QUANTUM TRANSPORT IN LOW DIMENSIONAL ELECTRONIC SYSTEMS

by

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Abstract

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by

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Spin and charge transports of low dimensional electronic systems are studied. The general idea of low dimensional physics is briefly introduced. We studied the spin transport of a spin polarized Luttinger liquid in the presence of a single impurity. We found that, by considering the extended Hubbard interaction, a spin-polarized wire is obtained in response to an external magnetic field. In addition, we studied the charge transport of a confined two-dimensional electron gas in the presence of an insulating region. We considered the two-dimensional electron gas as being confined by a harmonic potential. The insulating region, which causes the electronic wave function to vanish, is replaced by a static vortex. We found that a rectified voltage is induced in response to a microwave.

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Physics in Low Dimension

1.1 Electron States in Various Dimensions

1.1.1 Density of States in 3D

Consider a single electron of mass m_e confined in a box of sides L_x, L_y and L_z, the one particle Schrödinger equation in three dimension is given by

$$\left[-\frac{\hbar^2}{2m_e}\nabla^2 + V(\vec{r})\right]\psi_{lmn}(\vec{r}) = E_{lmn}\psi_{lmn}(\vec{r})$$

where $V(\vec{r})$ is the confining potential, ψ_{lmn} is the wave function, E_{lmn} is the eigen-energy, and (l, m, n) are the quantum numbers. The presence of the confining potential restricts the electron to move only inside of the box, namely $V(\vec{r}) = 0$ inside of the box and $V(\vec{r}) = \infty$ outside of the box. By satisfying the condition that $\psi_{lmn}(\vec{r}) = 0$ on the boundary of the box, one

can obtain the analytical solution for the wave function $\psi_{lmn}(\vec{r})$ that

$$\psi_{lmn}(\vec{r}) = \sqrt{\frac{8}{L_x L_y L_z}} \sin(\frac{l\pi x}{L_x}) \sin(\frac{m\pi y}{L_y}) \sin(\frac{n\pi z}{L_z})$$

with the eigen-energies being

$$E_{lmn} = \frac{\hbar^2 \pi^2}{2m_e} \left[(\frac{l}{L_x})^2 + (\frac{m}{L_y})^2 + (\frac{n}{L_z})^2 \right].$$

In practice, the sides of the box L_x, L_y and L_z are on the scale of millimeters and the kinetic energy is in the order of a few meV. Let's take $L_x = 1\ mm$ for example, the x-component kinetic energy becomes $\frac{\hbar^2\pi^2}{2m_e}(\frac{1}{L_x})^2 = 3.4\times10^{-10}\ meV$ for $m_e = 9.1\times10^{-31}\ kg$ and $\hbar = 1.05\times10^{-34}\ J\cdot s$. Since the Fermi energy for most of the metallic materials is in the range of a few electron volts, the quantum numbers (l, m, n) are thus very large. For this reason, the $\vec{k}$ vector and the eigen-energy E_{lmn} are both quasi-continuous, namely $\vec{k} = (k_x, k_y, k_z)$ and $E_{lmn} = E(\vec{k}) = \frac{\hbar^2}{2m_e}(k_x^2 + k_y^2 + k_z^2)$. For the ground state of N free electrons, a sphere in $\vec{k}$-space is occupied up to the maximum radius k_F such that $E_F = E(k_F) = \frac{\hbar^2}{2m_e}k_F^2$. We obtain $k_F = \left(\frac{3\pi^2 N}{V}\right)^{1/3} = (3\pi^2 n_3)^{1/3}$, where $n_3 = N/V$ is the electron density in 3D, and $E_F = \frac{\hbar^2}{2m_e}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$.

We can thus find the density of states $D(E)$, which is defined as the number of electrons per unit energy range per unit volume, i.e. $D(E) \equiv \frac{1}{V}\frac{dN}{dE}$, and

$$D(E) \equiv \frac{1}{V}\frac{dN}{dE} = \frac{1}{2\pi^2}\left(\frac{2m_e}{\hbar^2}\right)^{3/2}\sqrt{E}. \tag{1.1}$$

It is noted that the density of state $D(E)$ in three dimension is proportional to $\sqrt{E}$, as illustrated in figure 1.1(a).

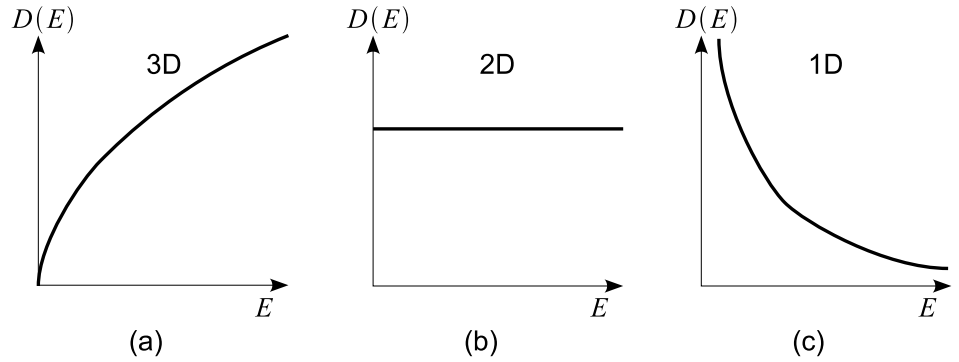

Figure 1.1: Density of states in various dimensions.

1.1.2 Density of States in 2D

Let's now consider the situation where L_z is of the order of nanometers. By taking $L_z = 1\ nm$, we obtain that the energy term $\frac{\hbar^2}{2m_e}(\frac{1}{L_z})^2 \simeq 0.34\ eV$, which allows the quantum number n to be very small. We can no longer consider the z-component of the $\vec{k}$ vector as quasi-continuous. On the contrary, the z-component of $\vec{k}$ is quantized into a few discrete quantum levels. The total energy of such a system reveals a mixed continuous-discrete energy spectrum, namely

$$E = E_n(k_x, k_y) = \frac{\hbar^2}{2m_e}\left(k_x^2 + k_y^2\right) + E_n \tag{1.2}$$

where energy $E_n = \frac{\hbar^2}{2m_e}\left(\frac{n}{L_z}\right)^2$ is the subband energy. For simplicity, we can consider that the electrons move only in the n-th subband.

Now, we want to calculate the density of state in two dimensions. Similar to the discussion given above, a total number of electrons N is occupied in the k-space up to a circle of area πk_F^2. Accordingly, we have $k_F = \left(\frac{2\pi N}{S}\right)^{1/2} = (2\pi n_2)^{1/2}$, where $S = L_x L_y$, and $n_2 = N/S$ is the electron density in 2D, and

$E_F = \frac{\hbar^2}{2m_e}\left(\frac{2\pi N}{S}\right) + E_n$.

By choosing the sub-band energy E_n as the zero point energy, the number of electrons N occupied below energy E is given by $N = \frac{S}{2\pi}\left(\frac{2m_eE}{\hbar^2}\right)$. Since we are interested in calculating the density of states in 2D, we define the 2D density of state as the number of electrons per energy range per unit area, i.e. $D(E) \equiv \frac{1}{S}\frac{N}{E}$. As a result, we have

$$D(E) \equiv \frac{1}{S}\frac{dN}{dE} = \frac{1}{2\pi}\frac{2m_e}{\hbar^2}, \tag{1.3}$$

which is a constant, as illustrated in figure 1.1(b). The density of states as being constant in 2D is the origin of many special electronic properties in 2D.

1.1.3 Density of States in 1D

Following the same idea, we can further confine the electronic motion in the y-direction, such that the side L_y is also in the order of nanometers. The total energy can now be written as

$$E = E_{mn}(k_x) = \frac{\hbar^2 k_x^2}{2m_e} + E_{mn} \tag{1.4}$$

where energy $E_{mn} = \frac{\hbar^2}{2m_e}\left((\frac{m}{L_y})^2 + (\frac{n}{L_z})^2\right)$ is the subband energy. A total number of electrons N is occupied in a line in k-space with a length $2k_F$. Accordingly, we have $k_F = \frac{\pi N}{2L_x} = \frac{\pi}{2}n_1$, where $n_1 = N/L_x$ is the electron density in 1D, and $E_F = \frac{\hbar^2}{2m_e}\left(\frac{\pi N}{2L_x}\right)^2 + E_{mn}$.

The number of electrons N occupied below energy E for one particular sub-band is given by $N = \frac{2L_x}{\pi}\left(\frac{2m_eE}{\hbar^2}\right)^{1/2}$. In 1D, we define the density

of states as the number of electrons per energy range per unit length, i.e. $D(E) \equiv \frac{1}{L}\frac{dN}{dE}$. As a result, we have

$$D(E) \equiv \frac{1}{L_x}\frac{dN}{dE} = \frac{1}{\pi}\left(\frac{2m_e}{\hbar^2}\right)^{1/2} E^{-1/2}. \tag{1.5}$$

This shows that the density of states in 1D is proportional to $E^{-1/2}$, as illustrated in figure 1.1(c).

The difference of density of states of various dimensions will give rise to different transport properties for electronic systems in different dimensions. The importance of density of states will become clear when one calculates the current in different dimensions.

1.2 Experimental Realizations

1.2.1 Two Dimensional Electron Gas

The development of modern semiconductor technology has made it possible to grow thin layered structures onto a semiconductor substrate. One of the most important applications of this technology is to confine electronic motions in a two dimensional plane, thereby forming a two dimensional electron gas (2DEG).

Basically, two types of two dimensional electron gas have been developed and studied. One such system is formed on a metal-oxide-semiconductor (MOS) structure, see figure 1.2. As shown in this band diagram, a gate voltage V_g is applied to the metallic gate that is separated with the semiconductor by a dielectric (SiO_2) layer. As a result, a depleting band bending in

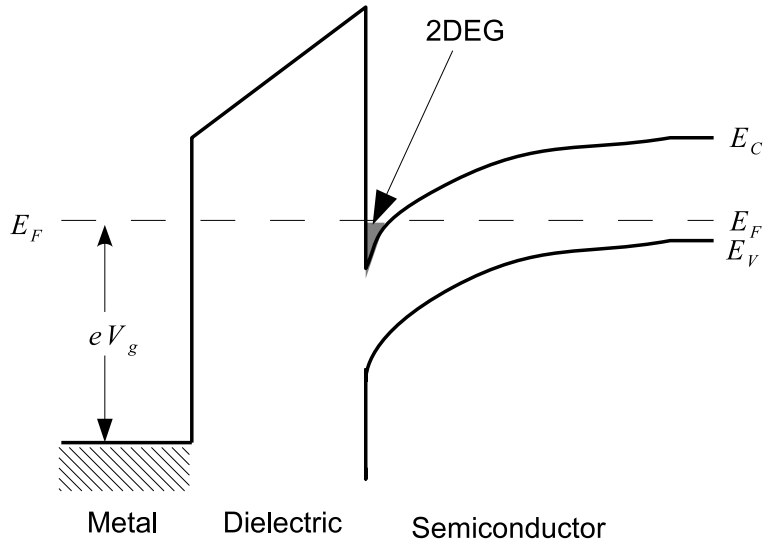

Figure 1.2: This band diagram shows the formation of 2DEG at the interface of dielectric and semiconductor.

the semiconductor layer is created. For a large enough gate voltage V_g, this bending can be of the order of the semiconductor band gap. Consequently, an inversion layer with carriers of opposite sign than in the bulk is formed near the interface of the dielectric and the semiconductor, thus forming a two dimensional electron gas. Consider that the MOS structure forms a capacitor of parallel plates which has a capacitance of $C = \epsilon_D A/d$, where ϵ_D and d are the dielectric constant and the thickness of the dielectric layer respectively, and A is the area of the capacitor. The surface electron density n_2 in the inversion layer is then given by

$$n_2 = \frac{\epsilon_D}{ed}(V_g - V_0),$$

where V_0 is the threshold voltage for opening of a conducting channel at

the dielectric-semiconductor interface. For silicon structures, the maximum possible value for n_s is of the order of 10^{13} cm^2.

Another type of two dimensional electron gas is formed in the modulation-doped $GaAs-AlGaAs$ heterojunction, where the band gap of the $AlGaAs$ is wider than that of the $GaAs$, see figure 1.3. The formation of the modulation-doped heterojuction gives rise to an inversion layer. By properly doping the $AlGaAs$ layer, one can move the Fermi level in the forbidden gap such that the 2DEG is formed within the inversion layer. At present, 2DEG formed in the modulation-doped heterojunction is the most commonly used structure for studying quantum transport in low dimensional systems. One advantage of the heterojunction over the MOS-structure is in that the quality of the heterojuction interface is better than that of the MOS-structure. By properly matching the lattice structure of the $GaAs$ and the $AlGaGs$ layers, one can obtain a 2DEG that the electron density n_s thereof is as low as $10^8 cm^{-2}$. This gives rise to an extremely high carrier mobility μ. For example, the mobility of the heterojunction can easily exceed 10^7 $cm^2/V\cdot s$, while the mobility of the MOS-structure is typically in the range of 10^4-10^5 $cm^2/V\cdot s$. In addition, the 2DEG formed by modulation doping can be squeezed into narrow channels by using a patterned metallic gate grown on top of the 2DEG. By applying a gate voltage V_g to the metallic gate, the electrons under the metallic gate will be depleted, thereby forming the desired narrow conducting channel.

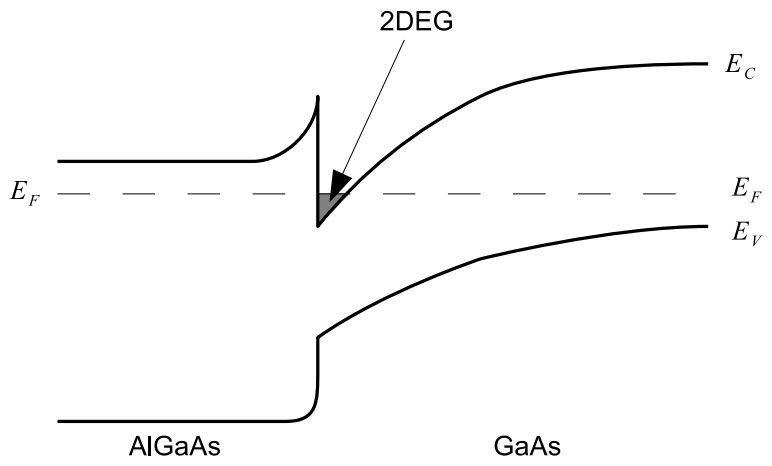

Figure 1.3: This band diagram shows the formation of 2DEG at the modulation-doped $AlGaAs - GaAs$ heterojection.

1.2.2 One Dimensional Electron Gas

Once the creation of 2DEG is realized, attempts have been made to further confine the 2DEG in one of the two in-plane directions, thereby forming a one-dimensional conducting wire. One way to achieve this spatial confinement is to deposit a layer of patterned metallic gate above the 2DEG. By applying a gate voltage to the patterned gate, electrons beneath the patterned gate are depleted. As illustrated in figure 1.4, one can make the separation width W of the patterned gate so small that only one conducting channel exists between the left and right leads. As a result, a one-dimensional electron gas is obtained.

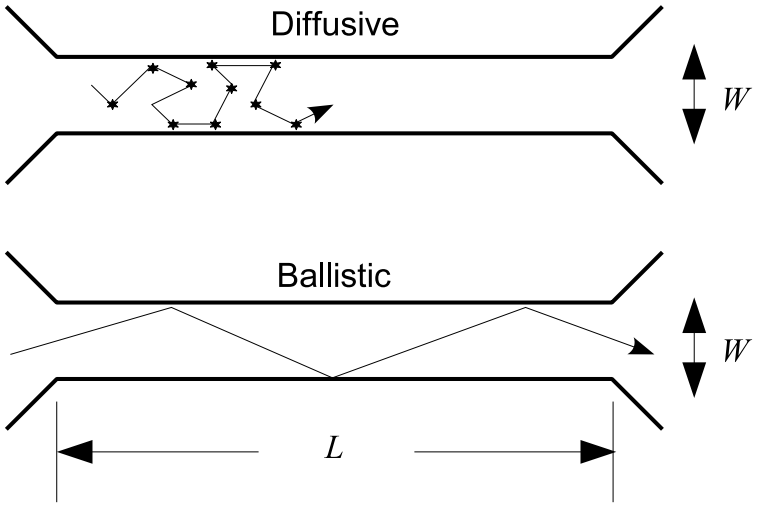

Figure 1.4: Schematic diagram illustrating diffusive transport and ballistic transport in a quasi-one-dimensional narrow wire

1.3 Important Physical Quantities

In order to study quantum transport in a proper regime, we need to use certain space, time and energy scales to characterize a physical system. Many of the physical quantities discussed here are summarized in table 1.1 for two dimensional electron gas of $GaAs - AlGaAs$ heterojunction and MOS-structure.

1.3.1 Fermi Wavelength (λ_F)

The Fermi wavelength λ_F is defined as the de Broglie wavelength on the Fermi surface. Since the Fermi surface is characterized by the wave vector

k_F, we can write

$$\lambda_F = \frac{2\pi}{k_F}. \tag{1.6}$$

Following the discussions in section 1.1, we know that the Fermi wave vectors k_F is sensitive to dimensionality. Accordingly, we obtain the Fermi wavelength in different dimensions as

$$1D: \quad \lambda_F = 2\left(\frac{\pi}{3n_3}\right)^{1/3} \tag{1.7}$$

$$2D: \quad \lambda_F = \sqrt{\frac{2\pi}{n_2}} \tag{1.8}$$

$$3D: \quad \lambda_F = \frac{4}{n_1} \tag{1.9}$$

In quantum mechanics, electrons are described as a matter wave traveling in space. For Fermi gas at low temperature, electrons are moving on the Fermi surface. Therefore, the Fermi wavelength λ_F can be used as a measure of whether quantum effect may occur at certain length scale. For length scales comparable to the same order of magnitude of the Fermi wavelength λ_F, the problem should be treated quantum mechanically. On the other hand, for length scales much larger than λ_F, we can treat the problem semi-classically or classically.

1.3.2 Scattering Time and Length

The scattering time (also known as momentum relaxation time) τ is the average time between two elastic scattering events. According to the Drude model, the conductivity is given by

$$\sigma = \frac{n_e e^2 \tau}{m_e}, \tag{1.10}$$

where e is the electric charge, n_e is the electron density and m_e is the electron mass. Since conductivity σ is an experimentally measurable quantity, the scattering time τ follows from the conductivity σ once the electron density n_e is known. The scattering time τ is related to the scattering length ℓ via

$$\ell = v_F \tau, \tag{1.11}$$

where v_F is the Fermi velocity defined as $mv_F = \hbar k_F$. The scattering length ℓ is often referred to as mean free path. One can understand the scattering length ℓ as the average distance that the electrons travel at the Fermi surface without being scattered.

Another important physical quantity that is often used to characterize transport is the mobility μ, which is defined as the drift velocity v_d per applied electric field E, i.e.

$$\mu = \frac{|v_d|}{E}. \tag{1.12}$$

For an electron gas of high mobility, applying a weak electric field will generate a large drift velocity. According to the Ohm's law $J = \sigma E$, where $J = n_e(-e)v_d$ and σ is the Drude conductivity given in eq. (1.10), we find

$$\mu = \frac{e}{m_e}\tau \tag{1.13}$$

which is proportional to the momentum relaxation time τ.

1.3.3 Thermal Length

An electron undergoes a ballistic motion when it travels shorter than the mean free path ℓ. However, the electronic motion becomes diffusive when

the electron travels extensively further. The diffusive motion of electrons is characterized by the diffusion constant, which is given as

$$D \sim v_F^2 \tau = v_F \ell = \frac{\ell^2}{\tau}. \tag{1.14}$$

An electron traveling in a plane wave of wave vector k has a broadening in k-space, i.e. Δk. According to the uncertainty principle, $\Delta x \Delta p \sim \Delta \epsilon \Delta t \sim \hbar$, the broadening in k-space Δk is found to be $\Delta k \sim \frac{1}{\Delta x}$, which corresponds to an energy width of $\Delta \epsilon \sim \hbar / \Delta t$. At nonzero temperature, the electron loses its phase memory when the thermal energy $k_B T$ is larger than the energy width $\Delta \epsilon$, where k_B denotes the Boltzmann constant and T denotes the temperature. That is, the electron loses its phase memory after a time interval $\Delta t \sim \hbar / k_B T$. The length that the electron travels during this time interval Δt is given as

$$\ell_T = \sqrt{\frac{\hbar D}{k_B T}} \tag{1.15}$$

This quantity ℓ_T is often referred to as thermal length, which is useful for characterizing the thermal effect to a quantum system.

1.3.4 Phase Relaxation Time and Length

In quantum mechanics, the motion of electron is characterized by the evolution of wave function, which includes a phase. The phase of the wave function is especially important when one is interested in the observation of quantum interference phenomena. Since single particle motions in quantum mechanics are time-reversible, the only way to destroy the phase of wave function is

Quantity(Symbol)	GaAs	Si	Units
Effective Mass (m)	0.067	0.19	$m_e = 9.1 \times 10^{-31} kg$
Dielectric Constant (ϵ)	13.1	11.9	$\epsilon_0 = 8.9 \times 10^{-12} Fm^{-1}$
Density of States ($D(E)$)	0.28	1.59	$10^{11} cm^{-2} meV^{-1}$
Fermi Wave Vector (k_F)	1.58	$0.56 - 1.77$	$10^6 cm^{-1}$
Fermi Velocity (v_F)	2.7	$0.34 - 11$	$10^7 cm/s$
Fermi Energy (E_F)	14	$0.63 - 6.3$	meV
Fermi Wavelength ($\lambda_F = \frac{2\pi}{k_F}$)	40	$112 - 35$	nm
Electronic Mobility (μ_e)	$10^4 - 10^6$	10^4	$cm^2/V \cdot s$
Scattering Time ($\tau = m\mu_e/e$)	$0.38 - 38$	1.1	ps
Mean Free Path ($\ell = v_F\tau$)	$10^2 - 10^4$	$37 - 118$	nm
Diffusion Constant ($D = v_F^2\tau/2$)	$140 - 14000$	$6.4 - 64$	cm^2/s
Phase Coherence Length ($\ell_\phi = (D\tau_\phi)^{1/2}$)	$200 - \ldots$	$40 - 400$	$nm(T/K)^{-1/2}$
Thermal Length ($\ell_T = (\hbar D/k_B T)^{1/2}$)	$330 - 3300$	$70 - 240$	$nm(T/K)^{-1/2}$

Table 1.1: Electronic properties of two dimensional electron gas for $GaAs - AlGaAs$ heterojunction and MOS-structure. [BvH91]

to break the time reversal symmetry by inelastic scattering. We can define the phase relaxation time τ_ϕ as the time of two *inelastic* scattering events. Therefore, the corresponding phase relaxation length ℓ_ϕ is given as

$$\ell_\phi = \sqrt{D\tau_\phi}. \tag{1.16}$$

where D is the diffusion constant of eq. (1.14).

