Mahler formula for morphisms on the n-dimensional projective space

by

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Abstract

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The Mahler formula expresses the height of an algebraic number as the integral of the log of its equation with respect to the Haar measure on the circle. The height is in fact the canonical height associated to the monomial maps x^n and the Haar measure is nothing but the invariant measure associated to those maps. We show in this work that for "good" morphisms φ on $\mathbb{P}^n$ the canonical height of a hypersurface can be expressed as the integral, with adelic terms, of the log of its equation.

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Contents

Introduction

The classical formula of Mahler in complex variable states that if $\lambda \in \mathbb{Q}$, then $\log(\sup(|\lambda|, 1)) = \int_{\mathbb{C}} \log|z - \lambda| dz$. The left hand side can be read as the naive height $h((\lambda, 1))$ of the point $(\lambda, 1) \in \mathbb{P}^1_{\mathbb{Q}}$ and the right hand side as the integral of the log of the minimal equation of λ over $\mathbb{Q}$. In general it is classically known [28], that for any point $P = (\lambda, 1) \neq \infty$ in $\bar{\mathbb{Q}}$, the naive height $h(P)$ can be related to the integral of the log of the minimal equation F of λ, in the following way

$$h(P) = \frac{1}{\deg(F)} \int_{\mathbb{P}^1_{\mathbb{C}}} \log|F(z)| dz = \frac{1}{\deg(F)} \int_{S^1} \log|F(\theta)| d\theta.$$

Now the naive height is the canonical height (c.f. later) associated to the morphisms $\phi_n : t \to t^n$ on $\mathbb{P}^1$. This means that $h(\phi_n(t)) = nh(t)$ and $h(t) \geq 0$. On the other hand the Haar measure $d\theta$ on S^1 is invariant under the action of this endomorphisms, in the sense that $\phi_n^* d\theta = n d\theta$ and $\phi_{n*} d\theta = d\theta$. We prove in the following work that these phenomena are general for dynamical systems on $\mathbb{P}^1$ as well as for dynamical systems on $\mathbb{P}^n$ provided that the morphism admits a "good model". The general Mahler formula we prove states that if $\varphi = (p_0, .., p_n) : \mathbb{P}^n \to \mathbb{P}^n$ has a model such that $p_0, .., p_n$ represents a regular sequence inside $\mathcal{O}_K[T_0, ..., T_n]$ then:

$$h_\varphi(D_\mathbb{Q}) = \sum_{v/\infty} \int_{\mathbb{P}^n_{\bar{K}_v}} \log |F|_v d\mu_{\varphi,v} + E(F, v\,\text{finite}) + \deg(F) h_\varphi(\infty),$$

where we have the following:

(i) F is a polynomial in the n-variables $T_0/T_n, .., T_{n-1}/T_n$.

(ii) $h_\varphi(D_\mathbb{Q})$ represents the canonical height of the cycle $D = \text{div}(F) - \deg(F)\infty - \sum_{v\,\text{finite}} v(F) X_v$.

(iii) The divisor ∞ is nothing but $\text{div}(T_n)$.

(iv) For every place v at infinity $d\mu_{\varphi,v}$ represents an invariant measure relative to φ on X_v.

(v) $E(F, v\,\text{finite})$ is an error term arising from the blow-up we need to do when the map can not be extended to an integral model. It depends in fact only on a finite number of finite places, which we call places of bad reduction. When the map can be extended to an integral model, the term $E(F, v\,\text{finite}) = 0$.

Suppose now that our map admits a model such that $h_\varphi(\infty) = 0$, then we can eliminate one term on the right and get:

$$h_\varphi(D_{\mathbb{Q}}) = \sum_{v/\infty} \int_{\mathbb{P}^n_{\bar{K}_v}} \log |F|_v d\mu_{\varphi,v} + E(F, v \,\text{finite})$$

For the discussion that follows in this introduction we assume that we have $h_\varphi(\infty) = 0$. Under certain conditions, which we call positivity conditions, we have $E(F, v\,\text{finite}) < 0$. If we pick the equation F such that $v(F) = 0$ for every finite place v, we get the inequality:

$$h_\varphi(D_{\mathbb{Q}}) \leq \sum_{v/\infty} \int_{\mathbb{P}^n_{\bar{K}_v}} \log |F|_v d\mu_{\varphi,v}.$$

When the map φ has good reduction everywhere (which is the case of the maps $t \to t^n$ associated to the naive height) and $v(F) = 0$ for every v, we obtain the equality

$$h_\varphi(D_{\mathbb{Q}}) = \sum_{v/\infty} \int_{\mathbb{P}^n_{\bar{K}_v}} \log |F|_v d\mu_{\varphi,v}.$$

In case that we are working with $\varphi = (p_0, p_1) : \mathbb{P}^1 \to \mathbb{P}^1$ we can always change coordinates to get T_1/p_1 (this makes $h_\varphi(\infty) = 0$) and assume by base change that the sequence (p_0, p_1) is regular. The term $E(F, v\,\text{finite})$ can be interpret as the sum of "integrals" over the finite places of K, $E(F, v\,\text{finite}) = \sum_{v\,\text{finite}} \int_{\mathbb{P}^n_{\bar{K}_v}} \log |F|$. The measure $d\mu_\varphi$ at each place over infinity is nothing but the Brolin measure [10], further studied by other authors such as, Lyubich

[27] and Freire, Lopez, and Mañe [18]. Taking a point $P = (\lambda, 1) \neq \infty$ in $\mathbb{P}^1_{\bar{K}}$ and F the minimal equation of λ over K (which we assume with no common factors $v(F) = 0$ at every finite valuation), the formula we found takes a more symmetric shape,

$$h_\varphi(P) = \frac{1}{\deg(F)} \sum_v \int_{\mathbb{P}^1_{\bar{K}_v}} \log |F|_v d\mu_{\varphi,v}.$$

When we pick $\lambda \in \bar{\mathbb{Q}}$, F the minimal equation of λ over $\mathbb{Q}$ and the map $\varphi = (p_0, p_1)$ satisfying positivity conditions (wich reduces to impose $A_d = 1$ to the homogeneous polyr

$$h_\varphi(P) = \frac{1}{\deg(F)} \int_{\mathbb{P}^1(\mathbb{C})} \log |F| d\mu_\varphi,$$

$$h_\varphi(P) = \frac{1}{\deg(F)} \int_{\mathbb{P}^1(\mathbb{C})} \log |F| d\mu_\varphi,$$

which is very similar to the classical Mahler formula.

1.0.1 Notation and conventions

All rings are assumed to be commutative. Unless otherwise stated K will denote a number field with ring of integers $\mathcal{O}_K$. $\mathbb{P}^n_K$ will denote the n-dimensional projective space over K and similar for $\mathbb{P}^n_{\mathcal{O}_K}$. In the section on endomorphisms on projective varieties, X will denote a projective variety of dimension n, $\mathcal{L}$ a line bundle on X and φ a map from X to itself. The term $c_1(\mathcal{L}, \|.\|)$ will denote a $1-1$ current similar to the first Chern form of $\mathcal{L}$. In the rest of this work X will denote an arithmetic variety of absolute dimension $n+1$ over $\operatorname{Spec}(\mathcal{O}_K)$, in such a way that the fibres X_v over the places v of K are algebraic varieties of dimension n. For an arithmetic variety X, $\mathcal{L}$ will denote a line bundle on X and $\mathcal{L}_v = \mathcal{L} \otimes_{\mathcal{O}(X)} \operatorname{Spec}(K_v)$ the restriction of $\mathcal{L}$ to the fibre X_v. The line bundle $\mathcal{L}$ will comes sometimes equipped with hermitian metrics $\|.\|_{P,v}$ on the fibres $\mathcal{L}_{P,v}$ over a point $P \in X_v$. We will be often interested in the arithmetic variety $X = \mathbb{P}^n_{\mathcal{O}_K}$ and models X_k that arise from blowing-up subschemes Y_k of X. We will use $\mathcal{L}_1, .., \mathcal{L}_i$ to denote several line bundles on the arithmetic variety X and the expression $\widehat{\deg}(c_1(\mathcal{L}_1)..c_1(\mathcal{L}_i)|Z)$ will represent the arithmetic intersection degree of the line bundles $\mathcal{L}_1, .., \mathcal{L}_i$ over a cycle $Z \subset X$ of dimension i. The ideal generated by the polynomials $p_0, .., p_n$ will be denoted by $\langle p_0, ..., p_n \rangle$ and the symbol $\sqrt{I}$ will be used to denote the radical of any ideal I.

Mahler formula

1.1 Endomorphisms on projective varieties

In this section we will give some examples of endomorphisms on projective varieties. That is, maps $\varphi : X \to X$, of a projective variety X to itself. Under certain conditions on the variety X (existence of a line bundle $\mathcal{L}$ with good properties), we will associate to X and $\mathcal{L}$ a canonical height function and a canonical measure. The canonical height will be a generalization of both the Neron-Tate on abelian varieties and the naive height on $\mathbb{P}^1$. The points with finite forward orbit by φ will be characterized as the points of canonical height zero. The canonical measure will be a generalization of Brolin's measure for maps on $\mathbb{P}^1$.

Example 1.1.1. *Suppose that K is a field. A degree d map $\varphi : \mathbb{P}^n_K \to \mathbb{P}^n_K$ is given by a set of homogenous polynomials $p_0(T_0, ..., T_n), ..., p_n(T_0, ..., T_n) \in K[T_0, .., T_n]$ of degree d, such that $\sqrt{\langle p_0, ..., p_n \rangle} = \langle T_0, ..., T_n \rangle$. Here we are using $\sqrt{I}$ to denote the radical of the ideal I.*

Here are some examples:

(i) Take $p_n = T_n^d$ and for $0 < i < n-1$, $p_i = T_i^d + T_{i+1}q_i$, where the q_i can be taken to be zero or homogenous polynomials of degree $d-1$.

(ii) Take any permutation of the polynomials above.

(iii) Consider for $0 \leq i \leq n$ homogeneous polynomials $p_i = \sum_j a_{i,j}T_j^d$ of degree d, where $\{a_{i,j}\}$ is an invertible matrix.

Example 1.1.2. *Consider a prime number p and a polynomial $G(x) \in K(x)$ where K is an algebraically closed field. The family of plane algebraic curves*

$$C_{p,G} : y^p = G(x)$$

has automorphisms given by $(x, y) \to (x, \xi_p y)$, where ξ_p is any $p-$root of unity.

Example 1.1.3. *The following example was studied by Wheler and later used by Silverman in [38]. Consider the family of $K3$ surfaces $S_{a,b} \subset \mathbb{P}^2 \times \mathbb{P}^2$ determined by the two equations with coefficients in a number field K,*

$$\sum_{i,j=1}^{3} a_{i,j}x_iy_j = \sum_{i,j,k,l=1}^{3} b_{i,j,k,l}x_ix_ky_jy_l = 0. = 0.$$

The projections p_1 and p_2 represents double coverings of $\mathbb{P}^2_K$ and determine morphisms σ_1 and σ_2 in each of the members of the family $S_{a,b}$. The free *product* $\sigma_1 * \sigma_2$ *is contained in the group of automorphism of each element of* $S_{a,b}$.

Example 1.1.4. *Consider an elliptic curve* $E = \mathbb{C}/\mathbb{C} + \tau\mathbb{C}$ *given by Weierstrass equation* $y^2 = x^3 + px + q$. *The multiplication by* n *in* E *induces, as quotient by the action of* $[-1]$, *a map* $\varphi_n : \mathbb{P}^1 \to \mathbb{P}^1$. *For example for* $n = 2, 3$ *we have:*

$$\varphi_2(t) = \frac{x^4 - 2px^2 - 8qx + p^2}{4x^3 + 4px + 4q},$$

$$\varphi_3(t) = x - \frac{8(x^3 + px + q)(x^6 + 5px^4 + 20qx^3 - 5p^2x^2 - 4pqx - 8q^2 - p^3)}{(3x^4 + 6px^2 + 12qx - p^2)^2}.$$

The above example can be interpreted in the language of Lattès examples as follows: Suppose that $\tau \in \mathbb{C}$ *is in the upper half plane. Let* $G_1 = \{(\pm 1, m + n\tau) | m, n \in \mathbb{Z}\}$ *be a subset of the complex motion group* $\{(u, a) | u \in U(1), a \in \mathbb{C}\}$ *acting on* $A^1_{\mathbb{C}}$. *Suppose that we take* $N : \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ *to be the* $[n]$*−th map, then the quotient* $\mathbb{A}^1/G_1$ *is isomorphic to* $\mathbb{P}^1$ *and* N *descends to the morphism* $\varphi_n : \mathbb{P}^1 \to \mathbb{P}^1$.

Example 1.1.5. *Consider the plane conic* C *over* $\mathbb{Z}$ *defined by the equation* $C : X_0X_1 + pX_2^2 = 0$, *where* p *is an odd prime. The reduction of* C *mod any other prime* $l \neq p$ *is smooth and connected. The fiber over* p *is the union of two lines and the surface* C *is regular. The map from* $\mathbb{P}^2$ *to* $\mathbb{P}^1$ *defined by* $\phi([X_0 : X_1 : X_2]) = [X_0 + X_1, X_2]$ *is well defined on* C. *Projecting from*

$[0:1:0]$ *yields an isomorphism between our conic and* $\mathbb{P}^1$, *composing this last one with* ϕ *gives rise to a map* $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$, *which can be checked to be* $\varphi = \frac{t^2-p}{t}$ *(See the notes [3]). This example is an illustration of a blowing up allowing to define the map* φ *on* $\mathbb{P}^1_{\mathbb{Z}}$.

Example 1.1.6. *Any elliptic curve with complex multiplication admits an endomorphisms ring larger than* $\mathbb{Z}$. *For example, multiplication by* $1+i$ *on the elliptic curve* $E : y^2 = x^3 + x$. *In this case when we pass to the quotient by the action of* $[-1]$ *on* E, *we get a new map on* $\mathbb{P}^1$, *namely*

$$\varphi_{1+i}(t) = \frac{1}{(1+i)^2}\frac{t^2+1}{t}.$$

Example 1.1.7. *For a set of morphisms* $\varphi_i : X_i \to X_i$ *when* $0 < i \leq k$, *we can consider the product morphism* $\varphi = (\varphi_1, .., \varphi_k) : X_1 \times ... \times X_k \to X_1 \times .... \times X_k$.

Example 1.1.8. *(Another Lattès example). Take the morphism* $\varphi_1 \times ... \times \varphi_1 : \mathbb{P}^1 \times ... \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times ... \times \mathbb{P}^1$ *(n times) and consider the symmetric group* S_n *acting on* $\mathbb{P}^1 \times ... \times \mathbb{P}^1$ *by permutation of the coordinates. Then the quotient* $\mathbb{P}^1 \times ... \times \mathbb{P}^1 / S_n$ *is isomorphic to* $\mathbb{P}^n$ *and* $\varphi_1 \times ... \times \varphi_1$ *descend to a morphism* $\varphi : \mathbb{P}^n \to \mathbb{P}^n$.

1.1.1 Canonical height

Suppose that X is a projective variety defined over a number field K, and $\varphi : X \longrightarrow X$ a morphism, with the property that there exist a line bundle $\mathcal{L}$ on X and a real number $\alpha > 1$ such that $\varphi^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}^\alpha$. Then we can find (see for example [37] or ([21]) a positive height function h_φ on $X(\bar{K})$, defined as the limit $h_\varphi(P) = \lim_{k\to\infty} \frac{h_{\mathcal{L}}(\varphi^k(P))}{\alpha^k}$ with the properties:

(i) h_φ satisfies Northcott's theorem: points with coordinates in $\bar{K}$ with bounded degree and bounded height are finite in number.

(ii) $h_\varphi(\varphi(P)) = dh_\varphi(P)$.

(iii) h_φ is a non-negative function.

(iv) $h_\varphi(P) = 0$ if and only if P has a finite forward orbit under iteration of the map φ.

(v) $|h_\varphi(P) - h_{\mathcal{L}}(P)|$ is bounded on $X(\bar{\mathbb{Q}})$.

Remark 1.1.9. *The canonical height h_φ is uniquely determined by conditions (ii) and (v), and this represents a way of checking that certain formulas for h_φ are in fact correct.*

Let's see now how the properties of the canonical height can be used to generate one more example of a map on a projective variety.

