0-10%	1									
10-20%		1								
20-30%			1							
30-40%					1					
40-50%										
50-60%										
60-70%						2	1	1		
70-80%						1		2		
80-90%									3	3
90-100%			2		1	1			3	57
Total:	1	1	3	0	2	4	1	3	6	60

Table 6

Probit Models

The following Probit models regress the binary outcome of default, as a dependent variable, against the implied default probabilities, derived from the two models, as a continuous independent variable. The Merton model is described in panel A, and the multinationals model in panel B. The Probit models take the form:

$$Default\_Event_i = \alpha + \beta Default\_Probability_i + \varepsilon_i$$

where

$Default\_Event_i = 0$ *if no default and 1 if default*

Panel A

Variables in the Merton Model	Coefficient	Standard Error	t-value	Prob. > t
Intercept	- 0.00829	0.00136		
Implied Default Probability	0.08440	0.00428	19.74	< 0.0001

F-value = 389.70 [(Prob. > F) < 0.0001]; R^2 = 0.0670; Adjusted R^2 = 0.0668

Panel B

Variables in the Multinationals model	Coefficient	Standard Error	t-value	Prob. > t
Intercept	- 0.00989	0.00147		
Implied Default Probability	0.07452	0.00417	17.89	< 0.0001

F-value = 319.94 [(Prob. > F) < 0.0001]; R^2 = 0.0557; Adjusted R^2 = 0.0555

Homogeneous Markov Chain, Stochastic Economic, and Non-Homogeneous Models for Measuring Corporate Credit Risk

Abstract

This study examines new quantitative and comparative measurements of credit risk using the homogeneous Markov chain and stochastic economic models, and offers several innovative non-homogeneous dynamic models to explain credit ratings migration. The primary objectives are to discover some of the factors affecting the survivability of different debt portfolios, to present quantitative and comparative measurements for default rate and distance to default within a portfolio perspective, and to suggest various approaches for simulating ratings migration. The data sample used to test the models contains the S&P long-term credit ratings of industrial companies in North America from 1985 to 2004. A comparative analysis of the alternative models reveals that the density-dependent model is the most realistic approach, outperforming the homogeneous model in describing empirically observed ratings transitions. The measurements presented in this study can be used to compare and sort different debt portfolios with respect to their credit risk components, and to price credit-sensitive securities.

1. Introduction

Credit ratings are often assigned by rating agencies to public debt at the time of issuance, and are periodically revised, typically once every quarter. A change in the credit rating reflects an improvement or deterioration in the credit quality, from the rating agency's perspective. These ratings have a major effect on interest payments issued by the firm hereafter, thus companies frequently attempt to upgrade their credit ratings.[27] Although credit ratings are fundamentally meant to shed light on the current credit quality of a firm, several theories use the rating drift phenomenon to predict default events, and to price defaultable bonds.

Numerous studies explore credit ratings migration using the Markov chain process. Jarrow, Lando and Turnbull (1997) propose a discrete-space finite-state time-homogeneous Markov chain of credit ratings to describe the term structure of credit risk spreads. The authors develop a contingent claims model, incorporating finite-state Markov dynamics of bankruptcy into the valuation methodology of a risky zero-coupon bond. Using an absorbing time-homogeneity transition matrix, and assuming that the risk-free spot interest rate and the firm's credit ratings migration follow uncorrelated stochastic processes, they formulate the price of risky bonds. Kijima and Komoribayashi (1998) point to possible numerical problems that arise due to low default probabilities,

[27] A conceptual debate whether companies take continuous steps to upgrade their credit ratings, or discretely proceed only prior to new debt issuances, is still thriving. Löffler (2005) explores the common assertion that credit rating agencies take a rating action only when it is unlikely to be reversed shortly after, and concludes that this is the major rationale for why rating agencies are slow to react to new information. Therefore, companies may attempt to raise their credit ratings at all time.

for highly rated bonds, within relatively short periods, and offer different risk premium adjustments than that suggested by Jarrow, Lando and Turnbull (1997). Instead of the assumption of homogeneity for all transition matrix entries, the authors propose to consider transition probabilities as functions of time, and by that to allow them to change. Kijima (1998) discusses how prior ratings changes may help to predict future rating changes over time, conditional on a firms' survival. Using the first-order stochastic dominance in the Markov chain dynamic, the author explains how firms with low credit ratings are more likely to be upgraded while firms with high credit ratings are more likely to be downgraded, as the time horizon lengthens, given that they do not default.

Arvanitis, Gregory and Laurent (1999) contribute to the pricing of risky bonds and bond options, allowing memory in credit rating changes, stochastic jumps, and mean reversion in credit spreads. Similar to previous studies, the authors consider a two-state world, of default and non-default, a deterministic credit ratings migration matrix, along with the restriction for credit spread independent of the interest rates, but permit sudden jumps, memory and mean-reverting diffusion processes in credit rating changes. Thomas, Allen and Morkel-Kingsbury (1999) propose a Markov chain model for the term structure and credit spreads of risky bonds, allowing dependency between two stochastic processes, interest rate movements and credit ratings migrations. That establishes a link between credit rating movements and the state of economy. The authors also offer a linear programming algorithm to strip bonds from its coupons for better pricing accuracy. Bahar and Nagpal (2001) have proposed a non-homogeneous model to accommodate observed momentum. Their model enhances the standard Markov model with a partial, short-term memory. Each state describes not only the current rating category, but also the latest transitions. Dividing companies into three sets according to their rating over the previous year: upgraded, stable, or downgraded, is meant to capture the effect of past transitions as well. This methodology leads to three different transition matrices; one for each set of upgraded, stable, and downgraded companies. In the upgrade transition matrix, ratings migration probabilities are more prominent above the main diagonal, describing a common tendency of improvement in creditworthiness, while in the downgrade transition matrix, ratings migration probabilities are more pronounced below the main diagonal, suggesting a general propensity of deterioration in creditworthiness. Each of these three separate transition matrices is a Markov chain model by itself.

Bielecki and Rutkowski (2000) derive stochastic processes for pricing defaultable bonds and other credit derivatives from rating transition matrices as well. They model the intensities of credit migrations among various credit ratings classes by combining available data for credit spreads and recovery rates, and conditioning the martingale probabilities from the Markov chain model on interest rate risk. Israel, Rosenthal and Wei (2001) identify the conditions for a true generator of Markov transition matrices to exist, present several techniques for estimating an approximate generator when the true one is not feasible, and apply those methods to rating migration matrices.[28] This methodology helps to create different transition matrices through time, however it contains a major drawback; the generator does not correlate the transition matrices to any meaningful economic variable. Following Koyluoglu and Hickman (1998), Lucas,

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[28] A generator matrix G for a transition matrix T has a column-sums zero and non-negative off diagonal entries such that $exp(G)=T$.

Klaassen, Spreij and Straetmans (2001) derive an analytical approximation to the credit loss distribution of large portfolios by incorporating the Markov chain approach for credit ratings migration, along with several characteristics, such as credit quality, the degree of systematic risk, and the maturity profile. Assuming that rating transitions are normally distributed, the authors find that the rate of convergence of the actual credit loss quantiles to their analytic counterparts is influenced by the degree of portfolio heterogeneity.

The impact of business cycles on credit risk in general and on credit ratings migration in particular is well documented. Williamson (1987) provides support for real business cycle theory at the expense of monetary theories by constructing a model for credit supply that is affected by fluctuations in real output. Denis and Denis (1995) find that macroeconomic factors may trigger defaults, and that the relationship varies across business cycles. Kiyotaki and Moore (1997) develop a theoretical model for systematic credit risk factors correlated with business cycles. Nickell, Perraudin and Varotto (2000) quantify the dependency of rating transition probabilities both on the industry and domicile of the obligor and on the stage of the business cycle. By defining three categories of business cycles, peak, normal times and trough, depending on whether real GDP growth is at the top, middle or bottom third, the authors investigate the volatility of credit ratings transitions. Employing a probit analysis of Moody’s long-term corporate and sovereign bond ratings from 1970 to 1997, they discover that the business cycles variability is the most influential in explaining the volatility of transition probabilities. The authors conclude that applying the homogeneous transition matrix over high or low quality loans is likely to under- or overestimate the risk respectively.

Bangia, Diebold, Kronimus, Schagen and Schuermann (2002) empirically examine the homogeneous Markov chain model, and condition transition matrices on business cycles. They use the National Bureau of Economic Research (NBER) database to identify certain months as expansionary or contractionary stages. This identification is constructed by examining a large number of business indicators, and may reflect improving or deteriorating business cycles. Excluding the rating modifiers and considering only seven rating categories, the authors provide strong evidence for the assumption of Markov properties. Furthermore, the results support the momentum hypothesis, as most downgrade probabilities for the downward-momentum matrix are found to be larger than the corresponding values in the homogeneous matrix. Distinguishing between expansionary and contractionary stages, leads to a dramatic reduce in the level of uncertainty in transition estimations. However, the differences between the stochastic economic model and the unconditional simulation are relatively minor over short-term time horizons, though the differences gradually increase over time.

Kwark (2002) explores business cycles based on the interest rate spread between risky loan yields and risk-free rates. The author concludes that fluctuations of the interest rate spread are highly related to movements in default risk over the business cycle, and contain information about future fluctuations in output. Also considering business cycles, Wei (2003) relaxes the assumption of homogeneous Markov chain, and allows for rating inter-variations within the Markov chain states to price various credit derivatives. This analysis suffers from a reliance on some problematic assumptions. Deviations from the average credit score matrix are assumed to be mutually independent and normally distributed thus, ignoring the different frequencies, sequences, magnitudes and lengths of business cycles. Furthermore, the assumption that transition rates among different states are normally distributed excludes other likely patterns of cyclical behavior or mean reversion, found in several empirical studies. Farnsworth and Li (2004) develop a bond-pricing model where a firm's instantaneous probability of default is conditional on its credit ratings as well as on a latent systematic factor. Examining the model under constant transition rates and with stochastic ratings changes probabilities, linked to the systematic factor, over the Lehman Brothers fixed income proprietary database from 1985 to 1998, the authors find strong evidence supporting the non-homogeneous transition matrix.

The credit ratings migration methodology assumes the following: the rating agencies know how to tag the right rating label, all firms tagged within a given rating label share the same default risk, and historical sample data on credit rating migration has predictive strength regarding the future. These assumptions are presumed to hold throughout this study. The study proceeds as follows. Section 2 outlines the theoretical framework of the homogeneous Markov chain model and derives several comparative, quantitative, and sensitivity measurements. Section 3 discusses the theoretical arguments behind the dynamic of a stochastic economy then further explores default time distributions from a portfolio perspective. Section 4 reinvestigates some well-known non-homogeneous ratings migration dynamics, and introduces several new models. Section 5 briefly presents the data and examines the validity of the methods. Section 6 summarizes the findings and discusses their economic meaning, and Section 7 concludes and points to lines of possible future research.

2. Homogeneous Markov Chain Model for Measuring Credit Risk

Describing the life cycle of a company requires the distinguishing among different survival states and, in case of a default, an absorbing state. Credit rating agencies tag these states with different credit ratings while transitions among states are called *credit ratings migration*. Movements of companies through these credit ratings can be described as a *Markov chain* as follows. *Transition matrix T* is a *non-negative square matrix* of dimensions $(s+1)*(s+1)$, where its entries represent probabilities of all credit ratings migration. Matrix T contains a *survival matrix S* of dimensions $s*s$, representing transitions among livelihood states, an *absorbing row vector D* of dimensions $1*s$, representing transitions from livelihood states into a default state, and an arbitrary column vector of zeros and one.

$$T = \begin{matrix} S & 0 \\ D & 1 \end{matrix} \quad \text{of dimensions} \quad \begin{matrix} s*s & s*1 \\ 1*s & 1*1 \end{matrix} \tag{2.1}$$

The model considers default as chapter seven and not as chapter eleven. State $s+1$ is an absorbing state of default. Once a company enters this state, it cannot leave. The assumption that it is feasible to reach the default state from every survival state in S

guarantees that the *dominant eigenvalue* of S is strictly less than 1.[29] Iosifescu (1980, corollary of Theorem 2.4) proves that in this case, $lim\ S^t = 0$. Thus, probabilities to survive decay to zero when time lengths. Therefore, for every initial probability distribution, all companies eventually reach the absorbing state and default. This assumption seems more realistic than the *going concern* approach. Some companies will default faster than others, some will stay at the survival states longer than others, but the absorption state is inevitable for all companies. Let x be a column vector that contains the portfolio weights distribution of states with $x_i \in [0, 1]$ and $\Sigma_i x_i = 1$. Thus, the transition matrix T and the current distribution of states can project future distributions of states. In general

$$x(t + n) = T^n x(t) \qquad (2.2)$$

where

$$T^n = \begin{matrix} S^n & 0 \\ D\Sigma_{j=0}^{n-1} S^j & 1 \end{matrix}$$

The (i, j) entry in the matrix S^n represents the probability for a company in state j at time 0 to be in state i at time n, for every $i, j \in \{1, 2, \ldots, s\}$. Each column of T^n sums up to 1, since its entries represent all possible transitions from a given state, including the option of staying at the same state. This attribute remains true for every n.

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[29] The *Perron - Frobenius theorem* describes characteristics of eigenvalues and eigenvectors of non-negative matrices. A *dominant eigenvalue* is greater than or equal to the others in magnitude.

Define v_{ij} as the number of visits at survival state i before absorption, given that the company starts in state j. The expected values of v_{ij} can be summarized in the following fashion:

$$N = (E[v_{ij}]) = I + S + S^2 + \ldots = (I - S)^{-1} \qquad (2.3)$$

where I is the *identity matrix*, all matrices are of dimensions $s*s$ and the superscript of minus one represents *inverse*. Matrix N is often referred as the *fundamental matrix* of the Markov chain. Since S is a square matrix and $\lim S^t = 0$, then the fundamental matrix N exists and is equal to $(I - S)^{-1}$. This can easily be seen from the identity: $(I - S)(I + S + S^2 + \ldots S^{t-1}) = I - S^t$. The right-hand side of the equation converges to the identity matrix with a *determinant* equal to 1. Therefore, if t is large enough, $det(I - S^t) \neq 0$. Since a determinant of a product of two matrices is the product of their determinants, $det(I - S) \neq 0$, so $(I - S)^{-1}$ does exist. Multiplying both sides of the above identity by $(I - S)^{-1}$ yields: $I + S + S^2 + \ldots S^{t-1} = (I - S)^{-1}(I - S^t)$. Hence when t becomes large, equation (2.3) is obtained.[30]

Iosifescu (1980, Theorem 3.1) proves that higher moments, of any order, of the random variable v_{ij} are finite and correlated by a recursion equation. The second moment is then given by

[30] An attempt to first build the actual fundamental matrix, with actual rather than expected number of visits at different states before absorption, and then derive the real survival matrix backward trough $S = I - N^{-1}$, does not contribute much. This model relies on relatively few observations (in this study, 419 companies defaulted out of 4,510 companies from 1985 to 2004) and more important, this methodology is not verifiable.

$$(E[v^2_{ij}]) = (2N_{dg} - I)N \tag{2.4}$$

where N_{dg} is a square matrix resulting from setting all the entries off the main diagonal equal to zero. Hence, the variance of the number of visits at each survival state, before absorption is given by the matrix

$$(Var(v_{ij})) = (2N_{dg} - I)N - N \circ N \tag{2.5}$$

where ○ denotes the *Hadamard product.*[31] Common credit ratings migration analyses often calculate default probabilities, but the analysis of the fundamental matrix N provides additional important information about the *time to default*.

Define η_i as the time to default (i.e. the absorption state), conditional on starting at a credit rating (survival state) i. The time to default will be measured in units of quarters, since credit ratings are observed once a quarter. The mean of η_i is actually the number of visits to all survival states before bankruptcy, or the sum of entries in column i of the fundamental matrix N. Iosifescu (1980, Theorem 3.2) presents important properties of the distribution of time to default. The first moment is given by

$$(E[\eta_1], E[\eta_2], \ldots, E[\eta_S]) = Ne \tag{2.6}$$

[31] The *Hadamard product* between any two matrices of the same dimensions (not necessarily square matrices) is an entry-by-entry product.

where e is a column vector of ones. Thus, a right multiplication of e (or a left multiplication with the vector *e transpose*) gives the column sums. The second moment of times to default is

$$(E[\eta^2_1], E[\eta^2_2], \ldots, E[\eta^2_S]) = (2N - I)Ne \tag{2.7}$$

From the first and second moments the variance of time to default is derived as

$$(Var(\eta_1), Var(\eta_2), \ldots, Var(\eta_S)) = e^T(2N^2 - N) - e^T N \circ e^T N \tag{2.8}$$

In addition to the mean and the variance of η_i the complete distribution of times to default can be postulated. Matrix S contains all the transitions among survival states. Hence, the sum of entries of column j of S^t is actually the probability that default *has not* occurred until time t for a company that started at credit rating j. Therefore,

$$(P(\eta_1>t), P(\eta_2>t), \ldots, P(\eta_S>t)) = e^T S^T \tag{2.9}$$

Using the fact that $(P(\eta_i=t)) = (P(\eta_i>t-1)) - (P(\eta_i>t))$, the complete probability distribution for times to default is given by

$$(P(\eta_1=t), P(\eta_2=t), \ldots, P(\eta_S=t)) = e^T(S^{t-1} - S^t) \tag{2.10}$$

The credit risk of a *portfolio of companies* depends upon the transition probabilities among the different states. Hence, the sensitivity of the transition rate of a portfolio towards its inevitable outcome of default to the movements between any two credit ratings should be examined. Equation (2.2) implies that the survival of a portfolio of companies is determined by the behavior of S^n. The submatrix S^n contains all the information within T^n, since entries of the row vector for the absorption state are the complements to 1 for the corresponding columns in S^n. The survival matrix S is not a *diagonal matrix*, but if it has linearly independent eigenvectors, it is *similar* to a *diagonal matrix* Θ whose main diagonal entries contain the eigenvalues λ_i. Considering the survival matrix S as diagonalizable, gives[32]

$$S^n = \Sigma_i \lambda_i^n w_i v_i^* \tag{2.11}$$

where w_i are the *right eigenvectors* and v_i^* are the *complex conjugate transpose* (or *adjoint*) of the *left eigenvectors* corresponding to the *eigenvalues* λ_i.[33] Thus, examining the sensitivity of the eigenvalues, λ_i, of the survival matrix S, to its entries s_{ij}, may point to a group of credit ratings that are heavily responsible for changes in the credit risk of the whole portfolio. In fact, it is enough to examine the sensitivity of the *dominant*

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[32] For further discussion see Appendix 1.

[33] Eigenvalues may be complex numbers containing both real and imaginary parts. If $\lambda = a + bi$ is a complex number, its complex conjugate is $\hat{\lambda} = a - bi$ and its magnitude is $|\lambda| = \sqrt{(a^2 + b^2)}$.

eigenvalue, λ_1, to the entries s_{ij} since the dominant eigenvalue gives the asymptotic portfolio transition rate.[34]

When the characteristic equation is known, this sensitivity is explored, among others, by Hamilton (1966), Demetrius (1969), Emlen (1970), Goodman (1971), Keyfitz (1971), and Mertz (1971).[35] Caswell (1978) introduces a more general approach by estimating sensitivity of eigenvalues for all types of matrices. Cohen (1978) presents an alternative approach, which is used here. Equation (2.11) further suggests the following relationships between the survival matrix S, the eigenvalues λ_i, the right eigenvectors w_i and the complex conjugate transpose of the left eigenvectors v_i^*:

$$Sw_i = \lambda_i w_i \tag{2.12}$$

and

$$v_i^*S = \lambda_i v^*_i \tag{2.13}$$

Differentiating both sides of equation (2.12) while omitting the subscript *i*, yields

$$S(dw) + (dS)w = \lambda(dw) + (d\lambda)w \tag{2.14}$$

34 More on this in Appendix 2.

35 The *characteristic equation* is expressed as: $det(S-\lambda I)=0$.

where $dS = (ds_{ij})$ is a matrix with entries containing the differentials of s_{ij}. A *scalar product* of both sides with the left eigenvector v yields[36]

$$<S(dw),v> + <(dS)w,v> = \lambda<(dw),v> + <(d\lambda)w,v> \tag{2.15}$$

Opening the scalar products in equation (2.15) and canceling mutual terms yields

$$d\lambda = <(dS)w,v> / <w,v> = v^*dSw / v^*w \tag{2.16}$$

Examining the sensitivity to only one of the entries in the survival matrix S, while holding all the other entries constant, means that there is only one nonzero entry, ds_{ij}

$$d\lambda = \hat{v}_i w_j ds_{ij} / <w,v> \tag{2.17}$$

where $\hat{v}_i$ represents the complex conjugate of the left eigenvector v_i. Rearranging equation (2.17) gives the required sensitivity formula

$$\partial\lambda / \partial s_{ij} = \hat{v}_i w_j / <w,v> \tag{2.18}$$

Calculations based on this formula can be simplified, since the scalar product in the denominator is independent of row i and of column j and hence can be scaled such

[36] A scalar product of real vectors is the sum of the products of the corresponding entries.

that $\langle w,v \rangle = 1$. Finally, the sensitivities of the transition rate to all entries, s_{ij}, can be displayed within the *sensitivity matrix Q*:

$$Q = (\partial\lambda / \partial s_{ij}) = \hat{v} w^T / v^*w \tag{2.19}$$

Caswell et al. (2000) suggest that these derivatives point only to the *magnitude* of the change in the dominant eigenvalue, and neither to the *direction* nor to any *preconditions* for its variability. Those are open to interpretation.

To examine how sensitivities themselves respond to transition probabilities among the various life cycle states, the *second derivative* of the dominant eigenvalue, λ_1, to the entries s_{ij} can be found. This analysis may point to factors affecting the sensitivity itself, and could help to explore sensitivity behavior in different scenarios.

After scaling the denominator in equation (2.18) so that $\langle w,v \rangle = 1$, the sensitivity of λ_1 to the entry s_{ij} is given by

$$\partial\lambda_1 / \partial s_{ij} = \hat{v}_{i,1} w_{j,1} \tag{2.20}$$

where $w_{j,1}$ denotes the j^{th} element of eigenvector w_1 corresponding to the dominant eigenvalue λ_1. Differentiating equation (2.20) with respect to the survival entry, s_{kl} gives the second derivatives

$$(\partial/\partial s_{kl})(\partial\lambda_1 / \partial s_{ij}) = \partial^2\lambda_1 / \partial s_{ij}\partial s_{kl} = \hat{v}_{i,1}(\partial w_{j,1}/\partial s_{kl}) + w_{j,1}(\partial\hat{v}_{i,1}/\partial s_{kl}) \quad (2.21)$$

Hence, finding the *first derivatives* of the *eigenvectors* with respect to the survival matrix entries, s_{kl}, is necessary. The two solutions below follow Desoer (1967).[37]

$$\partial w_1 / \partial s_{kl} = w_{l,1}\Sigma_{m\neq 1}^{S}(\hat{v}_{k,m} / (\lambda_1 - \lambda_m))w_m \quad (2.22)$$

and

$$\partial v_1 / \partial s_{kl} = v_{k,1}\Sigma_{m\neq 1}^{S}(\hat{w}_{l,m} / (\hat{\lambda}_1 - \hat{\lambda}_m))v_m \quad (2.23)$$

Substituting equations (2.22) and (2.23) into equation (2.21) gives

$$\partial^2\lambda_1 / \partial s_{ij}\partial s_{kl} = \hat{v}_{i,1}w_{l,1}\Sigma_{m\neq 1}^{S}(\hat{v}_{k,m}/(\lambda_1-\lambda_m))w_{j,m} + \hat{v}_{k,1}\hat{w}_{j,1}\Sigma_{m\neq 1}^{S}\,\hat{v}_{i,m}/(\lambda_1-\lambda_m)w_{l,m} \quad (2.24)$$

Denoting $q_{il,m}$ as the *(i, l)* entry in the *sensitivity matrix* $Q_m = \hat{v}_m w^T_m$ corresponding to eigenvalue λ_m yields

$$\partial^2\lambda_1 / \partial s_{ij}\partial s_{kl} = q_{il,1}\Sigma_{m\neq 1}^{S}q_{kj,m}/(\lambda_1-\lambda_m) + q_{kj,1}\Sigma_{m\neq 1}^{S}q_{il,,m}/(\lambda_1-\lambda_m) \quad (2.25)$$

The discussion now turns to an examination of the *rate of convergence* towards the stable state of bankruptcy for a portfolio as a whole. Several scholars have examined credit risk from a portfolio perspective. Gollinger and Morgan (1993) consider default likelihoods while estimating default correlations across 42 industry indices. Stevenson

[37] The complete derivations are in Appendix 3.

and Fadil (1995) correlate default events among 33 industry groups. Both studies reveal the difficulty of estimating default correlations. Carey (1998) and Kealhofer (1998) discuss credit risk issues in private portfolio management. Froot and Stein (1998) show that the price of a marginal credit exposure depends upon several risk correlations within a bank's portfolio. Koyluoglu and Hickman (1998) analyze the correlations between market factors and credit risk factors within portfolios. These correlations are called *background factors*. Andersson et al. (2001) provide a *conditional value-at-risk* (CVaR) optimization criterion for credit risk management. Lucas et al. (2001) present analytical methods for measuring credit risk in large portfolios. Barnhill and Maxwell (2002) examine credit risk for fixed income portfolios. Frey and McNeil (2003) provide a cyclical correlation-based model of portfolio credit risk. Giesecke and Weber (2004) examine credit contagion within portfolios, and Acharya et al. (forthcoming 2006) discuss the costs related to diversification in a bank's loan portfolios.

Jafry and Schuermann (2004) consider in their analysis the complete transition matrix, as opposed to the survival matrix in this investigation. Since the complete transition matrix has a unit-dominant eigenvalue, they judge the rate at which the system decays towards equilibrium merely with the second-largest eigenvalue. In this study, the transition rate towards the absorption state of default and its oscillations are determined both by the magnitude of the dominant eigenvalue and the subdominant eigenvalue.

The strong ergodic theorem (in Appendix 2) implies that holding everything else constant, the convergence rate towards default is faster with a larger dominant eigenvalue, λ_1, relative to the subdominant eigenvalues. A *damping ratio* can then be defined as:

$$\delta = \lambda_1 / | \lambda_2 | \tag{2.26}$$

This transition rate towards default is fixed and independent of the initial portfolio construction. A convenient way to compare the credit risk among portfolios is by using their distances to default.

The *distance to default* for a portfolio can be defined as the distance between the current observed state distribution, x_0, and the absorption state of default. The last equation in Appendix 2 states that eventually x_t / λ^t_1 will converge to $c_1 w_1$. If the vector w were scaled so that $\Sigma_i w_i = 1$, it would represent the proportions of holdings within the portfolio in the stable state. Hence, measuring the distance between the two vectors along the path of default will imply the distance to default. Cohen (1979b) introduces two indices that can be used for that purpose. Those indices are obtained by accumulating the differences between x_t / λ^t_1 and $c_1 w_1$, where $c_1 = v^T{}_1 x_0$, with and without absolute values, as follows:

$$\textit{First Index } (S, x_0, t) = \Sigma_{i=0}{}^t (x_i / \lambda^i{}_1 - c_1 w_1) \tag{2.27}$$

$$\textit{Second Index } (S, x_0, t) = \Sigma_{i=0}{}^t |(x_i / \lambda^i{}_1 - c_1 w_1)| \tag{2.28}$$

To measure the cumulative distance between the initial portfolio probability distribution of states, x_0, and the stable portfolio probability distribution, Cohen (1979b) also offers the following distances

$$D_1 = \Sigma_j \lim \mid First\ Index\ (S, x_0, t)_j \mid \tag{2.29}$$

$$D_2 = \Sigma_j \lim \mid First\ Index\ (S, x_0, t)_j \mid \tag{2.30}$$

where the indices entries are first converted to their absolute values and then summed. An analytical expression to solve the limit in equation (2.29) is also proposed by Cohen (1979b) as follows. Define $Y = w_1 v^T{}_1$ and also define $Z = (I + Y - S / \lambda_1)^{-1}$, then

$$\lim First\ Index\ (S, x_0, t) = (Z - Y)x_0 \tag{2.31}$$

The first comparative measure for the distance to default, D_1, as described in equations (2.29) and (2.31), is fairly easy to implement. It is examined in Section 6. Unfortunately, no analytical expression has yet been found for the limit in equation (2.30); it can be found only through numeric calculations.

3. Stochastic Economic Model for Measuring Credit Risk

Perhaps a more realistic model for credit ratings migration should include different transition rates for different economic stages.[38] The homogeneous Markov chain approach assumes independent multinomial transitions, as opposed to the analysis in this section, which considers two separate stochastic processes. One dynamic describes the transition rates among credit ratings within each business cycle. The second dynamic describes the probabilistic rules for the occurrence of those business cycles. A transition matrix is a useful tool to describe the second dynamic as well. For simplicity, this analysis considers two possible quarterly economic stages, *expansionary* and *contractionary*, that may occur in a semi-random sequence with respect to the transition matrix P of dimensions $2*2$ as follows:[39]

$$P = \begin{matrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{matrix} \tag{3.1}$$

where $\alpha,\ \beta \in (0,\ 1]$.[40] The dominant eigenvalue of matrix P is always unit and the corresponding right eigenvector $w = (w_1,\ w_2)^T$ satisfies the following three equations:

$$(1-\alpha)w_1 + \beta w_2 = w_1 \tag{3.2}$$

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[38] A chi-square test validates the homogeneous Markov chain model assumptions in Section 6.1.

[39] It is possible to define more than two economic stages by considering various sublevels of expansion and contraction. Nickell, Perraudin and Varotto (2000) define three such categories. A higher level of separation will get closer to regress ratings migration on macroeconomic shocks as performed by Cantor and Falkenstein (2001).

[40] Both α and β are constrained to be strictly larger than zero to avoid an absorbing economic stage.

$$\alpha w_1 + (1 - \beta) w_2 = w_2 \tag{3.3}$$

$$w_1 + w_2 = 1 \tag{3.4}$$

Hence, the system yields the two-equation solution for the right eigenvector corresponding to the unit dominant eigenvalue as

$$w_1 = \beta / (\alpha + \beta) \tag{3.5}$$

$$w_2 = \alpha / (\alpha + \beta) \tag{3.6}$$

Entries of this right eigenvector determine the stationary probabilities of various economic stages. Since the dominant eigenvalue of P is always 1, the rate at which the stationary state is approached depends only on the magnitude of the *subdominant eigenvalue* of P, which is also the *correlation* ρ between any two consecutive economic stages. Thus, the characteristic equation $(1 - \alpha - \lambda)(1 - \beta - \lambda) = 0$ admits the subdominant eigenvalue as

$$\rho = 1 - \alpha - \beta \tag{3.7}$$

Hence, the correlation between economic stages at time t and at time $t + n$ is ρ^n. It may be more convenient to describe the model in terms of the first entry of the stationary distribution w_1 and the serial autocorrelation ρ as

$$\alpha = 1 - \rho - w_1(1 - \rho) \tag{3.8}$$

$$\beta = w_1(1 - \rho) \tag{3.9}$$

Positive autocorrelation leads to longer runs of economic stages, while negative autocorrelation triggers faster alternation between economic stages. Setting a periodic model, where economic stages constantly replace each other, yields $\alpha = 1$, $\beta = 1$, the autocorrelation $\rho = -1$ and the stationary distribution is $w_1 = 0.5$, $w_2 = 0.5$.

