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Colimits in the proper homotopy category

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Colimits in the Proper Homotopy Category

by

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Abstract

Colimits In The Proper Homotopy Category

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Dasarat Misir

Adviser: Professor Alex Heller

We study the proper homotopy category of the Category $\mathcal{L}$ of locally finite CW complexes. An object in $\mathcal{L}$ can be considered as being bi-filtered. This leads us to consider functor categories $C^{\mathcal{J}}$, where C is a category of CW complexes and $\mathcal{J}$ a small category that encodes the bifiltered character of an object in $\mathcal{L}$. We are able to show that $C^{\mathcal{J}}$ has enough structure to enable us to calculate left homotopy Kan extensions in appropriate fraction categories of $C^{\mathcal{J}}$. Also, we show that the proper homotopy category of $\mathcal{L}$ is equivalent to a fraction category of K^{W}, where K is the category of finite CW complexes and cellular maps, and W an appropriate subcategory of $\mathcal{J}$.

These results enable us to show that certain constructions in the ordinary homotopy category which are homotopy colimits, when considered in the proper homotopy category, turnout to be left homtopy Kan extensions. Further, the Milnor's classifying space construction lifts to the proper homotopy category when the group is finite.

Acknowledgments

Blessed are YOU, LORD God of Israel forever and ever. Yours, O LORD, is the greatness, the power and the glory, the victory and the majesty; for all that is in heaven and in the earth is Yours; Yours is the kingdom, O LORD, and you are exalted as head over all. Bothh riches and honor come from you, and might; in Your hand it is to make great and to give strength to all. Now therefore, our God, we thank you and praise your glorious name. The Bible, Chronicles 29:11,12 and 13,.... Blessed be the God and Father of our Lord Jesus Christ who has blessed us with every spiritual blessing in the heavenly places in Christ. The Bible Ephesians 1:3.

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This dissertation is dedicated to my loving and caring mother and to the memory of my father who as my high school principal was an inspiration and challenge to me.

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INTRODUCTION

Our aim is to study the proper homotopy category of the category $\mathcal{L}$ of locally finite CW complexes [Edwards & Hastings]. This proper homotopy category first got our attention because even pairwise simple products fail to exist. For example, consider the space X, a circle attached to $\mathcal{R}^+$ at 0, and the space Y a circle attached to each integer point of $\mathcal{R}^+$. The product of X and Y does not exist in the proper homotopy category [Misir]. An object in $\mathcal{L}$ can be considered as being "bi-filtered": an exhaustion by finite subcomplexes provides an increasing filtration; the smallest subcomplexes containing their complements gives a decreasing filtration. This leads us to consider functor categories $\mathcal{C}^{\mathcal{J}}$, where $\mathcal{C}$ is a category of CW complexes and $\mathcal{J}$ a small category that encodes the bi-filtered character of an object in $\mathcal{L}$. $\mathcal{C}^{\mathcal{J}}$ does not have a Quillen model structure. However, we are able to show in chapter 1 that $\mathcal{C}^{\mathcal{J}}$ has enough structure to enable us to calculate left homotopy Kan extensions in appropriate fraction categories of $\mathcal{C}^{\mathcal{J}}$. In chapter 2, we show that the proper homotopy category of $\mathcal{L}$ is equivalent to a fraction category of K^W, where K is the category of finite CW complexes and cellular maps, and W an appropriate subcategory of $\mathcal{J}$. The methods used in dealing with homotopy Kan extensions are adopted from those of [Heller 2]. Armed with the results of chapters 1 and 2, we explicate two a priori mysterious constructions in the proper homotopy category. First, the disjoint union of a sequence of locally compact spaces is locally compact, but certainly not a coproduct. We shall see that this is a Kan extension when described in terms of the richer structure we have introduced. Secondly, the Milnor telescope [Milnor] of a sequence of locally compact spaces and proper maps is again locally compact, but not as in the standard case a homotopy colimit.

Furthermore we shall see that the Milnor's classifying space construction lifts to the proper homotopy category when the group is finite. Thus in particular, the classifying space of a finite group has a canonical proper homotopy structure. Milnor's construction in fact produces in ordinary homotopy theory a homotopical orbit-space functor. We shall try to explicate its role in proper homotopy theory.

We remark that the "bi-filtered" character which is canonical for locally compact spaces, may nevertheless be of interest for more general spaces, where it would be an additional structure. We have not as yet, however, explored this question.

CHAPTER 1

Homotopy Theory & Homotopy Left Kan Extensions in $C^{\mathcal{J}}$

Let C be either the category K of finite CW complexes and cellular map or that of locally finite CW complexes and proper cellular maps, and $\mathcal{J}$ a small category. The functor category $C^{\mathcal{J}}$ does not have a Quillen model structure. However, for nice categories $\mathcal{J}$, we shall show that $C^{\mathcal{J}}$ has enough structure to enable us to calculate some proper homotopy Kan extensions.

A partially ordered set $(\mathcal{J}, \leq)$ is said to be of finite descent if for any $j \in \mathcal{J}$, the set $A(j) = \{m \in \mathcal{J} \mid m < j\}$ is finite. We shall say $f : X \longrightarrow Y$ in C is a cofibration, if f maps X isomorphically to a subcomplex of Y.

Definition 1.1: An object X in the functor category $C^{\mathcal{J}}$ is said to be cofibrant if for $j' \leq j$, $X_{j'} \longrightarrow X_j$ is a cofibration, $\underset{j' < j}{colim}\ X_j$ exists and $\underset{j' < j}{colim}\ X_{j'} \longrightarrow X_j$ is a cofibration.

We now establish some properties of cofibrant objects. For $\mathcal{J}' \subset \mathcal{J}$, we say $\mathcal{J}'$ is initial in $\mathcal{J}$ if for $j \in \mathcal{J}, j' \in \mathcal{J}'$ and $j < j'$ then $j \in \mathcal{J}'$. If $X : \mathcal{J} \longrightarrow C$ and $\mathcal{J}'$ initial in $\mathcal{J}$, then the restriction $X \mid \mathcal{J}'$ will be denoted by $X_{\mathcal{J}'}$. Also, if $\underset{\mathcal{J}'}{colim}\ X_{\mathcal{J}'}$ exists, we denote it by $\underset{\mathcal{J}'}{colim}\ X$.

Lemma 1.1 Let X be a cofibrant object of $C^{\mathcal{J}}$. Suppose $\underset{\mathcal{J}}{colim}\, X$ and $\underset{\overline{\mathcal{J}}}{colim}\, X$ exist, where $\overline{\mathcal{J}}$ is initial in $\mathcal{J}$. Then the induced map $\underset{\overline{\mathcal{J}}}{colim}\, X \longrightarrow \underset{\mathcal{J}}{colim} X$ is a cofibration.

Proof: Let the collection $\mathcal{L} = \{X_{\mathcal{J}'}\}$ consists of those $X_{\mathcal{J}'}$, such that $\overline{\mathcal{J}} \subseteq \mathcal{J}'$, $\mathcal{J}'$ initial in $\mathcal{J}$, $\underset{\mathcal{J}'}{colim}\, X$ exists and the induced map $\underset{\mathcal{J}''}{colim}\ X \longrightarrow \underset{\mathcal{J}'}{colim}\, X$ is a cofibration for $\overline{\mathcal{J}} \subseteq \mathcal{J}'' \subset \mathcal{J}'$. Clearly, $\mathcal{L}$ is partially ordered by inclusions on the $\mathcal{J}'$'s. Now for any ascending chain $S = \{X_{\mathcal{J}'}\} \subset \mathcal{L}$ the union $\bigcup_{X_{\mathcal{J}'} \in S} \mathcal{J}' = \mathcal{J}''$ is initial in $\mathcal{J}$, and $\underset{\mathcal{J}''}{colim}\ X = \bigcup_{X_{\mathcal{J}'} \in S} \underset{\mathcal{J}'}{colim}\, X$ exists. It is clear, for any initial $\mathcal{J}'$ of $\mathcal{J}$ such that

$\overline{\mathcal{J}} \subseteq \mathcal{J}' \subset \mathcal{J}''$ and $\underset{\mathcal{J}'}{colim}\, X$ exits that the induced map $\underset{\mathcal{J}'}{colim}\, X \longrightarrow \underset{\mathcal{J}''}{colim}\, X$ is a cofibration ($\mathcal{J}'$ is contained in some $\mathcal{J}'_\alpha$ where $X_{\mathcal{J}'_\alpha} \in S$). Hence $X_{\mathcal{J}''} \in \mathcal{L}$. By Zorn's Lemma there is a maximal $X_{\mathcal{J}_m}$ in $\mathcal{L}$. If $\mathcal{J}_m = \mathcal{J}$ we are done. Let's assume $\mathcal{J}_m \neq \mathcal{J}$. Since $\mathcal{J}_m$ is initial in $\mathcal{J}$, there is a minimal $j \in \mathcal{J}$ such that $j \notin \mathcal{J}_m$

Now consider the pushout:

$$\begin{array}{ccc} \underset{j'<j}{colim}\, X_j & \longrightarrow & X_j \\ \downarrow & & \downarrow \\ \underset{\mathcal{J}_m}{colim}\, X & \longrightarrow & \underset{\mathcal{J}_m \cup \{j\}}{colim}\, X \end{array}$$

.