Proposition 1.1.10. *Suppose that two maps $\varphi, \psi : X \to X$ that commute ($\varphi \circ \psi = \psi \circ \varphi$) satisfy the following properties: For some line bundle $\mathcal{L}$ on X and real numbers $\alpha, \beta > 1$, we have $\varphi^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}^\alpha$ and $\psi^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}^\beta$, then we have $h_\varphi = h_\psi$ and $h_{\varphi \circ \psi} = h_\varphi$.*

Proof. Let $\varepsilon > 0$, we can find n big enough such that we get all the inequalities:

$$|h_\varphi(x) - h_\mathcal{L}(\varphi^n(x))/\alpha^n| < \varepsilon$$

$$|h_\mathcal{L}(\varphi^n(x))/\alpha^n - h_\psi(\varphi^n(x))/\alpha^n| < \varepsilon$$

$$|h_\psi(\varphi^n(x))/\alpha^n - h_\mathcal{L}(\psi^n \circ \varphi^n(x))/\alpha^n \beta^n| < \varepsilon$$

and the first part of the proposition is a consequence of $\varphi \circ \psi = \psi \circ \varphi$. For the second part observe that

$$h_\varphi(\varphi(\psi(x))) = \alpha h_\varphi(\psi(x)) = \alpha\beta h_\varphi(x).$$

□

Example 1.1.11. *The maps $\varphi_{1+i}(t) = \frac{1}{(1+i)^2}\frac{t^2+1}{t}$ and $\varphi_{1-i}(t) = -\frac{1}{(1+i)^2}\frac{t^2+1}{t}$ commute and their composition $\varphi_{1+i}(\varphi_{1-i}(t)) = \varphi_{1-i}(\varphi_{1+i}(t)) = \varphi_2(t) = \frac{t^4-2t^2+1}{4(t^3+t)}$. We know the canonical height associated to φ_2, because φ_2 is the projection on the first coordinate of the multiplication by 2 on the elliptic curve E with Weierstrass equation $E : y^2 = x^3 + x$. In fact φ_{1+i} and φ_{1-i}*

are respectively the projection on the first coordinate of the multiplication by $1+i$ *and* $1-i$ *on* E.

Let's go back again to our discussion of the canonical height. Under the same conditions ($\varphi^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}^\alpha$) for φ, $\mathcal{L}$ and α; we can build the canonical multi-height $h_\varphi(Y_\mathbb{Q})$ of a $p-$cycle Y inside $X_{\mathcal{O}_K}$. Similar properties will be fulfilled (Check for example [46]),

(i) $h_\varphi(Y_\mathbb{Q}) \geq 0$.

(ii) $h_\varphi(Y_\mathbb{Q})$ satisfy the functional equation $h_\varphi(\varphi_* Y_\mathbb{Q}) = \alpha h_\varphi(Y_\mathbb{Q})$.

(iii) If the orbit $\{Y, f(Y), ...\}$ is finite, then $h_\varphi(Y_\mathbb{Q}) = 0$.

Conjecture 1.1.12. *(Zhang) The orbit* $\{Y, f(Y), ...\}$ *of a cycle* Y *is finite if and only if the* $h_\varphi(Y_\mathbb{Q}) = 0$.

In a more general context, we are able to build multi-heights $h_{\mathcal{L}_1,..,\mathcal{L}_n}$ attached to metrized line bundles $(\mathcal{L}_i, ||.||_i)$; the canonical height will then arise associated to a special kind of metric called the canonical metric.

1.1.2 Canonical metrics

Consider as before X defined over a number field K, $\mathcal{L}$ a line bundle on X such that $\phi : \mathcal{L}^\alpha \xrightarrow{\sim} \varphi^*\mathcal{L}$ for some $\alpha > 1$. Assume that for every infinite place

v we have chosen an smooth metric $||.||_v$ on each fibre $\mathcal{L}_v$ of $\mathcal{L}$. The following theorem is due to Shouwu Zhang [46]:

Theorem 1.1.13. *The sequence defined recurrently by* $||.||_{v,1} = ||.||_v$ *and* $||\cdot||_{v,n} = (\phi^*\varphi^*||\cdot||_{v,n-1})^{1/a}$ *for* $n > 1$*, converge uniformly on* $X(K_v)$ *to a metric* $||.||_{v,\varphi}$ *(independent of the choice of* $||.||_{v,1}$*) on* $\mathcal{L}_v$ *which satisfies the equation* $||\cdot||_{\varphi,v} = (\phi^*\varphi^*||\cdot||_{\varphi,v})^{1/d}$.

Proof. See theorem (2.2) in S. Zhang [46]. Denote by h the continuous function $\log \frac{||\cdot||_2}{||\cdot||_1}$ on $X(K_v)$. Then

$$\log ||.||_n = \log ||.||_1 + \sum_{k=0}^{n-2} (\frac{1}{d}\phi^*\varphi^*)^k h.$$

Since $||(\frac{1}{d}\phi^*\varphi^*)^k h||_{sup} \leq (\frac{1}{d})^k ||h||_{sup}$, it follows that the series given by the expression $\sum_{k=0}^{\infty}(\frac{1}{d}\phi^*\varphi^*)^k h$, converges absolutely to a bounded and continuous function $h_{\varphi,v}$. Let $||.||_{\varphi,v} = ||.||_1 \exp(h_{\varphi,v})$, then $||.||_n$ converges uniformly to $||\cdot||_{\varphi,v}$ and its not hard to check that $||.||_{\varphi,v}$ satisfies

$$||\cdot||_{\varphi,v} = (\phi^*\varphi^*||\cdot||_{\varphi,v})^{1/d},$$

which was the result we wanted to prove. □

Definition 1.1.14. *The metric* $||.||_{v,\varphi}$ *is called the canonical metric on* $\mathcal{L}_v$.

Example 1.1.15. *Consider the line bundle* $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(1)$ *on* $\mathbb{P}^n_{\mathbb{Q}}$*. The Fubini-Study metric* $||(\lambda_0 T_0 + ... + \lambda_n T_n)(a_0, ..., a_n)||_{FS} = \frac{|\sum \lambda_i a_i|}{\sqrt{\sum_i a_i^2}}$ *is a smooth metric* *on $\mathcal{L}$. If we take $\|.\|_1 = \|.\|_{FS}$, the limit metric we get is*

$$\|(\lambda_0 T_0 + \ldots + \lambda_n T_n)(a_0, \ldots, a_n)\|_\varphi = \frac{|\sum \lambda_i a_i|}{\sup_i(a_i)}.$$

Proposition 1.1.16. *Suppose that two maps φ and ψ commute, and for some line bundle $\mathcal{L}$ on X we have $\varphi^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}^\alpha$ and $\psi^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}^\beta$, then the canonical metrics $\|.\|_\varphi = \|.\|_\psi$.*

Proof. The key idea is that the canonical metric associated to a morphism does not depend on the metric with which we start the iteration. Let $s \in \Gamma(X)$ be a non-zero section of $\mathcal{L}$. We are going to consider two metrics $\|.\|_{v,1} = \|.\|_\varphi$ and $\|.\|'_{v,1} = \|.\|_\psi$ on the line bundle $\mathcal{L}$. By our definition of canonical metric for φ, we can start with $\|.\|'_{v,1}$ and obtain $\|s(x)\|_\varphi = \lim_k \|s(\varphi^k(x)\|_\psi^{1/\alpha^k}$, but also by our definition of canonical metric for ψ starting with $\|.\|_{v,1} = \|.\|_\varphi$ we get $\|s(x)\|_\psi = \lim_l \|s(\varphi^l(x)\|_{v,1}^{1/\beta^l}$. So using the uniform convergence and the commutativity of the maps, we have

$$\begin{aligned}\|s(x)\|_\varphi &= \lim_{k,l} \|s(\varphi^k \circ \psi^l(x)\|_{v,1}^{1/\alpha^k\beta^l} \\ &= \lim_{l,k} \|s(\psi^l \circ \varphi^k(x)\|_{v,1}^{1/\beta^l\alpha^k} = \|s(x)\|_\psi,\end{aligned}$$

which was the result we wanted to prove. □

1.1.3 Canonical measure and integral at infinite places

In this part we set up what will be called the integral at infinite places attached to a map $\varphi : X \to X$ and a line bundle $\mathcal{L}$ on X.

Suppose that we have equipped a line bundle $\mathcal{L}$ on X with a canonical metric $\|.\|_\varphi$ on the fibres and $U \subset X$ is an open set. The function $x \mapsto -\log\|s(x)\|^2_\varphi$ for a non-zero holomorphic section s on U, is not necessarily smooth but maybe only plurisubharmonic, and due to this fact, the first Chern "form" $c_1(\mathcal{L}, \|.\|) = \frac{1}{(\pi i)}\partial\overline{\partial}\log\|s_1(P)\|_\varphi$ may be no more than a distribution. We would like to define the product $c_1(\mathcal{L}, \|.\|)\wedge \ldots \wedge c_1(\mathcal{L}, \|.\|)$. Unfortunately for general currents we do not have a product as we do for smooth currents. Results of Bedford, Taylor and Demailly [5], [14], [15] [16], allow us to consider a product of currents with good properties.

Definition 1.1.17. *(Lelong). Let U be an open set of complex manifold M of dimension n. A current $T \in D^{p,p}(U)$ is said to be positive ($T \geq 0$) if for every choice of C^∞ $(1,0)$-forms $\alpha_1, .., \alpha_{n-p}$ with compact support on U, the distribution $T \wedge (i\alpha_1 \wedge \bar{\alpha}_1) \wedge \ldots \wedge (i\alpha_{n-p} \wedge \bar{\alpha}_{n-p})$ is a positive measure on U.*

Remark 1.1.18. *The positive volume forms on X are positive multiples of $(iz_1 \wedge \bar{z}_1) \wedge \ldots \wedge (iz_n \wedge \bar{z}_n)$.*

Remark 1.1.19. *A (p,p) current T can also be identified with an expression $T = \sum_{|I|=|J|=p} T_{I,J} dz_I d\bar{z}_J$ where the coefficients $T_{I,J}$ are distributions. To say that $T \geq 0$ is to say that $\sum T_{I,J}\alpha_I\bar{\alpha}_J \geq 0$, where $\alpha_I = \prod_{i\in I}\alpha_i$ and similarly for α_J. The positivity of T forces the components $T_{I,J}$ to be complex measures* *with* $T_{I,J} = \bar{T}_{J,I}$.

Example 1.1.20. *A locally integrable function* u *on* X *is says to be plurisubharmonic if the hessian* $i\partial\bar{\partial}u = i\sum \partial^2 u/\partial z_j \partial z_m z_j \wedge \bar{z}_m \geq 0$ *on* X. *For basic properties of plusibharmonic functions see for example [14] or [15].*

Example 1.1.21. *Let* Y *be an algebraic cycle on* X. *The current* δ_Y *of integration on* Y *is closed and positive. To see this, let* $\alpha_1, ..., \alpha_p$ *be* p *forms of type* $(1,0)$ *and* f *a continuous positive function on* X. *Then we have*

$$\int_X f\alpha_1 \wedge \bar{\alpha}_1 ... \alpha_p \wedge \bar{\alpha}_p = i^{p^2} \int_X f|\alpha_1|...|\alpha_p| dz_1 \wedge d\bar{z}_1 ... dz_p \wedge d\bar{z}_p \geq 0.$$

on the other hand, Y *is a compact subvariety of* X *and:*

$$d\delta_Y(u) = \int_X d\delta_Y \wedge u = \int_X \delta_Y \wedge du = 0.$$

Definition 1.1.22. *(Bedford-Taylor). Let* T *be a positive closed current of type* (p,p) *and* u *a plurisubharmonic function locally bounded on* U. *We define the product* $(dd^c u) \wedge T = dd^c(uT)$.

Remark 1.1.23. *The product* Tu *is well defined because the function* u *is locally bounded and the current* T *has measures as coefficients. In general we can define*

$$(dd^c u_1) \wedge (dd^c u_2) ... (dd^c u_q) \wedge T = dd^c(u_1 \wedge (dd^c u_2) ... (dd^c u_q) \wedge T).$$

By prop 1.2 in [16], the current $(dd^c u_1) \wedge (dd^c u_2)...(dd^c u_q) \wedge T$ *is a positive closed current of bidegree* $(p+q, p+q)$.

Lemma 1.1.24. *Let* $\mathcal{L}$ *be a line bundle on* X *and* $\varphi : X \to X$ *with the conditions of prop. 1.1.13. Let* $||.||_\varphi$ *be the canonical metric on* $\mathcal{L}$. *The function* $x \mapsto -\log ||s(P)||^2_\varphi$ *is plurisubharmonic on the open set* $U = X - \mathrm{div}(s)$ *and therefore the current* $i\partial\bar{\partial}(-\log ||s(P)||^2_\varphi)$ *is a positive current on* U.

Proof. The proof is basically taken from [24]. Consider the continuous and positive function $H(x) = \frac{||s(P)||_2}{||s(P)||_1}$ on X. Define $c = \min_{x \in X} H(x)$. By changing φ by $c\varphi$ if necessary, we can have $H > 1$. In general

$$\frac{||s||_n}{||s||_{n-1}} = \frac{||s \circ \varphi||_{n-1}}{||s \circ \varphi||_{n-2}}$$

and then $-\log ||s(P)||^2_n \leq -\log ||s(P)||^2_{n-1}$ for every $n > 2$ and the sequence $\{-\log ||.||^2_n\}_{n=1}^{\infty}$ is a non increasing sequence of plurisubharmonic function converging to $-\log ||s(P)||^2_\varphi$. □

Proposition 1.1.25. *Let* s_i $(i = 1..q)$ *be sections of the line bundles* $\mathcal{L}_i$ *respectively, such that the divisors* $\mathrm{div}(s_i)$ *meet properly on* X. *Let us denote* $c_1(\mathcal{L}_i) = c_1(\mathcal{L}_i, ||.||_\varphi)$ *the Chern "form" associated to the canonical metric studied in proposition 1.1.13, then the current*

$$(-\log ||s_i||^2_\varphi) c_1(\mathcal{L}_1)....c_1(L_{i-1}).\delta_{\mathrm{div}(s_{i+1})}...\delta_{\mathrm{div}(s_q)}$$

is a well defined current and

$$\int_X (-\log \|s_i\|_n^2) c_1(\mathcal{L}_1, \|.\|_n)....c_1(L_{i-1}, \|.\|_n).\delta_{\mathrm{div}(s_{i+1})}...\delta_{\mathrm{div}(s_q)}$$

tends to

$$\int_X (-\log \|s_i\|_\varphi^2) c_1(\mathcal{L}_1)....c_1(L_{i-1}).\delta_{\mathrm{div}(s_{i+1})}...\delta_{\mathrm{div}(s_q)}.$$

Proof. We have $c_1(\mathcal{L}_i, \|.\|_\varphi)$ is a positive current that can be written locally in the form $dd^c u$ where $u = -\log \|.\|_\varphi$ is plurisubharmonic on X. On the other hand $\delta_{\mathrm{div}(s_i)}$ are closed and positive. The second part follows because the currents $-\log \|s_i\|_n^2 c_1(\mathcal{L}_1, \|.\|_n)....c_1(L_{i-1}, \|.\|_n).\delta_{\mathrm{div}(s_{i+1})}...\delta_{\mathrm{div}(s_q)}$ converge weakly to $-\log \|s_i\|_\varphi^2 c_1(\mathcal{L}_1)....c_1(L_{i-1}).\delta_{\mathrm{div}(s_{i+1})}...\delta_{\mathrm{div}(s_q)}$. When we denote by $L(u)$ the set of point where the plurisubharmonic function u is not locally bounded, the following general proposition is proved in [16]. □

Proposition 1.1.26. *(Demailly). Let U be an open set of X and $T \in D_+^{p,p}(U)$ a positive closed current of type (p,p). Let also $u_1, .., u_q$ be plurisubharmonic functions on U, such that for every choice of indices $j_1 < j_2 .. < j_m$ inside $\{1, 2, .., q\}$ the intersection $L(u_{j_1}) \cap ... \cap L(u_{j_m}) \cap Supp(T)$ is contained in an analytic set of complex dimension $\leq n - p + m$. One can then construct the currents $u_1(dd^c u_2) \wedge ... \wedge (dd^c u_q) \wedge T$ and $(dd^c u_1) \wedge ... \wedge (dd^c u_q) \wedge T$ of mass locally finite over U and uniquely characterized by the fact: for every* *non-increasing sequences* $(u_1^k),..,(u_q^k)$ *of plurisubharmonic functions converging simply to* $u_1, ...u_q$ *respectively, we have that* $u_1^k(dd^c u_2^k) \wedge ... \wedge (dd^c u_q^k) \wedge T$ *and* $(dd^c u_1^k) \wedge ... \wedge (dd^c u_q^k) \wedge T$ *tend weakly on* U *to* $u_1(dd^c u_2) \wedge ... \wedge (dd^c u_q) \wedge T$ *and* $(dd^c u_1) \wedge ... \wedge (dd^c u_q) \wedge T$ *respectively.*

Proof. For the proof we refer to [16], Thm. 3.4.5 and Pro. 3.4.9. or [15]. Thm. 2.5 and Pro. 2.9. □

Definition 1.1.27. *Suppose that, on each of the fibres* X_v *at infinity (*v *denotes places of* $\mathcal{O}_K$*), the line bundle* $\mathcal{L}_v$ *provided with the canonical metric* $||\cdot||_{v,\varphi}$ *is an ample hermitian line bundle (i.e. the metric* $||.||_{P,v,\varphi}$ *is associated to a hermitian product on the fibre* $\mathcal{L}_{P,v}$ *at* P*). Then the canonical current associated to* φ *is defined as* $T_\varphi = c_1(\mathcal{L}_v, ||.||_{v,\varphi})$*. The canonical distribution associated to* φ *is* $d\mu_\varphi = c_1^n(\mathcal{L}_v, ||.||_{v,\varphi})$ *and we set* $\mu(A) = \int_A d\mu_{\varphi,v}$*.*

Proposition 1.1.28. *The canonical distribution is in fact a measure, which we call the canonical measure.*

Proof. The current T_{vfi} is positive, meaning that when we write $T = T_{i,j} dz_i \wedge d\bar{z}_j$, the coefficients $T_{i,j}$ are in particular real measures. In the same way $d\mu_{\varphi,v} = \bigwedge_{i=1}^n T_\varphi$ has also coefficients that are measures. In particular the normalized measure $\frac{\bigwedge_{i=1}^n T_\varphi}{c_1(\mathcal{L})^n}$ is a probability measure on X. □

The fact that the canonical measure satisfies the required invariant properties will be proved in the next proposition.