Defining two economic stages with probabilistic rules for occurrence requires the distinguishing between two credit ratings transition matrices. Each transition matrix is idiosyncratic with respect to the economic stage. In this setting, there are two survival submatrices $S^{(E)}$ and $S^{(C)}$, corresponding to the *expansionary* and *contractionary* economic stages. They appear in a stochastic sequence $S_0, S_1, S_2, \ldots$ with respect to the probabilistic rules in matrix P. Since both $S^{(E)}$ and $S^{(C)}$ describe ergodic properties, a *portfolio structure* at time t may be expressed as

$$||x_t|| = ||S_{t-1}S_{t-2}\ldots S_1 S_0 x_0|| \tag{3.10}$$

where the column vector x (as defined in Section 2) is the portfolio weights distribution of states, with $x_i \in [0, 1]$ and $\Sigma_i x_i = 1$. The vertical lines denote a *norm* of a vector.[41] Furstenberg and Kesten (1960) prove the existence of a unique value, λ^*, that satisfies

[41] A *norm* is a measure of size and can be defined in a verity of ways: a *Taxicab norm*: $||x||_1 = \Sigma_i |x_i|$, etc.

$$lim\ 1/t\ ln||x_t|| = lim\ 1/t\ ln||S_{t-1}S_{t-2}...S_1S_0x_0|| = ln(\lambda^*) \quad (3.11)$$

where the *stochastic transition rate* $ln(\lambda^*)$ is independent of the portfolio weights distribution at the initial economic stage. If the economy were fixed, with no separation of expansionary and contractionary stages, $ln(\lambda^*)$ would be equal to $ln(\lambda_1)$, where λ_1 is the dominant eigenvalue of the *homogeneous survival matrix* as defined in Section 2. However, in the context of two economic stages, $ln(\lambda^*)$ differs from $ln(\lambda_1)$ and is estimated in a different way. By considering deviations from the *mean survival matrix*, Tuljapurkar (1982) presents an approximation for $ln(\lambda^*)$ with a second-order expansion as follows:[42]

$$ln(\lambda^*) = ln(\lambda_1) - \tau^2 / 2\lambda^2{}_1 + \theta / 2\lambda^2{}_1 \quad (3.12)$$

where

$$\tau^2 = (v_1 \otimes v_1)^T C_0 (w_1 \otimes w_1) \quad (3.13)$$

$$\theta = \Sigma_{i=2}{}^n (v_1 \otimes v_1)^T (\Sigma_{j=1}{}^\infty (\lambda_i / \lambda_1)^{j-1} C_j)(w_i \otimes w_1) \quad (3.14)$$

Here the matrix C_0 is the *variance-covariance matrix* of the multivariate distribution with respect to entries of the survival matrices across all cycles, while matrix C_j contains the autocorrelations among economic stages in period j. λ_i are the eigenvalues, w and v are the right and left eigenvectors of the *mean survival matrix* after scaling, so that $<w_i, v_i> = 1$ for every i, and $\otimes$ denotes the *Kronecker product*, which

[42] Tuljapurkar's approximation has been used in several ecological studies and is found to be highly accurate. See Benton et al. (1995), Fieberg and Ellner (2001), and Morris and Doak (2004, 2005).

contains all possible multiplications among the entries. Assuming economic stages have zero autocorrelation, then $C_j = 0$ for every $j > 0$, and therefore $\theta = 0$. In this case, Tuljapurkar's approximation reduces to

$$ln(\lambda^*) = ln(\lambda_1) - \tau^2 / 2\lambda^2{}_1 \tag{3.15}$$

The assumption of zero autocorrelation between economic stages simplifies the calculations, but even without this assumption, the impact of C_j decays at an exponential rate of $(\lambda_i / \lambda_1)^{j-1}$, and thus the long-term autocorrelations' effect diminishes and can be ignored.[43] In this setting, λ_1 is strictly less then unit, $ln(\lambda^*) \leq 0$, and all portfolios will converge towards the absorption states of default.

After scaling $<w_i,v_i> = 1$ for every i, the sensitivity analysis in Section 2 concludes that $v_i w_j = \partial\lambda_1 / \partial S_{ij}$ hence, equation (3.13) can be calculated through the sensitivity matrix S as

$$\tau^2 = \Sigma_{ij}\Sigma_{kl}(\partial\lambda_1 / \partial S_{ij})(\partial\lambda_1 / \partial S_{kl})\ Cov(S_{ij}, S_{kl}) \tag{3.16}$$

Therefore, Tuljapurkar's approximation becomes

$$ln(\lambda^*) = ln(\lambda_1) - (1 / 2\lambda^2{}_1)\Sigma_{ij}\Sigma_{kl}(\partial\lambda_1 / \partial S_{ij})(\partial\lambda_1 / \partial S_{kl})\ Cov(S_{ij}, S_{kl}) \tag{3.17}$$

[43] Other ecological studies include serial autocorrelation, but find similar results to the model under the assumption of zero autocorrelation. For examples, see Silva et al. (1991) and Canales et al. (1994).

When the stochastic economic transition matrices do not deviate much from the mean transition matrix, as is the case for credit rating migrations, τ^2 is a linear approximation for the variance of the dominant eigenvalue, λ_1. This can be obtained by applying the *delta method*, which is a series of expansions for non-linear functions. In this case, Tuljapurkar's approximation can also be described as

$$ln(\lambda^*) = ln(\lambda_1) – Var(ln(\lambda_1) / 2\lambda^2{}_1) \tag{3.18}$$

The analysis thus far lays the foundations for a model that measures the time distribution for different portfolios to reach a certain threshold on their way to default, while considering stochastic economic stages. The following analysis is inspired by a series of environmental studies on population extinction dynamics, starting from Capocelli and Ricciardi (1974), Tuckwell (1974), and Ricciardi (1977).

Feller (1968) describes the stochastic Markov chain model as a continuous time *Wiener process*. Tuljapurkar and Orzack (1980) use the *central limit theorem* for stationary products of nonnegative matrices to demonstrate that, as t becomes large, $ln(||x_t||)$ will have an approximate normal distribution with a mean of $ln(||x_0||) + \mu t$ and a variance of $\sigma^2 t$.[44] In this setting, the *infinitesimal drift* of the process is $\mu = ln(\lambda_1) – \sigma^2 / 2$ and the *infinitesimal diffusion* of the process is given by

[44] Lande and Orzack (1988) use a computer simulation to show that the Wiener process approximation is highly accurate when perturbations in the transition matrix are either small or moderate.

$\sigma^2=(1/\lambda^2{}_1)\Sigma_{ij}\Sigma_{kl}(\partial\lambda_1/\partial S_{ij})(\partial\lambda_1/\partial S_{kl})Cov(S_{ij},S_{kl})$, both considered as constants.[45] The process follows a transition *probability density function* (PDF) corresponding to the normal distribution as follows:

$$P(ln||x_t||,t \text{ given } ln||x_0||) = 1 / \sqrt{(2\pi\sigma^2 t)}\ exp(-(ln||x_t|| -(ln||x_0|| +\mu t))^2 / 2\sigma^2 t) \qquad (3.19)$$

Financial institutions may define a portfolio as *quasi-default* if the portfolio structure, $||x_t||$, deteriorates to a certain point, γ, which is predetermined.[46] In this case, the *quasi-default threshold* can be expressed as

$$\zeta = \gamma / ||x_0|| \qquad (3.20)$$

The probability for quasi-default is the probability that $||x_t||$ drops below γ, or that $||x_t|| / ||x_0||$ ever falls below ζ. This probability can be postulated as[47]

$$P_\gamma = \begin{cases} 1 & \text{if} \quad ln(\lambda^*) \le 0 \\ Exp(2ln(\lambda^*)ln(\zeta) / \sigma^2) & \text{if} \quad ln(\lambda^*) > 0 \end{cases} \qquad (3.21)$$

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[45] Heyde and Cohen (1985) develop a different estimator for σ^2 that converges to the true value even when large deviations from the mean transition matrix occur. However, their estimator involves complicated computations and is difficult to implement.

[46] Ginzburg et al. (1982) use the term “quasi-extinction” in the context of live populations.

[47] Cox and Miller (1965) discuss the first-passage problems in the Wiener process in great detail.

When $ln(\lambda^{*}) > 0$, this probability distribution is improper, since some paths will never touch the quasi-default threshold, and thus the distribution universe is less the 1. A proper probability distribution can be constructed from the conditional probability distribution, given that the threshold is attained. Dennis et al. (1991) show that given that the threshold is attained, the *quasi-default time* T_γ is a positive, real-valued random variable with a *cumulative distribution function* (CDF) that can be described as

$$P(T_\gamma \leq t)=\Phi((ln(\zeta)+|ln(\lambda^*)|t)/\sigma\sqrt{t})+exp(-2ln(\zeta)|ln(\lambda^*)|/\sigma^2)\Phi((ln(\zeta)+|ln(\lambda^*)|t)/\sigma\sqrt{t}) \quad (3.22)$$

where Φ is the CDF of the *standard normal distribution* and $0 < t < \infty$. The probability density function (PDF) for the quasi-default time, T_γ, is then the derivative of equation (3.22) with respect to t:

$$P(T_\gamma = t) =(-ln(\zeta) / \sqrt{(2\pi\sigma^2 t^3)})\ exp(-(ln(\zeta)+|ln(\lambda^*)|t)^2/2\sigma^2 t)) \quad (3.23)$$

This distribution is known as the *Inverse Gaussian distribution* of first passage times for Brownian motion with a drift. Initially derived by Schrödinger (1915) and Smoluchowsky (1915), it is now widely used to model nonnegative asymmetric data, mainly for first-passage time analyses. Tweedie (1957a, 1957b) investigates its basic characteristics and discovers more of its statistical merits. Folks and Chhikara (1978) review its properties and many of its applications. Whitmore and Seshadri (1987) support intuitive derivations for the first-passage time results, and Mudholkar and Natarajan (2002) reexamine several analogies to the normal distribution.

The Inverse Gaussian distribution is part of the exponential family of distributions. It has two shape parameters constructing the mean μ and a critical value δ. The time required to reach the value δ for the first time is a random variable with a PDF as follows: $f(t|\delta, \mu) = \delta/(\sigma\sqrt{(2\pi t^3)})exp(-(\delta-\mu t)^2/2\sigma^2 t)$, where $t, \mu, \delta > 0$. A small δ dictates symmetry, but a large δ may cause high positive skeweness.

Various quantities pertaining to these results are of potential interest to financial institutions. The expected time until the threshold is attained and its variance, conditional on a quasi-default dynamic, are given by:

$$E[T_\gamma] = -ln(\zeta) / |ln(\lambda^*)| \quad (3.24)$$

$$Var(T_\gamma) = -ln(\zeta)\sigma^2 / |ln(\lambda^*)|^3 \quad (3.25)$$

The results show that expected time to quasi-default decreases when the asymptotic transition rate towards default, $ln(\lambda^*)$, increases. These quantities may apply to the expected time of quasi-default for various portfolios when examined over a long time period. However, the inverse Gaussian distribution is skewed to the right, with a thick right tail, and thus the mean might describe a biased picture. In this case, the *median* and the *mode* could assist in explanation. The median represents the fixed time t' at which the CDF of the quasi-default time T_γ is equal to 0.5. The mode of the distribution is the most probable time t^* of touching the threshold. This is done by maximizing the PDF of the quasi-default time T_γ and is given by

$$t^* = (-ln(\zeta) / |ln(\lambda^*)|)(\sqrt{(1 + 9 / 4\varphi^2)} - 3 / 2\varphi) \tag{3.26}$$

where $\varphi = -ln(\zeta)|ln(\lambda^*)| / \sigma^2$. The mode here is a product of the mean and a value between 0 and 1. This implies that the mode is smaller than the mean, due to the third moment of the Inverse Gaussian distribution: $-3ln(\zeta)\, \sigma^4 / |ln(\lambda^*)|^5$.

A structure of a portfolio $||x_t||$ as measured simply by any vector norm considers only the holdings' weights but ignores their location on the credit ratings spectrum. To distinguish between various portfolio structures, one needs to assign different factors to each rating category and then multiply by the corresponding holding weight. A simple way of doing that is to assign powers of ten to each rating category. Thus, the sum of the products of the weights and the powers of ten will map the portfolio holdings vector into a unique decimal number. These powers of ten should increase with higher ratings categories. For example, say that a portfolio that initially contains 80% bonds rated CCC- and 20% bonds rated CC deteriorates to its quasi-default point with 20% bonds rated CCC- and 80% bonds rated CC. Both holding weights are the same, but sorted in a different order, and a simple vector norm cannot distinguish between them. By assigning factors of 10^0 and 10^1 to the rating categories CC and CCC- respectively, $||x_0|| = 82$, $||x_t|| = 28$ and $\zeta = 0.3415$. This mapping procedure guarantees that any two dissimilar portfolio structures will receive two different decimal numbers. The drawback of this method is that assigning factors hurts the interpretation of the quasi-default time distribution. It may also change the economic meaning of the threshold. Thus, the procedure may be used as a comparative measure between different portfolios, but to maintain a quantitative measure of the distribution of the time to default, one should preserve the original portfolio structure.

It is worth mentioning that a perturbation analysis within a stochastic economy is feasible, but does not add much to the discussion. Several sensitivities could be examined here. A sensitivity analysis of the eigenvalues of each of the two transition matrices to changes in each transition matrix entries does not present the complete picture. Neither the stochastic transition rate, $ln(\lambda^*)$, nor an entire economy dynamic is described in this way. Similarly, a sensitivity analysis of the eigenvalues of each of the two transition matrices to changes in the mean transition matrix entries does not describe the real stochastic transition rate, and is difficult to interpret. A sensitivity analysis for the stochastic transition rate, $ln(\lambda^*)$, to changes in the mean transition matrix entries challenges the economic meaning, since the stochastic economic process is truly evolved from two different transition matrices. Finally, a sensitivity of the stochastic transition rate, $ln(\lambda^*)$, with respect to changes in entries within each transition matrix, is the only meaningful perturbation analysis, however an economic cycle is defined in practice as expansionary or contractionary only ex post hence, this type of sensitivity does not add to the predictive power of the model.[48]

[48] Analytical methods for performing sensitivity analysis within environmental stochasticity are discussed by Tuljapurkar (1990), Caswell (1996), and Grant and Benton (2000).

4. Non-Homogeneous Models for Measuring Credit Risk

To date, few attempts have been made to step away from the homogeneity assumption of the Markov chain transition matrix. Those models consider the credit ratings migration not to be a memory-less process. Solutions of non-homogeneous models can no longer be written in terms of eigenvalues and eigenvectors, and ergodic properties are often described in the weak form.[49]

This section starts with describing three known non-homogeneous models; the CreditMetrics™ offered by J. P. Morgan, the generator matrix for Markov chain investigated by Israel, Rosenthal and Wei (2001), and the momentum model proposed by Bahar and Nagpal (2001). Then, three new models are proposed; a simulated mean reversion model aiming to track the natural tendency to revert back to some mean credit migration probabilities, an internal correlations model tracking time-series movements in credit migration entries, and a density-dependent model, relating survivability and transitivity to ad hoc cumulative default rate. The advantages and disadvantages of the models are discussed and compared in Section 6.

4.1 CreditMetrics™ Credit Quality Correlations

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[49] Ergodicity of a weakly stationary process is the property required for the distance between any two vectors to decay to zero, in the limit, although neither converges to a fixed structure.

In 1997, J. P. Morgan, along with Bank of America, Bank of Montreal, BZW, Deutsche Morgan Grenfell, KMV Corporation, Swiss Bank Corporation, and Union Bank of Switzerland introduced a new approach for measuring credit risk. This led to the CreditMetrics™ methodology of Gupton, Finger and Bhatia (1997).

One of the important issues raised in this document is the evidence of joint credit-rating correlations that can be examined from data of default histories. The hypothesis suggests that if defaults were uncorrelated, then default rates would have been uniformly distributed along the years, but this is not the case in reality. Although this finding can be interpreted in other ways as well – say, of default correlations not to one another, but to time-varied macroeconomic conditions, it cannot be ignored.

Gupton, Finger and Bhatia (1997) illustrate a direct estimation of joint credit changes while examining credit ratings time series across companies, which are synchronized with each other. Thus, all possible pair combinations between firms are created. In theory, this method avoids explicitly specifying correlation estimation between defaults. However, the technique is difficult to implement over a large sample of firms, and becomes almost impossible when also considering ratings modifiers over a long time period.[50]

4.2 Generators for Markov Chains

[50] In the CreditMetrics™ document, 1,234 firms are examined over 40 quarters with only seven non-default categories. This leads to 1.13 million pair combinations with 28 unique joint likelihood tables.

A generator matrix G for a transition matrix T has a column-sums zero and non-negative off diagonal entries such that $exp(G)=T$.[51] In this case, $T(t)=exp(tG)$ thus, the discrete-time Markov chain model can be translated into a continuous-time process, and future transition matrices can be obtained. Israel, Rosenthal and Wei (2001) present a technique for creating such a generator matrix as

$$\tilde{G} = (T-1) - (T-1)^2/2 + (T-1)^3/3 - (T-1)^4/4 + \ldots \qquad (4.2.1)$$

This generator matrix quickly converges geometrically to the transition matrix while having a column-sums zero and non-negative off diagonal entries so that $exp(\tilde{G})=T$ exactly. However, a violation of any of the preconditions for using this technique leads to other approximate generator matrix methods. A simple method to keep non-negative off-diagonal entries is to replace negative off-diagonal entries with zero, and to add the missing values back to the corresponding diagonal entry thus, maintaining the property of column-sums zero. In this setting, the approximate generator matrix becomes

$$g_{ij} = Max(\tilde{g}_{ij}, 0) \text{ for every } i \neq j \text{ and } g_{ii} = \tilde{g}_{ii} + \Sigma_{i \neq j} Min(\tilde{g}_{ij}, 0) \qquad (4.2.2)$$

---

[51] *Conservative generators* have column-sums zero and *essential generators* require also negative diagonal entries. For further discussion, see Brémaud (1999). The term *exp* here does not represent a simple element-by-element exponential, but rather a *Padé approximation* (a rational function that uses a quotient of two polynomials) along with scaling and squaring the matrix. For additional information, see Moler and Van Loan (1979).

This methodology helps to create different transition matrices over time, but does not correlate them to any meaningful economic variable. The transition matrices generated here do not occur due to stochastic business cycles, other exogenous macroeconomic conditions, the mean reversion phenomenon, or default correlations of any kind.

The main use of a generator matrix is to convert the discrete Markov process into a continuous time dynamic, and by that, to allow valuation of credit derivatives with tick-size time intervals. Continuous changes in the stochastic process within tick size time intervals are difficult to relate to tick-size time movements in macroeconomic variables and are therefore somewhat less intuitive.

4.3 Momentum Effect in Ratings Migration

Bahar and Nagpal (2001) have proposed a different approach to accommodate for observed momentum. Their model enhances the standard Markov model with a partial, short-term memory. Each state describes not only the current rating category, but also the latest transitions. Dividing companies into three sets according to their rating over the previous year: upgraded, stable, or downgraded, is meant to capture the effect of past transitions as well.

This methodology leads to three different transition matrices; one for each set of upgraded, stable, or downgraded companies. In the upgrade transition matrix, ratings migration probabilities are more prominent above the main diagonal, while in the downgrade transition matrix; ratings migration probabilities are more pronounced below the main diagonal. Each of these three separate transition matrices is a Markov chain model by itself.

To compare the models, the cumulative default rate over time can be calculated by considering a starting point of equally weighted portfolios of upgraded, stable, and downgraded companies, and then iterating each transition matrix independently. The cumulative default rate measure accumulates the defaults in all three dynamics.

4.4 Ratings Migration with Mean Reversion

In this model the ratings tend to remain in the neighborhood of their initial category, and to converge to the long-run average rating. A computer simulation is artificially generating the mean reversion phenomenon by assigning small changes in every quarter to the transition probabilities, and matching the same small changes to the mirror entries from the other side of the main diagonal.

Random changes are simulated from the normal distribution with a zero mean and a standard deviation of $\sigma_{ij}(t) = \sigma_{ji}(t) \leq Min(s_{ij}(t), s_{ji}(t)) / 3$. This constraint assures that no transition probability will receive a negative value. Since the sum of the entries for each column must remain 1, entries along the main diagonal will be corrected appropriately.

$s_{ij}(t+1) = s_{ij}(t) + \Delta s_{ij}(t)$ for every $i \neq j$ (4.4.1)

$\Delta s_{ij}(t) = \Delta s_{ji}(t) \sim N(0, \sigma^2_{ij}(t) = \sigma^2_{ji}(t))$ for every $i \neq j$ (4.4.2)

$\Delta s_{ii}(t) = -\Sigma_{i \neq j} \Delta s_{ji}(t)$ for every i (4.4.3)

By assigning the same changes to mirror entries in the transition matrix, a sudden tendency of downgrading will be compensated by an upgrading propensity of the same magnitude. This simulation intends to mimic the natural phenomenon of mean reversion incorporated with stochastic transition probabilities. To compare the various models, the homogeneous matrix is chosen as the starting point.

4.5 Ratings Migration with Internal Correlations

This model makes an attempt to track trends in transition probabilities over time, and to generate future transition probabilities that follow the same affine trend. An evidence for time series correlation in credit ratings migration can be found in Altman and Kao (1992a, 1992b). The authors divide the examined time frame into two sub-periods, from 1970 to 1979, and from 1980 to 1985, and reveal a tendency for a downgrade in rating to be followed by a second downgrade, indicating on a serial autocorrelation when the initial rating changes was a downgrade. However, no autocorrelation is found when the initial change is an upgrade.

The model explicitly assumes that each transition probability is univariate autoregressive correlated with previous transition probabilities within the same entry. By that, the model implicitly assumes that transition probabilities are cross-correlated with other transition probabilities within the same column. The cross correlation evolves from the fact that the sum of entries within each column always remains at 1. When a specific entry in the transition matrix tends to increase (decrease), by definition, other entries in the same column tend to decrease (increase). The model estimates AR(1) time-series regressions for each entry in the transition matrix as

$$s_{ij}(t+1) = \alpha_{ij} + \beta_{ij} s_{ij}(t) + \varepsilon_{ij}(t) \text{ for every } i, j \tag{4.5.1}$$

where the β coefficient of the regression equation determines whether a transition probability is stationary. Transition probabilities must remain in their domain, thus, unless α is close to 1 and β is positive, or α is close to 0 and β is negative, the model requires no constrains.[52] To validate the significance of the trends over time, the transition matrix entries follow, a *Dickey-Fuller test* is performed in the empirical section.

4.6 Density-Dependent Survivability and Transitivity

In a competitive market, rating migrations may be correlated through another mechanism: when resources are limited by a finite market share, or a restricted number of potential customers, the credit rating migration of one company affects the others. A default of one company leaves more space for the others to thrive. An economic

[52] The empirical investigation has found no evidence for these extreme situations.

improvement for a different company narrows the market for the others, forcing them to reduce in size. When some firms have already defaulted, more market share is allocated to the survived ones allowing them to prosper. With fewer surviving companies supplying the fixed customers' demand, the survivability of those remaining companies should improve. Market density may affect companies' survivability as well as their transition probabilities.

The model assumes a closed system with a finite number of competitive companies and limited resources, customers, and market share.[53] It also assumes that a large *cumulative default rate* triggers an increase in the survivability of the remaining firms. In addition, it is assumed that larger *cumulative default rate* triggers slower transition rates towards the absorption state of default.

The density-dependent model considers the following definitions:

σ_i = survival probability within the next quarter for a company in state i at zero cumulative default rate (the starting point)

γ_{ji} = unconditional probability to migrate from credit rating i to credit rating j at zero cumulative default rate (the starting point)

s_{ji} = conditional transition probability

CDR = Cumulative Default Rate, the percentage of firms already defaulted as a measure of density

α_i = coefficients of density-dependent survivability for companies in state i

[53] This approach might be more applicable for specific industries rather than for the whole market. However, data on default rates within specific industries are scarce and relatively noisy.

β_i = coefficients of density-dependent transitivity for companies from state i

Without correlating the survivability and transitivity of credit ratings to the cumulative default rate, the model would have taken the following form:

$$s_{ii} = \sigma_i(1 - \Sigma_{j \neq i}\gamma_{ji}) \text{ for every } i \quad (4.6.1)$$

$$s_{ji} = \sigma_i\gamma_{ji} \text{ for every } i \neq j \quad (4.6.2)$$

Correlating the survivability and transitivity of credit ratings to the cumulative default rate, the model also considers the following dependencies:

$$\gamma_{ji}(CDR) = \gamma_{ji} \exp(\beta_1 CDR) \quad \text{if } j < i \quad \text{and}$$
$$= \gamma_{ji} \exp(\beta_2 CDR) \quad \text{if } j > i \quad (4.6.3)$$

$$\sigma_i(CDR) = \sigma_i \exp(\alpha_i CDR) \quad (4.6.4)$$

The cumulative default rate always lags one quarter behind the conditional transition probability. The conditions $\beta_1 > 0$, $\beta_2 < 0$, $\alpha_i \geq 0$, dictate that for a zero cumulative default rate, there is no dependency on the market density, but when the cumulative default rate increases over time, conditional downgrade (upgrade) migration probabilities decelerate (accelerate), while survival probabilities grow. A different set of conditions can also be

$$\gamma_{ji}(CDR) = \gamma_{ji}(1 / (1 + \beta_i CDR)) \quad (4.6.5)$$

$$\sigma_i(CDR) = \sigma_i(1 / (1 - \alpha_i CDR)) \tag{4.6.6}$$

Each set of conditions dictates different coefficients.[54] An illustration of the model under the first set of conditions is presented in Section 6, while considering the coefficients of density dependent transitivity as $\beta_1 = 10$ and $\beta_2 = -15$, and different coefficients of density dependent survivability for different groups of credit rating categories as follows: $\alpha_i = 0$ for the group of the four highest credit ratings (AAA to AA-); $\alpha_i = 0.0001$ for the group of the next eight credit ratings (A+ to BB); $\alpha_i = 0.001$ for the group of the next four credit ratings (BB- to B-); and $\alpha_i = 0.01$ for the group of the five lowest non-default credit ratings (CCC+ to C). These coefficients are chosen to reflect the growing impact of *CDR* on survivability for lower groups of credit ratings under the constraint of $0 \leq \sigma_i \leq 1$. To compare the models, the starting point for this simulation, the unconditional migration probabilities, γ_{ji}, and the survival probabilities within the next quarter, $\sigma_i = 1 - D_i$, are also collected from the homogeneous transition matrix.

In density-dependent models solutions do not grow exponentially. Instead, they tend to converge to limited subsets of the state space, which are often called *attractors*. These can be *fixed points* or *equilibria*, various cycles or *periodics*, or sometimes even more complicated structures. Density-dependent models, in general, are more sensitive to small perturbations of initial conditions, depending on the stability of the equilibria. The next stability analysis relies on Beddington (1974).

[54] The conditions described here are called *over-compensatory* and *compensatory* functions. Somewhat similar conditions are introduced by Ricker (1954) and Beverton-Holt (1957) in their life-science studies.

The market starts from a zero cumulative default rate. After a while, some companies have defaulted and the market has reached an equilibrium point of $n^{\wedge} = 1 - CDR$. This point represents the percentage of companies that survived out of the initial pull. If the equilibrium is unstable, there exists a deviation vector x such that

$$x(t) = n(t) - n^{\wedge} \text{ or } n(t) = x(t) + n^{\wedge} \tag{4.6.7}$$

In this setting the equilibrium in the next time period will take the form

$$x(t+1) + n^{\wedge} = S_{x(t)+n^{\wedge}}(x(t) + n^{\wedge}) \tag{4.6.8}$$

Applying the *Taylor series expansion* on the transition matrix around the equilibrium point, $n^{\wedge}$, gives

$$x(t+1) + n^{\wedge} = (S_{n^{\wedge}} + \Sigma_i x_i\, \partial S / \partial n_i)|_{n^{\wedge}}\, (x(t) + n^{\wedge}) \tag{4.6.9}$$

Decomposing the right-hand side of equation (4.6.9) yields

$$x(t+1) + n^{\wedge} = S_{n^{\wedge}} n^{\wedge} + S_{n^{\wedge}} x(t) + (\Sigma_i x_i\, \partial S / \partial n_i)|_{n^{\wedge}}\, x(t) + (\Sigma_i x_i\, \partial S / \partial n_i)_{n^{\wedge}}\, n^{\wedge} \tag{4.6.10}$$

The first term on the right-hand side equals $n^{\wedge}$ and canceled out. The third term on the right-hand side is of order x^2_i and can be ignored for small deviations. Thus, a linear approximation is obtained as

$$x(t+1) = S_{n^\wedge} x(t) + (\Sigma_i x_i\ \partial S / \partial n_i)|_{n^\wedge}\, n^{\wedge} \tag{4.6.11}$$

Define matrices H_i with $n^{\wedge}$ in column i and with zeros elsewhere. Then the above equation can also be expressed as

$$x(t+1) = (S_{n^\wedge} + (\Sigma_i \partial S / \partial n_i)|_{n^\wedge} H_i)x(t) = Jx(t) \tag{4.6.12}$$

The *Jacobian matrix J* depends on the equilibrium point and can be decomposed to[55]

$$J = S_{n^\wedge} +((\partial S / \partial n_1)|_{n^\wedge}\, n^{\wedge}\ \ (\partial S / \partial n_2)|_{n^\wedge}\, n^{\wedge}\ (\partial S / \partial n_3)|_{n^\wedge}\, n^{\wedge}\ \ldots) \tag{4.6.13}$$

Equation (4.6.12) is linear so its solution depends on the dominant eigenvalue of the Jacobian matrix *J*. Deviations from the equilibrium point grow over time (i.e. an unstable equilibrium) when the dominant eigenvalue of *J* falls outside the unit circle, but decay to zero (for an asymptotically stable equilibrium) when $|\lambda_1^{(J)}| < 1$. Considering

[55] The Jacobian is the matrix of all first-order partial derivatives of a vector-valued function. Its importance lies in the fact that it represents the best linear approximation of a differentiable function near a given point. The Jacobian matrix *J* in this context describes the deviations dynamic and neither transition nor survival probabilities; therefore, it may contain negative values.

constant survivability σ_i, and recalling that $n_t = 1 - CDR$, the Jacobian matrix J can be found by taking the partial derivatives with respect to

$$s_{ii} = \sigma_i(1 - \Sigma_{j\neq i}\gamma_{ji}\, exp(\beta_i(1 - n_t))) \text{ for every } i \tag{4.6.14}$$

$$s_{ji} = \sigma_i\gamma_{ji}\, exp(\beta_i(1 - n_t)) \text{ for every } i \neq j \tag{4.6.15}$$

In practice however, constantly changing market demand and supply, along with a dynamic market size and an irregular number of competitive firms, may dictate only a periodic equilibrium. These circumstances often dictate pro-cyclical patterns. Allen and Saunders (2004) survey pro-cyclicality effects on operational risk, credit risk, and market risk measures.

