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GRAPH REALIZABILITY AND CONNECTIVITY OF MATROIDS

by

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CHAPTER 1 INTRODUCTION

The axiomatic study of independence structures was originated by Hassler Whitney in his distinguished paper, "On the Abstract Properties of Linear Dependence", presented to the American Mathematical Society in 1935 [Wh 5]. He axiomatized a few properties of linear independence in vector spaces and named the axiom system a matroid. Since then, a large number of investigations have been made on matroid structure and properties and on the application of matroids to other fields of mathematics, including combinatorial theory and graph theory. For a detailed discussion on this matter, one may refer to the work of Crapo and Rota [Cr 1], Mirsky [Mi 2], and Tutte [Tu 6]. Practical applications of matroid structures are also considered by Bruno [Br 1], Iri [Ir 2], and Narayanan [Na 1] in the field of electrical network theory.

In this dissertation we regard a matroid as a generalization of a graph and generalize known results of graphs to matroid theorems. The principal emphasis is put on connectivity and graph-realizability of matroids. Below, we summarize the contents of each chapter of this dissertation.

Chapter 2 This chapter provides basic definitions and theorems in graph and matroid theories, which serve to make this dissertation self-contained.

Whitney's definition of graph connectivity is commonly accepted by most researchers. However, we employ the connectivity definition of Tutte, which can be immediately generalized to matroid connectivity, to develop a consistent theory of graphs and matroids.

In defining matroids we use the circuit axioms, since our main interest is the relation of graphs and matroids.

Chapter 3 This chapter consists of five independent sections on matroid connectivity. The concept of maximally distant bases was introduced by Kishi and Kajitani [Ki 1, 2] in graph theory and generalized to matroids by Bruno and Weinberg [Br 3]. We evaluate the maximum value of the connectivity function of a matroid in terms of maximally distant bases. Although there is no efficient algorithm for determining the connectivity of matroids, for a matroid of finite-connectivity the connectivity is always less than or equal to the maximum value of a connectivity function. Thus we obtain a non-trivial upper bound on the connectivity of matroids.

Tutte found the graphs of infinite connectivity [Tu 9]. In Section 3.2 we show that the matroids of infinite connectivity are binomial matroids. Since graphs are a subclass of matroids, Tutte's result may be deduced from our theorem.

Binomial matroids have a simple structure and are often given as examples and counterexamples of matroid theorems. We find the connectivity of binomial matroids and show, as a corollary, that there exists a matroid with a prescribed connectivity.

Tutte's theorem [Tu 8] on essential cells plays a crucial role in the graph-realizability of matroids. When we reduce a 3-connected matroid to an irreducible matroid $\underline{M}_n$, if the cardinality of the cell set of $\underline{M}_n$ is not less than six and all cells of $\underline{M}_n$ are essential, the $\underline{M}_n$ is a wheel or a whirl matroid. Since, as we show in this chapter, any 3-connected matroid can be reduced to a wheel, a whirl, or a matroid which has six cells and is neither a wheel nor a whirl, it is important to clarify the structure of irreducible matroids and obtain the irreducible matroids which have six cells and are neither wheel nor whirl matroids. We also list all the 3-connected matroids which have no more than six cells. These matroids are in some aspect related to binomial matroids.

In the last section of this chapter we prove a theorem on nonplanar matroids: If a matroid $\underline{M}$ is 4-connected and has at least 6 cells, then $\underline{M}$ is nonplanar. Using Whitney connectivity this theorem would be stated that if the connectivity of a graph G is not less than six, then G is nonplanar [Ha 1]. A graph of Tutte connectivity 4 which has the least number of vertices is identified as a bipartite graph $K_{4,4}$.

<u>Chapter 4</u> As shown by MacLane [Mc 1], the study of graph structures may be simplified, in particular for large graphs, by considering 3-connected subgraphs of a given graph. We generalize MacLane's theorem to matroids and obtain interesting new results.

In Section 4.2 we clarify the structure of matroids of connectivity two.

In the next section we introduce the concept of series-reduced matroids and state a relationship of 2-separators defined in the previous section and 3-connected minors of a matroid.

In the last two sections we define a decomposition of matroids called C-decomposition, which is a direct generalization of MacLane's decomposition of graphs. The non-decomposable matroids by C-decomposition are called minimal blocks, and the series-reduced matroids of the minimal blocks are termed atoms. MacLane's theorem on graphs is extended to matroids, and we show that every atom is a maximal 3-connected minor and every maximal 3-connected minor is isomorphic to an atom. The nullity of a matroid is also calculated from the nullities of atoms.

The dual concept of the operation of series-reduction is parallel-reduction. Using this operation we prove important theorems on matroid structures, namely, the binary, regular, and graphic (cographic) characteristics of the original matroid are completely determined by the corresponding characteristics of atoms.

<u>Chapter 5</u> C-decomposition introduced in Chapter 4 yields two minors at each step of the decomposition process. However, the number of minors produced at a decomposition step may be maximized in defining split decomposition. Two kinds of split decompositions are introduced in this chapter. These decompositions are called P- and S-decompositions, and they are dual to each other. C-decomposition is

a special case of these decompositions.

An application of P- and S-decompositions to network theory is also included in this chapter. We define a planar network, which is a slightly different concept from that of planar graphs. In practice, for instance, in printed circuits, we often require a planar graph to remain planar after adding edges representing external voltage and current sources. A set of necessary and sufficient conditions for planar networks is stated in terms of atoms of P-decomposition of the polygon matroid.

Chapter 6 The graph-realizability problem of abstract matroids has been considered by Tutte [Tu 3], Welsh [We 6], and Fournier [Fo 1, 2]; the first two authors generalized graph theorems to matroids. A special case of the problem occurs when matroids are binary; this gives rise to the problem of realizability of matrices, and there exist numerous articles on this topic.

In this chapter we obtain a new result on graph-realizability of abstract matroids by generalizing the theorem of Bruno, Steiglitz, and Weinberg [Br 2]. We first decompose a matroid into atoms by methods discussed in the previous chapter. Each atom, if it contains non-essential cells, is reduced to an irreducible matroid by reduction and contraction operations. In defining the admissible inverse operations, we state a condition for a matroid to be graph-realizable.

We present algorithms for determining whether the irreducible matroid is a wheel and whether the inverse operations are admissible.

Two examples are included to illustrate the algorithms.

Chapter 7 Since a matroid is a generalization of a graph, many graph theorems may be modified to matroid theorems. One such theorem, obtained by Welsh [We 5], is that a binary matroid is bipartite if and only if its dual is Euler. In Section 7.1 we prove that every non-trivial connected binary matroid has a curcuit of even cardinality. From this result one can always find a non-trivial bipartite minor, which contains at least one circuit. The maximal bipartite contractions of a given matroid may be constructed from the even sets which are uniquely obtained from the bases of the matroid. In Section 7.2 we show that the even sets also determine the maximal bipartite reductions of a binary matroid. By Welsh's theorem, we can state the dual theorems on maximal bipartite minors in terms of Euler minors, which are explained in Section 7.3.

Chapter 8 In this chapter we present a new definition of matroid connectivity and compare it with that of Tutte. Whitney's definition is stated in algebraic terms and generalized to matroids by showing that a generalized matroid connectivity of the polygon matroid coincides with Whitney graph connectivity if a graph is connected. An immediate consequence of this generalization is that Tutte connectivity of a matroid can not exceed Whitney connectivity. A simple sufficient condition for a matroid and its dual having the same connectivity is established, this condition being stated in terms of circuits and cocircuits of the matroid.

CHAPTER 2 PRELINIMARIES

2.1 FUNDAMENTAL CONCEPTS IN GRAPH THEORY

Although the reader may find a standard language of graph theory in the books by Harary [Ha 2] and Ore [Or 1], in this dissertation we will employ somewhat modified graph terminology for our convenience.

A graph G is defined as a pair of finite sets V and E together with an incidence relation by which each member of E is associated with two unordered members of V; these two members are not necessarily distinct. If G is a graph defined by V and E, we write $G=(V, E)$, $V=V(G)$, and $E=E(G)$, and the members of V and E are referred to as the vertices and edges of G. If edge e is associated with vertices v_1 and v_2, these vertices are called the ends of e, and we denote $e=(v_1, v_2)=(v_2, v_1)$. For a given pair of vertices v_1 and v_2, if there exists an edge e such that $e=(v_1, v_2)$, then v_1 and v_2 are said to be adjacent, and e is incident to v_1 and v_2. An edge which has the same ends is a loop, and a graph is a loop-graph if it consists of one vertex and one loop. The valence $\rho(v)$ of a vertex v is the number of edges incident to v in G, where a loop is counted twice. If $V(G)= \phi =E(G)$, then $G=(V, E)$ is called a null-graph.

Let $v_1, v_2, \ldots, v_n$, where $n \geqslant 2$, be a sequence of distinct vertices of G such that $e_i=(v_i, v_{i+1})$ is an edge for $1 \leqslant i \leqslant n-1$. Then a subset $\{e_1, e_2, \ldots, e_{n-1}\}$ of E is a path of G, and v_1, v_n are its ends. The vertices of a path which are not the ends are called the internal vertices of the path. A direct path of G is a path of which all the internal vertices have valence two in G. A direct path P is called <u>maximal</u> if there is no direct path which properly contains P. If the ends of a path are adjacent, i.e., $v_n = v_1$, then $\{e_1, e_2, \ldots, e_{n-1}\}$ is a <u>polygon</u> of G. A graph is a <u>polygon graph</u> if all the vertices have valence two. Thus the edge set of a polygon graph contains precisely one polygon.

Distinct edges are said to be in <u>parallel</u> if none of them are loops and they have the same ends in G, and are in <u>series</u> if they form a direct path in G. A <u>branch graph</u> is a graph which consists of at least three direct paths with the same ends. A <u>θ-graph</u> is a branch graph which consists of three parallel edges.

Graph G is <u>connected</u> if there is a path for each pair of distinct vertices of G which are the ends of the path; otherwise G is <u>not connected</u>.

Let G=(V, E) be connected. A non-null subset S of E is called a <u>tree</u> if S contains no polygons of G, and a maximal tree is a <u>spanning tree</u> of G. A <u>spanning cotree</u> of G is a non-null subset S of E such that E-S is a spanning tree of G. A non-null subset of a spanning cotree of G is called a <u>cotree</u> of G. If G is not connected, then "tree" and "cotree" in the above sentences should be replaced by "<u>forest</u>" and "<u>coforest</u>", respectively.

Let G=(V, E) be a graph. G'=(V', E') is called a <u>subgraph</u> of G if $V' \subsetneq V$, $E' \subsetneq E$, and each edge of G' has the same ends as in G. If G' is a subgraph of G, we also say that G is a <u>supergraph</u> of G'. Two subgraphs of G are <u>edge-disjoint</u> if the edge sets have no common elements.

If S is a subset of E(G), G·S is the subgraph of G formed by the edge set S and the ends of the members of S in G. G·S is called the reduction of G to S. By definition, if $S=\phi$, then $V(G \cdot S)=\phi$. Let S be a non-null proper subset of E(G). $\eta(G;S,\bar{S})$ denotes the number of common vertices of G·S and $G \cdot \bar{S}$. Tutte [Tu 4, 7] defines the connectivity $\lambda(G)$ of a graph G as the least integer k, such that $\eta(G;S,\bar{S})=k$ and $\min(|S|, |\bar{S}|) \geq k$, where η is evaluated over all non-null subsets of E. If there is no integer which satisfies the above condition, then $\lambda(G) = \infty$. The graphs with infinite connectivity are shown in Fig. 3.2 for $|E| \leq 6$. We say that G is n-connected if $1 \leq n \leq \lambda(G)$. It is obvious that G is 1-connected if and only if it is connected. A connected graph is separable if it is not 2-connected. If G is not connected, G can be partitioned into its maximally connected subgraphs, called the connected components or simply components of G. Similarly, a separable graph may be partitioned into maximal 2-connected subgraphs called nonseparable components.

Even though we shall use the above definition of graph connectivity throughout the dissertation, we note a more standard definition of graph connectivity due to Whitney [Wh 1]. According to his definition, a graph G is n-connected if G contains at least n+1 vertices and the deletion of any n-1 or fewer vertices and their incident edges results in a connected graph, and the connectivity of G is n if G is n-connected, but not (n+1)-connected. Generally the connectivity of a graph in accordance with Whitney's definition is greater than that of

Tutte connectivity. For instance, 3-connected graphs in Tutte's sense cannot have parallel edges, whereas Whitney's definition permits parallel edges.

Another definition of graph connectivity is that of MacLane [Ma 1]. Let G be a graph, and G' be the graph obtained from G by replacing all its maximal direct paths by single edges. Then the connectivity of G in MacLane's definition is equal to the Whitney connectivity of G'. Therefore, MacLane's connectivity of a graph is never less than that of Tutte.

Let $G=(V, E)$ be a graph and $S \subsetneq E$. Let H_i, $1 \leq i \leq n$, be the components of the graph which results from the deletion of edges S in G. Then we define graph H by the vertex set $V(H)=\{H_1, H_2 \ldots H_n\}$ and edge set $E(H)=S$, and the ends of $e \in E(H)$ are the components H_i, H_j which contain the ends of e in G. The <u>contraction</u> of G to S, designated by $G \times S$, is defined by $G \times S = H \cdot S$. A <u>cut-set</u> of a graph G is a subset S of $E(G)$ such that $G \times S$ consists of two vertices and all its edges are in parallel between these two vertices. Thus a cut-set can also be defined as a minimal set of edges whose deletion from G increases the number of components by one. If the edges of a cut-set have a common end in G, then we call it a <u>star cut-set.</u>

<u>Example 2.1</u> Let $G = (V, E)$ be the graph in Fig. 2.1. Let $S = \{e_1, e_2, e_6, e_7, e_{10}, e_{12}\}$ and $T = \{e_6, e_{10}, e_{12}\}$. $G \times S$ and $G \times T$ are the graphs in Fig. 2.2 and Fig. 2.3, respectively. T is a star cut-set of G, but S is not a cut-set.

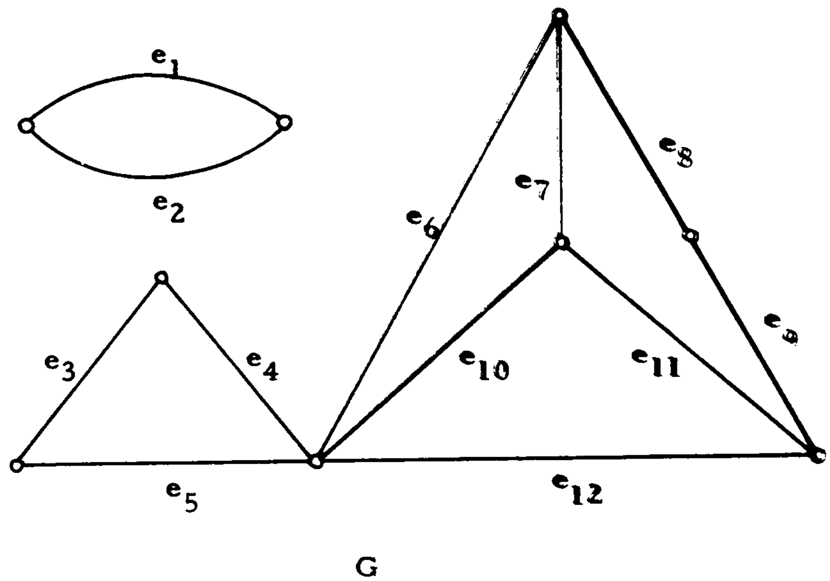


Figure 2. 1 Graph of Example 2. 1

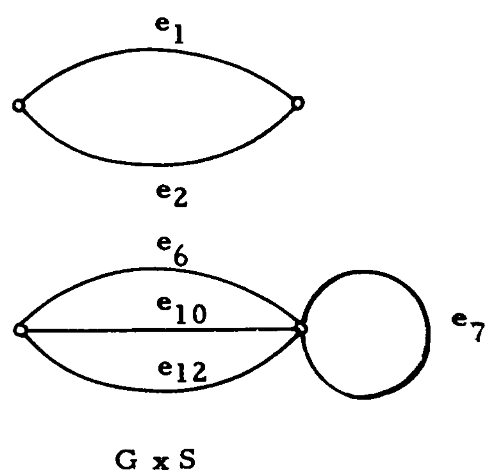


Figure 2. 2 Graph of Example 2. 1

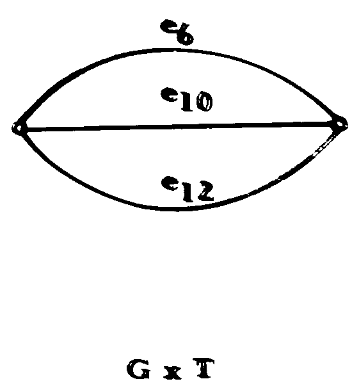


Figure 2. 3 Graph of Example 2. 1

Two graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ are said to be isomorphic if there is a one-to-one correspondence between V_1 and V_2 which preserves the incidence relations. G_1 and G_2 are homeomorphic if they can be made isomorphic with the other through a sequence of replacements of an edge by a series connection of edges. Two graphs are called 2-isomorphic if they can be made isomorphic under repeated applications of either or both of the following operations: (1) Separation of a separable component, and (2) When the graph consists of two edge-disjoint subgraphs having precisely two vertices in common, the interchange of the vertices of one of the subgraphs.

The following two theorems provide a characterization of 2-connected and 3-connected graphs in terms of polygons or cut-sets.

Theorem 2.1 [Wh 4] Two graphs are 2-isomorphic if and only if there is a one-to-one correspondence between the edges of the two graphs so that polygons (cut-sets) correspond to polygons (cut-sets). For 3-connected graphs, the word "2-isomorphic" can be replaced by "isomorphic".

Theorem 2.2 [Wh 1] If a graph G is 3-connected, then G is uniquely determined, within isomorphism, by its polygons or cut-sets.

A graph G* is said to be a dual of a graph G if there is a one-to-one correspondence between the edges of G and the edges of G*, such that a set of edges in G is a polygon of G if and only if its corresponding set of edges in G* is a cut-set of G*. For a given graph G a dual of G, if one exists, is not always unique.

<u>Example 2.2</u> Consider the graph $G=(V,E)$ shown in Fig. 2.4. $G_1{*}$ and $G_2{*}$ are dual graphs of G which are not isomorphic (Fig. 2.5).

A graph G is said to be <u>planar</u> if it can be drawn or embedded in the plane without crossing edges. A <u>plane</u> graph $\mathcal{G}(G)$ is an embedded graph of G in the plane. A plane graph of a connected graph divides the plane into a number of areas. These areas are called <u>regions</u>, and the unbounded area is the <u>infinite region</u>. The boundaries of the regions are <u>meshes</u> and the <u>outer mesh</u> corresponds to the infinite region. A mesh of a dual graph is called a <u>comesh</u> of G.

The following properties of planar graphs, basic in graph theory, are stated without proof:

<u>Theorem 2.3</u>

(a) A graph is planar if and only if it has a dual [Wh 2].

(b) If $G*$ is a dual of a connected graph G, then G is a dual of $G*$ [Wh 2].

(c) A connected planar graph can be embedded in the plane G_1, that any given region is the infinite region [Wh 2].

(d) If $G_1{*}$ is a dual of G, then $G_2{*}$ is a dual of G if and only if $G_1{*}$ and $G_2{*}$ are 2-isomorphic [Wh 3].

(e) A 3-connected planar graph has a unique dual [Wh 1].

(f) A 3-connected planar graph has a unique set of meshes [Ma 1].

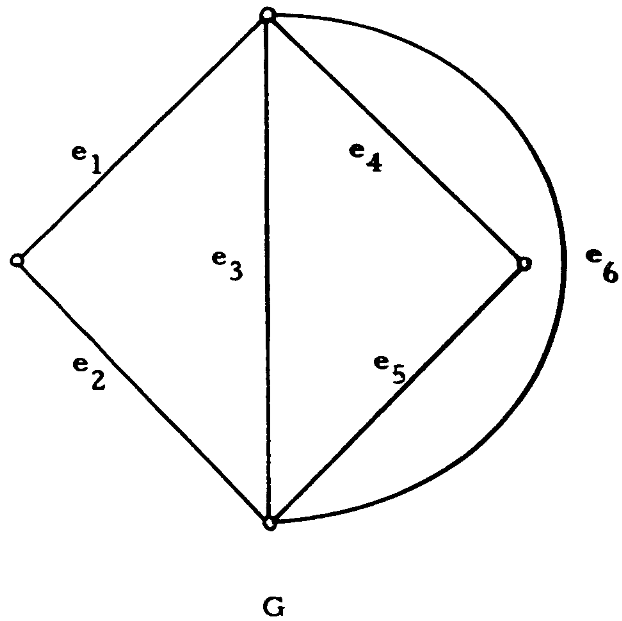


Figure 2.4 Graph of Example 2.2

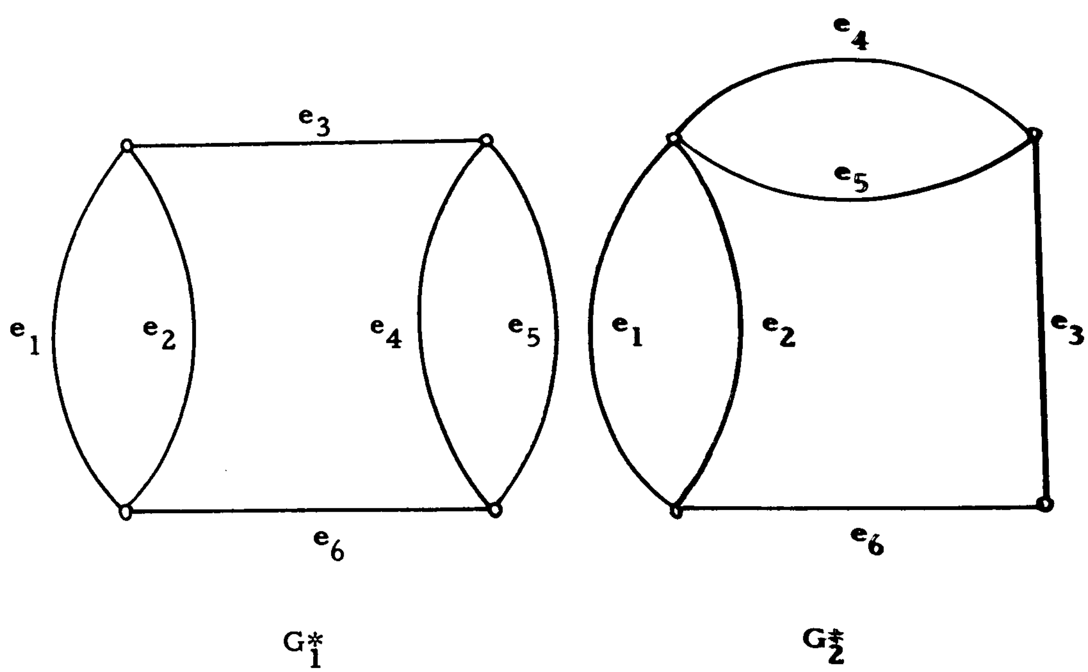


Figure 2.5 Graphs of Example 2.2

2.2 MATROIDS

A matroid may be defined as an axiomatic system which abstracts basic properties of a graph.

Let $\underline{C}$ be a class of non-null subsets of a finite set E. Then a matroid $\underline{M} = (\underline{C}, E)$ is defined on the set E if the following two axioms are satisfied:

Axiom I No member of $\underline{C}$ contains a member of $\underline{C}$ as a proper subset.

Axiom II If $C_1, C_2 \in \underline{C}$, $e \in C_1 \cap C_2$, and $e' \in C_1 - C_2$, then there exists a member $C_3 \in \underline{C}$ such that $e' \in C_3 \subseteq C_1 \cup C_2 - \{e\}$.

The members of E and $\underline{C}$ are referred to as cells and circuits of $\underline{M}$, respectively. Three other equivalent characterizations of a matroid are given by Whitney in terms of "independence sets", "bases", and "rank" [Wh 5].

A subset S of E is independent if S contains no circuit of $\underline{M}$; otherwise, S is dependent. A maximal independent subset of E is called a base of $\underline{M}$. If G is a graph and T_1 and T_2 are spanning trees of G, then for any $e_1 \in T_1$ there exists $e_2 \in T_2$ such that $(T_1 - \{e_1\}) \cup \{e_2\}$ is a spanning tree of G. This also holds for bases of matroids.

Theorem 2.4 Let B_1 and B_2 be bases of $\underline{M}$. If $e_1 \in B_1$, then there exists $e_2 \in B_2$ such that $(B_1 - \{e_1\}) \cup \{e_2\}$ is a base of $\underline{M}$.

This property of bases is called the exchange property of bases of a matroid. From this theorem it follows that:

Theorem 2.5 If B_1 and B_2 are bases of M, then $|B_1| = |B_2|$.

Proof. Suppose the theorem does not hold for some pairs of bases. Choose a pair of such bases B_1 and B_2 so that $|B_1 - B_2|$ is minimum. Let $B_1 - B_2 = \{e_1, e_2, \ldots, e_k\}$. By hypothesis, $k \neq 0$. Since B_1 and B_2 are bases, by Theorem 1.4 there exists $e_1' \in B_2$ such that $B_1' = (B_1 - \{e_1\}) \cup \{e_1'\}$ is a base of M. If $e_1' \in B_1$, then $B_1 - \{e_1\}$ is a base, which is a contradiction, since a base is a maximally independent set. Therefore $e_1' \notin B_1$. $|B_1'| = |B_1 - \{e_1\}| + 1 = |B_1| \neq |B_2|$ and $B_1' - B_2 = \{e_2, \ldots, e_k\}$. This is contrary to $|B_1 - B_2|$ being minimum. Thus the theorem follows. ∎

The cardinality of a base is an invariant and is called the rank of M denoted by r(M). The nullity of M is defined by $\mu(M) = |E| - r(M)$.

Two subsets S and T of E are said to be orthogonal if $|S \cap T| \neq 1$. Let C^* be the class of non-null minimal members of the power set of E which are orthogonal to every member of C. Then $M^* = (C^*, E)$ satisfies the matroid axioms, and is called the dual matroid of M.

Theorem 2.6 $(M^*)^* = M$.

The circuits and bases of M^* are called cocircuits and cobases of M. Cobases are related to bases of M as spanning cotrees are related to spanning trees in graphs.

Theorem 2.7 A subset B of E is a base of $\underline{M}$ if and only if E - B is a base of $\underline{M}^*$.

Two matroids $\underline{M}_1 = (\underline{C}_1, E_1)$ and $\underline{M}_2 = (\underline{C}_2, E_2)$ are called isomorphic if there exists a one-to-one mapping f of E_1 onto E_2 such that C is a circuit of $\underline{M}_1$ if and only if f(C) is a circuit of $\underline{M}_2$. If $\underline{M}_1$ and $\underline{M}_2$ are isomorphic, we write $\underline{M}_1 \cong \underline{M}_2$.

We shall now define wheel, whirl, and binomial matroids, which are essential for our discussion in Chapter 6.

The wheel graph of order n, denoted by W_n, is a graph with n+1 vertices and 2n edges. The outer polygon forms the rim and consists of n edges and n vertices connected in a closed path. Each of these vertices is joined by a single edge (spoke) to the hub vertex at the center. Let $\underline{C}_n$ consist of all the polygons of the wheel graph of order n. Then $\underline{W}_n = (\underline{C}_n, E)$ is called a wheel matroid where E is the edge set of W_n.

Suppose $\underline{C}_{rn}$ is obtained from the polygons of W_n by deleting the polygon of the rim and by adding as polygons all the sets formed by the union of the rim and a single spoke. $\underline{W}_{rn} = (\underline{C}_{rn}, E)$ satisfies the matroid axioms and is called a whirl matroid [Tu 8].

Let E be a set with n elements. The class of circuits ${}_n\underline{C}_k$ is the class of all subsets of E with k elements. Then ${}_n\underline{M}_k = ({}_n\underline{C}_k, E)$

has been called a binomial matroid by Weinberg [We 3] because of its relation to the binomial coefficients.

2.3 CONTRACTIONS AND REDUCTIONS

Let $\underline{M} = (\underline{C}, E)$ be a matroid and $S \subseteq E$. Define $\underline{C} \times S = \{C \mid C \in \underline{C} \text{ and } C \subseteq S\}$. The class $\underline{C} \times S$ satisfies the matroid axioms, and the matroid $\underline{M} \times S = (\underline{C} \times S, S)$ is called the contraction of $\underline{M}$ to S. Let $\underline{C} \cdot S$ be the class of non-null minimal intersections of the members of $\underline{C}$ with S. Then $\underline{M} \cdot S = (\underline{C} \cdot S, S)$ is a matroid called the reduction of $\underline{M}$ to S.

Tutte [Tu 3] has proved the following identities for contractions and reductions of matroids.

Theorem 2.8 Let $T \subseteq S \subseteq E$.

(a) $(M \cdot S)^* = M^* \times S$

(b) $(M \times S)^* = M^* \cdot S$

(c) $(M \times S) \times T = M \times T$

(d) $(M \cdot S) \cdot T = M \cdot T$

(e) $(M \cdot S) \times T = (M \times (E - (S - T))) \cdot T$

(f) $(M \times S) \cdot T = (M \cdot (E - (S - T))) \times T$

A matroid of the form $(\underline{M} \cdot S) \times T$ is called a minor of $\underline{M}$. By Theorem 2.8 (e), $(\underline{M} \times S) \cdot T$ is also a minor of $\underline{M}$. The next theorem results from Theorem 2.8.

Theorem 2.9

(a) Every minor of a minor of $\underline{M}$ is a minor of $\underline{M}$.

(b) The minors of $\underline{M}^*$ are the duals of the minors of $\underline{M}$.

We have stated that matroids are a generalization of graphs. We now make this statement precise by showing that with every graph we may associate two matroids, each unique for its type. Let $G=(V,E)$ be a graph and $\underline{C}_P$ be the class of polygons of G. The system $\underline{P}(G)=(\underline{C}_P, E)$ satisfies the matroid axioms and is called the polygon matroid of G. The dual of $\underline{P}(G)$ is the bond matroid of G, denoted by $\underline{B}(G)$, and the circuits of $\underline{B}(G)$ are the cut-sets of G. From Theorems 2.1, 2.3(d), and 2.6, we can state the following theorem:

Theorem 2.10 Let G be a planar graph and G^* its dual graph. Then

$$\underline{B}(G)=\underline{P}^*(G)=\underline{P}(G^*)=\underline{B}^*(G^*).$$

A matroid $\underline{M}$ is called graphic or cographic if there is a graph such that $\underline{M}=\underline{B}(G)$ or $\underline{P}(G)$, respectively. If $\underline{M}$ is both graphic and cographic, then $\underline{M}$ is planar and there exists a planar graph G so that $\underline{M}=\underline{B}(G)=\underline{P}(G^*)$.

Contractions and reductions in graphs are simply related to matroid contractions and reductions.

Theorem 2.11 Let $G=(V,E)$ be a graph and let $S \subsetneq E$. Then

(a) $\underline{P}(G \cdot S)=\underline{P}(G) \times S$, $\underline{P}(G \times S)=\underline{P}(G) \cdot S$

(b) $\underline{B}(G \cdot S)=\underline{B}(G) \cdot S$, $\underline{B}(G \times S)=\underline{B}(G) \times S$

Proof.

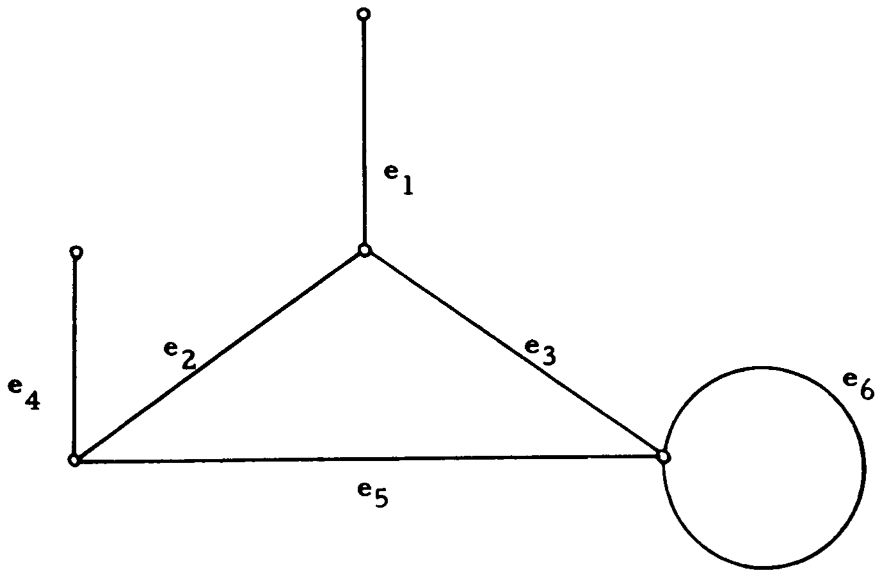
(a) The first part of (a) follows from the definition. We will prove the second part of (a). Let H_i, $1 \leq i \leq n$, be the components of the graph which is obtained from G by the deletion of the edges in S. Let P be a polygon of $G \times S$. If one edge in P is incident with two vertices, v_1 and v_2, of a component, H_i, then P is a loop of $G \times S$, and $P \cup P_i$ is a polygon of G, where P_i is a path of H_i with v_1 and v_2 as its ends. Since $(P \cup P_i) \cap S = P$ consists of a single element, P is a circuit of $\underline{P}(G) \cdot S$. If $|P| \geq 2$, each edge in P is incident with two distinct components, each of which contains no more than two distinct ends of P. Let $H_{i(1)}, H_{i(2)}, \ldots, H_{i(j)}$ be components containing two distinct ends of the edges in P. Since $H_{i(k)}$, $1 \leq k \leq j$, are connected, we can find a path P_k of $H_{i(k)}$, whose ends are those distinct ends of the edges. Clearly, $C = P \cup P_{i(1)} \cup \ldots \cup P_{i(j)}$ is a polygon of G, and $C \cap S = P$. If $P' \subset P$ for some circuit P' of $\underline{P}(G) \cdot S$, there exists a polygon C' of G such that $P' = C' \cap S$. Since the ends in G of the edges in P are also ends of the edges in P, we can deform some paths in $H_{i(k)}$, $1 \leq k \leq j$, so that $P_{i(k)} = C' \cap E(H_{i(k)})$. This deformation results in $C' \subset C$. This contradicts the fact that C' and C are polygons of G. Thus P is a circuit of $\underline{P}(G) \cdot S$.

Suppose P is a circuit of $\underline{P}(G) \cdot S$. Then there exists a polygon C of G such that $P = C \cap S$. By the definition of contractions, $C \cap S$ is an edge-disjoint union of polygons of $G \times S$. If P is one of the polygons of $G \times S$ in $C \cap S$, then we can construct, as above, a polygon of G containing P and can show P is a circuit of $\underline{P}(G) \cdot S$. By Axiom I, P contains only one polygon. Hence, P is a polygon of $G \times S$.

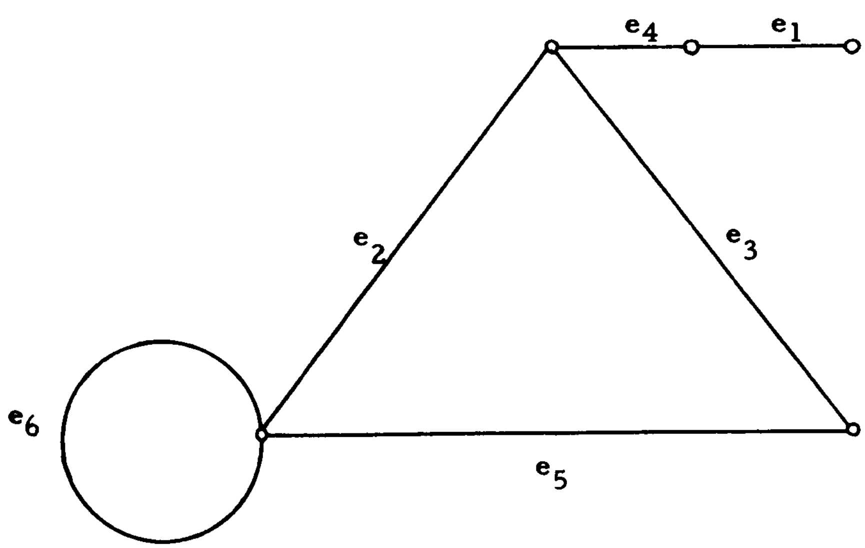
(b) By part (a) and by Theorem 2.8, $\underline{B}(G \cdot S) = [\underline{P}(G \cdot S)]^* = [\underline{P}(G) \times S]^* = \underline{P}^*(G) \cdot S = \underline{B}(G) \cdot S$, and $\underline{B}(G \times S) = [\underline{P}(G \times S)]^* = [\underline{P}(G) \cdot S]^* = \underline{P}^*(G) \times S = \underline{B}(G) \times S$. ∎

According to Theorem 2.1, if the polygon matroids of two graphs are isomorphic, then the graphs are 2-isomorphic. Thus matroid isomorphism does not coincide with graph isomorphism.