Proposition 1.1.29. *Suppose that A is a subset of X_v such that $\mu(A) = \int_A d\mu_{\varphi,v} < \infty$, then*

(i) $\mu_{v,\varphi}(\varphi(A)) = (\deg \varphi)^n \mu_{v,\varphi}(A) < \infty$ *whenever $\varphi|A$ is injective,*

(ii) $\mu_{v,\varphi}(\varphi^{-1}(A)) = \mu_{v,\varphi}(A) < \infty$.

Proof. Take an open set V with $\mu(V) < \infty$. Assume that we have $\varphi^{-1}(V) = \bigcup_{i=1}^{\deg(\varphi)^n} U_i$, $\varphi : U_i \to V$ is injective for each i and that U denotes any of the U_i. Take n local sections $s_i \neq 0$ of $\mathcal{O}(1)$ holomorphic on V, in this case $d\mu_{\varphi,v} = \frac{1}{(\pi i)}\partial\overline{\partial}(g_1)\ldots\frac{1}{(\pi i)}\partial\overline{\partial}(g_n)$, where $g_i = \log(\|s_i(P)\|_{\varphi,v})$. Denoting also $deg(\varphi) = d$ we have,

$$\begin{aligned}
\mu_{v,\varphi}(V) &= \int_V \frac{1}{(\pi i)}\partial\overline{\partial}\log(\|s_1(P)\|_{\varphi,v})..\frac{1}{(\pi i)}\partial\overline{\partial}\log(\|s_n(P)\|_{\varphi,v}) \\
&= \int_U \frac{1}{(\pi i)}\partial\overline{\partial}\log(\|s_1(\varphi(P))\|_{\varphi,v})..\frac{1}{(\pi i)}\partial\overline{\partial}\log(\|s_n(\varphi(P))\|_{\varphi,v}) \\
&= d^n \int_U \frac{1}{(\pi i)}\partial\overline{\partial}\log(\|(\varphi^* s_1)(P)\|_{\varphi,v})..\frac{1}{(\pi i)}\partial\overline{\partial}\log(\|(\varphi^* s_n)(P)\|_{\varphi,v}) \\
&= d^n \mu_{v,\varphi}(U),
\end{aligned}$$

and

$$\begin{aligned}
\mu_{v,\varphi}(\varphi^{-1}(V)) &= d^n \mu_{v,\varphi}(U) \\
&= \mu_{v,\varphi}(V).
\end{aligned}$$

Which is the result we wanted to prove. □

Corollary 1.1.30. *The probability measure $d\mu_{\varphi,v}/c_1^n(\mathcal{L})$ is an invariant probability measure on X_v. It is the unique measure of maximal entropy. As with the Brolin measure in $\mathbb{P}^1$ it can be obtained as the limit of the average of Dirac functions supported on back-iterate orbits of points outside a proper algebraic subset (in the dimension one case just the exceptional points).*

Proposition 1.1.31. *If two maps φ and ψ on X commute and for some line bundle $\mathcal{L}$ on X we have $\varphi^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}^\alpha$ and $\psi^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}^\beta$, then $T_\varphi = T_\psi$ and $d\mu_\varphi = d\mu_\psi$.*

Proof. This is a consequence of $||.||_\varphi = ||.||_\psi$ proved in 1.1.16. □

1.1.4 Canonical measures in dimension one

We restrict ourself in this subsection to maps φ on the Riemann Sphere $\mathbb{P}^1$. Recall that a point $x \in \mathbb{P}^1$ is called periodic for φ if it is a fixed point for the map φ^k for some k. A point is called pre-periodic if its image by φ^m for some m is a periodic point for φ. Assume that φ is differentiable as a map $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$, then a fixed point x of φ^k is called repelling (resp. attracting, resp. indifferent) if $|(\varphi^k)'(x)| > 1$ (resp. $|(\varphi^k)'(x)| < 1$, resp. $|(\varphi^k)'(x)| = 1$). The closure in $\mathbb{P}^1$ of the repelling pre-periodic points is called the Julia set of the map φ. We will show now that $d\mu_{v,\varphi}$ extends to a linear functional on the space of continuous functions on $\mathbb{P}^1(\mathbb{C})$ and that this linear functional is the unique φ-invariant probability measure on $\mathbb{P}^1(\mathbb{C})$ with support on the Julia set of φ. Following [27], we define A to be the operator on the space of continuous functions of $\mathbb{P}^1(\mathbb{C})$ which sends a continuous function f on $\mathbb{P}^1(\mathbb{C})$ to

$$(Af)(z) := \frac{1}{d} \sum_{\varphi(w)=z} e_w f(w),$$

where $z \in \mathbb{P}^1(\mathbb{C})$ and e_w is the ramification index of φ at w.

Lemma 1.1.32. *Let V be an open set in $\mathbb{P}^1(\mathbb{C})$. Then*

$$\int_{\varphi^{-1}(V)} f \, d\mu_{v,k+1} = \int_V Af \, d\mu_{v,k} \tag{1.1}$$

for any $k \geq 1$.

Proof. Since the set of ramification points of φ is finite, it suffices to show that (1.1) holds when V contains no ramification points; since any open subset can be written as a union of simply connected open subsets, we may further assume that V is simply connected. We may then decompose $\varphi^{-1}(V)$ into d branches U_i, $1 \leq i \leq d$ such that φ is bijective on each U_i with analytic inverse φ_i^{-1}.

Choose a section s of $\mathcal{O}_{\mathbb{P}^1}(1)$ that does not vanish on V or $\varphi^{-1}(V)$. Let $\rho = \frac{\log \|s\|_{v,k}}{\pi i}$. Then, on U, we have $d\bar{d}\rho = d\mu_{v,k}$ and on U_i, we have $d\bar{d}(\rho \circ \varphi) =$

$(\deg\varphi)(d\mu_{k+1,v})$. By change of variables, we then have

$$\begin{aligned}\int_{U_i} f\,d\mu_{v,k+1} &= \frac{1}{d}\int_{U_i} f(z)\,d\bar{d}(\rho(\varphi(z)))\\ &= \frac{1}{d}\int_{V} f(\varphi_\lambda^{-1}(v))\,d\bar{d}(\rho(v))\\ &= \frac{1}{d}\int_{V} f\circ\varphi_\lambda^{-1}\,d\mu_{v,k}.\end{aligned}$$

Since $(Af)(v) = \frac{1}{d}\sum_{\lambda=1}^{d} f\circ\varphi_\lambda^{-1}(v)$, we thus obtain

$$\begin{aligned}\int_{\varphi^{-1}(V)} f\,d\mu_{v,k+1} &= \sum_{\lambda=1}^{d}\int_{U_i} f\,d\mu_{v,k+1} = \frac{1}{d}\sum_{\lambda=1}^{d}\int_{V} f\circ\varphi_\lambda^{-1}\,d\mu_{v,k}\\ &= \int_{V}\frac{1}{d}\sum_{\lambda=1}^{d} f\circ\varphi_\lambda^{-1}\,d\mu_{v,k} = \int_{V} Af\,d\mu_{v,k}.\end{aligned}$$

□

An *exceptional point* ξ for φ is a point such that $\varphi^2(\xi)=\xi$ and φ^2 ramifies completely at ξ. An exceptional point ξ is a super-attracting fixed point for φ^2 (see J. Milnor [31]).

Now take $\epsilon > 0$. We may choose an open set V_ϵ containing the exceptional points of φ for which

$$\int_{\mathcal{U}_\epsilon} d\mu_{v,1} \le \frac{\epsilon}{2\sup_{z\in\mathbb{P}^1(\mathbb{C})}(|f(z)|_v)}.$$

Such a set exists since $d\mu_{v,1}$ is a continous form. Let $W_\epsilon = \mathbb{P}^1(\mathbb{C})\setminus V_\epsilon$. By Theorem 1 of [27], there is a constant C_f such that $(A^k f)(w)$ converges uniformly to C_f for $w\in W_\epsilon$. Thus, there is some M such that for any $k\ge M$, we have

$$|(A^k f)(w) - C_f|_v < \epsilon/2$$

for all $w \in W_\epsilon$. Using Lemma 1.1 and the fact that $\int_{W_\epsilon} d\mu_{v,1} \leq 1$, we then see that for all $k \geq M$ we have

$$\begin{aligned}
\left| \int_{\mathbb{P}^1(\mathbb{C})} f \, d\mu_{v,k} - C_f \right|_v &= \left| \int_{\mathbb{P}^1(\mathbb{C})} (A^k f) \, d\mu_{v,1} - C_f \right|_v \\
&\leq \left| \int_{W_\epsilon} (A^k f) \, d\mu_{v,1} - C_f \right|_v + \int_{U_\epsilon} |A^k f|_v \, d\mu_{v,1} \\
&\leq \int_{W_\epsilon} \frac{\epsilon}{2} \, d\mu_{v,1} \quad + \quad \int_{U_\epsilon} \left(\sup_{z \in \mathbb{P}^1(\mathbb{C})} (|f(z)|_v) \right) d\mu_{v,1} \\
&\leq \epsilon/2 \quad + \quad \epsilon/2 \quad = \quad \epsilon.
\end{aligned}$$

Thus, $d\mu_{v,\varphi}$ extends to a measure such that

$$\int_{\mathbb{P}^1(\mathbb{C})} f \, d\mu_{v,\varphi} = \lim_{k \to \infty} (A^k f)(z),$$

where z is any point in W_ϵ. Freir, Lopes, and Mane ([18]) have shown that the map sending a continuous function f to $\lim_{k \to \infty} (A^k f)(z)$, where z is a not an exceptional point of φ, is the unique φ-invariant probability measure on $\mathbb{P}^1(\mathbb{C})$ that is supported on the Julia set of φ.

1.1.5 Examples

Here we analyze what we know about the canonical height and the measure associated to each of the examples previously introduced:

Example 1.1.33. *Suppose that K is a field and $\varphi = (p_0 : .. : p_n) : \mathbb{P}^n_K \to \mathbb{P}^n_K$ is a map on $n-$dimensional projective space. In general it is hard to get a closed form for the iterate of such a map. In case that for some natural k we have $p_i(T_0, .., T_n) = T_i^k$ for each $0 \leq i \leq n$, we obtain the so-called naive height on $\mathbb{P}^n_K$:*

$$h([t_0 : .. : t_n]) = \frac{1}{[K : \mathbb{Q}]} \log \prod_{\text{places } v \text{ of } K} \sup(|t_0|_v, ..., |t_n|_v)^{\mathrm{N}_v},$$

where $\mathrm{N}_v = [K_v : \mathbb{Q}_w]$ and w is the place of $\mathbb{Q}$ such that $v \mid w$. The conditions ii) and v) for the canonical height can be easily checked in this case. The associated measure $d\mu_\varphi$ is the normalized Haar measure on the Torus $S^1 \times ... \times S^1$. In the case of dimension one we check that the preperiodic points for the maps $t \mapsto t^k$ on $\mathbb{P}^1$ are always $0, \infty$ and the roots of unity (the roots of unity are the repelling ones). The closure in $\mathbb{P}^1$ of the set of repelling preperiodic points in this example is S^1 and we observe how $d\mu_\varphi$ is the Haar measure supported on the Julia set. If T_0, T_1 represent projective coordinates in $\mathbb{P}^1$, the canonical metric whose curvature gives the canonical measure is

$$\|(\lambda_1 T_0 + \lambda_2 T_1)(a : b)\| = \frac{|\lambda_1 a + \lambda_2 b|}{\sup(|a|, |b|)}.$$

Example 1.1.34. *Suppose that we have our prime number p and the polynomial $G(x) \in K(x)$ where K is an algebraically closed field. As we mention* *before the family of plane algebraic curves* $C_{p,G} : y^p = G(x)$ *has automorphisms given by* $(x, y) \to (x, \xi_p y)$*, where* ξ_p *is a* $p-$*root of unity. Now, the generic member of this family is a curve of genus* $g > 1$*, for such curves the formula of Hurwitz, will tell us that we can not satisfy the conditions to build the canonical height with* $\alpha = d =$*degree of the map* φ *(for a proof see [3]).*

Example 1.1.35. *Consider the family of* $K3$ *surfaces* $S_{a,b} \subset \mathbb{P}^2 \times \mathbb{P}^2$ *determined by the two equations with coefficients in the number field* K*,*

$$\sum_{i,j=1}^{3} a_{i,j} x_i y_j = \sum_{i,j,k,l=1}^{3} b_{i,j,k,l} x_i x_k y_j y_l = 0.$$

The projections p_1 *and* p_2 *represents double coverings of* $\mathbb{P}^2_K$ *and determine morphisms* σ_1 *and* σ_2 *in each of the members of the family* $S_{a,b}$*. Following [38], a new canonical height can be found in this case. Denote by* $L_i = p_i^* \mathcal{O}_{\mathbb{P}^2}(1)$*,* $E^+ = (2+\sqrt{3})L_1 - L_2$ *and* $E^- = -L_1 + (2+\sqrt{3})L_2$*, the geometry of the family* $S_{a,b}$ *can be used to prove that* $(\sigma_1 \circ \sigma_2)^{\pm 1 *}(E^\pm) = (7+4\sqrt{3})E^\pm$ *and we obtain two canonical heights* $\hat{h}^\pm$ *associated to* $E^\pm$ *and* $(\sigma_1 \circ \sigma_2)^{\pm 1}$*. A new canonical height was then introduced by Silverman as* $\hat{h} = \hat{h}^+ + \hat{h}^-$*. In a similar way closed and positive* $(1,1)-$*currents* T^+ *and* T^- *can be defined associated to the morphisms* $(\sigma_1 \circ \sigma_2)^{\pm 1}$ *in such a way that* $T = T^+ + T^-$ *is an invariant current associated to the system of automorphisms* $(\sigma_1 \circ \sigma_2)^{\pm 1}$*. For generalities about canonical heights associated to systems of morphisms,* *as well as for the details on the construction of T^+ and T^- see [24].*

Example 1.1.36. *Consider an elliptic curve $E = \mathbb{C}/\mathbb{C}+\tau\mathbb{C}$ given by Weierstrass equation $y^2 = x^3 + px + q$. The multiplication by n in E induces, as quotient by the action of $[-1]$, a map $\varphi_n : \mathbb{P}^1 \to \mathbb{P}^1$. If $P = (x, y) \in E$, the canonical height $\hat{h}_{\varphi_n}(x)$ is the Neron-Tate height on E of the point P. The preperiodic points for φ_n are the image of the torsion points in E. To see this note that $n^k P = n^{\pm l} P$ for $k \neq l$ implies that P is torsion and conversely if $kP = 0$ we write $n^l = qk + r_l$ and $r_l < k$, then there have to be two indices l, l' with $r_l = r_l'$ and we get $n^l P = n^{l'} P$. The multiplication by n^k is etale on E, the derivative of this map is n^k everywhere and similarly for the map φ_n, more precisely we have that $|(\varphi_n)'(x)| = n^k$ for every $x \in \mathbb{P}^1$ and therefore the Julia set of φ_n is the entire $\mathbb{P}^1$. In [32] it is established that the Haar measure on E gives the curvature of the canonical metric associated to the Neron-Tate height. The projection onto $\mathbb{P}^1$ of this measure gives the $d\mu_{\varphi_n}$,*

$$d\mu_{\varphi_n} = \frac{idt \wedge \bar{d}t}{Im(\tau)|t^3 + pt + q|}.$$

Example 1.1.37. *For a set of k morphisms $\varphi_i : \mathbb{P}^n \to \mathbb{P}^n$ of degree d_i respectively, we can consider the diagonal morphism $\varphi = (\varphi_1, .., \varphi_k) : \mathbb{P}^n \times ... \times \mathbb{P}^n \to \mathbb{P}^n \times ... \times \mathbb{P}^n$. In this case $\varphi_i^* \mathcal{O}(1) \cong \mathcal{O}(d_i)$ and $\varphi^* \mathcal{L} \cong \mathcal{L}^{d_1 + ... + d_n}$, where $\mathcal{L} = p_i^*(\mathcal{O}(1))$. The canonical height satisfies $h_\varphi(x_0, .., x_k) = \sum_i h_{\varphi_i}(x_i)$ and* *for the invariant currents* $T_\varphi = \sum_i T_{\varphi_i}$.