Chapter 2

Transport of Spin-Polarized Luttinger Liquid

Recently, a new type of electronics called "spintronics" has attracted tremendous attentions. The study of spintronics focuses on the use of spin polarized electrons to build spin devices, such as spin diodes and spin transistors. In order to make use of the electron spins to build up devices, one needs to understand the principles of spin ballistic transport.

In this chapter, we will investigate the possibility for having polarized electrons in a quantum wire. We will compute the conductance of a spin-polarized Luttinger liquid in the presence of a non-magnetic impurity with two different Fermi velocities $v_F^{\uparrow} \neq v_F^{\downarrow}$.

2.1 Model Hamiltonian

2.1.1 Free Part

We consider the 1D Fermion representation with spin [GNT98],

$$\psi_\sigma(x) = e^{ik_F^\sigma x} R_\sigma(x) + e^{-ik_F^\sigma x} L_\sigma(x), \tag{2.1}$$

where $\sigma = \uparrow, \downarrow$, and $R_\sigma(x)$ and $L_\sigma(x)$ represent the right and left movers of spin σ. Here, the Fermi wave vector k_F^σ is also spin dependent. We consider a situation that due to the Zeeman interaction the Fermi gas is described by

$$v_F^\uparrow = v_F + \frac{\Delta}{2}, \tag{2.2}$$

$$v_F^\downarrow = v_F - \frac{\Delta}{2} \tag{2.3}$$

where $v_F = (v_F^\uparrow + v_F^\downarrow)/2$, $\Delta = v_F^\uparrow - v_F^\downarrow$, $v_F^\sigma = \hbar k_F^\sigma / m$ and $\sigma = \uparrow, \downarrow$. The parameter Δ represents the Zeeman energy.

We can substitute the 1D Fermion representation given in eq. (2.1) into the free Hamiltonian

$$H_0 = \sum_\sigma \int dx \psi_\sigma^+(x) \left(-\frac{\hbar^2}{2m} \partial_x^2 \right) \psi_\sigma(x).$$

We obtain the spin dependent free Hamiltonian

$$H_0 = -\sum_{\sigma=\uparrow,\downarrow} \int dx v_F^\sigma \left[R_\sigma^\dagger(x) i\partial_x R_\sigma(x) - L_\sigma^\dagger(x) i\partial_x L_\sigma(x) \right]. \tag{2.4}$$

It has been shown that the non-interacting hopping Hamiltonian

$$H_0 = -t \sum_x \sum_{\sigma=\uparrow,\downarrow} \left[\psi_\sigma^+(x) \psi_\sigma(x+a) + H.C. \right] \tag{2.5}$$

is equivalent to eq. (2.4) in its continuous limit [Eme79], where $\psi_\sigma^+(x)$ creates a particle at site x, $\psi_\sigma(x+a)$ destroys a particle at site $x+a$ (a being the inter-site distance), and $v_F = 2t$ [Voi92]. By following the method discussed in Appendix A and define the spin-dependent left and right movers as

$$R_\sigma(x) = \frac{1}{\sqrt{2\pi\alpha}} e^{i\sqrt{4\pi}\theta_{R,\sigma}(x)},\ L_\sigma(x) = \frac{1}{\sqrt{2\pi\alpha}} e^{-i\sqrt{4\pi}\theta_{R,\sigma}(x)}, \tag{2.6}$$

we can bosonize the free Hamiltonian in eq. (2.4) and obtain

$$H_0 = \sum_{\sigma=\uparrow,\downarrow} \int dx\, v_F^\sigma \left[\left(\partial_x \theta_{R,\sigma}(x)\right)^2 + \left(\partial_x \theta_{L,\sigma}(x)\right)^2 \right]. \tag{2.7}$$

2.1.2 Electron-Electron Interaction

It is known that a one-dimensional wire does not polarize by considering only the Hubbard interaction [KF92a]. In order to study the polarization effect, we need to consider the extended Hubbard interaction. The extended Hubbard interaction term is given by

$$H_I = U \sum_x n_\uparrow(x) n_\downarrow(x) + V \sum_x n(x) n(x+a). \tag{2.8}$$

Here, U is the on-site (Hubbard) interaction and $n_\sigma(x) = \psi_\sigma^+(x)\psi_\sigma(x)$, while V is the nearest neighbor (extended Hubbard) interaction and $n(x) = \sum_\sigma n_\sigma(x)$. In the continuum limit of the extended Hubbard interaction [Voi92], we have

$$\begin{aligned} H_I \;=\; & \sum_{\sigma,\sigma'=\uparrow,\downarrow} \int dx \Big\{ g_1 R_\sigma^\dagger(x) L_{\sigma'}^\dagger(x) R_{\sigma'}(x) L_\sigma(x) + g_2 R_\sigma^\dagger(x) L_{\sigma'}^\dagger(x) L_{\sigma'}(x) R_\sigma(x) \\ & + g_4 (1-\delta_{\sigma,\sigma'}) \left[R_\sigma^\dagger(x) R_{\sigma'}^\dagger(x) R_{\sigma'}(x) R_\sigma(x) + \ (R \leftrightarrow L) \ \right] \Big\} \end{aligned} \tag{2.9}$$

where R_σ and L_σ are the right and left mover, respectively. Here, the g_1 term represents backward scattering which transfers momentum from $(k_F, -k_F)$ to $(-k_F, k_F)$; the g_2 term represents forward scattering which transfers momentum from $(-k_F, k_F)$ to $(-k_F, k_F)$; and the g_4 term represents Umklapp scattering process. [Sch02, LE74]

We can now bosonize the extended Hubbard interaction term given in eq. (2.9). We find [Sch02]

$$\begin{aligned} H_I &= \int dx \left\{ \left(\frac{g_2^{\parallel} - g_1^{\parallel}}{2\pi} \right) [(\partial_x \theta_{R\uparrow}(x))(\partial_x \theta_{L\uparrow}(x)) + (\partial_x \theta_{R\downarrow}(x))(\partial_x \theta_{L\downarrow}(x))] \right. \\ &\quad + \left(\frac{g_2^{\perp}}{2\pi} \right) [(\partial_x \theta_{R\uparrow}(x))(\partial_x \theta_{L\downarrow}(x)) + (\partial_x \theta_{R\downarrow}(x))(\partial_x \theta_{L\uparrow}(x))] \\ &\quad + \left(\frac{g_4}{2\pi} \right) [(\partial_x \theta_{R\uparrow}(x))(\partial_x \theta_{R\downarrow}(x)) + (\partial_x \theta_{L\uparrow}(x))(\partial_x \theta_{L\downarrow}(x))] \\ &\quad \left. + \left(\frac{g_1^{\perp}}{2(\pi d)^2} \right) \cos \left[\sqrt{4\pi}(\theta_{R\uparrow} + \theta_{L\uparrow} - \theta_{R\downarrow} - \theta_{L\downarrow}) + \Delta(\frac{k_F}{v_F})x \right] \right\} \quad (2.10) \end{aligned}$$

Here, the parameters $g_1^{\parallel}$, $g_1^{\perp}$, $g_2^{\parallel}$, $g_2^{\perp}$, and g_4 are related to the on-site interaction U and the inter-site interaction V of the extended Hubbard interaction [Voi92], i.e. $g_2^{\parallel} = g_2^{\perp} = U + 2V$, $g_1^{\parallel} = g_1^{\perp} = U + 2V\cos(k_F a) \simeq U - 2V$, and $g_2^{\parallel} - g_1^{\parallel} = 4V$. Accordingly, the $g_2^{\parallel} - g_1^{\parallel}$ term in eq. 2.10 is the only term that relates solely with the inter-site interaction V. We note that this $g_2^{\parallel} - g_1^{\parallel}$ term is the interaction without spin flips, which gives rise to a spin polarized Luttinger liquid.

For a Hubbard model ($V = 0$), we have $g_2^{\parallel} - g_1^{\parallel} = 4V = 0$. The presence of the parameter $\Delta(k_F/v_F)x$, $k_F = (k_F^{\uparrow} + k_F^{\downarrow})/2$, gives rise to strong oscillation which allows us to neglect the last term in eq. (2.10) at large distances [Sch02].

For a pure extended Hubbard model ($V \neq 0$), $g_2^{\parallel} - g_1^{\parallel} = 4V \neq 0$. We denote $g_2^{\parallel} - g_1^{\parallel} \equiv g_0$ for simplicity. Notice that this g_0 term is the only parameter relevant to the polarization of the magnetic wire. All other terms are not sensitive to the externally applied magnetic field. Therefore, only the inclusion of the extended Hubbard interaction can give rise to a spin-polarized quantum wire.

2.1.3 Impurity Potential

The impurity potential for fermions are given as

$$H_{Imp} = \sum_{\sigma} \int dx V_{\sigma}(x) \left[\psi_{\sigma}^{\dagger}(x)\psi_{\sigma}(x) + h.c. \right], \tag{2.11}$$

where $V_{\sigma}(x)$ is non-zero in a region of a finite distance d and zero otherwise. We shall consider $V_{\sigma}(x) = V_{\sigma}\delta(x)$, which represents a non-magnetic impurity located at $x = 0$. By substituting the linearized Fermion field given in eq. (2.1) into eq. (2.11), we obtain

$$\begin{aligned} H_{imp} &= H_{imp}^{(f)} + H_{imp}^{(b)} && (2.12)\\ H_{imp}^{(f)} &= \sum_{\sigma} \int dx V_f(x) \left(R_{\sigma}^{\dagger}(x)R_{\sigma}(x) + L_{\sigma}^{\dagger}(x)L_{\sigma}(x) \right) && (2.13)\\ H_{imp}^{(b)} &= \sum_{\sigma} \int dx \left[V_b(x)L_{\sigma}^{\dagger}(x)R_{\sigma}(x) + H.C. \right] && (2.14) \end{aligned}$$

where $V_f(x) = V_{\sigma}(x)$ is the amplitude of the forward scattering potential and $V_b(x) = e^{2ik_F x}V_{\sigma}(x)$ is the amplitude of the backward scattering potential. Since the forward scattering potential can be absorbed to the free Hamiltonian by a canonical transformation [GNT98], we can consider the impurity

potential as if it includes only the backward scattering potential. Therefore, we will omit the discussion of the forward scattering potential, and focus only on the the backward scattering potential.

Using the standard bosonization method:

$$R_\sigma^\dagger = \frac{1}{\sqrt{2\pi\alpha}}\, e^{i\sqrt{4\pi}\theta_{R,\sigma}(x)}; \quad L_\sigma = \frac{1}{\sqrt{2\pi\alpha}}\, e^{-i\sqrt{4\pi}\theta_{L,\sigma}(x)}, \tag{2.15}$$

we have,

$$\begin{aligned} & V_b(x)R_\sigma^\dagger L_\sigma + V_b^\star(x)L_\sigma^\dagger R_\sigma \\ = \;& \frac{1}{2\pi\alpha}\left[e^{2ik_Fx}V_\sigma(x)e^{-i\sqrt{4\pi}(\theta_{R,\sigma}(x)+\theta_{L,\sigma}(x))} + e^{-2ik_Fx}V_\sigma(x)e^{i\sqrt{4\pi}(\theta_{R,\sigma}(x)+\theta_{L,\sigma}(x))}\right] \\ = \;& V_\sigma(x)\frac{1}{\pi\alpha}\cos(\sqrt{4\pi}\theta_\sigma(x) - 2k_Fx) \end{aligned} \tag{2.16}$$

where $\theta_{R,\sigma}(x) + \theta_{L,\sigma}(x) = \theta_\sigma(x)$.

We can now redefine $U_\sigma(x) = V_\sigma(x)/\pi\alpha$, and rewrite the backward scattering potential in its bososic form as

$$H_{Imp} = \sum_\sigma \int dx U_\sigma(x) \cos\left(\sqrt{4\pi}(\theta_{R,\sigma}(x) + \theta_{L,\sigma}(x))\right) \tag{2.17}$$

where $U_\sigma(x) = \lambda_\sigma\delta(x)$, and $\lambda_\uparrow = \lambda_\downarrow$. It is noted that the impurity potential is non-zero only at $x = 0$. We can thus require $2k_Fx = 0$ in eq. (2.16)

2.2 Spin Polarized Luttinger Liquid

We can now transform the right and left moving bosonic fields into spin-up field θ_σ and spin-down field ϕ_σ, namely

$$\theta_\sigma = \theta_{R\sigma} + \theta_{L\sigma},$$

$$\phi_\sigma = \theta_{R\sigma} - \theta_{L\sigma}.$$

We can further transform the spin-up and spin-down fields (θ_σ and ϕ_σ) into spin and charge fields, namely

$$\theta_c = \frac{1}{\sqrt{2}}(\theta_\uparrow + \theta_\downarrow),$$
$$\theta_s = \frac{1}{\sqrt{2}}(\theta_\uparrow - \theta_\downarrow),$$
$$\phi_c = \frac{1}{\sqrt{2}}(\phi_\uparrow + \phi_\downarrow),$$
$$\phi_s = \frac{1}{\sqrt{2}}(\phi_\uparrow - \phi_\downarrow).$$

Using the transformations given above, we can write the free Hamiltonian H_0 in eq. (2.7) and the interaction Hamiltonian H_I in eq. (2.10) in terms of the spin and charge fields (θ_c, θ_s, ϕ_c, ϕ_s). Because $\partial_x\theta_c(x)$ and $\partial_x\theta_s(x)$ are canonically conjugated with $\partial_x\phi_c(x)$ and $\partial_x\phi_s(x)$, we define $P_c(x) = \partial_x\phi_c(x)$, and $P_s(x) = \partial_x\phi_s(x)$. The total Hamiltonian becomes $H = H_0 + H_I = H_c + H_s + H_{c/s}$, where

$$H_c = \int dx \frac{v_c}{2}\left[K_c P_c^2(x) + \frac{1}{K_c}(\partial_x\theta_c)^2\right], \tag{2.18}$$
$$H_s = \int dx \frac{v_s}{2}\left[K_s P_s^2(x) + \frac{1}{K_s}(\partial_x\theta_s)^2\right], \tag{2.19}$$
$$H_{c/s}^Z = \frac{\Delta}{2}\int dx\,[P_c(x)P_s(x) + (\partial_x\theta_c)(\partial_s\theta_s)] \tag{2.20}$$

and

$$K_c = \sqrt{\frac{1 - \frac{1}{4\pi}(g_0 + g_2^\perp - g_4)}{1 + \frac{1}{4\pi}(g_0 + g_2^\perp + g_4)}}, \tag{2.21}$$
$$K_s = \sqrt{\frac{1 - \frac{1}{4\pi}(g_0 - g_2^\perp + g_4)}{1 + \frac{1}{4\pi}(g_0 - g_2^\perp - g_4)}}, \tag{2.22}$$

$$v_c = \sqrt{(1-\frac{1}{4\pi}(g_0+g_2^{\perp}-g_4))(1+\frac{1}{4\pi}(g_0+g_2^{\perp}+g_4))}, \quad (2.23)$$

$$v_s = \sqrt{(1-\frac{1}{4\pi}(g_0-g_2^{\perp}+g_4))(1+\frac{1}{4\pi}(g_0-g_2^{\perp}-g_4))} \quad (2.24)$$

We shall keep H_{Imp} in its original form of eq. (2.17) and perform Renormalization Group analysis. We observe that the Hamiltonian given in eqs. (2.18, 2.19, 2.20) is non-diagonal in the conjugate momenta, $P_c(x)P_x(x)$, and densities, $\partial_x\theta_c(x)\partial_x\theta_s(x)$. Therefore, we can write the total Hamiltonian in matrix form, namely

$$H = \frac{1}{2}\int dx \left[(P_c(x), P_s(x)) \cdot \mathbf{M}^{-1} \cdot \begin{pmatrix} P_c(x) \\ P_s(x) \end{pmatrix} + (\partial_x\theta_c(x), \partial_x\theta_s(x)) \cdot \mathbf{B} \cdot \begin{pmatrix} \partial_x\theta_c(x) \\ \partial_x\theta_s(x) \end{pmatrix} \right] \quad (2.25)$$

where

$$\mathbf{M}^{-1} = \begin{pmatrix} v_cK_c & \frac{\Delta}{2} \\ \frac{\Delta}{2} & v_sK_s \end{pmatrix},$$

and

$$\mathbf{B} = \begin{pmatrix} v_c/K_c & \frac{\Delta}{2} \\ \frac{\Delta}{2} & v_s/K_s \end{pmatrix}.$$

To diagonalize the Hamiltonian in eq. (2.25), we need to find two unitary matrices $\mathbf{U}$ and $\mathbf{V}$, such that $\mathbf{U}^+\mathbf{M}^{-1}\mathbf{U}$ and $\mathbf{V}^+\mathbf{B}\mathbf{V}$ are diagonal. In order to preserve the commutation rules $[\partial_x\theta_\alpha(x), P_\beta(x')] = i\delta_{\alpha,\beta}\delta(x-x')$, where $\alpha,\beta = c,s$, it is required that $\mathbf{U}\mathbf{V}^+ = 1$. However, this is not possible for an arbitrary set of parameters v_c, K_c, v_s, K_s and Δ. Therefore, the method described in [KKA96] is not applicable for an arbitrary set of parameters.

2.3 Renormalization Analysis

In order not to be restricted, we will use the Lagrangian formalism, which allows us to calculate in the Fourier transformed space (ω, k). The Lagrangian density $\mathcal{L}$ is defined as

$$\mathcal{L}(\theta_c, \dot{\theta}_c, \theta_s, \dot{\theta}_s) = P_c\dot{\theta}_c + P_s\dot{\theta}_s - h(P_c, \theta_c, P_s, \theta_s) \tag{2.26}$$

Since $\dot{\theta}_c = \frac{\delta}{\delta P_c}H$ and $\dot{\theta}_s = \frac{\delta}{\delta P_s}H$, we have $\dot{\theta}_c = v_cK_cP_c + \frac{\Delta}{2}P_s$ and $\dot{\theta}_s = v_sK_sP_s + \frac{\Delta}{2}P_c$. We can write

$$\begin{pmatrix} \dot{\theta}_c \\ \dot{\theta}_s \end{pmatrix} = \mathbf{M}^{-1} \begin{pmatrix} P_c \\ P_s \end{pmatrix},$$

where

$$\mathbf{M}^{-1} = \begin{pmatrix} v_cK_c & \frac{\Delta}{2} \\ \frac{\Delta}{2} & v_sK_s \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} P_c \\ P_s \end{pmatrix} = \mathbf{M} \begin{pmatrix} \dot{\theta}_c \\ \dot{\theta}_s \end{pmatrix} = \frac{1}{\det \mathbf{M}^{-1}} \begin{pmatrix} v_sK_s & -\frac{\Delta}{2} \\ -\frac{\Delta}{2} & v_cK_c \end{pmatrix} \begin{pmatrix} \dot{\theta}_c \\ \dot{\theta}_s \end{pmatrix}.$$

Let $z^{-1} = \det \mathbf{M}^{-1} = (v_cK_c)(v_sK_s) - (\Delta/2)^2$, we should require that $z^{-1} \neq 0$. Notice that the $z^{-1} = 0$ case corresponds to the fact that the fields $\dot{\theta}_c$ and $\dot{\theta}_s$ (or P_c and P_s) are linearly dependent.

We restrict our calculation to $\det \mathbf{M}^{-1} > 0$. ($\det \mathbf{M}^{-1} < 0$ describes an unstable liquid with negative masses.) Provided the above discussion with the aid of Appendix C.2, we obtain the Lagrangian density as

$$\mathcal{L}(\theta_c, \dot{\theta}_c, \theta_s, \dot{\theta}_s) = \frac{1}{2}(\dot{\theta}_c, \dot{\theta}_s) \cdot \mathbf{M} \cdot \begin{pmatrix} \dot{\theta}_c \\ \dot{\theta}_s \end{pmatrix} - \frac{1}{2}(\partial_x\theta_c, \partial_x\theta_s) \cdot \mathbf{M}^{-1} \cdot \begin{pmatrix} \partial_x\theta_c \\ \partial_x\theta_s \end{pmatrix}.$$

We can write the action in terms of Fourier components of the above Lagrangian density as

$$S = \int \frac{d\omega}{2\pi}\frac{dq}{2\pi}\mathcal{L}(\omega, q)$$

where

$$\mathcal{L}(\omega, q) = \frac{1}{2}\,\vec{\theta}(\omega, q)\cdot\left(\mathbf{A}^{-1}\right)\cdot\vec{\theta}(-\omega, -q) \tag{2.27}$$

and

$$\mathbf{A}^{-1} = \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix} \tag{2.28}$$

Here we denote $\vec{\theta} = (\theta_c, \theta_s)$, and the elements of matrix $\mathbf{A}^{-1}$ are given as

$$\begin{aligned} \alpha(\omega, q) &= -(zv_sK_s)\omega^2 + (\frac{v_c}{K_c})q^2 \\ \beta(\omega, q) &= -(zv_cK_c)\omega^2 + (\frac{v_s}{K_s})q^2 \\ \gamma(\omega, q) &= \frac{\Delta}{2}(z\omega^2 + q^2) \end{aligned} \tag{2.29}$$

In order to compute the conductance due to the impurity, we add to the total Hamiltonian the impurity potential

$$H_{Imp} = \sum_\sigma \int dx\, \lambda_\sigma \delta(x) \cos\left(\sqrt{2\pi}(\theta_c(x) + \sigma\theta_s(x))\right) \tag{2.30}$$

with the condition of $\lambda_\uparrow = \lambda_\downarrow$. Due to the electron-electron interaction, the bare impurity strength get renormalized. At length scale $a' = ba$ where $b = e^\ell > 1$, we find that $\lambda_\sigma(\ell) = \hat{\lambda}_\sigma\Lambda\exp[(1 - K_\sigma)\ell]$, i.e. when $K_\sigma < 1$, $\lambda_\sigma(\ell)$ increases. On the contrary, λ_σ decrease when $K_\sigma > 1$. The scaling equation of $\lambda_\sigma = \hat{\lambda}_\sigma\Lambda$ is [Ami78]

$$\frac{d\hat{\lambda}_\sigma}{d\ell} = \hat{\lambda}_\sigma(1 - K_\sigma) \tag{2.31}$$

and K_σ is given by

$$\begin{aligned} d\ell K_\sigma &\equiv \frac{1}{2}(\sqrt{2\pi})^2 \left[\langle \delta\theta_c^2(x)\rangle + \langle \delta\theta_s^2(x)\rangle + 2\sigma\langle \delta\theta_c(x)\delta\theta_s(x)\rangle\right] \\ &\equiv 2\pi\langle \delta\theta_\sigma(x)\delta\theta_\sigma(x)\rangle \end{aligned} \tag{2.32}$$

Using the Lagrangian in eq. (2.27), we will compute the correlation functions $\langle \delta\theta_\sigma(x)\delta\theta_\sigma(x)\rangle$. These correlation functions will be expressed in terms of the parameters α,β,γ (see eq. (2.29)).