By the definition of cofibrant objects $\underset{j'<j}{colim}\, X_{j'} \longrightarrow X_j$ is a cofibration, then $\underset{\mathcal{J}_m}{colim}\, X \longrightarrow \underset{\mathcal{J}_m \cup \{j\}}{colim}\, X$ is a cofibration. Hence we have extended $X_{\mathcal{J}_m}$ to $X_{\mathcal{J}_m \cup \{j\}}$. This contradicts the maximality of $X_{\mathcal{J}_m}$ so $\mathcal{J}_m = \mathcal{J}$.

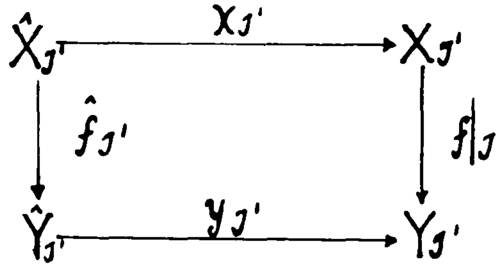
<u>Lemma 1.2</u>: If X is a cofibrant object of $C^{\mathcal{J}}$ then $\underset{\mathcal{J}}{colim}\, X$ exists.

<u>Proof</u>: The proof is similar to that of Lemma 1.1. Let $\mathcal{L} = \{X_{\mathcal{J}'}\}$, $\mathcal{J}'$ initial in $\mathcal{J}$, be the collection of objects such that $\underset{\mathcal{J}'}{colim}\, X$ exist. As before, $\mathcal{L}$ is partially ordered. Lemma 1.1 guarantees that the union of any ascending chain is in $\mathcal{L}$. Zorn's Lemma asserts there is a maximal $X_{\mathcal{J}_m}$ in $\mathcal{L}$. If $\mathcal{J}_m \neq \mathcal{J}$, since $\mathcal{J}_m$ is initial in $\mathcal{J}$, there is a minimal $j \in \mathcal{J}$ such that $j \notin \mathcal{J}_m$. Diagram 1 shows that $X_{\mathcal{J}_m}$ can be extended to $X_{\mathcal{J}_m \cup \{j\}}$. This contradicts the maximality of $X_{\mathcal{J}_m}$. Hence $\mathcal{J}_m = \mathcal{J}$.

<u>Definition 1.2</u>: A map $f : X \longrightarrow Y$ in $C^{\mathcal{J}}$ is a <u>weak homotopy equivalence</u> if for each $j \in \mathcal{J}$, f_j is a homotopy equivalence.

<u>Theorem 1.1</u>: There is a functor $(\ )^\wedge : C^{\mathcal{J}} \longrightarrow C^{\mathcal{J}}$ and a natural transformation $(\ )^\wedge \longrightarrow 1_C$ such that $\hat{X} \longrightarrow X$ is a w.h.e. and $\hat{X}$ is a cofibrant object.

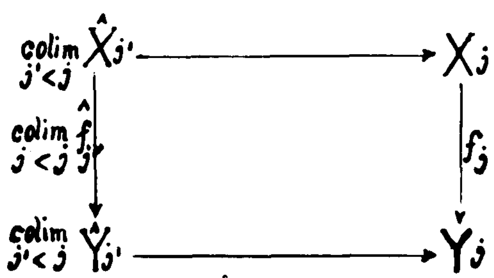
<u>Proof</u>: We construct the functor $(\ )^\wedge$ simultaneously on objects and maps. For $f : X \longrightarrow Y$ in $C^{\mathcal{J}}$, consider the commutative diagram



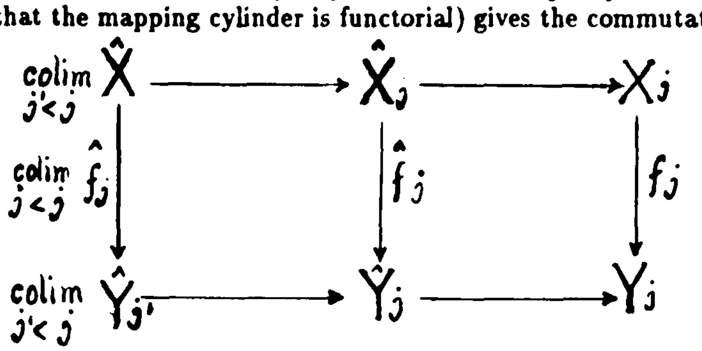
where $\mathcal{J}'$ is initial in $\mathcal{J}$; $\hat{X}_{\mathcal{J}'}$ and $\hat{Y}_{\mathcal{J}'}$ are cofibrant objects of $\mathcal{C}^{\mathcal{J}}$, $x_{\mathcal{J}'}$ and $y_{\mathcal{J}'}$ are w.h.e.'s $\hat{f}_{\mathcal{J}'} : \hat{X}_{\mathcal{J}'} \longrightarrow \hat{Y}_{\mathcal{J}'}$ makes the diagram commutative in $\mathcal{C}^{\mathcal{J}'}$.

Let $\mathcal{L} = \{(\hat{f}_{\mathcal{J}'}, x_{\mathcal{J}'}, y_{\mathcal{J}'})\}$ be the collection of all these commutative diagrams. $\mathcal{L}$ is partially ordered as follows: For the inclusion $\mathcal{J}'' \subset \mathcal{J}'$, $\hat{X}_{\mathcal{J}'} | \mathcal{J}'' = X_{\mathcal{J}''}$, $x_{\mathcal{J}'} | = x_{\mathcal{J}''}$ and $\hat{f}_{\mathcal{J}'} | \mathcal{J}'' = \hat{f}_{\mathcal{J}''}$, thus $(\hat{f}_{\mathcal{J}''}, x_{\mathcal{J}''}, y_{\mathcal{J}''}) \leq \hat{f}_{\mathcal{J}'}, x_{\mathcal{J}'}, y_{\mathcal{J}'})$ if we have the inclusion $\mathcal{J}'' \subset \mathcal{J}'$

Clearly, the union of any ascending chain in $\mathcal{L}$ is in $\mathcal{L}$. By Zorn's Lemma $\mathcal{L}$ has a maximal element: $(\hat{f}_{\mathcal{J}_m}, x_{\mathcal{J}_m}, y_{\mathcal{J}_m})$. If $\mathcal{J}_m = \mathcal{J}$ we are done. Let's assume that $\mathcal{J}_m \neq \mathcal{J}$. Since $\mathcal{J}_m$ is initial in $\mathcal{J}$, there is a minimal $j \in \mathcal{J}$ such that $j \notin \mathcal{J}_m$. By Lemma 1.2 $\underset{j'<j}{colim}\, \hat{X}_{j'}$ and $\underset{j'<j}{colim}\, Y_{j'}$ exist. Consider the commutative diagram induced by the colimits.



Taking the mapping cylinders of $\underset{j'<j}{colim}\, \hat{X}_j \longrightarrow X_j$ and $\underset{j'<j}{colim}\, \hat{X}_j \longrightarrow Y_j$.
(Recalling that the mapping cylinder is functorial) gives the commutative diagram



We have thus extended $(\hat{f}_{\mathcal{J}_m}, x_{\mathcal{J}_m}, y_{\mathcal{J}_m})$. This contradicts the maximality. Thus $\mathcal{J}_m = \mathcal{J}$.