5. Data and Methodologies

The Compustat database reports 130,559 quarterly S&P long-term credit ratings for 4,510 industrial companies in North America from 1985 to 2004. Each credit rating is tagged by a number between 2 for 'AAA' and 27 for default. Within this period, there are 123,849 valid credit rating transitions in consecutive quarters.[56] These transitions are collected and assigned to a *frequency matrix F*. Only few transitions out of the default state back to the survival submatrix are observed; hence, the assertion of an absorption

[56] The 'SPDRC' data field assigns a unique credit rating based on the issuer overall credit worthiness, apart from its ability to service individual debt. Thus intra-company effects among different obligations are eliminated, yet allowing for cross-ratings correlations to exist.

state of default is reasonable. Since there are $s = 21$ states in the life cycle of a company and an additional absorbing state of default, matrix F is of dimensions $(s+1)*s = 22*21$, where an entry f_{ij} represents the number of credit ratings migrations observed from state j to state i.

Several observations are reported as 'Not Rated' (NR). These observations are classified as 1, 3, 22, 25, 28, 29 and 90 in the database. This study adopts the industry standard and treats transitions to and from NR status as missing information, by excluding them from the data sample. Bangia et al. (2002) discuss other alternatives, but eventually choose to exclude NR observations as well. Conceptually, transition matrices can be estimated over a variety of time horizons. Nevertheless, those transition matrices estimated over relatively short time intervals might best represent credit migration dynamics. For short-time rating migrations, more observations are available in the data sample, and rating migrations tend to follow a smoother shape due to fewer extreme jumps. For these reasons, the transitions time interval is chosen to be a quarter of a year, the shortest possible time frame in a discrete space, since agencies report their credit ratings once every quarter.

The *homogeneous Markov chain* model assumes that transitions among different states are independent. Thus, entries in matrices S and D can be calculated by dividing the corresponding entry in matrix F by the sum of its column. This procedure guarantees that every column in the transition matrix T will sum to 1, and every survival or default state will receive the proportional probability of transition. The *maximum likelihood* *estimation* validates this procedure. The likelihood of matrix T is defined as the product of the likelihoods of its columns, as follows

$$L(T) = \Pi_{j=1}^{s} L(T_{.j}) \tag{5.1}$$

Assuming that transitions of companies within a specific credit rating are independent, since columns of the transition matrix T are derived from random variables in the frequency matrix F, the likelihood of a column in matrix T is

$$L(T_{.j}) = \Pi_{j=1}^{s+1} t_{ij}^{fij} \tag{5.2}$$

Maximizing the likelihood of the whole matrix T can be done by maximizing the natural logarithm of $L(P_{.j})\ \forall j$. So the optimization problem becomes

$$Max\ ln(L(T_{.j})) = \Pi_{i=1}^{s+1} f_{ij} ln(t_{ij}) \tag{5.3}$$

s.t

$$\Sigma_{i=1}^{s+1} t_{ij} = 1 \tag{5.4}$$

Let λ be the *Lagrange multiplier* then the first order conditions are

$$(\partial/\partial t_{ij})(\Sigma_{i=1}^{s+1} f_{ij} ln(t_{ij})) + \lambda\partial/\partial t_{ij}\ (\Sigma_{h=1}^{s+1} t_{hj} - 1) = 0 \text{ for every } i \tag{5.5}$$

and

$$(\partial/\partial\lambda)\lambda(\Sigma_{h=1}^{s+1} t_{hj} - 1) = 0 \tag{5.6}$$

Hence, the necessary conditions reduce to

$$f_{ij} / t_{ij} + \lambda = 0 \quad \text{and} \quad \Sigma_{h=1}^{s+1} t_{hj} = 1 \tag{5.7}$$

And the maximum likelihood estimation is then

$$t_{ij} = f_{ij} / \Sigma_{h=1}^{s+1} f_{hj} \text{ for every } i \blacksquare \tag{5.8}$$

This is not the case for the stochastic economic model or the non-homogeneous models. For the stochastic economic model, business cycles are defined in several ways. The first uses chained-dollars real GDP per quarter data, from 1985 to 2004. The data are collected from the National Bureau of Economic Research (NBER), a private research organization, and from the Bureau of Economic Analysis (BEA), an agency of the U.S. department of commerce. Quarterly GDP growth rates are then sorted and allocated into two categories, expansionary and contractionary quarters, based on their position within the GDP growth rate spectrum. The allocation into the two categories is done with respect to the *mean* or the *median* of the total GDP growth rate data set. Using the mean, there are 33 quarters of expansion and 47 quarters of contraction. Using the median, the entire period from 1985 to 2004 is equally divided into expansionary and contractionary cycles, 40 quarters each.

The second definition considers the NBER's own allocation of expansionary and contractionary business cycles, constructed by examining a larger number of business indicators in addition to the GDP. This distinction only covers up to November 2001. However, the NBER committee noted that "*a trough in business activity occurred in the U.S. economy in November 2001 and the trough marks the end of the recession that began in March 2001 and the beginning of an expansion*". Hence, it is assumed here that 2002 to 2004 were expansionary years. This definition indicates towards highly favorable economic development during the past twenty years, with 74 quarters of expansion and only 6 quarters of contraction.

The third definition involves the *Consumer Confidence Index* (CCI). This index, made by the *Conference Board*, an independent non-profitable organization, is almost identical to the *Consumer Sentiment Index*, prepared by the University of Michigan.[57] It is derived from a survey of 5,000 households and may point to shifts in consumption patterns, and thus a progression in business cycles. The percentage change in the CCI for each consecutive month is calculated and then accumulated for each quarter. Positive changes in the CCI are tagged as expansionary quarters, 43 in total. Negative changes in the CCI are considered as contractionary cycles, 37 in total.

6. Results

[57] The only difference between the two indices is that the consumer sentiment index has two monthly releases, a preliminary and a final reading.

This section reports the empirical findings and summarizes their economic meaning. **Tables 1 – 6** relate to the homogeneous Markov chain model. **Table 1** presents the frequency matrix *F* for the credit ratings migrations for all companies in the data sample within the past 20 years, from the beginning of 1985 to the end of 2004. **Table 2** describes the left part of the transition matrix *T*, or submatrices *S* and *D* as derived directly from the frequency matrix *F*.

The first two tables point to a greater stability among highly rated debt issuers, relative to low credit categories. Examining the main diagonal of matrix *S* in Table 2 shows that the probabilities of remaining at the same credit rating within a quarter declines from 98.04% for AAA-rated bonds down to 50% for C-rated bonds. After remaining above 90% for the credit ratings of AAA to B+, stability declines at a faster rate for the low credit ratings, from 89.82% for B to 50% for C. These findings support previous results.

Several scholars discover that higher-rated bonds tend to be more stable than lower-rated bonds, with respect to the time of remaining at the same credit rating. Altman and Kao (1992b) examine major credit ratings of more than 7,000 bonds issued from 1970 to 1988. They find that AAA-rated issues had the greatest stability, in terms of retaining their initial ratings, up to five years after issuance. BB-rated bonds were found to be the least stable. Carty and Fons (1994) test approximately 4,700 long-term public debt issuers over a 70-year period and 2,400 short-term public debt issuers over a 22-year period from the Moody's proprietary database. The authors discover, among other things, that higher ratings are relatively more stable, as reflected by their longer average length of time holding the same rating, given that it subsequently changes. Fons (1994) investigates broad rating categories out of the Moody's long-term default studies covering the years 1970 through 1993. He also finds that default rates are not particularly stable, especially at low rating levels.

The second conclusion from the first two tables is that by including the probabilities of default, almost all firms are more likely to be downgraded over time. For each and every column in Table 2 (submatrices *S* and *D* together), the sum of the probabilities below the main diagonal of matrix *S*, is larger than the sum of the entries above it, except for a slight opposite trend for rating category BB+ and a more pronounced opposite trend for C-rated companies. These findings assess the hypothesis of a finite life cycle for all companies.

When *excluding* the absorbing vector *D*, low credit ratings have higher chances to be upgraded, where high credit ratings have higher chances to be downgraded. These findings match previous results as well. Carty and Fons (1994) and Fons (1994) report that firms with low (high) credit ratings are more likely to be upgraded (downgraded), *conditional on surviving*. This phenomenon of cyclical behavior could be explained by the presence of complex eigenvalues for matrix *S*. If all eigenvalues were positive and real, the transition rate would have taken an exponential form, either growing or decaying, conditional on whether the eigenvalues are larger or smaller than unit, respectively. Negative eigenvalues may cause damped or diverging oscillations, depending on whether the eigenvalues are larger or smaller than negative unit, respectively. Complex eigenvalues appear in complex conjugate pairs and generate harmonic process. Since the survival matrix S is a non-negative matrix with all eigenvalues strictly less the 1, and it contains 19 real and one pair of complex conjugate eigenvalues, a cyclical behavior is observed.

The dominant eigenvalue for matrix S is *0.9964*, and the second-largest eigenvalue in magnitude is *0.98405*. It turns out that the damping ratio is: $\delta = 1.0126$, which represents a very slow convergence rate. When δ is close to *1*, the convergence rate is relatively slow, and vice versa. This should not be a surprise, since a number of firms in a well-diversified portfolio in the data sample may survive for many years, while the transition matrix T measures this convergence in units of quarters of a year.

Table 3 describes the fundamental matrix N. **Table 4** shows the time to default for individual credit ratings and their *coefficient of variation* for comparison. **Figure 1** presents changes in the coefficient of variations graphically. The coefficient of variation is calculated as the standard deviation divided by the mean of the time to default. The smaller the numerator, or the larger the denominator, the longer time a company has before absorption. The coefficient of variation is almost entirely a monotonic increasing function of the credit ratings, with a slight bias at credit rating C. As observed in Table 2, C-rated companies have higher chances to be upgraded rather than downgraded hence their coefficient of variation does not follow the general trend. Due to the small number of observations of C-rated companies, this phenomenon could be considered as a noise in the data set. In any event, these quantitative measures can be used to compare individual companies tagged in different credit ratings.

Table 5 presents various portfolios and their cumulative distances to default, as measured by D_1. **Figure 2** describes the growth rate of the cumulative distances to default throughout the selected equally weighted portfolios. Measuring the cumulative distances to default is a powerful tool for comparing the credit risk of different portfolios. Not only do the cumulative distances to default take an exponential form in the selected portfolios, but one can also estimate by how much one portfolio is more risky than another. By tagging portfolios with quantitative measures of their cumulative distances to default, a comparative analysis becomes feasible. For example, an equally weighted portfolio of seven debt issuers tagged from A- to AAA (portfolio number 15) with $D_1=205.850$ is about *as twice as safe* as an equally weighted portfolio with 12 debt issuers tagged from BB to AAA (portfolio number 10) with $D_1=104.500$. Or a single investment in one debt issuer tagged AAA (portfolio number 21) with $D_1=446.040$ is about *as eight times as safe* as an equally weighted portfolio with 18 debt issuers tagged from CCC to AAA (portfolio number 4) with $D_1=56.199$.

This technique can help to analyze more complicated portfolios, or portfolios that are difficult to compare. Even a simple risk comparison between an equally weighted portfolio of two debt issuers, tagged at A- and A, with a second equally weighted portfolio with two debt issuers tagged at BBB+ and A+, is not a trivial comparison. Neither of these two portfolios strictly dominates the other. The technique described here solves the issue by attaching unique values to each portfolio representing their cumulative distances to default.

Table 6 shows matrix Q representing the sensitivity the transition rate to movements among survival states. **Figure 3** emphasizes the highly sensitive area through a surface graph. The transition rate towards default of a well-diversified portfolio is found to be highly sensitive to the number of companies that will be *upgraded* from the credit ratings area of BBB-, BBB, and BBB+ to the area of AA, AA+, and AAA. By examining the frequency matrix in Table 1, one can observe few such transitions. They are rare but feasible. This means that any change in this *specific area* has a relatively large effect on the default rate of the whole portfolio. Increments in this specific group of transitions will slow down the default rate of the whole portfolio. Decrements will trigger a more rapid default rate. The transition rate towards the inevitable outcome of default, for the portfolio as a whole, heavily depends on this group of transitions.

Figures 4 – 6 illustrate various second derivatives of the dominant eigenvalue to several entries in the survival submatrix. This analysis can help to investigate changes in the sensitivity itself. One may take a portfolio with several companies from the same industry, all tagged with the same credit rating. When a macroeconomic shock affects this sector, or when other radical changes take place within this industry, all the companies are likely to be upgraded or downgraded together. Examining the second derivative of the dominant eigenvalue may help to predict sudden changes in the transition rate sensitivity in advance.

Tables 7 – 8 relate to the stochastic economic Markov chain model. **Table 7** discusses the probabilities of being in an expansionary or a contractionary economic stage in the next quarter, conditional on the current business cycle. The four definitions of business cycles used here dictate different transition matrices between economic stages.

Figure 7 illustrates a *Monte Carlo simulation* of the cumulative default rates for the homogeneous Markov chain model, and for the four different business cycles definitions. The simulation describes the incremental percentage of holdings that reach the absorption state and default within a time frame of 40 quarters. There are some differences between the models. The homogeneous Markov chain model describes a similar cumulative default rate to the stochastic economic models based on the GDP growth rates and the consumer confidence index. However, the stochastic economic model, based on the NBER definition for business cycles (the lowest curve along most of the analysis), is somewhat separate from the others. After 10 years, there is a gap of 2.1% between the model based on the NBER definition and the others. The source of the difference evolves from the NBER definition for business cycles, a highly optimistic view with expansionary cycles dominating the last decades.[58]

[58] Figure 13 in Bangia et al. (2002) is mistakenly labeled and therefore incorrectly interpreted. Even in the more pessimistic time frame (from 1959 to 1988), the regime-switching behavior is still more optimistic than the homogeneous Markov chain model (see their Table 5). The analysis here also includes ratings modifiers and thus, it is more accurate.

Another interesting observation is the higher volatility of the stochastic economic model based on the mean GDP growth rate, relative to the other smoother curves. The second-highest volatility is observed for the model based on the median GDP growth rate. This phenomenon is associated with the negative autocorrelation between economic stages as defined in equation (3.7) and calculated in Table 7.

Table 8 demonstrates distribution calculations of time to quasi-default for 15 different portfolios with one dollar of investment in each holding. Each portfolio structure is defined with a taxicab norm, while the quasi-default point is set to be a portfolio of six equally weighted holdings. **Figure 8** illustrates a complete distribution for portfolio number 15 from Table 8. This *histogram* is constructed from the density function in equation (3.23). A small positive skewness is observed from the inverse Gaussian distribution, and hence, the mode falls slightly below the mean.

Table 9 presents the first transition matrix as created from the generator matrix *G*. Slight differences from the homogeneous transition matrix are spotted for most of the entries. More significant differences are observed for the lower credit ratings. This may be related to the additional correction which took place in matrix *G* before generating the dynamics $T(t)=exp(tG)$ as described in equation (4.2.2). **Figure 9** illustrates the tick-by-tick cumulative default rate using the generator matrix. Increments in the CDR create an exponential form in the continuous time process, similar to the discrete Markov chain.

Figure 10 illustrates the cumulative default rates evolving from the momentum transition matrices. As expected, downgraded companies create detrimental momentum that leads to the highest CDR among all the examined models. Upgraded companies generate beneficial momentum, which leads to the lowest CDR. Stable companies are located somewhere in between. The total CDR reflects an equally weighted portfolio of the three sets of companies.

Figure 11 compares the homogeneous Markov chain model and the three new non-homogeneous models. As opposed to previous assertions in the literature, the mean reversion has only a minor effect on the cumulative default rate, and thus slightly deviates from the homogeneous model. The internal correlations and the density-dependent models obtain a much lower CDR than the other models.

6.1 A Comparative Analysis of the Alternative Models

To identify which of the models explored in this study best represents the actual ratings migration dynamic, a comparative analysis is now taken. The literature discusses several popular model validation techniques, yet all of them require constant classification of parameters. The *intracohort analysis* presented by Miller (1998), the *power curve* suggested by Kealhofer (2003), as well as the *Receiver Operating Characteristics* (*ROC*) methodology discussed, among others, by Stein (2003) and Jarrow and Van Deventer (2004), all entail fixed characterization of the involved default probabilities. This is not the case within the non-homogeneous models, and thus different approaches are undertaken.

A relatively naïve yet robust method to examine the various models is through a *back-testing*, by comparing the expected cumulative default rates from the theoretical models to the actual default rate, as observed from the data sample, all with the same investment horizon. By comparing the expected cumulative default rates from the different models, it is clear that results vary. While the homogeneous Markov chain model predicts a 28.9% CDR within 10 years, the stochastic economic model based on the NBER definition for business cycles points to 26.8% CDR in the same time frame. The mean reversion non-homogeneous model only slightly deviates from the homogeneous model by predicting 29.47% CDR in 10 years, but a much lower estimation is obtained from the internal correlations and the density-dependent non-homogeneous models: 23.1% CDR and 12.15% CDR, respectively. The momentum effect model projects the highest CDR, a 39.6% within a ten-year period.

Those models aim to trace 308 defaults out of 3,692 companies, or an actual cumulative default rate of 8.3% from 1995 to 2004. **Figure 12** presents a comparative analysis of the expected cumulative default rates, within a ten-year period, from the various models, and the actual default rate as observed in the past ten years. The results show that the existing models overestimate the credit ratings migration probabilities, and project higher cumulative default rates than is observed in practice. The two new models, the internal correlations as evolved from the autoregressive time-series regressions as well as the density-dependent non-homogeneous dynamic, describe much more realistic scenarios. Nevertheless, all models thus far generate smooth CDR curves, and miss changes in patterns and slopes in the actual cumulative default rate. Correlating the models to exogenous macroeconomic variables may capture these fluctuations in the default rate.

Figure 13 illustrates the difference of the various models from the actual CDR. Actual cumulative default rate, from 1995 to 2004, is subtracted from those predicted by the various models. While the density-dependent model tend to correct itself over time, with a downward sloping difference from the actual CDR, the internal correlations, the stochastic economic and the homogeneous Markov chain models, maintain relatively steady gaps. The momentum model increases its over-estimation, with a monotonic increasing gap curve. This analysis emphasizes by how much the models deviate from the observed data, and through that, accentuates the superiority of the density-dependent model predictive power.

Another effective way to compare the models is by directly examining their assumptions. A suitable test to validate the homogeneity of the transition matrix is the *chi-square test*. The null hypothesis of homogeneity of credit rating migrations asserts that the *multinomial probability distribution* dictates independent transitions among different states. The *chi-square statistic* sums the differences of actual credit rating migrations from the expected values as follows: $\chi^2 = \Sigma_{i=1}^{r}\Sigma_{j=1}^{c}(A_{ij} - E_{ij})^2 / E_{ij}$, where r represents the number of rows and c denotes the number of columns. The degrees of freedom are set to be the number of non-zero columns minus 1 multiplied by the number of non-zero rows minus 1. The chi-square statistic yields a value of 349.485, suggesting a strong rejection of the null hypothesis of the homogeneous Markov chain model, with a significance level of 0.999. By that, it refutes the assertion of independent credit rating transitions.

In essence, the momentum model, proposed by Bahar and Nagpal (2001), represents a high-order Markov chain. Thus, it can be validated through a stratified version of the chi-square test for homogeneity, with each possible rating category separating any two credit ratings, together forming a stratum. Sequences of three, four, and even higher number of credit ratings can be examined through this method. However, a simpler way to examine this model is through a chi-square test for each of the three separated Markov chains, the upgraded, the stable and the downgraded transition matrices. This method leads to a strong rejection of the null hypothesis of the momentum model as well.

To examine whether the transition matrix entries follow some trends over time, and by that to validate the time-series internal correlations model, the *Dickey-Fuller test* is now undertaken. Dickey and Fuller (1981) present a likelihood ratio statistical test for autoregressive time-series regression, examining whether the dynamic follows a *random walk* with zero drift. The AR(1) internal correlation model suggests that the transition matrix entries follow time-series trends:

$$s_{ij}(t+1) = \alpha_{ij} + \beta_{ij}s_{ij}(t) + \varepsilon_{ij}(t) \text{ for every } i, j \tag{6.1.1}$$

To examine whether the time-series dynamic, within a specific entry, follows a random walk with zero drift, or some trends occur, the likelihood ratio null hypothesis admits the coefficients: H_0: $\alpha_{ij}=0; \beta_{ij}=1$, against the alternative hypothesis H_1: not H_0. The likelihood ratio statistic is

$$(1+ \Phi_1 / 2(n-3))^{(n-1)/2} \tag{6.1.2}$$

where n is the number of intervals in the time-series analysis, and the statistics

$$\Phi_1 =(2S^2_{e\mu})^{-1}((n-1)\sigma^2_0 - (n-3)S^2_{e\mu}) \tag{6.1.3}$$

$$S^2_{e\mu} = (n-3)^{-1}\Sigma_{t=2}^{n}(s_{ij}(t) - \alpha_{ij} - \beta_{ij}s_{ij}(t-1))^2 \tag{6.1.4}$$

$$\sigma^2_0 = (n-1)^{-1}\Sigma_{t=2}^{n}(s_{ij}(t) - s_{ij}(t-1))^2 \tag{6.1.5}$$

The notations Φ_1, $S^2_{e\mu}$ and σ^2_0 are consistent with Dickey and Fuller (1981), while $s_{ij}(t)$ denote the transition probabilities at time t. The likelihood ratio test rejects the null hypothesis for larger values of Φ_1, relative to critical values from Dickey-Fuller tables.[59]

Table 10 presents the values of Φ_1, derived from the likelihood ratio tests, for all the entries in the transition matrix. **Table 11** summarizes the statistical significance levels of each entry, after comparing the values from Table 10 to the critical values from

[59] The Dickey-Fuller tables are described in Dickey and Fuller (1981) page 1,063.

Dickey-Fuller tables. Out of the 462 possible entries in the transition matrix, 168 are classified as not meaningful (N/M), since no transitions were observed throughout the whole time frame, from 1985 to 2004. Among the remaining 294 meaningful entries, 194 entries, or 66.0%, reject the null hypothesis with a significance level of 0.99, 29 entries, or 9.9%, reject the null hypothesis with a significance level of 0.975, 18 entries, or 6.1%, reject the null hypothesis with a significance level of 0.95, and 21 entries, or 7.1%, reject the null hypothesis with a significance level of 0.90. Only 32 entries, or 10.9%, cannot reject the null hypothesis of a random walk with zero drift. These findings assess the claim that transition probabilities may follow trends over time, and thus, validate the internal correlations model.

To validate the assumptions of the density-dependent model, the relations between the actual CDR and the survivability as well as the transitivity of companies tagged with different credit ratings are now explored. To investigate the relationship between the cumulative default rate and the density-dependent survival and transition probabilities, a closer look at the period from 1999 to 2004 is taken. Most of the defaults in the data sample took place within this time frame, and hence it can be used as an intensive test-field for validating the assumptions. To remain as close as possible to reality, all companies' debt tagged throughout the entire time frame are examined, as well as those tagged only in parts of this period. Including these partly tagged companies allows the description of a more realistic scenario, with a variable number of companies, rather than a closed system. Clearly this adds some noise to the test on top of the existing noise, considering that not all companies tagged in this time frame are necessarily in direct competition with each other. However, the results remain robust.

Table 12 presents sample results for survivability and transitivity measures from 1999 to 2004. A clear picture of increasing survivability functions with the CDR arises from the first panel. For lower credit ratings, survivability increases at a higher pace when the CDR rises, relative to higher credit ratings. For the highest credit ratings, survivability remains constant, as predicted by the different coefficients α_i in the theoretical model. The transitivity measures describe a somewhat less clear picture, but it is still possible to identify in the second and third panels several credit ratings with increasing transition probabilities above the main diagonal, as well as decreasing transition probabilities below the main diagonal when CDR arises, as predicted by the coefficients $\beta_1>0$ and $\beta_2<0$ in the theoretical model.

A word of caution is required here. This test does not prove any causality between the cumulative default rate and the survivability or transitivity of the remaining companies. The test purely aims to validate the assumptions of the theoretical model, and it clearly points to a plausible correlation between the mechanisms, and thus validates the assumptions.

The question as to whether or not the density of companies in the market affects their default rate can be directly examined through several known statistical tests; some are more robust than others. Morris (1959 and 1963) discusses a *key factor analysis*, which is a scatter plot test of the departure of the slope of a fitted line from 1. This nonparametric test ignores the direction of deviations from 1. The *autocorrelation test* of Bulmer (1975) also suffers from a lack of accuracy when a trend is observed. Slade (1977) and Vickery and Nudds (1984) have reviewed a variety of methods for this purpose, yet they undermine the effectiveness of those techniques.

In contrast, Pollard, Lakhani and Rothery (1987) present a vigorous assessment also known as the *randomization test*. This technique is used here to verify whether companies are indeed density-dependent or not. The method measures a population size, $N(t)$, at time t, computes $X(t)=ln(N(t))$, and compares three time series processes:

$$X_{t+1} = X_t + \varepsilon_t \qquad (6.1.6)$$

$$X_{t+1} = \mu + X_t + \varepsilon_t \qquad (6.1.7)$$

$$X_{t+1} = \mu + \beta X_t + \varepsilon_t \qquad (6.1.8)$$

Equation (6.1.6) describes a random walk in time and thus density-independent. Equation (6.1.7) presents also a random walk, but with a drift. Equation (6.1.8) illustrates a density-dependent dynamic. The true slope β is estimated by the regression coefficient b as follows:

$$b = \Sigma_{t=1}^{n-1}(X_t - m_1)(X_{t+1} - m_2) / \Sigma_{t=1}^{n-1}(X_t - m_1)^2 \qquad (6.1.9)$$

where

$$m_1 = \Sigma_{t=1}^{n-1}(X_t / (n-1))$$

$$m_2 = \Sigma_{t=1}^{n-1}(X_{t+1} / (n-1))$$

For the null hypothesis of density-independence, since equation (6.1.6) is a special case of equation (6.1.7), it is always preferable to examine the second and third time series processes. In this event, the likelihood ratio test-statistic is

$$T = [\Sigma_{t=1}^{n-1}(X_{t+1} - m_2)^2 - b\Sigma_{t=1}^{n-1}(X_t - m_1)(X_{t+1} - m_2)] \qquad (6.1.10)$$
$$/ [\Sigma_{t=1}^{n-1}(X_{t+1} - X_t)^2 - (X_n - X_1)^2 / (n-1)]$$

To test the null hypothesis that observations arise from a density-independent population, the following steps are taken. First, the number of companies in the database at the beginning of the period and the number of defaults throughout the period are used to estimate the observed values X_t, and the test-statistic T. Second, changes in values, $D_t = X_{t+1} - X_t$ are permutated and the corresponding X_t values computed. Next, for each possible permutation, the corresponding T value is calculated. Finally, the percentages of T values that fall below (or exactly at) the test-statistic T are calculated. If, for example, less than 5% of the permutations' T values are smaller than or equal to the observed test-statistic T, then the null hypothesis of density-independence is rejected, at a 5% level of significance.

Table 13 presents the results of four randomization tests. Each test starts at a different point in time and lasts for seven consecutive half-year intervals. [60] The first two tests suggest that some density dependence exists, but the results lack significance. However, the last two tests imply a strong rejection of the null hypothesis of density-independence. Both of the last two tests are statistically significant at a 1% level. These robust results clearly point to various levels of density-dependency among companies' default rate.

7. Summary and Conclusions

The methods presented in this study provide a powerful tool to compare and sort different portfolios with respect to their credit risk components. Both of the models, the time-homogeneous and the stochastic economic Markov chain models, yield quantitative as well as comparative measures for credit risk within a portfolio perspective. Some of the non-homogeneous models improve predictive power, while others are found to have very little effect. The mean reversion phenomenon model is found to have merely a minor impact on transition rates, opposing claims in the literature. However, the internal correlations and the density-dependent models are significantly better at telling the true story of credit ratings migration. In particular, the density-dependent model is found to be

[60] There are *(n-1)!* possible permutations, and thus sets of X_t values. Taking a large number of short consecutive periods creates computation difficulties, with either the *lexicographic order* or the *recursive algorithms* for permutations. On the other hand, considering a small number of longer time intervals may miss the statistical significance of the test. For these reasons, each of the four tests described here uses seven consecutive half-year intervals.

highly pragmatic, and may improve the predictive strength of the credit ratings migration theory.

Different portfolios may have different transition matrices. Some may have different weights of holdings within the same transition matrix. Each portfolio should be examined separately, and its default rate, idiosyncratic distance to default, cyclical behavior of transitions, and, unique sensitivities should be explored independently.

Further research may examine those risk components and their corresponding returns for various portfolios. The quantitative and comparative risk measures described in this study may be used along the horizontal axis of an efficient frontier for a portfolio of bonds. Other lines of investigation could explore the impact of various macroeconomic shocks, along the history, on transition dynamics and their sensitivities. A different line of exploration might correlate the density-dependent model to macroeconomic variables to capture changes in the pattern of the cumulative default rate.

Appendix 1

If the survival matrix S were a diagonal matrix, the behavior of S^n was determined by raising each entry along the main diagonal to the power of n. However, S is not a diagonal matrix, but assumed to be diagonalizable (as it truly is in this study), hence it is *similar* to matrix Θ whose diagonal entries are the eigenvalues λ_i.