Denote that $i, j \equiv (c, s)$, we can transform the correlation function back to the space-time coordinate. We have

$$\begin{aligned} \langle \theta_i(x,t)\theta_j(0,0)\rangle &= \int \frac{d\omega}{2\pi}\frac{dq}{2\pi} e^{-i(qx-\omega t)}\langle \theta_i(\omega,q)\theta_j(-\omega,-q)\rangle \\ &= i\int \frac{d\omega}{2\pi}\frac{dq}{2\pi} e^{-i(qx-\omega t)}\mathbf{A}_{ij}(\omega,q) \equiv i\mathbf{A}_{ij}(x,t) \end{aligned} \tag{2.33}$$

Since for any function $f(|q|)$, the integral in the momentum shell gives $\int_{\Lambda/s\le|q|\le\Lambda} f(|q|)dq = \int_{-\Lambda}^{-\Lambda/s} + \int_{\Lambda/s}^{\Lambda} = 2f(\Lambda)$, we can obtain the correlation functions at the same space point as

$$\langle \delta\theta_i(x,\tau)\delta\theta_j(x,0)\rangle = \frac{id\Lambda}{\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi} e^{i\omega\tau}\mathbf{A}_{ij}(\omega,\Lambda) \tag{2.34}$$

Since $\delta\theta_\uparrow = \frac{1}{\sqrt{2}}(\delta\theta_c + \delta\theta_s)$ and $\delta\theta_\downarrow = \frac{1}{\sqrt{2}}(\delta\theta_c - \delta\theta_s)$, we have

$$\langle \delta\theta_\uparrow\delta\theta_\uparrow\rangle = \frac{1}{2}\left[\langle \delta\theta_c\delta\theta_c\rangle + \langle \delta\theta_s\delta\theta_s\rangle + \langle \delta\theta_c\delta\theta_s\rangle\right], \tag{2.35}$$

$$\langle \delta\theta_\downarrow\delta\theta_\downarrow\rangle = \frac{1}{2}\left[\langle \delta\theta_c\delta\theta_c\rangle + \langle \delta\theta_s\delta\theta_s\rangle - \langle \delta\theta_c\delta\theta_s\rangle\right] \tag{2.36}$$

This is equivalent to

$$\langle \delta\theta_\uparrow\delta\theta_\uparrow\rangle = \frac{id\Lambda}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi} e^{i\omega\tau} f_{\uparrow\uparrow}(\omega,\Lambda) \tag{2.37}$$

and

$$\langle \delta\theta_\downarrow \delta\theta_\downarrow \rangle = \frac{id\Lambda}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega\tau} f_{\downarrow\downarrow}(\omega, \Lambda) \tag{2.38}$$

where $f_{\uparrow\uparrow}(\omega, \Lambda) \equiv \frac{\alpha+\beta-2\gamma}{\alpha\beta-\gamma^2}$, and $f_{\downarrow\downarrow}(\omega, \Lambda) \equiv \frac{\alpha+\beta+2\gamma}{\alpha\beta-\gamma^2}$.

Now we need to carry out the definite integrals in eq. (2.37) and eq. (2.38). These can be done by using the Cauchy Residue Theorem. By following the requirement that $\alpha\beta - \gamma^2 = 0$, we can obtain four different poles. They are $\omega_1 = \frac{\Lambda}{2}\sqrt{u_1 + u_2}$, $\omega_2 = -\frac{\Lambda}{2}\sqrt{u_1 + u_2}$, $\omega_3 = \frac{\Lambda}{2}\sqrt{u_1 - u_2}$, $\omega_4 = -\frac{\Lambda}{2}\sqrt{u_1 - u_2}$, where $u_1 = 2(v_c^2 + v_s^2) + \Delta^2$ and $u_2 = 2[(v_c^2 - v_s^2)^2 + (K_c K_s v_c + v_s)(K_s K_c v_s + v_c)\Delta^2/(K_c K_s)]^{1/2}$. Notice that these poles are located on the real axis and that $\omega_2 = -\omega_1$ and $\omega_4 = -\omega_3$. Due to the factor $e^{i\omega\tau}$ inside of both integrals, $\tau > 0$, we need to choose the contour around the upper hemisphere and shift ω_1, ω_3 by a small positive imaginary number and ω_2, ω_4 by a small negative imaginary number. Thus, only the poles at ω_1 and ω_3 are included in the chosen contour. We get

$$\langle \delta\theta_\uparrow \delta\theta_\uparrow \rangle = -\frac{d\Lambda}{2\pi} \left[Res\{f_{\uparrow\uparrow}(\omega_1)\} + Res\{f_{\uparrow\uparrow}(\omega_3)\} \right]$$
$$\langle \delta\theta_\downarrow \delta\theta_\downarrow \rangle = -\frac{d\Lambda}{2\pi} \left[Res\{f_{\downarrow\downarrow}(\omega_1)\} + Res\{f_{\downarrow\downarrow}(\omega_3)\} \right]$$

By carefully calculating the residues of $f_{\uparrow\uparrow}$ and $f_{\downarrow\downarrow}$, one can obtain that

$$\begin{aligned} \langle \delta\theta_\uparrow \delta\theta_\uparrow \rangle &= \frac{d\Lambda}{2\pi\Lambda} K_\uparrow \\ &= \frac{d\Lambda}{2\pi\Lambda} \left[\frac{a}{2u_2}(\sqrt{u_1 + u_2} - \sqrt{u_1 - u_2}) - \frac{2b}{u_2}\left(\frac{1}{\sqrt{u_1 + u_2}} - \frac{1}{\sqrt{u_1 - u_2}}\right) \right] \end{aligned} \tag{2.39}$$

and

$$\langle \delta\theta_\downarrow \delta\theta_\downarrow \rangle = \frac{d\Lambda}{2\pi\Lambda} K_\downarrow \tag{2.40}$$

$$= \frac{d\Lambda}{2\pi\Lambda}\left[\frac{c}{2u_2}(\sqrt{u_1+u_2}-\sqrt{u_1-u_2}) - \frac{2d}{u_2}(\frac{1}{\sqrt{u_1+u_2}}-\frac{1}{\sqrt{u_1-u_2}})\right]$$

where

$$a = (K_c v_c + K_s v_s + \Delta), \tag{2.41}$$

$$b = (\frac{v_s}{K_s} + \frac{v_c}{K_c} - \Delta)/z, \tag{2.42}$$

$$c = (K_c v_c + K_s v_s - \Delta), \tag{2.43}$$

$$d = (\frac{v_s}{K_s} + \frac{v_c}{K_c} + \Delta)/z. \tag{2.44}$$

We can now write the $K_\uparrow$ and $K_\downarrow$ as

$$\begin{aligned} K_\uparrow &= \frac{a}{2u_2}(\sqrt{u_1+u_2}-\sqrt{u_1-u_2}) \\ &- \frac{2b}{u_2}(\frac{1}{\sqrt{u_1+u_2}}-\frac{1}{\sqrt{u_1-u_2}}), \text{ and} \end{aligned} \tag{2.45}$$

$$\begin{aligned} K_\downarrow &= \frac{c}{2u_2}(\sqrt{u_1+u_2}-\sqrt{u_1-u_2}) \\ &- \frac{2d}{u_2}(\frac{1}{\sqrt{u_1+u_2}}-\frac{1}{\sqrt{u_1-u_2}}). \end{aligned} \tag{2.46}$$

By increasing Δ (the magnetic field), we have found that $K_\uparrow \neq K_\downarrow$ when $g_0 \neq 0$. The conductances for spin up and spin down electrons due to a weak impurity (i.e. $\hat{\lambda}_\sigma$ small) is then given by [KF92b]

$$G_\uparrow = \frac{e^2}{h}\left(1 - |\hat{\lambda}_\uparrow|^2 \left(\frac{T}{T_F}\right)^{2(K_\uparrow - 1)}\right), \tag{2.47}$$

$$G_\downarrow = \frac{e^2}{h}\left(1 - |\hat{\lambda}_\downarrow|^2 \left(\frac{T}{T_F}\right)^{2(K_\downarrow - 1)}\right). \tag{2.48}$$

When the impurity potential $\hat{\lambda}_\sigma$ is large, the result given in eqs. (2.47) and (2.48) are not valid. Using the fact that the scattering exponents computed in eqs. (2.45) and (2.46) holds for any $\hat{\lambda}_\sigma$, we can map the impurity into a

weak link. Assuming that the problem on the lattice has an hopping constant $t \sim 1$, the matrix element through the impurity will be $t' \sim \frac{t^2}{|\hat{\lambda}_\sigma|} \sim \frac{1}{|\hat{\lambda}_\sigma|}$. The conductance for the weak link $t' \sim \frac{1}{|\hat{\lambda}_\sigma|}$ has been considered in ref. [Sch02]. Using the result in ref. [Sch02] plus the results for the exponents $K_\uparrow$ and $K_\downarrow$ given in eqs. (2.45) and (2.46), we find:

$$G_\uparrow = \frac{e^2}{h}|\frac{1}{\hat{\lambda}_\uparrow}|^2 \left(\frac{T}{T_F}\right)^{2(\frac{1}{K_\uparrow}-1)}, \tag{2.49}$$

$$G_\downarrow = \frac{e^2}{h}|\frac{1}{\hat{\lambda}_\downarrow}|^2 \left(\frac{T}{T_F}\right)^{2(\frac{1}{K_\downarrow}-1)}. \tag{2.50}$$

Due to the fact that $K_\uparrow \neq K_\downarrow$, at low temperatures, i.e. $T_F/T \sim 100 - 1000$, we find one of the conductances is strongly suppressed, which gives rise to a situation that only one spin polarization is conducting. Consequently, a polarized wire is obtained.

2.4 Results and Discussions

Based on the exact results from eq. (2.45) and eq. (2.46), we have plot our resultant $K_\uparrow$ and $K_\downarrow$ as a function of the external magnetic field Δ in figures 2.1, 2.2 and 2.3, and figures 2.4, 2.5 and 2.6. Since the Umklapp scattering term is negligible, we have set $g_4 = 0$ in both figures. In order to demonstrate that only the nearest neighbor interaction $g_0 \equiv g_2^{\parallel} - g_1^{\parallel} \equiv 4V$ term is important for causing $K_\uparrow \neq K_\downarrow$, we set $g_2^\perp = 0.0$ in figures 2.1, 2.2 and 2.3, and $g_2^\perp = 1.0$ in figure 2.4, 2.5, and 2.6.

As shown in figure 2.1, the solid line represents $K_\uparrow$ and the dotted line represents $K_\downarrow$. In this figure, the value of g_0 ($g_0 = 4V$) is set to be 0.1. One

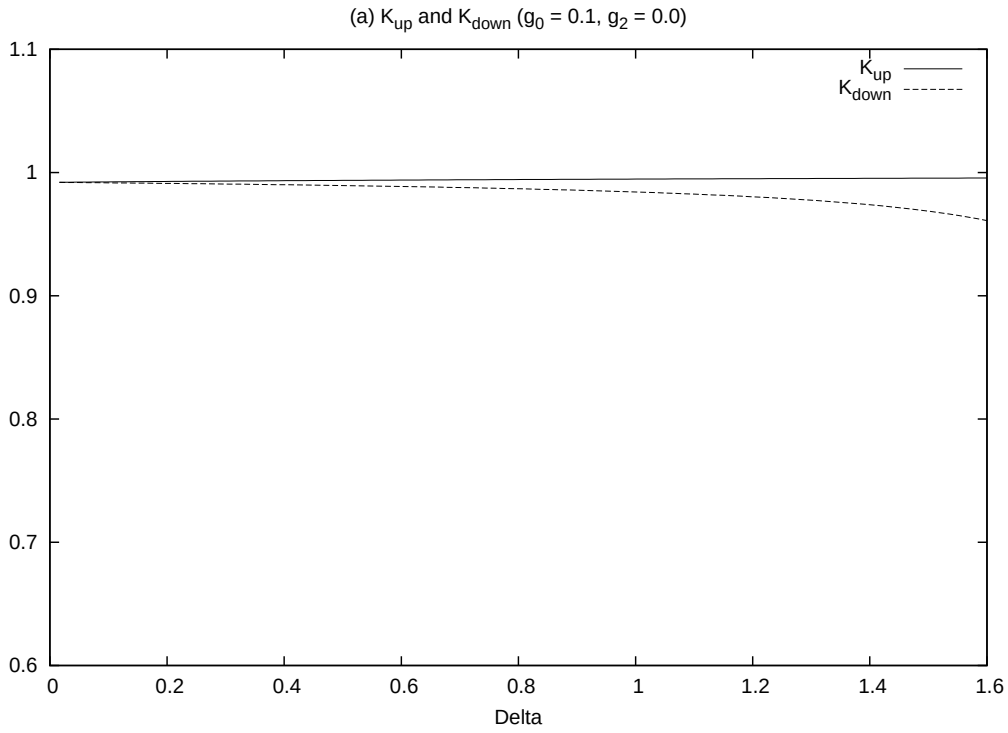

Figure 2.1: The results of $K_\uparrow$ and $K_\downarrow$ by increasing the external magnetic field Δ at $g_2^\perp = 0.0$ and $g_0 = 0.1$.

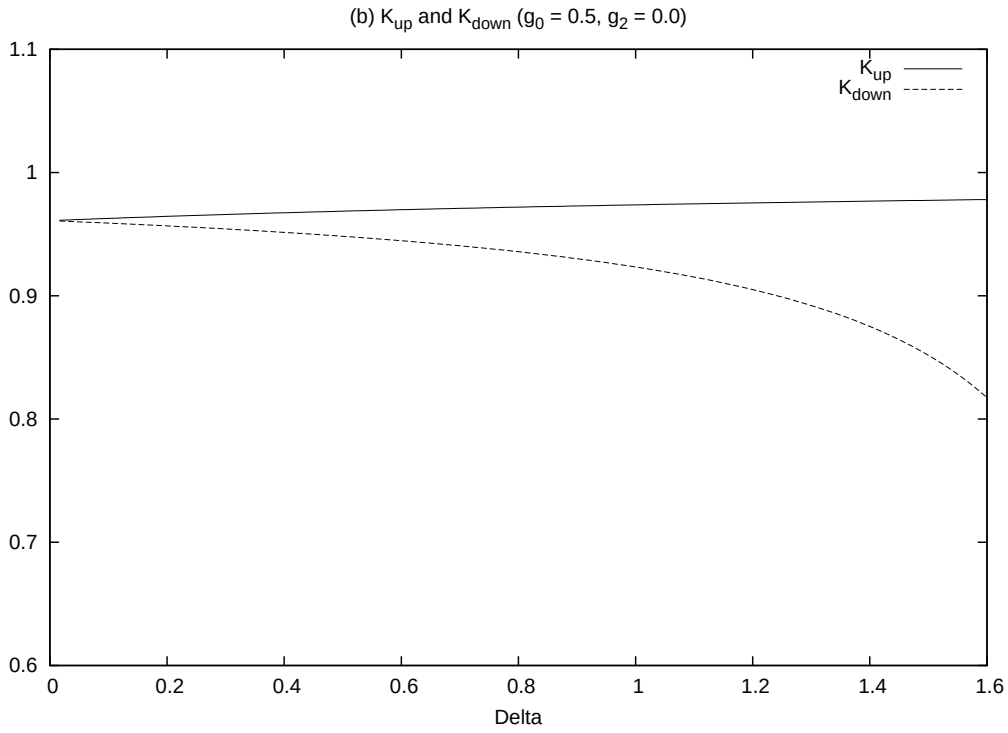

Figure 2.2: The results of $K_\uparrow$ and $K_\downarrow$ by increasing the external magnetic field Δ at $g_2^\perp = 0.0$ and $g_0 = 0.5$.

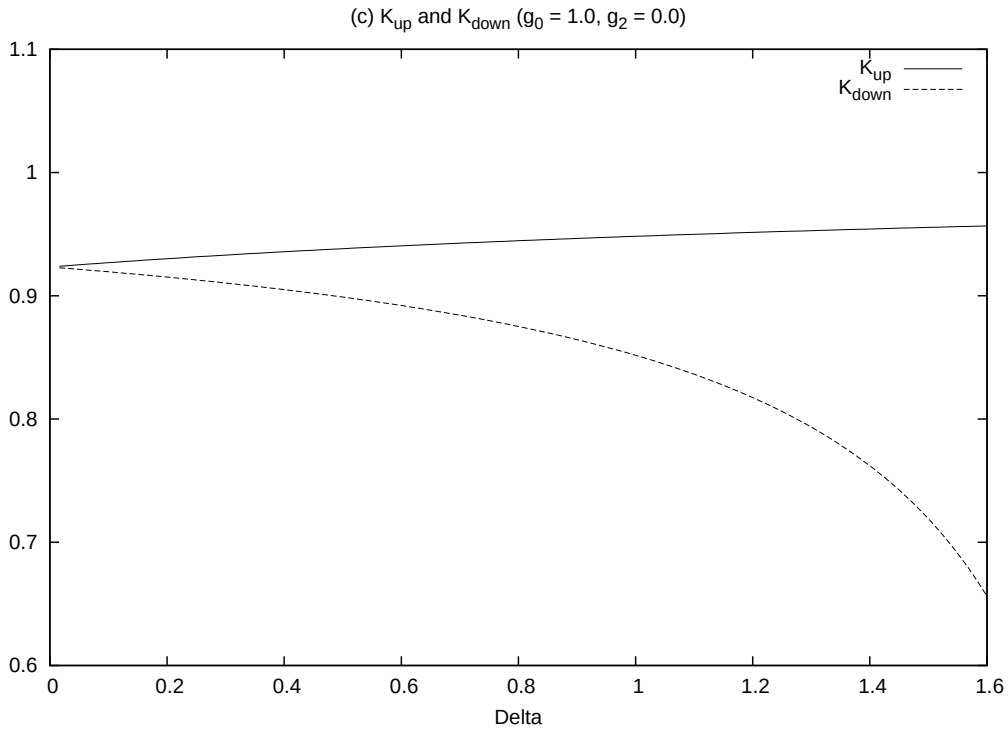

Figure 2.3: The results of $K_\uparrow$ and $K_\downarrow$ by increasing the external magnetic field Δ at $g_2^\perp = 0.0$ and $g_0 = 1.0$.

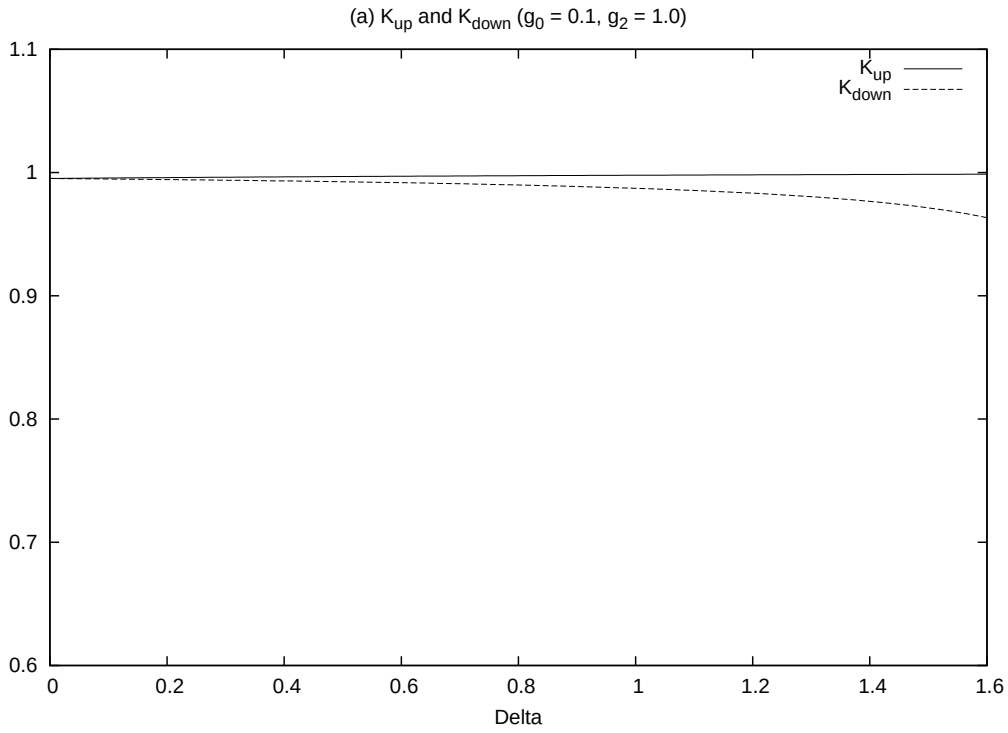

Figure 2.4: The results of $K_\uparrow$ and $K_\downarrow$ by increasing the external magnetic field Δ at $g_2^\perp = 1.0$ and $g_0 = 0.1$.