If $f : X \longrightarrow Y$ in $C^{\mathcal{J}}$ is a w.h.e. between cofibrant objects, then for each $j \in \mathcal{J}$ there is a map $g_j : Y_j \longrightarrow X_j$ which is a homotopy inverse to f_j. Also $g : Y \longrightarrow X$ in $C^{\mathcal{J}}$ is a homotopy inverse to the w.h.e. $f : X \longrightarrow Y$ if there exist $H : X \times I \longrightarrow X$ and $L : Y \times I \longrightarrow Y$ such that $H \mid X \times 1 = f \circ g$, $H \mid X \times o = 1_X$, $\times L \mid Y \times 1 = g \circ f$ and $L \mid Y \times o = 1_Y$

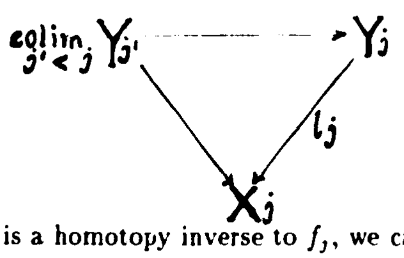
This implies that for each $j \in \mathcal{J}$, f_j and g_j are homotopy inverses.

<u>Theorem 1.2</u>: $f : X \longrightarrow Y$ in $C^{\mathcal{J}}$ is a w.h.e. between cofibrant objects iff there exists a homotopy inverse g to f.

<u>Proof:</u> Consider the set $\mathcal{L} = \{g_{\mathcal{J}'}\}$ consisting of weak homotopy inversees to $f_{\mathcal{J}'}$, where $\mathcal{J}'$ is initial in $\mathcal{J}$. Set $g_{\mathcal{J}''} \leq g_{\mathcal{J}'}$ if $\mathcal{J}'' \subset \mathcal{J}'$, $g_{\mathcal{J}''} = g_{\mathcal{J}'} \mid \mathcal{J}''$, $H_{\mathcal{J}''} = H_{\mathcal{J}'} \mid \mathcal{J}''$ and $L_{\mathcal{J}''} = L_{\mathcal{J}'} \mid \mathcal{J}''$. Then $\mathcal{L}$ is partially ordered.

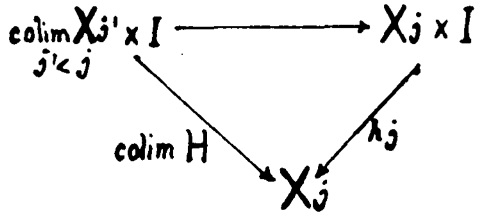
Clearly, the union of any ascending chain of $g'_{\mathcal{J}'}s$ is in $\mathcal{L}$. Hence by Zorn's Lemma there is a maximal $g_{\overline{\mathcal{J}}}$, so there exist $H_{\overline{\mathcal{J}}}$ and $L_{\overline{\mathcal{J}}}$ such that $H_{\overline{\mathcal{J}}}$ is a homotopy between $f_{\overline{\mathcal{J}}} \circ g_{\overline{\mathcal{J}}}$ and $1_{X_{\mathcal{J}}}$, $L_{\overline{\mathcal{J}}}$ is a homotopy between $g_{\overline{\mathcal{J}}} \circ f_{\overline{\mathcal{J}}}$ and $1_{Y_{\overline{\mathcal{J}}}}$, if $\overline{\mathcal{J}} = \mathcal{J}$ we are done, so assume not. There is a minimal $j \in \mathcal{J}$ such that $j \notin \overline{\mathcal{J}}$, since $\overline{\mathcal{J}}$ is initial in $\mathcal{J}$. We shall extend $g_{\overline{\mathcal{J}}}$ to $g_{\overline{\mathcal{J}} \cup \{j\}}$ and the homotopies $H_{\overline{\mathcal{J}}}$ to $H_{\overline{\mathcal{J}} \cup \{j\}}$. $L_{\overline{\mathcal{J}}}$ extends to $L_{\overline{\mathcal{J}} \cup \{j\}}$ in analogy to the extension of $H_{\overline{\mathcal{J}}}$, which we leave out. Thus we show that the w.h.e. $g_{\overline{\mathcal{J}}}$ is extended to the w.h.e. $g_{\overline{\mathcal{J}} \cup \{j\}}$ which would contradict maximality of $g_{\overline{\mathcal{J}}}$. Hence $\overline{\mathcal{J}} = \mathcal{J}$.

We now extend $g_{\overline{\mathcal{J}}}$. Y being cofibrant implies that $\operatorname{colim}_{j' < j} Y_{j'}$ exists and $\operatorname{colim}_{j' < j} Y_{j'} \longrightarrow Y_{j'}$ is a cofibration. Since the diagram



commutes up to homotopy, where l_j is a homotopy inverse to f_j, we can change l_j up to homotopy to a map g_j so that the above diagram commutes. We have therefore, extended $g_{\mathcal{J}}$ to $g_{\mathcal{J}\cup\{j\}}$. (Note g_j is a homotopy inverse of f_j.) We now extend $H_{\mathcal{J}}$

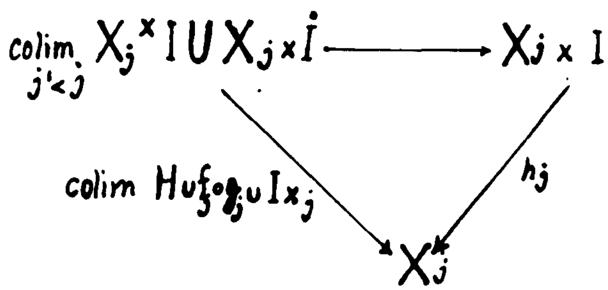
The diagram



commutes up to homotopy where h_j is a homotopy between $f_j \circ g_j$ and 1_X,

Note, $\operatorname{colim}_{j'<j} X_{j'} \times I \longrightarrow X_j \times I$ and $X_j \times \dot{I} \longrightarrow X_j \times I$ are cofibrations. Also $\operatorname{colim}_{j'<j} X_{j'} \times \dot{I}$ is cofibered in both $X_j \times \dot{I}$ and $\operatorname{colim}_{j'<j} X_{j'} \times I$, thus $\operatorname{colim}_{j'<j} X_{j'} \times I \cup X_j \times \dot{I} \longrightarrow X_j \times I$ is a cofibration.

The diagram :



commutes up to homotopy We can change h_j up to homotopy so that the diagram commutes. We have therefore extended $H_{\mathcal{J}}$ to $H_{\mathcal{J}\cup\{j\}}$. The converse of the theorem is obvious.

We now give examples of some computations of Kan extensions.

The set $\mathcal{I} = \{[m,n] \mid m \leq n, m \text{ and } n \in \mathcal{N}\}$, where $[m,n] = \{m, m+1, \ldots, n\}$, are partially ordered by inclusion. We regard $\mathcal{I}$ as an ordered category. Observe $\mathcal{I}$ and $\mathcal{I} \times \mathcal{I}$ are finite descent categories. We now consider the functor categories $C^{\mathcal{I}\times\mathcal{I}}$ and $C^{\mathcal{I}}$. Recall, C is the category of locally finite CW complexes and proper cellular maps or C is the category of finite CW complexes and cellular maps.

The sum functor $S : \mathcal{I} \times \mathcal{I} \longrightarrow \mathcal{I}$, where $S([m,n] \times [i,j]) = [m+i, n+j]$, gives rise to $S^* : C^{\mathcal{I}} \longrightarrow C^{\mathcal{I}\times\mathcal{I}}$.

Let $Ho(\mathcal{C}^{\mathcal{I}})$ and $Ho(\mathcal{C}^{\mathcal{I}\times\mathcal{I}})$ be the fraction categories gotten by inverting their w.h.e.'s. Obviously, S^* preseves w.h.e.'s hence $HoS^* : Ho(\mathcal{C}^{\mathcal{I}}) \longrightarrow Ho(\mathcal{C}^{\mathcal{I}\times\mathcal{I}})$

Lemma 1.3: HoS^* has a left adjoint $Holan_s$

Proof: Theorem 1.1 gives the following construction: for $X \in \mathcal{C}^{\mathcal{I}\times\mathcal{I}}$, there is a cofibrant $\hat{X}$ and a w.h.e. $\hat{X} \longrightarrow X$. Therefore, it suffices to define $Holan_s$ on cofibrant objects. For X a cofibrant object, we define $(Holan_s X)_{[m,n]}$ where $[m,n]$ is any interval of $\mathcal{I}$. Note, $S \downarrow [m,n]$ is a finite, partially ordered set. Hence, the composition $S \downarrow [m,n] \longrightarrow \mathcal{I}\times\mathcal{I} \longrightarrow \mathcal{C}$ is cofibrant, and $\operatorname{colim}_{S\downarrow[m,n]} X = \operatorname{colim}(S \downarrow [m,n] \longrightarrow \mathcal{I}\times\mathcal{I} \longrightarrow \mathcal{C})$ exists.

Define $(Holan_s X)_{[m,n]} = (Lan_S X)_{[m,n]} = \operatorname{colim}_{S\downarrow[m,n]} X$.