Example 2.3 Let G_1 and G_2 be the graphs in Fig. 2.6. Although $\underline{P}(G_1) = \underline{P}(G_2)$, the two graphs are not isomorphic.



G_1



G_2

Figure 2.6 Graphs of Example 2.3

$- \{e_1\}$. Then $|(C_3 - B_S) \cap S| < |(C_1 - B_S) \cap S|$. T

contradiction, and hence $e \in C_1 \subseteq B_S \cup B_{\overline{S}} \cup \{e\}$. C

2.4 CONNECTIVITY OF MATROIDS

Whitney's rank functions [Wh 5] play an important role in defining connectivity of matroids. Therefore, we summarize formulas on rank functions in the next theorem and will refer to them often.

Theorem 2.12 Let $\underline{M} = (\underline{C}, E)$ be a matroid and $S, T \subseteq E$.

(a) $r(\underline{M}) = \mu(\underline{M}^*)$, $\mu(\underline{M}) = r(\underline{M}^*)$

(b) $r(\underline{M}) + \mu(\underline{M}) = |E|$

(c) $r(\underline{M} \times S) + r(\underline{M} \cdot \overline{S}) = r(\underline{M})$, $\mu(\underline{M} \times S) + \mu(\underline{M} \cdot \overline{S}) = \mu(\underline{M})$

(d) $r(\underline{M} \times (S \cup T)) + r(\underline{M} \times (S \cap T)) \leq r(\underline{M} \times S) + r(\underline{M} \times T)$

$\mu(\underline{M} \times (S \cup T)) + \mu(\underline{M} \times (S \cap T)) \geq \mu(\underline{M} \times S) + \mu(\underline{M} \times T)$.

Proof.

(a) and (b) follow immediately from the definition.

(c) Let B_S and $B_{\overline{S}}$ be bases of $\underline{M} \times S$ and $\underline{M} \cdot S$, respectively. We show $B = B_S \cup B_{\overline{S}}$ is a base of $\underline{M}$. B is obviously independent in $\underline{M}$. We show that $B \cup \{e\}$ contains a circuit of $\underline{M}$ for every $e \in E - B$. If $e \in S$, $B \cup \{e\}$ ($\supseteq B_S \cup \{e\}$) contains a circuit of $\underline{M} \times S$. Now suppose $e \in E - S$ and $e \notin B_{\overline{S}}$. Let C_1 be a circuit of $\underline{M}$ such that $e \in C_1 \cap \overline{S} \subseteq B_{\overline{S}} \cup \{e\}$ and $|(C_1 - B_S) \cap S|$ is minimum. If $(C_1 - B_S) \cap S \neq \phi$, then there exists a circuit C_2 of $\underline{M}$ such that $e_1 \in C_2 \subseteq B_S \cup \{e_1\}$, where e_1 is a member of $(C_1 - B_S) \cap S$. By Axiom II, we can find a circuit C_3 which satisfies $e \in C_3 \subseteq C_1 \cup C_2 - \{e_1\}$. Then $|(C_3 - B_S) \cap S| < |(C_1 - B_S) \cap S|$. This is a contradiction, and hence $e \in C_1 \subseteq B_S \cup B_{\overline{S}} \cup \{e\}$. Consequently, B is a base of $\underline{M}$. The second part of (c) follows from the first part of (c) and (a).

(d) Let B' be a base of $\underline{M} \times (S \cap T)$. Choose a base B of $\underline{M} \times (S \cup T)$ so that $B' \subseteq B$. Then $B \cap S$ and $B \cap T$ are independent in $\underline{M} \times S$ and $\underline{M} \times T$, respectively. $r(\underline{M} \times (S \cup T)) + r(\underline{M}\ (S \cap T)) = |B| + |B'| = |B \cap S| + |B \cap T| \leqslant r(\underline{M} \times S) + r(\underline{M} \times T)$. The second part of (d) is obtained from the first part of (d) and (b). ∎

Let $\underline{M} = (\underline{C}, E)$ be a matroid and $S \subseteq E$. Define a function $\xi(\underline{M}; S, \bar{S})$ on the subsets of E by $\xi(\underline{M}; S, \bar{S}) = -r(\underline{M}) + r(\underline{M} \times S) + r(\underline{M} \times \bar{S}) + 1$. From Theorem 2.12 the following relations are proved:

<u>Theorem 2.13</u> Let $S \subseteq E$. Then

$$\begin{aligned} \xi(\underline{M}; S, \bar{S}) &= r(\underline{M} \times S) - r(\underline{M} \cdot S) + 1 \\ &= -|S| + r(\underline{M} \times S) + r(\underline{M}^* \times S) + 1 \\ &= \mu(\underline{M}) - \mu(\underline{M} \times S) - \mu(\underline{M} \times \bar{S}) + 1 \\ &= \mu(\underline{M} \cdot S) - \mu(\underline{M} \times S) + 1 \\ &= |S| - \mu(\underline{M} \times S) - \mu(\underline{M}^* \times S) + 1. \end{aligned}$$

Let S and $\bar{S} = E - S$ be non-null complementary subsets of E. Tutte [Tu 8] defines the <u>connectivity</u> $\lambda(\underline{M})$ of a matroid $\underline{M} = (\underline{C}, E)$ as the least integer k over all non-null proper subsets S such that $\xi(\underline{M}; S, \bar{S}) = k$ and $\min(|S|, |\bar{S}|) \geqslant k$. If there is no such integer, then $\lambda(\underline{M}) = \infty$. Matroid $\underline{M}$ is <u>n-connected</u> if $1 \leqslant n \leqslant \lambda(\underline{M})$, and $\underline{M}$ is <u>connected</u> if $\lambda(\underline{M}) \geqslant 2$: otherwise $\underline{M}$ is separable.

The connected components of a separable matroid are the maximal connected minors of $\underline{M}$.

From Theorem 2.13 the connectivity of the dual matroid is equal to that of the original matroid.

Theorem 2.14 $\lambda(\underline{M}^*) = \lambda(\underline{M})$.

Tutte [Tu 8] has proved that if a graph G is connected, then the matroid connectivity is equal to the graph connectivity. Thus we can develop a unified connectivity theory of graphs and matroids.

Theorem 2.15 [Tu 8] If a graph G is connected, then $\lambda(\underline{P}(G)) = \lambda(\underline{B}(G)) = \lambda(G)$. If a graph is not connected and contains no isolated vertices, then $\lambda(\underline{P}(G)) = \lambda(\underline{B}(G)) = 1$ and $\lambda(G) = 0$.

If a graph contains an isolated vertex, certainly the connectivity of the graph is 0. However, the connectivity of the polygon matroid can be any positive integer.

Example 2.4 Let $G = (V, E)$ be the graph shown in Fig. 2.7. Let $S = \{e_1, e_2, e_3\}$ and $\bar{S} = \{e_4, e_5\}$. Clearly $\eta(G;S,\bar{S}) = 0$ since $G \cdot S$ and $G \cdot \bar{S}$ have no common vertex. However, $\xi(\underline{P}(G);S,\bar{S}) = -r(\underline{P}(G)) + r(\underline{P}(G) \times S) + r(\underline{P}(G) \times \bar{S}) + 1 = -r(\underline{P}(G)) + r(\underline{P}(G \cdot S)) + r(\underline{P}(G \cdot \bar{S})) + 1 = -3 + 2 + 1 + 1 = 1$. Accordingly, $\lambda(G) = 0$ and $\lambda(\underline{P}(G)) = 1$.

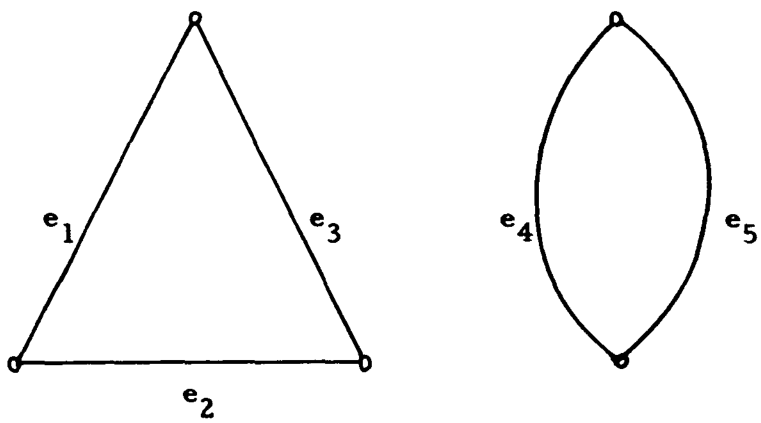


Figure 2.7 Graph G of Example 2.4

Triply-connected matroids are of particular interest to us since they will provide a characterization for graph-realizable matroids in Chapter 4. Let $\underline{M} = (\underline{C}, E)$ be 3-connected and $e \in E$. If neither $\underline{M} \times (E - \{e\})$ nor $\underline{M} \cdot (E - \{e\})$ is 3-connected, then e is called an essential cell of $\underline{M}$. The following theorem on essential cells was proved by Tutte [Tu 8].

Theorem 2.16 Each cell of a 3-connected matroid $\underline{M}$ is essential if and only if $\underline{M}$ is a wheel or a whirl matroid.

Let $\underline{M} = (\underline{C}, E)$ be a matroid. When $\underline{M}$ is connected, as Lehman [Le 1] proved, all circuits containing a fixed element of E can uniquely determine the matroid $\underline{M}$.

Theorem 2.17 Let $\underline{M} = (\underline{C}, E)$ be connected and $e \in E$. Then the set of all circuits containing e uniquely determines $\underline{M}$.

According to this theorem, the condition for isomorphism of matroids may be weakened as follows:

Theorem 2.18 Let $\underline{M}_1 = (\underline{C}_1, E_1)$ and $\underline{M}_2 = (\underline{C}_2, E_2)$ be connected and $e \in E_1$. Then $\underline{M}_1$ is isomorphic to $\underline{M}_2$ if and only if there exists a one-to-one mapping f of E_1 onto E_2, such that C is a circuit of $\underline{M}_1$ containing e if and only if f(C) is a circuit of $\underline{M}_2$ containing f(e).

2.5 BINARY MATROIDS AND GRAPH REALIZABILITY

In a graph, the intersection of any polygon (circuit) and cut-set (cocircuit) has even cardinality. However, not every matroid has this property. Consider the matroid $\underline{M} = (\underline{C}, E)$ defined by $E = \{1, 2, 3, 4\}$ and $\underline{C} = \{123, 124, 134, 234\}$. This matroid is self-dual, that is, $\underline{M}^* = \underline{M}$, and clearly the even cardinality property does not hold. If for every $C \in \underline{C}$ and $C^* \in \underline{C}^*$, $|C \cap C^*| =$ even, then a matroid $\underline{M}$ is called binary. From the symmetry of the binary condition, we have:

Theorem 2.19

(a) $\underline{M}$ is binary if and only if $\underline{M}^*$ is binary.

(b) A minor of a binary matroid is binary.

Even though general binary matroids are not graph realizable, they have many graph-theoretic properties. The following characterization of binary matroids is often useful.

Theorem 2.20 $\underline{M}$ is binary if and only if any symmetric difference of circuits is a disjoint union of circuits of $\underline{M}$.

In this theorem, by the symmetric difference of A and B we mean $A \oplus B = A \cup B - A \cap B$. This operation is associative and commutative.

Let B be a base of $\underline{M} = (\underline{C}, E)$. If $e \in E - B$, $B \cup \{e\}$

contains a unique circuit of $\underline{M}$ called the <u>fundamental circuit</u> determined by B and e. Then the third characterization of binary matroids may be stated in terms of fundamental curcuits.

<u>Theorem 2.21</u> $\underline{M}$ is binary if and only if, for any base B and circuit C of $\underline{M}$, if $e_1, e_2, \ldots, e_k$ are the members of C - B, then $C = C_1 \oplus C_2 \oplus \ldots . \oplus C_k$, where C_i is the fundamental circuit determined by B and e_i.

The concept of binary matroids was first introduced by Whitney [Wh 5] as a matroid associated with matrices whose elements are 0 or 1. The (0,1) matrices are extremely important in their application to science and engineering. Applications of binary matroids have been made in generalized networks [Br 1, Na 1], linear programming and network flow problems [Ga 1, Mi 1], and game theory [Le 1], and some research on binary matroids has been directed to graph realizability [Tu 4, We 6] and generalization of graph theorems [We 5].

A binary matroid closely related to graph realizability is the <u>Fano matroid</u> [Tu 6, Ve 1]. This matroid $\underline{M} = (\underline{C}, E)$ consists of seven cells and seven circuits: $E = \{e_1, e_2, \ldots, e_7\}$ and $\underline{C} = \{\{e_1, e_4, e_5, e_7\}, \{e_2, e_5, e_6, e_7\}, \{e_3, e_4, e_6, e_7\}, \{e_1, e_2, e_4, e_6\}, \{e_1, e_3, e_5, e_6\}, \{e_2, e_3, e_4, e_5\}, \{e_1, e_2, e_3, e_7\}\}$. The dual of a Fano matroid is called a <u>heptahedron matroid</u>, and these two matroids are associated with the following binary matrices:

Fano						
1	0	0	1	1	0	1
0	1	0	0	1	1	1
0	0	1	1	0	1	1

Heptahedron						
1	0	1	1	0	0	0
1	1	0	0	1	0	0
0	1	1	0	0	1	0
1	1	1	0	0	0	1

If a binary matroid $\underline{M}$ contains neither Fano nor heptahedron matroids as minors, $\underline{M}$ is said to be regular. These matroids are not graph realizable [Tu 6]. Therefore it is necessary, if $\underline{M}$ is to be realizable as a graph, that $\underline{M}$ not contain either of these matroids as a minor. A set of necessary and sufficient conditions for graph-realizability that contains this condition follows.

Theorem 2.22 [Tu 6] A matroid is graphic (cographic) if and only if it is regular and contains as a minor neither $\underline{P}(K_5)$, $(\underline{B}(K_5))$ nor $\underline{P}(K_{3,3})$ $(\underline{B}(K_{3,3}))$, where K_5 is the complete graph on five vertices, and $K_{3,3}$ is the complete bipartite graph on (3, 3) vertices.

Another graph realizability condition has been obtained by Welsh as a generalization of MacLane's theorem on graph planarity. A family of circuits is said to be complete if it is a basis of the vector space of circuits modulo 2. It is 2-complete if it is complete and no element of E lies in more than two of the circuits. A similar definition holds for cocircuits. Welsh states the following theorem:

Theorem 2.23 [We 6] A matroid $\underline{M}$ is graphic (cographic) if and only if it is binary and has a 2-complete family of circuits (cocircuits).

CHAPTER 3 CONNECTIVITY OF MATROIDS

3.1 MAXIMUM VALUE OF CONNECTIVITY FUNCTION

Though a procedure for determining connectivity of matroids is needed because it would have interesting applications, at present there exists no efficient algorithm for accomplishing this, except for separable matroids.

If a matroid $\underline{M} = (\underline{C}, E)$ is separable, then its connected components can be easily determined by consideration of the circuits of $\underline{M}$. Let e be a fixed member of E. Then the union S of all the circuits of $\underline{M}$ containing e is a connected component of $\underline{M}$. If $S = E$, then $\underline{M}$ is connected; otherwise, $\underline{M}$ is separable. If $S \neq E$, choose another element $e' \in E - S$ and find all the circuits containing e', whose union yields S', a second connected component of $\underline{M}$. We now choose $e'' \in E - (S \cup S')$ and repeat this process; we continue it until all the elements of E are exhausted, that is, $E - (S \cup S' \cup S'' \cup \ldots)$ is null. At each step of the process a maximally connected minor of $\underline{M}$ will be determined.

For a non-separable matroid, a direct calculation of $\lambda(\underline{M})$ requires in the most extreme case 2^r or 2^μ computations of the ranks of minors, where r and μ are rank and nullity of $\underline{M}$. However, since $\lambda(\underline{M}) \leqslant \max_{S \subseteq E} \xi(\underline{M}; S, \bar{S})$ for a matroid of finite-connectivity, it might be useful to determine the maximum value of a connectivity function ξ.

A trivial upper bound of ξ is obtained from the definition of a connectivity function.

<u>Theorem 3.1</u> $\max_{S \subseteq E} \xi(\underline{M};S,\overline{S}) = \min(r(\underline{M}) + 1, \ \mu(\underline{M}) + 1).$

<u>Proof.</u> $\xi(\underline{M};S,\overline{S}) = -r(\underline{M}) + r(\underline{M} \times S) + r(\underline{M} \times \overline{S}) + 1$

$$\leqslant -r(\underline{M}) + r(M) + r(\underline{M}) + 1 \leqslant r(\underline{M}) + 1,$$

and $\xi(\underline{M};S,\overline{S}) = \mu(\underline{M}) - \mu(\underline{M} \times S) - \mu(\underline{M} \times \overline{S}) + 1$

$$\leqslant \mu(\underline{M}) - 0 - 0 + 1 \leqslant \mu(\underline{M}) + 1. \blacksquare$$

Even though $\max_{S \subseteq E} \xi(\underline{M};S,\overline{S}) \leqslant n$, where n is an integer, it does not mean $\lambda(\underline{M}) \leqslant n$. For finite connectivity, the condition $\min(|S|,|\overline{S}|) \geqslant \xi(\underline{M};S,\overline{S})$ must be satisfied. The class of matroids which do not satisfy this condition, that is the class of matroids with infinite connectivity, has been found and will be discussed in the next section.

Now we shall find the maximum value of $\xi(\underline{M};S,\overline{S})$ for a given matroid $\underline{M}$. Let B_1 and B_2 be bases of $\underline{M}$. Then the <u>distance</u> between B_1 and B_2 is defined by $d(B_1, B_2)=|B_1-B_2|=|B_2-B_1|$. A pair of bases which have the maximum distance are called <u>maximally distant bases.</u> The concept of maximally distant bases was first introduced by Kishi and Kajitani [Ki 1, 2] in graph theory with an application to a partition of graphs. Bruno and Weinberg [Br 3] generalized the concept and applied it to a matroid partition. Here we give another application of maximally distant bases.

<u>Lemma 3.1</u> Let B_1 and B_2 be maximally distant bases of $\underline{M}$. Then

$$r(\underline{M} \times (E-B_1))=r(\underline{M} \times (E-B_2))=d_0,$$

where $d_0 = |B_1-B_2|$.

Proof. Since $B_2-B_1 \subseteq E-B_1$, B_2-B_1 is an independent set of $\underline{M} \times (E-B_1)$. If B_2-B_1 is not a base of $\underline{M} \times (E-B_1)$, we can find $e \in E-B_1$ such that $e \notin B_2$ and $\{e\} \cup (B_2-B_1)$ is independent in $\underline{M} \times (E-B_1)$. $B_2 \cup \{e\}$ contains a circuit $C \in \underline{C}$ satisfying $e \in C$ and $C \cap B_1 \neq \phi$, because if $C \cap B_1 = \phi$, then $C \subseteq (B_2-B_1) \cup \{e\}$, which is a contradiction. Let $e' \in C \cap B_1$. Define $B_2' = (B_2 - \{e'\}) \cup \{e\}$. B_2' is clearly a base of $\underline{M}$. Then $|B_1-B_2'| = |B_1-((B_2-\{e'\}) \cup \{e\})| = |B_1-B_2| + 1 = d_0 + 1$. Therefore, B_1 and B_2 are not maximally distant, which is contrary to the hypothesis. Thus B_2-B_1 is a base of $\underline{M} \times (E-B_1)$. With a similar argument B_1-B_2 is a base of $\underline{M} \times (E-B_2)$. Consequently, we have $r(\underline{M} \times (E-B_1)) = r(\underline{M} \times (E-B_2)) = |B_2-B_1| = |B_1-B_2| = d_0$. ∎

Theorem 3.2 $\max_{S \subseteq E} \xi(\underline{M};S,\overline{S}) = d_0 + 1$, where $d_0 = |B_1-B_2|$ and B_1, B_2 are maximally distant bases of $\underline{M}$.

Proof. Let S be any subset of E, and $B_S, B_{\overline{S}}$ be bases of $\underline{M} \times S$ and $\underline{M} \times \overline{S}$, respectively. Suppose B_1' and B_2' are bases of $\underline{M}$ such that $B_1' \supseteq B_S$ and $B_2' \supseteq B_{\overline{S}}$. Then

$$\begin{aligned}\xi(\underline{M};S,\overline{S}) &= -r(\underline{M}) + r(\underline{M} \times S) + r(\underline{M} \times \overline{S}) + 1 \\ &= -r(\underline{M}) + |B_S| + |B_{\overline{S}}| + 1 \\ &= -r(\underline{M}) + |B_S \cup B_{\overline{S}}| + 1 \\ &\leq -r(\underline{M}) + |B_1' \cup B_2'| + 1 \\ &= -r(\underline{M}) + |B_2'| + |B_1'-B_2'| + 1 \\ &= |B_1'-B_2'| + 1 \leq d_0 + 1.\end{aligned}$$

If B_1 and B_2 are maximally distant bases, then by Lemma 3.1, we have

$$\xi(\underline{M};B_1,\overline{B}_1) = -\,r(\underline{M}) + r(\underline{M} \times B_1) + r(\underline{M} \times \overline{B}_1) + 1$$
$$= r(\underline{M} \times \overline{B}_1) + 1 = d_0 + 1.$$

Therefore, $\max_{S \subseteq E} \xi(\underline{M};S,\overline{S}) = d_0 + 1.$ ∎

Since we know an efficient algorithm for finding a pair of maximally distant bases [Br 3], the maximum value of a connectivity function can be easily determined.

Example 3.1 Let G be the graph given in Fig. 3.1, and $\underline{M}=\underline{P}(G)$. A pair of maximally distant bases of $\underline{M}$ are

$$B_1=\{e_1, e_5, e_6, e_7, e_9\}, \qquad B_2=\{e_2, e_5, e_7, e_8, e_{10}\}.$$

In addition, $\min(r(\underline{M}) + 1, \mu(\underline{M}) + 1)=\min(6,6)=6$. However,

$$\max_{S \subseteq E} \xi(\underline{M};S,\overline{S}) = d_0 + 1 = 3 + 1 = 4.$$

In concluding this section we propose one problem related to a connectivity function. Let $\underline{M}=(\underline{C},E)$ be a matroid and B a base of $\underline{M}$. The following question arises in applications: what is the minimum value of $r(\underline{M} \times \overline{B})$ for all the bases B of $\underline{M}$? Since $\xi(\underline{M};B,\overline{B}) = r(\underline{M} \times \overline{B}) + 1$, this problem is equivalent to finding the value of $\min_{S:\text{base}} \xi(\underline{M};S,\overline{S})$. The corresponding graph theory problem is called the central tree problem: given any connected graph G, what is the minimum rank of spanning cotrees of G? This problem is unsolved, and at present there exists no efficient algorithm for finding a spanning tree which satisfies the above condition.

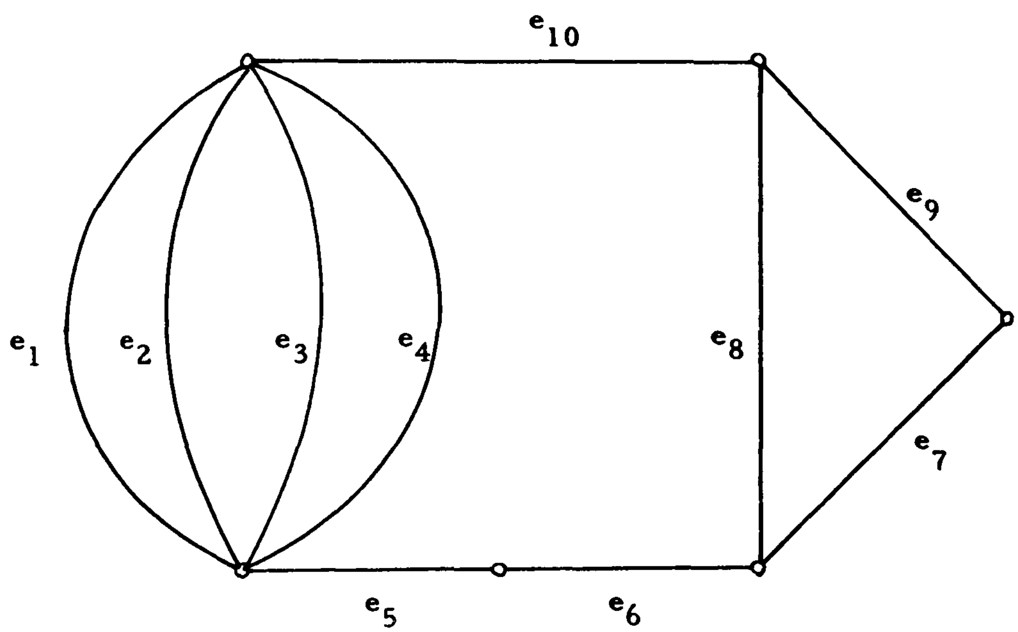


Figure 3.1 Graph G of Example 3.1

3.2 MATROIDS OF INFINITE CONNECTIVITY

In Theorem 3.2 we found the maximum value of a connectivity function. For a finite-connectivity matroid the connectivity is never greater than this value. However, there are matroids with infinite connectivity. Consider the following example: $\underline{M}=(\underline{C}, E)$ is a matroid defined by $E=\{e_1, e_2\}$ and $\underline{C}=\{E\}$. By the definition of connectivity, the only possible partition of E is $S=\{e_1\}$ and $\bar{S}=\{e_2\}$. However,

$$\xi(\underline{M};S,\bar{S}) = -r(\underline{M}) + r(\underline{M} \times S) + r(\underline{M} \times \bar{S}) + 1$$
$$= -1+1+1+1=2,$$

and $\min(|S|, |\bar{S}|)=1$. Therefore this matroid does not satisfy the condition of finite-connectivity, and we then say, $\lambda(\underline{M})=\infty$.

In the following theorem we show that the class of matroids of infinite connectivity are binomial matroids.

<u>Theorem 3.3</u> Let $\underline{M}=(\underline{C}, E)$ be a matroid and $|E|=n$. If $\lambda(\underline{M})=\infty$ and $|E| \geq 2$, then $\underline{M} \cong {}_n\underline{M}_r$, where $2r-1 \geq n \geq 2r-3$. For $|E|=1$, $\underline{M} \cong {}_1\underline{M}_0$, or ${}_1\underline{M}_1$.

In the above theorem if $|E|=n$ is odd, then there are two possible matroids, $\underline{M} \cong {}_n\underline{M}_{(n+1)/2}$ and ${}_n\underline{M}_{(n+3)/2}$, whereas if n is even, there is only one, namely, ${}_n\underline{M}_{(n+2)/2}$.

<u>Proof.</u> If n=1, the theorem is obvious. For $n \geq 2$ we consider the two cases of n even and odd.

Case 1. $n=2k$, where $k \geq 1$.

Without loss of generality, we can assume $|S| \leq k$. First we show that every set consisting of k elements is independent in $\underline{M}$. Suppose a subset S of E contains k elements and is dependent in $\underline{M}$. Since $\underline{M} \times S$ contains a circuit of $\underline{M}$, $r(\underline{M} \times S) \leq |S| - 1$.

$$\xi(\underline{M};S,\overline{S}) = -r(\underline{M}) + r(\underline{M} \times S) + r(\underline{M} \times \overline{S}) + 1$$
$$\leq -r(\underline{M}) + |S| - 1 + r(\underline{M} \times \overline{S}) + 1$$
$$= |S| - r(\underline{M}) + r(M \times \overline{S}) \leq |S| = k,$$

and $\min(|S|,|\overline{S}|)=k$. Therefore $\lambda(\underline{M}) \leq k$, which is contrary to the hypothesis. Thus every subset of E consisting of k elements is independent in $\underline{M}$ and hence, $r(\underline{M}) \geq k$.

Now suppose $r(\underline{M}) \geq k + 1$. Then for any subset S of E we have

$$\xi(\underline{M};S,\overline{S}) = -r(\underline{M}) + r(\underline{M} \times S) + r(\underline{M} \times \overline{S}) + 1$$
$$\leq -(k + 1) + |S| + |\overline{S}| + 1 = k.$$

If we again choose S so that $|S|=|\overline{S}|=k$, then $\lambda(\underline{M}) \leq k$, which is once more a contradiction. Thus, $r(\underline{M})=k$ for $\lambda(\underline{M})=\infty$. Consequently, we have $\underline{M} \cong_{2k} \underline{M}_{k+1}$ and $\lambda(\underline{M})=\infty$.

Case 2. $n=2k+1$, where $k \geq 1$.

As in Case 1 we can show that every subset of E consisting of k elements is independent in $\underline{M}$. Suppose there exists a dependent set S of $\underline{M}$ consisting of k + 1 elements of E. Since $|S|=k+1$ and $|\overline{S}|=k$, $\overline{S}$ is independent in $\underline{M}$. If $r(\underline{M}) \geq k + 1$, then

$$\xi(\underline{M};S,\overline{S}) = -r(\underline{M}) + r(\underline{M} \times S) + r(\underline{M} \times \overline{S}) + 1$$
$$\leq -(k + 1) + k + k + 1 = k,$$

and $\min(|S|,|\overline{S}|)=k$, which contradicts $\lambda(\underline{M})=\infty$. Thus either $r(\underline{M})=k$ or there is no dependent set consisting of k + 1 elements. If $r(\underline{M})=k$, then $\underline{M} \cong {}_{2k}\underline{M}_{k+1}$ and $\lambda(\underline{M}) = \infty$.

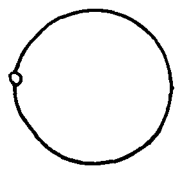
Suppose all the subsets of E consisting of k + 1 elements are independent. If $r(\underline{M}) \geq k + 2$, then

$$\xi(\underline{M}; S, \bar{S}) \leq -(k+2) + |S| + |\bar{S}| + 1 = k.$$

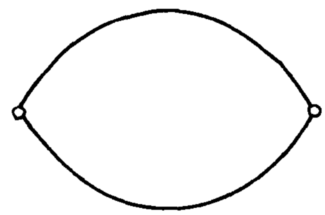
If we choose S so that $|S| = k$ and $|\bar{S}| = k + 1$, then $\lambda(\underline{M}) \leq k$, which is contrary to the hypothesis. Thus $r(\underline{M}) = k + 1$ and the corresponding matroid is isomorphic to ${}_{2k+1}\underline{M}_{k+2}$, and $\lambda(\underline{M}) = \infty$. Consequently, the proof is complete. ∎

From this theorem we can deduce these results for graph realizable matroids: if $\lambda(G) = \infty$, then $\underline{P}(G) \cong {}_1\underline{M}_0$, ${}_1\underline{M}_1$, ${}_2\underline{M}_2$, ${}_3\underline{M}_2$, or ${}_3\underline{M}_3$. The same statement holds for $\underline{B}(G)$. The corresponding graphs are shown in Fig. 3. 2. We see that this result coincides with that of Tutte for graph connectivity [Tu 9].

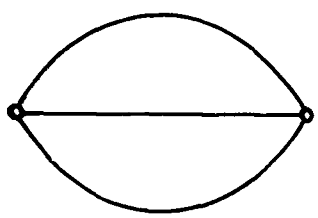
${}_1\underline{M}_0$



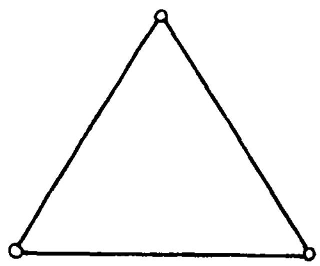
${}_1\underline{M}_1$



${}_2\underline{M}_2$



${}_3\underline{M}_2$



${}_3\underline{M}_3$

Figure 3.2 Graphs Corresponding to Graph-Realizable Matroids of Infinite Connectivity, Where Matroids are Considered as Polygon Matroids

3.3 CONNECTIVITY OF BINOMIAL MATROIDS

The structure of binomial matroids is exceedingly simple and has been referred to often in articles giving examples and counter-examples. This simplicity, however, has resulted in a possible oversight on the important role played by binomial matroids in matroid theory. Therefore, it might be worthwhile to uncover more properties of binomial matroids. In the following theorem we determine the connectivity of binomial matroids.

Theorem 3.4 The connectivity of the binomial matroid ${}_n\underline{M}_r$, $n \geq r \geq 0$, is given by:

$$\lambda({}_n\underline{M}_r) = \begin{cases} 1 & \text{for } n \geq 2 \text{ and } r=0. \\ r & \text{for } n \geq 2r. \\ \infty & \text{for } 2r-1 \geq n \geq 2r-3, \\ & \text{and for } n=0,1. \\ n+2-r & \text{for } 2r-4 \geq n. \end{cases}$$

Proof. Since the theorem is obvious for $r=0$, we assume $n \geq r \geq 1$. Let S and $\bar{S}$ be complementary subsets of E. From Theorem 2.13 we have

$$\xi({}_n\underline{M}_r; S, \bar{S}) = \mu({}_n\underline{M}_r) - \mu({}_n\underline{M}_r \times S) - \mu({}_n\underline{M}_r \times \bar{S}) + 1$$
$$= n - r + 2 - \mu({}_n\underline{M}_r \times S) - \mu({}_n\underline{M}_r \times \bar{S}),$$

and $\Delta(S, \bar{S}) = \min(|S|, |\bar{S}|)$. For convenience we use Δ for $\Delta(S, \bar{S})$. Without loss of generality we assume $1 \leq |S| = k \leq n/2$ and $|\bar{S}| = n - k$.

Then $\xi({}_n\underline{M}_r; S, \bar{S}) = n - r + 2 - (k - r + 1)_+ - (n - k - r + 1)_+$ and $\Delta = \min(k, n-k) = k$, where $(i)_+ = i$ for $i \geq 0$ and 0 for $i < 0$.

Four possible cases can be considered:

(i) $r \leqslant k \leqslant n - r$,

(ii) $r,\ n - r + 1 \leqslant k$,

(iii) $k \leqslant r - 1,\ n - r$,

(iv) $n - r + 1 \leqslant k \leqslant r - 1$.

We shall find the conditions for finite-connectivity: $\xi \leqslant k$.

Case (i). $\xi = n - r + 2 - (k - r + 1) - (n - k - r + 1) = r \leqslant k$ for finite-connectivity. If $2r \leqslant n$, then $\lambda(\underline{M}) = r$.

Case (ii). Since $k = \min(r, n - r + 1) \geqslant n/2 + 1$, by assumption we do not have to consider this case.

Case (iii). $\xi = n - r + 2 - (n - k - r + 1) = k + 1 \leqslant k$, which is a contradiction.

Case (iv). $\xi = n - r + 2 \leqslant k$. For ${}_n\underline{M}_r$ having finite connectivity it must satisfy the following conditions: $n - r + 2 \leqslant k \leqslant r - 1$ and $n - r + 2 \leqslant k \leqslant n/2$. Thus we have $n \leqslant 2r - 4$ and $\lambda(\underline{M}) = n - r + 2$.

From Cases (i) and (iv) we obtain the condition for infinite connectivity; $\lambda({}_n\underline{M}_r) = \infty$ for $2r - 1 \geqslant n \geqslant 2r - 3$. Accordingly, the theorem follows. ∎

The binomial matroids are one of a very few classes of matroids whose connectivity is completely known. Another use of this theorem is to provide examples of matroids with prescribed connectivities.