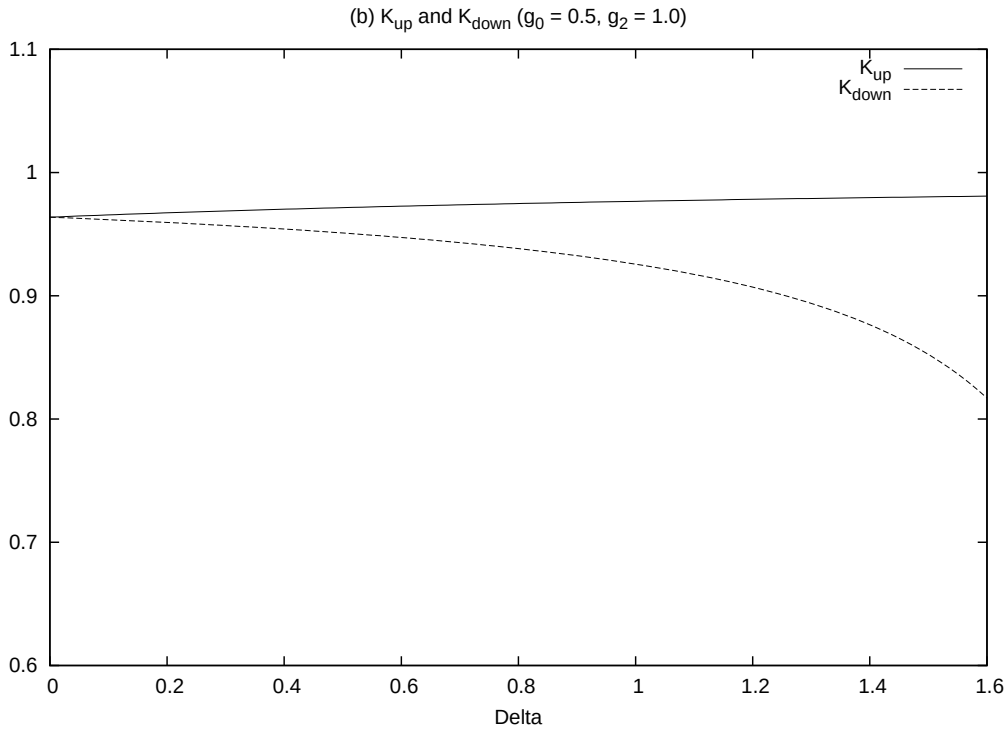

Figure 2.5: The results of $K_\uparrow$ and $K_\downarrow$ by increasing the external magnetic field Δ at $g_2^\perp = 1.0$ and $g_0 = 0.5$.

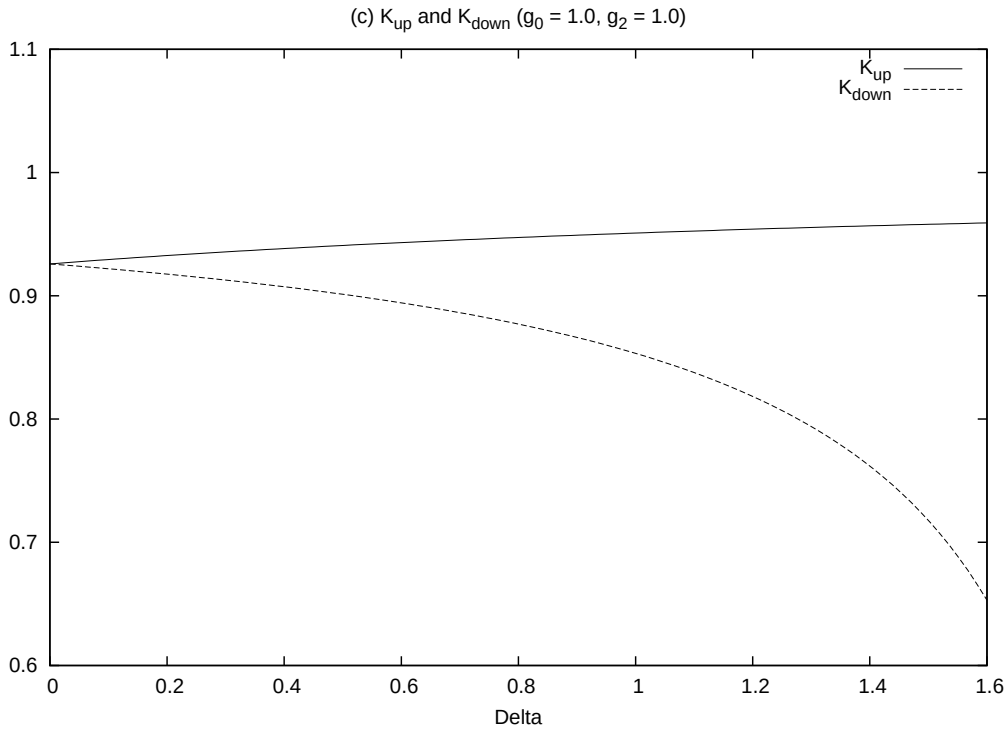

Figure 2.6: The results of $K_\uparrow$ and $K_\downarrow$ by increasing the external magnetic field Δ at $g_2^\perp = 1.0$ and $g_0 = 1.0$.

can clearly see that, as the external magnetic field Δ increases, $K_\uparrow$ and $K_\downarrow$ split only slightly. This indicates that the spin polarization of the quantum wire in this case is rather small. In figure 2.2, we increase the g_0 value to 0.5 and plot again $K_\uparrow$ and $K_\downarrow$ as a function of Δ. We find that $K_\uparrow$ and $K_\downarrow$ split more, which gives rise to a more polarized wire. Now, we increase further the g_0 value to 1.0 in figure 2.3. We find that $K_\uparrow$ and $K_\downarrow$ split even further, giving rise to a even more polarized wire.

In order to show that the $g_2^\perp$ term is indeed not playing a significant role in the polarization effect, we plot figures 2.4, 2.5, and 2.6 by taking $g_2^\perp = 1.0$. Similar to figures 2.1, 2.2, and 2.3, we set: $g_0 = 4V = 0.1$ in figure 2.4; $g_0 = 4V = 0.5$ in figure 2.5; and $g_0 = 4V = 1.0$ in figure 2.6. By simply comparing these two figures, one can easily see that the polarization effect is NOT affected by the inclusion of the $g_2^\perp$ term. For this reason, we can conclude that only in the presence of the $g_0 = 4V$ term and $\Delta \neq 0$ can we obtain a polarized wire.

Furthermore, in order to show that the difference between $G_\uparrow$ and $G_\downarrow$ increases when an external magnetic field Δ is applied, we have prepared figure 2.7 to figure 2.10 illustrating the dependency of polarization $P \equiv |\frac{G_\uparrow - G_\downarrow}{G_\uparrow + G_\downarrow}|$ versus the external magnetic field Δ and the nearest neighbor interaction $g_0 = 4V$.

In the case of a weak impurity potential, the resultant polarization is shown in figure 2.7 and figure 2.8. In figure 2.7, a 3D diagram of the polarization P is plot. The external magnetic field Δ is taken as the x-axis, while the nearest neighbor interaction $g_0 = 4V$ is taken as the y-axis. The range

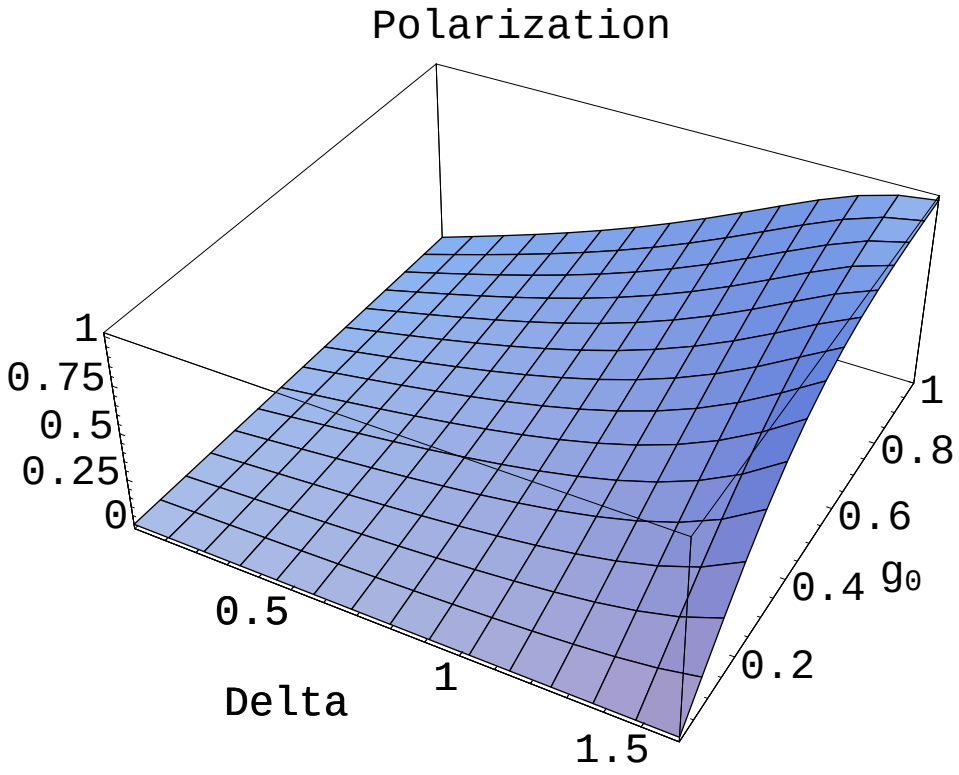

Figure 2.7: 3D diagram of the polarization P in the weak impurity potential limit. Here, we have take the external magnetic field Δ ranging from 0. and 1.6, and the nearest neighbor interaction $g_0 = 4V$ ranging from 0. to 1., where $T = T_F/500$ and $\hat{\lambda}_\sigma = 0.1$ are taken as typical values.

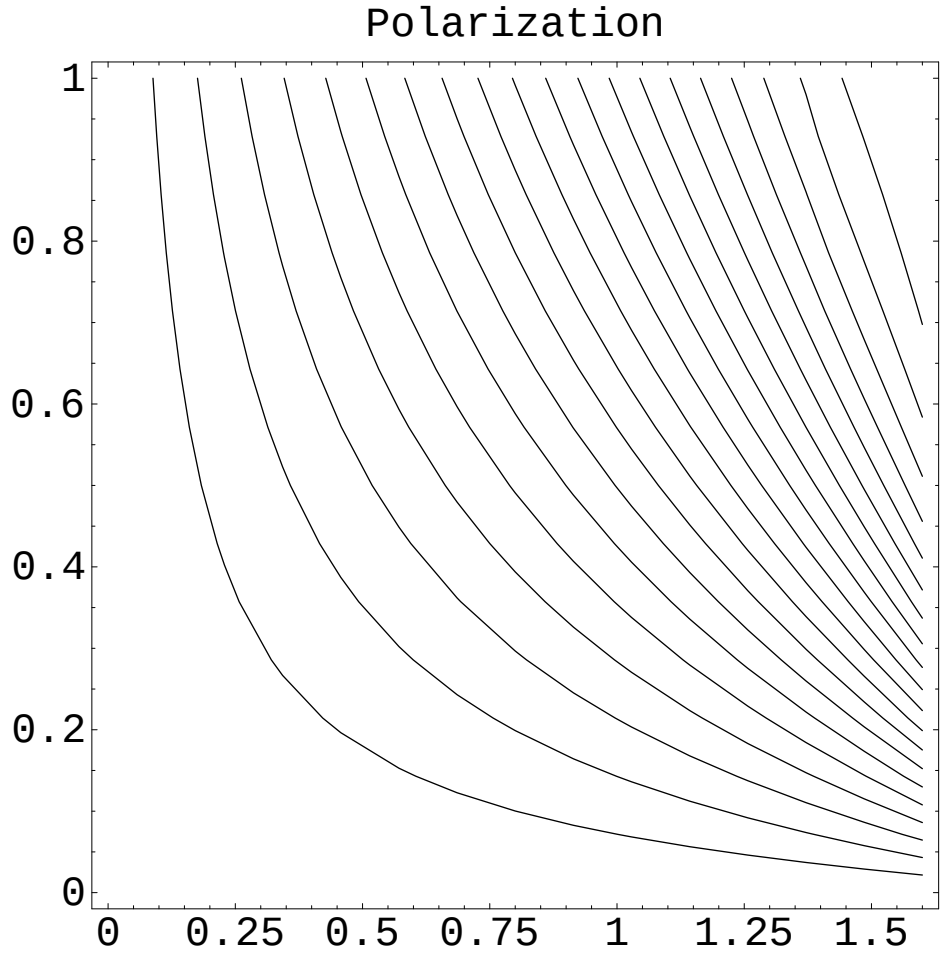

Figure 2.8: Contour diagram of polarization P shown in figure 2.7. The contour line starting from the lower-left side represents the polarization of 5 percent. The next line represents a polarization of 10 percent, and so on.

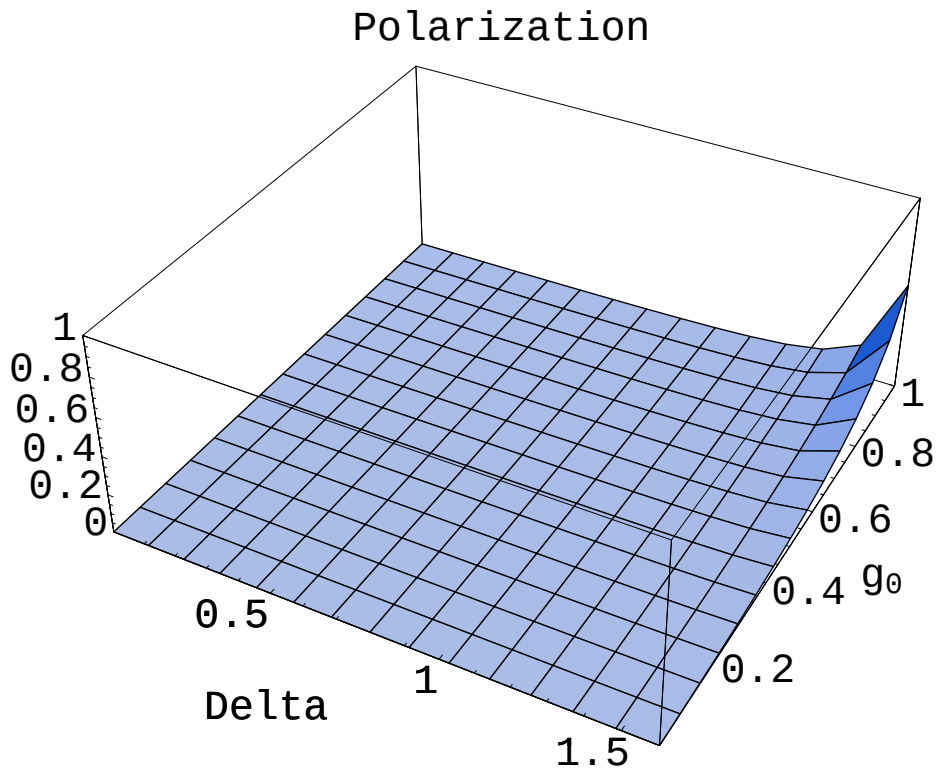

Figure 2.9: 3D diagram of the polarization P in the weak link limit. Here, we have taken the external magnetic field Δ ranging from 0. and 1.6, and the nearest neighbor interaction $g_0 = 4V$ ranging from 0. to 1., where $T = T_F/500$ is taken as a typical value.

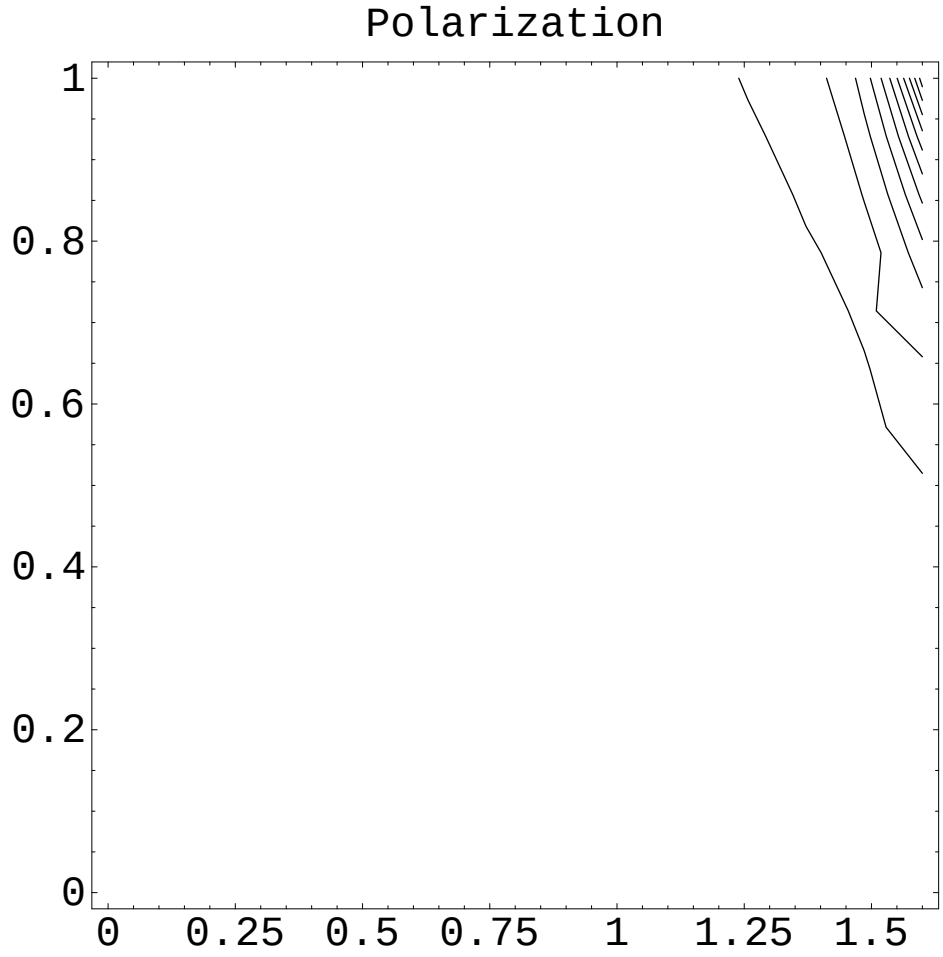

Figure 2.10: Contour diagram of polarization P shown in figure 2.9. The contour line starting from the lower-left side represents the polarization of 5 percent. The next line represents a polarization of 10 percent, and so on.

of Δ is between 0. to 1.6, while the range of g_0 is between 0. and 1., where $g_2^{\perp} = 0$, $T = T_F/500$ and $\hat{\lambda}_\sigma = 0.1$ are taken as typical values. As shown, the larger the external magnetic field Δ is applied to the wire and the more the nearest neighbor interaction is turned on, the more the wire is polarized. Note that the external magnetic field Δ here is taken in the unit of $v_F = 1$. In figure 2.8, a contour plot of figure 2.7 is provided. The x-axis represents the external magnetic field Δ and the y-axis represents the nearest neighbor interaction $4V$. Starting from the lower left side of the figure, the contour lines represents the polarization of 5 percent, 10 percent, 15 percent, and so on. According to our result, the electron gas in the weak impurity potential case is significantly polarized only when the the external magnetic field Δ and the nearest neighbor interaction $4V$ are large.

On the other hand, the resultant polarization for the case of a weak link is shown in figure 2.9 and figure 2.10. Similarly, we take the external magnetic field Δ as the x-axis and the nearest neighbor interaction $g_0 = 4V$ as the y-axis. The range of Δ is between 0. to 1.6, while the range of g_0 is between 0. and 1., where $g_2^{\perp} = 0$ and $T = T_F/500$. In figure 2.9, the 3D diagram clearly shows that the electron gas is significantly polarized when the external magnetic field Δ or the nearest neighbor interaction $4V$ is greater than zero. In figure 2.10, a contour plot of figure 5 is provided. Starting from the lower left side of the figure, the contour lines represent 5 percent, 10 percent, 15 percent, etc., of polarization, respectively. By comparing the results of the weak potential case and that of the weak link case, we can conclude that the stronger the impurity potential, the more the electron gas is polarized.

In real systems and large temperatures, inelastic scattering will give rise to spin relaxation. As a result, our effect will be drastically reduced. However, in the ballistic regime, where the length of the wire L is comparable to the thermal length L_T, $L \approx L_T \approx 10^{-6}m$ ($T \approx 2K^{\circ}$), a "spin polarizer" can be realized.

To conclude, the possibility for a spin polarized quantum wire and the possibility for a "spin polarizer" caused by magnetic field, interaction and impurity are suggested. This effect might play a significant role in spintronics. The results of this chapter are published in [CS05], which is cited by [KTK06].

Chapter 3

Rectification Induced by Microwave in a Confined 2DEG with an Insulating Region

We investigate the effect of a microwave field on a confined two dimensional electron gas which contains an insulating region comparable to the Fermi wavelength. The insulating region causes the electron wave function to vanish in that region. We describe the insulating region as a static vortex, which gives rise to non-commuting kinetic momenta. As a result, a rectified voltage is obtained. The proposed theory may explain some aspects of the experimental results reported by J. Zhang et al. [ZVK$^+$05].

3.1 Introduction

The topology of ground state wave function plays a crucial role in determining the physical properties of many particle systems. These properties are revealed through the quantization rules. It is known that fermions and bosons obey different quantization rules, while the quantized Hall conductance [Koh85] and the value of the spin-Hall conductivity are resulted from the non-commutativity of the Cartesian coordinates [Sch06]. The phenomena of quantum pumping observed in one-dimensional electronic systems [Sch00, Bro98, FSV05] is a consequence of the non-commuting frequency (i.e. $\hat{\omega} = i\frac{\partial}{\partial t}$) and coordinate (i.e. $R = i\frac{\partial}{\partial K}$) [Tho83].