Example 1.1.38. *Take the morphism* $\varphi_1 \times ... \times \varphi_1 : \mathbb{P}^1 \times ... \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times ... \times \mathbb{P}^1$ *(n times) and consider the symmetric group* S_n *acting on* $\mathbb{P}^1 \times ... \times \mathbb{P}^1$ *by permutation of the coordinates. Then the quotient* $\mathbb{P}^1 \times ... \times \mathbb{P}^1/S_n$ *is isomorphic to* $\mathbb{P}^n$ *and* $\varphi_1 \times ... \times \varphi_1$ *descend to a morphism* $\varphi : \mathbb{P}^n \to \mathbb{P}^n$. *In this case* $h_\varphi(\bar{x}) = \sum_i h_{\varphi_1}(p_i(x))$, *where* p_i *is representing the* $i-th$ *projection on* $\mathbb{P}^1 \times ... \times \mathbb{P}^1$ *and* $\bar{x}$ *is the class of* $x \in \mathbb{P}^1$ *under the action of* S_n.

1.2 Geometric intersection in a Macaulay variety

Suppose that X is Cohen-Macaulay arithmetic variety of dimension n+1. The geometric intersection theory, usually developed for non-singular varieties, can be extended to Cohen-Macaulay varieties for certain kind of cycles.

Definition 1.2.1. *Suppose that the* q_1-*cycle* D_1 *is a locally complete intersection and* D_2 *is any* q_2-*cycle. Assume that they have no common components and* $q_1 + q_2 \leq n + 1$. *If* D_1 *and* D_2 *are locally given by coordinate rings* A_1 *and* A_2 *the intersection cycle is defined by a sum of components:*

$$(D_1.D_2) = \sum_{A_1,A_2} n_{A_1,A_2} C_{A_1,A_2},$$

where

$$n_{A_1,A_2} = \sum_{i=0}^{\dim X} (-1)^i \operatorname{length}(\operatorname{Tor}_i(A_1, A_2)).$$

Definition 1.2.2. *If a multiple kD_1 is a locally complete intersection we define $(D_1.D_2) = 1/k(kD_1, D_2)$.*

Definition 1.2.3. *The degree of the intersection is defined as:*

$$\deg(D_1.D_2) = \sum_{A_1,A_2} n_{A_1,A_2}.$$

Remark 1.2.4. *Let P be a point on X. Suppose D_1 is given in a neighborhood of P by the zero set of $f_1, ..., f_q$ and that D_2 is given by complete intersection of certain equations $f_{q+1}, ..., f_{n+1}$ in such a way that the sequence $f_1, .., f_{n+1}$ is a regular sequence in $\mathcal{O}_P$, then the degree of the intersection $\deg(D_1.D_2)$ is expressed as a sum of terms of the form $\operatorname{length}(\mathcal{O}_P/(f_1, .., f_{n+1}))$. ∎ Even more, if only multiples k_1D_1 and k_2D_2 are expressed as before, the intersection number will be a sum of $1/k_1k_2 \operatorname{length}(\mathcal{O}_P/(f_1, .., f_{n+1}))$.*

Proposition 1.2.5. *If D_2 has codimension $n+1$ and D_1 is locally given by one equation, then $(D_1.D_2) = 0$.*

Proof. Suppose that f is the equation defining D_1 and I is the ideal of D_2 in the local ring $\mathcal{O}_x = A$. The assumption on the dimension of D_2 implies that the modules A/I and $\operatorname{Tor}_i(A/I, A/f)$ are of finite length for all $i \leq n+1$.

Now the result follows because the length is an additive function and we have the exact sequence:

$$0 \to \mathrm{Tor}_{n+1}(A/I, A/f) \to \dots \to \mathrm{Tor}_2(A/I, A/f) \to \mathrm{Tor}_1(A/I, A/f) \to A/I$$

$$\to A/I \to A/(I + (f)) \to 0.$$

□

The symmetry and bilinearity can be proved extending the methods of [36] to rings which are Cohen-Macaulay instead of regular. In the dimension 2 case we will give complete proofs of a lot of desirable properties for an intersection pairing:

Definition 1.2.6. *Assume that X is an Arithmetic Surface. We define the schematic (or geometric) intersection number of a Cartier divisor D with a Weil cycle C when they have no common components as*

$$(D.C) = \mathrm{length}(\mathcal{O}_D \otimes \mathcal{O}_C) - \mathrm{length}(\mathrm{Tor}_1(\mathcal{O}_D, \mathcal{O}_C)).$$

Proposition 1.2.7. ***(bilinearity and symmetry)*** *The pairing $(D.C)$ we just defined is bilinear and symmetric when both sides are Cartier divisors.*

Proof. If C is a Weil divisor, it is a linear combination with integral coefficients of reduced and irreducible Weil divisors C_i (the coefficient of C_i

is equal to length$((\mathcal{O}_C)_{\wp_i})$. The sheaf $\mathcal{O}_D$ being of Tor dimension 1 the pairing is linear on the right by devissage. To see the linearity on the left it is enough to look at the case where C is reduced and irreducible. The proof will be complete after the reader checks the following lemma:

Lemma 1.2.8. *Let A be a commutative ring, I an ideal in A and f a non zero divisor in A. Then one has the following exact sequence:*

$$0 \to A/I \to A/fI \to A/fA \to 0$$

Proof. The symmetry is clear when D and C are Cartier divisors with no common components. □

If C' is a $\mathbb{Q}$-Cartier divisor, i.e. a Weil divisor with an integral multiple nC' which is Cartier, we will define

$$(C'.C) = \frac{1}{n}((nC').C)$$

The following propositions are classical:

Proposition 1.2.9. ***(linear equivalence)*** *If C is a Cohen-Macaulay projective curve then*

$$(D.C) = \deg_C \mathcal{O}_X(D)_{|C}.$$

Proof. One needs only to check that the degree of a line bundle is a well behaved notion on Cohen-Macaulay projective curves. This is the Riemann-Roch theorem for curves.

If C is a projective curve and L is a line bundle on X a projective surface one can then speak of $(L.C)$ for L is the difference in $\mathrm{Pic}(X)$ between two very ample line bundles each of them having sections with no common components with C. □

Corollary 1.2.10. *When C is a projective curve and D is a $\mathbb{Q}$-Cartier divisor, the intersection $(D.C)$ is well-defined by bilinearity and linear equivalence even when D and C have a common component.*

Corollary 1.2.11. *Let F be a rational function on a reduced, irreducible surface X that is projective and generically smooth over B. Then for any Weil divisor C contained in a fiber over B, we have*

$$(\mathrm{div}(F).C) = 0.$$

Proof. This is clear for the line bundle $\mathcal{O}_X(\mathrm{div}(F))$ is equal to $\mathcal{O}_X$ and C is a projective curve. □

Proposition 1.2.12. ***(projection formula)*** *Let $\varphi : Y \to X$ be a map between surfaces X and Y that are projective over B. If L is a line bundle* *on X and C closed subscheme of Y one has*

$$(\varphi^*(L).C) = (L.\varphi_*(C)).$$

In particular if C is contracted by φ to a subscheme of X of codimension 2 the intersection number $(\varphi^*(L).C)$ is zero.

Proof. By additivity we can suppose C is a reduced irreducible curve in X. There are two cases: $\varphi_*(C)$ is of dimension 1 and $\varphi_*(C)$ is of dimension zero. In the first case $C \to \varphi_*(C)$ is finite and by Lemma 1.2.8 we have

$$\begin{aligned}\text{length}(\mathcal{O}_C/(f\mathcal{O}_C)) &= \text{length}(\mathcal{O}_X/(f\mathcal{O}_X) \otimes \mathcal{O}_C) \\ &= \text{length}(\mathcal{O}_Y/(f\mathcal{O}_Y) \otimes \mathcal{O}_C).\end{aligned}$$

In the second case L can be realized as the line bundle associated to the difference of two very ample divisors on X each of them having no intersection with $\varphi_*(C)$. The reciprocal images of these divisors in Y do not meet C, so both side of the projection formula vanish as it is required. □

The following proposition shows that intersection theory for $\mathbb{Q}$-Cartier divisors does not change when we pass to the normalization.

Proposition 1.2.13. ***(invariance under normalization)*** *Let A be a Cohen-Macaulay integral domain of dimension 2 and let $\tilde{A}$ be its integral closure. Let (f, g) a regular sequence in A then supposing $\tilde{A}$ is a finitely* *generated A-module (f,g) is a regular sequence in $\tilde{A}$ and*

$$\text{length}(A/(f,g)) = \text{length}(\tilde{A}/(f,g)).$$

Proof. We shall note that $\tilde{A}$ is finitely generated over A when we are in a geometric or arithmetic situation. When (f,g) is a regular sequence in a module M the only non vanishing Tor is the tensor product $M \otimes A/(f,g)$. One proves by induction that the alternating sum $\sum_i (-1)^i \text{length}(\text{Tor}_i(A/(f,g), M))$ is a non negative additive function on the set of finitely generated A-modules. This sum is is equal to zero on modules of the form A/h or $A/(h,k)$ for a system of parameters (h,k) in A and thus is zero on $A/\wp$ for any prime ideal $\wp$ containing h or (h,k) by devissage. Hence, it is equal to zero on any module of dimension less than or equal to one. Any non-zero prime ideal of A being of the previous form our assertion of vanishing is proved. The module $\tilde{A}/A$ being of dimension at most one and (f,g) remaining a regular sequence in $\tilde{A}$ the proposition follows by additivity. □

The following proposition is proved in P. Deligne [13] with the additional assumption that X is normal:

Proposition 1.2.14. *Let X be a generically smooth, reduced, irreducible and locally Cohen-Macaulay surface. The geometric intersection product on* *a fiber of $X \to \mathrm{Spec}(B)$ when X is projective over B, is negative. Only combinations of full fibers have zero self-intersection.*

By Proposition 1.2.13, the assumption that X is normal may be replaced with the weaker assumption that X is Cohen-Macaulay.

1.3 The blow up.

Let $\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ be a map with $\varphi^*\mathcal{O}(1) \cong \mathcal{O}(d)$ and defined over a number field K. A model for φ is a set $q_0, q_1, ..., q_n$ of elements of $\mathcal{O}_K[T_0, T_1, ..., T_n]_d$, such that $\varphi = (q_0, q_1, ..., q_n)$. We will be interested in models of φ such that the set of elements $q_0, q_1, ..., q_n$ form a regular sequence.

Lemma 1.3.1. *Let K be a field, $A = K[T_0, T_1, ..., T_n]$ and $p_0, p_1, .., p_n \in A$. Then, the following two statements are equivalent:*

(i) the sequence $p_0, p_1, .., p_n$ is a regular sequence.

(ii) the $(n+1)-$tuple $(p_0, p_1, .., p_n)$ represents a map from $\mathbb{P}^n \to \mathbb{P}^n$.

Proof. *i)* $\Rightarrow$ *ii)* The sequence $(p_0, p_1, .., p_n)$ is regular so $V(\langle p_0, p_1, .., p_n \rangle) = 0$, which is equivalent to *ii)*. *ii)* $\Rightarrow$ *i)* If we have *ii)* then $\sqrt{\langle p_0, p_1, .., p_n \rangle} = \langle T_0, T_1, .., T_n \rangle$, this says that $p_0, .., p_n$ is a system of parameters in A and therefore a regular sequence, because A is Cohen-Macaulay. □

Corollary 1.3.2. *Let K be a field, $A = K[T_0, T_1, ..., T_n]$ and $p_0, p_1, .., p_n$ a regular sequence in A. Then the sequence $p_k = (p_{k0}, p_{k1}, .., p_{kn})$, defined recursively by*

$$p_0 = (p_0, p_1, .., p_n) \quad p_{ki} = p_{k-1i}(p_0, p_1, .., p_n) \quad 0 \leq i \leq n \quad k > 0$$

is also a regular sequence for all k.

Proof. p_0 is regular sequence by assumption. On the other hand if p_{k-1} is regular, then by 1.3.1 $(p_{k-10}, p_{k-11}, .., p_{k-1n})$ represents a map $\mathbb{P}^n \to \mathbb{P}^n$ and then by definition of p_{ki} the sequence $p_k = (p_{k0}, p_{k1}, .., p_{kn})$ also represents a map on the n-projective space. Again by 1.3.1, the sequence p_k is a regular sequence. □

Lemma 1.3.3. *Let $A = O_K[T_0, T_1, ..., T_n]$ and $p_0, p_1, .., p_n \in A$. Then, the following two statements are equivalent:*

(i) *the sequence $p_0, p_1, .., p_n$ is a regular sequence.*

(ii) $\dim(V(\langle p_0, p_1, .., p_n \rangle)) = 0$.

Proof. The proof is very similar to the one of 1.3.1. The ring A is Cohen-Macaulay of dimension $n + 1$, therefore the sequence $p_{k,0}, ..., p_{k,n}$ is regular if and only if it is a maximal system of parameters. □

Corollary 1.3.4. *Let* $A = \mathcal{O}_K[T_0, T_1, ..., T_n]$ *and* $p_0, p_1, .., p_n$ *a regular sequence in A. Then the sequence* $p_k = (p_{k0}, p_{k1}, .., p_{kn})$*, defined recursively by*

$$p_0 = (p_0, p_1, .., p_n) \quad p_{ki} = p_{k-1i}(p_0, p_1, .., p_n) \quad 0 \le i \le n \quad k > 0$$

is also a regular sequence for all k.

Proof. If p_{k-1} is a regular sequence, $\dim(V(\langle p_{k-1,0}, p_{k-1,1}, .., p_{k-1,n}\rangle)) = 0$, because φ is a finite map, $\dim(V(\langle p_{k,0}, p_{k,1}, .., p_{k,n}\rangle)) = 0$ and p_k is also a regular sequence. □

In the rest of this subsection we will work with $X = \mathbb{P}^n_{\mathcal{O}_K}$. Denote by Y_k, the closed subscheme of X defined by the ideal $I_{\varphi^k} = \langle p_{k,0}, p_{k,1}, ..., p_{k,n}\rangle$.

Definition 1.3.5. *The model* X_k *is defined by the property that* $\sigma^k : X_k \to X$ *is the blowing up of* Y_k*. The exceptional divisor will be denoted by* E_k *and its irreducible components by* $C_{v,i,k}$*, in such a way that we have a finite sum* $E_k = \sum_{v,i>0} r_{v,i,k} C_{v,i,k}$.

Proposition 1.3.6. *The scheme* X_k *is Macaulay and* Y_k *is a subscheme of codimension* $n+1$ *in* X_k *and does not meet the generic fiber* X_K*. The component* $C_{v,i,k}$ *is a projective space of dimension* n *over the the residual field field* $K_{v,i,k}$ *of the close point image of* $C_{v,i,k}$.

Proof. These are consequences of the fact that $p_{k,0}, p_{k,1}, ..., p_{k,n}$ is a regular sequence in $\mathcal{O}_K[T_0, T_1, ..., T_n]$. The scheme $X_k = \text{proj}(\bigoplus I^n_{\varphi^k})$ is locally complete intersection in $\mathbb{P}^n_{\mathcal{O}_K}$. The intersection $\hat{\deg}(\mathcal{L}^n_k | C_{i,v,k}) = [K_{v,i,k} : K] \log |N(v)|$ and by definition of the blow-up we get $C_{v,i,k} = \mathbb{P}^n_{K_{v,i,k}}$. □

Definition 1.3.7. *The projection from* Y_1 *to* Spec $\mathcal{O}_K$ *will be called the places of bad reduction.*

Remark 1.3.8. *The only places* v *appearing in the exceptional divisors* $E_k = \sum_{v,i>0} r_{v,i,k} C_{v,i,k}$ *are the places of bad reduction.*

Assume that the polynomial $F \in K[z_1, ..., z_n]$ has $v(F) = 0$ for every v of bad reduction. We can do this because there is only finitely many places of bad reduction and any Dedekind domain with finitely many primes (like $\cap_v \mathcal{O}_v(z)$) is unique factorization domain. Then the irreducible horizontal divisor D in X is defined by the equation:

$$div(F) = D - \deg(F)\infty + \sum_{\text{finite } v} v(F) X_v.$$

On the other hand we also let $X_{k,v}$ denote the fibre of X_k at the finite place v and $F_k = \sigma_k^* F$. We are going to consider the line bundle $\mathcal{L}_0 = \mathcal{O}(1)$ on X and $\mathcal{L}_k = \sigma_k^* \mathcal{L}_0$ on X_k. An homogeneous polynomial $F_1 \in K[T_0, T_1, ..., T_n]$ defines a hypersurface $D_{\mathbb{Q}} \subset X_{\mathbb{Q}}$ and we consider the polynomial $F = F_1 / T_n^{\deg(F)}$.

The symbol ∞_k is denoting the divisor of X_k defined by the equation $\sigma_k^* T_n = 0$ and in particular $\infty = \operatorname{div}(T_n)$.