3.4 REFERENCE TO WHEELS AND WHIRLS

Let $\underline{M}=(\underline{C}, E)$ be a 3-connected matroid. It is known that if every cell of $\underline{M}$ is essential, then $\underline{M}$ is a wheel or a whirl matroid (Theorem 2.16). If $\underline{M}$ contains a non-essential cell e, then the removal of e by the reduction or contraction of $\underline{M}$ yields again a 3-connected matroid. Repeating this process, we finally obtain a wheel or a whirl matroid, or neither. Since the former two matroids contain at least six cells, our task is to find the matroids with six cells which contain a non-essential cell.

Lemma 3.2 If a matroid $\underline{M}=(\underline{C}, E)$ is 3-connected, where $|E| \geq 4$, then $|C| \geq 3$ for every $C \in \underline{C}$.

Proof. If $\underline{M}$ has a circuit consisting of a single cell, then the connectivity of $\underline{M}$ is one. Suppose $\underline{M}$ contains no circuit consisting of a single cell, and let $C=\{e_1, e_2\}$ be a circuit of $\underline{M}$. Then by Theorem 2.13,

$$\begin{aligned}\xi(\underline{M};C,\bar{C}) &= \mu(\underline{M}\cdot C) - \mu(\underline{M}\times C)+1\\ &= \mu(\underline{M}\cdot C) - 1 + 1 \leq |C| = 2,\end{aligned}$$

and $\min(|C|, |\bar{C}|) = |C| = 2$, which contradicts $\lambda(\underline{M}) \geq 3$. Thus the lemma follows. ∎

Theorem 3.5 If $\underline{M}=(\underline{C}, E)$ is 3-connected, where $|E|=4$, then $\underline{M} \cong {}_4\underline{M}_3$.

Proof. Let $E=\{e_1, e_2, e_3, e_4\}$. Since $\underline{M}$ is 3-connected every cell is a member of some circuit, and E is not a circuit of $\underline{M}$; otherwise $\underline{M} \cong {}_4\underline{M}_4$ and $\lambda({}_4\underline{M}_4)=2$. By Lemma 3.2 without loss of generality, let $C_1=\{e_1, e_2, e_3\}$ and $C_2=\{e_1, e_2, e_4\}$ be circuits of $\underline{M}$. By Axiom II:

$C_3 = C_1 \cup C_2 - \{e_1\} = \{e_2, e_3, e_4\}$,

$C_4 = C_1 \cup C_2 - \{e_2\} = \{e_1, e_3, e_4\}$

are members of $\underline{C}$. Hence $\underline{M} \cong {}_4\underline{M}_3$. ∎

Lemma 3.3 Let $\underline{M} = (\underline{C}, E)$ be 3-connected, where $|E| = 5$. If there exists a circuit C of $\underline{M}$ such that $|C| = 4$, then $\underline{M} \cong {}_5\underline{M}_4$.

Proof. Let $E = \{e_1, e_2, \ldots, e_5\}$. Suppose C_1 is the only circuit of $\underline{M}$ containing four cells. Then $|C| = 3$ for every $C (\neq C_1) \in \underline{C}$ by Axiom I. We shall show a contradiction. Let $C_1 = \{e_1, e_2, e_3, e_4\}$ and $C_2 = \{e_1, e_2, e_5\}$ be circuits of $\underline{M}$. By Axiom II and by Lemma 3.2, we may suppose that $C_3' = \{e_2, e_3, e_5\}$ or $C_3'' = \{e_3, e_4, e_5\}$ is a circuit of $\underline{M}$. If $C_3' \in \underline{C}$, then $C_4' = C_2 \cup C_3' - \{e_5\} = \{e_1, e_2, e_3\} \in \underline{C}$ by Lemma 3.2. This contradicts Axiom I. Thus $C_3'' \in \underline{C}$. Then $\underline{M} = (\underline{C}, E)$, where $\underline{C} = \{C_1, C_2, C_3''\}$ is a matroid of connectivity 2 since

$$\xi(\underline{M}; \{e_1, e_2\}, \{e_3, e_4, e_5\})$$
$$= \mu(\underline{M}) - \mu(\underline{M} \times \{e_1, e_2\}) - \mu(\underline{M} \times \{e_3, e_4, e_5\}) + 1$$
$$= 2 - 0 - 1 + 1 = 2,$$

and $\min(|\{e_1, e_2\}|, |\{e_3, e_4, e_5\}|) = 2$. This is contrary to the hypothesis. Since C_2 has been chosen without loss of generality, $\underline{M}$ has at least two circuits which contain four cells of $\underline{M}$.

Let $C_2 = \{e_1, e_2, e_3, e_5\}$ be another circuit of $\underline{M}$, and $|C| = 3$ for every $C (\neq C_1, C_2) \in \underline{C}$. By Axiom II, we may suppose ($e_5 \in$) $C_3' = \{e_1, e_4, e_5\}$ ($\subseteq C_1 \cup C_2 - \{e_3\}$) is a member of $\underline{C}$. Since $C_1 \cup C_3' - \{e_1\} = \{e_2, e_3, e_4, e_5\}$ must contain a circuit C_4' which consists of three cells, we choose $C'_4 = \{e_2, e_4, e_5\}$ without loss of generality. $C'_3 \cup C'_4 - \{e_5\} = \{e_1, e_2, e_4\} \in \underline{C}$ by Axiom II, which is contrary to Axiom I.

Now let C_1, C_2 given above be circuits of $\underline{M}$, and $C_3 = \{e_1, e_2, e_4, e_5\} \in \underline{C}$. By Axiom II, $C_1 \cup C_3 - \{e_1\} = \{e_2, e_3, e_4, e_5\}$ contains a circuit of $\underline{M}$. Two cases are to be considered: $\{e_3, e_4, e_5\}$ and $\{e_2, e_3, e_4, e_5\}$. If $C'_4 = \{e_3, e_4, e_5\} \in \underline{C}$, then $\underline{M} = (\underline{C}, E)$, where $\underline{C} = \{C_1, C_2, C_3, C'_4\}$, satisfies the matroid axioms and its connectivity is two, which is contrary to the hypothesis.

Let $C_4 = \{e_2, e_3, e_4, e_5\} \in \underline{C}$. By Axiom II $C_3 \cup C_4 - \{e_2\} = \{e_1, e_3, e_4, e_5\}$ contains a circuit of $\underline{M}$ and hence, $C_5 = \{e_1, e_3, e_4, e_5\}$ is a circuit of $\underline{M}$ by Axiom I. Then $\underline{M} = (\underline{C}, E)$, where $\underline{C} = \{C_1, C_2, \ldots, C_5\}$, is a 3-connected matroid and is isomorphic to ${}_5\underline{M}_4$. Since there is no other subset consisting of four cells, the lemma follows. ∎

<u>Lemma 3.4</u> Let $\underline{M} = (\underline{C}, E)$ be 3-connected, where $|E| = 5$. If there is a circuit of $\underline{M}$ consisting of three cells, then $\underline{M} \cong {}_5\underline{M}_3$.

<u>Proof.</u> Let $E = \{e_1, e_2, \ldots, e_5\}$. By Lemma 3.3 $|C| = 3$ for every $C \in \underline{C}$. Let $C_1 = \{e_1, e_2, e_3\} \in \underline{C}$. Since $\underline{M}$ is connected $\underline{C}$ must contain circuits of the forms $\{e_i, e_j, e_4\}$ and $\{e_i, e_k, e_5\}$, where $e_i, e_j, e_k \in \{e_1, e_2, e_3\}$. Thus without loss of generality we may, by Axiom II, suppose $C_2 = \{e_1, e_4, e_5\} \in \underline{C}$. Also by Axiom II, $C_1 \cup C_2 - \{e_1\} = \{e_2, e_3, e_4, e_5\}$ contains a circuit C_3. Let $C_3 = \{e_2, e_3, e_4\}$, maintaining generality. From these three circuits we can identify other members of $\underline{C}$ by Axiom II.

$$C_4 = C_1 \cup C_3 - \{e_2\} = \{e_1, e_3, e_4\}$$

$C_5 = C_1 \cup C_3 - \{e_3\} = \{e_1, e_2, e_4\}$

$C_6 = C_2 \cup C_4 - \{e_1\} = \{e_3, e_4, e_5\}$

$C_7 = C_2 \cup C_4 - \{e_4\} = \{e_1, e_3, e_5\}$

$C_8 = C_2 \cup C_5 - \{e_1\} = \{e_2, e_4, e_5\}$

$C_9 = C_2 \cup C_5 - \{e_4\} = \{e_1, e_2, e_5\}$

$C_{10} = C_7 \cup C_9 - \{e_1\} = \{e_2, e_3, e_5\}$

The C_i's that have been obtained are all the subsets of E consisting of three elements. Clearly $\underline{M} \cong {}_5\underline{M}_3$, and $\underline{M}$ is 3-connected. ∎

From Lemmas 3.3 and 3.4 we obtain the next theorem.

Theorem 3.6 If $\underline{M}=(\underline{C}, E)$ is 3-connected and $|E|=5$, then $\underline{M} \cong {}_5\underline{M}_4$ or ${}_5\underline{M}_3$.

Proof. By Lemma 3.2 every circuit of $\underline{M}$ must contain at least three cells. Suppose $\underline{C} = \{E\}$, where $E = \{e_1, e_2, \ldots, e_5\}$. If we choose $S = \{e_1, e_2\}$, then

$$\xi(\underline{M}; S, \bar{S}) = \mu(\underline{M}) - \mu(\underline{M} \times S) - \mu(\underline{M} \times \bar{S}) + 1$$

$$= 1 - 0 - 0 + 1 = 2,$$

therefore the connectivity of the matroid is two, which is contrary to the hypothesis. Thus the theorem follows from Lemmas 3.3 and 3.4. ∎

Lemma 3.5 Let $\underline{M}=(\underline{C}, E)$ be 3-connected, where $|E|=6$, and e is a member of E. If $\underline{M} \times (E-\{e\}) \cong {}_5\underline{M}_3$, then $\underline{M} \cong {}_6\underline{M}_3$.

Proof. Let $E = \{e_1, e_2, \ldots, e_6\}$ and $\underline{M}' = \underline{M} \times (E - \{e\})$, where $e_1 = e$. Since all the circuits of $\underline{M}'$ consist of three cells, we then have by Axiom I that no circuit of $\underline{M}$ contains more than three cells. Let $C_{ijk} = \{e_i, e_j, e_k\}$

be the circuits of $\underline{M}'$, where $2 \leq i, j, k \leq 6$ and $i \neq j \neq k \neq i$. Since $\underline{M}$ is 3-connected, there exists a circuit of $\underline{M}$ containing cell e. Let $C_{123} = \{e_1, e_2, e_3\} \in \underline{C}$. By Axiom II, $C_{123} \cup C_{23i} - \{e_j\}$ is a circuit of $\underline{M}$, where $4 \leq i \leq 6$ and j=2, 3. Again applying Axiom II, the following subsets of E are circuits of $\underline{M}$.

$$C_{145} = C_{134} \cup C_{135} - \{e_3\}, \qquad C_{146} = C_{124} \cup C_{126} - \{e_2\},$$
$$C_{156} = C_{125} \cup C_{126} - \{e_2\}$$

Let $\underline{C} = \{C_{ijk} \mid 1 \leq i < j < k \leq 6\}$. Then $\underline{M} \cong {}_6\underline{M}_3$. ∎

Before proceeding further we shall define two matroids associated with the binomial matroid ${}_6\underline{M}_4$. Let $E = \{e_1, e_2, \ldots, e_6\}$. We use the following notational convention to avoid complexity: $C_{ij\ldots k} = \{e_i, e_j, \ldots, e_k\}$, where $1 \leq i < j < \ldots < k \leq 6$. Let ${}_6\underline{C}_{4(1)} = \{C_{123}, C_{1245}, C_{1246}, C_{1256}, C_{1345}, C_{1346}, C_{1356}, C_{2345}, C_{2346}, C_{2356}, C_{2456}, C_{3456}\}$. Then ${}_6\underline{M}_{4(1)} = ({}_6\underline{C}_{4(1)}, E)$ satisfies the matroid axioms and it can be easily shown that this matroid is isomorphic to its dual. The second matroid ${}_6\underline{M}_{4(2)} = ({}_6\underline{C}_{4(2)}, E)$ is defined by ${}_6\underline{C}_{4(2)} = \{C_{123}, C_{145}, C_{1245}, C_{1346}, C_{1356}, C_{2345}, C_{2346}, C_{2356}, C_{2456}, C_{3456}\}$. ${}_6\underline{M}_{4(2)}$ is a self-dual matroid. It should be noted that ${}_6\underline{M}_{4(1)}$ (${}_6\underline{M}_{4(2)}$) is obtained from ${}_6\underline{M}_4$ by deleting all circuits which properly contain $\{e_1, e_2, e_3\}$ ($\{e_1, e_2, e_3\}$ and $\{e_1, e_4, e_5\}$), and adding $\{e_1, e_2, e_3\}$ ($\{e_1, e_2, e_3\}$ and $\{e_1, e_4, e_5\}$) as a member of the circuits.

<u>Lemma 3.6</u> Let $\underline{M} = (\underline{C}, E)$ be 3-connected, where $|E| = 6$, and $e \in E$. If $\underline{M} \times (E - \{e\}) \cong {}_5\underline{M}_4$, then $\underline{M} \cong {}_6\underline{M}_4$, ${}_6\underline{M}_{4(1)}$, or ${}_6\underline{M}_{4(2)}$.

Proof. Let $E=\{e_1, e_2, \dots, e_6\}$ and $e_1=e$. By Axiom I, $\underline{M}$ cannot have a circuit consisting of more than four cells. Suppose $\underline{C}$ contains at least three circuits which consist of three cells. Since each of these circuits must contain e_1, let C_{1ij}, C_{1km}, $C_{1pq} \in \underline{C}$, where $2 \leqslant i, j, k, m, p, q$. Since $i, j, k, m, p, q \in \{n \mid 2 \leqslant n \leqslant 6$, at least two of these subscripts are the same. Let i=k. By Axiom II, $C_{1ij} \cup C_{1im} - \{e_1\} = \{e_i, e_j, e_m\}$ contains a circuit of $\underline{M}$. However, this is contrary to Axiom I, since $\{e_i, e_j, e_m\} \subseteq E-\{e_1\}$. Therefore, $\underline{C}$ contains at most two circuits consisting of three cells.

Case 1. $\underline{C}$ contains no circuit consisting of three cells. Since $\underline{M}$ is 3-connected, there exists a circuit containing e_1 and consisting of four cells. Let $C_{1234} \in \underline{C}$. By Axiom II, $C_{1234} \cup C_{234i} - \{e_j\}$ are circuits of $\underline{M}$, where i=5, 6 and j=2, 3, 4. Again by Axiom II, the following subsets of E are members of $\underline{C}$.

$$C_{1256}=C_{1245} \cup C_{1246} - \{e_4\}$$

$$C_{1356}=C_{1345} \cup C_{1346} - \{e_4\}$$

$$C_{1456}=C_{1345} \cup C_{1346} - \{e_3\}$$

Therefore $\underline{C}=\{C_{ijkm} \mid 1 \leqslant i < j < k < m \leqslant 6\}$, and $\underline{M}=(\underline{C}, E) \cong {}_6\underline{M}_4$.

Case 2. $\underline{C}$ contains exactly one circuit consisting of three cells. Let $C_{123} \in \underline{C}$. By Axiom II, $C_{123} \cup C_{23ij} - \{e_k\}$ are members of $\underline{C}$, where $4 \leqslant i < j \leqslant 6$ and k=2, 3. $C_{1456}=C_{1346} \cup C_{1356} - \{e_3\}$ is a circuit of $\underline{M}$ by Axiom II. Clearly $\underline{M} \cong {}_6\underline{M}_{4(1)}$. Now we show that ${}_6\underline{M}_{4(1)}$ is 3-connected. Let $S=\{e_1, e_2, e_3\}$ and $\overline{S}=\{e_4, e_5, e_6\}$. By Lemma 2.13

$$\zeta(\underline{M};S,\bar{S}) = \mu(\underline{M}) - \mu(\underline{M} \times S) - \mu(\underline{M} \times \bar{S}) + 1$$
$$= 3 - 1 - 0 + 1 = 3,$$

and $\min(|S|, |\bar{S}|) = 3$. It can be easily seen that no other choice for S satisfies the finite-connectivity condition. Accordingly, $\lambda(\underline{M}) = 3$.

Case 3. $\underline{C}$ contains exactly two circuits consisting of three cells. If $C_{1ij}, C_{1ik} \in \underline{C}$, then $C_{1ij} \cup C_{1ik} - \{e_1\} = \{e_i, e_j, e_k\}$ contains a circuit of $\underline{M}$, which is contrary to Axiom I. Thus, without loss of generality, we can assume $C_{123}, C_{145} \in \underline{C}$. By Axiom II, $C_{123} \cup C_{23ij} - \{e_k\}$ are circuits of $\underline{M}$, where $k=2,3$ and $4 \leq i < j \leq 6$, except $(i,j)=(4,5)$. Thus we have $\underline{M} \cong {}_6\underline{M}_{4(2)}$. Let $S=\{e_1, e_2, e_3\}$ and $\bar{S} = \{e_4, e_5, e_6\}$. Then

$$\zeta(\underline{M};S,\bar{S}) = \mu(\underline{M}) - \mu(\underline{M} \times S) - \mu(\underline{M} \times \bar{S}) + 1$$
$$= 3 - 1 - 0 + 1 = 3,$$

and $\min(|S|, |\bar{S}|) = 3$. It is not hard to show $\lambda(\underline{M}) = 3$.

Since we have considered all possible cases, the proof is complete. ∎

From Theorem 3.6 and Lemmas 3.5 and 3.6 we obtain the following theorem:

<u>Theorem 3.7</u> Let $\underline{M}=(\underline{C}, E)$ be 3-connected, where $|E|=6$ and $e \in E$. If $\underline{M} \times (E-\{e\})$ is 3-connected, then $\underline{M} \cong {}_6\underline{M}_3$, ${}_6\underline{M}_4$, ${}_6\underline{M}_{4(1)}$, or ${}_6\underline{M}_{4(2)}$.

Since ${}_6\underline{M}_4$, ${}_6\underline{M}_{4(1)}$, and ${}_6\underline{M}_{4(2)}$ are isomorphic to their duals, we obtain the next theorem, which is the dual of Theorem 3.7.

Theorem 3.8 Let $\underline{M}=(\underline{C}, E)$ be 3-connected, where $|E|=6$ and $e \in E$. If $\underline{M}(E-\{e\})$ is 3-connected, then $\underline{M} \cong {}_6\underline{M}_5$, ${}_6\underline{M}_4$, ${}_6\underline{M}_{4(1)}$, or ${}_6\underline{M}_{4(2)}$.

Proof. By Theorem 2.14, $\underline{M}^* \times (E-\{e\})$ is 3-connected, and hence, by Theorem 3.7 $\underline{M}^* \cong {}_6\underline{M}_3$, ${}_6\underline{M}_4$, ${}_6\underline{M}_{4(1)}$, or ${}_6\underline{M}_{4(2)}$. Therefore, the theorem follows from Theorem 2.6. ∎

By definition, if a 3-connected matroid $\underline{M}=(\underline{C}, E)$, where $|E|=6$, contains a non-essential cell, then there exists a cell $e \in E$, such that either $\underline{M} \times (E-\{e\})$ or $\underline{M} \cdot (E-\{e\})$ is 3-connected. In terms of essential cells, we state our main result in this section.

Theorem 3.9 If a 3-connected matroid $\underline{M}=(\underline{C}, E)$, where $|E|=6$, contains a non-essential cell, then $\underline{M} \cong {}_6\underline{M}_3$, ${}_6\underline{M}_4$, ${}_6\underline{M}_{4(1)}$, ${}_6\underline{M}_{4(2)}$, or ${}_6\underline{M}_5$.

Proof. The theorem immediately follows from Theorems 3.7 and 3.8. ∎

The result of Theorem 3.9 is useful in our consideration of graph-realizability of matroids. This theorem clarifies the structure of non-realizable matroids in addition to whirl matroids.

Making use of Theorem 2.16 and the results in this section, we list all the 3-connected matroids with six and fewer numbers of cells in Table 3.1.

$\|E\|$	$\underline{M}=(\underline{C}, E)$	$\lambda(\underline{M})$	Graph-Realizable
1	$_1\underline{M}_0$	∞	yes
	$_1\underline{M}_1$	∞	yes
2	$_2\underline{M}_2$	∞	yes
3	$_3\underline{M}_2$	∞	yes
	$_3\underline{M}_3$	∞	yes
4	$_4\underline{M}_3$	∞	no
5	$_5\underline{M}_3$	∞	no
	$_5\underline{M}_4$	∞	no
6	$_6\underline{M}_3$	3	no
	$_6\underline{M}_4$	∞	no
	$_6\underline{M}_{4(1)}$	3	no
	$_6\underline{M}_{4(2)}$	3	no
	$_6\underline{M}_5$	3	no
	$\underline{W}_6$	3	yes
	$\underline{W}_{r6}$	3	no

Table 3.1 List of 3-Connected Matroids for $|E| \leq 6$

3.5 A THEOREM ON NONPLANAR MATROIDS

In graph theory it is known [Ha 1] that if the Whitney connectivity of a graph G is greater than five, then G is nonplanar. In this section we will prove the matroid counterpart of this theorem.

Lemma 3.7 If $\underline{M}=(\underline{C}, E)$ is n-connected and $|E| \geq 2n - 2$, then $|C|$, $|C^*| \geq n$ for any $C \in \underline{C}$ and any $C^* \in \underline{C}^*$, where $\underline{M}^*=(\underline{C}^*, E)$ is the dual of $\underline{M}$.

Proof. Suppose there exists a circuit C, such that $|C| \leq n - 1$. By Theorem 2.13

$$\xi(\underline{M}; C, \bar{C}) = |C| - \mu(\underline{M} \times C) - \mu(\underline{M}^* \times C) + 1$$
$$= |C| - 1 - 0 + 1 = |C| \leq n - 1,$$

and $\min(|C|, |\bar{C}|) = |C|$. Therefore $\lambda(\underline{M}) \leq n - 1$, which is contrary to the hypothesis. Since $\lambda(\underline{M}^*) = \lambda(\underline{M}) \geq n$, the lemma follows. ∎

Theorem 3.10 If $\underline{M}=(\underline{C}, E)$ is 4-connected and $|E| \geq 6$, then $\underline{M}$ is nonplanar.

Proof. Suppose $\underline{M}$ is a planar matroid. By definition, there exists a corresponding planar graph G=(V, E). Let M be the meshes(including the outer mesh) of a plane graph G. Since $\underline{M}$, and hence G, is 4-connected, we have by Lemma 3.7,

$$\rho_V(v) \geq 4, \qquad \nu_M(m) \geq 4$$

for every $v \in V$ and $m \in M$, where $\rho_V(v)$ is the valence of v, and $\nu_M(m)$ is the number of edges of mesh m. The following equalities are basic in graph theory:

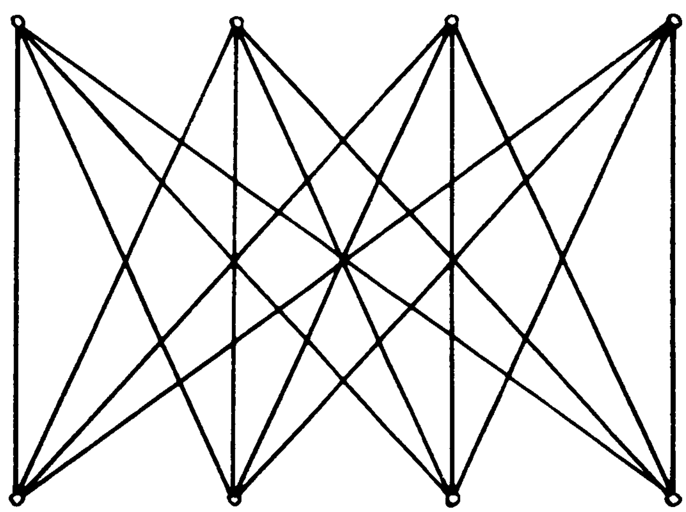
$$\sum_{v \in V} \rho_V(v) = 2|E|, \qquad \sum_{m \in M} \mu_M(m) = 2|E|.$$

Then by Euler's formula we have

$$|E| = |V| + |M| - 2 \leqslant \sum_{v \in V} (1/4) \ \rho_V(v) + \sum_{m \in M} (1/4)\mu_M(m)$$
$$- 2 = (1/4) \times 2|E| + (1/4) \times 2|E| - 2 = |E| - 2.$$

This is a contradiction and hence, $\underline{M}$ is not planar. ∎

This theorem implies that if the connectivity of a planar graph is finite, then the connectivity is not greater than three. This fact is not surprising since matroid connectivity of graphs is determined, not only by star cut-sets, (which define vertices), but also by polygons of the graph. In Fig. 3.3, we show bipartite graph $K_{4,4}$, which is a graph of connectivity 4 that possesses the least number of vertices.



$K_{4,4}$

Figure 3.3 Minimal Graph of Connectivity 4

CHAPTER 4 DECOMPOSITION OF MATROIDS

4.1 INTRODUCTION

The study of the structure of a matroid can be simplified by considering its connected components. Many structural characteristics of a matroid are determined by the corresponding characteristics of its components. The decomposition of a matroid into its components is particulartly useful in studying a large matroid, which has a large number of circuits or a large number of bases.

In graph theory the equivalent decomposition is described as a partition of a graph into a set of maximally 2-connected subgraphs. A further decomposition of a graph into a set of maximally 3-connected subgraphs has been proposed by MacLane [Ma 1]. He also noted that planarity of a given graph is determined by atoms, which are maximal 3-connected subgraphs obtained by the decomposition.

In this chapter we will generalize MacLane's results to matroid decomposition. A special case of our decomposition has been considered by Brylawski [Br 4, 5]. Let $\underline{M} = (\underline{C}, E)$ be a connected matroid. Then his decomposition process is carried out if either $\underline{M} \times (E - \{e\})$ or $\underline{M} \cdot (E - \{e\})$ is separable for some $e \in E$. The decomposition discussed in this chapter can be applied for any connected matroid and yields a set of 3-connected matroids; on the other hand non-decomposable matroids obtained by Brylawski's method may have connectivity 2. The decomposition of a matroid into its basic components, or maximal 3-connected minors, determines whether the original matroid is binary, regular, graphic, or cographic* An application of matroid decomposition to graph-realizability of matroids will be considered in Chapter 6.

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* A similar matroid decomposition is independently studied by Bixby [Bi 1] in making use of virtual cells, and is a direct generalization of Brylawski's idea, while the decomposition presented in this chapter is performed using circuits. Bixby notes that the non-decomposable matroids consist of 3-connected, graphic, and cographic matroids. However, in addition to this result new and interesting theorems will be found in this chapter. These are stated as Theorems 4.13, 4.14, 4.15, and 4.16 which characterize the structure of a matroid by its atoms.

4.2 2-SEPARATORS

In this section we investigate the structure of matroids of connectivity 2.

Let $\underline{M}=(\underline{C}, E)$ be a matroid of connectivity n, and $S \subset E$. If $\xi(\underline{M};S,\bar{S})=n$ and $\min(|S|,|\bar{S}|)=n$, then S is called an n-separator of of $\underline{M}$. A 1-separator is simply called a separator.

Theorem 4.1 Let $\underline{M}=(\underline{C}, E)$ be a matroid and $\phi \neq S \subset E$. Then the following statements are equivalent.

(a) S is a separator of $\underline{M}$.

(b) $r(\underline{M} \times S) + r(\underline{M} \times \bar{S}) = r(\underline{M})$ $[\mu(\underline{M} \times S) + \mu(\underline{M} \times \bar{S}) = \mu(\underline{M})]$

(c) Let B_S and $B_{\bar{S}}$ be bases of $\underline{M} \times S$ and $\underline{M} \times \bar{S}$, respectively. Then $B_S \cup B_{\bar{S}}$ is a base of $\underline{M}$.

(d) If $C \in \underline{C}$, then $C \subseteq S$ or $C \subseteq \bar{S}$.

(e) $\underline{M} \times S = \underline{M} \cdot S$.

Proof. (a)$\leftrightarrow$(b). By definition S is a separator of $\underline{M}$ if and only if $r(\underline{M} \times S) + r(\underline{M} \times \bar{S}) = r(\underline{M})$ and $|S|, |\bar{S}| \geq 1$.

(b)$\leftrightarrow$(c). Let e be any member of $E-(B_S \cup B_{\bar{S}})$. Then $B_S \cup B_S \cup \{e\} = (B_S \cup \{e\}) \cup (B_{\bar{S}} \cup \{e\})$ and hence, either $B_S \cup \{e\}$ or $B_{\bar{S}} \cup \{e\}$ contains a circuit of $\underline{M}$. Thus $B_S \cup B_{\bar{S}}$ contains a base of $\underline{M}$. Suppose $B_S \cup B_{\bar{S}} - T$ is a base of $\underline{M}$, where T is a non-null subset of $B_S \cup B_{\bar{S}}$. Then $r(\underline{M} \times S) + r(\underline{M} \times \bar{S}) = |B_S| + |B_{\bar{S}}| = |B_S \cup B_{\bar{S}}| > |B_S \cup B_{\bar{S}} - T| = r(\underline{M})$. This is contrary to (b). Therefore $B_S \cup B_{\bar{S}}$ is a base of $\underline{M}$.

(c)$\longleftrightarrow$(d). Let $C \not\subseteq (\underline{C} \times S) \cup (\underline{C} \times \overline{S})$ be a circuit of $\underline{M}$. Let B_S and $B_{\overline{S}}$ be bases of $\underline{M} \times S$ and $\underline{M} \times \overline{S}$, such that $B_S \supseteq C \cap S$ and $B_{\overline{S}} \supseteq C \cap \overline{S}$. However, $B_S \cup B_{\overline{S}}$ contains a circuit C of $\underline{M}$, which contradicts (c). Consequently, for any circuit C of $\underline{M}$, $C \subseteq S$ or $C \subseteq \overline{S}$.

(d)$\longleftrightarrow$(e). The intersection of any circuit of $\underline{M}$ with S is null or the circuit itself. Therefore $\underline{M} \times S = \underline{M} \cdot S$.

(e)$\longleftrightarrow$(b). If $\underline{M} \times S = \underline{M} \cdot S$, then $r(\underline{M} \times S) = r(\underline{M} \cdot S)$. Accordingly, $\xi(\underline{M}; S, \overline{S}) = 1$ and (b) follows. ∎

<u>Lemma 4.1</u> Let S be a separator of $\underline{M} = (\underline{C}, E)$. If $T \subseteq E$ and $S \cap T$ is not null, then $S \cap T$ is a separator of both $\underline{M} \cdot T$ and $\underline{M} \times T$.

<u>Proof.</u> This immediately follows from Theorem 4.1(d). ∎

<u>Lemma 4.2</u> Let $\underline{M} = (\underline{C}, E)$ be connected. Then $\lambda(\underline{M}) = 2$ if and only if there exists a subset $S \subset E$, such that $\min(|S|, |\overline{S}|) \geq 2$ and $B_S \cup B_{\overline{S}} - \{e\}$ is a base of $\underline{M}$ for some $e \in B_S \cup B_{\overline{S}}$, where B_S and $B_{\overline{S}}$ are bases of $\underline{M} \times S$ and $\underline{M} \times \overline{S}$, respectively.

<u>Proof.</u> Let S be a 2-separator of $\underline{M} = (\underline{C}, E)$, and let B_S and $B_{\overline{S}}$ be bases of $\underline{M} \times S$ and $\underline{M} \times \overline{S}$. By definition, we have

$$r(\underline{M}) = -\xi(\underline{M}; S, \overline{S}) + r(\underline{M} \times S) + r(\underline{M} \times \overline{S}) + 1$$
$$= |B_S| + |B_{\overline{S}}| - 1 = |B_S \cup B_{\overline{S}}| - 1.$$

Therfore, $B_S \cup B_{\overline{S}} - \{e\}$ is a base of $\underline{M}$ for some $e \in B_S \cup B_{\overline{S}}$.

Now suppose $\min(|S|, |\overline{S}|) \geq 2$ and that B_S and $B_{\overline{S}}$ are bases of $\underline{M} \times S$ and $\underline{M} \times \overline{S}$, such that $B_S \cup B_{\overline{S}} - \{e\}$ is a base of $\underline{M}$, where $e \in B_S \cup B_{\overline{S}}$. Then

$$r(\underline{M})=|B_S \cup B_{\overline{S}} - \{e\}| - 1 = |B_S| + |B_{\overline{S}}| - 1$$
$$= r(\underline{M} \times S) + r(\underline{M} \times \overline{S}) - 1.$$

Consequently, $\lambda(\underline{M})=2.$ $\blacksquare$

__Lemma 4.3__ Let B be a base of $\underline{M}=(\underline{C}, E)$ and $e \in E-B$. Then $B \cup \{e\}$ contains the only circuit $C_0=\{e' \in B \cup \{e\} \mid B \cup \{e\}-\{e'\}$ is a base of $\underline{M}\}$.

__Proof.__ By definition, $B \cup \{e\}$ contains a circuit of $\underline{M}$. Let C_1 and C_2 be distinct circuits of $\underline{M}$ contained in $B \cup \{e\}$. By Axiom II, $C_1 \cup C_2 - \{e\} (\subseteq B)$ contains the third circuit of $\underline{M}$. However, this is impossible because B is a base of $\underline{M}$. Accordingly, $B \cup \{e\}$ contains only one circuit C of $\underline{M}$.

To show $C_0=C$, we choose any $e' \in C$. Since $B \cup \{e\} - \{e'\}$ is a base of $\underline{M}$, we have $e' \in C_0$. Hence, $C \subseteq C_0$. Now if we choose any $e' \in C_0$, then $B'=B \cup \{e\} - \{e'\}$ is a base of $\underline{M}$. Thus $B' \cup \{e'\} = B \cup \{e\}$ contains only one circuit C', and hence, $e' \in C'=C$. Thus we also have $C_0 \subseteq C$. Therefore $B \cup \{e\}$ contains the only circuit C_0 of $\underline{M}$. $\blacksquare$

Let $\underline{M}=(\underline{C}, E)$ be a matroid and $S \subseteq E$. We use the following convention: $(\underline{C} \mid S)=\underline{C} - (\underline{C} \times S) \cup (\underline{C} \times \overline{S})$.

__Lemma 4.4__ Let S be a 2-separator of $\underline{M}$, and let B_S and $B_{\overline{S}}$ be bases of $\underline{M} \times S$ and $\underline{M} \times \overline{S}$. Then $B_S \cup B_{\overline{S}}$ contains exactly one circuit C of $\underline{M}$ and $C \in (\underline{C} \mid S)$.

__Proof.__ By Lemma 4.2, there always exists such a circuit C, and the uniqueness of C is obtained from Lemma 4.3. If the circuit C is a subset of S or $\bar{S}$, C has a common element with $B^*_S \cup B^*_{\bar{S}}$, where $B^*_S = S - B_S$ and $B^*_{\bar{S}} = \bar{S} - B_{\bar{S}}$ are cobases. Then $C \cap (B^*_S \cup B^*_{\bar{S}}) = C \cap (E - (B_S \cup B_{\bar{S}})) \neq \phi$. This contradicts the fact that C is a subset of $B_S \cup B_{\bar{S}}$. Thus $C \in (\underline{C}|S)$. ■

<u>Lemma 3.5</u> Let S be a 2-separator of $\underline{M}$ and $C \in (\underline{C}|S)$. Then $\mu(\underline{M} \times (S \cup C)) = \mu(\underline{M} \times S) + 1$ and $\mu(\underline{M} \times (\bar{S} \cup C)) = \mu(\underline{M} \times \bar{S}) + 1$.