Matrices S and Θ are *similar* if there exists a *nonsingular* (can be inverted) matrix W such that

$$S = W\Theta W^{-1} \tag{A1.1}$$

This transformation helps to find the dynamics of S^n, for example

$$S^2 = W\Theta W^{-1} W\Theta W^{-1} = W\Theta^2 W^{-1} \tag{A1.2}$$

and in general

$$S^n = W\Theta^n W^{-1} \tag{A1.3}$$

or in more explicit form

$$S^n = W \begin{matrix} \lambda^n{}_1 & 0 & \ldots & 0 \\ 0 & \lambda^n{}_1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda^n{}_1 \end{matrix} W^{-1} \qquad \text{(A1.4)}$$

Equation (A1.1) implies that $SW = W\Theta$, hence the columns of W are the *right eigenvectors* w_i of S. Equation (A1.1) also implies that $W^{-1}S = \Theta W^{-1}$, hence the rows of W^{-1} are the *complex conjugates of the left eigenvectors* v_i of S. Thus

$$S^n = \Sigma_i \lambda^n{}_i w_i v^*{}_i \qquad \text{(A1.5)}$$

where v_i^* are the *complex conjugate transpose* of the *left eigenvectors* corresponding to the *eigenvalues* λ_i ■

Appendix 2

Cohen (1979a) introduces the *strong ergodic theorem*, which states that the *dominant eigenvalue* determines the *ergodic* properties of a portfolio' transition rate. The relation between the survival matrix S and its eigenvalues, λ_i, and right eigenvectors, w_i, is: $Sw_i = \lambda_i w_i$. Assuming the eigenvalues are distinct (they truly are in this study), hence the eigenvectors are linear independent. Then the initial portfolio probability distribution of states, x_0, can be described as a linear combination

$$x_0 = c_1 w_1 + c_2 w_2 + \ldots + c_s w_s \tag{A2.1}$$

where c_i are a set of coefficients that can be found by

$$x_0 = (w_1 \ldots w_s)(c_1 \ldots c_s)^T = Wc \text{ or } c = W^{-1} x_0 \tag{A2.2}$$

To find the next period portfolio probability distribution of states, x_1, multiply x_0 by S as follows

$$x_1 = Sx_0 = \Sigma_i c_i S w_i = \Sigma_i c_i \lambda_i w_i \tag{A2.3}$$

Iterating this process for t times gives

$$x_t = \Sigma_i \lambda^t_i w_i = c_1 \lambda^t_1 w_1 + c_2 \lambda^t_2 w_2 + c_3 \lambda^t_3 w_3 + \ldots \tag{A2.4}$$

where λ_1 is the dominant eigenvlue and all the eigenvalues are placed in a decreasing order. Since λ_1 is strictly bigger than all the other eigenvalues, dividing both sides with $\lambda^t{}_1$ yields

$$x_t / \lambda^t{}_1 = c_1 w_1 + c_2(\lambda_2 / \lambda_1)^t w^2 + c_2(\lambda_3 / \lambda_1)^t w^3 + \ldots \qquad \text{(A2.5)}$$

Regardless of the initial portfolio, when taking large enough t, all terms on the right hand side disappear except the first one, therefore

$$Lim\ x_t / \lambda^t{}_1 = c_1 w_1 \qquad \text{(A2.6)}$$

■

Appendix 3

Assuming the eigenvalues are distinct (as they truly are in this study) and the eigenvectors have been scaled such that $<w_i, v_i> = 1$ and $<w_i, v_j> = 0$ for every $i \neq j$. To find the sensitivity of the eigenvectors to the entries in the survival matrix, start with

$$Sw_1 = \lambda_1 w_1 \tag{A3.1}$$

Differentiating both sides of equation (A3.1) to receive

$$(dS)w_1 + S(dw_1) = (d\lambda_1)w_1 + \lambda_1(dw_1) \tag{A3.2}$$

After switching the first term on the left hand side with the second term on the right hand side, since $d\lambda_1$ is known, (A3.2) can be written as a linear equation

$$(S - \lambda_1 I)dw_1 = (d\lambda_1 I - dS)w_1 \tag{A3.3}$$

Hence, any solution can be expressed as a linear combination of the eigenvectors:

$$dw_1 = \Sigma_{m=1}^{s} k_m w_m \tag{A3.4}$$

where k_m are unknown coefficients, but the value of k_1 is irrelevant, since while substituting (A3.4) into (A3.3), $k_1(S - \lambda_1 I)w_1 = 0$ for every k_1. Take $k_1 = 0$ and the scalar product of both sides of equation (A3.2) with v_j for $j \neq 1$ yields

$$<(dS)w_1, v_j> + <Sdw_1, v_j> = d\lambda_1<w_1, v_j> + \lambda_1<dw_1, v_j> \qquad \text{(A3.5)}$$

After simplifying $<Sdw_1, v_j> = \lambda_j<dw_1, v_j>$ and $d\lambda_1<w_1, v_j> = 0$ for $j \neq 1$ gives

$$<dw_1, v_j> = <(dS)w_1, v_j> / (\lambda_1 - \lambda_j) \qquad \text{(A3.6)}$$

Substitute equation (A3.4) into equation (A3.6) yields

$$\Sigma_{m\neq 1}k_m<w_m, v_j> = <(dS)w_1, v_j> / (\lambda_1 - \lambda_j) \qquad \text{(A3.7)}$$

This can be further simplified since $<w_m, v_j> = 0$ for every $j \neq m$ and $<w_m, v_j> = 1$ for every $j = m$, so that

$$k_j = <(dS)w_1, v_j> / (\lambda_1 - \lambda_j) \qquad \text{(A3.8)}$$

Substitute equation (A3.8) into equation (A3.4) to obtain the differential of the right eigenvector w_1 as follows

$$dw_1 = \Sigma_{m=1}^{s}[<(dS)w_1, v_m> / (\lambda_1 - \lambda_m)]w_m \qquad \text{(A3.9)}$$

Since the left eigenvector v_1 of the survival matrix S is the right eigenvector of the complex conjugate transpose matrix S^*, it follows that

$$dv_1 = \Sigma_{m=1}^{s}[<(dS)v_1, w_m> / (\hat{\lambda}_1 - \hat{\lambda}_m)]v_m \tag{A3.10}$$

Consider that only one entry s_{kl} is perturbed, the first derivatives of the eigenvectors are obtained as

$$\partial w_1 / \partial s_{kl} = \Sigma_{m=1}^{s}[w_{l,1}\, \hat{v}_{k,m} / (\lambda_1 - \lambda_m)]w_m = w_{l,1}\, \Sigma_{m=1}^{s}[\hat{v}_{k,m} / (\lambda_1 - \lambda_m)]w_m \tag{A3.11}$$

$$\partial v_1 / \partial s_{kl} = v_{k,1}\, \Sigma_{m=1}^{s}[\hat{w}_{l,m} / (\hat{\lambda}_1 - \hat{\lambda}_m)]v_m \tag{A3.12}$$

■

8. References

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Figure 1

Coefficient of Variation of Time to Default for Individual Credit Ratings

A graphic representation of the *coefficient of variation* (CV) of time to default for individual credit ratings, as derived from Table 4 is preseted below.

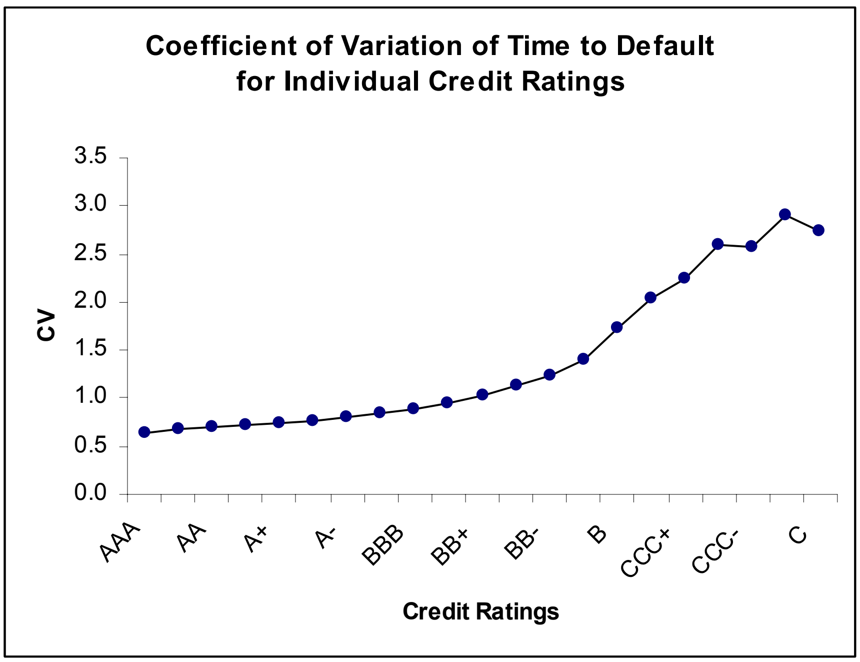


Figure 2

Distances to Default for Equally Weighted Portfolios

A graphic representation of the distances to default for the equally weighted portfolios, as described in Table 5 is presented below.

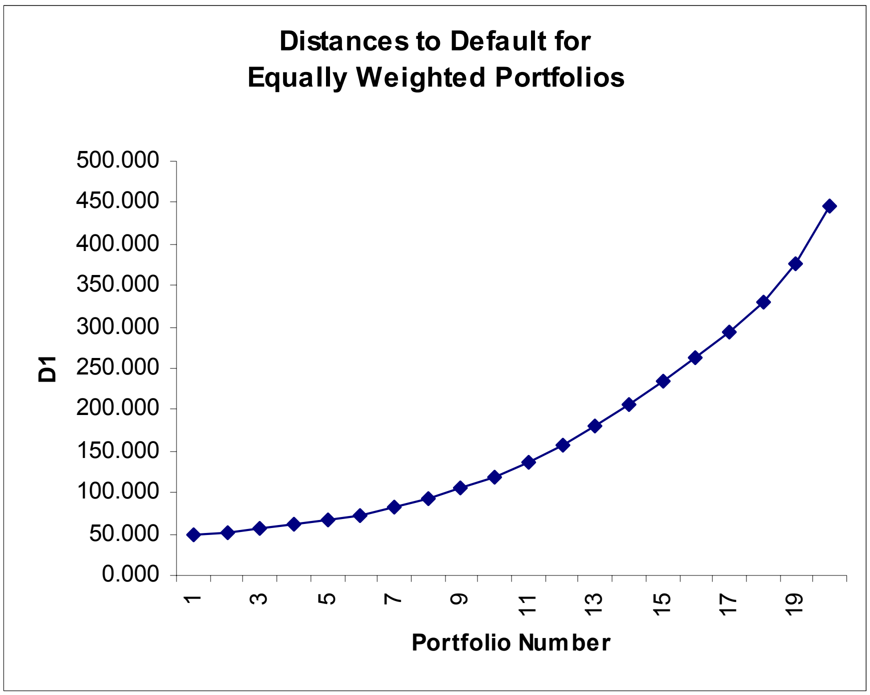


Figure 3

A Surface Graph for the Sensitivity Matrix

A surface graph illustrates the sensitivity matrix *Q* from Table 6.

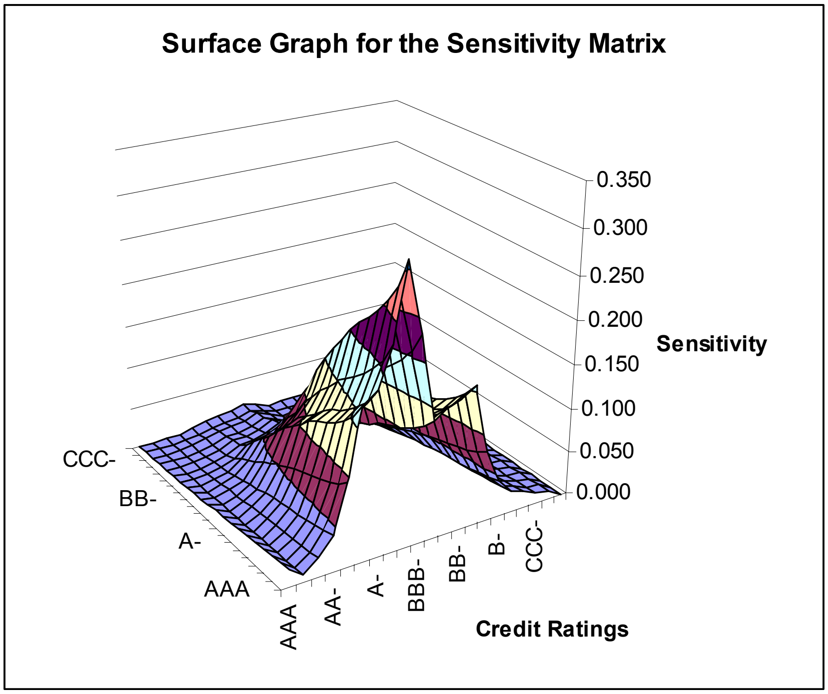


Figure 4

Surface Graph for the Second Sensitivity Derivative, Example I

An example of the second derivative of the dominant eigenvalue, with respect to entrie (4, 8) in the survival matrix S is presented below.

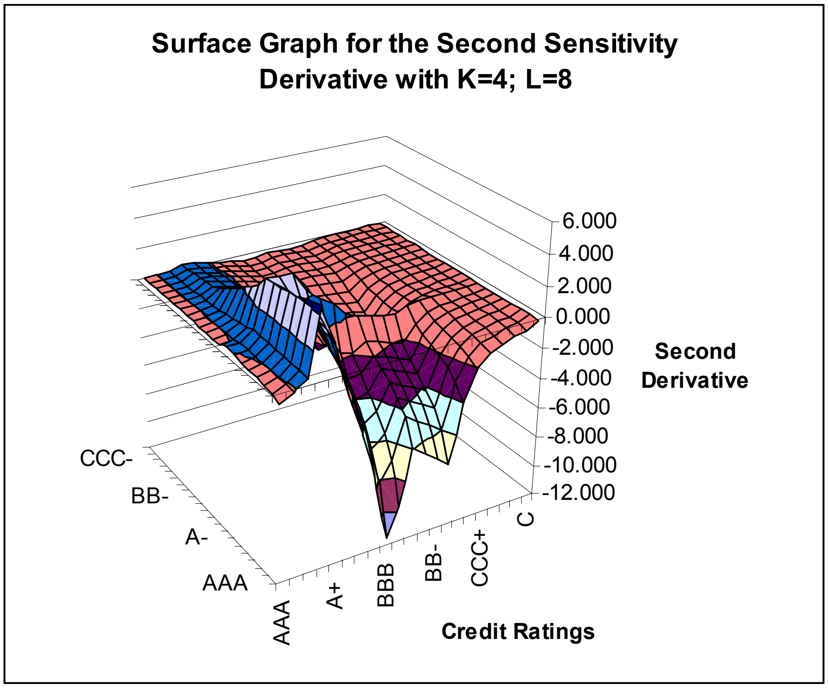


Figure 5

Surface Graph for the Second Sensitivity Derivative, Example II

An example of the second derivative of the dominant eigenvalue, with respect to entrie (11, 7) in the survival matrix S is presented below.

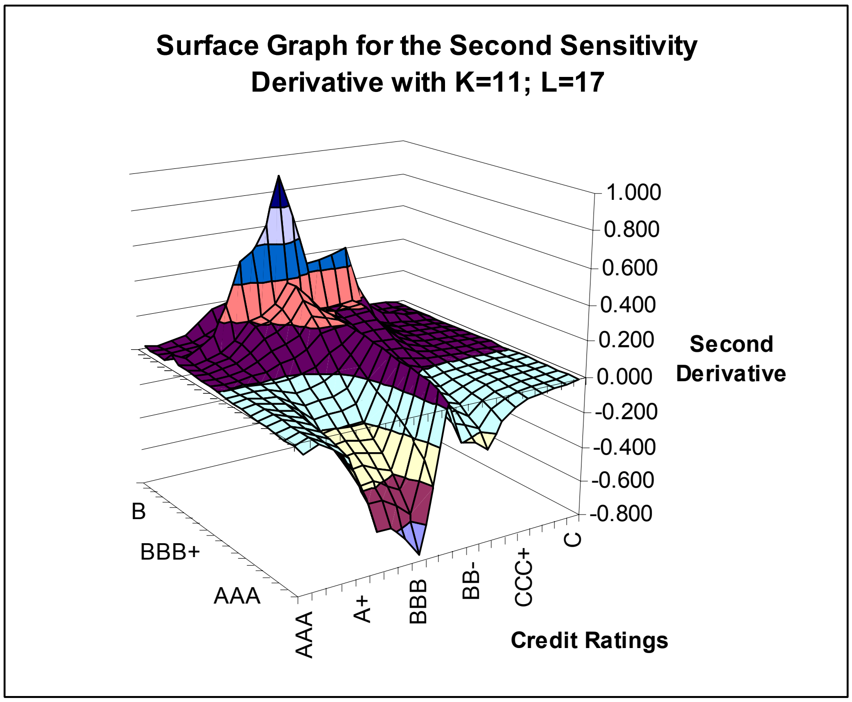


Figure 6

Surface Graph for the Second Sensitivity Derivative, Example III

An example of the second derivative of the dominant eigenvalue, with respect to entrie (13, 6) in the survival matrix *S* is presented below.

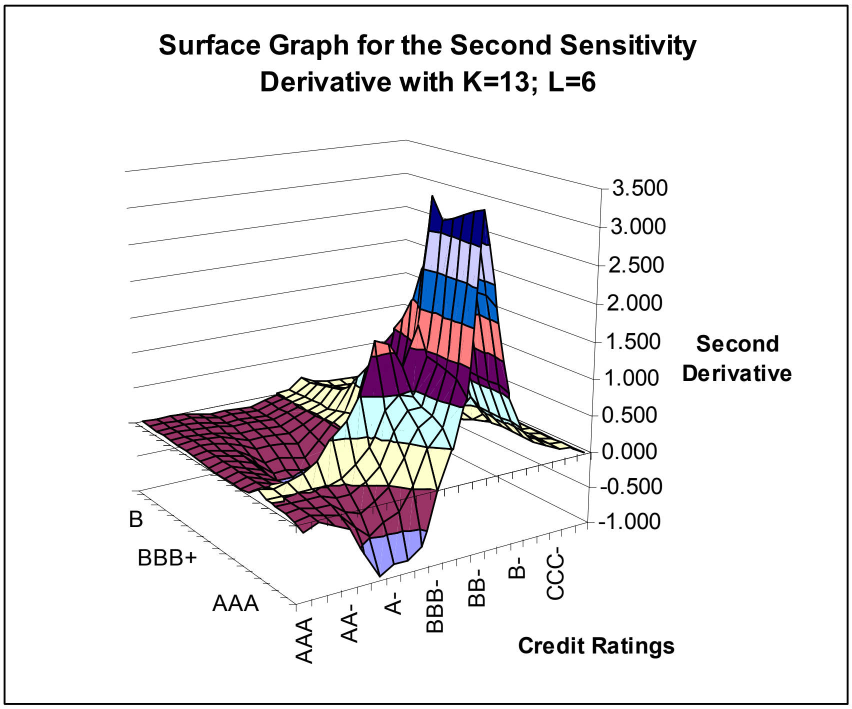


Figure 7

Comparative Analysis between the Homogeneous Markov Chain Model and the Four Stochastic Economic Models

A comparative analysis of the cumulative default rate (CDR) between the homogeneous Markov chain model and the four different definitions of business cycles, generated by Monte Carlo simulation is presented below.

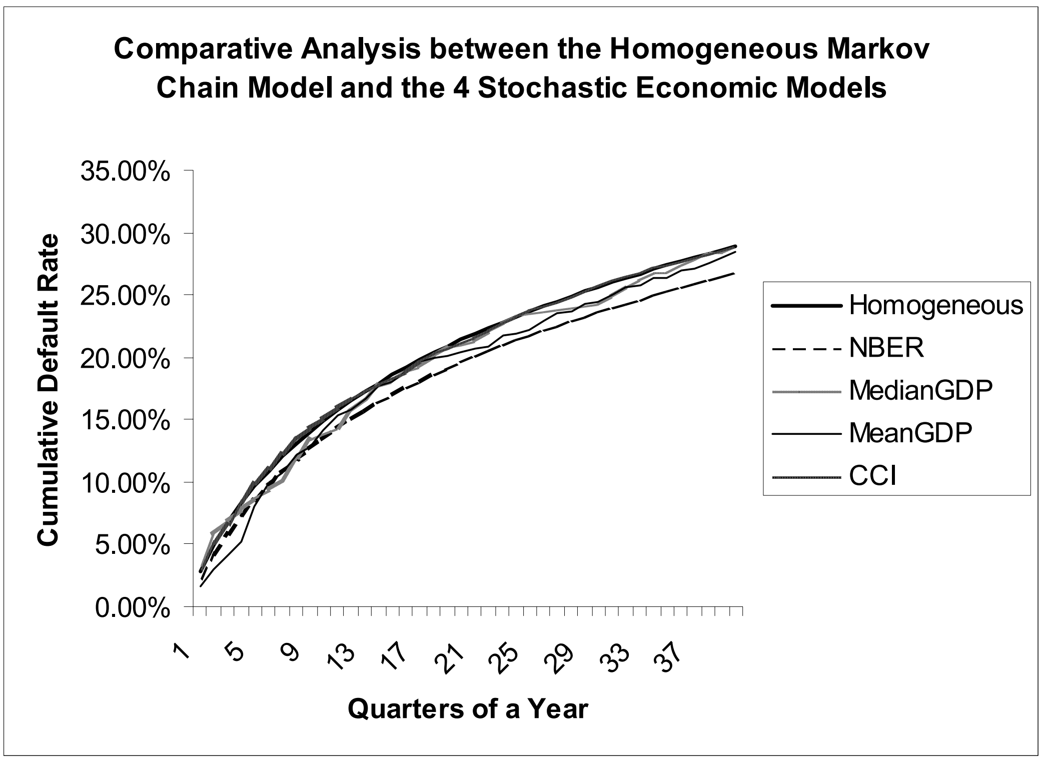


Figure 8

Quasi-Default Time Distribution for Portfolio 15

An illustration of a complete distribution of portfolio 15 from Table 8 is presented here. The inverse Gaussian distribution dictates here a slight positive skewness, while the mean stands on 24.88, and the mode on 24.09 quarters of a year.

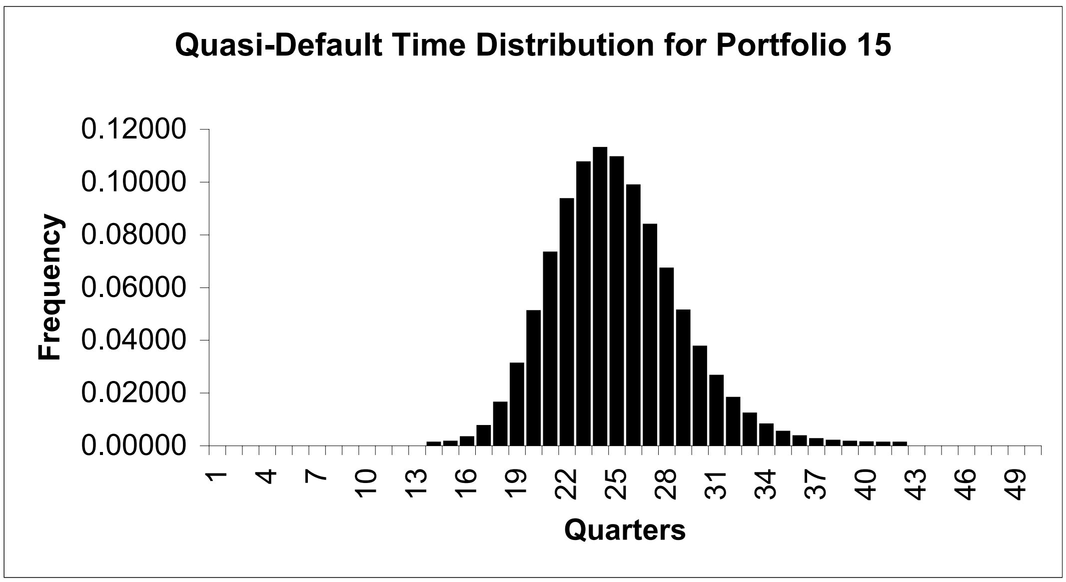


Figure 9

Cumulative Default Rate by Using the Generator Matrix

The generator matrix G is useful for measuring default rates within small time-intervals. By converting the Markov discrete process into a continuous time dynamic, the transition matrices can be found for any time $t \geq 0$ through the process of $T(t) = exp(tG)$. Here is an illustration for the first twenty tick intervals, and the corresponding Cumulative Default Rate (CDR).

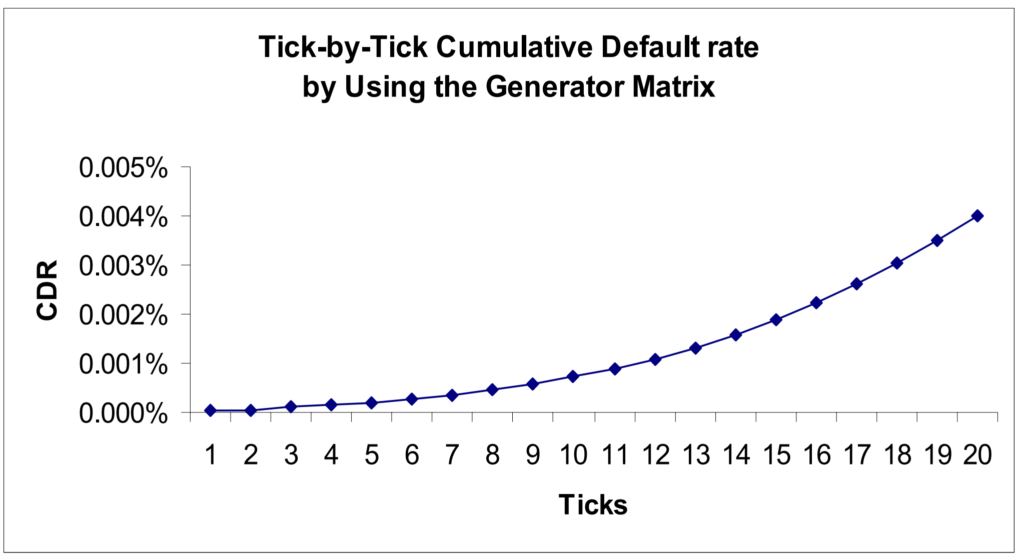


Figure 10

Momentum Credit Ratings Migration

Cumulative Default Rates are calculated for each of the three momentum transition matrices. Downgrade momentum leads to the highest CDR. Upgraded companies gain the lowest CDR and stable companies are in between. The total CDR reflects an equally weighted portfolio of the three sets of companies.

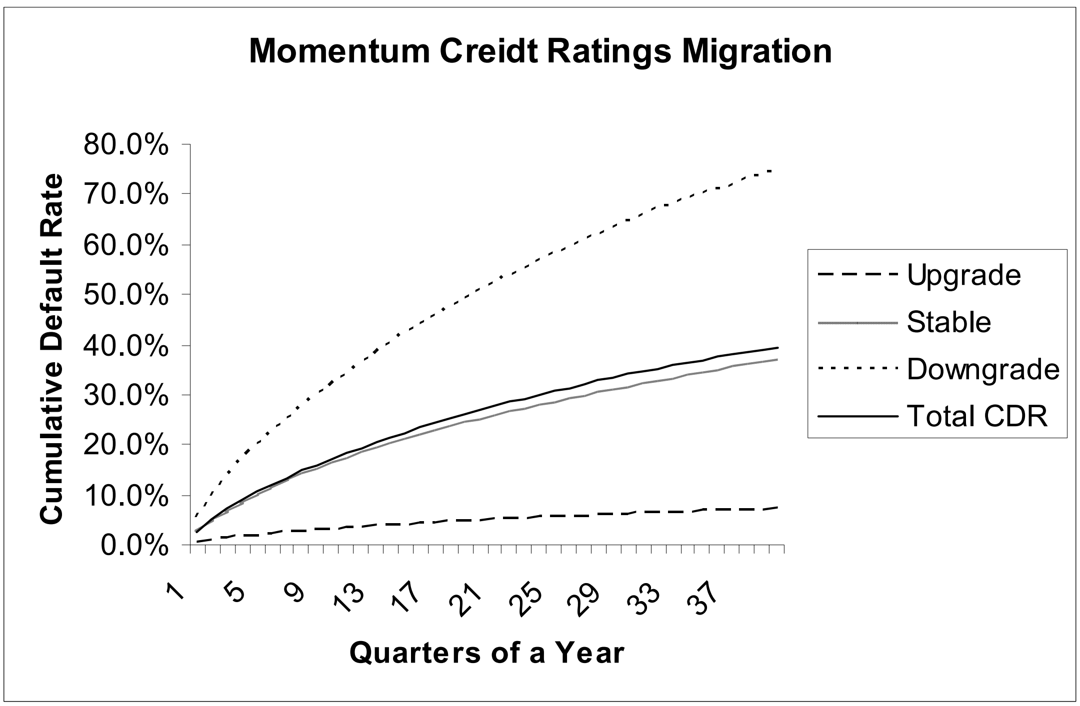


Figure 11

Comparative Analysis between the Homogeneous Markov Chain Model and the Non-Homogeneous Models

Cumulative Default Rates are calculated within the three new non-homogeneous models, and compared to the homogeneous Markov chain model over a ten-year period. The mean reversion has a minor effect, and thus slightly deviates from the homogeneous model. The internal correlations and the density-dependent models achieve much lower CDR.