All we need to show is that $Holan_s$ preserves w.h.e. Let $f : X \longrightarrow Y$ a w.h.e. between cofibrant objects. By theorem 1.2 there exists $g : Y \longrightarrow X$ such that $g \circ f$ is homotopic to 1_X and $f \circ g$ is hmotopic to 1_Y. If H is a homotopy between $g \circ f$ and 1_X, that is $H : X \times I \longrightarrow X$ such that $H|X \times I = g \circ f$ and $H|X \times o = 1_X$

then $\operatorname{colim}_{S\downarrow[m,n]} X \times I \longrightarrow \operatorname{colim}_{S\downarrow[m,n]} X$. Since $- \times I$ commutes with colimits, it follows that $\operatorname{colim}_{S\downarrow[m,n]} g \circ f \simeq 1 \operatorname{colim}_{S\downarrow[m,n]} X$ Hence, $Holan_s$ preserves w.h.e.'s

Clearly, $Holan_s \dashv HoS^*$.

We shall see in chapter 2 that the proper homotopy category embeds in a fraction category of $Ho(\mathcal{C}^{\mathcal{I}})$, which we now define by introducing the notion of a reindexing on $Ho(\mathcal{C}^{\mathcal{I}})$ and $Ho(\mathcal{C}^{\mathcal{I}\times\mathcal{I}})$.

The process is described first on $Ho(\mathcal{C}^{\mathcal{I}})$. Let $\Psi : \mathcal{I} \longrightarrow \mathcal{I}$ be an order preserving map such that $[m,n] \subset \Psi[m,n]$ for each interval $[m,n]$ of $\mathcal{I}$, and as $m \longrightarrow \infty$, $l(\Psi[m,n]) \longrightarrow \infty$, where $l[a,b] = a$. Each such Ψ and $X \in Ho(\mathcal{C}^{\mathcal{I}})$ give rise by composition to an object in $Ho(\mathcal{C}^{\mathcal{I}})$ which will denote by X_Ψ. It is clearly seen that there is a morphism $i_\Psi : X \longrightarrow X_\Psi$. We will refer to i_Ψ as a reindexing. Let $\epsilon_{\mathcal{I}}$ be the collection of all reindexings and form the fraction category $(Ho(\mathcal{C}^{\mathcal{I}}))[\epsilon_{\mathcal{I}}^{-1}]$.

Similarly, we introduce the notion of reindexing on $H_0(\mathcal{C}^{\mathcal{I}\times\mathcal{I}})$. Consider, $\Psi : \mathcal{I}\times\mathcal{I} \longrightarrow \mathcal{I}\times\mathcal{I}$ an order preserving map with the following three properties:

1) $\Psi(R) \supset R$ for $R \in \mathcal{I} \times \mathcal{I}$

2) if $S(R) = S(Q)$ then $S(\Psi(R)) = S(\Psi(Q))$ where S is the sum function defined above and $Q \in \mathcal{I} \times \mathcal{I}$

3) as $l(SR) \longrightarrow \infty$, $lS(\Psi R) \longrightarrow \infty$

As before, the morphism $i_\Psi : X \longrightarrow X_\Psi$ is called a reindexing. Let $\epsilon_{\mathcal{I}\times\mathcal{I}}$ be the collection of all reindexings, and form the fraction category $(Ho(C^{\mathcal{I}\times\mathcal{I}}))[\epsilon^{-1}_{\mathcal{I}\times\mathcal{I}}]$.

Theorem 1.3: HoS^* and $Holan_*$ factor through the fraction categories.

Proof: We shall show that $HoS^*(i_\Psi) \in \epsilon_{\mathcal{I}\times\mathcal{I}}$ for $i_\Psi : X \longrightarrow X_\Psi$ in $\epsilon_{\mathcal{I}}$. We do this by showing that $S^*(X_\Psi) = S^*(X))_\phi$ where the construction of ϕ is as follows. For R a rectangle of $\mathcal{I} \times \mathcal{I}$, let $\phi(R) \supset R$ such that $S(\phi(R)) = \Psi(S(R))$ where Ψ is the map associated with the reindexing i_Ψ. Clearly ϕ satisfies conditions 1, 2, and 3. Further it is easily seen from the construction of ϕ that $S^*(X_\Psi) = S^*(X)\phi$. Hence, $HoS^*(i_\Psi) \in \epsilon_{\mathcal{I}\times\mathcal{I}}$, yielding the factorization of HoS^*.

We now show for $i_\Psi : X \longrightarrow X_\Psi$ in $\epsilon_{\mathcal{I}\times\mathcal{I}}$ that $Holan_*(i_\Psi) : Holan_*X \longrightarrow Holan_*(X_\Psi)$ is an isomorphism in $(Ho(C^{\mathcal{I}}))[\epsilon_{\mathcal{I}}^{-1}]$. The approach is to construct a reindexing for $Holan_*X$ and show that the reindexing morphism factors through $Holan(X_\Psi)$.

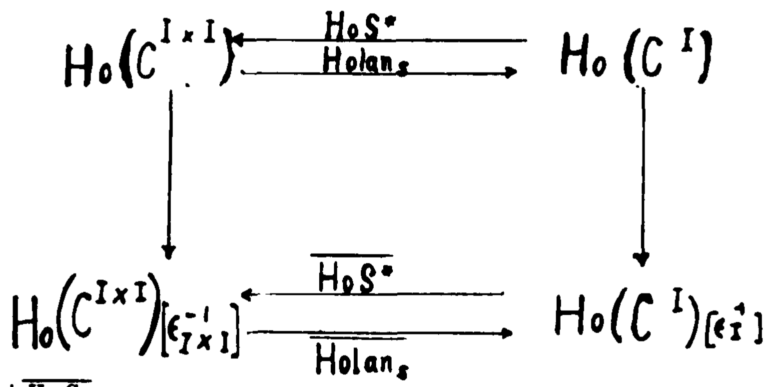
Let $\phi : \mathcal{I} \longrightarrow \mathcal{I}$ be the composition: $\mathcal{I} \xrightarrow{T} \mathcal{I} \times \mathcal{I} \xrightarrow{\Psi} \mathcal{I} \times \mathcal{I} \xrightarrow{S} \mathcal{I}$ where $T[m,n] = [m,n] \times [0,0]$ and Ψ is the map associated to i_Ψ. Observe that if $S(R) = S(Q)$ then $\Psi(R) = \Psi(Q)$ by condition 2. It follows then, if $S(R) \subseteq [m,n]$ that $S(\Psi(R)) \subseteq \phi[m,n]$.

We conclude $\operatorname*{colim}_{S(R) \subseteq [m,n]} X_R \longrightarrow \operatorname*{colim}_{S(R) \subseteq [m,n]} X_{\Psi(R)} \longrightarrow \operatorname*{colim}_{S(Q) \subseteq \phi[m,n]} X_Q$
which is $(Holan_*X)_{[m,n]} \longrightarrow (Holan_*X_\Psi)_{[m,n]} \longrightarrow (Holan_*X)_{\phi[m,n]}$

Similarly, we construct a reindexing for $Holan_*X_\Psi$ and show that the reindexing morphism factors through $(Holan_*(X))_\phi$

We conclude that $Holan_*X \longrightarrow Holan_*X_\Psi$ is an isomorphism.

Hence the factorization



Clearly $\overline{Holan_s} \dashv \overline{HoS^*}$

CHAPTER 2

The Proper Homotopy Category

We begin by defining the categories K, W and $\mathcal{L}$. K is the category of finite CW complexes and cellular maps. W is the full subcategory of $\mathcal{I}$ with objects $\{[m,n] \mid m = n$ or $m + 1 = n\}$. $\mathcal{I}$ is defined in chapter 1.

$\mathcal{L}$ is the category of CW complexes such that there exists a proper map into $\mathcal{R}^+$, the nonnegative reals. Such complexes are of course locally finite. Its morphisms are proper cellular maps.

In this chapter we shall compute the left homotopy Kan extension along the inclusion $W \subset \mathcal{I}$ between $Ho(K^W)$ and $Ho(K^{\mathcal{I}})$. We shall use this result to show the equivalence of the category $\mathcal{L}[phe^{-1}]$ and a fraction category of $Ho(K^W)$.

Let $J : W \longrightarrow \mathcal{I}$ be the inclusion. This gives rise to functors: $K^W \longleftarrow K^{\mathcal{I}}$ and $Top^W \longleftarrow Top^{\mathcal{I}}$. The functor $J^* : Top^{\mathcal{I}} \longrightarrow Top^W$ has a left Kan extension $Lan_J : Top^W \longrightarrow Top^{\mathcal{I}}$, i.e. $Lan_J \dashv J^*$.