We propose to show that a rectified voltage can be generated in the absence of a magnetic field when one applies a microwave field to a confined 2DEG which contains an insulating region. The proposed theory is inspired by the experimental work of J. Zhang et al. [ZVK$^+$05]. In the experiment, a 2DEG GaAs sample including three insulation regions (or anti-dots) is considered. By applying a polarized microwave field to the sample, a rectified DC voltage is observed in a direction perpendicular to the polarization direction. The experiment has been performed in an external magnetic field and in zero magnetic field. The major result of this experiment is the change of sign of the rectified voltage when the microwave frequency varies from 1.46 GHz to 17.41 GHz. This behavior, we believe, can be understood as being caused by the existence of the insulating regions, which create obstacles for the electrons.

In order to clarify the physical phenomena of the experiment, we consider a simplified situation. Instead of three insulating regions, we consider only one insulating (circular) region of radius D, which is caused by an infinite potential $U_I(r)$, in a confined 2DEG. The 2DEG is confined by a confinement potential causing the electrons to be confined in a restricted region in the 2DEG. The effect of the insulating region of radius D causes the electronic wave function to vanish for $|\vec{r} - \vec{R}| \leq D$. The effect of the vanishing wave function is described effectively by a static vortex with the center at $\vec{r} = \vec{R}$ [Nel02].

The confined 2DEG is characterized by the following features. A high mobility 2DEG GaAs typically has an electronic density of $n \simeq 10^{15} m^{-2}$, a Fermi energy of $E_F \simeq 0.01 eV$, and a Fermi temperature of $T_F \simeq 120 K$. This corresponds to a Fermi wavelength $\lambda_F \simeq 0.5 \times 10^{-7} m$. For high mobility GaAs, the typical scattering time is $\tau \simeq 10^{-11} sec$, which corresponds to the mean free path $\ell = v_F \tau$. The ratio between the mean free path and the Fermi wave length obeys the condition, $\frac{\ell}{\lambda_F} = \frac{v_F \tau}{\lambda_F} = \frac{h\tau}{\lambda_F^2 m} \simeq 30$. Accordingly, we can neglect multiple scattering effects. When the thermal length is comparable with the size of the system $L \simeq \lambda_{thermal} = (\frac{T_F}{T})^{1/2} \lambda_F$, one obtains a ballistic system with negligible multiple scattering.

We describe the confined 2DEG of size L as a system with a parabolic confining potential $V_c(\vec{r}) = \frac{m\omega_0^2}{2} \vec{r}^2$ which has a "classical turning point" L_F determined by the condition $\frac{m\omega_0^2 L_F^2}{2} = E_F$. This condition describes the effective physics of a free electron gas of size $L \leq L_F$. Demanding that L_F is of the order of the thermal wave length $L_F \simeq \lambda_{thermal}$ determines

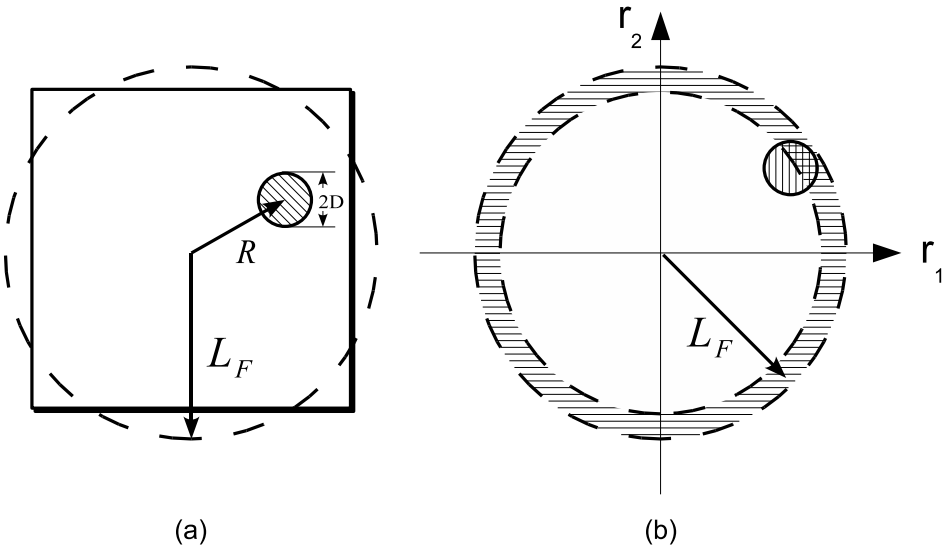

Figure 3.1: (a) A confined 2DEG of size $L \times L$ and classical turning point L_F contains an insulating region of radius D centered at R where the electronic wave function vanishes; and (b) particles close to the classical turning point L_F, which are represented by the shaded area, will satisfy the constraint $\Pi_1 \simeq \Pi_2 \simeq 0$.

the confining frequency $\omega_0^2 = \frac{\hbar}{m}(\frac{T}{T_F})\frac{1}{\lambda_F^2}$. Under this condition, we obtain a ballistic regime where $L < L_F \sim 10^{-7} - 10^{-6}m$, $T \sim 1 - 10K$, $\omega_0 \simeq 10^{10} - 10^{11}Hz$ and $\tau \simeq 10^{-11}sec$. In order to observe quantum scattering effects caused by the insulating region of radius "D", we require that the wavelength λ_F obeys the condition $D > \lambda_F \simeq 0.5 \times 10^{-7}m$. (For $D < \lambda_F$, the electrons will not experience the insulating region.)

3.2 Single Particle Hamiltonian

Based on the above analysis, we will investigate a 2DEG with a parabolic potential $V_c(\vec{r}) = \frac{m\omega_0^2}{2}\vec{r}^2$. The insulating region of radius "D," caused by the potential $U_I(\vec{r})$, causes the electronic wave function $\Phi(\vec{r};\vec{R})$ to vanish for $|\vec{r}-\vec{R}| \leq D$. The effect of the vanishing wave function is described effectively by a static vortex with the center at $\vec{r}=\vec{R}$ [Eza00]. We introduce the following parametrization for this wave function:

$$\Phi(\vec{r};\vec{R}) = \frac{|\vec{r}-\vec{R}|}{D} e^{i\theta(\vec{r},\vec{R})}\psi(\vec{r})$$

for $|\vec{r}-\vec{R}| < D$, and

$$\Phi(\vec{r};\vec{R}) = e^{i\theta(\vec{r},\vec{R})}\psi(\vec{r})$$

for $|\vec{r}-\vec{R}| > D$.

The wave function $\psi(\vec{r})$ is a regular function and $\theta(\vec{r};\vec{R})$ is a multivalued function. As a result, the single particle Hamiltonian

$$\hat{H}_0 = \frac{\hbar^2}{2m}\vec{K}^2 + V_c(\vec{r}) + U_I(\vec{r})$$

is replaced by the transformed Hamiltonian:

$$H_0 = \frac{\hbar^2}{2m}(\vec{K} - \vec{\partial}\theta(\vec{r};\vec{R}))^2 + V_c(\vec{r}) \tag{3.1}$$

The Hamiltonian in eq. (3.1) is obtained with the help of the gauge transformation of wave function $\psi(\vec{r})$. We obtain

$$e^{-i\theta(\vec{r};\vec{R})}\hat{H}_0\Phi(\vec{r};\vec{R}) \xrightarrow[|\vec{r}-\vec{R}|>D]{} H_0(\vec{\Pi},\vec{r})\psi(\vec{r})$$

where $\vec{\Pi} = \vec{K} - \vec{\partial}\theta(\vec{r};\vec{R})$ is the kinetic momentum [LL81]. The derivative of the multivalued phase $\theta(\vec{r};\vec{R})$ determines the vector potential

$$\vec{A}(\vec{r};\vec{R}) = \vec{\partial}\theta(\vec{r};\vec{R}).$$

Due to the existence of the multivalued phase, we obtain that the kinetic momenta do not commute with each other, i.e. $[\Pi_1, \Pi_2] \neq 0$, and the kinetic momentum obeys the commutation rule,

$$[\Pi_1(\vec{r}), \Pi_2(\vec{r})] = i\frac{\bar{B}(\vec{r})}{D^2} \tag{3.2}$$

where $\bar{B}(\vec{r})$ is an effective magnetic field due to the insulation region, and is defined as $\bar{B}(\vec{r}) = \nabla \times \vec{A}(\vec{r};\vec{R})$. We want to consider a situation where the vortex has a finite size "D." A good approximation to the vortex given in [Eza00] is to use a delta function with a finite width D, namely

$$\bar{B}(\vec{r}) \simeq 2\pi \int_{|u|\leq D} d^2u \delta^{(2)}(r_1 - R_1 + u_1, r_2 - R_2 + u_2).$$

This replaces the delta function with a step function.

In order to study the effect of a microwave field, we consider an external microwave potential,

$$H_{ext} = -eE_2(t)r_2 \tag{3.3}$$

for $t \geq 0$, where $E_2(t) = E_c \cos(\omega t + \alpha(t))$ is the microwave field with an amplitude E_0, an angular frequency ω, and a phase α. For simplicity, the microwave is chosen to be linearly polarized along the r_2 direction. The total Hamiltonian thus becomes

$$H_{total} = H_0 + H_{ext} \tag{3.4}$$

where H_0 is the transformed Hamiltonian given in eq. 3.1.

3.3 Many Particle Formulation

In order to compute the current, we introduce a continuous formulation of the 2DEG many particle system. We replace the single particle Hamiltonian $H_{total}(\vec{\Pi}, \vec{r})$ by a many electron formulation. This will be done by using the Lagrange formulation [YM79]. We introduce a continuous representation, namely $\vec{r}(\vec{u}, t)$, where $\vec{r}$ is a vector field of the initial position $\vec{u}$ and the time t. Here, $\vec{r}(\vec{u}, t)$ is the continuous form of $\vec{r}_\beta(t)$, where "β " denotes the particular particle, $\beta = 1, 2, \ldots, N$. We assume that $\vec{r}_\beta(t) \simeq \vec{r}(\vec{u}, t)$ with a multiplicative factor of the density function $\rho_0(\vec{r})$, which satisfies $N/L^2 = \int \rho_0(\vec{u}) d^2u$ in two dimensions (L^2 is the two dimensional area). Initially, at time $t = 0$, the coordinate and the momentum obey $\vec{r}(\vec{u}, t = 0) = \vec{u}$ and $\vec{\Pi}(\vec{u}, t = 0) = \vec{K} - \vec{\partial}\theta(\vec{u}, \vec{R})$. In addition, the coordinate $\vec{r}(\vec{u})$ and the momentum $\vec{\Pi}(\vec{u})$ obey canonical commutation rules,

$$[r_i(\vec{u}), \Pi_j(\vec{u}')] = i\delta_{ij}\delta^2(\vec{u} - \vec{u}').$$

As a result, eq. 3.2 is replaced by

$$[\Pi_1(\vec{u}), \Pi_2(\vec{u}')] = \frac{i\bar{B}(\vec{r}(\vec{u}))}{D^2}\delta^2(\vec{u} - \vec{u}'). \tag{3.5}$$

The equilibrium density is given by

$$\rho_0(\vec{u}) = \int \frac{d^2K}{(2\pi)^2} f_{F.D.}[\frac{\hbar^2}{2m}\vec{K}^2 + V_c(\vec{u}) - E_F] \tag{3.6}$$

and the deformed density is given by

$$\rho(\vec{u}) = \frac{\rho_0(\vec{u})}{\det|\frac{\partial r_i}{\partial u_j}|}, \tag{3.7}$$

where $f_{F.D.}[\frac{\hbar^2}{2m}\vec{K}^2 + V_c(\vec{u}) - E_F]$ is the Fermi-Dirac function and $\det|\frac{\partial r_i}{\partial u_j}|$ is the Jacobian, see ref. [YM79].

Using the above variables, we obtain the Hamiltonian of our system in the Lagrange formulation,

$$H_E = \int d^2u\rho_0(\vec{u})\{\frac{\vec{\Pi}^2(\vec{u})}{2m} + V_c(\vec{r}(\vec{u})) + (-e)r_2(\vec{u})E_2(t) + \lambda(\vec{u})Q(\vec{u})\} \quad (3.8)$$

where $Q(\vec{u})$ is a constraint which imposes the Fermionic statistics, namely the density is one below Fermi energy E_F and zero otherwise). H_E is the extended Hamiltonian which includes the constraints $Q(\vec{u})$ with the Lagrange multiplier $\lambda(\vec{u})$. We have

$$\rho_0 Q(\vec{u}) = \int \frac{d^2K}{(2\pi)^2}\{f_{F.D.}[h(\vec{\Pi}(\vec{u})) - E_F] - \frac{f_{F.D.}[h(\vec{\Pi}(\vec{r}(\vec{u}))) - E_F]}{|\det(\frac{\partial r^i}{\partial u^j})|}\} \quad (3.9)$$

where $Q(\vec{u})$ is the Dirac's constraint of the second class [Dir01] and obey $[Q(\vec{u}), Q(\vec{u}')] \neq 0$.

Making use of this continuous formulation, we can show that the kinetic momentum do not commute, i.e.$[\Pi_1(\vec{u}), \Pi_2(\vec{u}')] = \frac{i\bar{B}(\vec{r}(\vec{u}))}{D^2}\delta^2(\vec{u} - \vec{u}')$. We can also show that the coordinates do not commute, i.e. $[\vec{r}_1(\vec{u}), \vec{r}_2(\vec{u}')] \neq 0$. Therefore, a rectified current is possible. To see this, we concentrate on the electrons on the Fermi surface which have the kinetic momentum $\Pi_1 \simeq \Pi_2 \simeq 0$. This is possible for electrons close to the classical turning points L_F, see figure 3.1(b). For this case, the Hamiltonian in eq. 3.8 is replaced by

$$\begin{aligned} H = \int d^2u\rho_0(\vec{u})\{&\frac{\vec{\Pi}^2(\vec{u})}{2m} + V_c(\vec{r}(\vec{u})) + (-e)r_2(\vec{u})E_2(t) \\ &+\mu_1\Pi_1(\vec{u}) + \mu_2\Pi_2(\vec{u}) + \lambda(\vec{u})Q(\vec{\Pi}(\vec{u}), \vec{r}(\vec{u}))\} \end{aligned} \quad (3.10)$$

where μ_1, μ_2 are new Lagrangian multipliers.

Following Dirac's method of the second class constraints [Dir01], we obtain

$$\begin{aligned}
[r_1(\vec{u}), r_2(\vec{u}')]_D &= [r_1(\vec{u}), r_2(\vec{u}')] \\
&\quad -[r_1(\vec{u}), \Pi_1(\vec{u}'')] \left(C_{1,2}(\vec{u}'', \vec{u}''')\right)^{-1} [\Pi_2(\vec{u}'''), r_2(\vec{u}')] \\
&\simeq i\frac{D^2}{\bar{B}(\vec{r})}\delta(\vec{u}-\vec{u}') f_{F.D.}(|\vec{r}(u)-\vec{R}|-D) \\
&\equiv \Omega(\vec{u})\delta(\vec{u}-\vec{u}') \neq 0
\end{aligned}$$

Here, $C_{i,j}$ is defined as the commutation relation between all the constraint operators χ_i, i.e. $C_{i,j} = [\chi_i, \chi_j]$. In this case, we have $\chi_i = \Pi_i$, where $i = 1, 2$. Because this non-commutativity is given rise from the existence of the insulating regions, we have $[r_1, r_2] = 0$ for $|\vec{r} - \vec{R}| > D$.

When $[r_1, r_2]$ is non-zero, the semi-classical equation of motion of the electron is modified, i.e.

$$\frac{dr_1}{dt} = \frac{1}{\hbar}\frac{\partial E(\vec{K}, \vec{r})}{dK^1} + \Omega(\vec{K})\frac{dK_2}{dt}, \tag{3.11}$$

where $E(\vec{K}, \vec{r})$ is the energy in the semi-classical approximation.

As a result of the external electric field $E_2(t)$, a change of velocity in the $i = 1$ direction is generated, i.e. $\delta\dot{r}_1 \simeq \frac{e}{\hbar}\Omega(\vec{K})V_2(t)/L$, where $V_2(t)$ is the voltage caused by the external field $E_2(t)$. As a result, we obtain the current in the $i = 1$ direction,

$$J_1(r) \simeq e\int \frac{d^2K}{(2\pi)^2}\delta\dot{r}_1(\vec{K})\rho_0(\vec{K}, \vec{r}) = \frac{e}{\hbar}\int \frac{d^2K}{(2\pi)^2}\Omega(\vec{K})\rho_0(\vec{K}, \vec{r})\frac{V_2(t)}{L} \tag{3.12}$$

with $\rho_0(\vec{K}, \vec{r}) = f_{F.D.}[E(\vec{K}, \vec{r}) - eV_2(t) - E_F]$. In the absence of the external voltage, the density is given by the equilibrium Fermi-Dirac function

$\rho_0(\vec{K},\vec{r}) = f_{F.D.}[E(\vec{K},\vec{r}) - E_F]$. The rectified current is obtained by expanding the Fermi-Dirac function to first order of the voltage $V_2(t)$. Using Larmor's theorem, we can replace $\Omega(\vec{K})$ by an effective magnetic field $\Omega(\vec{K}) = \frac{e}{2mc}\bar{B}(\vec{r})$. This gives rise to the rectified current,

$$I_1 = \frac{e^2V_2^2}{2\hbar mc}\int\frac{d^2K}{(2\pi)^2}\int\frac{dr_1}{L}\bar{B}(\vec{K})\delta(E(\vec{K},\vec{r}) - E_F) \propto (V_2)^2\bar{B} \tag{3.13}$$

As a result, we find that a rectified current is possible. This argument shows that the presence of zeros in the wave function of the electron might lead to the violation of the Onsager symmetry [AG06], and gives rise to rectification. Once we have qualitatively established that a rectified current is possible, we will drop the restriction $\Pi_1 \simeq \Pi_2 \simeq 0$. For the remaining part, we will restrict ourselves to electrons which obey $E(\vec{K},\vec{R}) \simeq E_F$.

3.4 Equation of Motion

Using a simplified model with an insulating disk which is described by a static vortex embedded in a confining parabolic potential we have performed calculations using the extended Hamiltonian

$$H_E = \int \rho_0(\vec{u})\, d^2u \left[\frac{\hbar^2\vec{P}^2(\vec{u})}{2m} + \frac{m\omega_0^2}{2}(\vec{r}(\vec{u}))^2 + \lambda(\vec{u})Q(\vec{r}(\vec{u})) + (-e)r_2(\vec{u})E(t)\right] \tag{3.14}$$

where $E(t) = E_0\cos(\omega t + \alpha)$ is the microwave field and ω_0 is the confining frequency. The static vortex, which describes the insulating region, is incorporated in the kinetic momentum

$$\vec{P}(\vec{u}) = \vec{p}(\vec{u}) - \frac{e}{\hbar c}\vec{\partial}_u\vartheta(\vec{r}(\vec{u});\vec{R})$$

The current operator I_i in the i direction is determined by the equation of motion for $\frac{d\vec{r}(\vec{u},t)}{dt}$, namely

$$I_i = (-e)\int \rho_0(\vec{u})\, d^2u \left(\frac{dr_i(\vec{u},t)}{dt}\right)[J^{-1}(\vec{r}(\vec{u},t)) - 1] \tag{3.15}$$

where $J(\vec{r}(\vec{u},t))$ is the Jacobian defined as $J = \det|\frac{\partial r_i}{\partial u_j}|$. The current is given by the expectation value of I_i operator with respect to the Fermi sea $|F.S.>$, i.e. $< I_i >=< F.S.|I_i|F.S. >$. The current is computed using the Heisenberg equation of motion in the presence of the constraints .

We solve *iteratively* the Heisenberg's equation of motion as a function of the ratio ω/ω_0 and dimensionless parameter $\gamma = \frac{\hbar\omega_0}{m\omega_0^2 D^2}$ with D being the radius of the insulating region. According to the Hamiltonian given in eq. 3.14, we can write the equation of motion as

$$i\hbar\frac{d\vec{r}(\vec{u},t)}{dt} = [\vec{r}(\vec{u}), H_E] \tag{3.16}$$

$$i\hbar[\frac{d\vec{P}(\vec{u},t)}{dt} + \frac{\vec{P}(\vec{u},t)}{\tau}] = [\vec{P}(\vec{u}), H_E] \tag{3.17}$$

$$\frac{dQ(\vec{r}(\vec{u}))}{dt} = [Q(\vec{r}(\vec{u})), H_E] = 0 \tag{3.18}$$

In eq. 3.17, we have introduced a phenomenological relaxation time τ caused by impurity scattering. In order to solve the equations of motion, we have to satisfy the constraint at each time. The iterative solution is given as a series in γ and the microwave amplitude,

$$\vec{P}(\vec{u},t) = K(\vec{u}) + \gamma\vec{P}^{(1)}(\vec{u},t)\ldots$$

where $K(\vec{u}) = \sqrt{\frac{2m}{\hbar^2}(E - \frac{m\omega_0^2}{2}(\vec{u})^2)}$ for energies $E < E_F$ and the Jacobian $J = 1$. Here, $\vec{P}^{(1)}(\vec{u},t)$ is the first order correction in γ and the microwave

amplitude. For $E \approx E_F$, the correction is represented as a deformation of the Fermi Sea $\vec{P}^{(1)}(\vec{u},t) \approx \delta\vec{K}_F(\phi;\vec{u},t)$, where ϕ is the polar angle on the Fermi sea with $J \neq 1$ and $\vec{r}(\vec{u},t) - \vec{u} = \frac{\hbar}{m\omega_0}\int_0^t \delta\vec{K}_F(\phi;\vec{u},t')\,dt'$

We introduce a dimensionless time $s = t\omega_0$ and obtain from eq. 3.17 that $\delta\vec{K}_F(s)$ obeys the following differential equation,

$$\begin{aligned}(\frac{d^2}{ds^2}+1)\delta K_{F,i}(s) + \epsilon_{ij}\frac{\gamma}{2}\frac{d}{ds}\{\bar{B}(\vec{r}(s)),\delta K_{F,j}(s)\}_+ \\ = -\frac{1}{\omega_0\tau}\frac{d}{ds}\delta K_{F,i}(s) - \frac{e}{\hbar\omega_0}\frac{d}{ds}(E_2(s))\delta_{i,2}.\end{aligned} \tag{3.19}$$

The notation $\{A,B\}_+ = AB + BA$ represents the anti-symmetric combination. The solution of eq. (3.19) is given as a power series in the coupling constant γ. We find,

$$\begin{aligned}\delta K_{F,i}(s) &= E_0\{\delta_{i,1}[\gamma\delta K_1^{(1)} + \gamma^3\delta K_1^{(3)} + \cdots] + \delta_{i,2}[\delta K_2^{(0)} + \gamma^2\delta K_2^{(2)} + \cdots]\} \\ &\quad + E_0^2\{\delta_{i,1}[\gamma\delta Q_1^{(1)} + \cdots] + \delta_{i,2}[\gamma^3\delta Q_2^{(3)} + \cdots]\}\end{aligned} \tag{3.20}$$

The solution $\delta K_{F,i}(s)$ has two parts, i.e. a D.C. (time independent) part and an A.C. (time dependent) part. The zeroth order solution used to generate the series are given by $\delta K_1^{(0)}(s) = 0$, $\delta K_2^{(0)}(s) = A_2^{(0)}\sin[(\frac{\omega}{\omega_0})s + \varphi + \alpha]$ where $\tan\varphi = \frac{\omega/\tau}{\omega^2-\omega_0^2}$ and $A_2^{(0)} = -\frac{e\tau(\omega/\omega_0)}{\hbar\omega_0}\sin\varphi$.