Lemma 1.3.9. *There exist non-negative integers $x_{v,i,k}$ and $y_{v,i,k}$ depending only on D, such that $\operatorname{div}(F_k)$ can be written as*

$$\begin{aligned}\operatorname{div}(F_k) = D_k - \deg(F)\infty_k + \sum_{v,i} x_{v,i,k} C_{v,i} \\ - \deg(F) \sum_{v,i} y_{v,i,k} C_{v,i} + \sum_{\text{finite } v} v(F) X_v\end{aligned} \tag{1.2}$$

where D_k is the proper transform of D by σ_k.

Proof. We have the formula for the divisor $\operatorname{div}(F_k)$:

$$\operatorname{div} F_k = \operatorname{div} \sigma_k^* F = \sigma_k^* \operatorname{div} F = \sigma_k^*(D) - n\sigma_k^*(\infty) + \sum_{\text{finite } v} v(F)\sigma_k^*(X_v),$$

but now for certain non-negative integers $x_{v,i,k}$ the reciprocal image of the effective divisor D is $\sigma_k^*(D) = D_k + \sum_{v,i} x_{v,i,k} C_{v,i,k}$. Now also for certain non-negative integers $y_{v,i,k}$ we have $\sigma_k^*(\infty) = \infty_k + \sum_{v,i} y_{v,i,k} C_{v,i}$ and the proof is finished. □

Remark 1.3.10. *As a result of the proof we just did:*

$$\sigma_k^*(\infty) = \infty_k + \sum_{v,i} y_{v,i,k} C_{v,i} \quad \sigma_k^*(D) = D_k + \sum_{v,i} x_{v,i,k} C_{v,i,k}.$$

1.3.1 Positivity conditions

It is interesting to look at a particular type of models.

Definition 1.3.11. *We will say that a model* $(q_0, ..., q_n)$ *satisfies positivity conditions if for every* k *we have* $\sqrt{\langle p_{k,0}, p_{k,1}, ..., p_{k,n}, T_n \rangle} = \langle T_0, T_1, ..., T_n \rangle$ *in* $\mathcal{O}_K[T_0, ..., T_n]$.

Example 1.3.12. *This condition is satisfied for example if we are consider a map* $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$, *on the Riemann sphere and* p_0 *is a monic polynomial in the variable* T_0.

Lemma 1.3.13. *If the model have positivity conditions the proper transform* ∞_k *of* ∞ *in* X_k *is equal to the reciprocal image* $\sigma^k(\infty)$.

Proof. It is enough to show that ∞ does not meet Y_k and this is a consequence of the fact that the ideal $\sqrt{\langle p_{k,0}, p_{k,1}, ..., p_{k,n}, T_n \rangle} = \langle T_0, T_1, ..., T_n \rangle$. □

Lemma 1.3.14. *If the model has positivity conditions, there exist non-negative integers* $x_{v,i,k}$ *depending only on* D, *such that* $\mathrm{div}(F_k)$ *can be written as*

$$\mathrm{div}(F_k) = D_k - \deg(F)\infty_k + \sum_{v,i,k} x_{v,i,k} C_{v,i,k} + \sum_{\text{finite } v} v(F) X_v$$

where D_k *is the proper transform of* D *by* σ_k.

Proof. The results follows by Lemma 1.3.1 and Lemma 1.3.9. □

1.4 Integral at finite places

Let v denote a valuation on K, $\mathcal{O}_v$ the set of elements $z \in K$ such that $v(z) \geq 0$, x will denote a vector $x = (x_0, .., x_n) \in \bar{K}^n$ and $P \in \mathbb{P}^n(\bar{K})$ a point in the n-projective space over $\bar{K}$. The valuation v is assumed to be extended to an algebraic closure $\bar{K}$ of K. For a vector $x = (x_0, x_2, ..., x_n)$ we define $v(x) = min_i\{v(x_i)\}$. For a polynomial $p \in \bar{K}[T_0, .., T_n]$ we take $v(p)$ as the valuation of the vector formed by its coefficients. For a sequence of polynomials $(p_0, ..., p_n)$ we put $v(p_0, ..., p_n) = \min_i\{v(p_i)\}$. Suppose that $\varphi : \mathbb{P}^n \to \mathbb{P}^n$ is a rational map of degree d given by homogeneous polynomials $(p_0 : ... : p_n)$ over $\mathcal{O}_v$, then we can define a map

$$S_v : \bar{K}^{n+1} - (0, ..., 0) \times \mathcal{O}_v[T_0, .., T_n]_d^{n+1} \to \mathbb{R}$$

$$S_v(x, (p_o, ..., p_n)) = v(p_0(x), .., p_n(x)) - v(x_0^d, ..., x_n^d)$$

The map S_v, is in fact a well defined map $S_v : \mathbb{P}^n \times \mathcal{O}_v[T_0, .., T_n]_d^{n+1} \to \mathbb{R}$, which we still denote by $S_v(P, (p_0, .., p_n))$. To see this, take any two sets of homogenous coordinates for P, say $(x_0, ..., x_n)$ and $(y_0, ..., y_n) = \lambda(x_0, ..., x_n)$, then $v(p_0(\lambda x), .., p_n(\lambda x)) = dv(\lambda) + v(p_0(x), .., p_n(x))$ and $v(\lambda^d x^d) = dv(\lambda) + v(x^d)$, and the result follows.

Definition 1.4.1. *The polynomials $p_{k,i}$ are defined as before for each $0 \leq i \leq n$ as $p_{k,i} = p_i(p_{k-1,0}, ..., p_{k-1,n})$ for $k > 0$ and $p_{0,i} = p_i$.*

Definition 1.4.2. *Suppose that the polynomial F has divisor* $\operatorname{div}(F) = D - \deg(F)\infty - \sum_{\text{finite}\, v} v(F) X_v$, *then we define:*

$$E(F, v\,\text{finite}) = -\lim_k \sum_v \log|N(v)| \Big(\frac{\sum_{P in D} S_v(P, p_{k,0}, ..., p_{k,n})}{d^{nk}} - \deg(F) \frac{\sum_{P\in\infty} S_v(P, p_{k,0}, ..., p_{k,n})}{d^{nk}} - v(F) \Big).$$

Remark 1.4.3. *We have* $S_v(P, p_{k,0}, ..., p_{k,n}) > 0$ *only for finitely many* P, *because the sequence* $(p_{k,0}, ..., p_{k,n})$ *is regular. The existence of the limit* $E(F, v\,\text{finite})$ *in the most general case will be result of theorem 1.5.8.*

1.4.1 Convergence of each v-adic integral in dimension one.

We are going to work now in dimension one, with a map $\varphi = (p_0, p_1) : \mathbb{P}^1 \to \mathbb{P}^1$ on the Riemann Sphere. Let's keep the notation of the previous subsection, that is $S_v(x, (p_0, p_1)) = v(p_0(x), p_1(x)) - v(x_0^d, x_1^d)$.

Proposition 1.4.4. *The sequence*

$$h_k(P) = \{ \frac{S_v(P, (p_{k,0}, p_{k,1}))}{d^k} \}_k$$

(i) is bounded and increasing, and therefore convergent to a function, which we denote $h_{p_0,p_1,v}(P)$.

(ii) $h_{p_0,p_1,v}(\varphi(P)) = d h_{p_0,p_1,v}(P)$.

The proof will be the result of the application of two lemmas:

Lemma 1.4.5. *Suppose that we denote* $P_k = (p_{k,0}(P) : p_{k,1}(P))$, *then we have the equality*

$$S_v(P,(p_{k+1,0},p_{k+1,1})) = dS_v(P,(p_{k,0},p_{k,1})) + S_v(P_k,(p_0,p_1)).$$

Proof. Assume that $P = (x_0 : x_1)$ and $v(x) = v((x_0,x_1)) = 0$. If we set $x_k = (p_{k,0}(x), p_{k,1}(x))$ we have the equalities

$$\begin{aligned} S_v(P,(p_{k+1,0},p_{k+1,1})) &= v(p_{k+1,0}(x),p_{k+1,1}(x)) \\ &= v(p_0(x_k),p_1(x_k)) - v(x_k^d) + v(x_k^d) \\ &= dS_v(P,(p_{k,0},p_{k,1})) + S_v((P_k,(p_0,p_1)). \end{aligned}$$

□

Lemma 1.4.6. *The function* $S_v(P,(p_0,p_1))$ *is bounded on* $\mathbb{P}^n(\bar{K})$, *so we can define*

$$R_v(p_0,p_1) = \sup_P\{S_v(P,(p_0,p_1))\}.$$

Proof. There exist elements $b_i \in \mathcal{O}_v$, where $0 \leq i \leq 1$, such that $b_i x_i^d \equiv 0(p_0,p_1)$. Besides $b_i \neq 0$ because the polynomials p_0, p_1 do not have a common zero in K. If $P = (x_0 : x_1) \in \mathbb{P}^n$, with $x_i \neq 0$, then $S_v(P,(p_0,p_1)) \leq v(b_i)$. So in general $\sup_P\{S_v(P,(p_0,p_1))\} \leq \sup_i\{v(b_i)\}$. □

Now we can do the proof of the proposition

Proof. From lemma 1.4.5, we get $0 \le h_{k+1}(P) - h_k(P) \le R_v(p_0, p_1)/d^{k+1}$, so $\{h_k(P)\}$ is bounded by $R_v(p_0, p_1)/(d-1)$ and therefore converges. On the other hand

$$\begin{aligned} h_k(\varphi(P)) - dh_k(P) &= \frac{S(P_1, (p_{k,0}, p_{k,1}))}{d^k} \\ &\quad \frac{-dS(P, (p_{k,0}, p_{k,1}))}{d^k} \\ &= \frac{S(P, (p_{k+1,0}, p_{k+1,1})) - v(x_1)}{d^k} \\ &\quad \frac{-dS(P, (p_{k,0}, p_{k,1}))}{d^k} \\ &= \frac{S(P_k, (p_0, p_1)) - v(x_1)}{d^k} \\ &\le \frac{R_v(p_0, p_1) - v(x_1)}{d^k}. \end{aligned}$$

By passing to the limit we get $h_{v,p_0,p_n}(\varphi(P)) = dh_{v,p_0,p_1}(P)$. □

Definition 1.4.7. *Suppose that* $\varphi : \mathbb{P}^1_K \to \mathbb{P}^1_K$. *We will define the "integral" of* $\log|F|_v$ *over the finite place* v *of a polynomial* $F = K[z_1, .., z_n]$ *as*

$$\begin{aligned} \int_{\mathbb{P}^1_{\bar{K}^n_v}} \log|F|_v d\mu_{v,\varphi} = \log|N(v)| \Big(&\sum_{P \in D} h_{v,p_0,p_1}(P) \\ &- \deg(F) \sum_{P \in \infty} h_{v,p_0,p_1}(P) - v(F) \Big), \end{aligned}$$

and then

$$E(F, v\,\text{finite}) = \sum_v \int_{\mathbb{P}^1_{\bar{K}^n_v}} \log|F|_v d\mu_{v,\varphi}.$$

1.4.2 A remark on the use of integral notation in dimension one.

Assume in this subsection that we are working with a map $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$. In this case we can add a few words about why it makes sense to think of our definition of $\int_{\mathbb{P}^1(\mathbb{C}_v)} \log |F|_v \, d\mu_{v,\varphi}$ as an integral when v is finite. H. Brolin ([10]) and M. Lyubich ([27]) have shown that if v is an infinite place and θ is a continuous, bounded function on $\mathbb{P}^1(\mathbb{C}_v)$, then for any $\xi \in \mathbb{C}$ with an infinite backward orbit under φ (i. e. for which the set $\cup_{k=1}^{\infty}(\varphi^k)^{-1}(\xi)$ is infinite), one has

$$\lim_{k\to\infty} \sum_{\varphi(z)=\xi} \frac{\theta(z)}{d^k} = \int_{\mathbb{P}^1(\mathbb{C}_v)} \theta \, d\mu_{v,\varphi},$$

where $d\mu_{v,\varphi}$ is the unique φ-invariant measure (see [18]) on $\mathbb{P}^1$ (which is the same as our $d\mu_{v,\varphi}$). Our p-adic integrals can be written in a similar way. For example, suppose D is an irreducible divisor corresponding to a single point $[a : b]$ such that $p_{0,k}(a,b) \neq 0$. Then the polynomial $F(t) = bt - a$ defines D. Writing $p_{0,k} = \eta_k \prod_{j=1}^{d^k}(T_0 - u_j T_1)$, we then have

$$p_{0,k}(a,b) = \eta_k \prod_{j=1}^{d^k}(b - u_j a) = \eta_k \prod_{j=1}^{d^k} F(u_j).$$

Since $\varphi^k(z) = 0$ if and only if $[z : 1] \sim [u_j : 1]$ for some j, we thus have

$$\frac{\log |p_{0,k}(a,b)|_v}{d^k} = \frac{\log |\eta_k|_v}{d^k} + \sum_{\varphi^k(z)=0} \frac{\log |F(z)|_v}{d^k},$$

where the z with $\varphi^k(z) = 0$ are counted with multiplicities. Similarly, when p_1 is not a multiple of X_1^d, we have

$$\frac{\log|p_{1,k}(a,b)|_v}{d^k} = \frac{\log|\gamma_k|_v}{d^k} + \sum_{\varphi^k(z)=\infty} \frac{\log|F(z)|_v}{d^k},$$

where γ_k is the leading coefficient of $p_{1,k}(T_0, 1)$. Taking limits and subtracting off $\frac{\log|A_d|_v}{d-1}$, we see that $\int_{\mathbb{P}^1(\mathbb{C}_v)} \log|F|_v d\mu_{v,\varphi}$ is equal to

$$\lim_{k\to\infty} \max\left(\sum_{\varphi^k(z)=0} \frac{\log|F(z)|_v}{d^k}, \sum_{\varphi^k(z)=\infty} \frac{\log|F(z)|_v}{d^k}\right).$$

More generally, with a bit of diophantine geometry, we can show that for any point $\xi \in \mathbb{C}_v$ that has an infinite backwards orbit under φ, we have

$$\int_{\mathbb{P}^1(\mathbb{C}_v)} \log|F|_v d\mu_{v,\varphi} = \lim_{k\to\infty} \sum_{\substack{\varphi^k(z)=\xi \\ F(z)\neq 0}} \frac{\log|F(z)|_v}{d^k}.$$

The proof of the above proposition is an ongoing project, but if we accept it, our v-adic integrals at finite places would look quite analogous to our integrals at the infinite places. We should note, however, that we do not know what classes of functions we can expect to be able to "integrate" in this way at finite places.

1.4.3 Geometry of $E(F, v\,\text{finite})$

We are now in the position to relate $E(F, v\,\text{finite})$ with the geometry of the blow-up. Suppose that $\sigma_k : \mathbb{P}_k^n \to \mathbb{P}^n$ is the blow-up associated with the model $(p_{k,0}, ..., p_{k,n})$. Writing

$$\sigma_k^* D = D_k + \sum_{v,i} x_{v,i,k} C_{v,i,k} \qquad \sigma_k^*(\infty) = \infty_k + \sum_{v,i} y_{v,i,k} C_{v,i,k}. \tag{1.3}$$

where $C_{v,i,k}$ are the different components of the exceptional fibre above v and denoting by $K_{v,i,k}$ the field of definition of the close point corresponding to $C_{v,i,k}$, we can state

Proposition 1.4.8. *With the notation as above and with v a finite place of K we have:*

$$\sum_i x_{i,v,k}[K_{v,i,k} : K] = \sum_{P \in D} S_v(P, p_{k,0}, ..., p_{k,n}),$$
$$\sum_i \deg(F) y_{i,v,k}[K_{v,i,k} : K] = \sum_{P \in \infty} S_v(P, p_{k,0}, ..., p_{k,n}).$$

Proof. Consider the set of polynomials $p_{k,0}, ..., p_{k,n}$ defining a model for the map φ^k and the ideal $I_k = \langle p_{k,0}, ..., p_{k,n} \rangle$. The set $V(I_k)$ is finite, say $\{P_{i,\rho,k}\}_{i,\rho}$, where ρ moves through the finite places of K. Fix a finite place v. We have the divisor D_v, such that $\mathrm{div}(F)_v = D_v - \deg(F)\infty_v + v(F)$ and the points $P_{v,i,k} \in D_v \cap V(I_v)$. Take $P = (x_0 : ... : x_n)$ be one of them and build the local ring $\mathcal{O}_P$. The image $\bar{p}_{k,i}$ of $p_{k,i}$ under the reduction map $\mathcal{O}_K \to \mathcal{O}_P$ generate an ideal which we call $I_{k,P}$. So on one hand we have,

$$\begin{aligned}\sigma_k^* I_{k,P} =& Proj(\mathcal{O}_P[T_0, .., T_n] / \langle (\bar{p}_{k,i}(x_0 : ... : x_n) T_j - \bar{p}_{k,j}(x_0 : ... x_n) T_i \rangle_{i,j}) \\ \cong& Proj(\mathcal{O}_P / \pi^{r_{k,v}}[T_0, .., T_n])\end{aligned}$$

where $r_{k,v} = S_v(P, p_{k,0}, ..., p_{k,n})$ and π is a uniformizer parameter for $\mathcal{O}_P$ and on the other hand

$$\sigma_k^* P_{v,i,k} = [K_{v,i,k} : K] x_{v,i,k} C_{v,i,k}.$$

for $C_{v,i,k} \cong \mathbb{P}^n_{K_{v,i,k}}$. As a consequence of this two facts, for each $P_{v,i,k} \in X_v \cap V(I_k)$ we get

$$S_v(P_{v,i,k}, p_{k,0}, ..., p_{k,n}) = x_{i,v,k}[K_{v,i,k} : K]$$

and

$$\sum_i x_{i,v,k}[K_{v,i,k} : K] = \sum_{P \in D} S_v(P, p_{k,0}, ..., p_{k,n}).$$