There are also forgetful functors $U : K^W \longrightarrow Top^W$ and $V : K^{\mathcal{I}} \longrightarrow Top^{\mathcal{I}}$ and the diagram

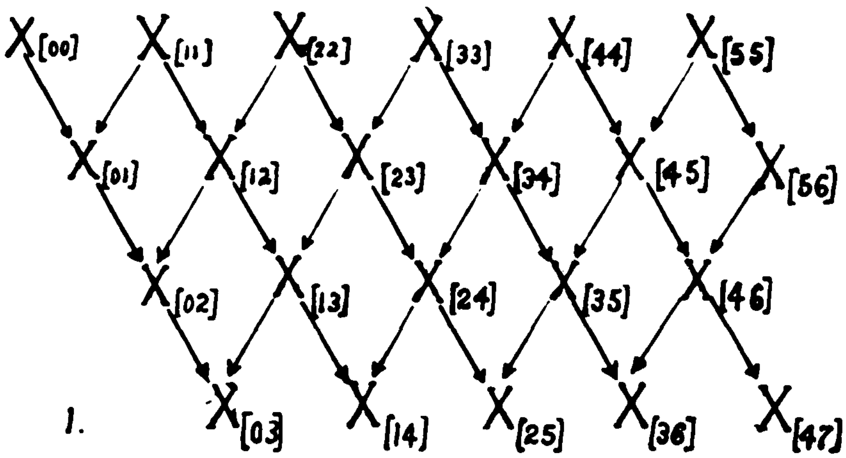
$$\begin{array}{ccc} K^W & \xleftarrow{\;J^*\;} & K^I \\ \Big\downarrow U & & \Big\downarrow V \\ Top^W & \xleftarrow{\;J^*\;} & Top^I \end{array}$$

commutes. Clearly Lan_J restricted to K^W does not land in $K^{\mathcal{I}}$. However, for a full subcategory of K^W it does. We now show this:

In chapter 1, we define cofibrant objects in $C^{\mathcal{J}}$ for $\mathcal{J}$ any finite descent category.

Clearly, W is a finite descent category. Thus, an object X is cofibrant in K^W if and only if $X_{[i,i]} \cup X_{[i+1,i+1]} \longrightarrow X_{[i,i+1]}$ is a cofibration for each i.

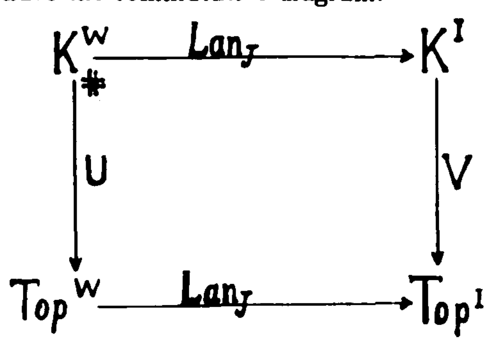
Since X has the form: $X_{[00]} \longrightarrow X_{[01]} \longleftarrow X_{[11]} \longrightarrow X_{[12]} \longleftarrow X_{[23]} \longrightarrow \ldots$ we can describe $Lan_J X$ as follows: taking repeated pushouts of the diagram of X we have



1.

If X is cofibrant in K^W, it is clearly seen that $(Lan_J X)_{[m,n]}$ is compact and for $[m', n'] \subset [m, n]$ that $(Lan_J X)_{[m',n']} \longrightarrow (Lan_J X)_{[m,n]}$ is a cofibration. It follows then, for X cofibrant in K^W, that $Lan_J X \in K^{\mathcal{I}}$.

We therefore, have the commutative diagram:



where $K^W_{\#}$ is the full subcategory of K^W consisting of the cofibrant objects. Now, if we view an object in $K^{\mathcal{I}}$ as a diagram in Top, then we have the functor $colim_{\mathcal{I}} : K^{\mathcal{I}} \longrightarrow Top$.

Proposition 2.1: The composition $colim_{\mathcal{I}} \circ Lan_J : K^W_{\#} \longrightarrow Top$ factors through $\mathcal{L}$.

Proof: Let τ denote the composition $colim_{\mathcal{I}} Lan_J$. we shall show for $X \in K^W_{\#}$ that

$\tau(X)$ is in $\mathcal{L}$, and for $f : X \longrightarrow Y$ in $K^W_\#$ that $\tau(f)$ is proper. From the description of $Lan_J X$, it is seen that $\tau(X) = colim(X_{[00]} \longrightarrow X_{[0,1]} \longrightarrow X_{[02]} \ldots)$. Since $X_{[0,n]} \longrightarrow X_{[0,n+1]}$ is a cofibration for each n, we conclude that $\tau(X)$ is a CW complex.

We need to show that $\tau(X)$ is locally finite, for $\tau(X)$ to be in $\mathcal{L}$. Before we show this, we shall show for $f : X \longrightarrow Y$ in $K^W_\#$ that $\tau(f)$ is proper. For any compact $A \subset \tau(Y) = colim(Y_{[0,0]} \longrightarrow Y_{[0,1]} \longrightarrow \ldots)$, there is a smallest n such that $A \subset Y_{[0,n]}$ and $A \not\subset Y_{[0,n-1]}$. It follows that A is contained in the image of a finite number of spaces

$Y_{[k-1,k]}, Y_{[k,k]}, \ldots, Y_{[n,n]}, Y_{[n,n+1]}$.

So $(\tau f)^{-1}(A)$ intersects only $X_{[k-1,k]}, X_{[k,k]}, \ldots X_{[n,n]}, X_{[n,n+1]}$. Since each $X_{[i,j]}$ is compact, it follows that $(\tau f)^{-1}(A)$ is compact, hence $\tau(f)$ is proper. To show that $\tau(X)$ is locally finite, is sufficient to show the existence of a proper map $\alpha : \tau(X) \longrightarrow \mathcal{R}^+$. Therefore, consider the object $\overline{\mathcal{R}} \in K^W_\#$ where $\overline{\mathcal{R}} : 0 \longrightarrow [0,1] \longleftarrow 1 \longrightarrow [1,2] \longleftarrow 2 \longrightarrow \ldots$.

Clearly, $\tau(\overline{\mathcal{R}}) = \mathcal{R}^+$. For $X \in K^W_\#$, $X_{[i,j]}$ is a normal space, and $X_{[i,i]} \cup X_{[i+1,i+1]} \longrightarrow X_{[i,i+1]}$ is a cofibration hence the existence of a map $X \longrightarrow \overline{\mathcal{R}}$ in $K^W_\#$. It follows, that there exists a proper map $\alpha : \tau(X) \longrightarrow \mathcal{R}^+$. Hence the theorem.

We saw in theorem 2 (chapter 1) that the fraction category $Ho(K^W_\#)$ is also a quotient category. Since $- \times I$ commutes with the functors Lan_* and $colim$, the following factorization follows:

$$\begin{array}{ccc} K^W_\# & \xrightarrow{\quad \tau \quad} & \mathcal{L} \\ \downarrow & & \downarrow \\ Ho(K^W_\#) & \xrightarrow{\quad Ho\tau \quad} & \mathcal{L}[phe^{-1}] \end{array}$$

Proposition 2.2: $Ho\tau : Ho(K^W) \longrightarrow \mathcal{L}[phe^{-1}]$

Proof: In theorem 1 (chapter 1) we showed that $(\ )^\wedge : Ho(K^W) \longrightarrow Ho(K^W_\#)$ is an equivalence of categories. The proposition follows from the above observation.

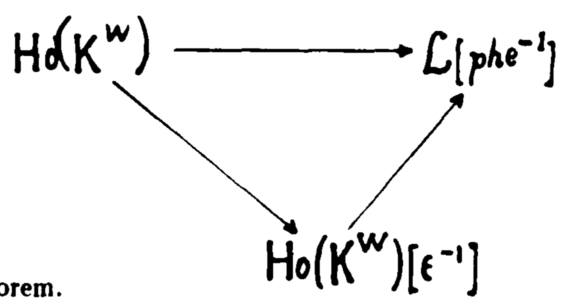
A further fraction category of $Ho(K^W)$ will give us the equivalence of categories we are seeking to achieve. To this end, we introduce a reindexing on $Ho(K_{\#}^{W})$. We do this reindexing by applying the functor $Holan_J$ to an object in $Ho(K_{\#}^{W})$. We reindex this object in $Ho(K^{\mathcal{I}})$ and then apply the functor J^{*} to get the reindexing in $Ho(K_{\#}^{W})$. That is, for $X \in K_{\#}^{W}$, consider the reindexing $i_{\Psi} : Lan_J X \longrightarrow (Lan_J X)_{\Psi}$ in $Ho(K^{\mathcal{I}})$. Now applying $J^{*}(Lan_J X \longrightarrow (Lan_J X)_{\Psi})$ we have $X \longrightarrow X_{\Psi}$, where X_{Ψ} is $J^{*}(Lan_J X)_{\Psi}$, and it is easily seen that $J^{*}(Lan_J X) = X$

Let $\epsilon = \epsilon_W$ be the collection of all reindexing maps $J^{*}(i)_{\Psi}$. As before we have the fraction category $Ho(K^{W})[\epsilon^{-1}]$.