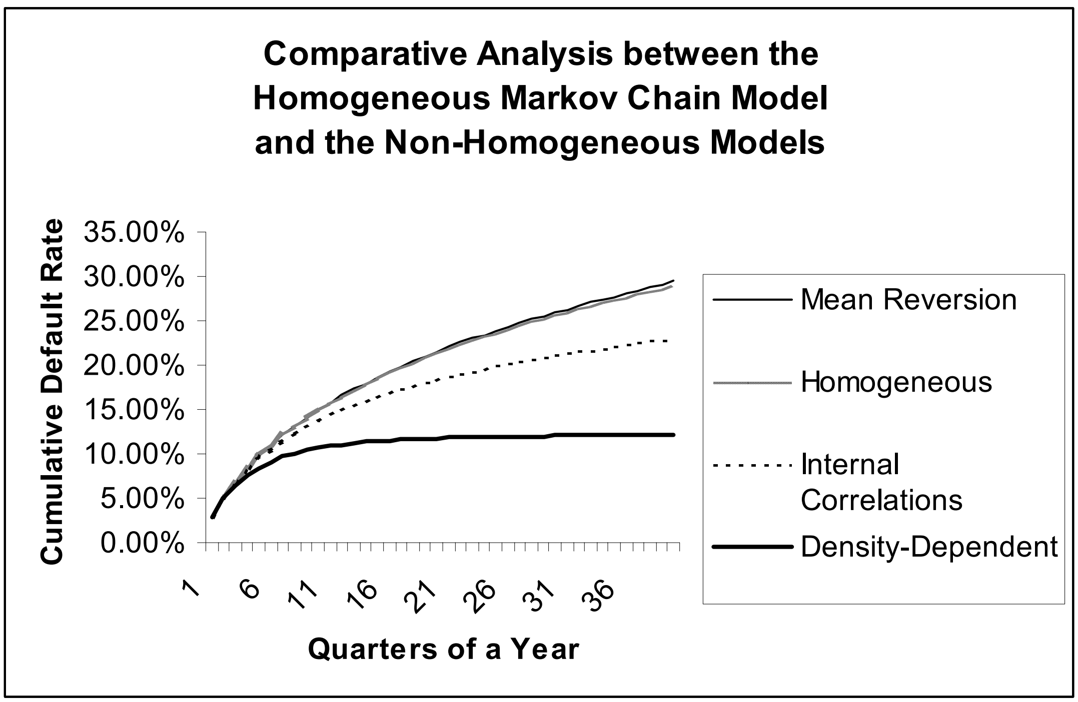


Figure 12

Comparative Analysis between the Models

Here is a Comparative Analysis between the homogeneous Markov chain model, the stochastic economic model based on the NBER definition for business cycles, the non-homogeneous models, and the actual cumulative default rates from the data sample.

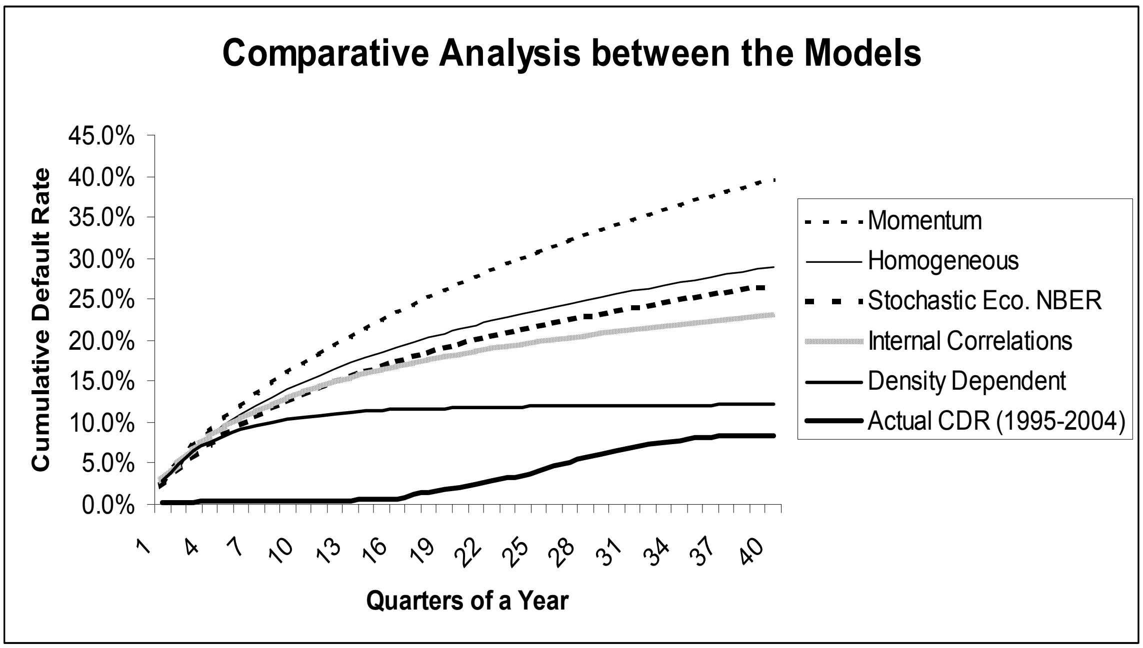
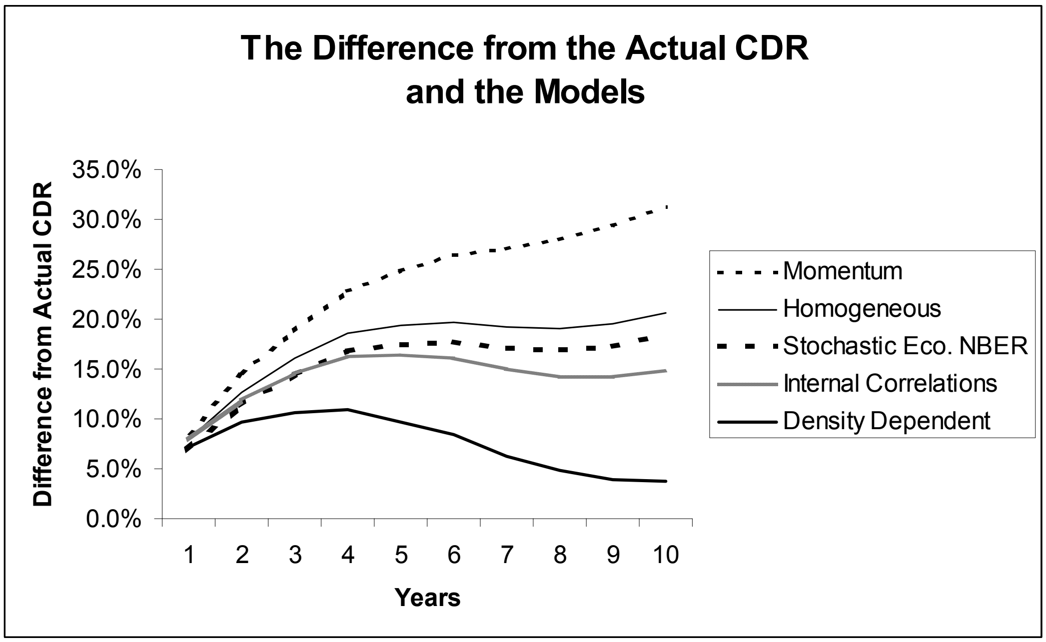


Figure 13

The Difference from the Actual CDR and the Models

The actual Cumulative Default Rate (CDR), from 1995 to 2004, is subtracted from those predicted by the various models. This is meant to emphasize by how much the models deviate from the observed data, and through that, to accentuate the differences in their predictive power.



Years	1	2	3	4	5	6	7	8	9	10
Momentum	8.6%	14.4%	18.9%	22.7%	24.8%	26.5%	27.1%	28.0%	29.4%	31.2%
Homogeneous	8.0%	12.7%	16.0%	18.6%	19.3%	19.7%	19.2%	19.1%	19.5%	20.6%
Stochastic Eco. NBER	6.9%	11.3%	14.5%	16.9%	17.5%	17.9%	17.2%	17.0%	17.4%	18.4%
Internal Correlations	8.0%	11.9%	14.5%	16.3%	16.4%	16.1%	15.0%	14.3%	14.2%	14.8%
Density Dependent	7.3%	9.7%	10.6%	10.9%	9.7%	8.4%	6.3%	4.8%	3.9%	3.8%

Table 1

The Frequency Matrix

The frequency matrix F for all quarterly data of credit ratings movements among industrial companies as recorded by Compustat North America, from the beginning of 1985 until the end of 2004. The horizontal axis describes the initial credit rating, and the vertical axis describes the credit rating in the next consecutive quarter. The credit ratings movements are measured within a quarter of a year. All together there are 123,849 valid credit ratings migrations representing 4,510 different industrial companies.

Frequency Matrix	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB	BBB-	BB+	BB	BB-	B+	B	B-	CCC+	CCC	CCC-	CC	C
AAA	2794	12	8	1	1	0	4	1	0	2	2	0	0	1	0	0	0	0	0	0	0
AA+	30	1372	14	2	1	2	1	1	0	0	0	0	0	1	0	0	0	0	0	0	0
AA	9	54	4671	44	5	5	4	0	2	1	0	0	0	0	1	0	0	0	0	0	0
AA-	9	11	147	5261	74	11	6	1	3	5	2	1	1	2	0	0	0	0	0	0	0
A+	3	2	32	186	8606	144	15	15	6	0	0	0	0	1	0	0	1	0	0	0	0
A	0	2	17	45	251	11458	176	34	14	6	4	0	2	3	0	0	0	1	0	0	0
A-	1	0	4	14	77	280	9089	175	50	15	6	1	4	3	5	0	0	1	0	0	0
BBB+	0	0	6	7	22	94	272	9494	212	42	7	6	4	10	1	1	0	1	0	0	0
BBB	2	1	3	2	7	33	109	311	10872	223	51	24	11	3	4	0	1	0	0	0	0
BBB-	0	0	1	2	1	6	12	78	268	8234	195	64	15	13	1	2	2	0	0	0	0
BB+	0	0	0	0	2	8	1	14	66	180	5208	177	62	18	4	2	0	2	1	0	0
BB	1	0	0	0	2	5	3	22	26	106	140	6783	206	73	9	4	3	1	1	0	0
BB-	1	0	0	0	1	5	1	3	12	29	72	222	8633	260	25	8	4	2	3	3	0
B+	0	0	0	1	4	2	4	4	18	8	21	87	279	12448	162	27	15	5	4	6	0
B	0	0	0	3	4	0	1	3	3	6	9	24	71	344	5407	60	20	9	3	2	1
B-	0	0	0	0	0	0	0	0	0	4	4	12	27	115	153	2399	32	12	4	9	0
CCC+	0	0	0	0	0	0	0	1	2	1	0	4	15	42	99	84	1125	14	5	4	1
CCC	0	0	0	0	1	0	0	0	1	0	2	5	9	29	40	68	34	706	9	7	1
CCC-	0	0	0	0	0	0	0	1	0	0	2	3	7	7	24	29	35	14	339	5	0
CC	0	0	0	0	0	2	0	0	0	0	2	4	4	11	26	31	27	18	14	238	0
C	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	4	1	0	5
D	0	0	0	0	2	1	1	1	6	1	5	2	12	22	58	54	56	70	33	49	2

Table 2

The Transition Matrix

The non-arbitrary part of the transition matrix *T* as calculated directly from the frequency matrix *F*. The upper submatrix is the survival matrix *S* and the lower row vector is the default state *D*. Entries inside the matrix stated as percentages. Each column sums to 100%. Like the frequency matrix, this table is of dimensions $(s + 1)*s$.

Transition Matrix (%)	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB	BBB-	BB+	BB	BB-	B+	B	B-	CCC+	CCC	CCC-	CC	C
AAA	98.04	0.83	0.16	0.02	0.01	0.00	0.04	0.01	0.00	0.02	0.03	0.00	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AA+	1.05	94.36	0.29	0.04	0.01	0.02	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AA	0.32	3.71	95.27	0.79	0.06	0.04	0.04	0.00	0.02	0.01	0.00	0.00	0.00	0.00	0.02	0.00	0.00	0.00	0.00	0.00	0.00
AA-	0.32	0.76	3.00	94.49	0.82	0.09	0.06	0.01	0.03	0.06	0.03	0.01	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
A+	0.11	0.14	0.65	3.34	94.98	1.19	0.15	0.15	0.05	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.07	0.00	0.00	0.00	0.00
A	0.00	0.14	0.35	0.81	2.77	95.04	1.81	0.33	0.12	0.07	0.07	0.00	0.02	0.02	0.00	0.00	0.00	0.12	0.00	0.00	0.00
A-	0.04	0.00	0.08	0.25	0.85	2.32	93.71	1.72	0.43	0.17	0.10	0.01	0.04	0.02	0.08	0.00	0.00	0.12	0.00	0.00	0.00
BBB+	0.00	0.00	0.12	0.13	0.24	0.78	2.80	93.45	1.83	0.47	0.12	0.08	0.04	0.07	0.02	0.04	0.00	0.12	0.00	0.00	0.00
BBB	0.07	0.07	0.06	0.04	0.08	0.27	1.12	3.06	94.04	2.52	0.89	0.32	0.12	0.02	0.07	0.00	0.07	0.00	0.00	0.00	0.00
BBB-	0.00	0.00	0.02	0.04	0.01	0.05	0.12	0.77	2.32	92.90	3.40	0.86	0.16	0.10	0.02	0.07	0.15	0.00	0.00	0.00	0.00
BB+	0.00	0.00	0.00	0.00	0.02	0.07	0.01	0.14	0.57	2.03	90.86	2.39	0.66	0.13	0.07	0.07	0.00	0.23	0.24	0.00	0.00
BB	0.04	0.00	0.00	0.00	0.02	0.04	0.03	0.22	0.22	1.20	2.44	91.43	2.20	0.54	0.15	0.14	0.22	0.12	0.24	0.00	0.00
BB-	0.04	0.00	0.00	0.00	0.01	0.04	0.01	0.03	0.10	0.33	1.26	2.99	92.21	1.94	0.42	0.29	0.29	0.23	0.72	0.93	0.00
B+	0.00	0.00	0.00	0.02	0.04	0.02	0.04	0.04	0.16	0.09	0.37	1.17	2.98	92.85	2.69	0.98	1.11	0.58	0.96	1.86	0.00
B	0.00	0.00	0.00	0.05	0.04	0.00	0.01	0.03	0.03	0.07	0.16	0.32	0.76	2.57	89.82	2.17	1.47	1.05	0.72	0.62	10.00
B-	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.05	0.07	0.16	0.29	0.86	2.54	86.64	2.36	1.40	0.96	2.79	0.00
CCC+	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.02	0.01	0.00	0.05	0.16	0.31	1.64	3.03	82.96	1.63	1.20	1.24	10.00
CCC	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.01	0.00	0.03	0.07	0.10	0.22	0.66	2.46	2.51	82.09	2.16	2.17	10.00
CCC-	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.03	0.04	0.07	0.05	0.40	1.05	2.58	1.63	81.29	1.55	0.00
CC	0.00	0.00	0.00	0.00	0.00	0.02	0.00	0.00	0.00	0.00	0.03	0.05	0.04	0.08	0.43	1.12	1.99	2.09	3.36	73.68	0.00
C	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.02	0.00	0.07	0.47	0.24	0.00	50.00
D	0.00	0.00	0.00	0.00	0.02	0.01	0.01	0.01	0.05	0.01	0.09	0.03	0.13	0.16	0.96	1.95	4.13	8.14	7.91	15.17	20.00

Table 3

The Fundamental Matrix

The fundamental matrix *N*. Entry *(i, j)* represents the expected number of visits at survival state *i*, before absorption, given that the company starts in state *j*.

Fundamental Matrix *N*	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB	BBB-	BB+	BB	BB-	B+	B	B-	CCC+	CCC	CCC-	CC	C
AAA	60.00	13.43	6.04	3.71	3.15	2.92	3.01	2.53	2.25	2.16	1.94	1.43	1.12	0.90	0.57	0.40	0.35	0.29	0.27	0.21	0.24
AA+	12.66	21.94	3.14	1.67	1.28	1.17	1.10	0.93	0.80	0.73	0.64	0.49	0.39	0.33	0.21	0.14	0.13	0.10	0.10	0.08	0.09
AA	19.05	23.48	29.41	8.07	4.64	3.85	3.42	2.81	2.50	2.22	1.88	1.46	1.16	0.93	0.64	0.43	0.38	0.31	0.29	0.23	0.26
AA-	22.32	23.57	23.86	30.28	11.25	7.98	6.72	5.58	4.95	4.42	3.74	2.91	2.30	1.80	1.19	0.83	0.75	0.61	0.56	0.44	0.51
A+	31.89	33.11	33.49	34.99	41.97	21.91	16.48	13.44	11.37	9.65	8.08	6.29	4.98	3.88	2.59	1.83	1.72	1.38	1.23	0.97	1.14
A	41.98	43.91	44.19	44.61	45.34	54.07	33.72	25.93	21.61	18.21	15.22	11.80	9.34	7.19	4.81	3.41	3.08	2.67	2.29	1.81	2.11
A-	37.79	38.87	39.15	39.39	39.85	40.56	48.64	31.29	25.48	21.16	17.53	13.62	10.73	8.19	5.55	3.88	3.47	2.99	2.61	2.06	2.40
BBB+	41.15	42.20	42.49	42.32	42.47	43.35	44.63	51.48	35.24	28.57	23.32	18.15	14.09	10.68	7.09	5.05	4.46	3.77	3.39	2.67	3.06
BBB	51.24	51.90	51.90	51.57	51.72	52.77	54.27	54.38	61.81	44.15	35.70	27.45	20.89	15.45	10.22	7.25	6.47	5.23	4.94	3.86	4.38
BBB-	35.06	35.49	35.56	35.35	35.37	36.04	36.76	37.30	37.98	45.97	31.84	23.90	17.76	12.96	8.42	6.07	5.39	4.24	4.16	3.23	3.61
BB+	19.41	19.55	19.56	19.47	19.52	19.87	20.04	20.26	20.57	21.37	28.49	17.06	12.54	8.85	5.69	4.11	3.57	2.89	2.94	2.21	2.43
BB	20.52	20.44	20.42	20.33	20.39	20.70	20.86	21.09	21.03	21.85	21.60	28.70	16.47	11.41	7.15	5.12	4.49	3.42	3.62	2.80	3.01
BB-	20.48	20.30	20.26	20.19	20.23	20.49	20.56	20.67	20.74	21.26	21.57	22.11	29.29	15.44	9.36	6.60	5.72	4.28	4.80	3.95	3.87
B+	21.27	21.23	21.22	21.19	21.20	21.30	21.39	21.40	21.45	21.54	21.55	22.10	22.64	29.85	14.15	9.29	7.94	5.72	6.20	5.43	5.56
B	10.74	10.73	10.73	10.75	10.69	10.66	10.69	10.70	10.68	10.78	10.76	10.92	11.06	11.62	16.73	6.44	5.09	3.70	3.64	3.05	5.11
B-	5.52	5.51	5.51	5.51	5.50	5.51	5.53	5.55	5.55	5.64	5.63	5.72	5.75	5.94	5.70	10.81	3.69	2.56	2.44	2.43	2.39
CCC+	3.23	3.23	3.23	3.23	3.22	3.23	3.24	3.25	3.25	3.27	3.25	3.31	3.35	3.42	3.49	3.30	7.71	1.83	1.68	1.40	2.61
CCC	2.49	2.49	2.49	2.49	2.49	2.49	2.49	2.50	2.50	2.52	2.53	2.57	2.57	2.61	2.53	2.74	2.32	6.75	1.74	1.39	2.32
CCC-	1.55	1.55	1.55	1.54	1.54	1.55	1.55	1.56	1.55	1.57	1.57	1.59	1.60	1.58	1.59	1.62	1.75	1.20	6.10	0.92	0.91
CC	1.25	1.24	1.24	1.24	1.24	1.25	1.24	1.24	1.24	1.25	1.26	1.27	1.26	1.27	1.27	1.30	1.28	1.04	1.25	4.32	0.72
C	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.05	0.04	0.04	0.04	0.07	0.05	0.02	2.03

Table 4

Time to Default Measures

The table presents different time-to-default measures for each credit rating. The units of η_i are quarters of a year, also converted to round years to obtain more meaningful results. The coefficient of variation (CV) is calculated as the standard deviation divided by the mean.

Time to Default for Individual Credit Ratings	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB	BBB-	BB+	BB	BB-	B+	B	B-	CCC+	CCC	CCC-	CC	C
mean Eta (Qarters)	459.6	434.2	415.5	397.9	383.1	371.7	356.4	333.9	312.6	288.3	258.1	222.9	189.3	154.4	109.0	80.6	69.8	55.0	54.3	43.5	48.8
var Eta	88054	85190	84069	83382	82627	81370	80105	78365	76472	73521	69672	62860	55519	47025	35337	26996	24420	20254	19521	15937	17773
mean Eta (Years)	114.9	108.6	103.9	99.5	95.8	92.9	89.1	83.5	78.1	72.1	64.5	55.7	47.3	38.6	27.2	20.2	17.5	13.8	13.6	10.9	12.2
Standard Deviation Eta	296.7	291.9	289.9	288.8	287.4	285.3	283.0	279.9	276.5	271.1	264.0	250.7	235.6	216.9	188.0	164.3	156.3	142.3	139.7	126.2	133.3
CV	0.6	0.7	0.7	0.7	0.8	0.8	0.8	0.8	0.9	0.9	1.0	1.1	1.2	1.4	1.7	2.0	2.2	2.6	2.6	2.9	2.7

Table 5

Portfolios' Distance to Default

The portfolios presented in this table are equally weighted, and contain different combinations of credit rating categories. Since there are 21 different credit ratings in the survival matrix, there could be 2^{21} different combinations of equally weighted portfolios, and infinite number while considering different weights within each portfolio. Clearly, only a small fraction of this number can be presented here. The numbers inside the table are the weights for each credit rating within each of the 21 different portfolios, where the corresponding distances to default are on the right-hand side of the table.

Portfolio Number	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB	BBB-	BB+	BB	BB-	B+	B	B-	CCC+	CCC	CCC-	CC	C	Distance to Default (D1)
1	0.048	0.048	0.048	0.048	0.048	0.048	0.048	0.048	0.048	0.048	0.048	0.048	0.048	0.048	0.048	0.048	0.048	0.048	0.048	0.048	0.048	46.748
2	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050	0.050		49.351
3	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053			52.414
4	0.056	0.056	0.056	0.056	0.056	0.056	0.056	0.056	0.056	0.056	0.056	0.056	0.056	0.056	0.056	0.056	0.056	0.056				56.199
5	0.059	0.059	0.059	0.059	0.059	0.059	0.059	0.059	0.059	0.059	0.059	0.059	0.059	0.059	0.059	0.059	0.059					60.738
6	0.063	0.063	0.063	0.063	0.063	0.063	0.063	0.063	0.063	0.063	0.063	0.063	0.063	0.063	0.063	0.063						66.311
7	0.067	0.067	0.067	0.067	0.067	0.067	0.067	0.067	0.067	0.067	0.067	0.067	0.067	0.067	0.067							73.038
8	0.071	0.071	0.071	0.071	0.071	0.071	0.071	0.071	0.071	0.071	0.071	0.071	0.071	0.071								81.371
9	0.077	0.077	0.077	0.077	0.077	0.077	0.077	0.077	0.077	0.077	0.077	0.077	0.077									91.878
10	0.083	0.083	0.083	0.083	0.083	0.083	0.083	0.083	0.083	0.083	0.083	0.083										104.500
11	0.091	0.091	0.091	0.091	0.091	0.091	0.091	0.091	0.091	0.091	0.091											119.260
12	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100												135.950
13	0.111	0.111	0.111	0.111	0.111	0.111	0.111	0.111	0.111													155.980
14	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.125														179.440
15	0.143	0.143	0.143	0.143	0.143	0.143	0.143															205.850
16	0.167	0.167	0.167	0.167	0.167	0.167																234.400
17	0.200	0.200	0.200	0.200	0.200																	262.640
18	0.250	0.250	0.250	0.250																		293.960
19	0.333	0.333	0.333																			330.610
20	0.500	0.500																				376.220
21	1.000																					446.040

Table 6

The Sensitivity Matrix

The sensitivity matrix Q as derived from the survival matrix S.

Sensitivity Matrix	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB	BBB-	BB+	BB	BB-	B+	B	B-	CCC+	CCC	CCC-	CC	C
AAA	0.020	0.007	0.022	0.042	0.096	0.172	0.185	0.229	0.312	0.232	0.134	0.144	0.150	0.160	0.081	0.043	0.025	0.019	0.012	0.010	0.000
AA+	0.018	0.006	0.020	0.038	0.085	0.154	0.165	0.205	0.279	0.207	0.120	0.129	0.134	0.143	0.073	0.038	0.022	0.017	0.011	0.009	0.000
AA	0.017	0.006	0.018	0.035	0.079	0.142	0.153	0.190	0.258	0.192	0.111	0.119	0.124	0.132	0.067	0.035	0.021	0.016	0.010	0.008	0.000
AA-	0.016	0.006	0.017	0.033	0.074	0.133	0.143	0.177	0.241	0.179	0.104	0.111	0.116	0.124	0.063	0.033	0.019	0.015	0.009	0.007	0.000
A+	0.015	0.005	0.016	0.031	0.070	0.125	0.135	0.167	0.227	0.169	0.098	0.105	0.109	0.116	0.059	0.031	0.018	0.014	0.009	0.007	0.000
A	0.014	0.005	0.015	0.029	0.066	0.119	0.128	0.159	0.216	0.161	0.093	0.100	0.104	0.111	0.056	0.029	0.017	0.013	0.008	0.007	0.000
A-	0.013	0.005	0.014	0.027	0.062	0.112	0.120	0.149	0.203	0.150	0.087	0.094	0.097	0.104	0.053	0.028	0.016	0.013	0.008	0.006	0.000
BBB+	0.012	0.004	0.013	0.025	0.056	0.102	0.109	0.135	0.184	0.137	0.079	0.085	0.088	0.094	0.048	0.025	0.015	0.011	0.007	0.006	0.000
BBB	0.011	0.004	0.012	0.023	0.051	0.093	0.100	0.123	0.168	0.125	0.072	0.078	0.081	0.086	0.044	0.023	0.013	0.010	0.006	0.005	0.000
BBB-	0.010	0.003	0.011	0.020	0.046	0.083	0.089	0.110	0.150	0.111	0.064	0.069	0.072	0.077	0.039	0.020	0.012	0.009	0.006	0.005	0.000
BB+	0.008	0.003	0.009	0.017	0.040	0.071	0.077	0.095	0.129	0.096	0.056	0.060	0.062	0.066	0.034	0.018	0.010	0.008	0.005	0.004	0.000
BB	0.007	0.002	0.007	0.014	0.032	0.058	0.063	0.078	0.106	0.079	0.045	0.049	0.051	0.054	0.028	0.014	0.008	0.007	0.004	0.003	0.000
BB-	0.005	0.002	0.006	0.011	0.026	0.047	0.050	0.062	0.085	0.063	0.036	0.039	0.041	0.044	0.022	0.012	0.007	0.005	0.003	0.003	0.000
B+	0.004	0.002	0.005	0.009	0.020	0.036	0.039	0.048	0.065	0.049	0.028	0.030	0.031	0.034	0.017	0.009	0.005	0.004	0.003	0.002	0.000
B	0.003	0.001	0.003	0.006	0.013	0.024	0.026	0.032	0.043	0.032	0.019	0.020	0.021	0.022	0.011	0.006	0.003	0.003	0.002	0.001	0.000
B-	0.002	0.001	0.002	0.004	0.009	0.017	0.018	0.023	0.031	0.023	0.013	0.014	0.015	0.016	0.008	0.004	0.002	0.002	0.001	0.001	0.000
CCC+	0.002	0.001	0.002	0.004	0.008	0.015	0.016	0.020	0.027	0.020	0.012	0.012	0.013	0.014	0.007	0.004	0.002	0.002	0.001	0.001	0.000
CCC	0.001	0.001	0.002	0.003	0.007	0.012	0.013	0.016	0.022	0.016	0.009	0.010	0.010	0.011	0.006	0.003	0.002	0.001	0.001	0.001	0.000
CCC-	0.001	0.000	0.001	0.003	0.006	0.011	0.012	0.015	0.021	0.015	0.009	0.010	0.010	0.011	0.005	0.003	0.002	0.001	0.001	0.001	0.000
CC	0.001	0.000	0.001	0.002	0.005	0.009	0.010	0.012	0.016	0.012	0.007	0.008	0.008	0.008	0.004	0.002	0.001	0.001	0.001	0.001	0.000
C	0.001	0.000	0.001	0.002	0.006	0.010	0.011	0.014	0.019	0.014	0.008	0.009	0.009	0.009	0.005	0.003	0.001	0.001	0.001	0.001	0.000

Table 7

Transition Matrices for Business Cycles

The four definitions of business cycles lead to different transition matrices between economic stages. The horizontal expansionary and contractionary business cycles are the starting economic stages, and the vertical are the end economic stages. Each column sums to 1. The correlation coefficient between business cycles is also presented for each of the four definitions.

Matrix *P* for NBER Business Cycles		
	Expansion	**Contraction**
Expansion	0.9726	0.3333
Contraction	0.0274	0.6667
Correlation	0.64	

Matrix *P* for CCI Business Cycles		
	Expansion	**Contraction**
Expansion	0.4419	0.3714
Contraction	0.5581	0.6286
Correlation	0.07	

Matrix *P* for Mean GDP Business Cycles		
	Expansion	**Contraction**
Expansion	0.3125	0.5106
Contraction	0.6875	0.4894
Correlation	-0.20	

Matrix *P* for Median GDP Business Cycles		
	Expansion	**Contraction**
Expansion	0.5128	0.5250
Contraction	0.4872	0.4750
Correlation	-0.01	

Table 8

Portfolios' Time to Quasi-Default

Illustration of the distribution of time to quasi-default for different portfolios when the quasi-default critical point is set to be a portfolio of six equally weighted holdings.

Portfolio Number	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB	BBB-	BB+	BB	BB-	B+	B	B-	CCC+	CCC	CCC-	CC	C	Portfolio Structure (Taxicab norm)	Quasi Default Threshold	Expected Time to Quasi Default (Quarters)	Expected Time to Quasi Default (Years)	Mode of Time to Quasi Default (Quarters)	St. Dev. of Time to Quasi Default
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	21	0.285714	202.19	50.55	201.39	10.42
2		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	20	0.300000	194.32	48.58	193.51	10.21
3			1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	19	0.315789	186.04	46.51	185.24	9.99
4				1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	18	0.333333	177.31	44.33	176.51	9.75
5					1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	17	0.352941	168.09	42.02	167.28	9.50
6						1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	16	0.375000	158.30	39.58	157.50	9.22
7							1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	15	0.400000	147.89	36.97	147.08	8.91
8								1	1	1	1	1	1	1	1	1	1	1	1	1	1	14	0.428571	136.75	34.19	135.95	8.57
9									1	1	1	1	1	1	1	1	1	1	1	1	1	13	0.461538	124.79	31.20	123.99	8.18
10										1	1	1	1	1	1	1	1	1	1	1	1	12	0.500000	111.87	27.97	111.07	7.75
11											1	1	1	1	1	1	1	1	1	1	1	11	0.545455	97.83	24.46	97.03	7.25
12												1	1	1	1	1	1	1	1	1	1	10	0.600000	82.45	20.61	81.64	6.65
13													1	1	1	1	1	1	1	1	1	9	0.666667	65.44	16.36	64.64	5.93
14														1	1	1	1	1	1	1	1	8	0.750000	46.43	11.61	45.63	4.99
15															1	1	1	1	1	1	1	7	0.857143	24.88	6.22	24.09	3.65
The Quasi Default Critical Point:																1	1	1	1	1	1						

Table 9

First Time Interval Transition Matrix

The first time interval transition matrix as created from the generator matrix *G*. Entries are similar to those of the homogeneous transition matrix for most of the table, but more pronounced differences are observed in the lower right corner.