The second part is analogous using $\deg(F)\infty$ instead of D. □

Remark 1.4.9. *The expression $E(F, v\,\text{finite})$ previously defined, takes the form*

$$E(F, v\,\text{finite}) = -\lim_k \sum_v \log|N(v)|(\frac{\sum_{i,v} x_{v,i,k}[K_{v,i,k} : K_v]}{d^{nk}} - \deg(F) \lim_k \frac{y_{v,i,k}[K_{v,i,k} : K_v]}{d^{nk}} - v(F)).$$

Remark 1.4.10. *In case we are working in dimension one, that is with a map $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$, we can do a more precise computation of the contributions $y_{i,v,k}$. In this case our model (p_0, p_1) satisfies T_1/p_1 and therefore $T_1/p_{k,1}$ for each k. So, having in mind that the point $\infty = (1, 0)$ and taking our* *polynomial* $p_0 = A_d T_0^d + ...$ *we get* $S_v((1,0), p_{0,k}, p_{1,k}) = v(p_{0,k}(1,0))$ *and* $y_{v,i,k} = v(p_{0,k}(1,0)) - v(p_{0,k}, p_{1,k})$. *Since* $p_{0,1}(1,0) = A_d$, *then for all* k *we have* $p_{0,k+1}(1,0) = A_d p_{1,0}(1,0)^d$ *and*

$$\lim_k \frac{y_{v,i,k}}{d^k} = \lim_k v(A_d) \sum_i^{\infty} \frac{1}{d^i} = \frac{v(A_d)}{d-1}.$$

The integral at a finite place v *take then the particular form*

$$\int_{\mathbb{C}_v^n} \log |F|_v d\mu_{v,\varphi} = -\lim_k \log |N(v)| (\lim_k \frac{x_{v,i,k}[K_{v,i,k} : K_v]}{d^k} - \deg(F) \frac{A_d[K_{v,i,k} : K]}{d-1} - v(F)).$$

and again

$$E(F, v\,\text{finite}) = \sum_v \int_{\mathbb{P}^n_{K_v^n}} \log |F| d\mu_{v,\varphi}.$$

Remark 1.4.11. *The geometry of* $E(F, v\,\text{finite})$ *allows to express in a better way the positivity conditions for a model, indeed If our model* $(p_0, .., p_n)$ *have positivity conditions and we pick the equation* F *such that* $v(F) = 0$ *for* v, *then,*

$$E(F, v\,\text{finite}) \leq 0.$$

1.5 Mahler formula

In this section we present the main result of this paper. We start by recalling some properties of the arithmetic degree of line bundles.

Proposition 1.5.1. *Let X be a Cohen-Macaulay arithmetic variety of dimension $n+1$ and $Z \in Z_k(X)$, a cycle on X. Denote the class $\hat{c}_1(\mathcal{L}_i)$ by $\hat{c}_1(\mathcal{L}_i) = (c_1(\mathcal{L}_i), c_1(\bar{\mathcal{L}}_i))$, then the number $\hat{\deg}_Z(\hat{c}_1(\mathcal{L}_1)...\hat{c}_1(\mathcal{L}_k)) \in \mathbb{R}$, is completely determined by the properties:*

(i) is $k-$linear.

(ii) is symmetric.

(iii) for $k=0$ and $Z = \sum_i n_i P_i$, we have $\hat{\deg}_Z = \sum_i n_i \log N_{P_i}$.

(iv) for $k \geq 1$ and $s_k \neq 0$ a section of $\mathcal{L}_k$ which meets Z properly we have

$$\hat{\deg}_Z(\hat{c}_1(\mathcal{L}_1)...\hat{c}_1(\mathcal{L}_k)|Z) = \hat{\deg}_Z(\hat{c}_1(\mathcal{L}_1)...\hat{c}_1(\mathcal{L}_{k-1})|Z.\operatorname{div}(s_k)) - \int_{X(\mathbb{C})} \delta_{Z(\mathbb{C})} \log \|s_k\|_k c_1(\bar{\mathcal{L}}_1)...c_1(\bar{\mathcal{L}}_{k-1}).$$

Proof. Conditions (i), (iii) and (iv) are in fact sufficient to uniquely determined the number $\hat{\deg}_Z(\hat{c}_1(\mathcal{L}_1)...\hat{c}_1(\mathcal{L}_k))$. Introducing the star product $g_1 * g_2 * ... * g_k$ of currents we can state a non-recursive definition of the arithmetic degree $\hat{\deg}_Z(\hat{c}_1(\mathcal{L}_1)...\hat{c}_1(\mathcal{L}_k))$ and prove (ii) with the use of Stoke's theorem. □

Proposition 1.5.2. *Suppose that $\mathcal{L}$ is a hermitian line bundle on X and $f \in K(X)$ is a rational function on X. Then:*

$$\hat{\deg}(\hat{c}_1(\mathcal{O}(f)).\hat{c}_1^n(\mathcal{L})) = 0.$$

Proof. The curvature of the trivial bundle $c_1(\mathcal{O}(f)) = 0$. Using this result and the symmetry of the arithmetic degree we can reduce to the case of dimension 1, which is nothing else but the product formula (see for example the treatment in [40]). □

Proposition 1.5.3. *Let Z be the algebraic cycle defined by the polynomial equation $g_v = 0$ contained in a finite fibre X_v of an arithmetic variety X_v, such that $X_{\mathbb{C}} = \mathbb{P}^n_{\mathbb{C}}$ and $\mathcal{L} = \mathcal{O}(1)$. Then*

$$\hat{\deg}_Z(\hat{c}_1^{n-1}(\mathcal{L})|Z) = \deg(g_v)\log N(v).$$

Proof. First of all we realize that because Z is contained in a finite fibre $\int_{X(\mathbb{C})} \delta_{Z(\mathbb{C})} \log\|s\| c_1^{k-1}(\mathcal{L}) = 0$, for any choice of the section s of $\mathcal{L} = \mathcal{O}(1)$. On the other hand the sections of $\mathcal{O}(1)$ represent linear conditions on the coordinates. The proposition is then a consequence of Bertini's theorem. □

Proposition 1.5.4. *Suppose that $\mathcal{L}$ is a hermitian line bundle on X and $f \in K(X)$ is a rational function on X then,*

$$\hat{\deg}(\hat{c}_1^n(\mathcal{L})|\operatorname{div}(f)) = \sum_\sigma \int_{X_\sigma} \log|f|_\sigma d\mu_\sigma.$$

Proof. We have that $\hat{\deg}(\hat{c}_1(\mathcal{O}(f)).\hat{c}_1^n(\mathcal{L})) = 0$ and also that

$$\hat{\deg}(\hat{c}_1(\mathcal{O}(f)).\hat{c}_1^n(\mathcal{L})) = \hat{\deg}(\hat{c}_1(\mathcal{L}_1)^n)|\operatorname{div}(f))- \sum_\sigma \int_{X_\sigma(\mathbb{C})} \log|f|_\sigma(c_1(\bar{\mathcal{L}}_1)...c_1(\bar{\mathcal{L}}_{k-1}).$$

which gives the formula we wanted. □

Definition 1.5.5. *Let $Y \in Z_q(X)$ be a $q-$cycle inside the arithmetic variety X_{O_k} and $\mathcal{L}_1, ...\mathcal{L}_q$ ample line bundle on X. The real number*

$$h_{\mathcal{L}_1,..,\mathcal{L}_q}(Y) = \hat{\deg}_Y(\hat{c}_1(\mathcal{L}_1)...\hat{c}_1(\mathcal{L}_q)|Y)$$

is called the multi-height of Y relative to $\mathcal{L}_1, ..., \mathcal{L}_q$.

Suppose that the arithmetic varieties $(X_k, \mathcal{L}_k)$ represents a models of a projective variety $(X, \mathcal{L})$ of dimension n together with a line bundle $\mathcal{L}$. Assume that $(\mathcal{L}_k, ||.||_k)$ are metrized line bundles on X_k and the metrics satisfy $||.||_{k+1} = (\varphi^*||.||_k)^{1/\alpha}$ where the map $\varphi : X \to X$ satisfies $\varphi^*\mathcal{L} \cong \mathcal{L}^\alpha$. Then the numbers $\hat{\deg}_Y(\hat{c}_1((\mathcal{L}_k, ||.||_k))^{\dim(Y)}|Y)$ converge for every cycle Y (see [46]) and the limit will be called $\hat{\deg}(c_1(\mathcal{L}, ||.||_\varphi)^{\dim Y}|Y)(Y)$.

Definition 1.5.6. *Under the conditions just discussed we define*

$$h_\varphi(Y) = \frac{\hat{\deg}(c_1(\mathcal{L}, ||.||_\varphi)^{\dim Y}|Y)(Y)}{c_1(\mathcal{L})^{\dim Y - 1}}.$$

Remark 1.5.7. *If we have a model $(X_1, \mathcal{L}_1)$ of $(X, \mathcal{L})$ such that the map $\varphi : X \to X$ extends to a map $\varphi : X_1 \to X_1$ we can put $\mathcal{L}_k = \varphi^{k*}\mathcal{L}_1$ and the normalized probability measure $d\mu_\varphi / c_1(\mathcal{L})^n$ from section 2.3 and the canonical height just defined will satisfy the relation*

$$h_\varphi(\mathrm{div}(F)) = \sum_{v/\infty} \int_{X_v} \log |F|_v \frac{d\mu_{\varphi,v}}{c_1(\bar{\mathcal{L}})^n}.$$

For the next proposition suppose that we are dealing with a map $\varphi : \mathbb{P}^n \to \mathbb{P}^n$. Recall the non-standard models $\sigma_k : X_k \to \mathbb{P}^n$ that we introduced before as the blows up of $\mathbb{P}^n$ at the subschemes Y_k. Denote by $K_{v,i,k}$ the field of definition of $C_{v,i,k}$ and by K_v the local field of K at v. Recall that $F_k = \sigma_k^* F$ for a polynomial $F = F(T_0/T_n, .., T_{n-1}/T_n)$ and $\mathcal{L}_k = \sigma_k^* \mathcal{L}_0$, for $\mathcal{L}_0 = \mathcal{O}(1)$. The divisor $\operatorname{div}(F) = D_{\mathbb{Q}} - \deg(F)\infty + \sum_{v \text{ finite}} v(F) X_v$ where $\operatorname{div}(T_n) = \infty$ in $\mathbb{P}^n$ and $\sigma_k^*(\infty) = \infty_k + \sum_{i,v} y_{v,i,k} C_{v,i,k}$.

Theorem 1.5.8. *We have the equality*

$$h_\varphi(D_{\mathbb{Q}}) = \sum_{v/\infty} \int_{\mathbb{P}^n_{K_v}} \log |F|_v d\mu_{\varphi,v} + E(F, v \text{ finite}) + \deg(F) h_\varphi(\infty).$$

Proof. We are going to make use of the arithmetic intersection theory on $\mathbb{P}^n_k$. Let's compute $\hat{\deg}(\hat{c}_1(L_k)^n | \operatorname{div}(F_k))$. We have the following:

(i) $\hat{\deg}(\hat{c}_1(L_k)^n | \operatorname{div}(F_k)) = d^{nk} \sum_{v|\infty} \int_{\mathbb{P}^1(\mathbb{C}_v)} \log |F|_v c_1(v, \|.\|_k)^n$ by proposition 1.5.4.

(ii) $\hat{\deg}(\hat{c}_1(L_k))^n | C_k) = [K_{v,i,k} : K] \log |N(v)|$ by proposition 1.3.6.

Let's recall the formula 1.5:

$$\operatorname{div}(F_k) = D_k - \deg(F)\infty_k + \sum_{v,i} x_{v,i,k} C_{v,i} - \deg(F) \sum_{v,i} y_{v,i,k} C_{v,i} + \sum_{\text{finite } v} v(F) X_v$$

Now we are going to let $\hat{\deg}(\hat{c}_1(L_k)^n|.))$ acts on each side,

$$\hat{\deg}(\hat{c}_1(L_k)^n|\operatorname{div}(F_k)) = h_{\mathcal{L}_k^n}(D_k) - (\deg(F))h_{\mathcal{L}_k^n}(\infty_k)$$

$$+\sum_{i,v} x_{v,i,k}\log|N(v)|[K_{v,i,k}:K] - \deg(F)\sum_{i,v} y_{v,i,k}$$

$$\log|N(v)|[K_{v,i,k}:K] + d^{nk}\sum_v v(F)\log|N(v)|,$$

dividing by d^{nk} and taking limits gives us that the limit

$$\lim_k \sum_{i,v}(x_{v,i,k}\log|N(v)|[K_{v,i,k}:K] - \deg(F)y_{v,i,k}\log|N(v)|[K_{v,i,k}:K])$$

exists and

$$\sum_{v|\infty}\int_{\mathbb{P}^1(\mathbb{C}_v)} \log|F|_v d\mu_{\varphi,v} = -\,E(F, v\,\text{finite})$$

$$+\,h_\varphi(D_{\mathbb{Q}}) - \deg(F)h_\varphi(\infty),$$

which was the result we wanted to prove. □

Corollary 1.5.9. *Suppose that* $\varphi = (p_0, p_1) : \mathbb{P}^1 \to \mathbb{P}^1$ *and we have choosen coordinates such that* T_n/p_1. *The integral of the log the minimal equation* F *of a point* $P = (\lambda, 1) \in \mathbb{P}^1$ *is related to the height of* P *by the formula*

$$h_\varphi(P) = \frac{1}{\deg(F)}\sum_v \int_{\mathbb{P}^1_{\bar{K}_v}} \log|F|_v d\mu_{\varphi,v}.$$

Corollary 1.5.10. *If* φ *has a model such that the divisor* ∞ *has a finite forward orbit* $\{\infty, \varphi(\infty), \ldots.\}$ *(which forces* $h_\varphi(\infty) = 0$*), then*

$$h_\varphi(D_{\mathbb{Q}}) = \sum_{v/\infty}\int_{\mathbb{P}^n_{\bar{K}_v}} \log|F|_v d\mu_{\varphi,v} + E(F, v\,\text{finite}).$$

Corollary 1.5.11. *Assume $h_\varphi(\infty) = 0$. Suppose that our model $(p_0, ..., p_n)$ for φ, satisfies positivity conditions and the $v(F) = 0$ for every finite place $v \in \mathcal{O}_k$, then*

$$h_\varphi(D_\mathbb{Q}) \leq \sum_{v/\infty} \int_{\mathbb{P}^n_{\bar{K}_v}} \log |F|_v d\mu_{\varphi,v}.$$

Example 1.5.12. *Consider a model for a map $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$. Suppose that $p_0(T_0, 1)$ is monic and that the minimal equation F of the point $(\alpha, 1)$ has coprime coefficients in $\mathcal{O}_K$. Also assume that $p_1(\alpha, 1) = 0$. Then we have*

$$\int_{\mathbb{P}^1_{\bar{K}^n_v}} \log |F|_v d\mu_{v,\varphi} = \frac{-v(p_0(\alpha, 1))}{d} \log |N(v)|$$

for each finite place v. If $-v(p_0(\alpha, 1)) > 0$ for some v we get the strict inequality

$$h_\varphi(D_\mathbb{Q}) < \sum_{v/\infty} \int_{\mathbb{P}^n_{\bar{K}_v}} \log |F|_v d\mu_{\varphi,v}.$$

Example 1.5.13. *Let E be an elliptic curve with Weierstrass equation $E : y^2 = G(x)$, where G is a polynomial of degree 3. Multiplication by 2 gives rise to the rational map*

$$\varphi(t) = \frac{G'^2(t) - 8tG(t)}{4G(t)}$$

on $\mathbb{P}^1$. Suppose that $G(t) = (t-a)(t-b)(t-c)$, for certain $a < b < c \in \mathbb{Q}$. Since the points $(a, 0), (b, 0), (c, 0)$ are preperiodic for φ we have, applying the *formula we just proved,*

$$\sum_v \int_{\mathbb{P}^1(\mathbb{Q})} \log |G|_v d\mu_{\varphi,v} = h_\varphi(a) + h_\varphi(b) + h_\varphi(c) = 0$$

so we use the product formula and the example above to get

$$\sum_{finite v} \int_{\mathbb{P}^1(\mathbb{Q})} \log |G|_v d\mu\varphi, v = \sum_{finite v} \frac{-2v(G'(a)G'(b)G'(c))}{4} \log |N(v)|$$
$$= \frac{-\log(|G'(a)||G'(b)||G'(c)|)}{2},$$

Now we know that the product $|G'(a)||G'(b)||G'(c)| = |\operatorname{Disc}(G)|$ *so our result is translated into*

$$\int_{\mathbb{P}^1(\mathbb{C})} \log |G| d\mu_{\varphi,\infty} = \frac{\log |\operatorname{Disc}(G)|}{2},$$

where ∞ *is the only archimedean place of* $\mathbb{Q}$.