Proposition 2.3: $Ho\tau$ factors through the fraction category.

Proof: For $\rho : X \longrightarrow X_{\Psi}$ a reindexing, it is easily seen from the discription of $Lan_J X$ in fig 1 that $colim_{\mathcal{I}}(Lan_J X) = colim_{\mathcal{I}}((Lan_J X)_{\Psi})$. It follows that $Ho\tau(X) \simeq Ho\tau(X_{\Psi})$.

Hence, we have the factorization



We may now state the main theorem.

Theorem 2.1: $q : (Ho(K^{W}))[\epsilon^{-1}] \longrightarrow \mathcal{L}[phe^{-1}]$ is an equivalence of categories.

Proof: We define a functor $T : \mathcal{L}[phe^{-1}] \longrightarrow (Ho(K^{W}))[\epsilon^{-1}]$ such that $q \circ T \simeq 1_{\mathcal{L}[phe^{-1}]}$ and $1_{Ho(K^{W})[\epsilon^{-1}]} \simeq T \circ q$.

For a space X in $\mathcal{L}$, choose a proper map $\alpha : X \longrightarrow \mathcal{R}^{+}$. This gives a sequence of subcomplexes $X_1, X_2, \ldots\ldots, X_m, \ldots$ where X_m is the smallest subcomplex of X that contains $\alpha^{-1}[m-1, m]$. We define T as follows: Let $T(X,\alpha)_{[0,1]} = X_1$. Since X is locally finite, X_1 is compact, and X_1 intersects a finite number of subcomplexes from the sequence $X_2, X_3, \ldots, X_m, \ldots$. Renumbering the sequence, we may assume $X_2, X_3, \ldots, X_l$ are the subcomplexes that intersect X_1 nontrivially. Set $T(X, \alpha)_{[1,2]} = X_2 \cup X_3 \cdots \cup X_l$. Clearly $T(X,\alpha)_{[1,2]}$ is compact. Repeat the process as with X_1 and the sequence $X_{l+1}, \ldots X_n, \ldots$.

What we have is sequence of subcomplexes of X,

$T(X,\alpha)_{[0,1]}$, $T(X,\alpha)_{[1,2]}$, $T(X,\alpha)_{[2,3]}\ldots$ Observe from the construction that

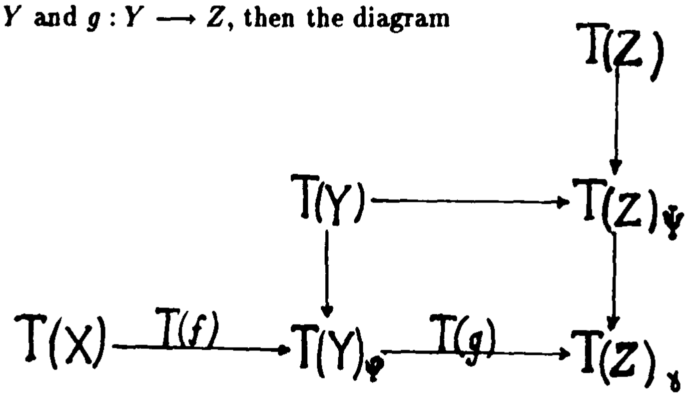
$T(X,\alpha)_{[n,n+1]} \cap T(X,\alpha)_{[m,m+1]} = \phi$ for $m \neq n \pm 1$ or n.

Let $T(X,\alpha)_{[n,n]} = T(X,\alpha)_{[n-1,n]} \cap T(X,\alpha)_{[n,n+1]}$. Clearly, $T(X,\alpha)$ is a cofibrant object in K^W.

We now show that $T(X,\alpha)$ is well defined up to isomorphism in $Ho(K^W)[\epsilon^{-1}]$. If $\beta : X \longrightarrow \mathcal{R}^+$ in $\mathcal{L}$ is also a proper map, then we obtain $T(X,\beta)$. Now consider the description of $Lan_J(T(X,\alpha))$ as in fig (1). Since $T(X,\beta)_{[n,n+1]}$ is compact, we can choose $[l,m]$ such that $[n,n+1] \subset [l,m]$ and $T(X,\beta)_{[n,n+1]} \subset (Lan_J T(X\alpha))_{[l,m]}$. This gives a reindexing $T(X,\alpha)_\Psi$ of $T(X,\alpha)$, such that $T(X,\beta) \subset T(X,\alpha)_\Psi$. Similarly, we can show that $T(X,\alpha) \subset T(X,\beta)_\phi$ for some reindexing of $T(X,\beta)$. We conclude that $T(X,\beta)$ is isomorphic to $T(X,\alpha)$ in $Ho(K^W)[\epsilon^{-1}]$

We can now write $T(X)$ instead of $T(X,\alpha)$. We may observe at this point, that $q(T(X)) = X$ since $colim_I(Lan_J T(X)) = X$.

We now define T on maps. If $f : X \longrightarrow Y$ in $\mathcal{L}$, it is clearly seen that $T(X) \longrightarrow T(Y)_\Psi \longleftarrow T(Y)$ for some reindexing $T(Y)_\Psi$ of $T(Y)$. Composition is also straightforward. If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, then the diagram



shows $Tg \circ Tf = T(g \circ f)$. Note $T(g)$ is defined on $T(Y)_\Psi$, since for each n, $(T(Y))_{\Psi[n,n+1]}$ is a subcomplex of Y.

Finaly, if $f : X \longrightarrow Y$ is a proper homotopy equivalence with inverse g, then there exist proper maps H and G such that H is a homotopy between 1_X and $g \circ f$; and G

is homotopy between 1_Y and $f \circ g$. Now from the construction of T it is clear that T commutes with $- \times I$, that is $T(X \times I) \simeq T(X) \times I$. Since H is proper we can clearly choose a reindexing $T(X)_*$ for $T(X)$ such that $T(H)$ is a homotopy between $T(g \circ f)$ and $T(1_X)$ Similarly, we have $T(G)$ a homotopy between $T(1_Y)$ and $T(f \circ g)$

Therefore, T preserves homotopy equivalences. We already saw that $q(T(X)) = X$ Also, it is easily seen that $qT(f) = f$.

Now we shall show for $Y \in (Ho(K^W))[\epsilon^{-1}]$ that $Y \longrightarrow Tq(Y)$ is a reindexing. For given Y there exists $\alpha : Y \longrightarrow \overline{\mathcal{R}}$ in $(Ho(K^W))[\epsilon^{-1}]$. Hence $q(Y) \longrightarrow \mathcal{R}^+$, and $q(\alpha)$ is a proper map. It follows that $q(\alpha)^{-1}[0,1] = Y_{[0,1]}$. In general $q(\alpha)^{-1}[n,m] = Y_{[n,m]}$ where $m = n$ or $n+1$. Hence, $Y \longrightarrow Tq(Y)$ is a reindexing.We conclude that q is an equivalence of categories.

CHAPTER 3

Constructions in the Proper Homotopy Category

We now use the description of the proper homotopy category embodied in Theorem 2.1 to explicate some puzzling constructions in that category. The disjoint union of the sequence of locally compact spaces is again locally compact, but is not their coproduct, either in $\mathcal{L}$ or in $\mathcal{L}[phe^{-1}]$, as may be seen in the case of a sequence of copies S^1: the map into S^1 which is the identity on each S^1 cannot be proper. But a family $\{X_n\}$ in $Ho(K^W)[\epsilon^{-1}]$ is also an object of $Ho(K^{W\times N})[\epsilon_N^{-1}] \simeq (Ho(K^W))^N[\epsilon_N^{-1}]$ where the reindexings ϵ_N are sequences of reindexings in $Ho(K^W)$

Let $S' : W \times N \longrightarrow W$ be given by $S'([m,n],K]) = [m+k, n+k]$. Then $Holan_{S'}$: $Ho(K^{W\times N})[\epsilon_N^{-1}] \longrightarrow Ho(K^W)[\epsilon^{-1}]$ left adjoint to $Ho(S'^*)$, is constructed in analogy to $Holan_S$ in Lemma 1.3

Theorem 3.1: In $HoTop$, $Holan_{S'}X \simeq \coprod_n X_n$.