Transition Matrix (%)	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB	BBB-	BB+	BB	BB-	B+	B	B-	CCC+	CCC	CCC-	CC	C
AAA	98.03	0.83	0.16	0.02	0.01	0.00	0.04	0.01	0.00	0.02	0.03	0.00	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AA+	1.05	94.35	0.29	0.04	0.01	0.02	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AA	0.32	3.71	95.27	0.79	0.06	0.04	0.04	0.00	0.02	0.01	0.00	0.00	0.00	0.00	0.02	0.00	0.00	0.00	0.00	0.00	0.00
AA-	0.32	0.76	3.00	94.48	0.82	0.09	0.06	0.01	0.03	0.06	0.03	0.01	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
A+	0.11	0.14	0.65	3.34	95.00	1.19	0.15	0.15	0.05	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.08	0.00	0.00	0.00	0.01
A	0.00	0.14	0.35	0.81	2.77	95.04	1.81	0.33	0.12	0.07	0.07	0.00	0.02	0.02	0.00	0.00	0.00	0.12	0.00	0.00	0.01
A-	0.04	0.00	0.08	0.25	0.85	2.32	93.72	1.72	0.43	0.17	0.10	0.01	0.04	0.02	0.08	0.00	0.00	0.12	0.00	0.00	0.01
BBB+	0.00	0.00	0.12	0.13	0.24	0.78	2.80	93.46	1.83	0.47	0.12	0.08	0.04	0.07	0.02	0.04	0.00	0.12	0.00	0.00	0.01
BBB	0.07	0.07	0.06	0.04	0.08	0.27	1.12	3.06	94.08	2.52	0.89	0.32	0.12	0.02	0.07	0.00	0.08	0.01	0.00	0.00	0.01
BBB-	0.00	0.00	0.02	0.04	0.01	0.05	0.12	0.77	2.32	92.90	3.40	0.86	0.16	0.10	0.02	0.07	0.15	0.01	0.01	0.00	0.01
BB+	0.00	0.00	0.00	0.00	0.02	0.07	0.01	0.14	0.57	2.03	90.92	2.39	0.66	0.13	0.07	0.07	0.02	0.24	0.25	0.01	0.02
BB	0.04	0.00	0.00	0.00	0.02	0.04	0.03	0.22	0.22	1.20	2.44	91.43	2.20	0.54	0.15	0.15	0.23	0.12	0.25	0.03	0.03
BB-	0.04	0.00	0.00	0.00	0.01	0.04	0.01	0.03	0.10	0.33	1.26	2.99	92.31	1.94	0.42	0.29	0.30	0.24	0.75	1.01	0.06
B+	0.00	0.00	0.00	0.02	0.04	0.02	0.04	0.04	0.16	0.09	0.37	1.17	2.98	92.96	2.70	0.99	1.13	0.61	1.00	2.02	0.29
B	0.00	0.00	0.00	0.05	0.04	0.00	0.01	0.03	0.03	0.07	0.16	0.32	0.76	2.58	90.59	2.20	1.51	1.10	0.75	0.68	11.26
B-	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.05	0.07	0.16	0.29	0.87	2.58	88.14	2.43	1.47	1.01	3.06	0.43
CCC+	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.02	0.01	0.01	0.06	0.16	0.32	1.69	3.12	86.36	1.74	1.28	1.38	11.47
CCC	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.01	0.00	0.04	0.07	0.10	0.23	0.70	2.59	2.67	89.40	2.35	2.46	11.74
CCC-	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.04	0.04	0.08	0.05	0.42	1.10	2.74	1.77	88.25	1.76	0.33
CC	0.00	0.00	0.00	0.00	0.00	0.02	0.00	0.00	0.00	0.00	0.04	0.06	0.05	0.09	0.47	1.23	2.21	2.38	3.81	87.58	0.34
C	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.02	0.01	0.08	0.55	0.28	0.01	63.96

Table 10

Dickey-Fuller Likelihood Ratio Test

The values of the statistics Φ_1 of the Dickey-Fuller likelihood ratio test, for a random walk with zero drift, are presented below. Entries classified as not meaningful (N/M) represent entries with zero observations in the transition matrix, throughout the whole time frame, from 1985 to 2004.

Dickey_Fuller Test	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB	BBB-	BB+	BB	BB-	B+	B	B-	CCC+	CCC	CCC-	CC	C
AAA	5.32	8.91	7.25	9.50	9.50	N/M	11.00	9.50	N/M	10.70	10.77	N/M	N/M	9.50	N/M	N/M	N/M	N/M	N/M	N/M	N/M
AA+	8.68	36.74	21.13	3.30	9.50	9.50	9.50	9.50	N/M	N/M	N/M	N/M	N/M	9.50	N/M	N/M	N/M	N/M	N/M	N/M	N/M
AA	8.32	2.15	92.08	4.15	6.49	4.45	10.56	N/M	10.53	9.50	N/M	N/M	N/M	N/M	9.50	N/M	N/M	N/M	N/M	N/M	N/M
AA-	10.01	12.28	8.15	21.24	8.13	11.68	15.10	9.50	11.91	14.27	10.47	9.50	9.50	10.61	N/M	N/M	N/M	N/M	N/M	N/M	N/M
A+	4.47	10.57	5.43	11.92	17.81	15.99	12.54	11.17	6.63	N/M	N/M	N/M	N/M	9.50	N/M	N/M	9.50	N/M	N/M	N/M	N/M
A	N/M	3.34	6.43	358.19	6.47	13.86	5.81	15.45	8.61	10.56	7.40	N/M	10.52	12.39	N/M	N/M	N/M	9.50	N/M	N/M	N/M
A-	9.50	N/M	10.29	11.13	5.07	4.25	18.86	6.55	4.67	4.31	17.07	9.50	11.96	10.74	13.59	N/M	N/M	9.50	N/M	N/M	N/M
BBB+	N/M	N/M	9.89	9.73	7.79	4.28	8.96	9.02	4.08	16.92	10.34	16.57	13.72	11.12	9.50	9.50	N/M	9.50	N/M	N/M	N/M
BBB	10.46	9.50	11.87	10.76	7.06	3.65	5.88	5.64	16.70	13.54	8.18	195.21	9.75	12.20	12.26	N/M	9.50	N/M	N/M	N/M	N/M
BBB-	N/M	N/M	9.50	10.76	9.50	3.89	9.54	6.67	10.33	3.89	12.00	4.05	6.51	4.19	9.50	9.50	10.71	N/M	N/M	N/M	N/M
BB+	N/M	N/M	N/M	N/M	1.99	13.48	9.50	12.85	15.06	11.12	18.91	9.84	3.43	12.37	13.97	10.55	N/M	10.74	9.50	N/M	N/M
BB	9.50	N/M	N/M	N/M	9.50	5.52	9.50	9.25	6.91	58.42	3.24	9.16	5.68	5.81	5.34	10.83	12.04	9.50	9.50	N/M	N/M
BB-	9.50	N/M	N/M	N/M	9.50	13.67	9.50	11.81	16.57	8.58	9.91	5.40	12.28	5.34	9.68	5.12	9.77	10.60	10.58	5.99	N/M
B+	N/M	N/M	N/M	9.50	7.90	10.76	11.37	4.16	6.98	16.90	14.32	6.50	3.74	8.99	6.08	8.31	2.25	8.51	7.25	8.38	N/M
B	N/M	N/M	N/M	10.48	4.30	N/M	9.50	12.41	3.90	10.43	15.17	5.06	12.59	2.35	16.74	10.62	6.94	7.07	9.73	4.44	0.00
B-	N/M	N/M	N/M	N/M	N/M	N/M	N/M	N/M	N/M	12.94	13.79	15.09	5.06	8.60	1.35	12.98	2.42	10.42	4.14	9.52	N/M
CCC+	N/M	N/M	N/M	N/M	N/M	N/M	8.50	-8.50	10.41	9.50	N/M	9.57	5.16	4.18	18.74	7.54	17.55	10.33	7.65	6.61	0.00
CCC	N/M	N/M	N/M	8.50	-8.50	N/M	N/M	N/M	9.50	N/M	10.77	13.12	15.02	2.70	11.28	7.69	17.04	21.85	10.83	4.46	0.00
CCC-	N/M	N/M	N/M	N/M	N/M	N/M	N/M	9.50	N/M	N/M	10.36	10.73	5.63	7.40	13.62	9.38	2.83	5.19	13.60	6.64	N/M
CC	N/M	N/M	N/M	N/M	8.50	-8.50	N/M	N/M	N/M	N/M	10.77	5.57	3.68	5.36	8.69	3.87	7.68	6.33	7.21	6.60	N/M
C	N/M	N/M	N/M	N/M	N/M	N/M	N/M	N/M	N/M	N/M	N/M	N/M	N/M	9.50	9.50	N/M	9.50	7.95	9.50	N/M	0.00
D	N/M	N/M	N/M	N/M	9.50	9.50	9.50	9.50	7.19	9.50	4.63	3.30	6.63	6.39	5.37	327.22	3.57	2.24	7.94	4.88	8.50

Table 11

Significance Levels for Dickey-Fuller Test

The likelihood ratio test rejects the null hypothesis for larger values of Φ_1, relative to critical values from Dickey-Fuller tables. This table presents the significance levels for each Φ_1 value in Table 10. The statistical levels are classified as 0.99, 0.975, 0.95, and 0.90. Not statistically significant entries are classified as either 0.10 or 0.00. Entries classified as not meaningful (N/M) represent entries with zero observations in the transition matrix, throughout the whole time frame, from 1985 to 2004.

Dickey_Fuller Statistical Significance	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB	BBB-	BB+	BB	BB-	B+	B	B-	CCC+	CCC	CCC-	CC	C
AAA	0.95	0.99	0.975	0.99	0.99	N/M	0.99	0.99	N/M	0.99	0.99	N/M	N/M	0.99	N/M	N/M	N/M	N/M	N/M	N/M	N/M
AA+	0.99	0.99	0.99	0.10	0.99	0.99	0.99	0.99	N/M	N/M	N/M	N/M	N/M	0.99	N/M	N/M	N/M	N/M	N/M	N/M	N/M
AA	0.99	0.10	0.99	0.90	0.975	0.90	0.99	N/M	0.99	0.99	N/M	N/M	N/M	N/M	0.99	N/M	N/M	N/M	N/M	N/M	N/M
AA-	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	N/M	N/M	N/M	N/M	N/M	N/M	N/M
A+	0.90	0.99	0.95	0.99	0.99	0.99	0.99	0.99	0.975	N/M	N/M	N/M	N/M	0.99	N/M	N/M	0.99	N/M	N/M	N/M	N/M
A	N/M	0.10	0.975	0.99	0.975	0.99	0.95	0.99	0.99	0.99	0.975	N/M	0.99	0.99	N/M	N/M	N/M	0.99	N/M	N/M	N/M
A-	0.99	N/M	0.99	0.99	0.90	0.90	0.99	0.975	0.90	0.90	0.99	0.99	0.99	0.99	0.99	N/M	N/M	0.99	N/M	N/M	N/M
BBB+	N/M	N/M	0.99	0.99	0.975	0.90	0.99	0.99	0.10	0.99	0.99	0.99	0.99	0.99	0.99	0.99	N/M	0.99	N/M	N/M	N/M
BBB	0.99	0.99	0.99	0.99	0.975	0.10	0.95	0.95	0.99	0.99	0.99	0.99	0.99	0.99	0.99	N/M	9.50	N/M	N/M	N/M	N/M
BBB-	N/M	N/M	0.99	0.99	0.99	0.10	0.99	0.975	0.99	0.10	0.99	0.10	0.975	0.90	0.99	0.99	0.99	N/M	N/M	N/M	N/M
BB+	N/M	N/M	N/M	N/M	0.10	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.10	0.99	0.99	0.99	N/M	0.99	0.99	N/M	N/M
BB	0.99	N/M	N/M	N/M	0.99	0.95	0.99	0.99	0.975	0.99	0.10	0.99	0.95	0.95	0.95	0.99	0.99	0.99	0.99	N/M	N/M
BB-	0.99	N/M	N/M	N/M	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.95	0.99	0.95	0.99	0.90	0.99	0.99	0.99	0.95	N/M
B+	N/M	N/M	N/M	0.99	0.99	0.99	0.99	0.90	0.975	0.99	0.99	0.975	0.10	0.99	0.95	0.99	0.10	0.99	0.975	0.99	N/M
B	N/M	N/M	N/M	0.99	0.90	N/M	0.99	0.99	0.10	0.99	0.99	0.90	0.99	0.10	0.99	0.99	0.975	0.975	0.99	0.90	0.00
B-	N/M	N/M	N/M	N/M	N/M	N/M	N/M	N/M	N/M	0.99	0.99	0.99	0.90	0.99	0.10	0.99	0.10	0.99	0.90	0.99	N/M
CCC+	N/M	N/M	N/M	N/M	N/M	N/M	0.99	0.00	0.99	0.99	N/M	0.99	0.90	0.90	0.99	0.975	0.99	0.99	0.975	0.975	0.00
CCC	N/M	N/M	N/M	0.99	0.00	N/M	N/M	N/M	0.99	N/M	0.99	0.99	0.99	0.10	0.99	0.975	0.99	0.99	0.99	0.90	0.00
CCC-	N/M	N/M	N/M	N/M	N/M	N/M	N/M	0.99	N/M	N/M	0.99	0.99	0.95	0.99	0.99	0.99	0.10	0.95	0.99	0.975	N/M
CC	N/M	N/M	N/M	N/M	0.99	0.00	N/M	N/M	N/M	N/M	0.99	0.95	0.10	0.95	0.99	0.10	0.975	0.975	0.975	0.975	N/M
C	N/M	N/M	N/M	N/M	N/M	N/M	N/M	N/M	N/M	N/M	N/M	N/M	N/M	0.99	0.99	N/M	0.99	0.99	0.99	N/M	0.00
D	N/M	N/M	N/M	N/M	0.99	0.99	0.99	0.99	0.975	0.99	0.90	0.10	0.975	0.975	0.95	0.99	0.10	0.10	0.99	0.90	0.99

Table 12

Survivability and Transitivity Measures

A sample of the survivability and transitivity measures, above and below the main diagonal from 1999 to 2004, with the corresponding Cumulative Default Rate (CDR) within this period.

Year	CDR	Sample of Survivability Measures (in %)					
		AAA	AA	BBB	BB+	B+	CCC
1999	1.44%	100.00	100.00	99.843505	99.685535	99.607843	72.000000
2000	2.90%	100.00	100.00	99.864682	99.737877	99.663866	77.631579
2001	5.07%	100.00	100.00	99.872881	99.829497	99.672012	77.697842
2002	6.67%	100.00	100.00	99.848439	99.811202	99.722145	80.660377
2003	7.66%	100.00	100.00	99.884793	99.851998	99.749602	85.937500
2004	7.72%	100.00	100.00	99.896737	99.867374	99.770355	87.958115

Year	CDR	Sample of the Transitivity Measures (in %) Above the Main Diagonal (j < i)			
		B+	B-	CCC+	CC
1999	1.44%	1.568627451	0	1.428571429	5.555555556
2000	2.90%	1.568627451	1.089918256	1.621621622	9.756097561
2001	5.07%	1.494169096	1.208459215	2.265372168	7.142857143
2002	6.67%	1.555987774	1.857923497	2.678571429	11.20689655
2003	7.66%	2.048713863	2.629272568	3.442340792	11.94968553
2004	7.72%	2.171189979	2.762430939	4.423380727	11.69590643

Year	CDR	Sample of the Transitivity Measures (in %) Below the Main Diagonal (j > i)		
		AAA	B-	CCC-
1999	1.44%	0	10.85271318	14.28571429
2000	2.90%	4.184100418	7.629427793	4.761904762
2001	5.07%	3.768115942	9.516616314	4.255319149
2002	6.67%	3.628117914	10.49180328	3.529411765
2003	7.66%	3.047619048	9.991235758	4.838709677
2004	7.72%	3.169014085	9.629044988	4.964539007

Table 13

Randomization Test

The final results of the randomization tests for seven consecutive half-year periods are given below. The first column describes the time frame of the test, including the first and the last half-years in each interval. The second column presents the number of companies in the database at the beginning of the test. The third column shows the sequence of defaults (from left to right) as reported by S&P. The next four columns are the statistics m_1, m_2, b and the test-statistic T. The last column describes the significance level of each test. These values are the percentages of *T-values* computed for the permutations, and are smaller or equal to the test-statistic T in the previous column. Values less than 5% suggest a rejection of the null hypothesis of density independence, at 5% level of significance. Values less than 1% indicate on a rejection of the null hypothesis of density independence, at 1% level of significance, etc.

Time Period	Number of Companies at the Starting Point	Number of Companies Defaulted in Consecutive Half-Year Periods	M1	M2	b	T	Significance Level
2000 H1 - 2003 H1	2,925	33,21,47,34,34,25,20	7.946691	7.924889	1.625152	0.663133	28.5714%
2000 H2 - 2003 H2	2,809	21,47,34,34,25,20,13	7.90569	7.895467	0.909386	0.679899	19.3254%
2001 H1 - 2004 H1	2,701	47,34,34,25,20,13,6	7.862521	7.852725	0.791643	0.056129	0.0198%
2001 H2 - 2004 H2	2,588	34,34,25,20,13,6,0	7.826733	7.819254	0.75175	0.110571	0.0397%

A Review of Developments in Credit Structural Models and Credit Ratings Migration Analysis

Abstract

This survey reviews past developments in structural models of credit risk, started with the Merton's framework in 1974, and credit ratings migration analysis, as well as introduces recent schemes in these two methodologies. The advantages and disadvantages of each approach are further discussed and compared.

1. The Merton Model – Where it all Began

It has been more than three decades since the first appearance of the Merton structural approach for measuring corporate credit risk. The pioneering formation for measuring default risk as a function of assets and debt values was introduced by Merton (1974), and is often referred to as the option theoretic valuation of debt. The perception underlying the model states that the default threshold is a function of the firm's stochastic total assets market value and a deterministic face value of debt.

Merton (1974) presents a model where a company's equity can be considered as a European call option on the company's assets. The model can be simplified by assuming that the company has one zero-coupon bond outstanding which matures at time *T*. The model also defines the following:

V_0: market value of the company's assets today

V_T: market value of the company's assets at time T

E_0: value of the company's equity today

E_T: value of the company's equity at time T

D: total amount of debt (interest and principal) due to be repaid at time T

σ_V: volatility of the company's assets

σ_E: volatility of the company's equity

r_f: risk-free interest rate

Hence, if $V_T < D$, the company defaults at time T and the value of the equity is then zero. If $V_T > D$, the company pays out its debt at time T, and the value of the equity becomes $V_T - D$. Therefore, the value of the firm's equity at time T can be written as

$$E_T = Max\ (V_T - D,\ 0) = (V_T - D)^+ \quad \forall V_T \in \mathbb{R} \quad \forall D \in \mathbb{R} \tag{1.1}$$

The company's equity is a call option on the value of the assets, with the total amount of debt to be paid at time T as the strike price. Black-Scholes (1973) model solves the value of the company's equity today as follows:

$$E_0 = V_0 \Phi(d_1) - De^{-r_f T} \Phi(d_2) \tag{1.2}$$

where

$$d_1 = (\ ln(V_0 / D) + (r_f + \sigma^2_V / 2)T\ ) / \sigma_V \sqrt{T}$$

$$d_2 = (\ ln(V_0 / D) + (r_f - \sigma^2_V / 2)T\ ) / \sigma_V \sqrt{T} = d_1 - \sigma_V \sqrt{T}$$

Black-Scholes model considers $\Phi(d_2)$ as the probability that the option will be exercised, or an in-the-money option, where Φ denotes the cumulative distribution function (CDF) of the normal distribution. In the Merton model context, $\Phi(d_2)$ is the risk-neutral probability that the company will not default, and thus $1 - \Phi(d_2) = \Phi(-d_2)$ is the risk-neutral probability that the company will default on the debt. More formally, default happens if and only if, at time t, the value of the firm's assets, V_t, is below its debt face value, D_t. The Geometric Brownian motion for the asset value of the firm is:

$$dV_t = \mu_V V_t dt + \sigma_V V_t dZ_{V,t} \tag{1.3}$$

where μ_V and σ_V are assumed to be constants. Merton model assumes that asset value at time t will follow a lognormal distribution, or that the natural logarithm of the asset value at a given time t is normally distributed as follows:

$$ln(V_t) = ln(V) + (\mu_V - \sigma^2{}_V / 2)t + \sigma_V \sqrt{t}\varepsilon \tag{1.4}$$

with mean $ln(V) + (\mu_V - \sigma^2{}_V / 2)t$ and variance $\sigma^2{}_V t$. Therefore, the probability of default equals

$$\begin{aligned} P(V_t < D_t) &= P(ln(V_t) < ln(D_t)) \\ &= P(ln(V) + (\mu_V - \sigma^2{}_V / 2)t + \sigma_V \sqrt{t}\varepsilon < ln(D_t)) \\ &= P((ln(V / D_t) + (\mu_V - \sigma^2{}_V / 2)t\,) / \sigma_V \sqrt{t} < -\varepsilon) \\ &= \Phi(-((ln(V / D_t) + (\mu_V - \sigma^2{}_V / 2)t\,) / \sigma_V \sqrt{t})) \\ &= \Phi(-d_2) \end{aligned} \tag{1.5}$$

An increase in σ_V causes a decrease in d_2, as its numerator declines and its denominator elevates. As a result $(-d_2)$ increases, so that the risk-neutral probability for the company to default on the debt, $\Phi(-d_2)$, increases.

The Merton risk neutral default probability, $\Phi(-d_2)$, increases with the face value of debt, D, and with the volatility of the firm's assets, σ_V, but decreases with the firm's value, V_0, and with the risk free interest rate, r_f. As the time to maturity T increases, the Merton risk neutral default probability converges in the limits to either 0 or 1. This depends on whether the risk free interest rate, r_f, is bigger or smaller than $\sigma^2_V / 2$.

If $r_f > \sigma^2_V / 2$ then $T \to +\infty \Rightarrow d_2 \to +\infty \Rightarrow (-d_2) \to -\infty \Rightarrow \Phi(-d_2) \to 0$.

If $r_f < \sigma^2_V / 2$ then $T \to +\infty \Rightarrow d_2 \to -\infty \Rightarrow (-d_2) \to +\infty \Rightarrow \Phi(-d_2) \to 1$.

Assuming that at any given time there is a positive probability for default, the default process is a sub-martingale[61]. When time passes, the debt time to maturity, T, gets smaller, therefore:

If $V_0 > D \Rightarrow ln(V_0 / D) > 0$ then $T \to 0 \Rightarrow d_2 \to +\infty \Rightarrow (-d_2) \to -\infty \Rightarrow \Phi(-d_2) \to 0$.

If $V_0 < D \Rightarrow ln(V_0 / D) < 0$ then $T \to 0 \Rightarrow d_2 \to -\infty \Rightarrow (-d_2) \to +\infty \Rightarrow \Phi(-d_2) \to 1$.

If the total assets value is larger than the debt face value, when the debt time to maturity gets smaller, the company's default risk decreases. But, when the debt face value is larger than the assets value, getting closer to maturity only increases the default risk.

When default occurs, assuming negligible distress and bankruptcy costs, Altman, Resti and Sironi (2001) define the recovery rate as a ratio between the assets value and the debt face value, V_t / D_t. Since the debt face value is fixed, the expected recovery rate is then: $E[V_t/D_t] = E[V_t]/D_t$. Since recovery rate has meaning only in case of a default,

[61] A process with zero sloping-trend is called a martingale. Martingale is a "fair" process in the sense that expected gains or losses are zero. A process with an upward or downward sloping trend is called a sub-martingale.

when $V_t < D_t$, the expected recovery rate can be written as the mean of a truncated lognormal variable divided by D_t as follows:

$$E[V_t / D_t \mid V_t < D_t] = E[V_t \mid V_t < D_t] / D_t \tag{1.6}$$

Liu, Lu, Kolpin and Meeker (1997) prove the following identity:[62]

$$E[V_t \mid V_t < D_t] = e^{ln(V)+\mu vt}\Phi(-((ln(V_t / D_t) + (\mu_V + \sigma^2{}_V / 2)t) / \sigma_V\surd t)) \tag{1.7}$$
$$/ \Phi(-((ln(V_t / D_t) + (\mu_V - \sigma^2{}_V / 2)t) / \sigma_V\surd t)) = Ve^{\mu vt}\Phi(-d_1) / \Phi(-d_2)$$
$$= E[V_t]\ \Phi(-d_1) / \Phi(-d_2)$$

where d_1 and d_2 are the same parameters defined in the Black–Scholes model. Therefore, the expected recovery rate turns out to be inversely related to default probability:

$$E[V_t / D_t \mid V_t < D_t] = E[V_t / D_t]\ \Phi(-d_1) / \Phi(-d_2) \tag{1.8}$$

Since all the parameters are positive, d_1 is always positive, but d_2 may be positive or negative. For every $\sigma_V > 0$ and $T > 0$, d_1 is always bigger than d_2, therefore, the relation $\Phi(d_1) > \Phi(d_2)$ is always true, thus, one may concludes that

$$1 - \Phi(d_1) < 1 - \Phi(d_2) \Rightarrow \Phi(-d_1) < \Phi(-d_2) \Rightarrow \Phi(-d_1) / \Phi(-d_2) < 1 \tag{1.9}$$

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[62] The complete proof is in Appendix 1.

This inequality holds regardless of the sign of d_2. Therefore, the expected recovery rate always satisfies

$$E[V_t / D_t]\ \Phi(-d_1) / \Phi(-d_2) < E[V_t / D_t] \tag{1.10}$$

In the standard Merton model V_t and σ_V are not directly observable, but can be postulated. Jones, Mason and Rosenfeld (1984), among others, suggest to find the probability for a public company to default on the debt, $\Phi(-d_2)$, by solving two equations with two unknowns.[63] The first equation is the interpretation of Black-Scholes formula for the value of company's equity

$$E_t = V_t\Phi(d_1) - De^{-rfT}\Phi(d_2) \tag{1.11}$$

The Black-Scholes formula claims that the equity price is a function of asset value, time to maturity and asset volatility

$$E_t = f(V_t, t) \tag{1.12}$$

Assuming a constant equity growth drift, μ_E, constant equity diffusion, σ_E, and a standard Wiener process dZ_t, applying Itô's lemma for equity movements yields

[63] The authors also concluded that the introduction of stochastic interest rates might improve the performance of contingent claims pricing models.

$$df(V_t, t) = \left((\partial f(V_t, t) / \partial V_t)\mu_V V_t + (\partial f(V_t, t) / \partial t) + \left(\tfrac{1}{2} (\partial^2 f(V_t, t)) / \partial V^2_t \right)\sigma^2_V V^2_t \right)dt + (\partial f(V_t, t) / \partial V_t)\sigma_V V_t dW_t \quad (1.13)$$

The equity value dynamic follows a Geometric Brownian motion of the form

$$dE_t = \mu_E E_t dt + \sigma_E E_t dW_t \quad (1.14)$$

where μ_E and σ_E are assumed to be constants. Hence, while comparing the diffusion coefficients from the equity value dynamic and the one from the Itô's lemma for equity movements, and since the second and the third partial derivatives on the right hand side of Itô's lemma vanished, the second necessary equation is obtained as:

$$\sigma_E E_t = (\partial E_t / \partial V_t)\sigma_V V_t = \Phi(d_1)\sigma_V V_t \quad (1.15)$$

The Merton model has been under critics due to some of its simplifying assumptions: liabilities of a firm consist only a single class of debt, debt yields no coupons and no embedded option features involved, interest rate is constant, firms may default only at debt maturity, and bankruptcy is considered as costless.[64] Later studies of structural models remove some of these problematic assumptions.

2. The Next Stage of Structural Models

[64] Although Merton model assumes only one zero-coupon bond outstanding, more complicated debt structures can be repositioned as a single liability asset. Macaulay Duration can be used to compose the face value of the company's multiple liabilities to a single duration for all outstanding liabilities.

Black and Cox (1976) introduce more complex capital structures along with subordinated debt, often found in practice. Their extension of the Merton model considers a safety covenants assumption, defining a specified level, where creditors are entitled to force the firm into bankruptcy and obtain ownership of the assets, and an additional assumption for subordinated bonds, where payments to senior debt holders must be retired prior to any junior payments. The authors show that in this setting, the probability that a firm has not reached the reorganization boundary is given by:

$$\Phi((ln(V) - ln(D) + (r - a - \sigma^2 / 2)(\tau - t)) / \surd(\sigma^2(\tau - t))) - (V / Ce^{-r(T-t)})^{1-(2(r-a-\gamma)/\sigma 2)} \quad (2.1)$$
$$\Phi((2ln(Ce^{-\gamma(T-t)}) - ln(V) - ln(D) + (r - a - \sigma^2 / 2)(\tau - t)) / \surd(\sigma^2(\tau - t))$$

where V = the value of the firm's assets, D = the debt face value, a = proportion of dividend payments out of the firm's value shareholders are entitled to receive, σ^2 = instantaneous variance of return on the firm's assets, τ = time of reorganization, t = current time, C = boundary level for reorganization, r = risk-free interest rate, T = time to debt maturity, and γ = the appropriate discount rate that specifies the bankruptcy level at $Ce^{-\gamma(T-t)}$.