<u>Proof.</u> By Theorem 2.12(d)

$$\mu(\underline{M} \times (S \cup C)) \geq \mu(\underline{M} \times S) + \mu(\underline{M} \times C) - \mu(\underline{M} \times (S \cap C))$$
$$= \mu(\underline{M} \times S) + 1.$$

Suppose $\mu(\underline{M} \times (S \cup C)) > \mu(\underline{M} \times S) + 1$. Since $\mu(\underline{M} \times (\bar{S} \cup C)) \geq \mu(\underline{M} \times \bar{S}) + 1$, we have

$$\mu(\underline{M}) = \mu(\underline{M} \times ((S \cup C) \cup (\bar{S} \cup C))$$
$$\geq \mu(\underline{M} \times (S \cup C)) + \mu(\underline{M} \times (\bar{S} \cup C)) - \mu(\underline{M} \times C)$$
$$= \mu(\underline{M} \times (S \cup C)) + \mu(\underline{M} \times (\bar{S} \cup C)) - 1$$
$$\geq \mu(\underline{M} \times S) + 1 + \mu(\underline{M} \times \bar{S}) + 1 - 1$$
$$= \mu(\underline{M} \times S) + \mu(\underline{M} \times \bar{S}) + 1 = \mu(\underline{M}) - \mu(\underline{M};S,\bar{S}) + 2.$$

However, S is a 2-separator of $\underline{M}$, i.e., $\xi(\underline{M};S,\bar{S}) = 2$. Accordingly, $\mu(\underline{M}) > \mu(\underline{M}) - 2 + 2 = \mu(\underline{M})$. This is a contradiction. Therefore $\mu(\underline{M} \times (S \cup C)) = \mu(\underline{M} \times S) + 1$. Similarly, we have $\mu(\underline{M} \times (\bar{S} \cup C)) = \mu(\underline{M} \times \bar{S}) + 1$. ■

This result leads to the next lemma.

<u>Lemma 4.6</u> Let S be a 2-separator of $\underline{M}$ and $C \in (\underline{C} \setminus S)$. If a circuit of $\underline{M} \times (S \cup C)$ has a non-null intersection with C-S, then it contains C-S.

<u>Proof.</u> Let $C' \in \underline{C} \times (S \cup C)$ and $C' \cap (C-S) \neq \phi$. Suppose there is an element $e \in (C-C') \cap S$. By Lemma 4.5

$$\mu(\underline{M} \times S) = \mu(\underline{M} \times (S \cup C)) - 1 = \mu(\underline{M} \times (S \cup C - \{e\}))$$
$$\geq \mu(\underline{M} \times (S \cup C')) = \mu(\underline{M} \times S) + 1.$$

This is a contradiction. Therefore $C' \supseteq C-S$. $\blacksquare$

<u>Lemma 4.7</u> Let S be a 2-separator of $\underline{M}=(\underline{C}, E)$ and $C \in (\underline{C} | S)$. Then $\underline{M} \times (S \cup C)$ and $\underline{M} \times (\bar{S} \cup C)$ are connected.

<u>Proof.</u> Suppose $\underline{M} \times (S \cup C)$ is separable. Let T be a separator of $\underline{M} \times (S \cup C)$. Since $\underline{M} \times (S \cup C)$ has no circuit which consists of one cell, $|T| \geq 2$. We can choose T so that $T \cap C = \phi$, because T is a separator of $\underline{M} \times (S \cup C)$ if and only if $S \cup C - T$ is a separator of $\underline{M} \times (S \cup C)$. By Lemma 4.1, T is also a separator of $\underline{M} \times S$. Then by Theorem 4.1(b) $\mu(\underline{M} \times T) + \mu(\underline{M} \times (S-T)) = \mu(\underline{M} \times S)$.

We show T is a separator of $\underline{M}$. By Theorem 2.13

$$\xi(\underline{M}; T, \bar{T}) = \mu(\underline{M}) - \mu(\underline{M} \times T) - \mu(\underline{M} \times \bar{T}) + 1$$
$$= \mu(\underline{M}) - \mu(\underline{M} \times S) + \mu(\underline{M} \times (S-T)) - \mu(\underline{M} \times \bar{T}) + 1.$$

By Theorem 2.12(d)

$$\mu(\underline{M} \times \bar{T}) = \mu(\underline{M} \times (\bar{S} \cup C) \cup (S-T))$$
$$\geq \mu(\underline{M} \times (\bar{S} \cup C)) + \mu(\underline{M} \times (S-T)) - \mu(\underline{M} \times ((\bar{S} \cup C) \cap (S-T)))$$
$$= \mu(\underline{M} \times \bar{S}) + \mu(\underline{M} \times (S-T)) + 1.$$

Since S is a 2-separator of $\underline{M}$, we have

$$\xi(\underline{M};T,\overline{T}) \leq \mu(\underline{M}) - \mu(\underline{M} \times S) - \mu(\underline{M} \times \overline{S})$$
$$= \xi(\underline{M};S,\overline{S}) - 1 = 2 - 1 = 1.$$

This shows T is a separator of $\underline{M}$, but it contradicts $\lambda(\underline{M})=2$. ∎

The following lemma is a direct result of the definition of 2-separators.

Lemma 4.8 A 2-separator of $\underline{M}$ is also a 2-separator of $\underline{M}^*$.

Proof. $\xi(\underline{M}^*;S,\overline{S}) = |S| - \mu(\underline{M} \times S) - \mu(\underline{M}^* \times S) + 1 = \xi(\underline{M};S,\overline{S})$, and $\min(|S|,|\overline{S}|) \geq 2$. ∎

Connected minors of a matroid with $\lambda=2$ will have an important role in subsequent sections. We will now prove a useful theorem for matroids, which will be frequently referred to later.

Theorem 4.2 Let S be a 2-separator of $\underline{M}=(\underline{C}, E)$, and let $\underline{M}_A=(\underline{C}_A, E)$, where $A \subseteq E$, be a connected minor of $\underline{M}$, such that $|S \cap A|$, $|\overline{S} \cap A| \geq 2$. Then $\lambda(\underline{M}_A)=2$ and $S \cap A$, $\overline{S} \cap A$ are 2-separators of $\underline{M}_A$.

Proof. Case 1. $\underline{M}_A = \underline{M} \times A$.

Let B_1 and B_2 be bases of $\underline{M} \times (A \cap S)$ and $\underline{M} \times (A \cap \overline{S})$, and let B_S and $B_{\overline{S}}$ be bases of $\underline{M} \times S$ and $\underline{M} \times \overline{S}$, such that $B_S \supseteq B_1$ and $B_{\overline{S}} \supseteq B_2$. Since $\underline{M} \times A$ is connected, $B_1 \cup B_2$ contains at least one circuit of $\underline{M} \times A$ by Theorem 4.1. And $B_S \cup B_{\overline{S}}$ ($\supseteq B_1 \cup B_2$) contains precisely one circuit of $\underline{M}$ by Lemma 4.4. Therefore $B_1 \cup B_2$ contains only one circuit of $\underline{M} \times A$. Since $B_1 \cup B_2$ contains a base of $\underline{M} \times A$, $B_1 \cup B_2 - \{e\}$ is a base of $\underline{M} \times A$ for some $e \in B_1 \cup B_2$ and $A \cap S$, $A \cap \bar{S}$ are 2-separators of $\underline{M} \times A$. Thus the theorem follows by Lemma 4.2.

Case 2. $\underline{M}_A = \underline{M} \cdot A$.

By Lemma 4.8 S is a 2-separator of $\underline{M}^* = (\underline{C}^*, E)$ and $\underline{M}^* \times A$ is a connected minor of $\underline{M}^*$ by Theorem 2.14. Therefore $(S \cap A)$ and $(\bar{S} \cap A)$ are 2-separators of $\underline{M}^* \times A$ as shown in Case 1, and they are also 2-separators of $\underline{M}_A = (\underline{M}^* \times A)^*$ by Lemma 4.8.

Case 3. $\underline{M}_A = (\underline{M} \times B) \cdot A$, where $A \subsetneq B \subsetneq E$.

Since $\underline{M}_A$ is connected, we can choose $\underline{M} \times B$ to be connected. From Case 1, $\lambda(\underline{M} \times B) = 2$, since $|S \cap B| \geq |S \cap A| \geq 2$ and $|\bar{S} \cap B| \geq |\bar{S} \cap A| \geq 2$. Since $S \cap B$ and $\bar{S} \cap B$ are 2-separators of $\underline{M} \times B$, apply Case 2 to show $\lambda(\underline{M}_A) = 2$. Here $(S \cap B) \cap A = S \cap A$ and $(\bar{S} \cap B) \cap A = \bar{S} \cap A$ are 2-separators of $\underline{M}_A$.

If a minor $\underline{M}_A$ is of the form $(\underline{M} \cdot B) \times A$, apply the same argument as in Case 3 for $\underline{M}^*_A = (\underline{M}^* \times B) \cdot A$. ∎

<u>Lemma 4.9</u> Let S be a 2-separator of $\underline{M} = (\underline{C}, E)$. Let $C_1, C_2 \in (\underline{C} \setminus S)$ be circuits of $\underline{M}$, such that $(C_1 - C_2) \cap S \neq \phi$. Then $(C_1 \cup C_2) \cap S$ contains a circuit of $\underline{M}$.

<u>Proof.</u> Since $\underline{M} \times (S \cup C_1)$ is connected, by Lemma 4.7, and $\bar{S} \cap C_2 \neq \phi$, a matroid $\underline{M} \times (S \cup C_1 \cup C_2)$ is connected. Suppose $|(C_1 \cup C_2) \cap \bar{S}| = 1$. Let $C_1 \cap \bar{S} = C_2 \cap \bar{S} = \{e\}$. By Axiom II, there exists a circuit C_3 of $\underline{M}$, such that $C_3 \subseteq C_1 \cup C_2 - \{e\} \subseteq (C_1 \cup C_2) \cap S$. The theorem follows.

Suppose $|(C_1 \cup C_2) \cap \bar{S}| \geq 2$. Then $(C_1 \cup C_2) \cap S$ is a

2-separator of $\underline{M} \times (C_1 \cup C_2 \cup \bar{S})$ by Theorem 4.2. Then

$$2 = \xi(\underline{M} \times (C_1 \cup C_2 \cup \bar{S}); (C_1 \cup C_2) \cap S, \bar{S})$$

$$= \mu(\underline{M} \times (C_1 \cup C_2 \cup \bar{S})) - \mu(\underline{M} \times ((C_1 \cup C_2) \cap S)) - \mu(\underline{M} \times \bar{S}) + 1$$

$$\geq \mu(\underline{M} \times C_1) + \mu(\underline{M} \times (C_2 \cup \bar{S})) - \mu(\underline{M} \times (C_1 \cap (C_2 \cup \bar{S})$$

$$- \mu(\underline{M} \times ((C_1 \cup C_2) \cap S)) - \mu(\underline{M} \times \bar{S}) + 1$$

$$= 1 + \mu(\underline{M} \times \bar{S}) + 1 - \mu(\underline{M} \times ((C_1 \cup C_2) \cap S))) - \mu(\underline{M} \times \bar{S}) + 1$$

$$= - \mu(\underline{M} \times ((C_1 \cup C_2) \cap S)) + 3.$$

Hence $\mu(\underline{M} \times (((C_1 \cup C_2) \cap S)) \geq 1$. Therefore $(C_1 \cup C_2) \cap S$ contains a circuit of $\underline{M}$. ∎

The following theorem is the main theorem in this section, and it will make possible our matroid decomposition.

<u>Theorem 4.3</u> Let S be a 2-separator of $\underline{M} = (\underline{C}, E)$, and let C_1, C_2 $(\underline{C} \| S)$ be circuits of $\underline{M}$. Then $(C_1 \cap S) \cup (C_2 \cap \bar{S})$ is a circuit of $\underline{M}$.

<u>Proof.</u> If $C_1 \cap S = C_2 \cap S$ or $C_1 \cap \bar{S} = C_2 \cap \bar{S}$, then $(C_1 \cap S) \cup (C_2 \cap \bar{S}) = (C_2 \cap S) \cup (C_2 \cap \bar{S}) = C_2$, or $(C_1 \cap S) \cup (C_2 \cap \bar{S}) = (C_1 \cap S) \cup (C_1 \cap \bar{S}) = C_1$. Either case satisfies the theorem.

Suppose neither $C_1 \cap S = C_2 \cap S$ nor $C_1 \cap \bar{S} = C_2 \cap \bar{S}$. We have $\|(C_1 \cup C_2) \cap S\|$, $\|(C_1 \cup C_2) \cap \bar{S}\| \geq 2$. We shall show that $\underline{M} \times (C_1 \cup C_2)$ is connected. Suppose $\underline{M} \times (C_1 \cup C_2)$ is separable and T is a separator of $\underline{M} \times (C_1 \cup C_2)$. By Theorem 4.1, T must contain C_1 or C_2, and also the third circuit $C_3 \subseteq (C_1 \cup C_2) \cap S$, by Lemma 4.9. Since $C_1 \cap C_3 \neq \phi$ and $C_2 \cap C_3 \neq \phi$, $T \supseteq C_1 \cup C_2$. However, this contradicts the fact that T is a separator. Thus $\underline{M} \times (C_1 \cup C_2)$ is connected.

Let $X=(C_1 \cap S) \cup (C_2 \cap \bar{S})$ and $Y=C_1 \cup C_2$. We shall show $\mu(\underline{M} \times X) \geq 1$. By Theorem 4.2 $\xi(\underline{M} \times Y; Y \cap S, Y \cap \bar{S})=2$. Since $X= (C_1 \cap S) \cup (C_2 \cap \bar{S}) = Y \cap (C_1 \cup \bar{S}) \cap (S \cup C_2)$, by Theorem 2.12(d), $\mu(\underline{M} \times X) \geq \mu(\underline{M} \times (Y \cap (C_1 \cup S))) + \mu(\underline{M} \times (S \cup C_2))$ $- \mu(\underline{M} \times ((Y \cap (C_1 \cup \bar{S})) \cup (S \cup C_2)))$. Since $(Y \cap (C_1 \cup \bar{S})) \cup (S \cup C_2)=$ $Y \cup S$, we have

$$\begin{aligned}\mu(\underline{M} \times X) \geq\ & \mu(\underline{M} \times Y) + \mu(\underline{M} \times (C_1 \cup \bar{S}) - \mu(\underline{M} \times (Y \cup \bar{S})) \\ & + \mu(\underline{M} \times (S \cup C_2)) - \mu(\underline{M} \times (Y \cup S)) \\ =\ & \mu(\underline{M} \times Y) + \mu(\underline{M} \times \bar{S}) + 1 - \mu(\underline{M} \times (Y \cup \bar{S})) \\ & + \mu(\underline{M} \times S) + 1 - \mu(\underline{M} \times (Y \cup S)) \\ =\ & \mu(\underline{M} \times Y) + \mu(\underline{M} \times S) + \mu(\underline{M} \times \bar{S}) - \mu(\underline{M} \times (Y \cup \bar{S})) \\ & - \mu(\underline{M} \times (Y \cup S)) + 2.\end{aligned}$$

However, $\underline{M} \times (Y \cup S)$ and $\underline{M} \times (Y \cup \bar{S})$ are connected because $\underline{M} \times Y$ is connected and $C_1 \cap S \neq \phi$, $C_2 \cap S \neq \phi$. Therefore $Y \cap S$ and $Y \cap \bar{S}$ are 2-separators of those matroids. Thus we have

$$\begin{aligned}\xi(\underline{M} \times (Y \cup \bar{S}); Y \cap S, \bar{S}) = 2 = & \mu(\underline{M} \times (Y \cup \bar{S})) - \mu(\underline{M} \times (Y \cap S)) \\ & - \mu(\underline{M} \times \bar{S}) + 1,\end{aligned}$$

and hence, $\mu(\underline{M} \times (Y \cup \bar{S})) = \mu(\underline{M} \times (Y \cap S)) + \mu(\underline{M} \times \bar{S}) + 1$.

Similarly, since

$$\begin{aligned}\xi(\underline{M} \times (Y \cup S); Y \cap \bar{S}, S)=2= & \mu(\underline{M} \times (Y \cup S)) - \mu(\underline{M} \times (Y \cap \bar{S})) \\ & - \mu(\underline{M} \times S) + 1,\end{aligned}$$

we have $\mu(\underline{M} \times (Y \cup S))= \mu(\underline{M} \times (Y \cap \bar{S})) + \mu(\underline{M} \times S) + 1$. Therefore,

$$\begin{aligned}\mu(\underline{M} \times X) \geq\ & \mu(\underline{M} \times Y) + \mu(\underline{M} \times S) + \mu(\underline{M} \times \bar{S}) - \mu(\underline{M} \times (Y \cap S)) \\ & - \mu(\underline{M} \times \bar{S}) - 1 - \mu(\underline{M} \times (Y \cap \bar{S})) - \mu(\underline{M} \times S) - 1 + 2\end{aligned}$$

$$= \mu(\underline{M} \times Y) - \mu(\underline{M} \times (Y \cap S)) - \mu(\underline{M} \times (Y \cap \overline{S}))$$

$$= \xi(\underline{M} \times Y; Y \cap S, Y \cap \overline{S}) - 1 = 2 - 1 = 1.$$

Consequently, X contains a circuit of $\underline{M}$. Suppose C_3 is a circuit of $\underline{M}$ which is properly contained in X. Then $(C_1 - C_3) \cap S \neq \phi$ or $(C_2 - C_3) \cap \overline{S} \neq \phi$. By Lemma 4.9, either $(C_1 \cup C_3) \cap S = C_1 \cap S$, or $(C_2 \cup C_3) \cap \overline{S} = C_2 \cap \overline{S}$ contains a circuit, which is a contradiction. Thus $X = (C_1 \cap S) \cup (C_2 \cap \overline{S})$ is a circuit of $\underline{M}$. ∎

4.3 ISOMORPHISM AND EQUIVALENCE

Since our aim in this chapter is the decomposition of matroids of connectivity 2 into 3-connected minors, it is necessary to clarify the relationship between 2-separators and 3-connected minors of matroids. This section is devoted to the study of isomorphic 3-connected minors of matroids.

Isomorphism of matroids, defined in Section 2.2, may also be defined by a 1-1 correspondence between the bases of two matroids.

Lemma 4.10 Let $\underline{M}_1=(\underline{C}_1, E_1)$ and $\underline{M}_2=(\underline{C}_2, E_2)$ be matroids. Then $\underline{M}_1 \cong \underline{M}_2$ if and only if there exists a 1-1 mapping f of E_1 onto E_2, such that B is a base of $\underline{M}_1$ if and only if f(B) is a base of $\underline{M}_2$.

Proof. Let $\underline{M}_1 \cong \underline{M}_2$. Let f be an isomorphic mapping of E_1 onto E_2, such that C is a circuit of $\underline{M}_1$ if and only if f(C) is a circuit of $\underline{M}_2$. Suppose B is a base of $\underline{M}_1$. If f(B) contains a circuit C_2 of $\underline{M}_2$, then $f^{-1}(C_2)$ ($\subseteq$ B) is a circuit of $\underline{M}_1$, which is contrary to the definition of B as a base. Therefore f(B) is independent in $\underline{M}_2$. Let e_2 be any element of E_2 - f(B). Then $f^{-1}(f(B) \cup \{e_2\}) = B \cup f^{-1}(e_2)$. Since $f^{-1}(e_2) \notin B$, $B \cup f^{-1}(e_2)$ contains a circuit of $\underline{M}_1$, and hence, $f(B) \cup \{e_2\}$ contains a circuit of $\underline{M}_2$. Thus f(B) is a base of $\underline{M}_2$. Similarly, if B_2 is a base of $\underline{M}_2$, then $f^{-1}(B_2)$ is a base of $\underline{M}_1$.

We leave the proof of sufficiency to the reader. ∎

The following lemma is immediately obtained from Lemma 4.10.

Lemma 4.11 If $\underline{M}_1 \cong \underline{M}_2$, then $r(\underline{M}_1)=r(\underline{M}_2)$ and $\mu(\underline{M}_1)=\mu(\underline{M}_2)$.

The next results may be proved without difficulty.

Lemma 4.12 Let $\underline{M}_1=(\underline{C}_1, E_1) \cong \underline{M}_2=(\underline{C}_2, E_2)$, and let f be an isomorphic mapping of E_1 onto E_2. Let $T \subsetneq S \subseteq E_1$. Then

(a) $\underline{M}_1^* \cong \underline{M}_2^*$

(b) $\underline{M}_1 \times S \cong \underline{M}_2 \times f(S)$

(c) $\underline{M}_1 \cdot S \cong \underline{M}_2 \cdot f(S)$

(d) $(\underline{M}_1 \times S) \cdot T \cong (\underline{M}_2 \times f(S)) \cdot f(T)$

(e) $(\underline{M}_1 \cdot S) \times T \cong (\underline{M}_2 \cdot f(S)) \times f(T)$

Lemma 4.13 If $\underline{M}_1 \cong \underline{M}_2$, then $\lambda(\underline{M}_1)=\lambda(\underline{M}_2)$.

Proof. Let f be an isomorphic mapping of E_1 onto E_2. Then by Lemmas 4.11 and 4.12,

$$\begin{aligned} \xi(\underline{M}_1; S, \overline{S}) &= \mu(\underline{M}_1 \cdot S) - \mu(\underline{M}_1 \times S) + 1 \\ &= \mu(\underline{M}_2 \cdot f(S)) - \mu(\underline{M}_2 \times f(S)) + 1 \\ &= \xi(\underline{M}_2; f(S), \overline{f(S)}), \end{aligned}$$

and $\min(|S|, |\overline{S}|) = \min(|f(S)|, |\overline{f(S)}|)$. Thus the theorem follows. ∎

Two cells e and e' of $\underline{M}$ are in series if no circuit of $\underline{M}$ contains only one of the cells. A subset S of E is called a series set of $\underline{M}$ if $|S| \geq 2$ and if every pair of members of S are in series. A series set S is maximal if no series set exists which properly contains S.

Let S be a 2-separator of $\underline{M}$. S is called a (maximal) series separator of $\underline{M}$ if S is a (maximal) series set of $\underline{M}$, and if S is not a series separator we call S a non-series separator of $\underline{M}$. Thus series and non-series separators are defined only for matroids of connectivity two.

Lemma 4.14 Any distinct series sets of $\underline{M}$ are mutually disjoint.

Let S_i, $1 \leq i \leq k$, be the maximal series sets of $\underline{M}=(\underline{C}, E)$ and $e_i \in S_i$ for $1 \leq i \leq k$. Then $\underline{M}_R=(\underline{C}_R, E_R)=\underline{M}\cdot(E-\bigcup_{i=1}^{k}(S_i-\{e_i\}))$ is called a series-reduced matroid of $\underline{M}$.

The next theorem shows that a series-reduced matroid of $\underline{M}$ is uniquely determined up to isomorphism.

Theorem 4.4 Two series-reduced matroids of $\underline{M}$ are isomorphic.

Proof. Let S_i, $1 \leq i \leq k$, be the maximal series sets of $\underline{M}=(\underline{C}, E)$. If k=0, then the theorem is obviously true. If $k \geq 1$, we divide the proof into steps:

Step 1. Let $\underline{M}_R=(\underline{C}_R, E_R)=\underline{M}\cdot E_R$ be a series-reduced matroid of $\underline{M}$, where $E_R=E-\bigcup_{i=1}^{k}(S_i-\{e_i\})$ and $e_i \in S_i$ for $1 \leq i \leq k$. We show that if $C \in \underline{C}$, then $C \cap E_R \in \underline{C}_R$. Suppose $C \in \underline{C}$ and $C \cap E_R \notin \underline{C}_R$. By definition, there exists a circuit C' of $\underline{M}$, such that $C' \cap E_R \subset C \cap E_R$. Since C' and C are distinct circuits of $\underline{M}$, we can find a cell e of $\underline{M}$ so that $e \in (C'-C) \cap (E-E_R)=(C'-C)\cap(\bigcup_{i=1}^{k}(S_i-\{e_i\}))$. If $e \in S_j$, by the definition of series sets $S_j \subset C'$, and hence, $e_j \in C' \cap E_R$.

However, by Lemma 4.14, $C \cap S_j = \phi$, which contradicts $C' \cap E_R \subset C \cap E_R$. Therefore, $C \cap E_R \in \underline{C}_R$ for every circuit C of $\underline{M}$. Clearly, if $C \in \underline{C}$ and if $C \cap (\bigcup_{i=1}^{k} S_i) = \phi$, then $C \in \underline{C}_R$. Since the converse of this statement is also true, for any circuit of $\underline{M}_R$, $C \cup (\bigcup_{i \in I} S_i)$ is a circuit of $\underline{M}$, where $I = \{ i \mid C \cap S_i \neq \phi \text{ and } 1 \leq i \leq k \}$.

Step 2. Let $\underline{M}_1 = (\underline{C}_1, E_1) = \underline{M} \cdot E_1$ and $\underline{M}_2 = (\underline{C}_2, E_2) = \underline{M} \cdot E_2$ be series-reduced matroids of $\underline{M}$, where $E_1 = E - \bigcup_{i=1}^{k} (S_i - \{e_i\})$, $E_2 = E - \bigcup_{i=1}^{k} (S_i - \{e'_i\})$, and $e_i, e''_i \in S_i$ for $1 \leq i \leq k$.

To prove $\underline{M}_1 \cong \underline{M}_2$, we define the following 1-1 mapping of E_1 onto E_2:

$$f(e) = \begin{cases} e & \text{for } e \in E_1 - \bigcup_{i=1}^{k} \{e_i\} \\ e''_i & \text{for } e = e_i \in \bigcup_{i=1}^{k} \{e_i\} \text{ and } 1 \leq i \leq k. \end{cases}$$

Let $C \in \underline{C}_1$ and $I = \{ i \mid C \cap S_i \neq \phi \text{ and } 1 \leq i \leq k \}$. As shown in Step 1, $C \cup (\bigcup_{i \in I} S_i)$ is a circuit of $\underline{M}$, and $(C \cup (\bigcup_{i \in I} S_i)) \cap E_2 = (C \cap E_2) \cup ((\bigcup_{i \in I} S_i) \cap E_2) = (C \cap E_2) \cup (\bigcup_{i \in I} \{e''_i\}) = f(C)$ is a circuit of $\underline{M}_2$. Since it is not hard to see that $f^{-1}(C) \in \underline{C}_1$, for any $C \in \underline{C}_2$, we have $\underline{M}_1 \cong \underline{M}_2$. Thus the theorem follows. ∎

Two matroids are called <u>equivalent</u> if their series-reduced matroids are isomorphic. Thus a series-reduced matroid of $\underline{M}$ is equivalent to $\underline{M}$. The concept of equivalence of matroids corresponds to 2-homeomorphism in graph theory.

<u>Theorem 4.5</u> Let $\underline{M}_1 = (\underline{C}_1, E_1)$ and $\underline{M}_2 = (\underline{C}_2, E_2)$ be equivalent, and let $\underline{S}_1$ and $\underline{S}_2$ be the class of the union of members of $\underline{C}_1$ and $\underline{C}_2$.

Then there is a 1-1 mapping g of $\underline{S}_1$ onto $\underline{S}_2$, such that $\mu(\underline{M}_1 \times S) = \mu(\underline{M}_2 \times g(S))$ for any $S \in \underline{S}_1$.

Proof. By definition, series-reduced matroids of $\underline{M}_1$ and $\underline{M}_2$ are isomorphic. Therefore it suffices to prove the thorem for a matroid $\underline{M}=(\underline{C}, E)$ and its series-reduced matroid $\underline{M}_R=(\underline{C}_R, E_R)$.

Let $E_R = E - \bigcup_{i=1}^{k}(S_i - \{e_i\})$, where S_i, $1 \leqslant i \leqslant k$, are the maximal series sets of $\underline{M}$ and $e_i \in S_i$ for $1 \leqslant i \leqslant k$. We will show that if C_1 and C_2 are distinct circuits of $\underline{M}$, then $C_1 \cap E_R$ and $C_2 \cap E_R$ are distinct in $\underline{C}_R$. We have shown $C \cap E_R \in \underline{C}_R$ for any $C \in \underline{C}$ in the proof of Theorem 4.4.

Suppose C_1 and C_2 are members of $\underline{C}$ and $C_1 \cap E_R = C_2 \cap E_R$ $\underline{C}_R$. Then $C_1 \cap (\bigcup_{i=1}^{k} S_i) = C_2 \cap (\bigcup_{i=1}^{k} S_i)$ by the definition of series sets. Therefore $C_1 = C_1 \cap (E_R \cup (\bigcup_{i=1}^{k} S_i)) = (C_1 \cap E_R) \cup (C_1 \cap (\bigcup_{i=1}^{k} S_i)) = (C_2 \cap E_R) \cup (C_2 \cap (\bigcup_{i=1}^{k} S_i)) = C_2 \cap (E_R \cup (\bigcup_{i=1}^{k} S_i)) = C_2$. Thus distinct members of $\underline{C}$ correspond to distinct members of $\underline{C}_R$. Similarly, it is easy to show that distinct members of $\underline{C}_R$ correspond to distinct members of $\underline{C}$.

Let $\underline{S}$ and $\underline{S}_R$ be the classes of all unions of members of $\underline{C}$ and $\underline{C}_R$, respectively. As shown above we can define a 1-1 mapping f of $\underline{S}$ onto $\underline{S}_R$ by

$$f(C) = C \cap E_R \text{ for } C \in \underline{C}, \quad f^{-1}(C) = C \cup (\bigcup_{i \in I} S_i) \text{ for } C \in \underline{C}_R,$$

where $I = \{i \mid C \cap S_i \neq \phi \text{ and } 1 \leqslant i \leqslant k\}$.

Let $S = \bigcup_x C_x$ be any member of $\underline{S}$. We define a mapping g of $\underline{S}$ onto $\underline{S}_R$ by $g(S) = \bigcup_x f(C_x) = \bigcup_x (C_x \cap E_R) = S \cap E_R \in \underline{S}_R$. We show that g is a 1-1 and onto mapping. Let $S_a = \bigcup_x C_x$ and $S_b = \bigcup_y C_y$ be distinct members of of $\underline{S}$. If $e \in (S_a - S_b) \cap (\bigcup_{i=1}^{k} S_i)$, then $e \in S_i$ for some i and $e \in \bigcup_x C_x$, $e \notin \bigcup_y C_y$. Thus we have $e \in E_R \cap (\bigcup_x C_x)$ and $e \notin E_R \cap (\bigcup_y C_y)$. If $e \in (S_a - S_b) \cap (E - \bigcup_{i=1}^{k} S_i)$, then $e \in E_R$, $e \in \bigcup_x C_x$, and $e \notin \bigcup_y C_y$. Hence $e \in E_R \cap (\bigcup_x C_x)$ and $e \notin E_R \cap (\bigcup_y C_y)$. Therefore $g(S_a) = S_a \cap E_R$ and $g(S_b) = S_b \cap E_R$ are distinct members of $\underline{S}_R$, that is, g is a 1-1 mapping.

Now let $T = \bigcup_x C_x \in \underline{S}_R$. Since we can write $C_x = C_x' \cap E_R = f(C_x')$, where $C_x' \in \underline{C}$, $T = \bigcup_x f(C_x')$, and hence, $\bigcup_x C_x' \in g^{-1}(T)$. Thus g is an onto mapping.

Now we prove the last part of the theorem. Let $S = \bigcup_x C_x \in \underline{S}$ and $S \cap (\bigcup_{i=1}^{k} S_i) = \bigcup_{i \in I} S_i$, where $I = \{i \mid S \cap S_i \neq \phi \text{ and } 1 \leq i \leq k\}$. By Theorems 2.8(e) and 2.12(c):

$$\begin{aligned}
\mu(\underline{M}_R \times g(S)) &= \mu(\underline{M}_R \times (E_R \cap S)) = \mu((\underline{M}\ E_R) \times (E_R \cap S)) \\
&= \mu((\underline{M} \times (E-(E_R - S)))\ (E_R \cap S)) \\
&= \mu\ (\underline{M} \times (E-(E_R - S))) \\
&\quad - \mu((\underline{M} \times (E-(E_R - S))) \times (\bigcup_{i \in I} (S_i - \{e_i\}))) \\
&= \mu(\underline{M} \times (S \cup (\bigcup_{i=1}^{k} (S_i - \{e_i\})))) - \mu(\underline{M} \times (\bigcup_{i \in I} (S_i - \{e_i\}))) \\
&= \mu(\underline{M} \times S) - 0 = \mu(\underline{M} \times S),
\end{aligned}$$

since no circuit of $\underline{M}$ contains $S_i - \{e_i\}$ but not e_i. Thus the proof is complete. ∎

The following theorem gives a relationship between 2-separators and 3-connected minors of a matroid.

<u>Theorem 4.6</u> Let S be a 2-separator of $\underline{M} = (\underline{C}, E)$ and $C \in (\underline{C} \mid S)$. Let $\underline{M}_A = (\underline{C}_A, A)$ be a 3-connected minor of $\underline{M}$. Then if $A \subseteq S$ or $A \subseteq \overline{S}$,

then $\underline{M}_A$ is a minor of $\underline{M} \times (S \cup C)$ or $\underline{M} \times (\bar{S} \cup C)$, and if $e \in S \cap A$, then there exists a minor $\underline{M}' = (\underline{C}', A')$ of $\underline{M} \times (S \cup C)$ such that $\underline{M}_A \cong \underline{M}'$, where $A' = (A - \{e\}) \cup \{e'\}$ and $e' \in \bar{S} \cap C$.

Proof. If $|A| = 1$, then the theorem is obviously true.

Suppose $|A| \geq 2$. By Theorems 2.8(e) and 4.2, without loss of generality, we can write $\underline{M}_A = (\underline{M} \times B) \cdot A$, where $|A \cap \bar{S}| \leq 1$. For a given $\underline{M}_A$, choose B to be minimal. Then $\underline{M} \times B$ is connected. If $B \cap \bar{S} = \phi$, the theorem is true.

Let us assume $B \cap \bar{S} \neq \phi$. Suppose $\underline{M} \times B$ contains distinct circuits C_1 and C_2 of $\underline{M}$, such that $(C_1 - C_2) \cap \bar{S} \neq \phi$ and $(C_2 - C_1) \cap \bar{S} \neq \phi$. We shall show a contradiction. Let $C_1 \supset (A \cap \bar{S})$, where $(A \cap \bar{S})$ is null or consists of a single element. Let $\underline{L}_1 = \{X \cap A \mid X \in \underline{C} \times B\}$ and $\underline{L}_2 = \{X \cap A \mid X \in \underline{C} \times ((B \cap S) \cup C_1)\}$. Clearly, $\underline{L}_1 \supseteq \underline{L}_2$. Suppose $Y \in \underline{L}_1$. Then there exists $X \in \underline{C} \times B$ such that $Y = X \cap A$. If $X \subseteq B \cap \bar{S}$, then $|Y| \leq |(B \cap \bar{S}) \cap A| = |\bar{S} \cap A| \leq 1$. Since $\underline{M}_A$ is 3-connected, $\underline{L}_1$ cannot contain a member consisting of one element. Thus $X \cap S \neq \phi$. If $X \subset S$, then $X \subset B \cap S$, and hence, $Y = X \cap A \in \underline{L}_2$. If $X \in (\underline{C} \mid S)$, by Theorem 4.3 $W = (X \cap S) \cup (C_1 \cap \bar{S})$ is a circuit of $\underline{M}$, and W is a member of $\underline{C} \times ((B \cap S) \cup C_1)$ since $W \subseteq (B \cap S) \cup C_1$. Since $C_1 \supset A \cap \bar{S}$, $Y = X \cap A = W \cap A \in \underline{L}_2$ and hence, $\underline{L}_1 \subseteq \underline{L}_2$. Thus we have $\underline{L}_1 = \underline{L}_2$, and $\underline{C}_A$ is the class of non-null minimal members of $\underline{L}_2$. This is contrary to B being minimal. Therefore we can write $\underline{M} \times B = \underline{M} \times (D \cup C')$, where $D \cap \bar{S} = \phi$ and $C' \in (\underline{C} \mid S)$ is a circuit of $\underline{M}$, such that $C' \cap S \subseteq D$ and $C' \supseteq (\bar{S} \cap A)$.

Let $\underline{L}_1 = \{C_1 \cap A \mid C_1 \in \underline{C} \times (D \cup C')$ and $\underline{L}_2 = \{C_2 \cap A' \mid C_2 \in$

$\underline{C} \times (D \cup (C \cap \overline{S}))\}$, where $A' \subseteq D \cup (C \cap \overline{S})$. Let $\underline{C}_1$ and $\underline{C}_2$ be the classes of all the non-null minimal members of $\underline{L}_1$ and $\underline{L}_2$.

Case 1. $A \cap \overline{S} = \phi$. Set $A'=A$.

By Theorem 4.3 $C_1 \cap A=((C_1 \cap S) \cup (C \cap \overline{S})) \cap A \in \underline{L}_2$ and $C_2 \cap A= ((C_2 \cap S) \cup (C' \cap \overline{S})) \cap A \in \underline{L}_1$. Hence $\underline{L}_1=\underline{L}_2$. Thus we have $\underline{C}_A=\underline{C}_1= \underline{C}_2$ and $\underline{M}_A=(\underline{C}_1, A)=(\underline{C}_2, A)=(\underline{M} \times (D \cup (C \cap \overline{S})))\cdot A=\underline{M}_2$. Since $\underline{M}_2$ is a minor of $\underline{M} \times (S \cup C)$, the theorem is proved.