The first order term in γ is given by $\delta K_1^{(1)}(s)$ with the D.C. (time independent) part being defined as

$$\delta K_{1,D.C.}^{(1)} = \frac{1}{2\pi}\int_0^{2\pi}\delta K_1^{(1)}(s')ds'.$$

For this reason, to leading order in $\gamma < 1$ and to the second order in the microwave amplitude, we obtain the current in the $i = 2$ direction. In addition,

we have a Hall contribution in the $i = 1$ direction which is the first order in the microwave field, Accordingly, we have $\delta K_{F,1}^{D.C.} \simeq E_0 \delta K_{1,D.C.}^{(1)}$. This is used to determine the current density given in eq. 3.15, because $\dot{\vec{r}}(\vec{u}, t) \simeq \frac{\hbar}{m\omega_0} \delta \vec{K}_F$.

We obtain that by applying a microwave field in the $i = 2$ direction, a rectified D.C. voltage $V_{1,D.C.}$ is induced in the $i = 1$ direction. This rectified voltage is defined as $V_{1,D.C.} = I_1/\sigma$ with the current I_1 given by eq. 3.15, and σ is the Drude conductivity. The microwave field is expressed in terms of an R.M.S. (effective) voltage $V_{R.M.S.} = E_0 L/\sqrt{2}$ which allows to define a *dimensionless voltage* induced in the $i = 1$ direction

$$v_{1,D.C.} = \frac{V_{1,D.C.}}{V_{R.M.S.}} = (\frac{D}{L})^2 \gamma G(\omega/\omega_0, \alpha) \tag{3.21}$$

where

$$G(\omega/\omega_0, \alpha) \simeq -\frac{\sin\varphi \cos(\varphi + \alpha)}{1 - (\gamma \sin\varphi)^2} \tag{3.22}$$

where $\tan\varphi = \frac{\omega/\tau}{\omega^2 - \omega_0^2}$ and α is a phase of the microwave which can be chosen as zero. Using typical values of $\gamma = \frac{\hbar\omega_0}{m\omega_0^2 D^2}$, D and L, as well as the experimental data taken from ref. [ZVK$^+$05], we plotted figure 3.2.

For the confined 2DEG, we use typical values given in the experiment [ZVK$^+$05], that is, electronic density $n \approx 10^{15} m^{-2}$, Fermi energy $E_F \approx 0.01 eV$, confining frequency $\omega_0 \approx 10^{10} - 10^{11} Hz$, momentum relaxation time $\tau \approx 10^{-11} sec$, and radius of the insulating region $D > \lambda_F \approx 0.5 \times 10^{-7} m$. These typical values gives rise to $\gamma \approx 0.7$. We make a single harmonic approximations by neglecting terms which oscillate with frequencies $2\omega_0$, $3\omega_0$, etc.

We used figure 3 of ref. [ZVK$^+$05] to extract the voltage changes as

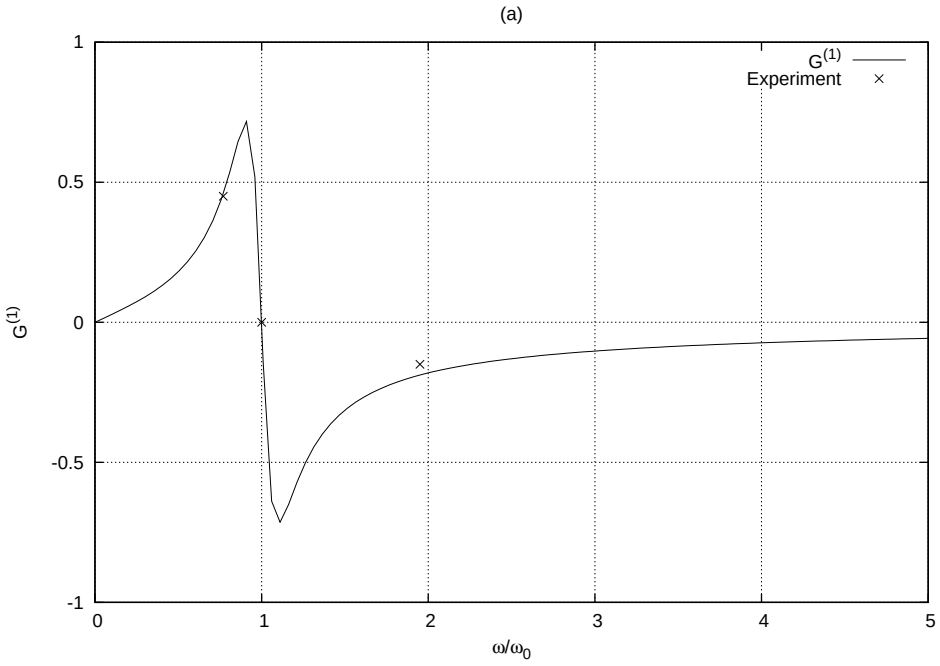

Figure 3.2: Dimensionless rectified voltage $G(\omega/\omega_0)$ as a function of ω/ω_0, which is compared with the rescaled experimental result of ref. [ZVK$^+$05] (plot as "$\times$").

a function of the microwave frequency for a zero magnetic field. Figure 3 in ref. [ZVK$^+$05] shows clearly a *change of sign* when the microwave varies between 1.46 GHz to 34 GHz, and vanishes at 17.41 GHz. As shown in figure 3.2, we find a good agreement of our theory with the experimental results once we choose $\omega_0 = 17.41$ GHz. We note that for frequencies which obeys $\frac{1.46}{17.41} < \frac{\omega}{\omega_0} < \frac{34}{17.41}$, we find good agreement with the experimental results. However, for low frequencies our theory is inadequate and does not fit the experiment.

3.5 Conclusion

To conclude, we have shown that when a microwave field is applied to a confined two dimensional electron gas which contains an insulating region, a rectified voltage is induced. Using this theory, we may explained some features of the experiment of [ZVK$^+$05]. This calculation is based on the assumption that the vanishing of wave function can be replaced by a static vortex, which gives rise to the non-commutativity of kinetic momenta. The results of this chapter are published in [SC07].

Chapter 4

Transverse Voltage Induced by Microwave in the Presence of an External Magnetic Field

In this chapter, we propose to understand the effect of an external magnetic field to the confined 2DEG as described in the previous chapter by introducing a phenomenological model. Because the application of the external magnetic field to the confined 2DEG gives rise to a transverse voltage, namely a Hall voltage, we will study in this chapter the Hall effect in response to the applied microwave.

4.1 Transverse Voltage Induced by Microwave

4.1.1 Fermi wavelength of the 2DEG

For 2DEG, the area of the Fermi circle is πk_F^2, and the volume element in k-space is $(\frac{2\pi}{L})^2$. The total number of electrons filled under the Fermi surface is $N = 2\frac{\pi k_F^2}{(\frac{2\pi}{L})^2} = \frac{k_F^2 L^2}{2\pi}$, where the factor 2 stands for the spin degree of freedom for electrons. The Fermi wavelength λ_F is given as

$$\lambda_F = \frac{2\pi}{k_F} = \sqrt{\frac{2\pi}{n}}, \tag{4.1}$$

where $n = N/L^2$ is the density of electrons. Referring to table (1.1), one can take $n \approx 3.0 \times 10^{11}$ (cm^{-2}) for GaAs/AlGaAs heterostructure as a typical value. This gives rise to $\lambda_F \approx 0.05\mu m$. Therefore, it is legitimate to consider the 2DEG semi-classically for samples of lateral sizes larger than $1\mu m$.

4.1.2 Semi-classical Equation of Motion

In order to study the response of 2DEG to an external electromagnetic wave, we consider a semiclassical 2DEG sample of area L^2 lying on the x-y plane. An electromagnetic wave, such as a microwave, that propagates on a wave guide of two parallel strips along the +x-direction is applied to the sample. This is equivalent to applying a time-dependent electric field $\vec{E}(t)$ in the y-direction.

We now consider a phenomenological time-dependent magnetic field $B(t)$ in the $\hat{z}$ direction to simulate the effective magnetic field originated from the static vortex in response to the microwave. In addition, a static magnetic

field $\vec{B}_s = B_s\hat{z}$ is also applied to the sample. Therefore, one can write the external electric field as

$$\vec{E} = \hat{y}E_0\cos(\omega t + \phi)$$

and the total magnetic field as

$$\vec{B} = \hat{z}(B_0\cos(\omega t) + B_s),$$

where ω is the frequency of the electromagnetic field, and ϕ is the phase difference between the electric field and the magnetic field of the microwave. The equation of motion thus becomes

$$m\left(\frac{d}{dt} + \frac{1}{\tau}\right)\vec{v} = (-e)\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right),$$

where we have considered the momentum relaxation time τ to simulate impurity scatterings.

We now consider a confining force $\vec{F}_c = -(K_{0,x}x\hat{x} + K_{0,y}y\hat{y})$, where $K_{0,x}$ and $K_{0,y}$ are the force constants. Because the time-dependent electric field is along the y-direction, we can disregard the $K_{0,x}$ term for the remaining part of our calculation. By writing the above equations of motion into x- and y-components, we obtain

$$\begin{aligned} m\dot{v}_x + \frac{1}{\tau}v_x &= -\frac{e}{c}v_y(B_0\cos(\omega t) + B_s) && (4.2)\\ m\dot{v}_y + \frac{1}{\tau}v_y &= -eE_0\cos(\omega t + \phi) - K_{0,y}y \\ &\quad + \frac{e}{c}v_x\left(B_0\cos(\omega t) + B_s\right). && (4.3)\end{aligned}$$

Since $m\vec{v} = \hbar\vec{k}$, we can rewrite the above equation of motion into the following form:

$$\dot{k}_x + \frac{1}{\tau}k_x = -\left(\omega_c \cos(\omega t) + \omega_s\right) k_y \tag{4.4}$$

$$\dot{k}_y + \frac{1}{\tau}k_y + \omega_0^2 \int^t dt' k_y = -\frac{e}{\hbar}E_0 \cos(\omega t + \phi) + \left(\omega_c \cos(\omega t) + \omega_s\right) k_x, \tag{4.5}$$

where $\omega_0^2 = \frac{K_{0,y}}{m}$ is the confining frequency, $\omega_c = \frac{eB_0}{mc}$ is the cyclotron frequency of the time-dependent magnetic field, and $\omega_s = \frac{eB_s}{mc}$ is the cyclotron frequency of the static magnetic field.

If current can not flow in the x-direction, we must have $\dot{k}_x = 0$. Therefore, eq. (4.4) gives rise to

$$k_x = -\tau(\omega_c \cos(\omega t) + \omega_s)k_y. \tag{4.6}$$

By substituting eq. (4.6) into eq. (4.5), we have

$$\dot{k}_y + \frac{1}{\tau}k_y + \omega_o^2 \int^t dt' k_y = -\frac{e}{\hbar}E_0 \cos(\omega t + \phi) - \tau(\omega_c \cos(\omega t) + \omega_s)^2 k_y \tag{4.7}$$

In order to solve the above integral-differential equation, we take an additional time derivative to eq. (4.7). This gives rise to the following second order differential equation.

$$\ddot{k}_y + \left[\frac{1}{\tau} + \tau\omega_s^2\right]\dot{k}_y + \omega_0^2 k_y = \frac{eE_0}{\hbar}\omega \sin(\omega t + \phi) \tag{4.8}$$

where a time average has been taken for the time dependent coefficients, and B_s is considered much larger than B_0. By solving the second order differential

eq. (4.8), we find

$$k_y(t) = e^{-\frac{t}{2\tau}(1+(\omega_s\tau)^2)}\left\{C_1 e^{-\frac{t}{2\tau}\alpha} + C_2 e^{+\frac{t}{2\tau}\alpha}\right\} + \frac{eE_0\tau}{\hbar}\cdot \tag{4.9}$$
$$\cdot\frac{(\omega\tau)^2\left[1+(\omega_s\tau)^2\right]\cos(\omega t+\phi) + (\omega\tau)\left[(\omega\tau)^2-(\omega_0\tau)^2\right]\sin(\omega t+\phi)}{\left[(\omega\tau)^2-(\omega_0\tau)^2\right]^2 + 2(\omega\tau)^2(\omega_s\tau)^2 + (\omega\tau)^2\left[1+(\omega_s\tau)^4\right]}$$

where $\alpha = \sqrt{(1+(\omega_s\tau)^2)^2 - 4(\omega_0\tau)^2}$, C_1 and C_2 are arbitrary constants to be determined by the initial conditions. However, since we are interested in large time limit of this solution, the C_1 and C_2 terms thus vanishes. By defining a few new dimensionless variables $\omega\tau = x$, $\omega_0\tau = y$, and $\omega_s\tau = s$, we rewrite the above solution as

$$k_y(t) = \frac{eE_0\tau}{\hbar}\frac{x^2\left[1+s^2\right]\cos(\omega t+\phi) + x\left[x^2-y^2\right]\sin(\omega t+\phi)}{\left[x^2-y^2\right]^2 + 2x^2s^2 + x^2\left[1+s^4\right]}. \tag{4.10}$$

Accordingly, we have

$$k_x(t) = -\tau(\omega_c\cos(\omega t)+\omega_s)k_y(t) \tag{4.11}$$

By taking a time average of $k_x(t)$, we have

$$\overline{k_x(t)} = -\frac{eE_0\omega_c\tau^2}{\hbar}\cdot\frac{x^2\left[1+s^2\right]\overline{CC} + x\left[x^2-y^2\right]\overline{CS}}{\left[x^2-y^2\right]^2 + 2x^2s^2 + x^2\left[1+s^4\right]} \tag{4.12}$$

where

$$\overline{CC} = \overline{\cos(\omega t)\cos(\omega t+\phi)} = \frac{\cos(\phi)}{2},$$

and

$$\overline{CS} = \overline{\cos(\omega t)\sin(\omega t+\phi)} = \frac{\sin(\phi)}{2}.$$

This gives rise to

$$\overline{k_x(t)} = -\frac{eE_0\omega_c\tau^2}{2\hbar}\frac{x^2\left[1+s^2\right]\cos(\phi) + x\left[x^2-y^2\right]\sin(\phi)}{\left[x^2-y^2\right]^2 + 2x^2s^2 + x^2\left[1+s^4\right]}. \tag{4.13}$$

Eq. (4.13) is an important result for calculating the Hall voltage V_H. According to the Ohm's law, $\overline{J}_x = n(-e)\overline{v}_x = \sigma\overline{E}_x$. Since $\overline{v}_x = \hbar\overline{k}_x/m$ and the Hall voltage is defined as $V_H = -\int_0^L \overline{E}_x dx$, we have

$$\begin{aligned} V_H &= -\int_0^L \overline{E_x} dx = -\overline{E_x} L = \frac{ne\hbar L}{m\sigma}\overline{k_x(t)} = \frac{ne^2\tau}{m\sigma}\frac{\omega_c\tau E_o L}{2}F(x,y,s,\phi) \\ &= \frac{\omega_c\tau E_o L}{2}F(x,y,s,\phi) \end{aligned} \tag{4.14}$$

where σ is the Drude conductivity, namely $\sigma = \frac{ne^2\tau}{m}$, and $F(x,y,s,\phi)$ is a form factor, which is defined as

$$F(x,y,s,\phi) = -\frac{x^2\left[1+s^2\right]\cos(\phi) + x\left[x^2-y^2\right]\sin(\phi)}{\left[x^2-y^2\right]^2 + 2x^2s^2 + x^2\left[1+s^4\right]}. \tag{4.15}$$

For a linearly polarized electromagnetic field, we set $\phi = \frac{3\pi}{2}$. The form factor $F(x,y,s,\frac{3\pi}{2})$ thus becomes

$$F(x,y,s,\frac{3\pi}{2}) = \frac{x\left[x^2-y^2\right]}{\left[x^2-y^2\right]^2 + 2x^2s^2 + x^2\left[1+s^4\right]}. \tag{4.16}$$

For $\phi = \frac{3\pi}{2} + \delta$, the form factor $F(x,y,s,\frac{3\pi}{2}+\delta)$ becomes

$$F(x,y,s,\frac{3\pi}{2}+\delta) = \frac{-x^2\left[1+s^2\right]\sin(\delta) + x\left[x^2-y^2\right]\cos(\delta)}{\left[x^2-y^2\right]^2 + 2x^2s^2 + x^2\left[1+s^4\right]}. \tag{4.17}$$

4.2 Results

In order to illustrate our results, we need to take some typical values of x, y, and s. First of all, we take the momentum relaxation time of a 2DEG sample as $\tau = 34ps$. For an electromagnetic wave, such as microwave, the ω is between 1. and 40. GHz, thus $x = \omega\tau$ is approximately between 0.03 and 1.38. By considering the confining frequency ω_0 between 12. and 14.

GHz, we have $y = \omega_0\tau$, which is approximately between 0.4 and 0.5. For the static external magnetic field between 0.1 to 0.5 Tesla, the cyclotron frequency ω_s is approximately between 50. and 200. GHz. Therefore, $s = \omega_s\tau$ is approximately between 1. and 7. In order to consider both signs of the static external magnetic, we can take s between $-7.$ and 7.

Provided the above typical values, we plot the form factor given in eq. 4.16 as a function of the external magnetic field s by choosing $-7 < s < 7$ and $y = 0.5$. The results are given in figures 4.1, 4.2, and 4.3, where the external microwave frequencies ω are taken as 7.38 GHz for fig. 4.1, 13.45 GHz for fig. 4.2, and 17.41 GHz for fig. 4.3. It is clearly seen from figures 4.1, 4.2, and 4.3 that increasing the external magnetic field will gradually suppress the transverse voltage. This result shows a qualitative agreement with the experimental results given in figure 3 of [ZVK$^+$05].

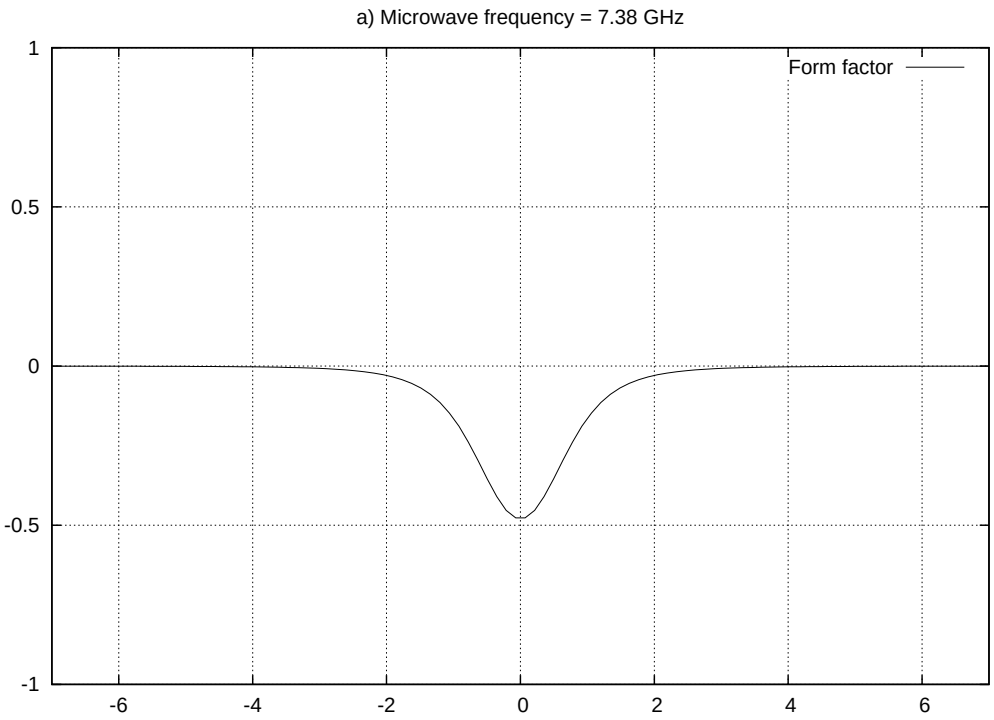

Figure 4.1: Form factor for microwave frequency at 7.38 GHz.