Geske (1977) generalizes the Merton model, allowing for multiple options to default on short-term debt, junior or subordinated debt, coupon payments, amortizing debts, sinking funds, safety covenants and other debt provisions. The formulation considers a company with both long-term debt, D_2, that matures at time T_2, and short-term debt, D_1, that matures at time T_1, where $T_2 \geq T_1$. If at time T_1 the market value of assets, V_{T1}, is greater than the par value of the short-term debt, D_1, plus the market value of the long-term debt at T_1, $D_{2,T1}$, then the company can service or refinance its debt without defaulting. A critical value, V^*_{T1}, of which the company avoids bankruptcy satisfies:

$$V^*_{T1} = D_1 + D_{2,T1} = D_1 + (V_{T1} + E_{T1}) \tag{2.2}$$

$$= D_1 + V_{T1} - V_{T1}\Phi(K_2 + \sigma_V\sqrt{(T_2 - T_1)}) + D_2e^{-rF,1(T2 - T1)}\Phi(K_2)$$

where

$$K_2 = (\ln(V / D_2) + (r_{F,2} - \sigma^2{}_V / 2)(T_2 - t)) / \sigma_V\sqrt{(T_2 - t)}$$

After substituting the terminal problem, at T_2, into equation (2.2), and solving for the value of the equity price at present, the solution becomes:

$$E_t = V\Phi_2(k_1 + \sigma_V\sqrt{(T_1 - t)},\ k_2 + \sigma_V\sqrt{(T_2 - t)};\ \rho) \tag{2.3}$$

$$- D_2e^{-rF,2(T2 - t)}\Phi_2(k_1, k_2;\ \rho) - D_1e^{-rF,1(T1 - t)}\Phi(k_1)$$

where in addition

$$k_1 = (\ln(V / V^*) + (r_{F,1} - \sigma^2{}_V / 2)(T_1 - t)) / \sigma_V\sqrt{(T_1 - t)}$$

$$\rho = \sqrt{((T_1 - t)(T_2 - t))}$$

and where $\Phi_2()$ denotes the CDF of a bivariate normal distribution. In this framework, several default probabilities may be considered; the joint probability to default at either T_1, or at T_2, is expressed as:

$$PD_{T1\cup T2} = 1 - \Phi_2(k_1, k_2; \rho) \tag{2.4}$$

the probability to default only at T_1 is expressed as:

$$PD_{T1} = 1 - \Phi_1(k_1) \tag{2.5}$$

and the probability to default at T_2, conditional on surviving at T_1 is:

$$PD_{T2} = 1 - (\Phi_2(k_1, k_2; \rho) / \Phi_1(k_1)) \tag{2.6}$$

Geske (1979) further extends the Merton framework, showing that if a stock is considered as a European option on the asset's market value, then an option on a stock can be considered as a compound option, i.e. an option on an option. A difficulty to apply the Black-Scholes differential equation arises in this framework due to the underlying assumption for a constant variance rate of return on the equity. With compound options this variance may not be constant, but rather depends on the stock price.

Holding the common assumptions for constant risk-free interest rate, and considering that security markets are perfect and competitive, short sales are unrestricted, firm's value follow a random walk, common information for all investors, and trading takes place in continuous time, the author solves for the value of a compound call option as:[65]

$$C = V\Phi_2(h + \sigma_V\sqrt{\tau_1}, k + \sigma_V\sqrt{\tau_2}; \sqrt{(\tau_1 / \tau_2)}) \qquad (2.7)$$

$$- \quad De^{-r_f\tau_2}\Phi_2(h, k; \sqrt{(\tau_1 / \tau_2)}) - ke^{-r_f\tau_1}\Phi(h)$$

where

$$h = (\ln(V / \hat{V}) + (r_f - \sigma^2_V / 2)\tau_1) / \sigma_V\sqrt{\tau_1}$$

$$k = (\ln(V / D) + (r_f - \sigma^2_V / 2)\tau_2) / \sigma_V\sqrt{\tau_2}$$

$\hat{V}$ is a critical value that makes

$$S_\tau - k = V\Phi(k + \sigma_V\sqrt{\tau}) - De^{-r_f\tau}\Phi(k) - k = 0$$

and where t = the current time, T = maturity date of the debt, t^* is the date that the value of the firm that makes the holder of the option on the stock indifferent between exercising and not exercising, i.e. an at-the-money option, $\tau = T - t^*$, $\tau_1 = t^* - t$, $\tau_2 = T - t$, C = current value of the call option, $\Phi_2()$ denotes the bivariate cumulative normal distribution with h and k as upper integral limits, $\sqrt{(\tau_1 / \tau_2)}$ as the correlation coefficient, S = current market value of the underlying stock, V = current market value of the firm's assets, D = face value of the debt, r_f = the risk-free interest rate, σ^2_V = the instantaneous variance of the return on the firm's assets, and K = the given strike price of the option.

[65] The solution can be postulated through Fourier transformations (integral transforms that re-express a function with its sinusodial basis functions multiplied by some coefficients), or by using the separation of variables technique (a method to solve ordinary and partial differential equations).

A comparison to the Black-Scholes model shows it to be a special case of the above compound option model. Furthermore, movements within the equity value change the firm's leverage, and the stock return variance is found to be a monotonic increasing function of this leverage. The author also discovers that the Merton model is consistent with the volatility skew observed in the equity market.[66]

Vasicek (1984) introduces a different distinction between short-term and long-term liabilities. If companies were forced into bankruptcy, they would service their debt first to the long-term creditors, then the short-term lenders would only receive a pro-rata compensation from the remaining assets. Thus, to avoid bankruptcy, the short-term debt-holders renew a partial credit to the firm, equal to the difference between the amount due and the firm's assets value. Assuming also that current liabilities constitute a claim senior to the short-term debt, and that the total asset value follows a geometric Brownian motion including some drift and diffusion, the solution to default probability becomes:

$$\begin{aligned} PD &= Prob.(V_T < D_T + C_T \mid V_0 = V) \\ &= Prob.(\ln(V_T) < \ln(D_T + C_T) \mid V_0 = V) \\ &= \Phi((\ln(D_T + C_T) - \ln(V - F) - \mu T + \sigma^2 T / 2) / \sigma\sqrt{T}) \end{aligned} \tag{2.8}$$

[66] Volatility skew, often called volatility smile, or volatility smirk, shows the variation of the implied volatility with the strike price. The observed pattern for options is usually with both tails fatter than the lognormal distribution. This means that the Black-Scholes model under-prices out-of the money calls, as well as out-of the money puts. From the put-call parity, this also implies under-pricing for in-the-money puts and for in-the-money calls.

where V = market value of total assets, C = market value of current liabilities, D = market value of short-term debt, F = total amount of dividends and bond interest over the time interval T, assumed to be prepaid at initiation, $\Phi()$ denotes the cumulative normal distribution function, and μ and σ^2 are the instantaneous mean and variance, respectively, of the stochastic rate of return of the firm's assets.

Shimko, Tejima and Van Deventer (1993) generalize Merton's risky debt pricing model to allow for stochastic interest rate. The authors examine the combined effect of term structure variables and credit variables on debt pricing, and discover that for reasonable parameter values, credit spread is an increasing function of the risk-free term structure volatility. However, term structure effects can cause the sign of the derivative to change. Furthermore, changes in the correlation between interest rates and asset value may have a positive or negative impact on the credit spread. For reasonable parameter values, the credit spread is an increasing function of this correlation. The value of a risky debt, F, when interest rates are stochastic is expressed as:

$$F = V - V\Phi(h_1) + BP(\tau)\Phi(h_2) \tag{2.9}$$

where

$$h_1 = (\ln(V / P(\tau)B) + T / 2) / \sqrt{T}$$

$$h_2 = h_1 - \sqrt{T}$$

$$T = \int_0^x v(s)^2 ds = \tau(\sigma^2_V + \sigma^2_\gamma / k^2 + 2\rho\sigma_V\sigma_\gamma / k)$$

$$+ (e^{-k\tau} - 1)(2\sigma^2_\gamma / k^3 + 2\rho\sigma_V\sigma_\gamma / k^2) - (\sigma^2_\gamma / 2k^3)(e^{-2k\tau} - 1)$$

$$v^2(s) = \sigma^2_V + \delta(s)^2 - 2\rho\sigma_V\delta(s)$$

$$\delta(s) = - (1 - e^{-ks} / k)\sigma_\gamma$$

and where V = asset value, B = the debt face value as the strike price, τ = remaining time to maturity, r = risk-free stochastic interest rate, σ denotes volatility, ρ = the correlation between interest rates and asset value, $\Phi()$ represents the CDF of the normal distribution, and the price of a zero coupon bond is priced as:

$$P(\tau) = exp[(1 - e^{-k\tau} / k)(R(\infty) - r) - \tau R(\infty) - (\sigma^2_\gamma / 4k^3)(1 - e^{-k\tau})^2] \tag{2.10}$$

where

$$R(\infty) = \gamma + (\sigma_\gamma / k)\lambda - \sigma^2_\gamma / 2k^2$$

and where λ is the market price of risk for such risk-free bond. To consider arbitrage-free environment, λ must be independent of the bond maturity, γ is the long-run mean of the risk-free term structure, and k denotes the speed of mean reversion.

Leland (1994) considers the bankruptcy decision as endogenous, rather than by the imposition of a positive net worth condition, while accounting for taxes and bankruptcy costs. The model assumes that the firm's asset value follows a stochastic process, which is independent from the financial structure, and if the firm is not otherwise constrained by covenants, bankruptcy will occur only when the firm cannot meet the instantaneous required coupon payment by issuing additional equity. Following Merton (1974) and Black and Cox (1976), the model also assumes the existence of a riskless asset that pays constant rate of interest. Under these assumptions, the author shows that the behavior of the firm at bankruptcy follows the next results:

$$D(V) = (C / r)[1 - (C / V)^x k] \tag{2.11}$$

$$v(V) = (1 - \alpha)V = V + (\tau C / r)[1 - (C / V)^x h]$$

$$E(V) = V - (1 - \tau)(C / r)[\,1 - (C / V)^x m]$$

where

$$m = [(1 - \tau)X / r(1 + X)]^x / (1 + X)$$

$$h = [1 + X + \alpha(1 - \tau)X / \tau]m$$

$$k = [1 + X - (1 - \alpha)(1 - \tau)X]m$$

$$X = 2r / \sigma^2$$

and where D = debt face value, V = total firm value of assets, E = equity value, C = continuously paid nonnegative coupon, r = risk-free interest rate, σ = constant volatility of rate of return on the assets value, the fraction $0 \leq \alpha \leq 1$of value denotes the lost of bankruptcy costs, and τ is the tax shelter bracket thus, τC is the tax shelter as value of interest payments. Leland and Toft (1996) extend this model to a richer class of possible debt structures, and permit study of the optimal maturity of debt, as well as optimal amount of debt.

Anderson and Sundaresan (1996) value debt contracts while incorporating strategic bargaining between shareholders and debt holders. In this setting, the firm's reorganization boundary is determined endogenously. The authors realize that strategic debt service results in significantly higher default premium at even small liquidation costs. This model tends to stress higher coupons and sinking funds when firms have a higher cash payout ratio. Zhou (1997) uses a jump-diffusion stochastic process for valuing the firm's assets. Accounting for possible shocks within the market value of assets, and assuming that the capital asset pricing model (CAPM) and the Modigliani and Miller theorem for irrelevancy of the capital structure hold, considering that the market is frictionless, and also regarding constant risk-free interest rate, along with the existence of a positive threshold at which financial distress occur, the author allows for large short-maturity spreads, and thus permits a surprise default.

Several studies use the option theoretic valuation of debt framework to price derivatives involving credit risk. Among the pioneers are Ho and Singer (1982), Johnson and Stulz (1987), and Chance (1990). Kim, Ramaswamy and Sundaresan (1993) propose a contingent claims model for pricing corporate bonds, while specifying a stochastic process for the short rate interest rates. Their model accommodates the risk of default in the presence of dividends, accounts for coupons and the attendant risk of default prior to debt maturity, and is further used to examine the call provision in a more realistic environment. The uncertainty in the term structure of interest rates is described with a stochastic process, which considers long-run mean reversion, as follows:

$$dr = k(\mu - r)dt + \sigma\sqrt{r}dZ \tag{2.12}$$

where r is the nominal short interest rate, μ is a scalar representing the long run mean, k denotes the speed of mean-reversion, and Z is a standard Wiener process.[67] The authors discover that although the yields on both Treasury and corporate issues are significantly influenced by the uncertainty of interest rates, the yield spreads are quite insensitive to stochastic interest rates. In addition, their unique setting reveals that the call feature is more valuable in Treasury issues than it is in corporate issues.

Jarrow and Turnbull (1995) consider a frictionless and arbitrage-free economy with finite-horizon, a bankruptcy dynamic which is uncorrelated with default-free spot interest rates, an exogenous recovery rate as a constant fraction smaller than 1, discrete or continuous trading in default-free zero-coupon bonds, and risky zero-coupon bonds, both can be of all maturities. Using these assumptions along with process trees to describe the evolution of the spot interest rate and the zero-coupon bond prices over time, the authors propose a technique for pricing credit derivatives, as well as derive the default martingale probabilities, and specify the distribution for the time of bankruptcy.

Hull and White (1995) consider a default, when the value of the firm's assets reaches a different threshold level than the debt face value, and take the recovery rate (RR), in the event of a default, to be exogenous and independent from the firm's asset value, to price options and other derivative securities. Both the probability of default and the size of proportional recovery are assumed to be stochastic. The authors demonstrate the valuation of a long position in a European option issued by counterparty subject to

[67] A mean reverting stochastic process describes a tendency to remain near, or to be pulled back over time, to the long-run average.

credit risk, and show how data on bonds issued by the same firm may provide input to the model. Their model solves the following:

$$C = exp(-(y - y^*)(T - t_0))S\Phi(d_1) - exp(-y(T - t_0))X\Phi(d_2) \tag{2.13}$$

$$P = exp(-y(T - t_0))X\Phi(-d_2) - exp(-(y - y^*)(T - t_0))S\Phi(-d_1)$$

where

$$d_1 = (ln(S / X) + (y^* + \sigma^2 / 2)(T - t_0)) / \sigma\sqrt{(T - t_0)}$$

$$d_2 = d_1 - \sigma\sqrt{(T - t_0)}$$

and where C = European call price, P = European put price, S = current stock price, σ is the volatility, X = strike price, T = time of promised payoff, t_0 = current time, y^* = the yield of a default-free zero-coupon bond, y = the yield of a discount bond ranking equally with the option in the event of default and maturing at time T, and $\Phi()$ denotes the CDF of the normal distribution.

Longstaff and Schwartz (1995) value defaultable corporate debt by relaxing the assumption for fixed interest rate. Incorporating stochastic interest rate, and allowing for a correlation between default and interest rate, while considering the recovery rate (RR) as a fraction of the par value at maturity bondholders receive if reorganization occurs during the life of the bond, the authors find that the correlation between default risk and the interest rate has a significant effect on the properties of credit spreads. By using Moody's corporate bond yield data, they discover that credit spreads are negatively correlated with interest rates.

KMV Corporation uses today a similar variant to the one proposed by Merton, as outlined by Crosbie (1999). However, the KMV approach does not rely solely on the analytical argument underlying the Merton model. Instead, the Merton framework for estimating distance-to-default (DD) is combined with historic proprietary database of over 250,000 U.S. companies, and more than 4,700 default events, yielding expected-default-frequency (EDF) assessments. These are proportions of companies within a given distance to default that have actually defaulted. Furthermore, KMV allows default to occur even before debt maturity. Sobehart, Stein, Mikityanskaya and Li (2000) present a hybrid credit risk model for U.S. non-financial public companies. By examining more than 14,000 public companies, where 923 of them defaulted, they propose an early warning methodology for changes in credit quality, within a one-year time frame. Their methodology combines the output of Merton model with logistic regressions, to obtain default probabilities. Higher accuracy is achieved by further adjustments to observed proportions of defaults obligors. Kealhofer and Kurbat (2002) review the KMV methodology. Further suggesting that by using more accurate methods for computing assets volatility, they find contradicting results to Sobehart et al. (2000). The authors conclude that the Merton traditional model outperforms the Moody’s rating and various accounting ratios in predicting defaults.

3. Empirical Findings on Structural Models

Bohn (2000) extends the tests of contingent-claims bond valuation models, to a sample populated with a representative sub-sample of sub-investment grade debt. Examining more than 600,000 observations, of U.S. corporate bond data, from 1992 to 1999, the author finds that higher credit classes are positively sloped term structures, while lower credit classes demonstrate negatively sloped term structures. This leads to the conclusion that since the market Sharpe ratio remains relatively stationary over time, credit spreads are driven primarily by changes in credit quality, suggesting that the structure found in yield spreads data adds considerable support to the contingent-claims approach, derived from the Black-Scholes-Merton formulation.

Jarrow, Van Deventer and Wang (2002) examine the Merton model for credit risk without depending on either estimated unobservable variables, such as the firm's market value of assets, or estimated default probabilities. Doing that, the authors are able to test the structural model itself, while defusing problems in the procedure of estimating relevant unobservable parameters. Relying solely on observable parameters from the equity market raises liquidity issues. The authors address these microstructure considerations by using only weekly and monthly observation intervals to reduce the impact of daily price illiquidity, and design a statistical methodology that explicitly incorporates microstructure observational error (noise). The procedure is employed over available debt and equity prices of five different corporations from 1992 to 2001. The proposed non-parametric statistical test leads to a strong rejection of the Merton structural approach.

Delianedis and Geske (2003) compute risk neutral probabilities of default (RNPD), from 1987 to 1996, using the diffusion models on Merton (1974) and Geske (1977), to confirm whether they contain significant information about credit events. An event study of the relation between RNPD and rating migrations is conducted, assessing significant information inherited in both diffusion models, at a very early stage. Testing the changes in implied default probabilities from the diffusion models, before the event of a rating migration or default, detect impending migrations to default with high accuracy. Furthermore, the term structure of default probabilities from the Geske model is found to contain additional information. Not only that it has significant information on default events, but also it provides information on the shape of term structure prior to the actual default.

Huang and Huang (2003) use a variety of structural models to examine how much of the historically observed corporate-Treasury yield spread is due to credit risk. To explore whether this spread can be explained by implied default probabilities from the structural models, they calibrate the probabilities derived from the structural models to be consistent with historical data of default experience. They discover that for investment grade bonds of all maturities, credit risk accounts for only a small fraction of the spread, and even smaller for shorter maturities. For junk bonds credit risk accounts for a larger fraction.

Leland (2004) compares the probabilities of default generated by two alternative structural models. The first type of models relies on Merton model, and defines 'exogenous default boundary' as a sufficiently low threshold of assets market value, for the firm to default on its debt, whenever the asset market value falls below this level. The second type of models considers 'endogenous default boundary', where management takes the optimal decision to default, acting to maximize the value of equity. In this setting, the default boundary depends on the expected return and volatility of assets, the risk free interest rate, leverage ratio, debt maturity and default costs. While the exogenous default threshold models are found to be more sensitive to asset volatility, the endogenous default boundary type of models fit real default frequencies for long time horizons, although the predicted default frequencies are too low for short maturities. The second type of models also predicts that default probabilities increase with default costs, and fall with bond time to maturity.

Several studies examine the accuracy of the KMV model, as derived from the Merton approach, and adjusted to agency ratings and other bond characteristics. Among them are Sobehart, Keenan and Stein (2000), and Stein (2002). Both studies find the KMV model to be incomplete. In contrast, Kealhofer and Kurbat (2002) discover that KMV model captures all the information contained in agency ratings migration theory and accounting ratios. Crosbie and Bohn (2003) discover that the most effective default measurement derives from models that utilize both market prices and financial statements. The authors empirically test the expected default frequency (EDF), derived from the KMV methodology, versus the credit rating analysis, and show that the EDF obtains a better power curve. Bharath and Shumway (2004) reanalyze the accuracy of default forecasting of the KMV model by comparing it to a much simpler alternative.

They find that implied default probabilities from credit default swaps and corporate bond yield spreads are only weakly correlated with KMV-Merton default probabilities. The authors conclude that the KMV-Merton model does not provide a sufficient statistic for default, which can be obtained using relatively naïve hazard models. Hillegeist, Keating, Cram and Lundstedt (2004) and Du and Sou (2004) compare the KMV model to other models, and conclude that the KMV model does not provide adequate predictive power. However, Duffie and Wang (2004) reveal a significant predictive strength over time within the KMV model. Campbell, Hilscher and Szilagyi (2004) use hazard models to conditional the KMV model on other relevant default variables, and discover a poor predictive power of the KMV model.

4. More Recent Issues

Hull, Nelken and White (2004) present alternative approach to estimate the unobservable asset log return volatility, σ_V. Considering the implied volatility of options on the company's stocks, the authors demonstrate a different approach than the one proposed by Jones, Mason and Rosenfeld (1984), to measure assets volatility. Geske (1979) suggests that since the equity of a company can be considered as an option on the firm's assets, an option on the firm's stock is a compound option, and further provides a valuation formula for such compound option. Using Geske (1979) formulation, the authors present two-equation system that can be solved with two implied volatilities, sampled from stock options. The first equation is:

$$L\Phi_2(-a_2, d_2; -\sqrt{(\tau / T)}) - \Phi_2(-a_1, d_1; -\sqrt{(\tau / T)}) + k\Phi(-a_2)(\Phi(d_1) - L\Phi(d_2)) \tag{4.1}$$

$$= (k\Phi(-d^*_2) - \Phi(-d^*_1))(\Phi(d_1) - L\Phi(d_2))$$

where

$d_1 = (-ln(L) / \sigma_V\sqrt{T}) + (\sigma_V\sqrt{T} / 2)$ and $d_2 = d_1 - \sigma_V\sqrt{T}$

$d^*_1 = (-ln(k) / \sigma_V\sqrt{\tau}) + (\sigma_V\sqrt{\tau} / 2)$ and $d^*_2 = d_1 - \sigma_V\sqrt{\tau}$

$a_1 = (-ln(\alpha) / \sigma_V\sqrt{\tau}) + (\sigma_V\sqrt{\tau} / 2)$ and $a_2 = a_1 - \sigma_V\sqrt{\tau}$

and where $\tau < T$ is the strike price expiration time, $\Phi_2()$ denotes the cumulative bivariate normal distribution, $L = De^{-rT} / V_0$ represents the level of leverage at time zero, and the parameter α is the ratio of the critical asset price for default, to the forward asset price, both observed at time zero. The second equation describes k as the ratio of the option strike price to the forward equity price, observed at time zero, often referred as the option's moneyness:

$$k = (\alpha\Phi(d_{1,\tau}) - L\Phi(d_{2,\tau})) / (\Phi(d_1) - L\Phi(d_2)) \tag{4.2}$$

where

$d_{1,\tau} = (-ln(L / \alpha) / \sigma_V\sqrt{(T - \tau)}) + (\sigma_V\sqrt{(T - \tau)} / 2)$ and $d_{2,\tau} = d_{1,\tau} - \sigma_V\sqrt{(T - \tau)}$

While testing the proposed alternative with credit default swaps (CDS) spread data, the authors find that this implementation of the Merton model outperforms the traditional methodology.[68]

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[68] A buyer of a credit default swap receives credit protection, whereas the seller guarantees the credit worthiness of a specific company. Such a derivative transfers the

The standard Merton model is argued to be applicable merely for domestic companies, where assets and debt are denominated in the same home currency. When assets and debt face value are denominated in different currencies, an extension of Merton model, incorporating stochastic face value of debt, is introduced by Parnes (2006b). When assets are denominated in home currency, and debt face value is denominated in foreign currency, multinational's credit risk can be postulated as:

$$E_{H,t} = V_{H,t}\Phi(d'_1) - D_F e^{-rF(T-t)}\Phi(d'_2)/S_t \tag{4.3}$$

where

$$d'_1 = (\ln(S_t V_{H,t} / D_F) + (r_F + \delta^2_{V,F} / 2)(T\text{-}t)) / \delta_{V,F}\sqrt{(T\text{-}t)}$$

$$d'_2 = (\ln(S_t V_{H,t} / D_F) + (r_F - \delta^2_{V,F} / 2)(T\text{-}t)) / \delta_{V,F}\sqrt{(T\text{-}t)} = d'_1 - \delta_{V,F}\sqrt{(T\text{-}t)}$$

$$\delta_{V,F} = \sqrt{(\sigma^2_V + \sigma^2_S + 2\rho_{V,S}\sigma_V\sigma_S)}$$

and where S_t denotes the indirect quote spot rate between home and foreign currencies at time t, σ^2_S is the exchange rate volatility, and $\rho_{V,S}$ is the assets-exchange rate correlation. In this setting, the second and the third necessary equations are:

$$\sigma_E E_{H,t} = V_{H,t}\Phi(d'_1)\sqrt{(|\sigma^2_{VH} + \sigma^2_S + 2\rho_{V,S}\,\sigma_{VH}\,\sigma_S|)} \tag{4.4}$$

$$\sqrt{(\sigma^2_V + \sigma^2_S + 2\rho_{V,S}\,\sigma_V\,\sigma_S)} = \Psi(d'_1) / (\Phi(d'_1)\sqrt{(T\text{-}t)}) \tag{4.5}$$

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default exposure from the buyer to the seller. A CDS spread is the premium paid for this protection, directly reflecting the firm's credit risk.

where *Ψ()* represents the probability density function (PDF) and *Φ()* denotes the cumulative distribution function (CDF) of a standard normal distribution. Although parametric and nonparametric tests point to some statistically significant difference in implied default probabilities, between the multinationals extension and the traditional Merton model, no real economic significance is found. Thus, exchange rate exposure is concluded to have almost no real impact on multinationals' credit risk, within the selected data sample. However, it is likely that under more severe exchange rate fluctuations, foreign exchange exposure might have a larger impact on multinationals' credit risk.

5. Credit Ratings Migration Analysis

During the 1980s central banks in the G10 countries became highly concerned about the default risk of bank assets, and particularly bank loans. In order to establish a common regulation, and to prevent banks from relocating to weak regulatory requirements areas, the G10 countries, along with the Bank of International Settlements (BIS) in Basle, introduced in 1988 the Basle Capital Accord, a set of minimum risk-based capital adequacy requirements dealing with credit risk. This regulator's view of relative default probabilities among different countries suffers from great subjectivity.

In 1997, J. P. Morgan along with Bank of America, Bank of Montreal, BZW, Deutsche Morgan Grenfell, KMV Corporation, Swiss Bank Corporation and Union Bank of Switzerland introduced a new approach for measuring credit risk. It later led to the CreditMetrics™ methodology of Gupton, Finger and Bhatia (1997). This approach differs from the Basle accord by its recognition of two important issues: the marking to market of assets and the diversification effect. The key concepts of this approach include calculating probabilities of credit ratings migrations, computing the bonds value in different credit ratings, measuring the likelihood of joint credit ratings migration, and estimating the *Value at Risk* (VaR) for a single or a portfolio of bonds. Other credit VaR models were published as CreditRisk+™ by Credit Suisse Financial Products (1997) and Gordy (2002), or as McKinsey's CreditPortfolioView by Wilson (1997a, 1997b and 1998). The CreditRisk+™ uses an actuarial approach and offers a closed form solution, considering that default rates are stochastic, and may significantly fluctuate with the credit cycles. Risk drivers themselves may vary as well, but recovery rates are taken as constants.[69]

Credit ratings are often assigned by rating agencies to public debt at the time of issuance, and are periodically revised, typically once every quarter. A change in the credit rating reflects an improvement or deterioration in the credit quality, from the rating agency's perspective. Rating agencies assign rating grades that convey information about the credit quality of the borrower as a service for investors mostly active in the bond markets. However, different agencies use credit ratings in various ways. Standard & Poor's identifies its ratings as likelihood of default, while Moody's ratings reflect expected loss, as probability of default times loss severity. These ratings have a major effect on interest payments issued by the firm hereafter, thus companies frequently

[69] For example, CreditRisk+™ requires default rates per country-industry segment as input, in addition to the average default rate within the credit category.

attempt to upgrade their credit ratings.[70] Although credit ratings are fundamentally meant to shed light on the current creditworthiness of a firm, several theories use the rating drift phenomenon to predict default events, to pinpoint probabilities of default, and to price defaultable bonds. In fact, rating agencies often declare that credit ratings should correspond to the long-term firm's prospect.

To date, a fair amount of studies investigate transition matrices and credit ratings migration. Pogue and Soldofsky (1969) argue that bond ratings, first appeared in the U.S. in 1909, are meant to reflect available statistics on a firm's operations and financial condition. However, the determinant of bond rating may suffer from subjectivity. Examining what portion of credit rating is derived from observed data, and how much judgment is involved, the authors discover that the probability of a bond, having a higher credit rating, is inversely related to the leverage and earnings instability of the issuing firm, and directly related to the firm's size and profitability. Furthermore, leverage and profitability appear to have the greatest impact on corporate bond rating. The authors further realize that differences in bond ratings can be explained to a significant degree by available operational and financial data, and judgment takes relatively a small portion in describing credit ratings. Thus, one may conclude that bond ratings are largely driven by past events, rather than by future prospects.

[70] A conceptual debate whether companies take continuous steps to upgrade their credit ratings, or discretely proceed only prior to new debt issuances, is still thriving. Löffler (2005) explores the common assertion that credit rating agencies take a rating action only when it is unlikely to be reversed shortly after, and concludes that this is the major rationale for why rating agencies are slow to react to new information. Therefore, companies may attempt to raise their credit ratings at all time.

Kaplan and Urwitz (1979) propose a statistical procedure appropriate to the ordinal nature of a bond rating. The authors offer a simple linear model with subordinated dummy variable, total assets, the long-term debt to total-assets ratio, and the common stock systematic risk measure to classify two-thirds of a holdout sample of newly issued bonds. A comparison to the credit ratings analysis reveals that the proposed model outperforms the ratings methodology, and demonstrates a superior predictive power in about half of the misclassifications.

Ederington, Yawitz and Roberts (1987) examine whether yields on industrial bonds indicate that market participants base their evaluations of a bond issue's default risk on agency ratings, or on publicly available financial data. Using non-linear least squares procedure, the authors find that credit rating-movements lag behind market pricing, and market yields are significantly correlated with both the ratings and a set of readily available financial and accounting statistics. They conclude that market participants do not rely merely on agencies' ratings, but base their bond issue's creditworthiness on more available data. However, these ratings add important information to the market beyond the public information.

Other scholars use ratings migration analysis to estimate expected losses given default (LGD). Austin (1992) demonstrates a procedure that can be used to extract useful information regarding future commercial loan losses from historical ratings migration. This technique involves the collection of five years data on losses of a given portfolio, and a computation of a weighted-average-loss percentage for the test date and the following period. The migration analysis can be further used by the lender, through dividing the portfolio according to loan purpose within risk grades. Meyer (1995) illustrates how banks can use quantitative information about expected losses to establish loan-loss reserves, price loans, value customer relationships, and improve performance measurement. The author demonstrates how to disaggregate a loan portfolio by risk rating, measure changes in balances from one period to the next, create a transition probability matrix, calculate total expected losses, and then compute the periodic risk charge, or set the necessary reserve. Smith and Lawrence (1995) use a logit model to find the variables that best predict default event for loans. The authors construct a forecasting model based on the Markov chain structure and non-stationary transition probabilities. The model illustrated to be effective in representing changes in probability of default that occur as individual loans mature, and accurate in forecasting aggregate defaults and losses on a nationwide portfolio of long-term loans. It also demonstrated the limited predictive power of macroeconomic variables.