Case 2. $A \cap \overline{S}=\{e\}$. Set $A'=(A-\{e\}) \cup \{e'\}$, where $e' \in C \cap \overline{S}$.

We show that for every member of $\underline{L}_1$ there corresponds a unique member of $\underline{L}_2$, and vice-versa. Let $e \in X=C_1 \cap A \in \underline{L}_1$. Then $(X-\{e\}) \cup \{e'\} =(C_1 \cap A \ -\{e\}) \cup \{e'\}=((C_1-\{e\}) \cap (A-\{e\})) \cup \{e'\} =(C_1-\{e\}) \cap A' =((C_1 \cap S) \cup (C \cap \overline{S})) \cap A' \in \underline{L}_2$ since $C_1 \cap S \subseteq D$. Suppose $X=C_1 \cap A$ and $Y=C_2 \cap A$ are distinct members of $\underline{L}_1$ containing e. Then $(X-\{e\}) \cup \{e'\}$ and $(Y-\{e\}) \cup \{e'\}$ are distinct members of $\underline{L}_2$. If $e \notin X=C_1 \cap A \in \underline{L}_1$, then $C_1 \subseteq D$, and hence, $C_1 \cap A=C_1 \cap A' \in \underline{L}_2$. If $e \notin X$, $Y \in \underline{L}_1$, then X and Y are clearly distinct in $\underline{L}_2$. The converse is likewise true.

Thus a mapping f of A onto A' defined by:

$$f(e'') = \begin{cases} e'' & \text{for } e'' \in A - \{e\} \\ e' & \text{for } e''=e \end{cases}$$

is a 1-1 and onto mapping, such that C_A is a circuit of $\underline{M}_A$ if and only if $f(C_A)$ is a circuit of $\underline{M}'=(\underline{C}_2, A')$. Therefore $\underline{M}_A \cong \underline{M}'$. Since $\underline{M}'= (\underline{M} \times (D \cup (C \cap \overline{S})))\cdot A'$ is a minor of $\underline{M} \times (S \cup C)$, the proof is complete. ∎

4.4 DECOMPOSITION OF MATROIDS

In this section we shall deal with matroid decomposition. Let $\underline{M}=(\underline{C}, E)$ be a connected matroid. Suppose S and $\overline{S}$ are complementary non-series separators of $\underline{M}$. By Lemma 4.4 there exists a circuit C of $\underline{M}$ such that $C \in (\underline{C}|S)$. Then the matroid $\underline{M}$ may be decomposed into two minors of $\underline{M}$ called blocks, namely, $\underline{M} \times (S \cup C)$ and $\underline{M} \times (\overline{S} \cup C)$. These minors are 2-connected, and they may have complementary non-series separators. If this is the case, we decompose the minors into finer minors. This process of decomposition is repeated for up-dated blocks until we obtain a class of matroids which have no pair of complementary non-series separators. The decomposition described above is called circuit decomposition or, simply, C-decomposition of $\underline{M}$.

A binomial matroid ${}_n\underline{M}_n$ is also called a 1-circuit matroid, where n is a positive integer. Let both S and $\overline{S}$ be series separators of $\underline{M}=(\underline{C}, E)$. Since $S \cup \overline{S}$ contains a circuit C, such that $C \cap S \neq \phi$ and $C \cap \overline{S} \neq \phi$, $S \cup \overline{S} = E$ is a circuit of $\underline{M}$. Thus, if $\underline{M}$ has both S and $\overline{S}$ as series separators, then it is a 1-circuit matroid.

A matroid is called semi-3-connected if its series-reduced matroid is 3-connected. If $\underline{M}$ is 3-connected and has a series set, then $\underline{M}$ is isomorphic to ${}_2\underline{M}_2$ or ${}_3\underline{M}_3$. In either case, the series reduced matroid is isomorphic to ${}_1\underline{M}_1$, which is 3-connected. Therefore every 3-connected matroid is also semi-3-connected. However, the converse is not always true.

Lemma 4.15 Let $\underline{M}=(\underline{C}, E)$ be connected. Then a series-reduced matroid of $\underline{M}$ is connected.

Proof. Let S_i, $1 \leqslant i \leqslant k$, be the maximal series sets of $\underline{M}$ and $e_i \in S_i$ for $1 \leqslant i \leqslant k$. Let $\underline{M}_R=(\underline{C}_R, E_R)=\underline{M}\cdot(E - \bigcup_{i=1}^{k} (S_i - \{e_i\}))$. It suffices to show that for any distinct cells e_1 and e_2 of $\underline{M}_R$, there is a circuit of $\underline{M}_R$ containing those cells.

Since e_1 and e_2 are cells of $\underline{M}$, there is a circuit C of $\underline{M}$ containing those cells. If $C \cap E_R \notin \underline{C}_R$, there is a circuit C' of $\underline{M}$, such that $C' \cap E_R \in \underline{C}_R$ and $C' \cap E_R \subset C \cap E_R$. If $C' \cap (\bigcup_{i=1}^{k} S_i) = \bigcup_{i \in I} S_i$, where $I \subseteq \{i \mid 1 \leqslant i \leqslant k\}$, then $\bigcup_{i \in I}\{e_i\} \subseteq C' \cap E_R$ and $\bigcup_{i \in I}\{e_i\} \subseteq C \cap E_R$. Thus $C'=(C' \cap E_R) \cup (\bigcup_{i \in I} S_i) \subset (C \cap E_R) \cup (\bigcup_{i \in I} S_i) \subseteq C$, which is contrary to Axiom I. Therefore $C \cap E_R$ is a circuit of $\underline{C}_R$ containing e_1 and e_2. ∎

Lemma 4.16 Let $\underline{M}$ be a connected matroid. If the complement of every non-series separator is a series separator, then $\underline{M}$ is semi-3-connected.

Proof. If $\lambda(\underline{M}) \geqslant 3$, then $\underline{M}$ is obviously semi-3-connected. If both S and $\bar{S}$ are series separators, then a series-reduced matroid is isomorphic to ${}_1\underline{M}_1$, and it follows that $\underline{M}$ is semi-3-connected.

Suppose there is no pair of complementary series separators of $\underline{M}$. Let $\underline{M}_R=(\underline{C}_R, E_R)$ be a series-reduced matroid of $\underline{M}$. By Lemma 4.15, $\underline{M}_R$ is connected. We assume $\lambda(\underline{M}_R)=2$. Then $\underline{M}_R$ has a pair of non-series separators S_R and $\bar{S}_R$ because there are no series sets of $\underline{M}_R$. Let S_i, $1 \leq i \leq k$, be the maximal series sets of $\underline{M}$ and $e_i \in S_i$ for $1 \leq i \leq k$. We partition $\{i \mid 1 \leq i \leq k\}$ into two sets $I_1 = \{i \mid e_i \in S_R\}$ and $I_2 = \{i \mid e_i \in \bar{S}_R\}$. Then by Theorem 2.13:

$$\begin{aligned} \xi(\underline{M}; S_R, \bar{S}_R) = 2 &= \mu(\underline{M}_R \cdot S_R) - \mu(\underline{M}_R \times S_R) + 1 \\ &= -\mu(\underline{M}_R) + \mu(\underline{M}_R \cdot S_R) + \mu(\underline{M}_R \cdot \bar{S}_R) + 1 \\ &= -\mu(\underline{M}_R) + \mu((\underline{M}_R \cdot E_R) \cdot S_R) + \mu((\underline{M} \cdot E_R) \cdot \bar{S}_R) + 1 \\ &= -\mu(\underline{M} \cdot E_R) + \mu(\underline{M} \cdot S_R) + \mu(\underline{M} \cdot \bar{S}_R) + 1. \end{aligned}$$

However, $\underline{M} \cdot E_R$, $\underline{M} \cdot S_R$, and $\underline{M} \cdot \bar{S}_R$ are series-reduced matroids of $\underline{M}$, $\underline{M} \cdot (S_R \cup (\bigcup_{i \in I_1} S_i))$, and $\underline{M} \cdot (\bar{S}_R \cup (\bigcup_{i \in I_2} S_i))$, respectively. By Theorem 4.5,

$$\begin{aligned} 2 &= -\mu(\underline{M}) + \mu(\underline{M} \cdot (S_R \cup (\bigcup_{i \in I_1} S_i))) + \mu(\underline{M} \cdot (\bar{S}_R \cup (\bigcup_{i \in I_2} S_i))) + 1 \\ &= \xi(\underline{M}; S_R \cup (\bigcup_{i \in I_1} S_i), \bar{S}_R \cup (\bigcup_{i \in I_2} S_i)), \end{aligned}$$

and also $\min(|S_R \cup (\bigcup_{i \in I_1} S_i)|, |\bar{S}_R \cup (\bigcup_{i \in I_2} S_i)|) \geq \min(|S_R|, |\bar{S}_R|) \geq 2$. Therefore $S_R \cup (\bigcup_{i \in I_1} S_i)$ and $\bar{S}_R \cup (\bigcup_{i \in I_2} S_i)$ are a pair of non-series separators of $\underline{M}$. Since this is contrary to the hypothesis $\lambda(\underline{M}_R) = 3$ and $\underline{M}$ is a semi-3-connected matroid. ∎

A C-decomposition of $\underline{M}$ finally yields a set of non-decomposable matroids which are semi-3-connected, according to Lemma 4.16. These semi-3-connected matroids are referred to as <u>minimal blocks</u> of $\underline{M}$, and their series-reduced matroids are 3-connected and called <u>atoms</u> of $\underline{M}$.

So far we have considered C-decomposition of connected matroids. However, this restriction is easily removed, and we can extend our decomposition to separable matroids. If a matroid $\underline{M}$ is separable, we define the C-decomposition of $\underline{M}$ as the C-decomposition of each connected component of $\underline{M}$. According to Lemma 4.16, the C-decomposition of each connected component of $\underline{M}$ decomposes $\underline{M}$ into minimal blocks. Thus we can state the following theorem for general matroids.

Theorem 4.7 Let $\underline{M}$ be a matroid. Then a C-decomposition of $\underline{M}$ yields a complete set of minimal blocks, which are semi-3-connected.

No minimal blocks and hence, no atoms of $\underline{M}$ are 1-circuit matroids unless some connected component of $\underline{M}$ is a 1-circuit matroid.

If a 3-connected matroid $\underline{M}$ contains series cells, then $\underline{M} \cong {}_2\underline{M}_2$ or ${}_3\underline{M}_3$, each of which is a 1-circuit matroid. We define the minimal block of a 1-circuit matroid $\underline{M}=(\underline{C}, E)$ by $\underline{M}$ itself. Nevertheless, the series-reduced matroids of the minimal blocks of 1-circuit matroids are isomorphic to ${}_1\underline{M}_1$. We define their atoms by $\underline{M}$ for $1 \leqslant |E| \leqslant 3$ and by ${}_3\underline{M}_3$ for $4 \leqslant |E|$. We make this exceptional definition of atoms of 1-circuit matroids for the sake of consistency in the following discussion.

A minor $\underline{M}_A=(\underline{C}_A, A)$ of a matroid $\underline{M}$ is called a maximal 3-connected minor (max. 3-conn. minor) if $\underline{M}_A$ is 3-connected and there is no 3-connected minor whose cell set properly contains A.

The minimal blocks and atoms obtained by a C-decomposition are not unique. However, we are able to characterize minimal blocks and atoms in terms of max. 3-conn. minors of $\underline{M}$.

<u>Lemma 4.17</u> In a C-decomposition of a matroid, atoms are max. 3-conn. minors of minimal blocks, and max. 3-conn. minors of minimal blocks are isomorphic to atoms.

<u>Proof.</u> For 1-circuit and 3-connected matroids the theorem is trivially true.

We consider a matroid of connectivity two which is not 1-circuit matroid. Let $\underline{M}=(\underline{C}, E)$ be minimal block and $\underline{M}_A=(\underline{C}_A, A)$ be a series-reduced matroid of $\underline{M}$. By Lemma 4.16, $\underline{M}$ is connected and $\underline{M}_A$ is 3-connected. If $\underline{M}_A$ is not maximal, then there exists a max. 3-conn. minor $\underline{M}_{A'}=(\underline{C}_{A'}, A')$, such that $A \subset A'$. Let $e_1 \in A' - A$. Since $\underline{M}_A$ is a series-reduced matroid of $\underline{M}$, there exists a cell e_2 of $\underline{M}_A$, such that e_1 and e_2 are in series in $\underline{M}$. Accordingly, e_1 and e_2 are series cells of $\underline{M}_{A'}$. Therefore $\underline{M}_{A'}$ is not 3-connected and, hence, $\underline{M}_A$ is a max. 3-conn. minor of $\underline{M}$.

Let $\underline{M}_B=(\underline{C}_B, B)$ be a max. 3-conn. minor of a minimal block

Let $\underline{M}_B=(\underline{C}_B, B)$ be a max. 3-conn. minor of a minimal block

$\underline{M}=(\underline{C}, E)$. Since $\underline{M}_B$ is 3-connected, it has no series cells of $\underline{M}$. Thus $\underline{M}_B$ is a minor of a series-reduced matroid $\underline{M}'$ of $\underline{M}$. However, $\underline{M}'$ is isomorphic to an atom $\underline{M}_A$ of $\underline{M}$ by Theorem 4.4, and hence $\underline{M}'$ is is a max. 3-conn. minor of $\underline{M}$ by Lemma 4.13 and by the first part of this proof. Accordingly, we have $\underline{M}_B=\underline{M}'\cong\underline{M}_A$. ∎

<u>Lemma 4.18</u> In the C-decomposition of a matroid $\underline{M}$, every atom is a max. 3-conn. minor of $\underline{M}$.

<u>Proof.</u> Since the lemma is obviously true for 1-circuit and 3-connected

matroids, it suffices to prove the lemma for a matroid of connectivity two. Let $\underline{M}=(\underline{C}, E)$ be a block and $S, \overline{S}$ be its non-series separators, and let $C \in (\underline{C} \mid S)$. Since atoms are max. 3-conn. minors of minimal blocks by Lemma 4.17, we have only to show that max. 3-conn. minors of blocks $\underline{M} \times (S \cup C)$ and $\underline{M} \times (\overline{S} \cup C)$ are also max. 3-conn. minors of $\underline{M}$.

Without loss of generality, let $\underline{M}_1=(\underline{C}_1, E_1)$ be a max. 3-conn. minor of a block $\underline{M} \times (S \cup C)$. Suppose $\underline{M}_1$ is not a max. 3-conn. minor of $\underline{M}$. Then, by definition, there exists a max. 3-conn. minor $\underline{M}_2=(\underline{C}_2, E_2)$ of $\underline{M}$, such that $E_1 \subset E_2$. If $E_1 \cap \overline{S} \subseteq E_2 \cap \overline{S} = \phi$, then $\underline{M}_2$ is a 3-connected minor of $\underline{M} \times (S \cup C)$ by Theorem 4.6. This is contrary to the hypothesis, and therefore $\underline{M}_1$ is a max. 3-conn. minor of $\underline{M}$.

Now suppose $E_1 \cap \overline{S} = \{e_1\} \neq \{e_2\} \in E_2 \cap \overline{S}$. By Theorem 4.6 we can find a minor $\underline{M}_3=(\underline{C}_3, E_3)$ of $\underline{M} \times (S \cup C)$ such that $\underline{M} \cong \underline{M}_3$ and $E_3=(E_2-\{e_2\}) \cup \{e_1\}$. However, $E_1 \cap \overline{S}=E_3 \cap \overline{S}=\{e_1\}$, $E_1 \cap S \subset E_2 \cap S = E_3 \cap S$ and hence, $E_1 \subset E_2$. This contradicts the hypothesis. Therefore $\underline{M}_1$ is a max. 3-conn. minor of $\underline{M}$. The lemma follows by induction. ∎

The following lemma is the converse of Lemma 4.18:

<u>Lemma 4.19</u> Every max. 3-conn. minor of a matroid is isomorphic to an atom.

<u>Proof.</u> Since a max. 3-conn. minor of a separable matroid is a max. 3-conn. minor of some connected component, we may assume, without

loss of generality, that a matroid is connected.

If a matroid is 3-connected or a 1-circuit matroid, the lemma is obviously true as shown previously.

Suppose matroid $\underline{M}=(\underline{C}, E)$ has a pair of non-series separators S and $\bar{S}$. Let $C \in (\underline{C}|S)$ be a circuit of $\underline{M}$, and $\underline{M}_1=(\underline{C}_1, E_1)$ be a max. 3-conn. minor of $\underline{M}$. We shall show $\underline{M}_1$ is isomorphic to a max. 3-conn. minor of either $\underline{M} \times (S \cup C)$ or $\underline{M} \times (\bar{S} \cup C)$.

By Theorem 4.6, $\underline{M}_1$ is isomorphic to a 3-connected minor $\underline{M}_2=(\underline{C}_2, E_2)$ of, say, $\underline{M}_S=\underline{M} \times (S \cup C)$, for which $E_1 \cap \bar{S}=E_2 \cap S$ and $|E_1 \cap \bar{S}| = |E_2 \cap \bar{S}| \leq 1$. If $E_1 \cap \bar{S} = E_2 \cap \bar{S}$, then $\underline{M}_1=\underline{M}_2$. If $\underline{M}_2$ is not a max. 3-conn. minor of $\underline{M}_S$, then there exists a max. 3-conn. minor $\underline{M}_3=(\underline{C}_3, E_3)$ of $\underline{M}_S$ so that $E_3 \supset E_2 = E_1$. Since $\underline{M}_3$ is also a minor of $\underline{M}$ by Theorem 2.9, $\underline{M}_1$ is not a max. 3-conn. minor of $\underline{M}$. This contradicts the hypothesis.

Now suppose $E_1 \cap \bar{S} = \{e\} \neq \{e'\} = E_3 \cap \bar{S}$. From Theorem 4.6 we can find a minor $\underline{M}_3'=(\underline{C}_3', E_3')$ of $\underline{M}$, such that $\underline{M}'$ is isomorphic to $\underline{M}_3$ and $E_3' = (E_3 - \{e'\}) \cup \{e\}$. However, $E_3' = (E_3 - \{e'\}) \cup \{e\} \supset (E_2 - \{e'\}) \cup \{e\} = E_1$, which is a contradiction, $\underline{M}_1$ being maximal. Thus $\underline{M}_2$ is a max. 3-conn. minor of $\underline{M} \times (S \cup C)$. By induction and Lemma 4.17, $\underline{M}_1$ is isomorphic to an atom. ■

From Lemmas 4.18 and 4.19 we deduce

<u>Theorem 4.8</u> Let $\underline{M}$ be a matroid. Then every atom is a max. 3-conn. minor of $\underline{M}$ and every max. 3-conn. minor of $\underline{M}$ is isomorphic to an atom.

Since maximal 3-connected minors are independent of a particular C-decomposition, a complete set of atoms is uniquely characterized by any C-decomposition. By this decomposition some important characteristics of the original matroid are preserved in the complete set of atoms. In the next section we will show that a graph-realizability condition of matroids can be formulated in terms of atoms.

Some invariants of a matroid are also calculated from the invariants of atoms. The following theorem shows a simple relationship between the nullities of a matroid and its atoms.

Theorem 4.9 Let $\underline{M}$ be a matroid, and let $\underline{M}_1, \underline{M}_2, \ldots, \underline{M}_n$ be a complete set of atoms of $\underline{M}$. Then

$$\mu(\underline{M}) = \sum_{i=1}^{n} \mu(\underline{M}_i) - n.$$

Proof. Let $\underline{M}' = (\underline{C}', E')$ be a block of $\underline{M}$ at some stage of the decomposition process. Suppose $\underline{M}'$ has complementary non-series separator S and $\bar{S}$. Let $C \in (\underline{C}' | S)$. Then

$$\mu(\underline{M}' \times (S \cup C)) = \mu(\underline{M}' \times S) + 1$$

$$\mu(\underline{M}' \times (\bar{S} \cup C)) = \mu(\underline{M}' \times \bar{S}) + 1.$$

By Theorem 2.13:

$$\begin{aligned}\mu(\underline{M}') &= \xi(\underline{M}'; S, \bar{S}) + \mu(\underline{M}' \times S) + \mu(\underline{M}' \times \bar{S}) - 1 \\ &= 2 + \mu(\underline{M}' \times (S \cup C)) - 1 + \mu(\underline{M}' \times (\bar{S} \cup C)) - 1 - 1 \\ &= \mu(\underline{M}' \times (S \cup C)) + \mu(\underline{M}' \times (\bar{S} \cup C)) - 1.\end{aligned}$$

Thus one step of decomposition increases the total nullity by one. Since the number of decomposition steps is equal to the number of atoms, a C-decomposition of $\underline{M}$ increases the total nullity by n. ∎

Lastly, we show that every max. 3-conn. minor of a matroid can be an atom of a C-decomposition.

Theorem 4.10 Let $\underline{M}$ be a matroid. For any given max. 3-conn. minor $\underline{M}_0$ of $\underline{M}$, there exists a C-decomposition of $\underline{M}$ of which $\underline{M}_0$ is an atom.

Proof. We prove the theorem for a connected matroid. If a matroid is separable, the same argument may be applied for each of it connected components.

Let $\underline{M}_0=(\underline{C}_0, E_0)$. Suppose $S, \overline{S}$ are non-series separators of $\underline{M}$. By Theorem 4.2, $|E_0 \cap \overline{S}| \leqslant 1$, without loss of generality. If $E_0 \cap \overline{S} = \phi$, then $\underline{M}_0$ is a minor of $\underline{M}$ x (S U C) by Theorem4.6, where $C \in (\underline{C}|S)$. Suppose $E_0 \cap \overline{S} = \{e\}$. Since $\underline{M}$ is connected, there always exists a circuit C of $\underline{M}$, such that $e \in C$ and $C \in (\underline{C}|S)$. By Theorem 4.6 $\underline{M}_0$ is a minor of $\underline{M}$ x (S U C). Thus at each stage of the decomposition process we choose a circuit C so that $\underline{M}_0$ is a minor of a block $\underline{M}$ x (S U C) or $\underline{M}$ x ($\overline{S}$ U C). Repeating this choice, we end up with a minimal block, $\underline{M}'=(\underline{C}', E')$, of which $\underline{M}_0$ is a minor.

Let S_i, $1 \leqslant i \leqslant k$, be the maximal series set of $\underline{M}'$. Since $\underline{M}_0$ is 3-connected, $S_i \cap E_0$ is null or consists of a single element. Suppose $S_i \cap E_0 = \{e_i\}$ for $1 \leqslant i \leqslant j < k$ and $S_i \cap E_0 = \phi$ for $j+1 \leqslant i \leqslant k$. Choose any element $e_i \in S_i$ for $j+1 \leqslant i \leqslant k$. Then $\underline{M}_A=(\underline{C}_A, A)=\underline{M}' \cdot A$ is an atom, where $A=E'-\bigcup_{i=1}^{k}(S_i-\{e_i\})$. Since $A \supset E_0$, $\underline{M}_0$ is not a max. 3-conn. minor of $\underline{M}'$, which contradicts the hypothesis. Thus j=k.

Similarly, we can show $A \cap (E' - \bigcup_{i=1}^{k} S_i) = E_0 \cap (E' - \bigcup_{i=1}^{k} S_i)$, and hence, $A = E_0$. Therefore, $\underline{M}_0 = \underline{M}_A$ and the proof is complete. ∎

4.5 GRAPHIC AND COGRAPHIC MATROIDS

This section is devoted to structural characterization of matroids in terms of atoms. We shall show that characteristics of graph-realizability and of being binary or regular are determined by corresponding properties of atoms.

We introduced series sets of matroids in Section 3.3. The dual concept of a series set is a parallel set. Let $\underline{M}=(\underline{C}, E)$ be a matroid. Two cells, e_1 and e_2, of $\underline{M}$ are referred to as being in parallel if $\{e_1, e_2\}$ is a circuit of $\underline{M}$.

Lemma 4.20 Let $\underline{M}=(\underline{C}, E)$ be a connected matroid and $\underline{M}^*=(\underline{C}^*, E)$ be its dual. Then two cells, e_1 and e_2, are in series in $\underline{M}$ if and only if e_1 and e_2 are in parallel in $\underline{M}^*$.

Proof. Let e_1 and e_2 be in series in $\underline{M}$. Neither $\{e_1\}$ nor $\{e_2\}$ can be a circuit of $\underline{M}^*$, because if $\{e_1\}$ is a circuit of $\underline{M}^*$, the intersection of $\{e_1\}$ and any circuit of $\underline{M}$ containing e_1 has one cell in common, which contradicts the definition of a dual matroid. If C contains e_1, then $e_2 \in C$ and $|C \cap \{e_1, e_2\}| = 2$. Thus $\{e_1, e_2\}$ is orthogonal to every member of $\underline{M}$ and is minimal. Therefore $\{e_1, e_2\}$ is a circuit of $\underline{M}^*$.

Similarly we can show the converse of the statement, and hence the theorem follows. ∎

Let $\underline{M}=(\underline{C}, E)$ be a matroid. A subset S of E is called a parallel set of $\underline{M}$ if $|S| \geq 2$ and any two members of S are in parallel. A parallel set S is maximal if no parallel set properly contains S.

The following lemma is obvious.

Lemma 4.21 Let $\underline{M}$ be a matroid and $\underline{M}^*$ its dual. Then S is a maximal series set of $\underline{M}$ if and only if S is a maximal parallel set of $\underline{M}^*$.

Let S_i, $1 \leqslant i \leqslant k$, be the maximal parallel sets of $\underline{M}=(\underline{C}, E)$, and let $e_i \in S_i$ for $1 \leqslant i \leqslant k$. Then $\underline{M}_R=(\underline{C}_R, E_R)=\underline{M} \times (E-\bigcup_{i=1}^{k}(S_i-\{e_i\}))$ is called a parallel-reduced matroid of $\underline{M}$. The following lemma shows this matroid to be the dual of a series-reduced matroid of $\underline{M}^*$.

Lemma 4.22 Let $\underline{M}=(\underline{C}, E)$ be a matroid and $\underline{M}^*=(\underline{C}^*, E)$ be its dual. Then $\underline{M}_R$ is a series-reduced matroid of $\underline{M}$ if and only if its dual, $\underline{M}_R^*$, is a parallel-reduced matroid of $\underline{M}^*$.

Proof. Let S_i, $1 \leqslant i \leqslant k$, be the maximal series sets of $\underline{M}$ and $e_i \in S_i$ for $1 \leqslant i \leqslant k$. $\underline{M}_R=\underline{M} \cdot E_R$ is a series-reduced matroid of $\underline{M}$. By Theorem 2.8(a) and Lemma 4.21, $\underline{M}_R^*=\underline{M}^* \times E_R$ is a parallel-reduced matroid of $\underline{M}$. ∎

From Lemma 4.22, every previous theorem on series-reduced matroids also holds for parallel-reduced matroids by simply replacing the matroid-theoretic terminology by its dual terminology. Thus Theorem 4.12 is obtained from Theorem 4.4 and Lemma 4.22.

Theorem 4.12 Two parallel-reduced matroids of a matroid are isomorphic.

Lemma 4.23 Let $\underline{M}_1=(\underline{C}_1, E_1)$ and $\underline{M}_2=(\underline{C}_2, E_2)$ be equivalent. Then

$\underline{M}_1$ is binary if and only if $\underline{M}_2$ is binary.

Proof. Let $\underline{M}=(\underline{C}, E)$ be a matroid and $\underline{M}_R=(\underline{C}_R, E_R)$ be its series-reduced matroid. Then it suffices to show that $\underline{M}$ is binary if and only if $\underline{M}_R$ is binary.

If $\underline{M}$ is binary, then $\underline{M}_R$ is binary by Theorem 2.19(b). Suppose $\underline{M}_R$ is binary. We shall show that $\underline{M}$ is also binary. Let S_i, $1 \leq i \leq k$, be the maximal series sets of $\underline{M}$, and $\underline{M}_R=\underline{M}\cdot E_R$ where $E_R= E-\bigcup_{i=1}^{k}(S_i-\{e_i\})$ and $e_i \in S_i$ for $1 \leq i \leq k$. By Lemma 4.21, S_i, $1 \leq i \leq k$, are the maximal parallel sets of the dual matroid $\underline{M}^*=(\underline{C}^*, E)$. Suppose $C \in \underline{C}$ and $C^* \in \underline{C}^*$. Our aim is to show $|C \cap C^*|$ is even. If $|C^* \cap S_i| \geq 2$ for some i, then by Lemma 4.20 C^* consists of two members of S_i, and $|C \cap C^*| =0$ or 2 according as $S_i \cap C = \phi$ or $S_i \subset C$, respectively. Now suppose $|C^* \cap S_i| \leq 1$ for every i, $1 \leq i \leq k$. If $C \cap (\bigcup_{i=1}^{k} \{e_i\})= \phi$ or $C^* \cap (\bigcup_{i=1}^{k} S_i)= \phi$, then it is not hard to show $|C \cap C^*|$ is even. We consider the remaining case. Let $C \cap (\bigcup_{i=1}^{k} e_i)=\{e_1, \ldots, e_r, e_{r+1}, \ldots, e_p, e_{p+1}, \ldots, e_q\}$ and $C^* \cap (\bigcup_{i=1}^{k} S_i)=\{e_1, \ldots, e_r, e'_{r+1}, \ldots, e'_p, e''_{q+1}, \ldots, e''_s\}$, where $e'_i \in S_i-\{e_i\}$ for $r+1 \leq i \leq p$ and $e''_i \in S_i$ for $q+1 \leq i \leq s$. $C_R=C-\bigcup_{i=1}^{q}(S_i-\{e_i\}) \in \underline{C}_R$ and by Lemma 4.22 $C^*_R=(C^*-(\bigcup_{i=r+1}^{p}\{e'_i\}) \cup (\bigcup_{i=q+1}^{s}\{e''_i\})) \cup (\bigcup_{i=r+1}^{p}\{e_i\}) \cup (\bigcup_{i=q+1}^{s}\{e_i\}) \in \underline{C}^*_R$.

Then:

$$\begin{aligned}
|C \cap C^*| &= |C \cap C^* \cap E_R| + |\bigcup_{i=r+1}^{p}\{e'_i\}| \\
&= |C_R \cap (C^* \cap E_R)| + p - r \\
&= |C_R \cap (C^*_R - (\bigcup_{i=r+1}^{p}\{e_i\}) \cup (\bigcup_{i=q+1}^{s}\{e_i\}))| + p - r \\
&= |C_R \cap C^*_R| - |\bigcup_{i=r+1}^{p}\{e_i\}| + p - r
\end{aligned}$$

$= |C_R \cap C_R^*| =$ even.

Accordingly, the proof is complete. ■

<u>Lemma 4.24</u> Let $\underline{M}=(\underline{C}, E)$ be a matroid and S a 2-separator of $\underline{M}$. If $C_1, C_2 \in (\underline{C} \backslash S)$, then $\underline{M} \times (S \cup C_1)$ and $\underline{M} \times (S \cup C_2)$ are equivalent.

<u>Proof.</u> From Lemma 4.6, $C_1 \cap \overline{S}$ and $C_2 \cap \overline{S}$ are series sets of $\underline{M} \times (S \cup C_1)$ and $\underline{M} \times (S \cup C_2)$, respectively. By Theorem 4.3, every circuit of $\underline{M} \times (S \cup C_1)$ which contains $C_1 \cap \overline{S}$ corresponds to a circuit of $\underline{M} \times (S \cup C_2)$, which contains $C_2 \cap \overline{S}$, and vice-versa. Thus the lemma follows. ■

We will give an equivalent binary condition for matroids, which will be useful in the following discussion.

<u>Lemma 4.25</u> A matroid $\underline{M}=(\underline{C}, E)$ is binary if and only if for any circuits $C_1, C_2 \in \underline{C}$, $C_1 \oplus C_2$ is a union of disjoint circuits of $\underline{M}$.

<u>Proof.</u> The necessary condition immediately follows from Theorem 2.20.

In proving the sufficient condition, let B be a base of $\underline{M}$ and $C \in \underline{C}$. We show that if $C \cap \overline{B} = \{e_1, e_2, \ldots, e_k\}$, then

$$C = C_1 \oplus C_2 \oplus \ldots \oplus C_k,$$

where C_i is the fundamental circuit determined by B and e_i. The proof follows by induction on k.

Let C be a circuit of $\underline{M}$ such that $|C \cap \overline{B}| = 2$. Suppose $C \cap \overline{B} = \{e_1, e_2\}$. By hypothesis, $C \oplus C_1$ contains a circuit of $\underline{M}$ which contains e_2, and otherwise only elements of B. Then, by definition, $C \oplus C_1 = C_2$

and hence, $C=C_1 \oplus (C \oplus C_1)=C_1 \oplus C_2$.

Suppose the above proposition is true for $\leq k-1$. Let $C \in \underline{C}$ and $C \cap \overline{B}=\{e_1, e_2, \ldots, e_k\}$. By Axiom I, there exists $e_k' \in C_k \cap B - C$, and $B'=(B-\{e_k\}) \cup \{e_k'\}$ is a base of $\underline{M}$ by Theorem 2.5 and Lemma 4.3. Let C_i', $1 \leq i \leq k$, be the fundamental circuits determined by B' and $e_1, e_2, \ldots, e_{k-1}, e_k'$. $C \cap (E-B')=\{e_1, e_2, \ldots, e_{k-1}\}$. Therefore, by the induction hypothesis, $C=C_1' \oplus C_2' \oplus \ldots \oplus C_{k-1}'$. Since $|C_i' \cap \overline{B}| \leq 2$ for $1 \leq i \leq k-1$, as shown above, C_i' is a fundamental circuit or is the symmetric difference of two fundamental circuits determined by B and elements of $\overline{B}$. Thus C can be expressed as a symmetric difference of C_i, $1 \leq i \leq k$, and it must contain an odd number of C_i's for each i; otherwise $e_i \notin C$, which contradicts the hypothesis. If C_i appears more than once in the expression, then we can deduce the symmetric difference of C_i's to C_i since $C_i \oplus C_i = \phi$. Consequently, $C=C_1 \oplus C_2 \oplus \ldots \oplus C_k$, and the lemma follows. ∎

A C-decomposition of a binary matroid yields a complete set of binary atoms. The converse of this statement is also true.

<u>Theorem 4.13</u> A matroid is binary if and only if a complete set of atoms is binary.

<u>Proof.</u> The necessary part follows from Theorem 2.19(b).

We shall prove the sufficient part of the theorem. Suppose all the atoms of a C-decomposition are binary. By Lemma 4.23 all the minimal blocks are binary. Let $\underline{M}=(\underline{C}, E)$ be a matroid and $S, \overline{S}$

its non-series separators. Suppose $C \in (\underline{C}|S)$, and blocks $\underline{M} \times (S \cup C)$ and $\underline{M} \times (\overline{S} \cup C)$ of $\underline{M}$ are binary matroids. We show that $\underline{M}$ will satisfy Lemma 4.25, and hence it is also binary. If $C_1, C_2 \in \underline{C}$ and $C_1 \cap C_2 = \phi$, then $C_1 \oplus C_2$ consists of the disjoint circuits C_1 and C_2.

Suppose $C_1 \cap C_2 \neq \phi$, $C_1 \in \underline{C} \times S$, and $C_2 \in (\underline{C}|S)$. By Lemmas 4.23 and 4.24, $\underline{M} \times (S \cup C_2)$ is binary, and hence $C_1 \oplus C_2$ is a union of the disjoint circuits of $\underline{M}$.

Now suppose $C_1 \cap C_2 \neq \phi$ and $C_1, C_2 \in (\underline{C}|S)$. By Lemmas 4.23 and 4.24, $\underline{M} \times (S \cup C_2)$ and $\underline{M} \times (\overline{S} \cup C_2)$ are binary. Let $X = (C_1 \cap S) \cup (C_2 \cap \overline{S})$ and $Y = (C_2 \cap S) \cup (C_1 \cap \overline{S})$. X and Y are circuits of $\underline{M}$ by Theorem 4.3. Since X and Y are circuits of $\underline{M} \times (S \cup C_2)$ and $\underline{M} \times (\overline{S} \cup C_2)$, respectively. $X \oplus C_2 = (C_1 \oplus C_2) \cap S$ and $Y \oplus C_2 = (C_1 \oplus C_2) \cap \overline{S}$ are unions of disjoint circuits of $\underline{M}$. Therefore $C_1 \oplus C_2$ is a union of disjoint circuits of $\underline{M}$. Accordingly, $\underline{M}$ is binary and, by induction, the theorem follows. ∎

We denote the Fano and heptahedron matroids by $\underline{M}_F$ and $\underline{M}_H$, respectively.