It seems appropriate to refer to $Holan_{S'}X$ as the "fake coproduct" of the sequence X.

Suppose next that $X : \omega \longrightarrow \mathcal{L}$, ω being the ordered set of natural numbers. The usual construction of the Milnor telescope gives the homotopy colimit in $HoTop$ of X, but in fact lies in $\mathcal{L}$, although in this category it is not a homotopy colimit. We may analyze this construction as follows.

First, we may suppose that X lies in $K^{W\times\omega}$. Notice that $l[m,n] = n$ is an order-preserving map $W \longrightarrow \omega$. Thus $(1_W \times l)^*X$ is in $K^{W\times W}$. Now define $S_W : W\times W \longrightarrow W$ by:

$([i,i] \times [j,j]) \longmapsto [i+j, i+j]$

$([i,i+1] \times [j,j]) \longmapsto [i+j, i+j]$

$([i,i] \times [j,j+1]) \longmapsto [i+j, i+j]$

$([i, i+1] \times [j, j+1]) \longmapsto [i+j, i+j+1]$

then $HoS_W^* : Ho(K^W) \longrightarrow Ho(K^{W \times W})$ has a left adjoint $Holan_{S_W}$ and, reindexing $Ho(K^{W \times W})$ in analogy with $Ho(K^W)$,

$\overline{HoS_W^*} : (Ho(K^W))[\epsilon^{-1}] \longrightarrow Ho(K^{W \times W})[\epsilon_{W \times W}^{-1}]$ has the left adjoint $\overline{Holan_{S_W}}$.

Theorem 3.2: The underlying space of $\overline{Holan}_{S_W}(1_W \times l)^* X$ is the homotopy colimit in Top of the underlying spaces of the X_n.

Proof: The underlying space of an object of $Ho(K^W)[\epsilon^{-1}]$ is gotten by taking the homotopy colimit over W. In order to compute the one in question we must take a cofibrant model Y of $(1_W \times l)^* X$ then

$(Holan_{S_W}(1_W \times l)^*)X_{[m,n]} = \bigcup_{S_W[i,j] \times [k,l] = [m,n]} Y_{[i,j] \times [k,l]}$ so that $Hocolim_W Holan_{S_W}(1_W \times l)^* X = \bigcup_{[i,j] \times [k,l]} Y_{[i,j] \times [k,l]} = \bigcup_k \bigcup_{[i,j]} Y_{[i,j] \times [k,k+1]} \simeq Hocolim_{k \in \omega} Hocolim_W X_k$

Finally, let G be a finite group and denote by C^G the category of right G-complexes. If $X \in C^G$ we denote the G operation on X by the map $op : X \times G \longrightarrow X$.

For a group G the map $p : G \longrightarrow *$, gives rise to the functor $p^* : C \longrightarrow C^G$ where $X \longmapsto X \times G$. We shall show that $Lan_p \dashv p^*$ on a subcategory of $K^{G \times W}$.

As before C is either the category of CW complexes and proper cellular maps or the category K of finite CW complexes and cellular maps.

Definition: Let $C_\#^G$ be the smallest subcategory of C^G with the following three properties.

1. $Y \times G \in C_\#^G$ for $Y \in C$
2. $C_\#^G$ is closed under coproducts (finite if $C = K$).
3. $C_\#^G$ is closed under pushouts of the form

$$\begin{array}{ccc} Z \times G & \longrightarrow & Y \times G \\ \downarrow op & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

where $Z \in C_\#^G$ and $Z \longrightarrow Y$ is a cofibration in C.

An object X in $C_\#^G$ is called a G-cofibrant object of C^G. The functor $p^* : C \longrightarrow C^G$ factors through $C_\#^G$.

Lemma 3.1: If X is a G-cofibrant space then X/G is a CW complex.

Proof: If X is of the form $Y \times G$ then $X/G \simeq (Y \times G)/G \simeq Y$.

If X is gotten from the pushout

$$\begin{array}{ccc} Z \times G & \longrightarrow & Y \times G \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

we may assume that Z/G ia a CW complex. From the pushout above, we mod-out the G-action on the spaces. We therefore have the pushout

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z/G & \longrightarrow & X/G \end{array}$$

Since $Z \longrightarrow Y$is a cofibration X/G is a CW complex.

Clearly, this is also functorial.

Proposition 3.1: $p^* : C \longrightarrow C^G_\#$ has a left adjoint, Lan_p.

Proof: By Lemma 1., $Lan_p X = X/G$, and since this is functorial we conclude that $Lan_p \dashv p^*$.

We now turn our attention to the functor categories $(C^G)^{\mathcal{J}} \simeq C^{G \times \mathcal{J}}$ where $\mathcal{J}$ is a finite descent category.

Definition: $X \in C^{G \times \mathcal{J}}$ is a G-cofibrant object if for each $j \in \mathcal{J}$, $X_j \in C^G_\#$

Let $C^{G \times \mathcal{J}}_\#$ be the subcategory of $C^{G \times \mathcal{J}}$ containing the G-cofibrant objects. As before:

Proposition 3.2: $p^* : C^{\mathcal{J}} \longrightarrow C^{G \times \mathcal{J}}_\#$ has a left adjoint, Lan_p

Proof: $(Lan_p X)_j = X_j/G$ which is a CW complex by Theorem 3.1. Since this is functorial we have $Lan_p \dashv p^*$

There are obvious forgetful functors $U : C^G \longrightarrow C$ and $V : C^{G \times \mathcal{J}} \longrightarrow C^{\mathcal{J}}$.

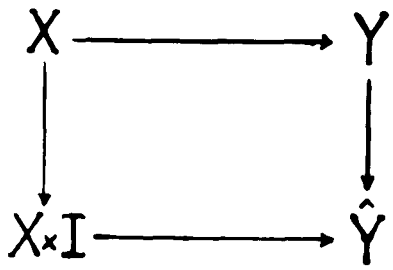
Definition: $f : X \longrightarrow Y$ in C^G is a weak homotopy equivalence if $U(f)$ is a homotopy equivalence.

Definition: $f : X \longrightarrow Y$ in C^G is a weak cofibration if $U(f)$ is a cofibration.

Lemma 3.2: If $f : X \longrightarrow Y$ in C^G then f can be factored as a weak cofibration followed by a weak homotopy equivalence.

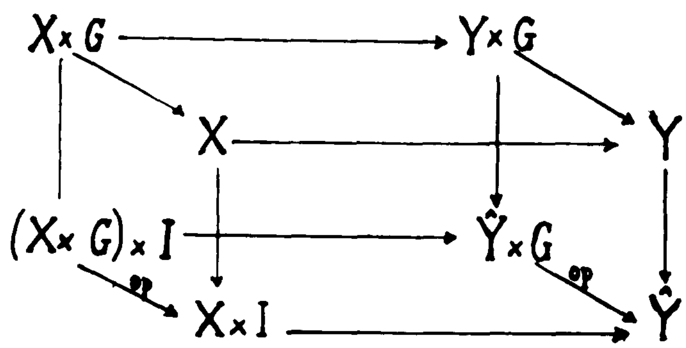
Proof: Consider the standard mapping cylinder $X \xrightarrow{i} \hat{Y} \xrightarrow{\hat{f}} Y$

Where

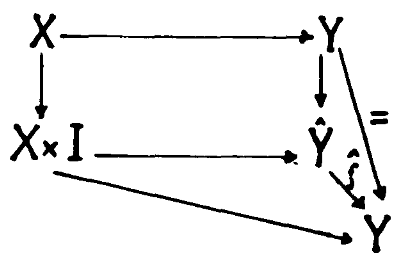


is a pushout.

Clearly, we see from the diagram



that $i : X \longrightarrow \hat{Y}$ is a weak cofibration and it follows from the diagram



(recalling that $- \times G$ commutes with pushout) that $\hat{f} : \hat{Y} \longrightarrow Y$ is a weak homotopy equivalence. This concludes the proof of the lemma.

<u>Definition</u>: $f : X \longrightarrow Y$ in $C^{G \times \mathcal{J}}$ is a w.h.e of G-spaces if Vf is a w.h.e.

<u>Lemma 3.3</u>: Let $A \neq \phi$, if $X : A \longrightarrow C^G$ and the colimit of the compostion $UX : A \longrightarrow C$ exists, then $colim_A X$ exists.