Example 1.5.14. *The map* $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ *given by the model* $\varphi(x, y, z) = (x^2+yx, y^2+zx+zy, z^2)$ *has good reduction everywhere, that is, the expression* $E(F, v\,\text{finite}) = 0$.

Example 1.5.15. *The map* $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ *given by the model* $\varphi(x, y, z) = (y^2 - 3z^2, x^2 - 3y^2, zy)$ *has bad reduction over* 3. *The reduced map is not defined at the point* $(0, 0, 1) \in \mathbb{P}^2_{\bar{K}_3}$. *The model satisfy positivity conditions and* $h_\varphi(\infty) = 0$. *If the equation* $F = c_m z^3 + \ldots$ *has* $c_m \equiv 0 (mod\ 3)$, *then* $E(F, v\,\text{finite}) \neq 0$ *and viceversa.*

Example 1.5.16. *The map* $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ *given by the model* $\varphi(x, y, z) = (y^2 - 5z^2, x^2 - 3y^2, zy)$ *has bad reduction over the places* 3 *and* 5. *It is not a well defined map at the points* $\{(\beta, 0, 1); (-\beta, 0, 1)\} \subset \mathbb{P}^2_{\bar{K}_3}$ *and also at* $\{(\alpha, 0, 1); (-\alpha, 0, 1)\} \subset \mathbb{P}^2_{\bar{K}_5}$, *where* β *is root of* $x^2 - 2 = 0$ *in* $\bar{K}_3$ *and* α *is root of* $x^2 - 3 = 0$ *in* $\bar{K}_5$

1.5.1 Generalized Mahler Measure

Suppose that a map φ has good reduction everywhere. It is natural to define:

Definition 1.5.17. *The Mahler measure with respect to* φ *and relative to an infinite place* v *of* K *of a polynomial* $F \in \mathcal{O}_K[T_0/T_n, .., T_{n-1}/T_n]$ *is defined as*

$$\mu_{v,\varphi}(F) = \int_{\mathbb{C}^n} \log|F| d\mu_{v,\varphi}.$$

1.5.2 Equidistribution theorems.

In recent results P. Autissier [2] has proved the following equidistribution theorem in dimension one:

Theorem 1.5.18. *Consider a map* $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ *on the Riemann sphere. Let* v *be any infinite place and* (α_n) *in* $\mathbb{P}^1(\bar{K})$ *a non-repeating sequence of points such that* $\lim_{n \to \infty} h(\alpha_n) = 0$ *then, the sequence of discrete measures* $\frac{1}{|\mathrm{Gal}(\alpha_n)|} \sum_{\sigma \in \mathrm{Gal}} \delta_{\sigma(\alpha_n)}$ *converges weakly to* $d\mu_\varphi$.

M. Baker and R. Rumely ([4]) have given a new proof of this theorem, using capacity theory. Their proof also gives equidistribution results at finite places. Suppose we are again in dimension one. One might also ask whether the Mahler measure should also be computable by equidistribution; more precisely, we conjecture:

Conjecture 1.5.19. *Let F be the minimal equation of point α not in the Galois orbit of any α_n and v an infinite place of K, one has*

$$\lim_{n\to\infty} \frac{1}{|\operatorname{Gal}(\alpha_n)|} \sum_{\sigma\in \operatorname{Gal}} \log |F(\sigma(\alpha_n))|_v = \int_{X(\mathbb{C}_v)} \log |F|_v d\mu_{v,\varphi}.$$

We can prove Conjecture 1.5.19 in the case that the points α are periodic points. This generalizes earlier work on "elliptic Mahler measure" by G. Everest and T. Ward in [17, Theorem 6.18]. Let's see now what we have for a map $\varphi : \mathbb{P}^n \to \mathbb{P}^n$. The following result is due to Briend and Duval.

Theorem 1.5.20. *The sequence of measures $\frac{1}{d^n}\sum_{\varphi^n(y)=y, y \text{ is repelling}} \delta_y$ converges to μ_φ as n goes to ∞. The exceptional set E of φ is defined as the biggest proper algebraic set $\mathbb{P}^n$ that is totally invariant under φ. Then for every $a \in \mathbb{P}^n - E$, the measures $\frac{1}{d^n}\sum_{y:\varphi^n(y)=a} \delta_y$ converges to μ_φ as n goes to ∞.*

Properties of the canonical height make the back iterated orbit a sequence of small points, indeed if $\varphi^n(x) = y$ then $h_\varphi(x) = h_\varphi(y)/d^n$. The above theorem is a particular result towards a desirable general statement about equidistribution of small points for maps on $\mathbb{P}^n$. When the map φ is a Lattés example, there exist such a theorem [24].

Appendix I: Arithmetic intersection theory

1.5.3 Currents on complex manifolds

Let X be a connected compact smooth projective complex manifold of dimension n. Let $A_c^{p,q}$ the space of smooth differential forms on X with compact support. A current of type (p,q) on X is a linear map

$$S : A_c^{n-p,n-q}(X) \longrightarrow \mathbb{C}$$

continuous for the Schwartz topology. We denote by $D^{p,q}(X)$ the space of currents of type (p,q).

Example 1.5.21. *If $Y \subset X$ is a closed irreducible subvariety of codimension p, we let δ_Y be the current acting on $A_c^{n-p,n-p}(X)$ as*

$$\delta_Y(\omega) = \int_{Y-Y_{sing}} \omega.$$

Example 1.5.22. *A (p,p) form ω_Z defines a (p,p) current $[\omega_Z]$ as*

$$[\omega_Z](\eta) = \int_X \omega_Z \wedge \eta.$$

If $S \in D^{p,q}(X)$, the derivatives $\partial S \in D^{p+1,q}(X)$ and $\bar{\partial} S \in D^{p,q+1}(X)$ are defined by

$$\partial S(\nu) = (-1)^{p+q+1} S(\partial \nu)$$

and

$$\bar{\partial} S(\nu) = (-1)^{p+q+1} S(\bar{\partial} \nu).$$

Remark 1.5.23. *There exist real operators $d = \delta + \bar{\delta}$ and $d^c = i(\bar{\delta} - \delta)$, which make the notations easier sometimes, for example in one dimension we have $dd^c f = \Delta f dx \wedge dy$.*

1.5.4 Green currents

In this part we define a good class of currents with respect to the integration on a complex manifold.

Definition 1.5.24. *Let Z be a cycle of codimension p on X. A Green current for Z is a current $g \in D^{p-1,p-1}(X)$ such that*

$$\frac{\bar{\partial}\partial}{\pi i} g + \delta_Z = \omega_Z \tag{1.4}$$

for some $\omega_Z \in A^{p,p}(X)$.

Definition 1.5.25. *Let X be a smooth quasi-projective complex variety and Y a proper algebraic subset. A C^∞ form ν on $X - Y$ is said to be of "log"* *type along Y when there exist a smooth quasi-projective variety $\pi : M \to X$ and a C^∞ form η on $M - \pi^{-1}Y$ such that:*

(i) $\pi^{-1}(Y)$ *is a divisor with normal crossing and* π *is smooth on* $X - Y$.

(ii) ν *is the direct image by* π *of* $\eta|Z - \pi^{-1}\nu$.

(iii) For a point $x \in M$, there is an open U of x and a system of coordinates $(z_1, .., z_n)$ such that $\pi^{-1}(Y) \cap U$ has equation $z_1 = z_2 = .. = z_k = 0$ and there exist δ and $\bar{\delta}$-closed forms α_i and a smooth form β on U such that $\eta|U = \sum_i \alpha_i \log|z_i| + \beta$.

Remark 1.5.26. *Suppose that $(\mathcal{L}, ||.||)$ is a hermitian line bundle on X, and $s \in \Gamma(X, \mathcal{L})$, with $s \neq 0$. The poincare-Lelong formula states*

$$dd^c[-\log||s||^2] + \delta_{\mathrm{div}(s)} = c_1(\mathcal{L}).$$

This is saying that a Green current for $Z = \mathrm{div}(s)$ is given by

$$g_Z(\omega) = \int_X -\log||s||\omega.$$

Once we have fix Z the solution for the differential equation 1.4 is not unique. In fact we can proof that we can find a candidate with logarithmic growth along Z with an explicit construction in the case of divisors given by the Poincare-Lelong lemma.

1.5.5 Product and *-product of currents

For general currents we do not have a product as we do for smooth currents. Results of Bedford, Taylor and Demailly [5], [14], [15] [16], allow us to consider a product of currents with good properties.

Definition 1.5.27. *(Lelong). Let U be an open set of complex manifold M of dimension n. A current $T \in D^{p,p}(U)$ is said to be positive ($T \geq 0$) if for every choice of C^∞ $(1,0)$-forms $\alpha_1, .., \alpha_{n-p}$ with compact support on U, the distribution $T \wedge (i\alpha_1 \wedge \bar{\alpha}_1) \wedge ... \wedge (i\alpha_{n-p} \wedge \bar{\alpha}_{n-p})$ is a positive measure on U.*

Example 1.5.28. *A locally integrable function u on X is said to be plurisubharmonic if the hessian $i\partial\bar{\partial}u = i\sum \partial^2 u/\partial zj \partial z_j \wedge \bar{z}_m \geq 0$ on X. For example if we take $\log(\rho(z))$, where ρ is an smooth function.*

Definition 1.5.29. *(Bedford-Taylor). Let T be a positive closed current of type (p,p) and u a plurisubharmonic function locally bounded on U. We define the product $(dd^c u) \wedge T = dd^c(uT)$.*

Remark 1.5.30. *The product Tu is well defined because the function u is locally bounded and the current T has measures as coefficients. In general we can define*

$$(dd^c u_1) \wedge (dd^c u_2)...(dd^c u_q) \wedge T = dd^c(u_1 \wedge (dd^c u_2)...(dd^c u_q) \wedge T).$$

By prop 1.2 in [16], the current $(dd^c u_1) \wedge (dd^c u_2)...(dd^c u_q) \wedge T$ *is a positive closed current of bidegree* $(p+q, p+q)$.

Definition 1.5.31. *Let* U *be an open set of* X *and* $u_1, ..., u_q$ *plurisubharmonic functions on* U. *The vector* $(u_1, ..., u_q)$ *is say to be admissible if for every* $1 \leq i \leq q$ *the set* $L(u_i)$ *is contained inside an analytic set* $A_i \subset U$ *and for every choice of index* $\{j_1, .., j_m\} \subset \{1, 2, .., q\}$ *we have codim* $\cap_i A_{j_i} \geq m$.

Definition 1.5.32. *Suppose that for certain locally bounded and continuous plurisubharmonic functions* $u_{i,1}, ...u_{i,r_i}$ $(i = 1..m)$, *admissible vectors* $(v_{j,1}, ..., v_{j,s_j})$ $(j = 1..l)$ *and forms* β_j *and* w_i, *the currents* x, y *are locally given (on an open set* U *of a covering of* X*) by expressions of the form:*

$$x = \sum_{i=1}^{m} w_i (dd^c u_{i,1}) \wedge ... \wedge (dd^c u_{i,r_i})$$

$$y = \sum_{j=1}^{l} \beta_j v_{j,1} (dd^c v_{j,2}) \wedge ... \wedge (dd^c v_{j,s_j})$$

then we can define a current of type $(p+q)$, $[x.y](U)$ *by the expression*

$$\sum_j \sum_i w_i \beta_j v_{j,1} (dd^c u_{i,1}) \wedge ... \wedge (dd^c u_{i,r_i}) \wedge (dd^c u_{j,1}) \wedge ... \wedge (dd^c v_{j,s_j}).$$

Definition 1.5.33. *Let* p *be a positive integer we will denote (following [29]) by* $\overline{A}^{p,p}(X) \subset D^{p,p}(X)$ *(resp.* $A_{\log}^{=p,p}(X) \subset D^{p,p}(X)$*) the vector space of the currents that can be expressed for every open of a sufficiently refine covering* *of X in the form:*

$$\sum_{i=1}^{m} w_i(dd^c u_{i,1}) \wedge ... \wedge (dd^c u_{i,r_i})$$

$$(\sum_{j=1}^{l} \beta_j v_{j,1}(dd^c v_{j,2}) \wedge ... \wedge (dd^c v_{j,s_j}))$$

for certain continuous and locally bounded plurisubharmonic functions u_{i,r_i} ($i = 1..m$), admissible vectors $(v_{j,1}, ..., v_{j,s_j})$ ($j = 1..l$) and forms β_j and w_i.

Remark 1.5.34. *The above definitions make sense after the results in th.2.5 and prop. 2.9 of [15] and th. 3.4.5 and prop. 3.4.9 of [16].*

Definition 1.5.35. *We say that a sequence $x_k \in \overline{A}^{p,p}(X)$ converge to $x \in \overline{A}^{p,p}(X)$ in the sense of Bedford and Taylor if:*

$$x_k = \sum_{i=1}^{m} w_i^k(dd^c u_{i,1}^k) \wedge ... \wedge (dd^c u_{i,r_i}^k)$$

$$x = \sum_{i=1}^{m} w_i(dd^c u_{i,1}) \wedge ... \wedge (dd^c u_{i,r_i})$$

and the sequence of plurisubharmonic functions $u_{i,j}^k$ converge uniformly to $u_{i,j}$ for each pair (i,j), as well as the forms w_i^k tend to w_i consider as C^∞ forms.

Proposition 1.5.36. *The product $[.,.] : \overline{A}^{p,p}(X) \times A_{\log}^{=q,q}(X) \to A_{\log}^{=p+q,p+q}(X)$ satisfies the properties:*

(i) $[x.y]$ extends the wedge product of forms $x \wedge y$.

(ii) $[x.y] = [y.x]$ *(commutativity).*

(iii) $[[x.y].z] = [x.[y.z]]$ *(associativity).*

(iv) If x_n *converge absolutely to* x*, then* $[x_n.y]$ *converge absolutely to* $[x.y]$*.*

(v) If $f : \mathbb{P}^n \to \mathbb{P}^n$ *is a morphism of the n-projective space, then* $f^*(x.y) = f^*x.f^*y$*.*

Proof. The proof is a combination of results in [15] and [16]. □

At this point a class of "good" metrics may be introduced:

Definition 1.5.37. *Let* $(\mathcal{L}, ||.||)$ *be hermitian metrized line bundle on* X*. The metric* $||.||$ *is said to be positive if and only if* $c_1(\mathcal{L}, ||.||) \geq 0$ *on* X*. A continuous and positive metric* $||.||$ *is said to be admissible on* X *if there exist a sequence of smooth and positive metrics* $||.||_n \to ||.||$ *on* X*.*

Example 1.5.38. *If we start with an smooth metric* $||.||_0$ *on a line bundle* $\mathcal{L}$ *on* X *and a map* $\varphi : X \to X$ *satisfies* $\varphi^*\mathcal{L} = \mathcal{L}^\alpha$*, the so called canonical metric* $||.||_\varphi$ *is admissible.*

Remark 1.5.39. *The curvature* $c_1(\mathcal{L}, ||.||)$ *for an admissible metric* $||.||$ *is a positive current. The expression* $c_1^n(\mathcal{L}_v, ||.||_{v,\varphi}) = c_1(\mathcal{L}_v, ||.||_{v,\varphi}) \times ... \times c_1(\mathcal{L}_v, ||.||_{v,\varphi})$ *can be consider for any admissible metric* $||.||$*.*

Definition 1.5.40. *Let* $Z \in Z^p(X)$ *a cycle of codimension* p*, and* $g \in D^{p-1,p-1}$ *a Green current for* Z*. Suppose that* $\mathcal{L}_1 = (L_1, \|.\|_1), ..., \mathcal{L}_q = (L_q, \|.\|_q)$ *are admissible line bundles on* X*, in such a way that the cycles* Z, $\mathrm{div}(s_0)$,..,$\mathrm{div}(s_q)$ *intersect properly on* X*. If we denote* $\delta_i = \delta_{\mathrm{div}(s_i)}$, $w_i = c_1(\mathcal{L}_i)$, $g_i = -\log\|s_i\|$ *and* $w = dd^c g + \delta_Z$*, we can define an element* $g * g_1 * ... * g_q \in D^{p+q-1,p+q-1}(X)$ *by the equality:*

$$g * g_1 * ... * g_q = g.\delta_1.\delta_2...\delta_q + g_1 w \delta_2....\delta_q + ... + g_q w w_1..w_{q-1}.$$

Remark 1.5.41. *Suppose that we have a line bundle* $\mathcal{L}$ *on an algebraic variety* X *of dimension* n *and* n*-sections* $s_1, s_2, .., s_n$ *meeting properly on* X*, denote also* $Z_i = \mathrm{div}(s_i)$ *and* $Z = \cap_{i=1}^n Z_i$*. The star product* $g_1 * ... * g_n$ *when* $g_i = \log\|s_i\|$ *can be chosen to be a Green current for* Z*, meaning that for some form* w_Z *we have*

$$\frac{1}{(\pi i)}\partial\bar{\partial}(g_1 * ... * g_k) + \delta_Z = w_Z.$$

1.5.6 Arithmetic Chow groups and intersection

Let K be a number field and X be a regular projective scheme flat over $\mathcal{O}_K$. Denote by $X_\sigma(\mathbb{C})$ the fibre at infinity over the place σ and $F_\sigma : X_\sigma(\mathbb{C}) \to X_\sigma(\mathbb{C})$ the complex conjugation.