<u>Lemma 4.26</u> $\lambda(\underline{M}_F) = \lambda(\underline{M}_H) = 3$.

<u>Proof.</u> Let $\underline{M}_F = (\underline{C}_F, E)$, where $E = \{e_1, e_2, \ldots, e_7\}$ and $\underline{C} = \{\{e_1, e_2, e_3, e_4\}, \{e_1, e_2, e_4, e_6\}, \{e_1, e_3, e_5, e_6\}, \{e_1, e_4, e_5, e_7\}, \{e_2, e_3, e_4, e_5\}, \{e_2, e_5, e_6, e_7\}, \{e_3, e_4, e_6, e_7\}\}$.

Let $S = \{e_1, e_2, e_3, e_7\}$ and $\overline{S} = E - S$. By Theorem 2.13,

$$\zeta(\underline{M}_F; S, \overline{S}) = \mu(\underline{M}_F) - \mu(\underline{M}_F \times S) - \mu(\underline{M}_F \times \overline{S}) + 1$$

$= 3 - 1 - 0 + 1 = 3,$

and min($|S|$, $|\bar{S}|$)=3. Thus $\lambda(\underline{M}_F) \leqslant 3$. If S =2, then $\underline{M}_F \times S$ contains no circuits and $\underline{M}_F \times \bar{S}$ contains only one circuit of $\underline{M}_F$. Therefore we have

$$\xi(\underline{M}_F; S, \bar{S}) = 3 - 0 - 1 + 1 = 3,$$

and min($|S|, |\bar{S}|$)=2. Hence $\underline{M}_F$ does not satisfy the finite connectivity condition, that is, $\lambda(\underline{M}_F) \nleq 2$. Since $\underline{M}_F$ is connected, $2 \leqslant \lambda(\underline{M}_F)$ and hence, $\lambda(\underline{M}_F)=3$.

The connectivity of $\underline{M}_H$ is obtained by Theorem 2.14.

$$\lambda(\underline{M}_H) = \lambda(\underline{M}_H^*) = \lambda(\underline{M}_F) = 3. \blacksquare$$

In the next theorem a complete set of atoms characterizes the regular property of a matroid.

<u>Theorem 4.14</u> A matroid is regular if and only if a complete set of atoms is regular.

<u>Proof.</u> Since every minor of a regular matroid is regular, all the atoms are regular.

If the original matroid, $\underline{M}$, is not regular, then by definition $\underline{M}$ is not binary, or $\underline{M}$ is binary and contains $\underline{M}_F$ or $\underline{M}_H$ as a minor.

If $\underline{M}$ is not binary, at least one atom is not binary by Theorem 4.13, and the theorem follows.

Suppose $\underline{M}$ is binary and contains $\underline{M}_F$ or $\underline{M}_H$ as a minor of $\underline{M}$. Since $\underline{M}_F$ and $\underline{M}_H$ are 3-connected by Lemma 4.26, one of the max. 3-conn. minors which contains $\underline{M}_F$ or $\underline{M}_H$ as a minor is isomorphic to an atom $\underline{M}_A$ by Lemma 4.19. Thus there exists at least one atom $\underline{M}_A$ which is not regular and hence the theorem follows. ∎

Lemma 4.27 $\lambda(\underline{P}(K_5)) = \lambda(\underline{B}(K_5)) = \lambda(\underline{P}(K_{3,3})) = \lambda(\underline{B}(K_{3,3})) = 3.$

Theorem 4.15 A matroid $\underline{M}$ is graphic if and only if a complete set of atoms is graphic, and $\underline{M}$ is cographic if and only if a complete set of atoms is cographic.

Proof. Since the proof of the second part of the theorem is similar to that of the first part, we need only prove the first part of the theorem.

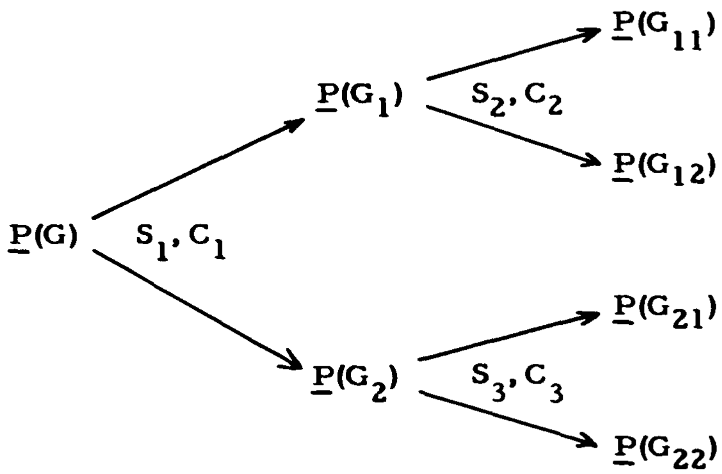
The necessary condition is obvious. Suppose $\underline{M}$ is not graphic. By Theorem 2.22, $\underline{M}$ is not regular or $\underline{M}$ contains $\underline{P}(K_5)$ or $\underline{P}(K_{3,3})$. If $\underline{M}$ is not regular, then some atom is not regular by Theorem 4.14, and hence, the theorem follows. If $\underline{M}$ contains $\underline{P}(K_5)$ or $\underline{P}(K_{3,3})$ as a minor, then, by Lemmas 4.19 and 4.27, there exists an atom $\underline{M}_A$ whose minor is isomorphic to $\underline{P}(K_5)$ or $\underline{P}(K_{3,3})$. Since $\underline{P}(K_5)$ and $\underline{P}(K_{3,3})$ are not graphic, $\underline{M}_A$ is not graphic. The theorem follows. ∎

As an immediate consequence of Theorem 4.15, we have the following theorem on planar matroids.

Theorem 4.16 A matroid is planar if and only if a complete set of atoms is planar.

There is no distinction among isomorphic matroids in their structures. Therefore a complete set of atoms may be classified into isomorphism classes. Let $\underline{A}$ consist of the isomorphism classes of atoms of $\underline{M}$. Then graph realizability of $\underline{M}$ is completely determined by the graph-realizability of representatives of the isomorphism classes in $\underline{A}$.

<u>Example 4.1</u> Let us consider the polygon matroid of the graph G in Fig. 4.1. Our C-decomposition process is performed as follows:



$S_1 = \{1, 2, 5, 6, 11, 12, 13\}$ $\quad$ $C_1 = \{2, 3, 4, 5\}$

$S_2 = \{11, 12, 13\}$ $\quad$ $C_2 = \{5, 6, 11, 12\}$

$S_3 = \{2, 3, 4, 5\}$ $\quad$ $C_3 = \{3, 4, 7, 10\}$

The resulting minimal blocks are $\underline{P}(G_{11})$, $\underline{P}(G_{12})$, $\underline{P}(G_{21})$, and $\underline{P}(G_{22})$ (see Fig. 4.3). A complete set of atoms is shown in Fig. 4.4 and its isomorphism classes in Fig. 4.5.

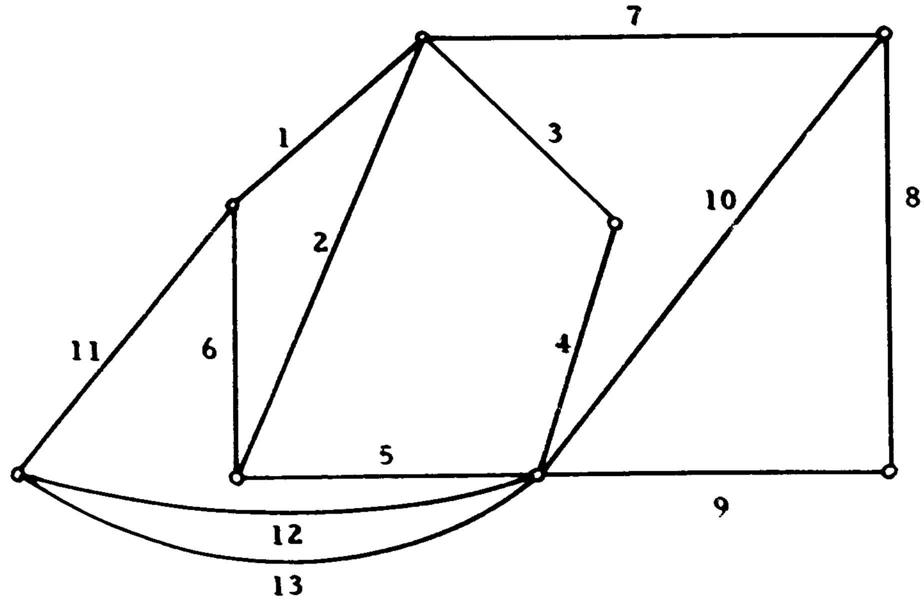


Figure 4.1 Graph G of Example 4.1

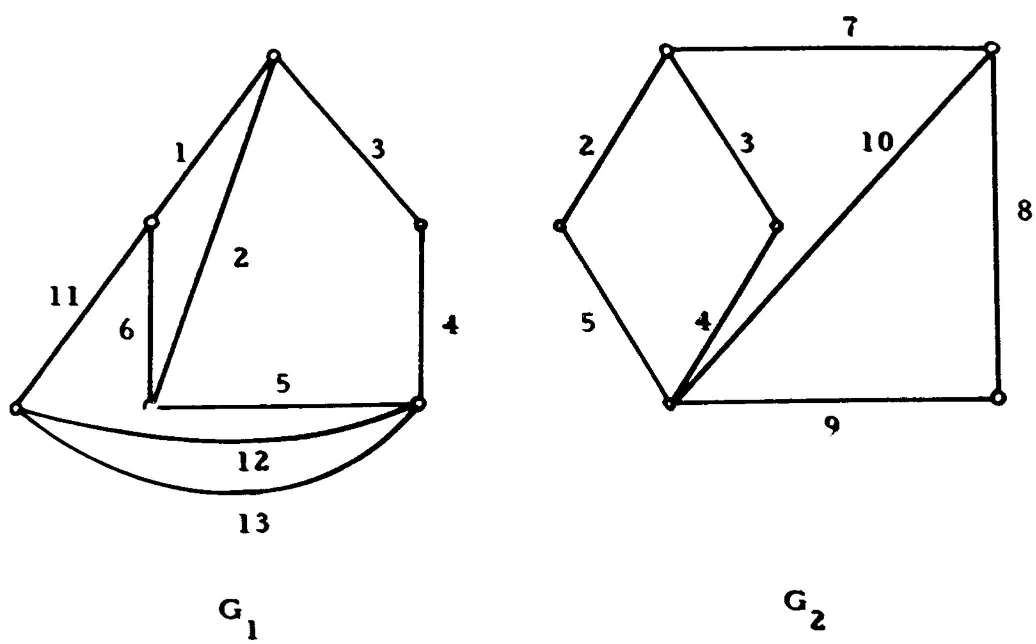


Figure 4.2 Graphs of Example 4.1

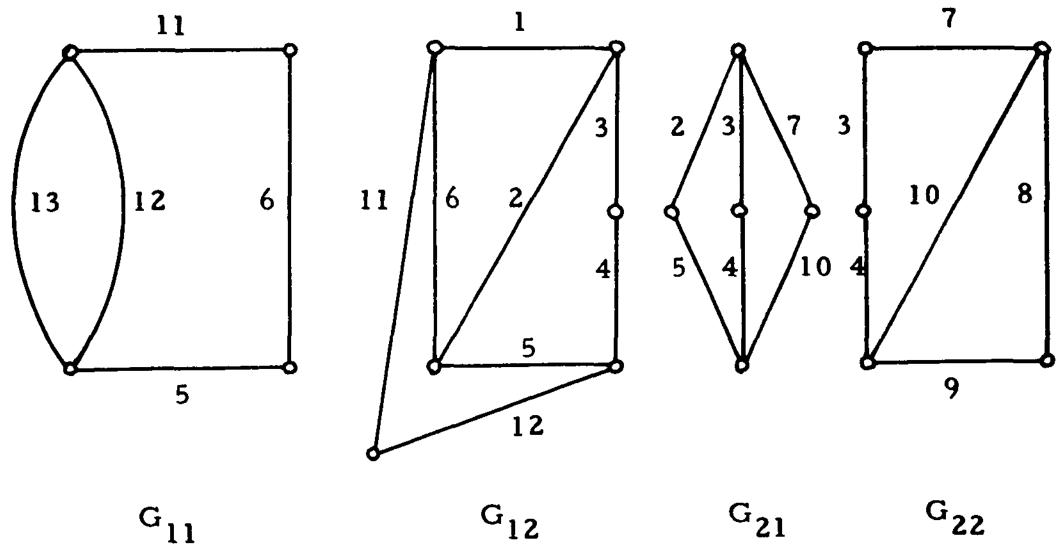


Figure 4. 3 Minimal Blocks of G

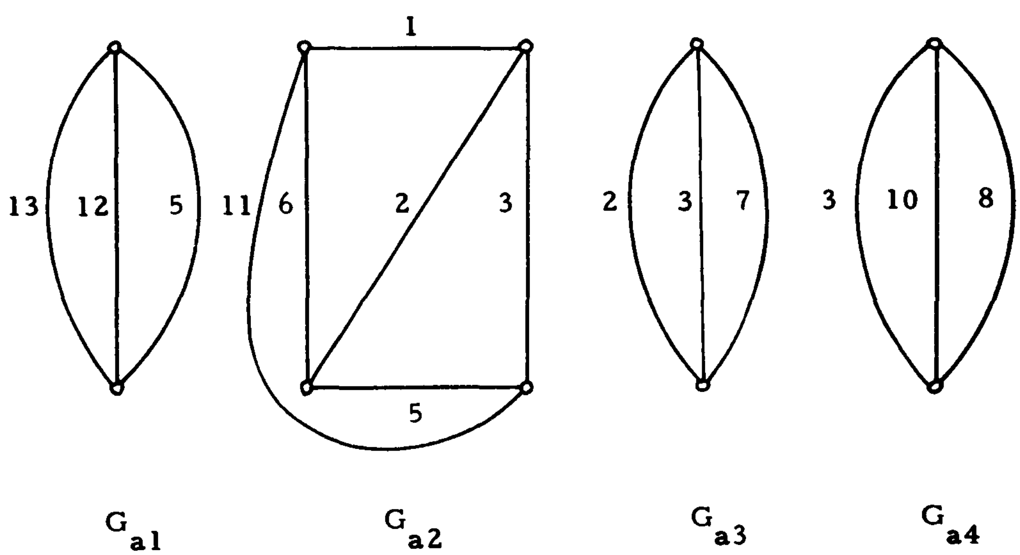
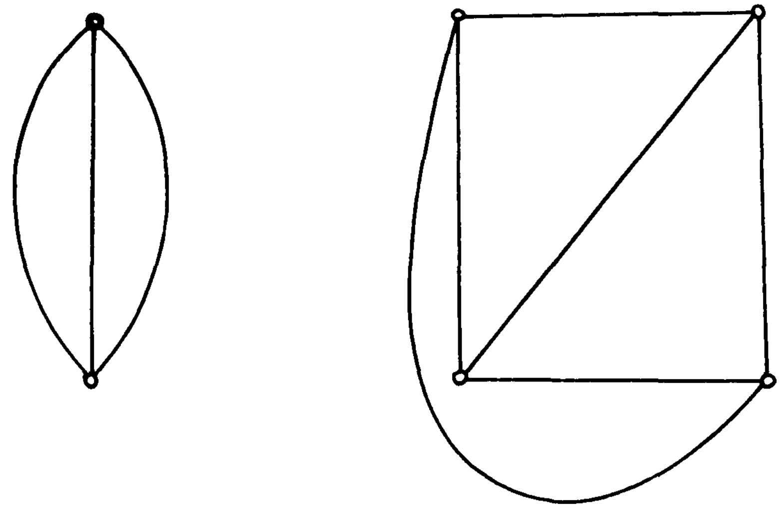


Figure 4. 4 Atoms of G



Representative of $\underline{A}_1$ Representative of $\underline{A}_2$

$$\underline{A} = \{\underline{A}_1, \underline{A}_2\}$$
$$\underline{A}_1 = \{G_{a1}, G_{a3}, G_{a4}\}$$
$$\underline{A}_2 = \{G_{a2}\}$$

Figure 4.5 Isomorphism Classes $\underline{A}$ of Atoms

CHAPTER 5 SPLIT DECOMPOSITIONS AND THEIR APPLICATION

5.1 P-DECOMPOSITION

C-decomposition discussed in the previous chapter yields two blocks at each step of the decomposition process. However, in defining so-called "split decomposition", we can maximize the number of blocks produced at each decomposition step. In this chapter two kinds of split decompositions, which are dual to each other, will be introduced with an application to planar n-port networks.

Let $\underline{M}=(\underline{C}, E)$ be a connected matroid, where $|E| \geq 3$. We partition E into non-null subsets S_i, $1 \leq i \leq k$, where $k \geq 2$. Then the sets S_i, $1 \leq i \leq k$, are called P-sets of $\underline{M}$ if they satisfy the following conditions:

(i) S_i is a 2-separator of $\underline{M}$ if $|S_i| \geq 2$

(ii) $\sum_{i=1}^{k} r(\underline{M} \times S_i)=r(\underline{M}) + k - 1$

(iii) S_i's are minimal with respect to conditions (i) and (ii).

Lemma 5.1 Let $\underline{M}=(\underline{C}, E)$ be connected and S_i, $1 \leq i \leq k$, be P-sets of $\underline{M}$. Then $\xi(\underline{M};S_i,\overline{S}_i) = 2$ for every i, $1 \leq i \leq k$.

Proof. If $|S_i| = 2$, then by definition S_i is a 2-separator of $\underline{M}$ and hence the lemma follows.

Suppose $|S_i| = 1$. Since $\underline{M}$ is connected and has no circuit consisting of a single cell, we have $\xi(\underline{M};S_i,\overline{S}_i) \geq 2$ and $r(\underline{M} \times S_i)=1$.

$$2 \leq \xi(\underline{M};S_i,\overline{S}_i)= - r(\underline{M}) + r(\underline{M} \times S_i) + r(\underline{M} \times \overline{S}_i) + 1$$
$$= - r(\underline{M}) + r(\underline{M} \times \overline{S}_i) + 2 \leq 2.$$

Thus, $\zeta(\underline{M};S_i,\overline{S}_i)=2.$ ■

Lemma 5.2 Let S_i, $1 \leq i \leq k$, be P-sets of $\underline{M}$. If $\{I_1, I_2\}$ is a partition of I, where $I_1, I_2 \neq \phi$ and $I \subseteq I_0=\{i \mid 1 \leq i \leq k\}$, then

$$r(\underline{M} \times \bigcup_{i\in I_1} S_i) + r(\underline{M} \times \bigcup_{i\in I_2} S_i)=r(\underline{M} \times \bigcup_{i\in I} S_i) + 1.$$

Proof. Case 1. $|I_1| = 1$.

Let $I_1=\{j\}$ and $I_2=I-I_1$. Set $S=\bigcup_{i\in I} S_i$ and $T=\bigcup_{i\in I_2} S_i$. By Theorem 2.12 and Lemma 5.1,

$$\begin{aligned}
&r(\underline{M} \times S_i) + r(\underline{M} \times T)=r(\underline{M} \times S_j) + r(\underline{M} \times (\overline{S}_j \cap S))\\
&\leq r(\underline{M} \times S_j) + r(\underline{M} \times \overline{S}_j) + r(\underline{M} \times S) - r(\underline{M} \times (\overline{S}_j \cup S))\\
&=r(\underline{M} \times S_j) + r(\underline{M} \times \overline{S}_j) + r(\underline{M} \times S) - r(\underline{M})\\
&= \zeta(\underline{M};S_j,\overline{S}_j) - 1 + r(\underline{M} \times S)\\
&=r(\underline{M} \times S) + 1.
\end{aligned}$$

Now suppose $r(\underline{M} \times S_j) + r(\underline{M} \times T)=r(\underline{M} \times S)$ for some I and j. Then by Theorem 2.12(d) and the above result we have:

$$\begin{aligned}
r(\underline{M})&=r(\underline{M} \times S) + r(\underline{M} \times \overline{S}) + 1 - \zeta(\underline{M};S,\overline{S})\\
&\geq r(\underline{M} \times S) + r(\underline{M} \times \overline{S}) - 1\\
&=r(\underline{M} \times S_j) + r(\underline{M} \times T) + r(\underline{M} \times \overline{S}) - 1\\
&\geq r(\underline{M} \times S_j) + \sum_{i\in I_2} r(\underline{M} \times S_i) - (|I_2| - 1) + \sum_{i\in I_0-I} r(\underline{M} \times S_i)\\
&\quad - (|I_0-I| - 1) - 1\\
&= \sum_{i=1}^{k} r(\underline{M} \times S_i) - (k - 2)\\
&=r(\underline{M}) + k - 1 - (k - 2)=r(\underline{M}) + 1.
\end{aligned}$$

This is a contradiction, and the lemma is true for $|I_1| = 1$.

Case 2. $|I_1| \geq 2$.

A proof is obtained by simply applying the result of Case 1.

$$r(\underline{M} \times \bigcup_{i\in I_1} S_i) + r(\underline{M} \times \bigcup_{i\in I_2} S_i)$$
$$= \sum_{i\in I_1} r(\underline{M} \times S_i) - (|I_1| - 1) + \sum_{i\in I_2} r(\underline{M} \times S_i) - (|I_2| - 1)$$
$$= \sum_{i\in I} r(\underline{M} \times S_i) - |I| + 2$$
$$= r(\underline{M} \times (\bigcup_{i\in I} S_i)) + |I| - 1 - |I| + 2$$
$$= r(\underline{M} \times \bigcup_{i\in I} S_i) + 1.$$

Since this is the required result, the proof is complete. ∎

In Lemma 5.2, if we choose $I=I_0$ and $|\bigcup_{i\in I_1} S_i|, |\bigcup_{i\in I_2} S_i| \geq 2$, then $\bigcup_{i\in I_1} S_i$ and $\bigcup_{i\in I_2} S_i$ are 2-separators of $\underline{M}$.

Lemma 5.3 Let S_i, $1 \leq i \leq k$, be P-sets of $\underline{M}$. Then there is no circuit of $\underline{M}$ which has a non-null intersection with more than two S_i's.

Proof. Suppose there exists a circuit C of $\underline{M}$ which has a non-null intersection with S_i, S_j, and $E-S_i \cup S_j$, where $i \neq j$.

Set $S=E-(S_i \cup S_j)$, $S'_j=S_j \cup (S \cap C)$, and $T=S_i \cup S'_j$. If $|S|=1$, then $\underline{M} \times T=\underline{M} \times E=\underline{M}$ is connected. If $|S|\geq 2$, then, by Lemma 5.2, $S_i \cup S_j$ and S are 2-separators of $\underline{M}$ and hence, $\underline{M} \times (S_i \cup S_j \cup C)=$ $\underline{M} \times T$ is connected by Lemma 4.7. Using Theorems 2.12(d), 2.13, and Lemma 5.1, we deduce

$$2 \leq \xi(\underline{M} \times T; S_i, S'_j) = \mu(\underline{M} \times T) - \mu(\underline{M} \times S_i) - \mu(\underline{M} \times S'_j) + 1$$
$$= \mu(\underline{M} \times T) - \mu(\underline{M} \times S_i) - \mu(\underline{M} \times (T \cap \overline{S}_i)) + 1$$
$$\leq \mu(\underline{M} \times T) - \mu(\underline{M} \times S_i) - [\mu(\underline{M} \times T) + \mu(\underline{M} \times \overline{S}_i)$$
$$- \mu(\underline{M} \times (T \cup \overline{S}_i))] + 1$$
$$= - \mu(\underline{M} \times S_i) - \mu(\underline{M} \times \overline{S}_i) + \mu(\underline{M}) + 1$$

$= \xi(\underline{M};S_i, \overline{S}_i)=2.$

Therefore, $\xi(\underline{M} \times T;S_i, S_j')=2$. By Lemma 4.5

$$\mu(\underline{M} \times (S_i \cup S_j))=\mu(\underline{M} \times (S_i \cup S_j \cup C)) - 1$$

$$=\mu(\underline{M} \times T;S_i,S_j') + \mu(\underline{M} \times S_i) + \mu(\underline{M} \times S_j') - 2$$

$$=\mu(\underline{M} \times S_i) + \mu(\underline{M} \times (S_j \cup (S \cap C))$$

$$=\mu(\underline{M} \times S_i) + \mu(\underline{M} \times S_j),$$

since $\overline{S}_j \cap C$ is a series set of $\underline{M} \times (S_j \cup C)$ and properly contains $S \cap C$. However, this contradicts Lemma 5.2. Consequently, $C \in \underline{C} \times (S_i \cup S_j)$, and the proof is complete. ∎

<u>Lemma 5.4</u> Let S_i, $1 \leqslant i \leqslant k$, be P-sets of $\underline{M}$. Then $\underline{M} \times (S_i \cup S_j)$ is connected for any i and j, $i \neq j$.

<u>Proof.</u> Suppose the lemma fails. Then there exist two distinct cells e and e', such that no circuit of $\underline{M} \times (S_i \cup S_j)$ contains these cells. If e and e' belong to different sets, say, $e \in S_i$ and $e' \in S_j$, then, by definition, there is a circuit C of $\underline{M}$ containing e and e', and $C \subseteq S_i \cup S_j$ by Lemma 5.3. Therefore, e and e' belong to the same set, say e, e' S_i. Let C and C' be circuits of $\underline{M} \times (S_i \cup S_j)$, such that $e \in C$, $e' \in C'$, and $C \cap C' \neq \phi$. Such circuits always exist. For example, pick any $e_j \in S_j$, and let C and C' be circuits containing e, e_j and e', e_j, respectively. Choose C and C' so that $C \cup C'$ is minimal, consistent with the above condition. Let $e_0 \in C \cap C'$. By Axiom II we can find circuits C_0 and C_0' of $\underline{M}$ so that $e \in C_0 \subseteq C \cup C' - \{e_0\}$ and $e' \in C_0' \subseteq C \cup C' - \{e_0\}$. By the minimality of $C \cup C'$, $C_0 \cap C_0' = \phi$ and by

Axiom I $C \cap C'_0 \neq \phi$, $(C' - C) \cap C_0 \neq \phi$. Then $C \cup C'_0 \subseteq C \cup (C \cup C' - C_0) = C \cup C' - (C' - C) \cap C_0 \subset C \cup C'$. However, this is contrary to the minimality of $C \cup C'$. Therefore, there always exists a circuit of $\underline{M} \times (S_i \cup S_j)$ containing any two elements of $S_i \cup S_j$. Thus $\underline{M} \times (S_i \cup S_j)$ is connected. ∎

Lemma 5.5 Let $\underline{M}=(\underline{C}, E)$ be connected and S_i, $1 \leq i \leq k$, be P-sets of $\underline{M}$. If $C_i \in (\underline{C}|S_i)$, then $\underline{M}_i=\underline{M} \times (S_i \cup C_i)$ is connected.

Proof. If $|S_i| = 1$, then $\underline{M}_i=\underline{M} \times (S_i \cup C_i)=\underline{M} \times C_i$ is a connected matroid consisting of only one circuit C_i.

For $|S_i| \geq 2$, S_i is a 2-separator of $\underline{M}$ and the lemma follows from Lemma 4.7. ∎

For given P-sets of $\underline{M}$, $\underline{M}_i$'s are uniquely obtained within equivalence. We define matroids $\underline{M}_i(e_a)$ by replacing the series set $C \cap \overline{S}_i$ of $\underline{M} \times (S_i \cup C_i)$ by a new symbol, e_a, called a supplementary cell of $\underline{M}_i(e_a)$. More specifically, the cell set of $\underline{M}_i(e_a)$ is $S_i \cup \{e_a\}$ and the circuits $\underline{C}_i(e_a)$ are the union of $\underline{C} \times S_i$ and $\{(C \cap S_i) \cup \{e_a\} \mid C \in (\underline{C}|S_i)\}$. Matroids $\underline{M}_i(e_a)$, $1 \leq i \leq k$, are referred to as P-blocks of $\underline{M}$.

The following lemma is an immediate consequence of Lemma 5.5.

Lemma 5.6 P-blocks of a matroid are connected.

In the following discussion we shall use the following notational convention: Let $\underline{M}_i=(\underline{C}_i, E_i)$, $1 \leq i \leq k$, be matroids and $E_i \cap E_j = \{e\}$ for every pair of i and j, $i \neq j$. We denote

$$\overset{k}{\underset{i=1}{\Delta}} \underline{C}_i = \underline{C}_1 \Delta \underline{C}_2 \Delta \ldots \Delta \underline{C}_k$$
$$= (\bigcup_{i=1}^{k} (\underline{C}_i \times (E_i - \{e\}))) \cup (\bigcup_{1 \leq i < j \leq k} \underline{C}_{ij}),$$

where $\underline{C}_{ij} = \{C_i \cup C_j - \{e\} \mid C_i \in (\underline{C}_i \mid \{e\}), C_j \in (\underline{C}_j \mid \{e\})\}$.

<u>Theorem 5.1</u> Let $\underline{M}=(\underline{C}, E)$ be connected, and $\underline{M}_i(e_a) = (\underline{C}_i(e_a), S_i \cup \{e_a\})$, $1 \leq i \leq k$, be its P-blocks. Then $E = \bigcup_{i=1}^{k} S_i$ and $\underline{C} = \overset{k}{\underset{i=1}{\Delta}} \underline{C}_i(e_a)$.

<u>Proof.</u> It is obvious that $E = \bigcup_{i=1}^{k} S_i$. We will prove $\underline{C} = \overset{k}{\underset{i=1}{\Delta}} \underline{C}_i(e_a)$. Let C be a circuit of $\underline{M}$. If $C \in \underline{C}_i(e_a) \times S_i$ for some i, then $C \in \overset{k}{\underset{i=1}{\Delta}} \underline{C}_i(e_a)$. Suppose $C \cap S_i \neq \phi$ and $C \cap S_j \neq \phi$ for some i and j, $i \neq j$. By Lemma 5.3 $C \subseteq S_i \cup S_j$ and $C_i=(C \cap S_i) \cup \{e_a\} \in (\underline{C}_i(e_a) \mid \{e_a\})$ and $C_j=(C \cap S_j) \cup \{e_a\} \in (\underline{C}_j(e_a) \mid \{e_a\})$. Therefore, $C=(C \cap S_i) \cup (C \cap S_j) = C_i \cup C_j - \{e_a\} \in \underline{C}_{ij} \subseteq \overset{k}{\underset{i=1}{\Delta}} \underline{C}_i(e_a)$.

Now let $C \in \overset{k}{\underset{i=1}{\Delta}} \underline{C}_i(e_a)$. If $C \in \underline{C}_i(e_a) \times S_i$, then clearly C is a circuit of $\underline{M} \times S_i$ and hence of $\underline{M}$. Let $C \in \underline{C}_{ij}$ and $|S_i|$, $|S_j| \geq 2$. Then there exist $(e_a \in)$ $C_i \in \underline{C}_i(e_a)$ and $(e_a \in)$ $C_j \in \underline{C}_j(e_a)$, such that $C=C_i \cup C_j - \{e_a\}$. By definition, for some $C_i' \in (\underline{C} \mid S_i)$ and $C_j' \in (\underline{C} \mid S_j)$, $C_i - \{e_a\} = C_i' \cap S_i$ and $C_j - \{e_a\} = C_j' \cap S_j$. We can choose C_i' and C_j' so that $C_i' \cap S_j \neq \phi$ and $C_j' \cap S_i \neq \phi$. Suppose this is impossible, and, say, $C_i' \cap S_i = C_i - \{e_a\}$ implies $C_i' \cap S_j = \phi$. Since $\underline{M}$ is connected, we can find $C' \in \underline{C}$ such that $C' \cap S_i \neq \phi$ and $C' \cap S_j \neq \phi$. Choose C_i' and C' so that $(C_i' \cup C') \cap S_i$ is minimal consistent with this condition.

By Lemma 4.9, $(C_i' \cup C') \cap S_i$ contains a circuit C'' of $\underline{M}$. Let $e \in C'' \cap (C' - C_i')$ and $e_j \in C' \cap S_j$. There exists, by Axiom II, a circuit C''' of $\underline{M}$, such that $e_j \in C''' \subseteq C'' \cup C' - \{e\}$. Then $C''' \cap S_i \neq \phi$, $e_j \in C''' \cap S_j \neq \phi$, and $(C_i' \cup C''') \cap S_i \subseteq (C_i' \cup C') \cap S_i - \{e\}$. This contradicts the minimality of $(C_i' \cup C') \cap S_i$. Consequently, there exists C_i' satisfying $C_i' \cap S_j \neq \phi$. By Lemma 5.3, $C_i', C_j' \in \underline{C} \times (S_i \cup S_j)$ and hence, $C = C_i \cup C_j - \{e_a\} = (C_i' \cap S_i) \cup (C_j' \cap S_j)$ is a circuit of $\underline{M} \times (S_i \cup S_j)$ by Theorem 4.3.

For $|S_i| = 1$ or $|S_j| = 1$ we assume, without loss of generality, $|S_i| = 1$. In this case $C_i' \cap S_i$ consists of a single element, say e. Since $C_j' \cap S_j$ is not a circuit of $\underline{M} \times (S_i \cup S_j)$, $e \in C_j'$. Thus $C_j' = (C_j' \cap S_j) \cup (C_j' \cap S_i) = (C_j - \{e_a\}) \cup \{e\} = (C_j - \{e_a\}) \cup (C_i - \{e_a\}) = C$ is a circuit of $\underline{M} \times (S_i \cup S_j)$. Accordingly, the proof is complete. ∎

Theorem 5.1 represents a unique reconstruction of $\underline{M}$ from its P-blocks. The process generating P-blocks from $\underline{M}$ is called a P-split of $\underline{M}$, and we write $\underline{M} = \underset{i=1}{\overset{k}{\Delta}} \underline{M}_i(e_a)$. Since P-blocks are connected by Lemma 5.6, repeated P-splits of P-blocks yield a collection of unsplittable P-blocks called P-atoms. Successive P-splits define the P-decomposition of $\underline{M}$. This decomposition is a general case of C-decomposition, where at each P-split we choose k=2 and S_1, S_2 to be complementary non-series separators.

We comment on P-atoms and atoms of C-decomposition. As we have seen in Example 4.1, ${}_3\underline{M}_2$ can be an atom of C-decomposition. However, P-decomposition never yields this matroid as a P-atom. P-atoms generally consist of matroids ${}_3\underline{M}_3$, ${}_2\underline{M}_2$, and matroids isomorphic to max. 3-conn. minors of the original matroid.

<u>Theorem 5.2</u> Let $\underline{M}$ be connected. Every P-atom of $\underline{M}$ is isomorphic to ${}_3\underline{M}_3$, ${}_2\underline{M}_2$, or a max. 3-conn. minor of $\underline{M}$.

Every max. 3-conn. minor of $\underline{M}$ which is not ${}_3\underline{M}_2$ is isomorphic to a P-atom.

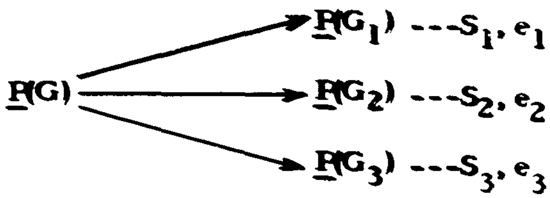
<u>Proof.</u> If $\underline{M}$ is a P-atom, by definition there is no P-split of $\underline{M}$, and therefore, $\underline{M}$ is 3-connected.