Proof: Given that $colim_A UX$ exists, all we need to show is that $colim_A(UX)$ is a G-

space. But $-\times G$ commutes with colimits. It follows that $colim_A UX$ is a G-space. Hence, $colim_A X$ exists, and is equal to $colim_A UX$.

Definition: $X \in C^{G\times\mathcal{J}}$ is called a weakly $\mathcal{J}$-cofibrant object, if $V(X)$ is cofibrant in $C^{\mathcal{J}}$.

Theorem 3.3: There is a functor $(\ )^\wedge : C^{G\times\mathcal{J}} \longrightarrow C^{G\times\mathcal{J}}$ and a natural transformation $(\ )^\wedge \longrightarrow 1_{C^{G\times\mathcal{J}}}$ such that $\hat{X} \longrightarrow X$ is a weak homotopy equivalence of G-spaces, and $\hat{X}$ is weakly $\mathcal{J}$-cofibrant.

Proof: Same as the proof of the existence of the cofibrant functor $(\ )^\wedge : C^{\mathcal{J}} \longrightarrow C^{\mathcal{J}}$ and natural transformation $(\ )^\wedge \longrightarrow 1_{C^{\mathcal{J}}}$ which is an equivalence. Only now, we use lemma 3.3 to conclude the forgetful functor $U : C^{G\times\mathcal{J}} \longrightarrow C^{\mathcal{J}}$ reflects colimits and lemma 3.2 for mapping cylinders. This concludes the proof of the theorem.

We now form the fraction category of $C^{G\times\mathcal{J}}$. Let $Ho(C^{G\times\mathcal{J}})$ be the fraction category gotten from the w.h.e's of G-spaces. Next we form the category of reindexings. Since $C^{G\times\mathcal{J}} \simeq (C^G)^{\mathcal{J}}$, we do the reindexings on $(C^G)^{\mathcal{J}}$. Let ϵ_G be the collection of all reindexings. As before, we have the fraction category $(Ho(C^G)^{\mathcal{J}}))[\epsilon_G^{-1}]$. We have also the isomorphism $(Ho(C^{G\times\mathcal{J}}))[\epsilon_G^{-1}] \simeq (Ho(C^G)^{\mathcal{J}}))[\epsilon_G^{-1}]$

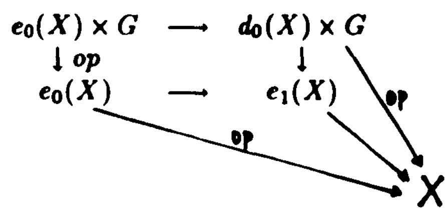
As in chapter 2, we construct also $(Ho(K^{G\times W}))[\epsilon_G^{-1}]$. Next we construct a functor $E : K^{G\times W} \longrightarrow K^{G\times W}$ with the property that $E(X)$ is a G-cofibrant object, and $E(X)$ becomes isomorphic to the telescope of X in $(Ho(K^{G\times W}))[\epsilon_G^{-1}]$. To define E, we first define a sequence of functors. The construction of these functors is similar to the Milnor's construction of a free G-space. By Theorem 3.3 we can replace $X \in K^{G\times W}$ by $\hat{X}$ a weakly W-cofibrant G-space. Thus we may assume that X is weakly W-cofibrant G-space.

Let $e_0(X) = X \times G$ and $e_0(X) = X \times G \longrightarrow X$ be the group operation on X. Since the construction of mapping cylinders is functorial, we may replace the map "*op*" by a weak cofibration followed by a w.h.e. Thus $e_0(X) \longrightarrow d_0(X) \simeq X$, where $d_0(X)$ is the mapping cylinder.

Note $e_0(X)$ is G-cofibrant and weakly W-cofibrant, $d_0(X)$ is weakly W-cofibrant. Also $(e_0(X))_{[m,n]} \longrightarrow (d_0(X))_{[m,n]}$ is a weak cofibration for each $[m,n]$ in W, where $n = m$ or $m+1$. We next construct $e_1(X)$ by means of the pushout

$$\begin{array}{ccc} e_0(X) \times G & \longrightarrow & d_0(X) \times G \\ \downarrow op & & \downarrow \\ e_0(X) & \longrightarrow & e_1(X) \end{array}$$

Clearly, $e_1(X)$ is G-cofibrant and weakly W-cofibrant, and $e_0(X) \longrightarrow e_1(X)$ is a map of G-cofibrant spaces. It follows from the commutative diagram



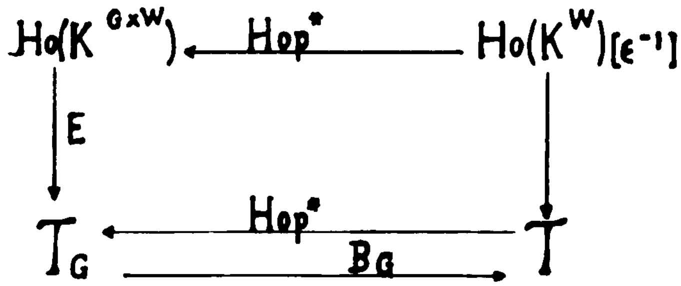
that $e_1(X) \longrightarrow X$ is in $K^{G \times W}$. Replacing the map $e_1(X) \longrightarrow X$ by its mapping cylinder gives $e_1(X) \longrightarrow d_1(X) \simeq X$, where $d_1(X)$ is the mapping cylinder. We can construct $e_2(X)$ using $e_1(X)$ and $d_1(X)$. Note $e_1(X)$ and $d_1(X)$ has the same properties of $e_0(X)$ and $d_0(X)$ respectively. It follows, if we iterate this process that we have the sequence:

$e_0(X) \longrightarrow d_0(X) \longrightarrow e_1(X) \longrightarrow d_1(X) \longrightarrow e_2(X) \longrightarrow d_2(X) \longrightarrow \cdots.$

From this sequence we can extract two sequences $e_0(X) \longrightarrow e_1(X) \longrightarrow e_2(X) \longrightarrow \cdots$ a sequence of G-cofibrant and weakly W-cofibrant objects, with $(e_i(X))_{[m,n]} \longrightarrow (e_j(X))_{[m,n]}$ a weak cofibration. Also $d_0(X) \longrightarrow d_1(X) \longrightarrow d_2(X) \longrightarrow \cdots$ a sequence of W-cofibrant objects. These sequences can be viewed as objects $e(X)$ and $d(X)$ of $K^{G \times W \times \omega}$. It is easily seen that $e(X) \longrightarrow d(X)$ is an isomorphism in $Ho(K^{W \times \omega})[\epsilon^{-1}_{W \times \omega}]$

Let $q : G \times W \times \omega \longrightarrow G \times W$ be the projection of $W \times \omega$ into W. This gives rise to $q^* : K^{G \times W} \longrightarrow K^{G \times W \times \omega}$. It is clear that $d(X) \longrightarrow q^*(X)$ is a w.h.e. in $K^{G \times W \times \omega}$. Hence $e(X) \longrightarrow q^*(X)$ is an isomorphism in $Ho(K^{W \times \omega})[\epsilon^{-1}_{W \times \omega}]$. Let $E(\ )$ denote the functor $\overline{Holan_{S_W}}(1_W \times l)^* e(\ )$, and let $\mathcal{T}_G$ be the full subcategory of $Ho(K^{G \times W})[\epsilon_G^{-1}]$ consisting of the w.h.e.'s of telescopes. Also, let $\mathcal{T}$ be the subcategory of $Ho(K^W)[\epsilon^{-1}]$

consisting of the w.h.e.'s of telescopes. Because $- \times G$ commutes with the telescopic functor $\overline{Holan_{S_W}}(1_W \times l)q^*$ we have the diagram



where $p^*X = X \times G$. Since $E(X) \simeq \overline{Holan_{S_W}}(1_W \times l)q^*(X)$ in $\mathcal{T}_G$, we have by poposition 3.2 $B_G : \mathcal{T}_G \longrightarrow \mathcal{T}$, where for a telescopic space Y of X $B_G(Y) = Lan_p(E(X))$, it follows that $B_G \dashv Hop^*$. We have proved the following theorem.

Theorem 3.4: $Hop^* : \mathcal{T} \longrightarrow \mathcal{T}_G$ has a left adjoint B_G.

B_G is thus the Milnor functor provided with a canonical locally compact structure. In particular the classifying space $B_G(\mathcal{R}^+)$ is thus provided with a canonical locally compact structure.

It seems likely that the last argument can be generalized by replacing G by any finite category.

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