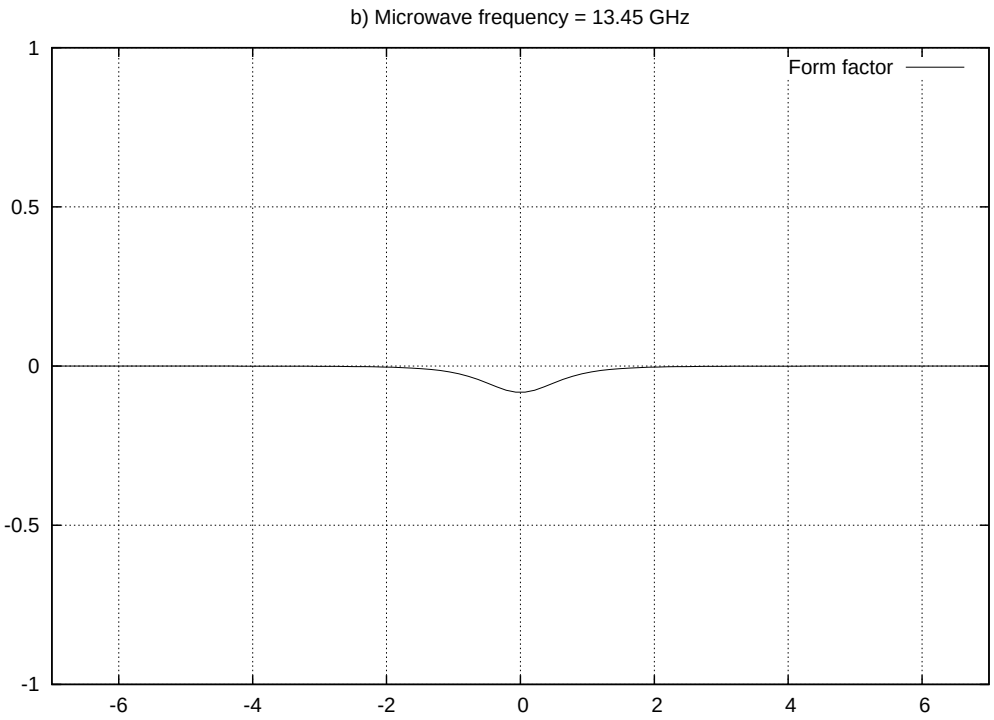

Figure 4.2: Form factor for microwave frequency at 13.45 GHz.

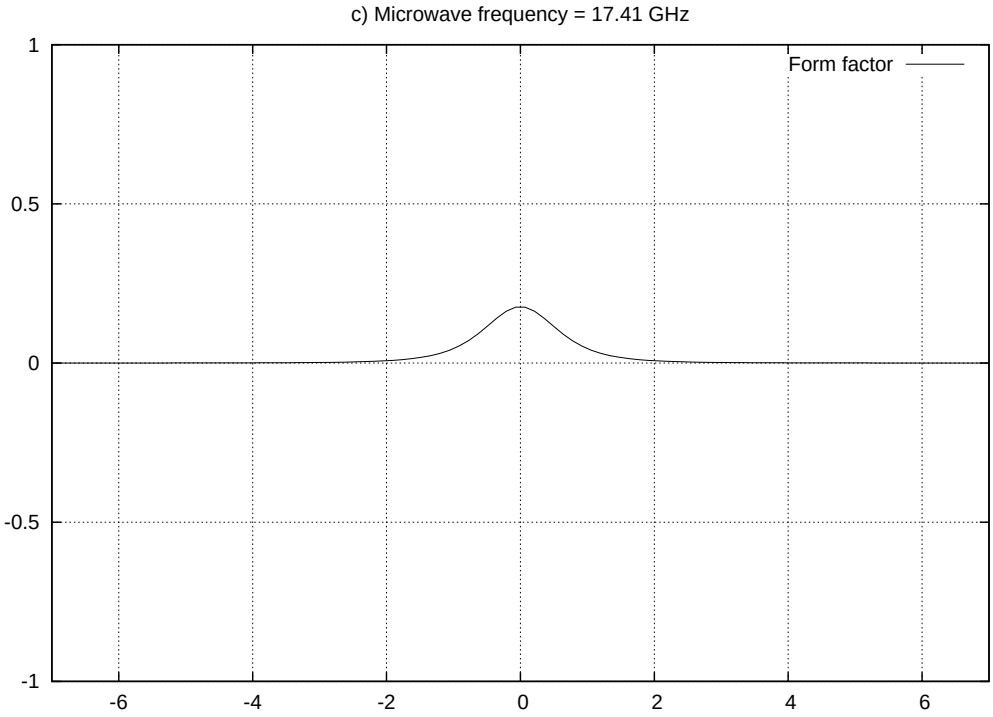

Figure 4.3: Form factor for microwave frequency at 17.41 GHz.

Appendix A

Bosonization Method

The necessity of bosonizing the fermionic fields is primarily due to the fact that for some certain types of interactions, it is intractable in fermionic language while it becomes trivial in its corresponding bosonic form. In principle, there are two approaches to understand the bosonization technique for 1D fermion fields. They are the "field-theoretical" and the "constructive" bosonization [vDS98]. Although, one can better understand *why* bosonization works in the constructive version of bosonization, it is so far adequate only to see *how* it works. For this reason, the basic ideas of field-theoretical bosonization for spinless fermions following Shankar's paper [Sha93]. One can easily generalize this for the case of spin-1/2 fermions.

Further, a bosonization method based on duality transformations is constructed by Schmeltzer [Sch93]. It is shown that one-dimensional fermion is equivalent to a bosonic field with a cubic potential. After linearization, one can obtain the bosonization results described as following. Moreover, a geo-

metrical non-abelian bosonization method is also constructed by Schmeltzer [Sch96]. Therefore, we note that the bosonization techniques introduced here and in Shankar's paper [Sha93] are limited.

A.1 Spinless Fermion Field

Since low energy modes in 1D are given by excitations near the Fermi points, we can linearize the dispersion relation around the Fermi points. The spinless Fermion field thus becomes

$$\psi(x) \simeq e^{ik_F x}\psi_R(x) + e^{-ik_F x}\psi_L(x) \tag{A.1}$$

where k_F is the Fermi wave vector, and $\psi_R(x)$ and $\psi_L(x)$ are the right and left movers, respectively. The right and left movers are often referred to as the chiral fields. By substituting the linearized Fermion field given in eq. (A.1) into the free Hamiltonian

$$H_0 = \int dx \psi^+(x) \left[-\frac{\hbar^2}{2m}\partial_x^2\right] \psi(x),$$

we can write the free Hamiltonian in the chiral basis as [Fra91]

$$H_0 = -\hbar v_F \int dx \left[\psi_R^\dagger(x) i\partial_x \psi_R(x) - \psi_L^\dagger(x) i\partial_x \psi_L(x)\right] \tag{A.2}$$

where $v_F = \hbar k_F/m$ is the Fermi velocity, $\psi_R(x)$ and $\psi_L(x)$ are the right and left movers, respectively. We can choose a unit such that $\hbar = 1$ and $v_F = 1$. (Notice that $\psi_R(x)$ and $\psi_L(x)$ are often denoted as $R(x)$ and $L(x)$.) The left and right movers obey the usual anti-commutation relation

$$\left\{\psi_{R/L}^\dagger(x), \psi_{R/L}(y)\right\} = \delta(x-y) \tag{A.3}$$

By Fourier transforming the left and right mover fields, namely

$$\psi_{R/L}(p) = \int_{-\infty}^{\infty} e^{-ipx}\psi_{R/L}(x)dx,$$

the anti-commutation relation becomes

$$\left\{\psi^{\dagger}_{R/L}(p), \psi_{R/L}(q)\right\} = 2\pi\delta(p-q) \tag{A.4}$$

and the Hamiltonian in eq. (A.2) becomes

$$H = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\psi^{\dagger}_{R}(p)(p)\psi_{R}(p) + \int_{-\infty}^{\infty} \frac{dp}{2\pi}\psi^{\dagger}_{L}(p)(-p)\psi_{L}(p) \tag{A.5}$$

It is clearly seen that the energy spectrum for the right mover is $E_p = p$ and the energy spectrum for the left mover is $E_p = -p$.

Consider now the equal time correlation function in the ground state:

$$\langle\psi_R(x)\psi^{\dagger}_R(0)\rangle = \int_{-\infty}^{\infty}\frac{dp}{2\pi}\int_{-\infty}^{\infty}\frac{dq}{2\pi}e^{ipx}\langle\psi_R(p)\psi^{\dagger}_R(q)\rangle \tag{A.6}$$

Since $\psi_R(p)|G\rangle = 0$ if $p > 0$ and $\psi_L(q)|G\rangle = 0$ if $q < 0$, by using the commutation relation in eq. (A.4), we can find that

$$\langle\psi_R(p)\psi^{\dagger}_R(q)\rangle = 2\pi\delta(p-q)\theta(q) \tag{A.7}$$

where $\theta(q)$ is a step function. Thus,

$$\langle\psi_R(x)\psi^{\dagger}_R(0)\rangle = \int_{0}^{\infty}\frac{dp}{2\pi}e^{ipx} \tag{A.8}$$

In order to properly carry out the above integral, we need to introduce a convergence factor $e^{-\alpha|p|}$. (We will need to set $\alpha = 0$ later for calculating physical quantities.) The integral is now tractable and becomes

$$\langle\psi_R(x)\psi^{\dagger}_R(0)\rangle = \int_{0}^{\infty}\frac{dp}{2\pi}e^{ip(x+i\alpha)} = \frac{i}{2\pi}\frac{1}{x+i\alpha} \tag{A.9}$$

Similarly, we can also show that

$$\langle \psi_L(x)\psi_L^\dagger(0)\rangle = -\frac{i}{2\pi}\frac{1}{x-i\alpha} \quad (A.10)$$

$$\langle \psi_R^\dagger(0)\psi_R(x)\rangle = -\frac{i}{2\pi}\frac{1}{x-i\alpha} \quad (A.11)$$

$$\langle \psi_L^\dagger(0)\psi_L(x)\rangle = \frac{i}{2\pi}\frac{1}{x+i\alpha} \quad (A.12)$$

A.2 Free Massless Scalar Field

Before we start to introduce the bosonization procedure, let us first review the properties of the free scalar field. The Hamiltonian of a massless free scalar field is given by

$$H_B = \frac{1}{2}\int dx \left[P^2(x) + (\partial_x\phi(x))^2\right] \quad (A.13)$$

where $\phi(x)$ is the bosonic field and $P(x)$ $(= \partial_t\phi)$ is its canonical momentum. They obey the usual commutation relation

$$[\phi(x), P(x)] = i\delta(x-y) \quad (A.14)$$

We can expand the bosonic field and its canonical momentum in terms of the normal modes:

$$\phi(x) = \int_{-\infty}^{\infty}\frac{dp}{2\pi\sqrt{2|p|}}\left[\phi(p)e^{ipx} + \phi^\dagger(p)e^{-ipx}\right]e^{-\alpha|p|/2} \quad (A.15)$$

$$P(x) = \int_{-\infty}^{\infty}\frac{dp}{2\pi\sqrt{2|p|}}\left[-i\phi(p)e^{ipx} + i\phi^\dagger(p)e^{-ipx}\right]e^{-\alpha|p|/2} \quad (A.16)$$

where we have introduced the convergence factor $e^{-\alpha|p|/2}$. One can easily show that the Fourier components obey the following commutation relation:

$$\left[\phi(p), \phi^\dagger(q)\right] = 2\pi\delta(p-q) \quad (A.17)$$

The Fourier transformed Hamiltonian then takes the form

$$H_B = \int_{-\infty}^{\infty} \frac{dp}{2\pi} |p| \phi^\dagger(p) \phi(p) \tag{A.18}$$

We now introduce the right and left mover

$$\phi_{R/L}(x) = \frac{1}{2}\left[\phi(x) \mp \int_{-\infty}^{x} dx' P(x')\right] = \pm \int_0^{\pm\infty} \frac{dp}{2\pi} e^{-\alpha|p|/2} \left[\phi(p) e^{ipx} + h.c.\right] \tag{A.19}$$

One can show that

$$\left[\phi_{R/L}(x), \phi_{R/L}(y)\right] = \pm \frac{i}{4} \theta(x-y) \tag{A.20}$$

$$\left[\phi_R(x), \phi_L(y)\right] = \frac{i}{4} \tag{A.21}$$

where $\theta(x-y)$ is the step function.

A.3 Bosonization

Provided the above results, we can now introduce the most important formula in bosonization:

$$\psi_{R/L}(x) = \frac{1}{\sqrt{2\pi\alpha}} e^{\pm i\sqrt{4\pi}\phi_{R/L}(x)} \tag{A.22}$$

where $\psi_{R/L}$ is the left and right mover of the fermionic field, $\phi_{R/L}$ is the left and right mover of the bosonic field and α is the cut-off. The claim is that provided the above transformation formula, one can obtain the same correlation function with respect to the fermi vacuum in both fermionic and bosonic languages, i.e. the two theories are equivalent.

For demonstration purpose, we consider here only the correlation function in eq. (A.9). The equivalence of other correlation functions can be proved by following a similar procedure.

Let's begin by plugging in the transformation in eq. (A.22) we have

$$\langle\psi_R(x)\psi_R^\dagger(0)\rangle = \langle\frac{1}{\sqrt{2\pi\alpha}}\, e^{i\sqrt{4\pi}\phi_R(x)}\frac{1}{\sqrt{2\pi\alpha}}\, e^{-i\sqrt{4\pi}\phi_R(0)}\rangle \tag{A.23}$$

In order to calculate the above correlation function, we need the following identity

$$e^A e^B =: e^{A+B} :\, e^{\langle AB+\frac{A^2+B^2}{2}\rangle} \tag{A.24}$$

where A and B are bosonic operators of the form $A = \bar{\alpha}\hat{a}^+ + \alpha\hat{a}$ and $A = \beta\hat{a}^+ + \bar{\beta}\hat{a}$. The proof of this important identity is given in appendix C. We can now write eq. (A.23) as

$$\begin{aligned}\langle\psi_R(x)\psi_R^\dagger(0)\rangle &= \frac{1}{2\pi\alpha}\langle :\, e^{i\sqrt{4\pi}(\phi_R(x)-\phi_R(0))}\, :\rangle\, e^{4\pi\langle\phi_R(x)\phi_R(0)-\phi_R^2(0)\rangle} \\ &= \frac{1}{2\pi\alpha}e^{4\pi G_R(x)}\end{aligned} \tag{A.25}$$

where $G_R(x) \equiv \langle\phi_R(x)\phi_R(0) - \phi_R^2(0)\rangle$. In deriving the above result, we have used the fact that the expectation value of any normal ordered operator is zero, namely $\langle :\, e^A\, :\rangle = 1$. The bosonic correlation function $G_R(x)$ can be derived using the definitions given in eq. (A.19),

$$\begin{aligned}G_R(x) &= \int_0^\infty \frac{dp}{2\pi\sqrt{2|p|}}\int_0^\infty \frac{dq}{2\pi\sqrt{2|q|}}\langle\phi(p)\phi^\dagger(q)\rangle(e^{ipx}-1)e^{-\alpha p} \\ &= \int_0^\infty \frac{dp}{4\pi p}(e^{ipx}-1)e^{-\alpha p} = \frac{1}{4\pi}\ln\frac{\alpha}{\alpha - ix}\end{aligned} \tag{A.26}$$

By substituting the result of eq. (A.26) back into eq. (A.25), we obtain the

same result as given in eq. (A.9), namely

$$\langle \psi_R(x)\psi_R^\dagger(0)\rangle = \frac{i}{2\pi}\frac{1}{x+i\alpha}. \tag{A.27}$$

This proves that the fermionic and bosonic descriptions are equivalent.

A.4 Some Examples

Since we have shown that the formula in eq. (A.22) can transform fermion fields into boson fields without changing the physics, we can use it to bosonize other physical quantities. In the following, we will show how one can bosonize some composite operators, which are useful for bosonizing the interaction Hamiltonian and the impurity potential.

First, we consider the composite operator of the form $\psi_R^+(x)\psi_L(x)$. By using the formula in eq. (A.22), we obtain

$$\begin{aligned}\psi_R^+(x)\psi_L(x) &= \frac{1}{2\pi\alpha}e^{-i\sqrt{4\pi}\phi_R(x)}e^{-i\sqrt{4\pi}\phi_L(x)} \\ &= \frac{1}{2\pi\alpha}e^{-i\sqrt{4\pi}\phi(x)}e^{-i\pi/2} \\ &= \frac{-i}{2\pi\alpha}e^{-i\sqrt{4\pi}\phi(x)}\end{aligned} \tag{A.28}$$

where $\phi(x) = \phi_R(x) + \phi_L(x)$. In the derivation given above, we have used eq. (A.21) and the fact that $e^Ae^B = e^{A+B}e^{[A,B]/2}$ for any operators A and B commuting with their commutator $[A, B]$. Similarly, we can bosonize the composite operator $\psi_L^+(x)\psi_R(x)$ and obtain

$$\psi_L^+(x)\psi_R(x) = \frac{i}{2\pi\alpha}e^{i\sqrt{4\pi}\phi(x)}. \tag{A.29}$$

In the discussions given above, we multiply two operators at the same point. Although the expectation value of each operator with respect to the ground state is well defined (finite or zero), there is no guarantee that the multiplication of two such operators is still well defined. Since the composite operator $\psi_L^+(x)\psi_R(x)$ and $\psi_R^+(x)\psi_L(x)$ create and annihilate different fermions in the ground state, the expectation values thereof are zero. However, there are certain composite operators that do not behave properly at the same point. If this is the case, we have to define the composite operator by a limiting process. For example, we should define $\psi_R^+(x)\psi_R(x)$ as

$$\begin{aligned}
\psi_R^+(x)\psi_R(x) &\equiv \lim_{\epsilon\to 0}\psi_R^+(x+\epsilon)\psi_R(x) \\
&= \lim_{\epsilon\to 0}\frac{1}{2\pi\alpha}e^{-i\sqrt{4\pi}\phi_R(x+\epsilon)}e^{i\sqrt{4\pi}\phi_R(x)} \\
&= \lim_{\epsilon\to 0}\frac{1}{2\pi\alpha}:e^{-i\sqrt{4\pi}(\phi_R(x+\epsilon)-\phi_R(x))}:e^{4\pi\left\langle\phi_R(x+\epsilon)\phi_R(x)-\phi_R^2(x)\right\rangle} \\
&= \lim_{\epsilon\to 0}\frac{1}{2\pi\alpha}:e^{-i\sqrt{4\pi}(\phi_R(x+\epsilon)-\phi_R(x))}:e^{4\pi G_R(\epsilon)} \\
&= \lim_{\epsilon\to 0}\frac{1}{2\pi\alpha}:\left(1-i\epsilon\sqrt{4\pi}\frac{\partial\phi_R(x)}{\partial x}+\cdots\right):\frac{i\alpha}{\epsilon+i\alpha} \\
&= \lim_{\epsilon\to 0}\frac{i}{2\pi(\epsilon+i\alpha)}+:\frac{1}{\sqrt{\pi}}\frac{\partial\phi_R(x)}{\partial x}+O(\epsilon):
\end{aligned} \tag{A.30}$$

where we have used the results given in eq. (A.24) and eq. (A.26). It is noted that the Taylor expansion is done inside of the normal ordering symbol, because only normal ordered operators are differentiable. In addition, we can disregard the higher order terms of ϵ because ϵ approaches zero. However, we have encountered an infinite c-number term in the first term of eq. (A.30). This infinite c-number is merely the infinite density of the right movers in the Dirac sea. Therefore, by normal ordering the composite

operator $\psi_R^+(x)\psi_R(x)$, we obtain

$$: \psi_R^+(x)\psi_R(x) := \frac{1}{\sqrt{\pi}}\frac{\partial\phi_R(x)}{\partial x} \tag{A.31}$$

Following the same spirit, we can bosonize the composite operator of $\psi_L^+(x)\psi_L(x)$, which is given by

$$: \psi_L^+(x)\psi_L(x) := \frac{1}{\sqrt{\pi}}\frac{\partial\phi_L(x)}{\partial x}. \tag{A.32}$$

Appendix B

Differential Hamiltonian Renormalization Group Theory

B.1 Introduction

The original Renormalization Group (RG) theory was an attempt to solve the ultraviolet divergence problem in Quantum Electrodynamics [IZ80]. However, the RG theory has been successfully applied to the study of phase transition and critical phenomena in statistical physics, under the hypothesis of scale invariance at the critical point, [Fis98]. Since for many physical problems, it is easier to construct a Hamiltonian rather than a Lagrangian, Schmeltzer [Sch95] developed a new version of RG theory based on the Hamiltonian formalism. We will discuss this version of RG here.

In order to describe the physics of phase transition, we need to specify first our order parameters. Good candidates of order parameters include

the magnetization, the electronic density, and the $U(1)$ phase. We always assume that there is an ultraviolet cutoff Λ. Since we are interested only in the physics at large distance, we can require that the measure

$$T \exp\left[-\frac{i}{\hbar}\int_{-\infty}^{\infty} dt \int d^d x h[\phi(x,t), P(x,t)]\right] \tag{B.1}$$

remains invariant if we reduce Λ to $\Lambda' = \Lambda/s$ for $s > 1$, where $h[\phi, P]$ is the Hamiltonian density, ϕ is the order parameter, and P is the canonical momentum of the order parameter. They satisfy the standard bosonic commutation relation

$$[\phi(x,t), P(x',t)] = i\delta_\Lambda(x - x'), \tag{B.2}$$

where $\delta_\Lambda(x - x') \equiv \int_{-\Lambda}^{\Lambda} dk e^{ik(x-x')}$. The long distance behavior is obtained from the existence of a fixed point, i.e. a set of coupling constants which are unchanged under $\Lambda \to \Lambda/s$.

We are interested in applying the present Hamiltonian RG theory to strongly correlated systems

$$H = \int d^d x \, h[\phi(x), P(x)],$$

where $h = h_0 + h_I$, h_0 is the free apart and h_I is the interacting part. Here we limited our discussion to $d = 1$ only, so we can write our free part explicitly in terms of bosonic fields as

$$h_0[\phi(x), P(x)] = \frac{v}{2}[KP^2(x) + \frac{1}{K}(\partial_x \phi)^2]. \tag{B.3}$$

We can write the interacting part as

$$h_I[\phi(x)] = \sum_i U_i O_i(x), \tag{B.4}$$

where $O_i(x)$ are some monomials of $\phi(x)$.