Definition 1.5.42. *An arithmetic cycle on* X *is a vector* $(Z, g_{\sigma_1}, ..., g_{\sigma_k})$*,* *where Z is a cycle on X and g_{σ_k} is a green current on $X_{\sigma_k}(\mathbb{C})$ associated to $Z \otimes_\sigma \mathbb{C}$, with the condition that g_σ is real and $F_\sigma^*(g_\sigma) = (-1)^p g_\sigma$ for every place σ.*

Example 1.5.43. *The vector $(0, \partial_{\sigma_1} u + \bar{\partial}_{\sigma_1} v, ... \partial_{\sigma_k} u + \bar{\partial}_{\sigma_k} v)$, where $u \in D^{p-2,p-1}(X_\sigma)$ and $v \in D^{p-1,p-2}(X_\sigma)$ for all σ.*

Example 1.5.44. *If $Y \subset X$ is an irreducible cycle of codimension $p-1$ and $f \in k(Y)^*$ is a non zero rational function on Y, we get the arithmetic cycle $(\mathrm{div}(f), -\log|f|_{\sigma_1}, ..., -\log|f|_{\sigma_k})$, where $-\log|f|_\sigma \in D^{p-1,p-1}(X_\sigma)$ maps ω to $-\int_{Y_\sigma} \log|f(\sigma)|\omega$.*

Definition 1.5.45. *Suppose that $f \in K(X)^*$ is a rational function on X. The arithmetic cycle $(\mathrm{div}(f), -\log|f|_{\sigma_1}, ..., -\log|f|_{\sigma_k})$, will be called the arithmetic cycle associated to the rational function f and will be denoted by $\mathrm{div}^a(f)$.*

Definition 1.5.46. *The arithmetic Chow group $CH^p(X)$ is defined by the quotient*

$$\frac{\{(Z, g_1, .., g_k), codim(Z) = p\}}{\langle (0, (\partial_\sigma u + \bar{\partial}_\sigma)_{\sigma=1}^k); (\mathrm{div}(f), -\log|f|), f \in k(Y)^*, cod(Y) = p-1 \rangle}$$

and it comes with the two projections $z(Z, g) = Z$ and $a(Z, g) = g$.

Definition 1.5.47. *The map ω_i is defined as $w_i(Z, g_1, ... g_k) = \frac{\bar{\partial}\partial}{2\pi i} g_i + \delta_Z$.*

Definition 1.5.48. *Let $\mathcal{L}$ be line bundle on X and s a non-zero section. The first arithmetic Chern class $\hat{c}_1(\mathcal{L})$ of $\mathcal{L}$ is given by $(\operatorname{div}(s), [-\log \|s\|_\sigma]_{\sigma=1}^k)$ as a class in $CH^1(X)$.*

Theorem 1.5.49. *The pairing $(Y, g_{Y_\sigma})_\sigma.(Z, g_{Z_\sigma})_\sigma = (Y \cap Z, g_{Y_\sigma} * g_{Z_\sigma})_\sigma$ defined as a function $CH^p(X) \times CH^q(X) \to CH^{p+q}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, for $p \geq 0$ and $q \geq 0$, is associative, bilinear and the maps z and ω are graded ring homomorphisms.*

Definition 1.5.50. *Let X be a Cohen-Macaulay arithmetic variety and $Z \in Z_k(X)$, a k cycle on X the number $\hat{\deg}_Z(\hat{c}_1(\mathcal{L}_1)...\hat{c}_1(\mathcal{L}_k)) \in \mathbb{R}$ is defined by:*

$$\hat{\deg}_Z(\hat{c}_1(\mathcal{L}_1)...\hat{c}_1(\mathcal{L}_k)) = \sum_i \deg(\operatorname{div}(s_1).... \operatorname{div}(s_k)) + \int_{Z(\mathbb{C})} g_1 * .. * g_k,$$

where s_i is a section of $\mathcal{L}_i$ for every $1 \leq i \leq k$ and we assume that the divisors $\operatorname{div}(s_i)$ meet properly on X.

Proposition 1.5.51. *The arithmetic degree $\hat{\deg}_Z(\hat{c}_1(\mathcal{L}_1)...\hat{c}_1(\mathcal{L}_k))$ is a symmetric function of the $\mathcal{L}_i$.*

Proof. Suppose that g_i (for $i = 1, 2$) are Green currents of "log" type along the cycles Z_1 and Z_2 (for the existence see lemma 1.2.2 of [7]). We have $g_1 * g_2 = g_2 * g_1 + \delta T_1 + \bar{\delta} T_2$ for some currents $T_1 \in D^{p-1,p}$ and $T_2 \in D^{p,p-1}$, then $\int_Z g_1 * g_2 = \int_Z g_2 * g_1 + \int_Z \delta T_1 + \int_Z \bar{\delta} T_2$ and by Stokes theorem for currents of "log" type, we obtain the symmetry for the arithmetic degree. □

Proposition 1.5.52. *Let X be a Cohen-Macaulay arithmetic variety and $Z \in Z_k(X)$, a cycle on X, the number $\hat{\deg}_Z(\hat{c}_1(\mathcal{L}_1)...\hat{c}_1(\mathcal{L}_k)) \in \mathbb{R}$, is completely determined by the properties:*

(i) is $k-$linear.

(ii) for $k = 0$ and $Z = \sum_i n_i P_i$, $\hat{\deg}_Z = \sum_i n_i \log N_{P_i}$.

(iii) for $k \geq 1$ and $s_k \neq 0$ a section of $\mathcal{L}_k$ which meets Z properly we have

$$\hat{\deg}_Z(\hat{c}_1(\mathcal{L}_1)...\hat{c}_1(\mathcal{L}_k)|Z) = \hat{\deg}_Z(\hat{c}_1(\mathcal{L}_1)...\hat{c}_1(\mathcal{L}_{k-1})|Z.\operatorname{div}(s_k)) - \int_{X(\mathbb{C})} \delta_{Z(\mathbb{C})} \log \|s_k\| c_1(\mathcal{L}_1)...c_1(\mathcal{L}_{k-1}).$$

Proof. The $*$ product of currents satisfies $g_Z * g_W = g_Z \wedge \delta_W + w_Z \wedge g_W$. This takes care of the recursion formula in property iii). The multilinearity is share by both, the geometric intersection and the star product of currents (definition 1.5.40). The starting point (property ii)) is a consequence of our definition of schematic intersection (definition 1.2.1). □

Proposition 1.5.53. *Suppose that $\mathcal{L}$ is a hermitian line bundle on X and $f \in K(X)$ is a rational function on X then:*

$$\hat{\deg}(\hat{c}_1(\mathcal{O}(f)).\hat{c}_1^n(\mathcal{L})) = 0.$$

Proof. The curvature of the trivial bundle $c_1(\mathcal{O}(f)) = 0$. Using this result and the symmetry of the arithmetic degree we can reduce to the case of dimension 1, which is nothing else but the product formula (see for example the treatment in [40]). □

Proposition 1.5.54. *Let Z be the algebraic cycle defined by the polynomial equation $g_v = 0$ contained in a finite fibre X_v of an arithmetic variety X_v, such that $X_{\mathbb{C}} = \mathbb{P}^n_{\mathbb{C}}$ and $\mathcal{L} = \mathcal{O}(1)$. Then*

$$\hat{\deg}_Z(\hat{c}_1^{n-1}(\mathcal{L})|Z) = \deg(g_v)\log N(v).$$

Proof. First of all we realize that because Z is contained in a finite fibre $\int_{X(\mathbb{C})} \delta_{Z(\mathbb{C})} \log \|s\| c_1^{k-1}(\mathcal{L}) = 0$, for any choice of the section s of $\mathcal{L} = \mathcal{O}(1)$. On the other hand the sections of $\mathcal{O}(1)$ represent linear conditions on the coordinates. The proposition is then a consequence of Bertini's theorem. □

Proposition 1.5.55. *Suppose that $\mathcal{L}$ is a hermitian line bundle on X and $f \in K(X)$ is a rational function on X then,*

$$\hat{\deg}(\hat{c}_1^n(\mathcal{L})|\operatorname{div}(f)) = \sum_\sigma \int_{X_\sigma} \log|f| d\mu_\sigma.$$

Proof. We have that $\hat{\deg}(\hat{c}_1(\mathcal{O}(f)).\hat{c}_1^n(\mathcal{L})) = 0$ and also that

$$\hat{\deg}(\hat{c}_1(\mathcal{O}(f)).\hat{c}_1^n(\mathcal{L})) = \hat{\deg}(\hat{c}_1(\mathcal{L}_1)^n)|\operatorname{div}(f))$$
$$- \sum_\sigma \int_{X_\sigma(\mathbb{C})} \log\|f\|(c_1(\mathcal{L}_1)...c_1(\mathcal{L}_{k-1}).$$

which gives the formula we wanted to prove. □

Definition 1.5.56. *Let Y be closed irreducible subscheme of X and $\mathcal{L}$ an ample line bundle on X. The real number*

$$h_{\mathcal{L}}(Y) = \frac{\hat{\deg}_Y(\hat{c}_1(\mathcal{L}|Y)^{\dim Y+1})}{\deg(c_1(\mathcal{L}_K|Y)^{\dim Y})}$$

is called the height of Y relative to $\mathcal{L}$.

Appendix II: Local Algebra

1.5.7 Dimension Theory

The main reference for this part is the book [25].

Definition 1.5.57. *The krull dimension of a ring A is the maximum number n, such that there is a $n-$links chain $p_0 \subset p_1 ... \subset p_n$ of primes ideals of A.*

Definition 1.5.58. *The height $h(p)$ of a prime ideal p is the maximum n, such that there is chain of prime ideals $p_0 \subset p_1 ... \subset p_n = p$. For I any ideal we take the infimum of the heights of all prime divisors of I.*

Definition 1.5.59. *An ideal q of a ring A is called primary if any zero divisor of A/q is nilpotent.*

Theorem 1.5.60. *(Generalized Krull principal ideal theorem)Let A be a noetherian ring, $I \neq A$ an ideal generated by m elements. For a any minimal prime divisor p of I, $h(p) \leq m$.*

Remark 1.5.61. *The radical of a primary ideal q is always prime p. We then said that q is $p-$primary. We should think of the concept of primary as* *an analogy to the prime powers. However powers of a prime ideal not need to be primary.*

Definition 1.5.62. *For I a finitely generated ideal in A, $\mu(I)$ denotes the minimum number of elements of A that generates I.*

Proposition 1.5.63. *Let (A, m) be a local noetherian ring and q an $m-$pri mary ideal, then $\mu(q) \geq \dim(A)$ and there exist q, $m-$primary ideal, with $\mu(q) = \dim(A)$.*

Definition 1.5.64. *A set $\{a_1, ..., a_n\}$ of elements of a $n-$dimensional noethe rian local ring (A, m) is called a system of parameters of A if it generates an $m-$primary ideal*

Proposition 1.5.65. *Let (A, m) be a local ring, $a_1, ..., a_m$ a system of elements of m. Then*

(i) $\dim(A) \geq \dim(A/(a_1, ..., a_m) \geq \dim(A) - m$.

(ii) $\dim(A/(a_1, ..., a_m) = \dim(A) - m$ *if and only if* $(a_1, .., a_m)$ *can be extended to a system of parameters of* A.

Definition 1.5.66. *The dimension of a module M over a ring A is the Krull dimension of $R/Ann(M)$.*

1.5.8 Primary Decomposition and Associated Primes

Let M be a module over a ring A.

Definition 1.5.67. *$p \in Spec(A)$ is said to be associated to M if there is an $m \in M$ such that $p = Ann(m)$. The set of associated primes to M will be denoted by $Ass(M)$.*

Definition 1.5.68. *A submodule $P \subset M$ is called primary if $Ass(M/P)$ consist of a single element. If p is this prime ideal then P is said to me $p-$primary.*

Proposition 1.5.69. *Any submodule $U \subset M$ of a finitely generated module M over a noetherian ring A, admits a reduce decomposition: $U = P_1 \cap ... \cap P_s$ where we have*

(i) P_i is p_i primary.

(ii) For $i \neq j$, $p_i \neq p_j$.

(iii) $\cap_{i \neq j} P_i \subset P_j$.

The P_i are called the primary components of U.

1.5.9 Blow-up

Let $I \subset R$ be an ideal. Consider the Riesz Ring $R_I = R \oplus It \oplus It^2 \oplus ... \subset R[t]$ (taking polynomials with coefficients in I). Now assume that $I = Rx_0 +$

$Rx_1 + ... + Rx_n$ and put

$$R[x_0, ..., x_n] \to R_I \to 0$$

with the maps $x_i \mapsto x_i t$ from $R[x_0, ..., x_n]$ to R_I.

Definition 1.5.70. *The blow-up $Bl(I, R)$ of R along I is defined as the scheme $Proj(R_I) \hookrightarrow \mathbb{P}^n_R$.*

Remark 1.5.71. *$Bl(I, R)$ is in fact defined by the ideal J satisfying*

$$J \to R[x_0, ..., x_n] \to R_I \to 0.$$

Theorem 1.5.72. *(Universal Property) The ideal generated by I in $Bl(I, R)$ is Cartier Divisor and $Bl(I, R)$ is universal for that property.*

1.5.10 Length of a Module

Definition 1.5.73. *A filtration of length n of a $A-$module M is a sequence*

$$0 = M_0 \subset M_1 \subset ... \subset M_n = M$$

of $A-$modules M_i.

Proposition 1.5.74. *Any two filtrations for M with successive simple quotients M_i/M_{i+1} has the same length, that number* length(M) *will be call the length of M.*

Proof. See the notes [40]. □

1.5.11 Tor functor

Let M be an $A-$module. The left derived functors of $M \otimes_A .$ are called $\mathrm{Tor}_i(M,.)$. The property $M \otimes_A N \cong N \otimes_A M$ carries over to Tor_i and we can think of $\mathrm{Tor}_i(.,.)$ as a functor in two variables.

Example 1.5.75. *If we take two ideals I and J inside $A[x_1, .., x_n]$ with $\dim(I) = k$ and $\dim(J) = n - k$, both I and J annihilate $\mathrm{Tor}_i(R/I, R/J)$ and therefore this last $A-$module has finite length for every i.*

1.5.12 Cohen-Macaulay Rings

Let A be a ring and M an $A-$module. An element $a \in A$ is said to be regular if $ax \neq 0$ for all $0 \neq x \in M$. A sequence $a_1, a_2, ..., a_n$ of elements of A is an $M-$regular sequence if the following two conditions hold:

(i) a_1 is $M-$regular, a_2 is M/a_1M-regular,..., a_n is $(M/\sum_1^{n-1} a_iM)$ -regular,

(ii) $(M/\sum_1^n a_iM) \neq 0$.

Proposition 1.5.76. *Let (A, m) be a local noetherian ring, M an $A-$module and $a_1, a_2, ..., a_r$ a $m-$regular sequence, then*

$$\dim(M/\langle a_1, a_2, ..., a_r\rangle) = \dim(M) - r.$$

Proposition 1.5.77. *Let A be a ring, M an $A-$module, and $a_1, a_2, ..., a_n \in A$ a regular sequence. If we write $I = (a_1, a_2, ..., a_n)A$, then*

$$(M/IM)[x] = gr_I M.$$

If M is finitely generated over a noetherian ring A and $I \subset A$ is an ideal with $IM \neq M$, then any M-regular sequence in I can be extended to a maximal $M-$regular sequence of elements also in I.

Proposition 1.5.78. *Any two maximal $M-$regular sequences in A have the same number of elements.*

Definition 1.5.79. *The maximal number of an $M-$regular sequence in I is what is called the $I-$depth of M. When A is local with maximal ideal m and $I = m$ then we simply call it the depth of M and denote it by $d(M)$.*

Remark 1.5.80. *The depth $d(M)$ of a module M over a regular ring A, can be also seen as the smallest i such that $H^i_m(M) \neq 0$.*

Proposition 1.5.81. *Let (A, m) be a noetherian local ring, $M \neq 0$ a finitely generated A module, then*

$$d(M) \leq \min_{p \in Ass(M)} \dim(R/p) \leq \dim(M).$$

Definition 1.5.82. *Let M be a finitely generated module over a noetherian ring A. If A is local we call M a Cohen-Macaulay (C.M.) module if $M = (0)$*

or if $d(M) = \dim(M)$. *In the general case* M *is defined to be C.M if* M_m *is C.M. as* A_m *module for every* $m \in SpecMax(A)$.

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