Let $\underline{M}=(\underline{C}, E)$ be a P-block and S_i, $1 \leq i \leq k$, be P-sets of $\underline{M}$. The P-blocks of $\underline{M}$ are denoted by $\underline{M}_i(e_a)=(\underline{C}_i(e_a), S_i \cup \{e_a\})$, $1 \leq i \leq k$. If $\underline{M}_i(e_a)$ is a 1-circuit matroid, then clearly its P-atoms are isomorphic to ${}_3\underline{M}_3$ or ${}_2\underline{M}_2$. Suppose $\underline{M}_i(e_a)$ is not a 1-circuit matroid. Then $|S_i| \geq 2$ and S_i is a 2-separator of $\underline{M}$ by the definition of P-sets. Let $C_i \in (\underline{C}\,|\,S_i)$. As shown in the proof of Lemma 4.18, a max. 3-conn. minor of $\underline{M} \times (S_i \cup C_i)$ is also a max. 3-conn. minor of $\underline{M}$. Since $\underline{M}_i(e_a)$ is obtained from $\underline{M} \times (S_i \cup C_i)$ by replacing $\bar{S}_i \cap C_i$ by supplementary cell e_a, every max. 3-conn. minor of $\underline{M}_i(e_a)$ is isomorphic to a max. 3-conn. minor of $\underline{M}$. By induction, the first part of the theorem follows.

In proving the second part of the theorem let $\underline{M}=(\underline{C}, E)$ be a P-block, and $\underline{M}_i(e_a)$, $1 \leq i \leq k$, be given as above. Suppose $\underline{M}'=(\underline{C}', E')$ is a max. 3-conn. minor of $\underline{M}$ and $\underline{M}' \not\cong {}_3\underline{M}_2$. Since the theorem is trivial for $|E'| \leq 3$, we assume $|E'| \geq 4$. By Theorem 4.2, there exists a 2-separator S_i of $\underline{M}$, such that $|\bar{S}_i \cap E'| \leq 1$, and $\underline{M}'$ is isomorphic to a minor of a P-block $\underline{M}_i(e_a)$ by Theorem 4.6. By induction $\underline{M}'$ is isomorphic to a minor of some P-atom. Since P-atoms which are not ${}_3\underline{M}_3$, ${}_2\underline{M}_2$ are isomorphic to max. 3-conn. minors of the original matroid, $\underline{M}'$ is isomorphic to a P-atom. The proof is complete. ∎

A P-atom is referred to as a <u>P-non-essential atom</u> if it is isomorphic to ${}_3\underline{M}_3$ or ${}_2\underline{M}_2$, and otherwise as a <u>P-essential atom.</u> A <u>complete set of P-atoms</u> consists of all the non-essential atoms and P-essential atoms of a P-decomposition.

<u>Example 5.1</u> Let $\underline{M}=(\underline{C}, E)$ be the polygon matroid of the graph G in Fig. 5.1. The first P-split is determined by P-sets $S_1=\{1, 2, \ldots, 8\}$, $S_2=\{9, 10, \ldots, 13\}$, and $S_3=\{14, 15, \ldots, 20\}$. In the figure e_1, e_2, e_3, e_4, and e_5 are supplementary cells at each P-split.



$P(G_2)$ is a $\underline{P}$-atom since it has no P-sets. Let $S_{11}=\{1, 2, 3, 4\}$, $S_{12}=\{5, 6\}$, $S_{13}=\{7\}$ and $S_{14}=\{8, e_1\}$. The corresponding P-blocks $\underline{P}(G_{1i})$, $1 \leq i \leq 4$, are shown in Fig. 5.3. For $\underline{P}(G_3)$ choose $S_{31}=\{14, 15, e_1\}$ and $S_{32}=\{16, 17, 18, 19, 20\}$. $\underline{P}(G_{31})$ and $\underline{P}(G_{32})$ are P-blocks of $\underline{P}(G_3)$ (Fig. 5.3).

Similarly $\underline{P}(G_{11})$ and $\underline{P}(G_{31})$ have further P-splits, and their blocks are the polygon matroids of G_{11i} and G_{31i}, $1 \leq i \leq 3$, given in Fig. 5.4.

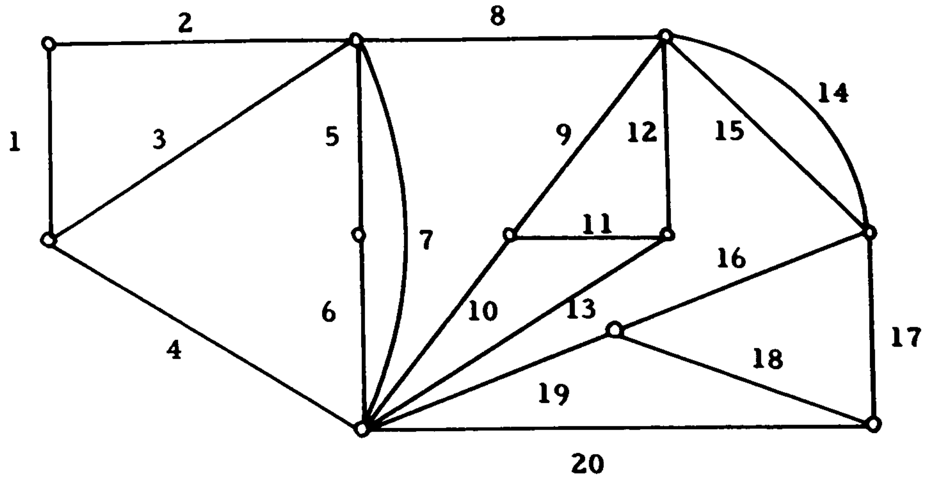


Figure 5.1 Graph G of Example 5.1

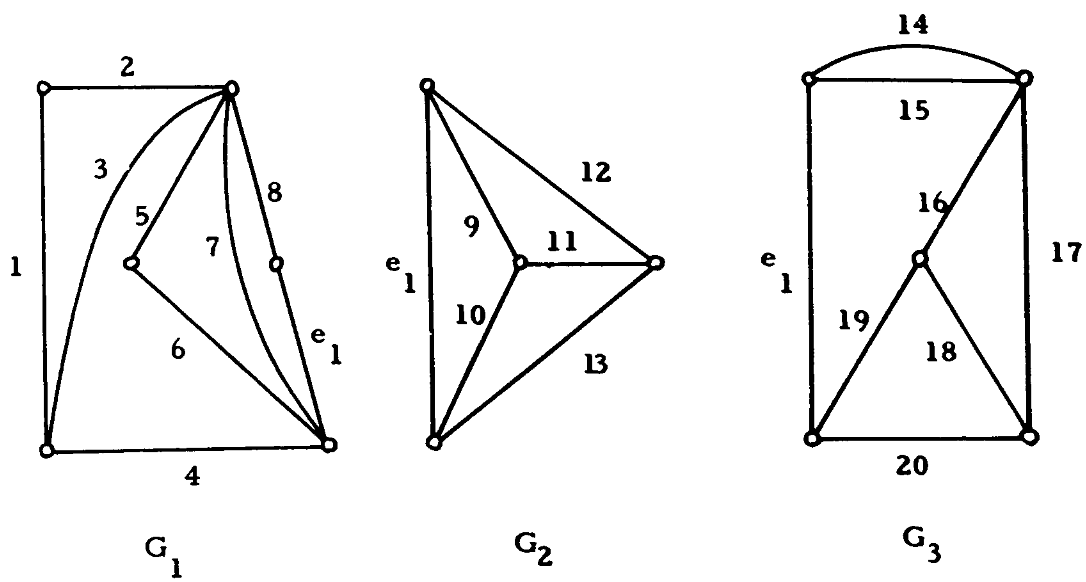


Figure 5.2 Graphs of Example 5.1

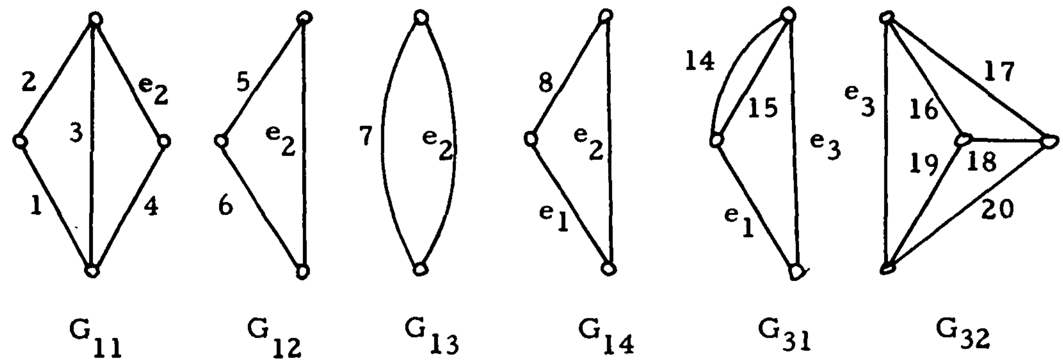


Figure 5. 3 Graphs of Example 5. 1

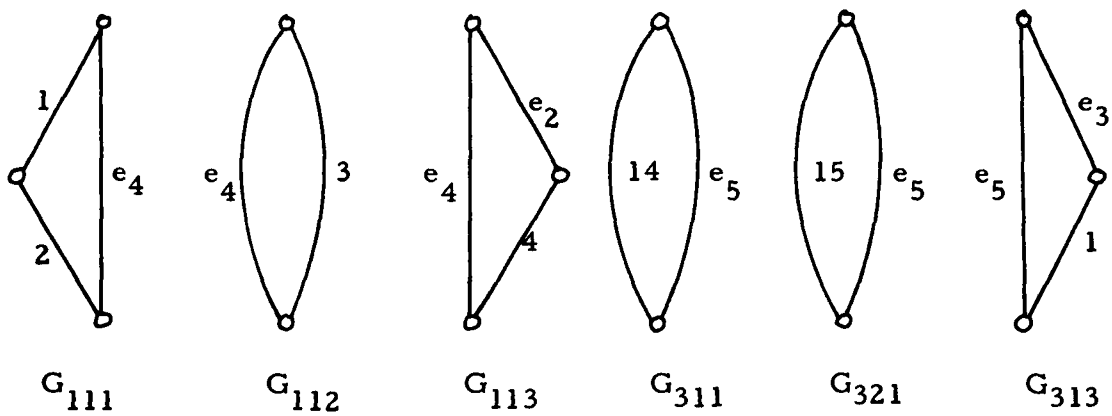


Figure 5. 4 Graphs of Example 5. 1

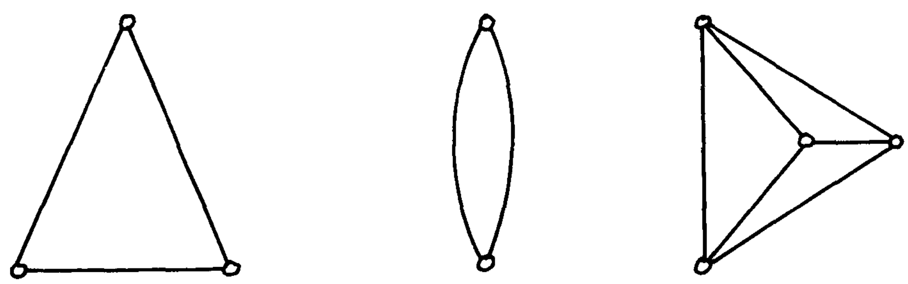


Figure 5. 5 Representatives of Isomorphism Classes of the P-Atoms

Graph G has three isomorphism classes of P-atoms, and their representatives are shown in Fig. 5. 5.

In Example 5. 1 we demonstrated the P-decomposition of a polygon matroid, where the subgraphs corresponding to P-sets are connected in parallel. Symbol " P " is used to stand for " parallel ". Thus at each step of the P-split of a polygon matroid the corresponding graph is decomposed into its parallel connected subgraphs. The dual results are obtained with a bond matroid. As seen in the next example, P-decomposition of a bond matroid decomposes a graph into its series connected subgraphs.

Example 5. 2 Consider the bond matroid $\underline{M}$ of the graph G in Fig. 5. 6. $S_1 = \{1, 2, \ldots, 8\}$, $S_2 = \{9, 10, \ldots, 13\}$, $S_3 = \{14, 15\}$, and $S_4 = \{16\}$ are P-sets of $\underline{M} = \underline{B}(G)$, and the corresponding P-blocks are the bond matroids of the graphs shown in Fig. 5. 7, where e is a supplementary cell.

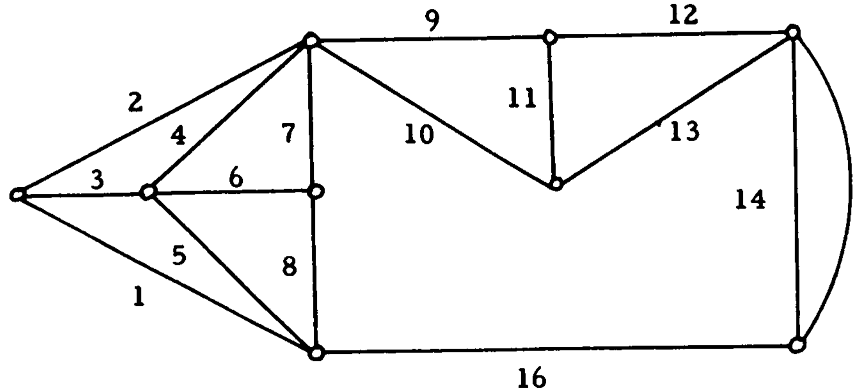


Figure 5.6 Graph G of Example 5.2

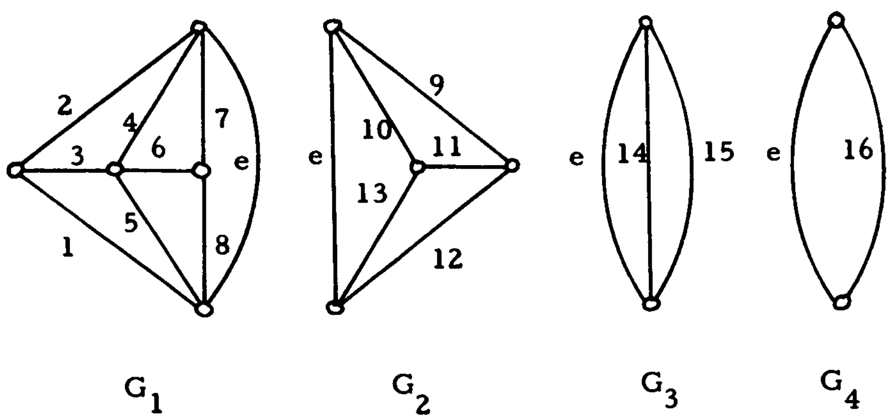


Figure 5.7 Graphs of Example 5.2

5.2 S-DECOMPOSITION

In this section we consider the dual of P-decomposition. Let $\underline{M}=(\underline{C}, E)$ be connected and S_i, $1 \leqslant i \leqslant k$, be a partition of E, where $|E| \geqslant 3$ and $k \geqslant 2$. Then the sets S_i's are called S-sets of $\underline{M}$ if they satisfy the following conditions:

(i) S_i is a 2-separator of $\underline{M}$ if $|S_i| \geqslant 2$

(ii) $\sum_{i=1}^{k} r(\underline{M} \times S_i) = r(\underline{M}) + 1$

(iii) S_i's are minimal with respect to conditions (i) and (ii).

As seen in the next lemma, S-sets of $\underline{M}$ are P-sets of $\underline{M}^*$, and vice-versa.

Lemma 5.7 Let $\underline{M}=(\underline{C}, E)$ be a matroid, and S_i, $1 \leqslant i \leqslant k$, be a partition of E. Then S_i's are S-sets of $\underline{M}$ if and only if they are P-sets of $\underline{M}^*$.

Proof. It suffices to show that the second condition of S-sets is equivalent to the second condition of P-sets for $\underline{M}^*$.

Let S_i, $1 \leqslant i \leqslant k$, be S-sets of $\underline{M}$. Set $R=\sum_{i=1}^{k} r(\underline{M} \times S_i) - r(\underline{M}) - 1$. By Theorem 2.12

$$R=\sum_{i=1}^{k} \left[|S_i| - \mu(\underline{M} \times S_i)\right] - \mu(\underline{M}^*) - 1$$
$$= -\sum_{i=1}^{k} \mu(\underline{M} \times S_i) + |E| - \mu(\underline{M}^*) - 1$$
$$= -\sum_{i=1}^{k} r(\underline{M}^* \cdot S_i) + r(\underline{M}^*) - 1.$$

Lemma 5.1 is valid after replacing "P-sets of $\underline{M}$" by "S-sets of $\underline{M}$", because only the first condition is used in the proof. Therefore

$$\xi(\underline{M}; S_i, \overline{S}_i) = \xi(\underline{M}^*; S_i, \overline{S}_i)$$
$$= r(\underline{M}^* \times S_i) - r(\underline{M}^* \cdot S_i) + 1 = 2,$$

or

$$r(\underline{M}^*.S_i)=r(\underline{M}^* \times S_i) - 1.$$

Then

$$R= - \sum_{i=1}^{k} r(\underline{M}^* \times S_i) - 1 \quad + r(\underline{M}^*) - 1$$

$$= - \sum_{i=1}^{k} r(\underline{M}^* \times S_i) + r(\underline{M}^*) + k - 1.$$

The lemma follows by setting R=0. ■

<u>Lemma 5.8</u> Let $\underline{M}=(\underline{C}, E)$ be connected, and S_i, $1 \leqslant i \leqslant k$, be S-sets of $\underline{M}$. If I is a non-null subset of $I_0=\{i \mid 1 \leqslant i \leqslant k\}$, then

$$r(\underline{M} \times \bigcup_{i \in I} S_i)= \sum_{i \in I} r(\underline{M} \times S_i).$$

<u>Proof.</u> Let $I'=I_0 - I$ and $T= \bigcup_{i \in I} S_i$, $\bar{T}= \bigcup_{i \in I'} S_i$. By Lemmas 5.2 and 5.7

$$r(\underline{M}^* \times T) + r(\underline{M}^* \times \bar{T})=r(\underline{M}^*) + 1,$$

or

$$r(\underline{M} \times T) + r(\underline{M} \times \bar{T})= \xi(\underline{M};T, \bar{T}) + r(\underline{M}) - 1$$

$$= \xi(\underline{M}^*;T, \bar{T}) + r(\underline{M}) - 1=r(\underline{M}) + 1.$$

Using this equality

$$r(\underline{M} \times T) \leqslant \sum_{i \in I} r(\underline{M} \times S_i)= \sum_{i \in I_0} r(\underline{M} \times S_i) - \sum_{i \in I'} r(\underline{M} \times S_i)$$

$$\leqslant r(\underline{M}) + 1 - r(\underline{M} \times \bar{T})=r(\underline{M} \times T).$$

Thus the lemma follows. ■

Lemma 5.8 implies that S_i, $i \in I \subset I_0$, are separators of $\underline{M} \times \bigcup_{i \in I} S_i$ and every circuit of $\underline{M} \times \bigcup_{i \in I} S_i$ is contained in some S_i, $i \in I$, by Theorem 4.1(d).

The following lemma is deduced from Lemma 5.8.

<u>Lemma 5.9</u> Let S_i, $1 \leqslant i \leqslant k$, be S-sets of $\underline{M}$. Then every circuit of $\underline{M}$ is either contained in some S_i or it has a non-null intersection with all S_i's.

Let $\underline{M}=(\underline{C}, E)$ be connected, where $|E|\geqslant 3$, and S_i, $1\leqslant i\leqslant k$, be S-sets of $\underline{M}$. Lemma 5.9 is equivalent to the condition:

$$(\underline{C}|S_1)=(\underline{C}|S_2)= \dots =(\underline{C}|S_k).$$

Let $\underline{C}_i(e_x)=(\underline{C} \times S_i) \cup \{(C\cap S_i) \cup \{e_x\} \mid C \in (\underline{C}|S_i)\}$ for $1\leqslant i\leqslant k$.

It can be easily shown that $\underline{M}_i(e_x)=(\underline{C}_i(e_x), S_i \cup \{e_x\})$, $1\leqslant i\leqslant k$, are connected matroids, where e_x is a supplementary cell of $\underline{M}_i(e_x)$. $\underline{M}_i(e_x)$, $1\leqslant i\leqslant k$, are referred to as <u>S-blocks</u> of $\underline{M}$.

Let $\underline{M}_i=(\underline{C}_i, E_i)$, $1\leqslant i\leqslant k$, be matroids and $E_i\cap E_j=\{e\}$ for every pair of i and j, where $i \neq j$. Define

$$\mathop{\square}_{i=1}^{k} \underline{C}_i = \underline{C}_1 \square \underline{C}_2 \square \dots \square \underline{C}_k$$
$$= (\bigcup_{i=1}^{k}(\underline{C}_i \times (E_i-\{e\}))) \cup \underline{C}_{12\dots k},$$

where $\underline{C}_{12\dots k}=\left\{\bigcup_{i=1}^{k} C_i - \{e\} \;\middle|\; C_i \in (\underline{C}_i|\{e\}),\ 1\leqslant i\leqslant k\right\}$.

The following theorem corresponds to Theorem 5.1 on P-blocks.

<u>Theorem 5.3</u> Let $\underline{M}=(\underline{C}, E)$ be connected and $\underline{M}_i(e_x)=(\underline{C}_i(e_x), S_i \cup \{e_x\})$, $1\leqslant i\leqslant k$, be its S-blocks. Then $E=\bigcup_{i=1}^{k} S_i$ and $\underline{C}=\mathop{\square}_{i=1}^{k}\underline{C}_i(e_x)$.

<u>Proof.</u> Since $E = \bigcup_{i=1}^{k} S_i$ is obvious, we have only to show that $\underline{C} = \mathop{\square}_{i=1}^{k} \underline{C}_i(e_x)$.

If $C \in \underline{C}$, then $C \subsetneq S_i$ for some i or C intersects with all S_i's. For $C\subsetneq S_i$, $C \in \underline{C}_i(e_x) \times S_i \in \mathop{\square}_{i=1}^{k} \underline{C}_i(e_x)$. If $C\in (\underline{C}|S_i)$, then, by Lemma 5.9, $C \in (\underline{C}|S_j)$ and $C_j' = (\underline{C}\cap S_j) \cup \{e_x\} \in \underline{C}_j(e_x)$ for $1 \leqslant j \leqslant k$. Therefore, $C = \bigcup_{j=1}^{k} (C\cap S_j) = \bigcup_{j=1}^{k} ((C_j' - \{e_x\})\cap S_j)$ $= \bigcup_{j=1}^{k} C_j' - \{e\} \in \mathop{\square}_{j=1}^{k} \underline{C}_j(e_x)$.

Let $C \in \bigsqcup_{i=1}^{k} \underline{C}_i(e_x)$. If $C \in \underline{C}_i(e_x) \times S_i$, then $C \in \underline{C} \times S_i \subseteq \underline{C}$. Suppose $C \in \underline{C}_{12\cdots k}$. There exists $C_i \in (\underline{C}_i \mid \{e_x\})$, $1 \leqslant i \leqslant k$, such that $C = \bigcup_{i=1}^{k} C_i - \{e_x\} = \bigcup_{i=1}^{k} (C_i \cap S_i)$. To show $C \in \underline{C}$, we assume $C \notin \underline{C}$. By hypothesis, there exists a circuit C' of $\underline{M}$ which has a non-null intersection with all S_i's. Choose C' so that $C \cup C'$ is minimally consistent with this condition. Let $C \cap S_i \neq C' \cap S_i$ for some i. By definition, $C_i' = (C' \cap S_i) \cup \{e_x\}$ is a circuit of $\underline{M}_i(e_x)$. Since no circuit of $\underline{M}_i(e_x)$ contains a circuit of $\underline{M}_i(e_x)$ properly, we can find $e \in C_i' - C_i$ and $C_i'' \in \underline{C}_i(e_x) \cap \underline{C}$ so that $e \in C_i'' \subseteq C_i \cup C_i' - \{e_x\}$. Let $e' \in (C' - C_i'') \cap \overline{S}_i$. By Axiom II, there exists $C'' \in \underline{C}$, such that $e' \in C'' \subseteq C' \cup C_i'' - \{e\}$. Then, $C \cup C'' \subseteq C \cup (C' \cup C_i'' - \{e\}) \subseteq C \cup C' - \{e\}$ and C'' has a non-null intersection with all S_i's. This is contrary to the hypothesis. Accordingly, C is a circuit of $\underline{M}$. ∎

From Theorem 5.3 we write $\underline{M} = \bigsqcup_{i=1}^{k} \underline{M}_i(e_x)$, and refer to the expression as an <u>S-split</u> of $\underline{M}$. <u>S-atoms</u> and <u>S-decomposition</u> are accordingly defined.

To show the connection of a P-split and an S-split, we shall prove some additional lemmas.

<u>Lemma 5.10</u> Let $\underline{M} = (\underline{C}, E)$ be connected, and S be a 2-separator of $\underline{M}$. If $C \in (\underline{C} \mid S)$, then $\underline{M} \cdot S = (\underline{M} \times (S \cup C)) \cdot S$.

<u>Proof.</u> Let $\underline{L}_1 = \{C' \cap S \neq \phi \mid C \in \underline{C}\}$ and $\underline{L}_2 = \{C' \cap S \neq \phi \mid C' \in \underline{C} \times (S \cup C)\}$. $\underline{M} \cdot S$ and $(\underline{M} \times (S \cup C)) \cdot S$ are the classes of non-null minimal members of $\underline{L}_1$ and $\underline{L}_2$. Therefore, it is sufficient to show

$\underline{L}_1 = \underline{L}_2$. Since $\underline{L}_1 \supseteq \underline{L}_2$, we will prove $\underline{L}_1 \subseteq \underline{L}_2$. Let $C_1 \cap S \in \underline{L}_1$, where $C_1 \in \underline{C}$. If $C_1 \subseteq S$, then $C_1 \cap S = C_1 \in \underline{C} \times (S \cup C)$ and hence, $C_1 \cap S \in \underline{L}_2$. Now suppose $C_1 \in (\underline{C}|S)$. By Theorem 4.3, $C_2 = (C_1 \cap S) \cup (C \cap \overline{S}) \in \underline{C} \times (S \cup C) \subseteq \underline{C}$, and $C_1 \cap S = C_2 \cap S \in \underline{L}_2$. Thus, $\underline{L}_1 \subseteq \underline{L}_2$, and the lemma follows. ∎

__Lemma 5.11__ Let S be a 2-separator of $\underline{M}$ and $C \in (\underline{C}\backslash S)$. Then $\underline{M}^* \times S = (\underline{M}^* \cdot (S \cup C)) \times S$.

__Proof.__ This lemma follows from Theorem 2.8 and Lemma 5.10. ∎

__Theorem 5.4__ Let $\underline{M} = (\underline{C}, E)$ be a connected matroid. Then $\underline{M} = \mathop{\square}\limits_{i=1}^{k} \underline{M}_i(e_x)$ if and only if $\underline{M}^* = \mathop{\Delta}\limits_{i=1}^{k} \underline{M}_i{}^*(e_x)$.

__Proof.__ Let S_i, $1 \leqslant i \leqslant k$, be S-sets of $\underline{M}$ and $\underline{M}_i(e_x) = (\underline{C}_i(e_x), S_i \cup \{e_x\})$, $1 \leqslant i \leqslant k$, be the corresponding S-blocks. From Lemma 5.7 $\underline{M}_i{}'(e_x) = (\underline{C}_i{}'(e_x), S_i \cup \{e_x\})$, $1 \leqslant i \leqslant k$, are P-blocks of $\underline{M}^*$. We establish the theorem by showing $\underline{C}_i{}'(e_x) = \underline{C}_i{}^*(e_x)$.

Let $C_i \in (\underline{C}|S_i)$, $\underline{M}_i = \underline{M} \times (S_i \cup C_i)$, and $e \in C_i \cap S_i$. Since $C_i \cap \overline{S}_i$ is a series set of $\underline{M}_i$, $\underline{M}_i(e_x)$ is obtained from $\underline{M}_i(e) = (\underline{C}_i(e), S_i \cup \{e\}) = \underline{M}_i \cdot (S_i \cup \{e\})$ by simply replacing e with e_x. Let $\underline{M}_i{}'(e) = (\underline{C}_i{}'(e), S_i \cup \{e\})$, where $\underline{C}_i{}'(e) = (\underline{C}^* \times (S_i \cup C_i{}^*))$. $(S_i \cup \{e\})$ and $e \in C_i{}^* \in (\underline{C}^*|S_i)$. It suffices to show $\underline{C}_i{}^*(e) = (\underline{C}^* (S_i \cup C_i)) \times (S_i \cup \{e\}) = \underline{C}_i{}'(e)$. Since $C_i{}^* \cap \overline{S}_i$ is a series set of $\underline{M}_i{}'(e)$, $\underline{C}^* \times S_i \subset \underline{C}_i{}'(e)$, and, by Lemma 5.11, $\underline{C}^* \times S_i \subset \underline{C}_i{}^*(e)$.

Let $e \in X^* \in \underline{C}_i{}^*(e)$. There exists $C^* \in \underline{C}^*$ such that

$e \in C^*$ and $X^* = C^* \cap (S_i \cup C_i) = C^* \cap (S_i \cup \{e\})$. $Y^* =$ $(C^* \cap S_i) \cup (C_i^* \cap \bar{S}_i)$ is a circuit of $\underline{M}^* \times (S_i \cup C_i^*)$ by Theorem 4.3 and hence, $Y^* \cap (S_i \cup \{e\}) = (C^* \cap S_i) \cup \{e\} = X^* \in \underline{C}_i'(e)$. Therefore, $\underline{C}_i^*(e) \subsetneq \underline{C}_i'(e)$.

Now let $e \in X^* \in \underline{C}_i'(e)$. $X^* = C^* \cap (S_i \cup \{e\})$ for some $C^* \in \underline{C}^* \times (S_i \cup C_i^*)$ containing e. Since $\underline{M}_i^*(e)$ is connected, there exists a circuit C'^* of $\underline{M}^*$ such that $e \in C'^* \cap (S_i \cup \{e\}) \in \underline{C}_i^*(e)$. Then $Y^* = (C^* \cap S_i) \cup (C'^* \cap \bar{S}_i)$ is a member of $\underline{C}^*$ by Theorem 4.3, and $X^* = Y^* \cap (S_i \cup C_i) \subseteq S_i \cup \{e\}$. If $X^* \notin \underline{C}_i^*(e)$, then by definition X^* properly contains a member Z^* of $\underline{C}_i^*(e)$. As shown above, $Z^* \in \underline{C}_i'(e)$ and this contradicts Axiom I. Consequently, $X^* \in \underline{C}_i^*(e)$ and $\underline{C}_i'(e) \subseteq \underline{C}_i^*(e)$. Thus $\underline{C}_i^*(e) = \underline{C}_i'(e)$ and $\underline{M}_i^*(e) = \underline{M}_i'(e)$. ∎

An S-atom isomorphic to ${}_3\underline{M}_2$ or ${}_2\underline{M}_2$ is called an <u>S-non-essential atom</u>, and otherwise an <u>S-essential atom</u>; together, they form a <u>complete set of S-atoms</u>.

From Theorem 5.4 we deduce the next theorem.

<u>Theorem 5.5</u> Let $\underline{M}$ be a connected matroid. The dual of P-atoms of $\underline{M}$ are S-atoms of $\underline{M}^*$, and vice-versa.

<u>Theorem 5.6</u> Let $\underline{M}$ be connected. Every S-atom of $\underline{M}$ is isomorphic to ${}_3\underline{M}_2$, ${}_2\underline{M}_2$, or a max. 3-conn. minor of $\underline{M}$.

Every max. 3-conn. minor of $\underline{M}$ which is not ${}_3\underline{M}_3$ is isomorphic to an S-atom.

Proof. The theorem follows from Theorems 2.14, 4.2, and 4.5. ∎

Example 5.3 Let $\underline{M}$ be the polygon matroid of graph G shown in Fig. 5.8. The first S-split is defined by S-sets of $\underline{M}=\underline{P}(G)$: $S_1=\{1,2,\ldots,7\}$, $S_2=\{8\}$, $S_3=\{9,10\}$, and $S_4=\{11,12,\ldots,15\}$. Then $\underline{P}(G)=\coprod_{i=1}^{4}\underline{P}(G_i)$, where G_i's are given in Fig. 5.9 and e_1 is a supplementary cell.

$\underline{P}(G_1)$ has a further S-split into $\underline{P}(G_{1i})$, $1\leqslant i\leqslant 3$, which are the polygon matroids of the graphs in Fig. 5.10, where e_2 denotes a supplementary cell.

S-blocks of $\underline{P}(G_{13})$ consist of the polygon matroids of G_{131} and G_{132} which are shown in Fig. 5.11.

This S-decomposition yields the following complete set of atoms: $\underline{P}(G_{11})$, $\underline{P}(G_{12})$, $\underline{P}(G_{131})$, $\underline{P}(G_{132})$, $\underline{P}(G_2)$, $\underline{P}(G_3)$, and $\underline{P}(G_4)$.

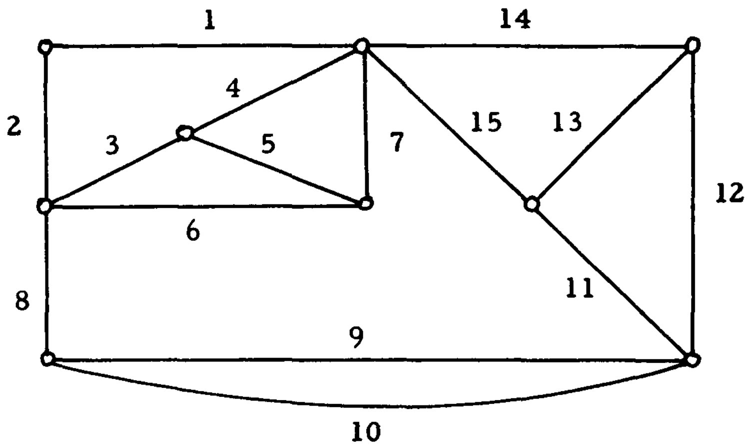


Figure 5.8 Graph G of Example 5.3

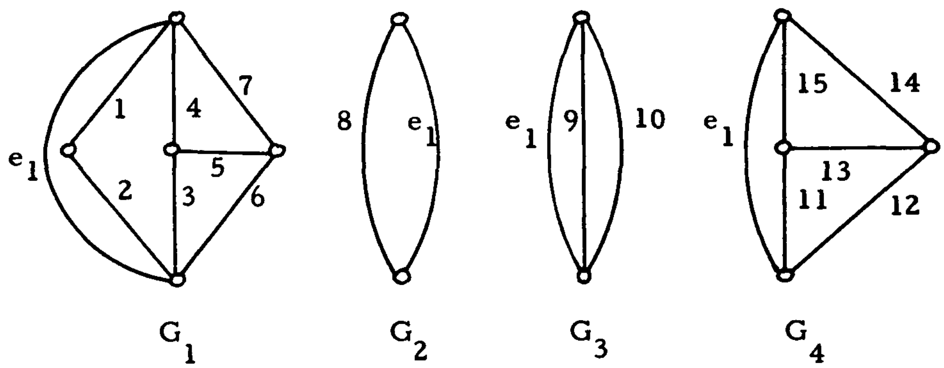


Figure 5.9 Graphs of Example 5.3

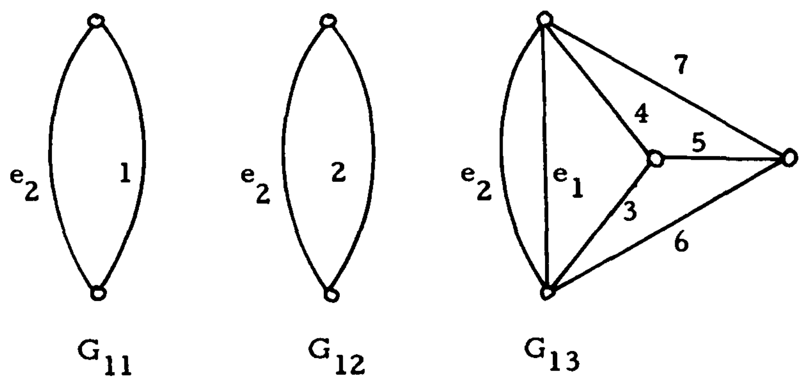


Figure 5.10 Graphs of Example 5.3

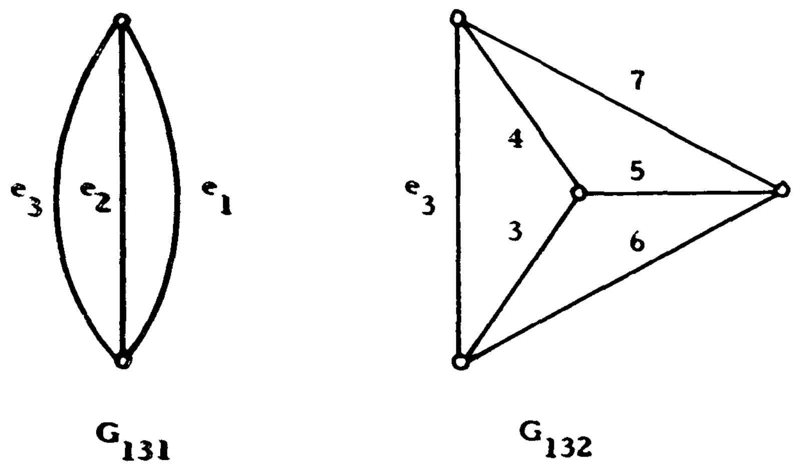


Figure 5.11 Graphs of Example 5.3

5.3 SPLIT DECOMPOSITION OF GRAPHS

The split decomposition of graphs was introduced by MacLane [Ma 1], and its generalization to matroids was considered in previous sections of this dissertation. MacLane's split decomposition of a graph can be interpreted as the P-decomposition of the polygon matroid or S-decomposition of the bond matroid.