B.2 Renormalization Group Construction

In order to perform Renormalization Group analysis, we need to separate out the fast modes from the slow ones for the field $\phi(x)$ as well as for its canonical momentum $P(x)$, i.e. $\phi(x) = \phi^<(x) + \delta\phi(x)$ and $P(x) = P^<(x) + \delta P(x)$. We have assumed that there is a momentum cut-off Λ. The slow parts, $\phi^<$ and $P^<$, contain those components with momenta $|q| \leq \Lambda/s$, while the fast parts, $\delta\phi$ and δP, hold the contributions from the momentum shell, $\Lambda/s < |q| \leq \Lambda$. For this reason, we can write the slow mode and the fast mode of $\phi(x)$ and $P(x)$ explicitly as:

$$\phi^<(x) = \int_{-\Lambda/s}^{\Lambda/s} dk e^{ikx}\phi(k), \;\; \delta\phi(x) = \int_{Shell} dk e^{ikx}\phi(k); and \tag{B.5}$$

$$P^<(x) = \int_{-\Lambda/s}^{\Lambda/s} dk e^{ikx}P(k), \;\; \delta P(x) = \int_{Shell} dk e^{ikx}P(k), \tag{B.6}$$

where "Shell"$\equiv \{\Lambda/s \leq |k| \leq \Lambda\}$. As a result, we can write the Hamiltonian as

$$\begin{aligned} H[\phi, P] &= H[\phi^< + \delta\phi, P^< + \delta P] \\ &\equiv H[\phi^<, P^<] + \delta H[\phi^<, \delta\phi; P^<, \delta P] \end{aligned} \tag{B.7}$$

where

$$\begin{aligned} \delta H[\phi^<, \delta\phi; P^<, \delta P] = \int_{-\infty}^{\infty} dx \Big\{ &\frac{vK}{2}[\delta P(x)]^2 + \frac{v}{2K}[\partial_x \delta\phi(x)]^2 \\ &+ h_I'[\phi^<]\delta\phi(x) + \frac{1}{2}h_I''[\phi^<][\delta\phi(x)]^2 + \cdots \Big\}. \end{aligned} \tag{B.8}$$

Here, $h_I'[\phi^<]$ and $h_I'[\phi^<]$ are the functional derivatives of first order and second order, defined as

$$h_I'[\phi^<] \equiv \left.\frac{\delta h_I[\phi(x)]}{\delta\phi(x)}\right|_{\phi^<(x)}, \quad h_I''[\phi^<] \equiv \left.\frac{\delta^2}{\delta\phi(x)^2} h_I[\phi(x)]\right|_{\phi^<(x)}.$$

We keep only terms of $(\delta\phi(x))^n$ for $n \leq 2$, since we are working in the limit of $s \to 1$. Notice that the crossing terms between slow and fast modes vanish because the absolute value of momentum in slow and fast modes can never be the same.

In order to find the effective Hamiltonian of the slow modes, we need to integrate out the fast modes in the momentum shell. This means that in the Hamiltonian language, we have to take the expectation value with respect to the "fast mode" ground state at $t = 0$, i.e. $|\Psi^>(t=0)\rangle \equiv |G^>\rangle$. For this reason, the operators $\delta\phi(x,t)$ and $\delta P(x,t)$ are now time dependent which corresponds to the "Heisenberg picture". Notice that only the fast mode fields $\delta\phi(x,t)$ and $\delta P(x,t)$ are in the "Heisenberg picture", the slow mode fields $\phi^<(x)$, $P^<(x)$ remain in the Schrödinger picture. We need now to find a unitary transformation U:

$$\delta P'(x) = U\delta P(x)U^\dagger = \delta P(x) + \Pi(x) \tag{B.9}$$

$$\delta\phi'(x) = U\delta\phi(x)U^\dagger = \delta\phi(x) + r(x) \tag{B.10}$$

where $r(x)$ and $\Pi(x)$ are chosen such that the linear coupling between the $\delta\phi$ field with the slow modes $\phi^<$ is removed from the transformed Hamiltonian. By using the Heisenberg equation of motion $i\partial_t\delta\phi(x) = [\delta\phi(x), \delta H]$, we have

$$\partial_t\delta\phi(x,t) = vK\delta P(x,t)$$

Demanding that $[\delta\phi(x,t), \delta P(x',t)] = [\delta\phi'(x,t), \delta P'(x',t)]$, we find that $\Pi(x)$ and $r(x)$ are not independent, they are related by $\partial_t r(x,t) = vK\Pi(x,t)$. Choosing

$$U = \exp\left[\frac{i}{2}\int_{-\infty}^{\infty} dx[r(x)\delta P(x) - \Pi(x)\delta\phi(x)]\right],$$

we get

$$\begin{aligned}\int dt\delta H' &= \int dt \int dx \left\{\frac{vK}{2}[\delta P(x,t)]^2 + \frac{v}{2K}\left[\partial_x\delta\phi(x,t)\right]^2\right. \\ &+ \frac{1}{2}h_I''[\phi^<(x)][\delta\phi(x,t)]^2 + \delta\phi(x,t)\left[\hat{L}r(x,t) + h_I'[\phi^<(x)]\right] \\ &+ \left. r(x,t)\left[\frac{1}{2}\hat{L}r(x,t) + h_I'[\phi^<(x)]\right]\right\} \qquad \text{(B.11)}\end{aligned}$$

In eq. (B.11), we have defined $\hat{L} = \frac{1}{vK}\partial_t^2 - \frac{v}{K}\partial_x^2 + h_I''[\phi^<(x)]$. To set the $\delta\phi$ term zero, we need $\hat{L}r(x,t) = -h_I'[\phi^<(x)]$. This gives the formal solution

$$r(x,t) = -\int dy \int dt' G_\Lambda(x,y;t-t')h_I'[\phi^<(y,t')]$$

where $G_\Lambda(x,y;t-t')$ is the Green's function of the operator $\hat{L}$ with a cut-off Λ, and

$$\hat{L}G_\Lambda(x,y;t-t') = \delta_\Lambda(x-y)\delta(t-t')$$

Since $\phi^<(x)$ is time independent, $r(x)$ is determined only by the spatial Green's function, $\hat{G}_\Lambda(x,y) \equiv \int_{-\infty}^{t} dt' G_\Lambda(x,y;t-t')$. By substituting the solution into eq. (B.11) we find

$$\begin{aligned}\int dt\delta H' = \int dt \int dx \left\{\frac{vK}{2}(\delta P(x,t))^2 + \frac{v}{2K}(\partial_x\delta\phi(x,t))^2\right. \\ \left. + \frac{1}{2}h_I''[\phi^<](\delta\phi(x,t))^2 - \frac{1}{2}\int dy h_I'[\phi^<(x)]\hat{G}_\Lambda(x,y)h_I'[\phi^<(y)]\right\} \qquad \text{(B.12)}\end{aligned}$$

The first two terms in eq. (B.12) are purely the free Hamiltonian in the fast modes which do not contribute to the correction of the slow mode Hamiltonian. However, based on the free Hamiltonian in the fast modes, we can expand the fast mode field and its canonical momentum by normal modes as

$$\delta\phi(x) = \int_{\text{Shell}} \frac{dp}{2\pi}\sqrt{\frac{K}{2|p|}}\left[\phi(p)e^{ipx} + \phi^\dagger(p)e^{-ipx}\right] \tag{B.13}$$

$$\delta P(x) = \int_{\text{Shell}} \frac{dp}{2\pi}\sqrt{\frac{|p|}{2K}}\left[-i\phi(p)e^{ipx} + i\phi^\dagger(p)e^{-ipx}\right] \tag{B.14}$$

where "Shell"$\equiv \{\Lambda/s \le |p| \le \Lambda\}$. Now we can take the average of eq. (B.12) over the ground state of fast modes. Let $s = e^\ell$, the third term in eq. (B.12) gives

$$\begin{aligned}\langle\delta\phi(x)\delta\phi(x')\rangle &= \int_{\text{Shell}} \frac{dp}{2\pi\sqrt{2|p|}}\frac{dq}{2\pi\sqrt{2|q|}} K e^{ipx-iqx'}\langle\phi(p)\phi^\dagger(q)\rangle \\ &= \frac{K}{4\pi}\int_{\text{Shell}}\frac{dp}{|p|}e^{ip(x-x')}\end{aligned} \tag{B.15}$$

In the above derivation, we have used the result $\langle\phi(p)\phi^\dagger(q)\rangle = 2\pi\delta(p-q)$. For $x = x'$, we have

$$\langle\delta\phi(x)\delta\phi(x)\rangle = \frac{2K}{4\pi}\int_{\Lambda/s}^{\Lambda}\frac{dp}{|p|} \cong \frac{K}{2\pi}\frac{d\Lambda}{\Lambda} \tag{B.16}$$

where $d\Lambda = \Lambda - \Lambda/s = \Lambda(1-e^{-\ell}) \cong \Lambda\ell$, or $\ell = d\Lambda/\Lambda$. For the fourth term in eq. (B.12), since the average is taken in the fast mode, $\hat{G}_\Lambda(x,y)$ is restricted to the momentum shell. Therefore, we can replace $\hat{G}_\Lambda$ by $\hat{G}_\Lambda^{(0)}$ (the free Green's function which corresponds to the operator $\hat{L}$ without the h_I'' term.) Finally, we can write the averaged $\delta H'$ as

$$\begin{aligned}\int dt\langle G^{>}|\delta H'|G^{>}\rangle &\equiv \int dt\Delta H \\ &= \int dtdx\left[\frac{K\ell}{4\pi}h_I''[\phi^{<}(x)]\right. \\ &\quad \left.-\frac{1}{2}\int dy h_I'[\phi^{<}(x)]\hat{G}_{\Lambda}^{(0)}(x,y)h_I'[\phi^{<}(y)]\right] \quad \text{(B.17)}\end{aligned}$$

This contributes to the correction of the slow modes.

Since in most cases, we are interested in short range interactions, we can take $x \simeq y$ and expand the $h_I'[\phi^{<}(y)]$ as

$$\begin{aligned}h_I'[\phi^{<}(y)] &= h_I'[\phi^{<}(x)] + (y-x)\partial_x h_I'[\phi^{<}(x)] \\ &\quad +\frac{1}{2}(y-x)^2\partial_x^2 h_I'[\phi^{<}(x)] + \cdots \quad \text{(B.18)}\end{aligned}$$

Plugging eq. (B.18) back into eq. (B.17), we can throw away the first two terms because: (1) $(y-x)G_{\Lambda}^{(0)}(x-y)$ is odd in y, thus the y integral vanishes for the first term, and (2) any integral over space (x or y) of integrand with both slow and fast modes vanishes, thus the second term vanishes. We have

$$\langle G^{>}|\Delta H|G^{>}\rangle = \int dx\left\{\frac{K\ell}{4\pi}h_I''[\phi^{<}(x)] - \frac{\xi}{4}h_I'[\phi^{<}(x)]\partial_x^2 h_I'[\phi^{<}(x)]\right\} \quad \text{(B.19)}$$

where $\xi \equiv \int dy' y'^2 G_{\Lambda}^{(0)}(y'; t-t') = \text{Const}$. We can integrate the second term by parts which gives

$$\langle G^{>}|\Delta H|G^{>}\rangle = \int dx\left\{\frac{K\ell}{4\pi}h_I''[\phi^{<}(x)] + \frac{\xi}{4}(\partial_x h_I'[\phi^{<}(x)])^2\right\} \quad \text{(B.20)}$$

B.3 An Example

In order to demonstrate how the Hamiltonian RG works, we choose the impurity Hamiltonian in section 2.1.3, $h_I[\phi^{<}(x)] = U(x)\cos(\sqrt{4\pi}\phi^{<}(x))$, as

an example. According to eq. (B.20), we need the following relations

$$h_I'[\phi^<(x)] = -\sqrt{4\pi}U(x)\sin(\sqrt{4\pi}\phi^<(x)) \quad \text{(B.21)}$$

$$h_I''[\phi^<(x)] = -4\pi U(x)\cos(\sqrt{4\pi}\phi^<(x)) \quad \text{(B.22)}$$

We also need

$$\begin{aligned}\partial_x h_I'[\phi^<(x)] &= -\sqrt{4\pi}\left[\partial_x(U(x))\sin(\sqrt{4\pi}\phi^<)\right. \\ &\quad \left.+\sqrt{4\pi}\delta(x)\cos(\sqrt{4\pi}\phi^<)\partial_x\phi^<\right] \quad \text{(B.23)}\\ (\partial_x h_I'[\phi^<(x)])^2 &= 4\pi\left[\left(\partial_x U(x)\sin(\sqrt{4\pi}\phi^<)\right)^2 + 4\pi\left(U(x)\cos(\sqrt{4\pi}\phi^<)\partial_x\phi^<\right)^2\right. \\ &+ \left. 2\sqrt{4\pi}\partial_x(U(x))U(x)\sin(\sqrt{4\pi}\phi^<)\cos(\sqrt{4\pi}\phi^<)\partial_x\phi^<\right] \quad \text{(B.24)}\end{aligned}$$

According to the identities that $2\sin(\phi)\cos(\phi) = \sin(2\phi)$, $\sin^2(\phi) = (1-\cos(2\phi))/2$ and $\cos^2(\phi) = (1+\cos(2\phi))/2$, since the doubling of ϕ corresponds to the doubling of power in the scaling equation, the $\sin(2\phi)$ and the $\cos(2\phi)$ terms are irrelevant, thus can be dropped. Also, the $(\partial_x U(x))^2/2$ term only contribute a constant to the Hamiltonian, we may always shift the energy such that this term is irrelevant. Under the above conditions, we have

$$(\partial_x h_I'[\phi^<(x)])^2 = (4\pi U(x))^2(\partial_x\phi^<(x))^2 \quad \text{(B.25)}$$

We can now find the effective Hamiltonian as

$$\begin{aligned}H &= H^< + \Delta H \\ &= \int_{-\infty}^{\infty} dx\left\{\frac{vK}{2}(P^<)^2 + \frac{v}{2K}\left[1+\frac{\xi}{4}(4\pi U(x))^2\right](\partial_x\phi^<(x))^2\right. \\ &\quad \left.+ U(x)(1-K\ell)\cos(\sqrt{4\pi}\phi^<(x))\right\} \quad \text{(B.26)}\end{aligned}$$

Following the scale invariance hypothesis, we require

$$\int dt'dx'h[P^<(x'),\phi^<(x')] = \int dtdxh[P^<(x),\phi^<(x)]$$

which leads to

$$\begin{aligned}
&\int dtdx\left\{\frac{vK}{2}(P^<(x))^2+\frac{v}{2K}\left[1+\frac{\xi}{4}(4\pi U(x))^2\right](\partial_x\phi^<(x))^2\right.\\
&\qquad\left.+U(x)(1-K\ell)\delta(x)\cos(\sqrt{4\pi}\phi^<(x))\right\}\\
&=\int dt'dx'\left\{\frac{v'K'}{2}\left(P^<(x')\right)^2+\frac{v'}{2K'}\left(\partial_{x'}\phi^<(x')\right)^2+U'(x')\cos(\sqrt{4\pi}\phi^<(x'))\right\}
\end{aligned}$$

It is known that for $x = x's$ and $t = t's$, the scaling dimension of $\phi(x)$ and $P(x)$ is given by $\phi_{\Lambda/s}(x,t) = s^{\Delta}\phi_{\Lambda}(x',t')$ and $P_{\Lambda/s}(x,t) = s^{\Delta-1}P_{\Lambda}(x',t')$, where $\Delta = 0$ in 1D. We can restore the cut-off from Λ/s to Λ and get the new set of parameters:

$$v'K' = vK \tag{B.27}$$

$$\frac{v'}{K'} = \frac{v}{K}\left[1+\frac{K}{v}(4\pi U\delta(x))^2\xi\right] \tag{B.28}$$

$$U' = U(1-K\ell)s \tag{B.29}$$

where $s = e^{\ell} \cong 1+\ell$

From eq. (B.29), one can see that

$$U' = U(1-K\ell)e^{\ell} = U(1-K\ell)(1+\ell) \simeq U - U(K-1)\ell \tag{B.30}$$

We can then find the β-function of U as

$$\frac{dU}{d\ell} = U(1-K) \tag{B.31}$$

where K is given in the previous discussion as $2\pi\langle\delta\phi(x)\delta\phi(x)\rangle/\ell$.

B.4 Summary

According to the discussions given above, we can summarize the RG procedures as follows.

(a) First, we shall project out the states in the momentum shell $\Lambda/s \leq |q| < \Lambda$ in Heisenberg picture, and choose $s = \exp \ell$ with $\ell \to 0$. The projection is done such that $T \exp[-\frac{i}{\hbar} \int dt \int d^d x h[\phi(x,t), P(x,t)]]$ remains invariant.

(b) Then, we shall restore the cutoff from Λ/s to Λ by increasing the unit length by a factor of s, $x' = x/s$, $t' = t/s$.

(c) Finally, we shall determine the scaling dimension of $\phi(x)$ and $P(x)$ such that $\int dt \int d^d x h_0[\phi(x,t), P(x,t)]$ remains invariant, i.e. $\phi_{\Lambda/s}(x,t) = \phi_{\Lambda/s}(x's, t's) = s^{\Delta}\phi_{\Lambda}(x',t')$ and $P_{\Lambda/s}(x,t) = s^{\Delta-1}P_{\Lambda}(x',t')$, where Δ is the scaling dimension of $\phi(x)$. From the invariance of the free part, one can obtain that $\Delta = \frac{1}{2}(d+1) - 1$. Using the scaling dimension of $\phi(x)$, we can find the scaling dimension of the monomials, $O_{i,\Lambda/s}(x,t) = s^{\Delta_i} O_{i,\Lambda}(x',t')$. This leads to the scaling of the coupling constants $U_i(s) = s^{d+1-\Delta_i} U_i(s=1)$. If $d+1-\Delta_i > 0$, U_i flows to ∞ for $s \to \infty$, we say U_i is relevant. If $d+1-\Delta_i < 0$, $U_i \to 0$, so we say U_i is irrelevant. If $d+1-\Delta_i = 0$, we say U_i is marginal.

Appendix C

Some Important Mathematical Results

C.1 Operator Identity

In this section, we will proof a very important identity used in appendix A, namely

$$e^A e^B =: e^{A+B} : e^{<AB+\frac{A^2+B^2}{2}>}. \tag{C.1}$$

Here A and B are bosonic operators of the form $A = \bar{\alpha}\hat{a}^+ + \alpha\hat{a}$ and $B = \bar{\beta}\hat{a}^+ + \beta\hat{a}$ where α, $\bar{\alpha}$, β and $\bar{\beta}$ are constants, and $[\hat{a}, \hat{a}^+] = 1$. The operators A and B satisfy the commutation relation $[A, B] = \alpha\bar{\beta} - \bar{\alpha}\beta$.

We shall start by considering a more familiar identity

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \tag{C.2}$$

provided that operators A and B commute with $[A, B]$. Now, substitute the

operators A and B into eq. (C.2), we have

$$\begin{aligned} e^{A+B} &= e^{A}e^{B}e^{-\frac{1}{2}[A,B]} = e^{A}e^{B}e^{-\frac{1}{2}(\alpha\bar{\beta}-\beta\bar{\alpha})} \\ &= e^{(\bar{\alpha}+\bar{\beta})\hat{a}^{+}+(\alpha+\beta)\hat{a}} = e^{(\bar{\alpha}+\bar{\beta})\hat{a}^{+}}e^{(\alpha+\beta)\hat{a}}e^{\frac{1}{2}(\bar{\alpha}+\bar{\beta})(\alpha+\beta)} \\ &= :e^{A+B}: e^{\frac{1}{2}(\bar{\alpha}\alpha+\bar{\beta}\beta+\bar{\beta}\alpha+\bar{\alpha}\beta)} \end{aligned} \tag{C.3}$$

Since $< \hat{a}^{+}\hat{a}^{+} >=< \hat{a}\hat{a} >=< \hat{a}^{+}\hat{a} >= 0$ and $< \hat{a}^{+}\hat{a} >= 1$, we can easily obtain that $< A^2 >= \bar{\alpha}\alpha$, $< B^2 >= \bar{\beta}\beta$ and $< AB >= \alpha\bar{\beta}$. By rearranging the result given in eq. (C.3), we can readily prove the result in eq. (C.1).

C.2 Legendre Transformation

In this section, the Legendre transformation that converts a Lagrangian to a Hamiltonian, and vice versa, is discussed. Let q be a n-dimensional coordinate vector, i.e. $q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$, the time derivative of the coordinate vector q (velocity vector) is defined as $\dot{q} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{pmatrix}$. Consider a Lagrangian of the form

$$L(q,\dot{q}) = \frac{1}{2}\sum_{i,j}\mathbf{M}_{ij}\dot{q}_i\dot{q}_j - V(q) = \frac{1}{2}\left(\dot{q}^{T}\cdot\mathbf{M}\cdot\dot{q}\right) - V(q), \tag{C.4}$$

one can obtain the corresponding Hamiltonian by performing a Legendre transformation. Here, we are restricted only to the coordinate dependent potentials, i.e. $V = V(q)$. As a result, the Hamiltonian is given as

$$H(q,p) = p^T \cdot \dot{q} - L(q,\dot{q}) \tag{C.5}$$

where p is an n-dimensional canonical momentum vector. The canonical momentum $p^T = (p_1, p_2, \ldots, p_n)$ is defined as $p_i = \frac{\partial L}{\partial \dot{q}_i} = \sum_j M_{ij}\dot{q}_j$. Here, we assume that the mass matrix $\mathbf{M}$ is invertible and $\det \mathbf{M} > 0$. Therefore, the relation between the canonical momentum p and the velocity vector $\dot{q}$ are given as

$$\dot{q}_i = (M^{-1})p_i. \tag{C.6}$$

By substituting equation (C.6) into equation (C.5), we have

$$\begin{aligned} H &= p^T(M^{-1})p - \left(\frac{1}{2}(p^T M^{-1})M(M^{-1}p) - V(q)\right) && \text{(C.7)} \\ &= \frac{1}{2}\left(p^T M^{-1} p\right) + V(q) && \text{(C.8)} \end{aligned}$$

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