Crabbe (1995) uses credit ratings migration techniques to estimate returns with regards to investors' horizons. Lucas (1995) examines the credit quality of letter of credit-backed debt and synthetic debt created from a debt obligation and a swap. The author derives a formula to account for default correlation when computing joint default probability. Lando (1998) uses historical data of credit ratings migration to price default bonds. The author shows how the Cox processes, also known as doubly stochastic Poisson processes, can be used to model credit sensitive securities. This approach allows models to account for dependency between the default-free term structure and the default characteristics of firms. Bank of International Settlements (2000) surveys rating techniques among 30 financial institutions in the G10 countries, and Treacy and Carey (2000) explore the ratings systems across 50 large U.S. banks. Both find that there is no single standard for the design and operation of an internal rating system, and a variety of rating models are in use.[71]

Shumway (2001) provides theoretical arguments and empirical evidence against the use of cross-sectional static credit rating models, arguing in favor of time-varying models, and demonstrating the benefits of hazard-rate models. Acharya, Das and Sundaram (2002) use ratings migration to price credit derivatives. Lando and Skodeberg (2002) review migration matrices techniques. Purhonen (2002) finds evidence of cyclicality in the *internal ratings-based* (*IRB*) approach to new capital accords. Christensen, Hansen and Lando (2004) examine continuous-time rating transition probabilities.

Jafry and Schuermann (2004) compare various approaches to estimating migration matrices; the cohort method imposes time homogeneity, and the two-variant duration relaxes it. Both matrices are compared over historic U.S. ratings from 1981 to 2002. The authors discover that relaxing the time-homogeneity assumption within the transition matrix has a minor effect, and in most cases, the cohort methodology cause more damage than forcing time-homogeneous transition rates. Gagliardini and

[71] Internal ratings refer to the in-house qualitative assessment process devised to identify the creditworthiness of a company. They often appear in letter-labeled categories, or in numbers. External ratings are provided by the credit rating agencies.

Gouriẻroux (2005) explain why cross-sectional estimated migration correlations can be inefficient, or sometimes even inconsistent, and discuss alternative framework involving stochastic migration and interest rates.

Numerous studies explore credit ratings migration using the Markov chain process. Jarrow, Lando and Turnbull (1997) propose a discrete-space finite-state time-homogeneous Markov chain of credit ratings to describe the term structure of credit risk spreads. The authors develop a contingent claims model, incorporating finite-state Markov dynamics of bankruptcy into the valuation methodology of a risky zero-coupon bond. Using an absorbing time-homogeneity transition matrix, and assuming that the risk-free spot interest rate and the firm's credit ratings migration follow uncorrelated stochastic processes, they formulate the price of defaultable bonds. Kijima and Komoribayashi (1998) point to possible numerical problems that arise due to low default probabilities, for highly rated bonds, within relatively short periods, and offer different risk premium adjustments than that suggested by Jarrow, Lando and Turnbull (1997). Instead of the assumption of homogeneity for all transition matrix entries, the authors propose to consider transition probabilities as functions of time, and by that to allow them to change.

Kijima (1998) discusses how prior ratings changes may help to predict future rating changes over time, conditional on firms' survival. Using the first-order stochastic dominance in the Markov chain dynamics, he explains how firms with low credit ratings are more likely to be upgraded while firms with high credit ratings are more likely to be downgraded, as the time horizon lengthens, given that they do not default. Arvanitis, Gregory and Laurent (1999) contribute to the pricing of risky bonds and bond options, allowing memory in credit rating changes, stochastic jumps, and mean reversion in credit spreads. Similar to previous work, they consider two-state world, of default and non-default, deterministic credit ratings migration matrix, along with the restriction for credit spread independent of the interest rates, but permit sudden jumps, memory and mean-reverting diffusion processes in credit rating changes. Thomas, Allen and Morkel-Kingsbury (1999) propose a Markov chain model for the term structure and credit spreads of risky bonds, allowing dependency between two stochastic processes, interest rate movements and credit ratings migrations. That establishes a link between credit rating movements and the state of economy. The authors also present a linear programming algorithm to strip bonds from its coupons for better pricing accuracy.

Bielecki and Rutkowski (2000) derive stochastic processes for pricing defaultable bonds and other credit derivatives from rating transition matrices as well. They model the intensities of credit migrations among various credit ratings classes by combining available data for credit spreads and recovery rates, and conditioning the martingale probabilities from the Markov model on interest rate risk. Bahar and Nagpal (2001) have proposed a non-homogeneous model to accommodate for observed momentum. Their model enhances the standard Markov model with a partial, short-term memory. Each state describes not only the current rating category, but also the latest transitions. Dividing companies into three sets according to their rating over the previous year: upgraded, stable, or downgraded, is meant to capture the effect of past transitions as well.

This methodology leads to three different transition matrices; one for each set of upgraded, stable, or downgraded companies. In the upgrade transition matrix, ratings migration probabilities are more prominent above the main diagonal, while in the downgrade transition matrix; ratings migration probabilities are more pronounced below the main diagonal. Each of these three separate transition matrices is a Markov model by itself.

Israel, Rosenthal and Wei (2001) identify the conditions for a true generator of Markov transition matrices to exist, present several techniques for estimating an approximate generator when the true one is not feasible, and apply those methods to rating migration matrices.[72] This methodology helps to create different transition matrices through time, however it contains a major drawback; the generator does not correlate the transition matrices to any meaningful economic variable. Following Koyluoglu and Hickman (1998) setup, Lucas, Klaassen, Spreij and Straetmans (2001) derive an analytical approximation to the credit loss distribution of large portfolios by incorporating the Markov chain approach for credit ratings migration, along with several characteristics, such as credit quality, the degree of systematic risk, and the maturity profile. Assuming that rating transitions are normally distributed, the authors find that the rate of convergence of the actual credit loss quantiles to their analytic counterparts is influences by the degree of portfolio heterogeneity.

[72] A generator matrix G for a transition matrix T has a column-sums zero and non-negative off diagonal entries such that $exp(G) = T$.

Frydman and Kadam (2002) propose a continuous-time non-homogeneous model for bond ratings migration to incorporate population heterogeneity with respect to the age of bonds, measured as the passing time since issuance. The proposed model, called mover-stayer, illustrates the likelihood estimators for the rating transitions. The authors discover that young bonds are relatively more stable within their credit ratings, relative to seasoned bonds, and thus contain lower probabilities to default. In addition, for young bonds with lowest credit ratings, the default probabilities predicted by the proposed mover-stayer model are significantly lower than those predicted by the standard Markov chain model.

The impact of business cycles on credit risk in general and on credit ratings migration in particular is well documented. Williamson (1987) provides support for real business cycle theory at the expense of monetary theories by constructing a model for credit supply that is affected by fluctuations in real output. Denis and Denis (1995) find that macroeconomic factors may trigger defaults, and that the relationship varies across business cycles. Kiyotaki and Moore (1997) develop a theoretical model for systematic credit risk factors correlated with business cycles.

Nickell, Perraudin and Varotto (2000) quantify the dependency of rating transition probabilities both on the industry and domicile of the obligor and on the stage of the business cycle. By defining three categories of business cycles, peak, normal times and trough, depending on whether real GDP growth is at the top, middle or bottom third, they investigate the volatility of credit ratings transitions. Employing a probit analysis of

Moody's long-term corporate and sovereign bond ratings from 1970 to 1997, they discover that the business cycles variability is the most influential in explaining the volatility of transition probabilities. The authors conclude that applying the homogeneous transition matrix over high or low quality loans is likely to under- or overestimate the risk respectively.

Bangia, Diebold, Kronimus, Schagen and Schuermann (2002) empirically examine homogeneous and conditioning transition matrices in business cycles. They use the National Bureau of Economic Research (NBER) database to identify certain months as expansionary or contractionary stages. This identification is constructed by examining a large number of business indicators, and may reflect improving or deteriorating business cycles. Excluding the rating modifiers and considering only seven rating categories, the authors provide strong evidence for the assumption of Markov properties. Furthermore, they support the momentum hypothesis, as most downgrade probabilities for the downward-momentum matrix are found to be larger than the corresponding values in the homogeneous matrix. Distinguishing between expansionary and contractionary stages, leads to a dramatic reduce in the level of uncertainty in transition estimations. However, the differences between the stochastic economic model and the unconditional simulation are relatively minor over short-term time horizons, though the differences gradually increase over time.

Kwark (2002) explores business cycles based on the interest rate spread between risky loan yields and risk-free rates. He concludes that fluctuations of the interest rate spread are highly related to movements in default risk over the business cycle, and contain information about future fluctuations in output. Considering business cycles, Wei (2003) relaxes the assumption of homogeneous Markov chains, and allows for rating inter-variations within the Markov chain states to price various credit derivatives. This analysis suffers from a reliance on some problematic assumptions. Deviations from the average credit score matrix are assumed to be mutually independent and normally distributed, ignoring the different frequencies, sequences, magnitudes and lengths of business cycles. In addition, the assumption that transition rates among different states are normally distributed excludes other likely patterns of cyclical behavior or mean reversion, found in several empirical studies. Farnsworth and Li (2004) develop a bond-pricing model where a firm's instantaneous probability of default is conditional on its credit ratings as well as on a latent systematic factor. Examining the model under constant transition rates and with stochastic ratings changes probabilities, linked to the systematic factor, over the Lehman Brothers fixed income proprietary database from 1985 to 1998, the authors find strong evidence supporting the non-homogeneous transition matrix.

6. Empirical Evidence on Credit Ratings

Altman and Kao (1992a) examine rating dynamics of 1,548 Standard & Poor's new bond issues from 1970 to1985. They measure serial dependency through a statistic defined as the frequency of subsequent rating changes in one direction divided by the frequency of subsequent rating changes in the opposite directions. If ratings may exhibit positive drift, the proposed statistic is larger than 1. If an upgrade rating is more likely to be followed by a downgrade rating, or a downgrade transition is more likely to be followed by a downgrade transition, the statistic is smaller than 1. The authors discover that original issue speculative-grade companies display no tendency either to rise or to decline in quality over time, but a positive serial autocorrelation is found within AAA, AA and A original categories, in most downgrades, but not necessarily in upgrade credit ratings.

Several scholars discover that higher-rated bonds tend to be more stable than lower-rated bonds, with respect to the time of remaining at the same credit rating. Altman and Kao (1992b) examine major credit ratings of more than 7,000 bonds issued from 1970 to 1988. They find that AAA-rated issues had the greatest stability, in terms of retaining their initial ratings, up to five years after issuance. BB-rated bonds are found to be the least stable. Dividing the time frame into two sub-periods, from 1970 to 1979 and from 1980 to 1985 sows a tendency for a downgrade in rating to be followed by a second downgrade, indicating on a serial autocorrelation when the initial rating changes was a downgrade. No autocorrelation is found when the initial change is an upgrade. Carty and Fons (1994) test approximately 4,700 long-term public debt issuers over a 70-year period, from 1923 to 1993, and 2,400 short-term public debt issuers over a 22-year period from the Moody's proprietary database. They discover that higher ratings are relatively more stable, as reflected by their longer average length of time holding the same rating, given that it subsequently changes. However, the likelihood that a rating will change increases with time for Aaa and Aa categories; approximately constant for A, Baa, and Ba; and decreases for B and Caa ratings. Fons (1994) investigates broad rating categories out of the Moody's long-term default studies from 1970 to 1993. He also finds that default rates are not particularly stable, especially at low rating levels. Both Carty and Fons (1994) and Fons (1994) report that firms with low (high) credit ratings are more likely to be upgraded (downgraded), conditional on surviving, suggesting an underlying mean reversion in company credit ratings.

Carty and Lieberman (1996a) extend the Moody's 1996 corporate bond default study, covering 58-year period from 1938 to 1995. The authors examine 833 corporate bond defaults, involving over 1,700 bond issues, aggregating more than 106 billion dollars, and discover that out of the 560 default issuers, which were rated by Moody's since 1938, only three carried investment-grade categories, at the time of default. Furthermore, only 22 issuers carried investment-grade ratings at the beginning of the year they defaulted, and 37 were rated investment grade at the start of the second year prior to default. The authors also realize that while holding constant the severity of loss in the event of default, the probability to default is higher among lower rating categories, and recovery decreases with the seniority of the bond. Carty and Lieberman (1996b) examine secondary market loan pricing and recovery rates among 58 U.S. bond issuers with one loan per borrower, where the earliest default occurred in September 1989, and the most recent in July 1996. Although a secondary market pricing does not accurately reflect the intrinsic value of bank loans in default, the authors consider default loan observed prices as proxies for loan recovery rates. Their analysis shows an average recovery rate of 71%, and a median of 77% for U.S. senior secured syndicated bank loans, with a standard deviation of 21%. The authors also discover that the higher recovery rates of loans relative to senior unsecured and subordinated debt obligations are on average captured in higher credit ratings.

Altman (1997) compares the Moody's and the S&P bond ratings, to examine the rating change experience of corporate bonds from the time of issuance, and from a static-pool if issuers of a given rating, both up to ten years hereafter. The impact of credit rating change on fixed income portfolio composition of investors is also tested, particularly on those with restricted credit quality strategies. The author realizes several differences in published reports on rating migrations, derived from different sample methodologies, various rating systems and periods of observation. Rating migration is found to have implications over financial institutions, with different policy of tolerance for credit quality changes, in their fixed income portfolio.

Hite and Warga (1997) examine the effect of changes in bond rating on bond returns. Using a database of month-end trader quotes from Lehman Brothers from 1985 to 1995, along with all Standard & Poor's and Moody's ratings, the authors discover that downgraded firms reveal a significant announcement effect in both the rating-change announcement month and preannouncement period. The magnitude of downgrading effects increases as the sample shifts from investment-grade to non-investment-grade firms. However, upgrade effects are much weaker in magnitude and significance. Blume, Lim and MacKinlay (1998) explore credit quality among U.S. corporations. Observing that for several years, the number of downgrades in corporate bond ratings exceeded the number of upgrades, the authors employ an ordered probit analysis, from 1978 to 1995, and discover that rating standards have become more stringent by the rating agencies, implying that some of the observed credit deterioration is due to changing standards.

Reisen (2000) and Ferri, Liu and Majnoni (2001) discover that rating agencies' behavior tends to be cyclical with respect to tagging sovereign country debt. The later also show that linking bank capital asset requirements to external ratings would have undesirable effects for non-high-income countries. Cantor and Falkenstein (2001) examine rating consistency around sector and macroeconomic shocks over time. Checking deviations within default rates from the normal distribution, and finding that sector and macroeconomic shocks inflate the sample standard deviations, compared to using a binomial default distribution, the authors conclude that historical default rates may vary across bond market sectors for a long time, without necessarily implying fundamental differences in underlying default probabilities.

Nagpal and Bahar (2001) examine the structure of correlation between defaults of U.S. corporations, from 1981 to 1999. Their historical study explores the correlation between default probabilities and joint default probabilities, as derived from economic conditions and industry-specific factors. Dividing the seven rated categories into two groups – investment grade, from AAA to BBB-, and non-investment grade, BB+ or lower, and assuming that individual and pair-wise default probabilities do not change over time, the authors find that the industry sectors and the credit ratings are major determinants for these correlations. In particular, default correlations for investment-grade companies are found to be lower than those of non-investment-grade companies. The default correlations between investment-grade companies generally hold over seven-year periods. However, over long-term periods, correlations diminished, due to variability of economic conditions.

De Servigny and Renault (2002) use the S&P CreditPro 5.20 database, focusing on the U.S. sub-sample, to provide new empirical evidence on default correlations. The impact of business cycles and investment horizon on the properties of credit correlations are examined using a Gaussian factor model and Student-t bivariate copula.[73] Results show that most large positive correlations are found within a given industry. Although the use of empirical correlations can generate several problems, such as zero observed joint default probabilities may not be valid ever after, the authors suggest that observed correlations should be used as the appropriate benchmark for credit portfolio model specifications.

7. Recent Developments within Credit Ratings Migration Analysis

Credit rating agencies declare that they take a rating action only when it is unlikely to be reversed shortly after. Löffler (2005) use a formal representation of the credit rating process to show that this agencies' policy provides a good explanation for several observed phenomena, including the infrequency of rating transitions, the serial

[73] The factor-based approach specifies the rating transition process as an outcome of both systematic (e.g. macroeconomic or market) and idiosyncratic (e.g. firm's asset market value and debt face value) factors.

dependency credit ratings contain, and the fact that ratings are often lag changes in companies' creditworthiness. By mapping credit quality continuous variables into discrete categories to account for credit ratings, and by using simulations, the author concludes that rating bounce avoidance can explain the above habits. Furthermore, since rating agencies are slow to react to new information, rating changes might become predictable.

Parnes (2006a) investigates several known models and adds some innovative non-homogeneous dynamics to explain credit ratings migration. While testing the homogeneous Markov chain model, and the stochastic economic dynamic, with various definitions for expansionary and contractionary cycles, against some non-homogeneous processes, including the credit quality correlations, offered by CreditMetrics™, the generator matrix, introduced by Israel, Rosenthal and Wei (2001), the momentum ratings migration, proposed by Bahar and Nagpal (2001), and three new dynamics, including a mean reversion effect, an autoregressive time series dynamic, and a density-dependent model, the last two innovative models are found to be highly realistic, and outperforming the homogeneous model in describing empirically observed ratings transitions.

Peura and Soininen (2005) classify four mutually exclusive types of default: a non-performing exposure, where interest or principal payments are more than 90 days past due, a distress decision, where the borrower is unable to get additional financing outside existing financial institutions, a legal corporate restructuring, also called chapter eleven, and a bankruptcy or chapter seven. This distinction extends the Markov chain transition matrix for credit ratings migration. The authors consider transition probabilities to be positive among the various default states, as well as from default to survival states, while the lowest default type, a bankruptcy, is the only absorbing state. This approach leads to a multinomial model of credit loss, which extends the binary 'default-no default' approach. Multiple default states may capture the variation in loss severities across defaults, and it allows to better examining the default process.

8. A Comparison between the Two Approaches

Both credit risk methodologies have their own pros and cons. This section discusses the advantages and disadvantages within the Merton structural model and the credit ratings migration approach. One of the problematic assumptions underlying the Merton structural model is a deterministic interest rate, although it may have merely a limited effect on default probabilities over a short-term interval. Jarrow and Turnbull (2000) describe many of the properties of the Merton approach, arguing that the CreditMetrics, the CreditRisk+ and the KMV methodologies cannot reproduce empirical observations given their constant interest rate assumption. On the other hand, the authors state that the KMV methodology relies on the market value of equity to estimate the firm's volatility, thus it incorporates market viable information on default probabilities. This information may not be included within the credit ratings migration analysis.

The Merton approach often leads to difficulties in computations of the unobservable variables. Although various methods exist for that purpose, market value of assets and log return volatility of the firm's assets are not easy to obtain. Since the derivations include nonlinear programming, special software is required to either solve the simultaneous equations, or to run an iterative procedure.[74] Furthermore, there is no way to verify the accuracy of these estimations. Repositioning complex structure of dent into a single liability can also be problematic. Infrequently traded debt contracts, with different time horizons and various covenants may seriously challenge this task.

An inherited problem within the Merton model, as illustrated in Section 1, is that when time to maturity converges to zero, the implied default probability approaches either 0 or 1. However, this is not the case in practice. In reality, companies often refinance their liabilities prior to maturity without been forced to default. Another observed bias within the Merton structural model, as discussed in Jones, Mason and Rosenfeld (1984), Franks and Torous (1989), and Fons (1994), is that predicted credit spreads are systematically smaller than actual spreads.

Credit migration analysis may sometimes capture changes in credit risk that could not be captured by the Merton approach. Transformations in the financial position of a firm can arise from different variables other than asset value, debt face value, equity value, remaining time to maturity and asset volatility. Sudden changes in demand or consumer taste, unhealthy industry trends, innovative technology in the hands of

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[74] Crosbie and Bohn (2003) state that Moody's KMV avoids solving the simultaneous equations, since in practice "*the model linking equity and assets volatility holds only instantaneously*". Bharath and Shumway (2004) illustrate the iterative procedure with an initial guess. Parnes (2006b) discusses more complicated iterative procedure that accounts also for unobserved correlation.

competitors, heavy reliance on a single supplier, changes in reputation, poor management skills, or other macroeconomic exogenous factors may also lead to a degradation of creditworthiness. All these are intrinsic elements already taken into consideration within the ratings migration probabilities.

Ratings migration analysis can be implemented, more easily than the structural approach, on a portfolio of traditional bank loans, or corporate bonds. Different techniques to apply the credit ratings migration from a portfolio perspective are discussed in the literature. Gollinger and Morgan (1993) consider default likelihoods by estimating default correlations across 42 industry indices. Stevenson and Fadil (1995) correlate default events among 33 industry groups. Both researches point to the difficulty to estimate real default correlations. Carey (1998) and Kealhofer (1998) discuss credit risk issues in private portfolios management. Froot and Stein (1998) show that the price of a marginal credit exposure depends upon several risk's correlations within bank's portfolio. Koyluoglu and Hickman (1998) analyze correlations, also called 'background factors', between market factors and credit risk factors within bond portfolios. Andersson, Mauser, Rosen and Uryasey (2001) provide a conditional value-at-risk (CVaR) optimization criterion for credit risk management. Lucas et al. (2001) present analytical method for measuring credit risk in large portfolios. Barnhill and Maxwell (2002) examine credit risk for fixed income portfolios. Frey and McNeil (2003) provide a cyclical correlation-based model of a portfolio. Giesecke and Weber (2004) examine credit contagion within portfolios. Parnes (2006a) present several measurements for distance to default and expected time to default from a portfolio view, and Acharya et al. (forthcoming 2006) discuss the costs related to diversification in a portfolio of bank loans.

Credit ratings methodologies carry several disadvantages. They are relatively recent hence estimates of transition probabilities often suffer from small samples. Since credit migration probabilities rely merely on few decades of experience, the credit ratings migration matrix sometimes assigns unrealistic transition rates. Many of the transition matrix entries receive zero probabilities among them are several high rated categories, which obtain zero default probabilities. However, one cannot disregard extreme situations thus, default probabilities can only converge to zero. This may cause biasness in estimating transition rates for extraordinary events. Further more, historical data may not always be a good predictor to future behavior. Past migrations are used to forecast future ones under the implicit assumption of stationarity.

The credit ratings migration methodology also assumes that the rating agencies know how to tag the right rating label, and all firms tagged within a given rating label share the same default risk. These assumptions could be problematic as well. Not all the companies rated with the same category behave in the same pattern. Furthermore, industry and geography homogeneity in credit ratings migration may be invalid assumptions. For example, companies from different industries could be affected in various ways, even under the same market conditions. Morgan (1997) shows that the level of consensus among rating agencies is significantly lower for financial institutions than it is for corporations. He concludes that the lower level of transparency between sectors can trigger this difference. Kealhofer, Kwok and Weng (1998) examine the validity of these assumptions, and test how a violation affects calculations of expected and unexpected loss. They discover that historical default rate can deviate significantly from the actual default rate, and there are substantial differences of default rate even within rating categories. Furthermore, since the mean default rate can remarkably exceed the median default rate, and historic default rates are statistics for the mean, default rates may be biased upward. Finally, the authors state that probabilities of remaining at the same credit rating are overstated by about double for most grades, while transitions from entries along the diagonal towards non-default grade are significantly understated. Ammer and Packer (2000) review these issues and conclude that geographic homogeneity is not questionable.

The pricing of defaultable bonds as well as credit risk modeling, have gained momentum in the last decades. Despite a growing interest in this subject, there is still a lack of solution that can fit both the dynamics and shape of credit spread curves. The debate over which of the above methodologies, the structural model and the credit ratings migration approach, can better predict default events will certainly continue over the next years.

9. Alternative Methodologies for Credit Risk Evaluation

Substitute approaches for measuring corporate credit risk can also be found in principal component analysis, feature maps, logistic regressions, hierarchical classification models, neural-networks and other reduced-form models. It is beyond the scope of this survey to go any further into these models, but the curious reader may find more explanation on the neural-networks of credit scoring in Dutta and Shekhar (1988), Kerling (1995), Tyree and Long (1995), and Yang, Platt and Platt (1999). The predictive power of some of these models is compared in Altman, Marco and Varetto (1993), Alici (1995), Episcopos, Pericli and Hu (1998) and Lopez (2000). Litterman and Iben (1991) are among the first to present reduced form models. Jarrow and Turnbull (1995) assume that when a firm defaults, its bond would have a market value of exogenously specified fraction of equivalent default free bond. Artzner and Delbaen (1995) present a model for default risk insurance. Jarrow, Lando and Turnbull (1997) allow various seniority of debt to trigger different recovery-rates. Duffie (1998) assumes that when a firm defaults, debt-holders receive a fixed payment, regardless of the interest payments, maturity, and seniority of the bond. Madan and Unal (1998a, 1998b) develop a hazard-rate theoretic model based on two principal components: a risk-free asset and a premium for default risk. Schönbucher (1998) specifies hazard-rate function as well. Duffie and Singleton (1999) allow for random recovery rate, depending on the pre-default value of the bond. Jarrow and Yu (2001) show that firm's interactions with each other trigger correlated jumps in rates of default, and Goldberg (2004) discusses incomplete information models for credit risk.

The reduced-form models differ from structural form models by not conditioning default on the firm's stochastic market value of assets. Reduced form models contain a different set of assumptions on default probabilities and recovery rates, assumed to be exogenous and independent of one another hence considering default as an unpredictable event. Their main disadvantage is by lacking predictability strength. However, reduced-form models are commonly used to price credit sensitive securities.

Some scholars use the structural models and credit ratings migration analysis, while incorporating corporate bond yield spreads. Rodriguez (1988) presents a theoretical model to show that yield spreads are complex function of maturity, and not necessarily monotonically increasing. The maturity-invariance property is shown to hold only under some restrictive conditions, as in a tax-free world, or when bonds sell at their par value. Litterman and Iben (1991) provide information about the default-free term structure, and develop a model that explicitly recognizes a term structure of credit risk. Their model measures the effective spread curve implied in the price of each bond, by isolating the impact of credit risk. The spreads on corporate bonds are found to vary with maturity, even when holding all other characteristics constant. For a given corporation, that variation reflects the changing probabilities of default, for the corporation in the future years. Das and Tufano (1996) use stochastic credit spreads within credit ratings to price credit sensitivity. Pedrosa and Roll (1998) show that firm-level credit spreads movements have a common source related to credit market conditions. Duffee (1999) estimates a separate model of jumps in bond's price in case of a default.

Helwege and Turner (1999) provide empirical evidence for a downward-sloping yield curve, in a setting of diversifiable default risk. Collin-Dufrense and Goldstein (2001) present a non-linear model to price default risk across firms. Collin-Dufrense, Goldstein and Martin (2001) consider time-varying correlations to economic factors to explain why it has been difficult to relate structural models to changes of credit spreads. Duffie and Lando (2001) use credit spreads to study single-firm incomplete accounting information credit models. They suggest that if the information available to investors were perfect, then observed credit spreads would be close to the theoretical ones, as implied by the Merton model. However, when information is not complete, observed spreads exhibit significant differences. Duffie and Liu (2001) show that using credit-default swap premium directly, as a measure of default component, could be biased. Elton, Gruber, Agrawal and Mann (2001) find that much of the information in credit spread is unrelated to default risk. Duffee (2002) allows for correlation between credit spreads and risk-free interest rates in a two-factor affine model. Guha and Hiris (2002) use credit spreads to predict macroeconomic developments and Eom, Helwege and Huang (2004) study the determinants of corporate yield spreads.

Appendix 1

If $V \sim Lognormal$ then: $ln(V) \sim N(\mu, \sigma^2)$. In addition: Z = $(ln(V)–\mu)/\sigma \sim N(0, 1)$. The conditional mean of V given $V < D$ (within the truncated log normal distribution), can be expressed as

$$E[V|V < D] = E[e^{\sigma Z+\mu}|\ e^{\sigma Z+\mu} < D] = E[e^{\sigma Z+\mu}|Z <(ln(D)-\mu)/\sigma]$$

This can be simplified but using the following notations

$$h_1(\mu, \sigma) = (ln(D)-\mu)/\sigma \sim N(0, 1)$$

$$h_2(\mu, \sigma) = \Phi(h_1(\mu, \sigma))$$

Where Φ is the Cumulative Distribution Function (CDF) of the standard normal distribution. Using the general theorems (Greene 2000)

$$E[x|x < a] = \int_{-\infty}^{a} xf(x|x < a)dx$$

$$f(x|x < a) = f(x) / Prob(x < a)$$

$$x \sim N(\mu, \sigma) \Rightarrow Prob(x < a) = \Phi((a-\mu)/\sigma)$$

The conditional mean becomes

$$E[V|V < D] = (\int_{-\infty}^{h2(\mu,\sigma)} exp(\sigma Z + \mu)exp(-z^2 / 2)dz) / (\sqrt{(2\pi)})h_2(\mu,\sigma)$$

$$= (exp(\mu + \sigma^2 / 2)\int_{-\infty}^{h1(\mu,\sigma)} exp(-(z-\sigma)^2 / 2)dz) / (\sqrt{(2\pi)})h_2(\mu,\sigma)$$

$$= (exp(\mu + \sigma^2 / 2)\Phi((ln(D)-\mu) / \sigma - \sigma) / \Phi((ln(D)-\mu) / \sigma)$$

By substituting the mean $ln(V) + (\mu_V - \sigma^2{}_V / 2)t$ instead of μ, and the standard deviation $\sigma_V\sqrt{t}$ instead of σ, after a simple algebra the required identity is obtained■

10. References

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Autobiographical Statement

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Education

2002 – 2006	Baruch College, CUNY	Ph.D. in Finance
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1998 – 1999	Baruch College, CUNY	Executive Master of Science in Finance
1994 – 1998	Tel-Aviv University, Israel	B.Sc. in Statistics, Operations Research and Computer Science

Dissertation Papers

1. "Homogeneous Markov Chain, Stochastic Economic, and Non-Homogeneous Models for Measuring Corporate Credit Risk"
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Professional Experience

Aug. 06 - …:	Assistant Professor in Finance at the University of South Florida
2002 – 2006:	Adjunct Lecturer and Substitute Instructor at Baruch College. Taught Financial Management (undergraduate level), and Advanced Investment, International Finance and Risk Management (graduate level)
2001 – 2002:	Portfolio manager in Psagot Mutual Funds, Israel
2000:	Research analyst at Tamir Fishman & Co., strategic partners with RBC
1999:	Software engineer in Advanced Technology Ltd, Real Time project
1997 – 1998:	Software engineer in Amdocs (DOX), Information Systems project
1990 – 1994:	Army service as a captain in a field unit, IDF

Presentations

1. "Homogeneous Markov Chain, Stochastic Economic, and Non-Homogeneous Models for Measuring Corporate Credit Risk" at the FMA conference, Chicago, Oct. 2005
2. "The Impact of Exchange Rate Exposure on Multinationals' Credit Risk" at the FMA conference, Chicago, Oct. 2005