Let $G = (V, E)$ be a 2-connected graph, which is not a polygon graph. A set $\{H_1, H_2, \ldots, H_k\}$ of non-null subgraphs of G is called a split of G at vertices v and v' if it satisfies the following conditions:

(i) H_i, $1 \leqslant i \leqslant k$, are edge-disjoint connected subgraphs of G, and the union of $E(H_i)$ is $E(G)$.

(ii) $V(H_i) \cap V(H_j) = \{v, v'\}$ for $i \neq j$, and the union of $V(H_i)$ is $V(G)$.

(iii) $k \geqslant 3$ if $|E(H_i)| = 1$ for some i.

Suppose that a 2-connected graph $G = (V, E)$ has a split $\{H_1, H_2, \ldots, H_k\}$ at v and v'. We define a set of new graphs by joining v and v' with a new edge e_x, called a supplementary edge. These graphs are denoted by $H_1(e_x)$, $H_2(e_x), \ldots, H_k(e_x)$, and are called blocks of G at v and v'. Graph G may be reconstructed from its blocks by deleting e_x of $H_i(e_x)$, $1 \leqslant i \leqslant k$, and joining the resulting graphs at the same ends of e_x.

A split is called least if none of $H_i(e_x)$, $1 \leqslant i \leqslant k$, has a split at v and v'. Our decomposition is performed based on the least split operation.

Let G be a 2-connected graph. A least split of G yields a collection of 2-connected graphs, i. e., its blocks. Some of these blocks might have further least splits. At each least split we repeatedly add a supplementary edge until the resulting graphs can no longer have a split. This decomposition process eventually yields a class of unsplittable graphs called atoms of G. Each of the atoms is either a polygon or a 3-connected graph, and is referred to as a polygon or an essential atom, respectively. All the polygons and essential atoms form a complete set of atoms. The class of essential atoms is independent of a splitting process, and the supplementary edges of an atom may be replaced by paths of G so that the resulting graph is a subgraph of G [Ma 1, Ta 1].

Theorem 5.7 A complete set of essential atoms is unique.

Lemma 5.12 The supplementary edges of an atom may be replaced by paths of G so that no vertices on these paths are in common with the atom except the ends of the supplementary edges, and the resulting graph is a subgraph of G.

Let e_x be a supplementary edge and $G_i = (V_i, E_i)$, $1 \leqslant i \leqslant k$, be graphs containing e_x, such that $V_i \cap V_j$ consists of two vertices which are the ends of e_x and $E_i \cap E_j = \{e\}$ for $i \neq j$. The merge graph

G of G_i' s is defined by [Ho 1].

$$G = (\bigcup_{i=1}^{k} V_i, \bigcup_{i=1}^{k} (E_i - \{e_x\})).$$

Let e_x be a supplementary edge and G_i, $1 \leq i \leq k$, be the atoms containing e_x. We form the merge graph G' of G_i's as above. Then choose any supplementtray edge e_y of G', and form the new merge graph G'' of G' and the remaining atoms which contain e_x. Repeating this merge operation for all the supplementary edges, we finally obtain the original graph.

Since a complete set of atoms is obtained by a set of splitting vertices and independent of a particular sequence of vertex pairs, the original graph may be uniquely reconstructed by any merging process, that is, any sequential choice of supplementary edges.

Theorem 5.8 The original graph is reconstructed from its atoms by any sequence of applications of the merge operation.

5.4 APPLICATION TO PLANAR n-PORT NETWORKS

In this section we consider an application of split decomposition of graphs to network theory. The electrical terminology used in the following discussion will be found in network theory books, such as Seshu and Reed [Se 1] or Weinberg [We 1].

An n-port network may be considered to be a graph of which each of n pairs of vertices is distinguished as a port, where a voltage or current source may be applied or a response measured. Each edge of an n-port network represents a physical element, such as resistor, inductor, or capacitor. An n-port network N is called planar if the network remains planar on adding an edge at each port and these newly added edges are on a mesh, when the network is embedded in the plane [Se 1]. With this definition, a dual n-port of a planar n-port network may not be a planar network. If a dual network is also planar, then the planar n-port network is called totally planar.

A planar n-port network is always a planar graph; however, the converse is not always true. Consider the two-port network N in Fig. 5.12(a), where the vertex pairs (1, 3) and (2, 4) are the ports. Though graph N', obtained from N by adding edges e_5 and e_6, is planar, it is not a planar two-port network (Fig. 4.12(b)). As seen in Fig. 4.13, if we try to pull out e_5 and e_6 on the infinite region, then the internal network becomes nonplanar.

Let N be an n-port network and N' be the graph obtained from N by adding an edge at each port of N. It is necessary that N'

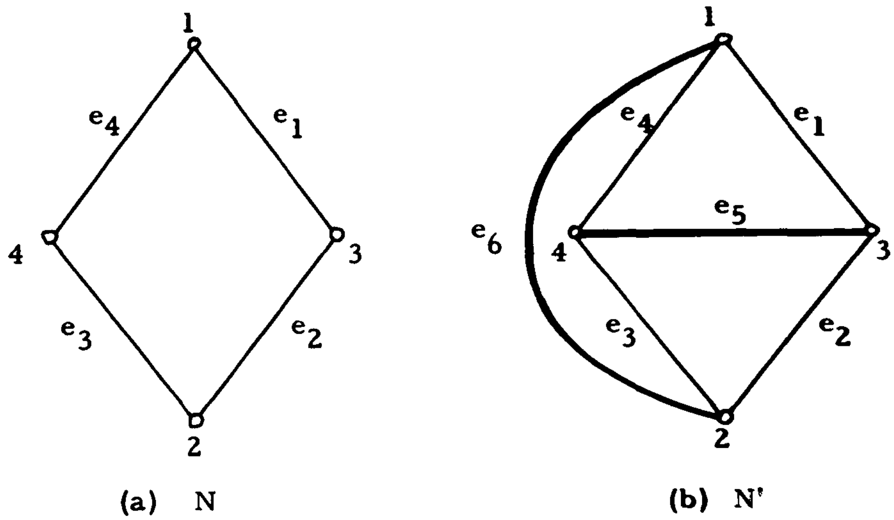


Figure 5.12 A Nonplanar Two-Port Network

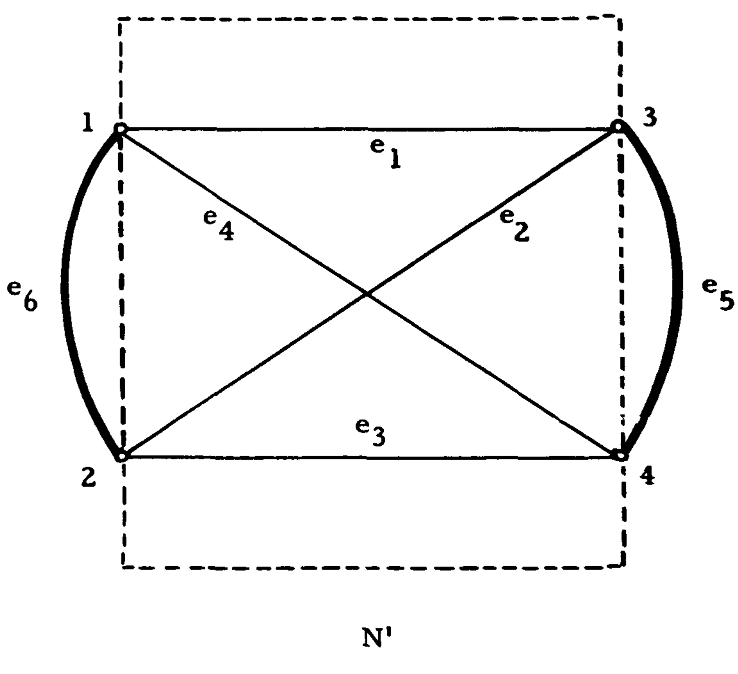


Figure 5.13 Plane Graph of N' whose Internal Network is Nonplanar

be a planar graph for N to be a planar n-port network. Given planar graph N', we will establish a necessary and sufficient condition for the newly added edges to be on a mesh of an embedded graph of N'.

Let N' be given above. We construct the graph N'' from N' by adding a new vertex and connecting all the port vertices of N' with it. Then, as it is easily proved, N is a planar n-port network if and only if N'' is a planar graph. Therefore, whether N is a planar network or not is determined by the planarity test of N''. However, the planarity condition for N established in this section will be stated in terms of atoms of N' due to split decomposition.

The theorems in this section can also be applied to planar n-terminal networks [We 2]. An <u>n-terminal network</u> is a graph N of which n vertices are identified as the <u>terminals</u>. It is <u>planar</u> if it remains planar on adding a new vertex and connecting the n terminals with it. Let N be a planar n-terminal network and N' be the graph obtained from N by adding an extra vertex and connecting the n terminals with it. Then N is a planar network if and only if those added edges are on a comesh of N'.

Let $G = (V, E)$ be a graph. A subset S of E is said to be <u>on a mesh</u> of G if G is planar and there exists a plane graph $\sigma(G)$ of which S is a subset of the edges of a mesh.

Let S be a subset of the edge set. We assign a color to the members of S and call them <u>colored edges</u>. Suppose G has a least split $\{H_1, H_2, \ldots, H_k\}$, and let $H_i(e_x)$, $1 \leq i \leq k$, be the blocks of the split. We color the supplementary edge e_x of $H_i(e_x)$ if some H_j, $j \neq i$, contains a colored edge. Thus, edge e_x may be colored in some blocks and not in others. If a block contains at least two colored edges, we call it a colored block. We continue this coloring process on successive least splits until a complete set of atoms is obtained. If S is not empty, then every atom contains at least one colored edge. An atom with at least two colored edges is called a colored atom.

Lemma 5.13 In Lemma 5.12, if supplementary edges are colored, then they may be replaced by paths of G which contain colored edges of G.

Proof. Suppose $H_i(e_x)$, $1 \leq i \leq k$, are blocks of a graph at some stage of the splitting process. If e_x is colored in $H_i(e_x)$, then, by definition, some block, say $H_j(e_x)$, $j \neq i$, contains a colored edge. Since $V(H_i) \cap V(H_j)$ consists of the ends of e_x and $H_j(e_x)$ is connected, there exists a path of H_j which contains a colored edge, and its ends are the ends of e_x. Thus a colored supplementary edge e_x of $H_i(e_x)$ is replaced by a path of H_j which contains a colored edge. The lemma follows from Lemma 5.12. ∎

Lemma 5.14 Let $G = (V, E)$ be a 2-connected planar graph, and S $(\neq \phi)$ be the set of colored edges of G. If S is on a mesh of G, then the colored edges of each atom are on a mesh of the atom, and each supplementary edge is contained by no more than two colored atoms.

Proof. Suppose atom A does not have a plane graph such that the colored edges of A are on a mesh. By Lemma 5.13, we can construct a subgraph A' of G by replacing every (colored) supplementary edge of A by a path (containing a colored edge) of G. Graph A' does not have a plane graph of which the colored edges of A' are on a mesh. Therefore, its supergraph G has no plane graph satisfying the required condition.

Suppose there is a supplementary edge e_x which is contained in more than two colored atoms. Let A_1, A_2, and A_3 be three of the colored atoms which contain e_x. A_1, A_2, and A_3 have neither common edges nor common vertices except e_x and its ends v and v'. Since each A_i, $1 \leq i \leq 3$, has at least two colored edges, we can find a path P_i of A_i whose ends are v and v', and which contains a colored edge which is not e_x. Joining these three paths at vertices v and v', construct a branch graph B. B contains at least one colored edge in each of its direct paths. We replace all the supplementary edges of B by paths of G (if a supplementary edge of B is colored, then replace it with a path containing a colored edge of G). The resulting graph is a branch graph B' which has a colored edge in each of three direct paths, and there exists no plane graph with the property: the colored edges of B' are on a mesh of B'. Since B' is a subgraph of G, G does not have the required embedding. This completes the proof of the lemma. ∎

By a constructive method, we now show that the condition given in Lemma 5.14 is also sufficient.

<u>Theorem 5.9</u> Let $G = (V, E)$ be a 2-connected planar graph, and $S\ (\neq \phi)$ be the set of colored edges of G. Then S is on a mesh of G if and only if the colored edges of each atom are on a mesh of the atom, and each supplementary edge is contained by no more than two colored atoms.

<u>Proof.</u> Since the necessary part of the theorem was proved in Lemma 5.14, we have only to prove the sufficient part.

Suppose the hypothesis is satisfied. Let e_x be a supplementary edge, and let $A_{x1}, A_{x2}, \ldots, A_{xk}$ be the atoms containing e_x. According to Theorem 2.3(c), we can embed the colored atoms on the plane so that the colored edges are on the outer meshes. Since $S \neq \phi$, there is at least one colored atom. Suppose we have only one colored atom. Then the rest of the atoms contain e_x as the only colored edge. Joining these atoms at the ends of e_x and removing e_x, we obtain a plane merge graph with all the colored edges being on the outer mesh. If there are two colored atoms, merge them at the ends of e_x, leaving the colored edges on the outer mesh. The rest of the atoms containing only colored edge e_x are merged at the ends of e_x, preserving the incidence relation so that the resulting plane graph $\mathcal{M}(G_x)$ has all the colored edges on the outer mesh.

Suppose e_y is an uncolored supplementary edge of G_x. Then all the atoms containing e_y are uncolored. Merging G_x and these atoms on the plane, a plane graph $\mathcal{M}(G_y)$, with all the colored edges on the outer mesh results. If e_y is a colored edge of G_x, then there exists precisely one colored atom A_y, in which e_y is a colored edge, and which does not contain e_x. Since all the colored edges of G_x and A_y are on the outer meshes, we can embed the merge graph of G_x and A_y in the plane so that the colored edges are on the outer mesh.

Repeating this merging process on the successive supplementary edges, by Theorem 5.8 we finally obtain the original graph G and all the colored edges are on the outer mesh of $\mathfrak{P}(G)$. $\blacksquare$

In the proof of Theorem 5.9 we have not used the concepts of polygon and essential atoms. However, the use of essential atoms gives a unique characterization of meshes. After we have reduced G to a complete set of atoms, it is a trivial job to test if the colored edges are on a mesh. By Theorem 2.3(f), whether the colored edges of an essential atom A are on a mesh is independent of a particular plane graph of A and is easily determined from any plane graph of A.

Even though we have restricted our discussion to 2-connected planar graphs, we can state Theorem 5.9 for a general graph. If a graph G is separable, then we first find the non-separable components. Each non-separable component is decomposed into atoms. By MacLane's theorem [Ma 1], the original graph is planar if and only if all the essential atoms are planar. If G is planar, Theorem 5.9 gives a criterion for a plane graph $\mathfrak{P}(G)$ of which a given subset of E(G) is on a mesh.

Example 5.5 Consider the graph G in Fig. 5.14, where $S = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ is the set of colored edges of G. The least split of G at vertices 1 and 2 yields two blocks G_b and G_{a1} in Fig. 5.15. G_b has a further split at vertices 1 and 3, and the resulting blocks are shown in Fig. 5.16. A complete set of atoms consists of G_{a1}, G_{a2}, and G_{a3}, and e_x and e_y are colored supplementary edges in the atoms. We see that the condition in Theorem 5.9 is satisfied. Therefore S is on a mesh of G. A plane graph of G, of which S is on a mesh, is shown in Fig. 5.17.

Now, let us consider a problem related to the previous discussion. Let G = (V, E) be a planar 2-connected graph and $\phi \neq S \subseteq E$. The question is: What is a necessary and sufficient condition for edge set S to be on a comesh of G? If G is 3-connected, then, by Theorem 2.3(e), S is on a comesh of G if and only if all the members of S have a common vertex of G. However, this problem is not trivial if a graph is not 3-connected.

Lemma 5.15 Let G = (V, E) be a 2-connected planar graph and $\phi \neq S \subseteq E$. Then S is on a comesh of G if and only if there exists a graph 2-isomorphic to G, of which the members of S have a common vertex.

Proof. Let S be on a comesh of G. By definition, there exists a dual graph G_1^* of G, of which S is on a mesh of G_1^*. Clearly, the members of S have a common vertex of a dual graph G_2 of G_1^*. By

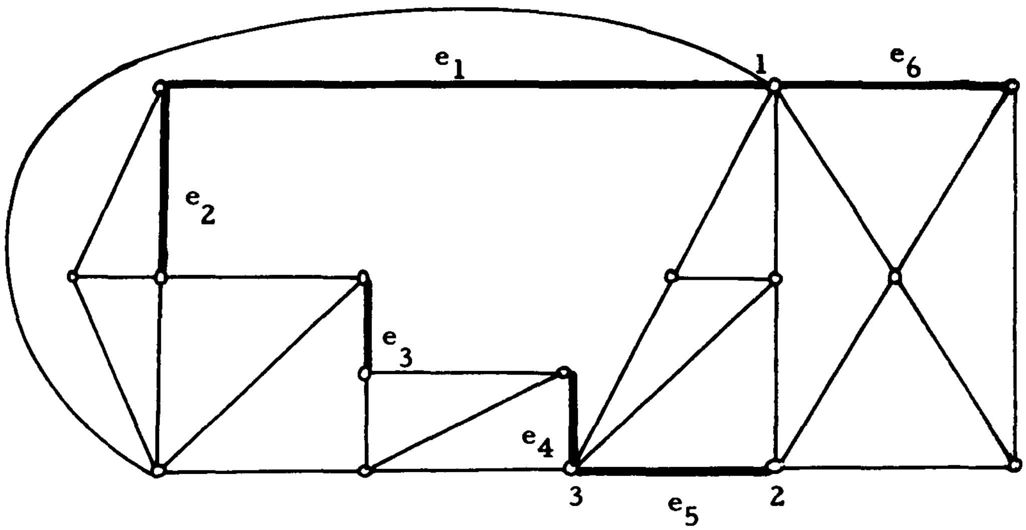


Figure 5.14 Graph G of Example 5.5

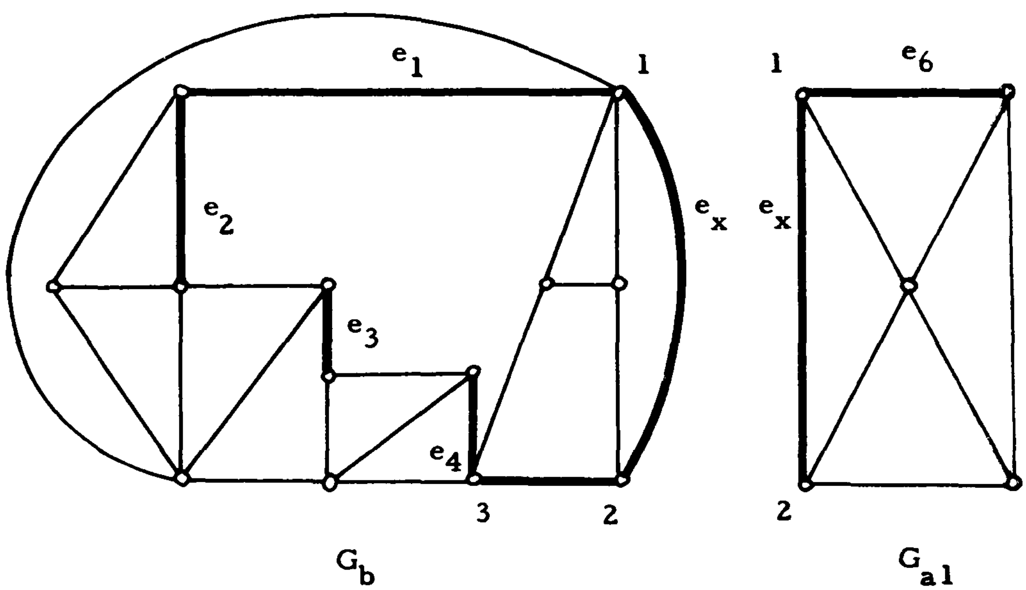


Figure 5.15 Blocks of G

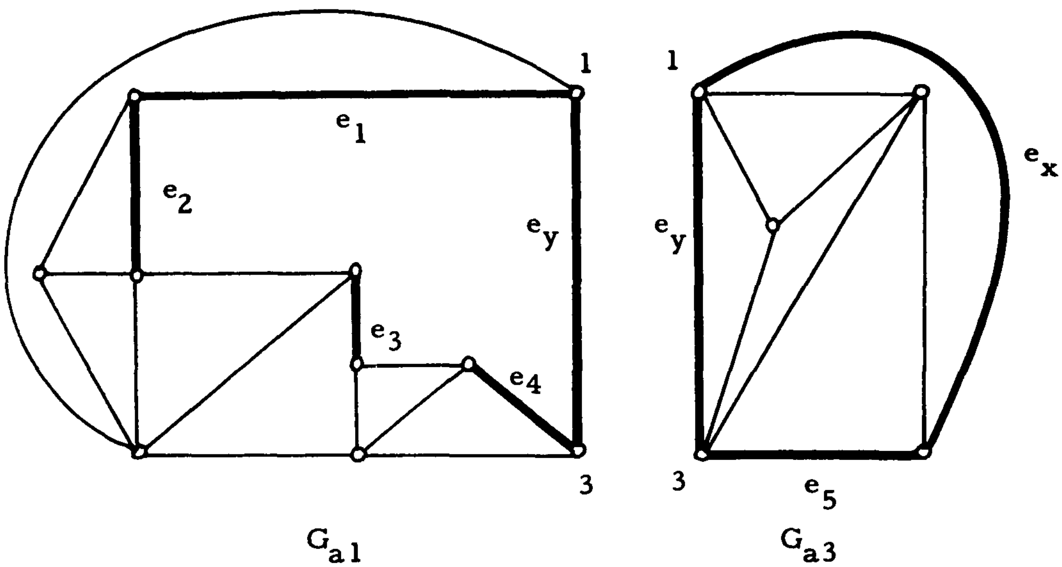


Figure 5.16 Blocks of G_b

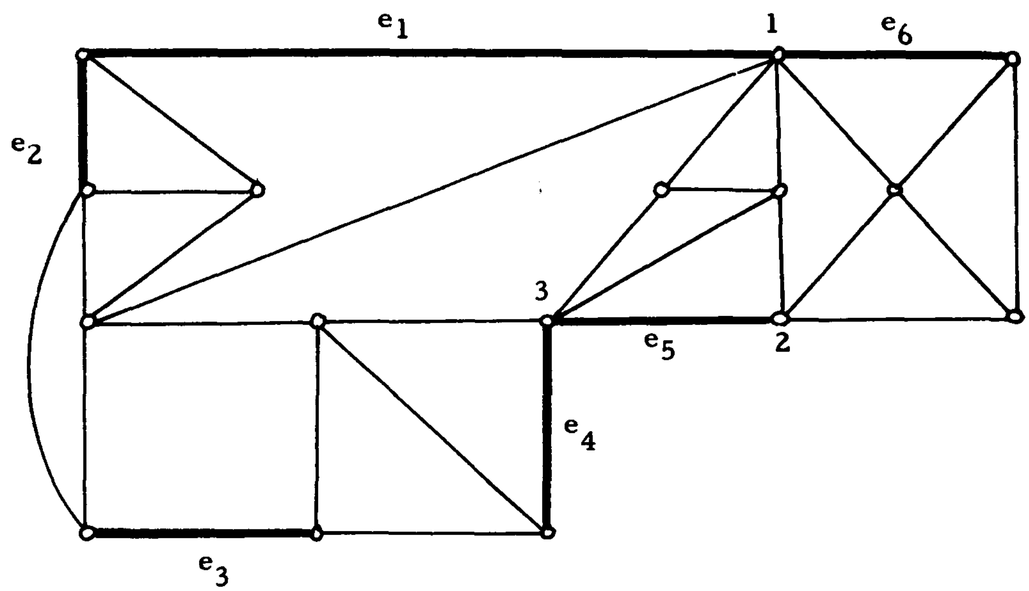


Figure 5.17 A Plane Graph of G of which S is on a Mesh

Theorem 2.3(d), G and G_2 are 2-isomorphic.

Now suppose G_1 is 2-isomorphic to G, and the members of S have a common vertex of G_1. Then, S is on a mesh of a dual graph G_1^* of G_1. By Theorem 2.3(d), G is a dual G_1^*, and, hence, G_1^* is a dual of G by Theorem 2.3(b). ■

<u>Lemma 5.16</u> Let $G = (V, E)$ be a 2-connected planar graph, and $G' = (V', E')$ be a 2-connected subgraph of G. If ($\phi \neq$) S ($\subseteq E'$) is on a comesh of G, then S is also on a comesh of G'.

<u>Proof.</u> If S is on a comesh of G, then, by Lemma 5.15, there exists a 2-isomorphic graph G_1 of G, such that the members of S have a common vertex of G_1. Let G_1' be a subgraph of G_1 which is 2-isomorphic to G'. Since S is an edge set of G_1', the members of S have a common vertex of G_1'. By Lemma 5.15, S is on a mesh of a dual graph $G_1'^*$ of G_1'. Since G_1' and G_1 are 2-isomorphic, the lemma follows from Theorem 2.3(d). ■

<u>Theorem 5.10</u> Let $G = (V, E)$ be a 2-connected planar graph and $S(\neq \phi)$ the set of colored edges of G. Then S is on a comesh of G if and only if every mesh of the atoms contains at the most two colored edges, and the colored edges of every essential atom have a common vertex of the atom.

<u>Proof.</u> <u>Necessity</u> Suppose some mesh M of an atom contains more than two colored edges. By Lemma 5.13, we may replace the supplementary edges of the atom by paths of G so that every colored

supplementary edge is replaced by a path containing a colored edge of G, and the resulting graph is a subgraph of G. By this replacement, M is transformed into a mesh M' of the subgraph. Let G' be the polygon graph consisting of the edges of M' and their adjacent vertices. G' is a subgraph of G and has more than two colored edges. If S' represents the colored edges of G', then S' is not on a comesh of G', for in the dual graph of G' the members of S' are in parallel. By Lemma 5.16, S' is not on a comesh of G. Clearly, $(S' \subseteq)$ S is not on a comesh of G.

Suppose some essential atom A does not have its colored edges on a common vertex. Replace the supplementary edges in A by paths of G so that every colored supplementary edge is replaced by a path of G containing a colored edge. Since A is 3-connected, by Theorem 2.3(e) and Lemma 5.15, the colored edges of A are not on a comesh of A and hence, the colored edges of the resulting graph G', by the above operation, are not on a comesh of G'. By Lemma 5.16, S is not on a comesh of G.

<u>Sufficiency</u> Suppose the conditions in the theorem are satisfied. Choose a supplementary edge e_x and the atoms A_{x1}, $A_{x2}, \ldots, A_{xk}$ containing e_x. If e_x is not colored in atom A_{xi}, then A_{xj}, $j \neq i$, has e_x as the only colored edge. The merge graph G_x of A_{xi}, $1 \leq i \leq k$, has its colored edges on a common vertex. By Lemma 5.15, a dual graph of G_x has its colored edges on a mesh.

Suppose e_x is colored in all the atoms A_{xi}, $1 \leq i \leq k$. By the hypothesis, the colored edges of each of these atoms has a common vertex which is an end of e_x. A_{xi}, $1 \leq i \leq k$, has the ends of e_x as the only common vertices. Construct G_x' by removing e_x from A_{xi}, $1 \leq i \leq k$, interchanging the ends of e_x of some atoms, and merging the atoms at the ends of e_x, so that all the colored edges of G_x' have a common vertex. The obtained graph G_x' is 2-isomorphic to the merge graph of A_{xi}, $1 \leq i \leq k$.

Choose another supplementary edge e_y and construct a graph G_y' as above from G_x' and the rest of the atoms which contain e_y. Clearly, G_y' is 2-isomorphic to the merge graph of the atoms which contain e_x or e_y, and the colored edges of G_y' have one vertex in common.

Repeating this construction process, we finally obtain a graph G' which is 2-isomorphic to the original graph and of which the members of S have a common vertex. The theorem follows from Lemma 5.15. ∎

Combining Theorems 5.9 and 5.10, we can state the following theorem:

<u>Theorem 5.11</u> Let G = (V, E) be a 2-connected planar graph and S ($\neq \phi$) be the set of colored edges of G. Then, S is on a mesh and comesh of G if and only if:

(i) each supplementary edge is contained by no more than two colored atoms;

and

(ii) every atom contains at most two colored edges, and they have a common vertex in an essential atom.

In Theorems 5.9 and 5.10, there is no constraint on the number of members of S. However, the next theorem shows that $|S|$ cannot be more than two for S being on a mesh and comesh.

Theorem 5.12 If S is on a mesh and comesh of G, then S contains at most two elements.

Proof. Suppose S is the colored edges of G, and it contains more than two elements. At a least split of G, S is divided into no more than two blocks, because if it is divided into more than two blocks, then at least three atoms contain an element of S, and this contradicts Theorem 5.11. If two of the blocks are colored, then one of them contains at least three colored edges, including a colored supplementary edge. Thus, each least split of blocks containing more than two colored edges yields a colored block containing more than two colored edges. Therefore, one atom contains more than two colored edges. This contradicts Theorem 5.11. ∎

An efficient algorithm to generate a complete set of atoms has been implemented by Hopcroft and Tarjan [Ho 1]. However, in practice, we don't have to find all the atoms to test the conditions in Theorems 5.9 and 5.10. Since least splits of uncolored blocks yield only uncolored blocks which are not important, once uncolored blocks are obtained, we terminate a splitting process for those blocks. Thus, we perform the least split operation on only colored blocks. For some colored blocks it might be intuitively determined without further splits whether the conditions in the theorems are satisfied.

Example 5.6 Let us consider the graph G in Fig. 5.18, where $S = \{e_1, e_2, e_3\}$ is the set of colored edges of G. The least split at vertices 1 and 2 yields G_b, G_{a1}, and G_{a2} as the blocks (Fig. 5.19). The graphs in Fig. 5.20 are the blocks of G_b. G_{b1} and G_{b2} are not atoms; however, clearly, the condition in Theorem 5.10 is satisfied for these graphs. Accordingly, S is on a comesh of G.

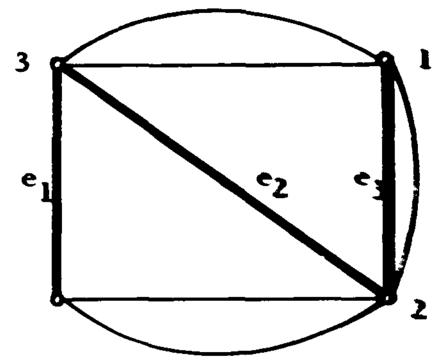


Figure 5.18 Graph G of Example 5.6

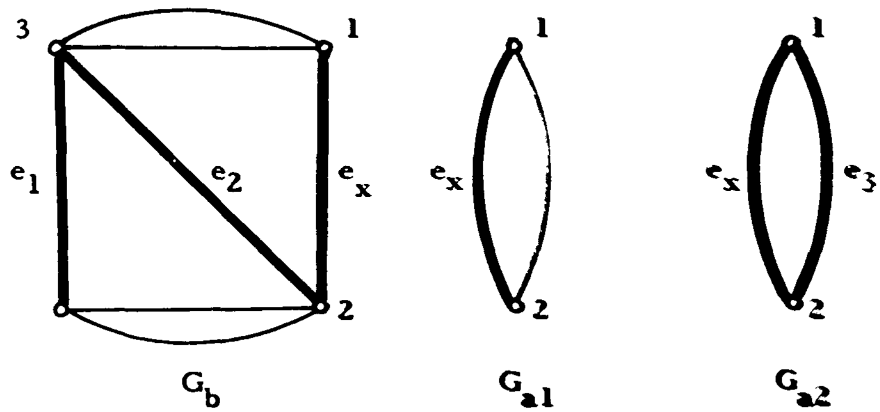


Figure 5.19 Blocks of G

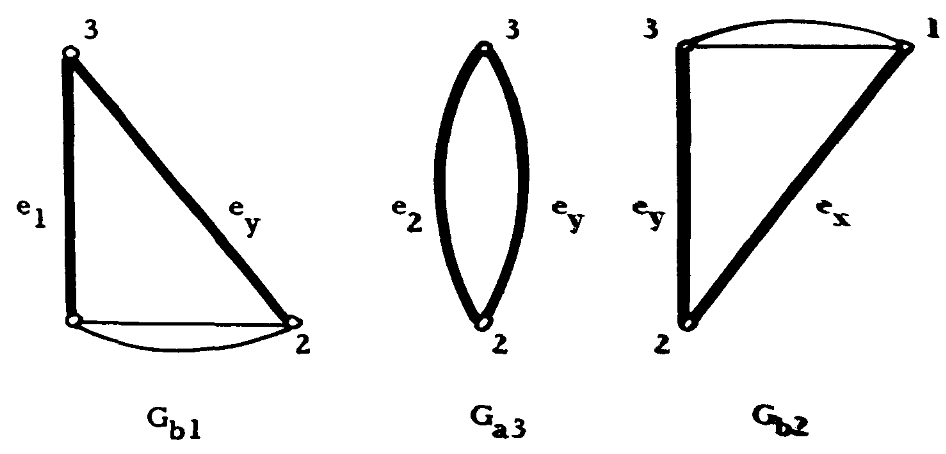


Figure 5.20 Blocks of G_b

CHAPTER 6 GRAPH REALIZABILITY OF MATROIDS

6.1 INTRODUCTION

The graph-realizability problem for matrices has been considered by a number of researchers. At present several general testing procedures exist for realizing cut-set and circuit matrices [Go 1, Ir 1, Lo 1, Ma 2, Tu 4]. If the given matrix is unrealizable, the procedure detects this property.

Since matrices are a subclass of matroids, a more general problem is the graph-realizability of matroids. This problem has been considered by Tutte [Tu 3], Welsh [We 6], and Fournier [Fo 1, 2]. Tutte and Welsh give necessary and sufficient conditions for realizability by generalizing graph theorems on planarity due to Kuratowski and MacLane, respectively.

The new realizability theorem presented in this chapter is obtained by generalizing the planarity theorem of Bruno, Steiglitz, and Weinberg [Br 2]. According to Theorem 2.16, if a 3-connected matroid $\underline{M}$ contains a non-essential cell, then a reduction or contraction of $\underline{M}$ can be performed to yield a 3-connected matroid with a smaller number of cells. This process of removing non-essential cells is continued until we obtain a wheel, a whirl, or a matroid related to a binomial matroid. For a given graphic or cographic matroid, it is necessary that a reduction sequence yields a wheel matroid. In the process of reconstructing the original matroid $\underline{M}$, if $\underline{M}$ is to be graphic or cographic, it must satisfy a certain condition. This condition will be stated in terms of circuits of minors.

6.2 REDUCTION SEQUENCE

Let $\underline{M}$ be a matroid with arbitrary connectivity. If $\underline{M}$ is not 3-connected, it may be decomposed into 3-connected minors (atoms) of $\underline{M}$ as shown in Chapter 4. By Theorem 4.15, the graph-realizability of $\underline{M}$ is determined by that of the atoms, that is, if the atoms of $\underline{M}$ are graphic (cographic), then $\underline{M}$ is also graphic (cographic). Therefore, without loss of generality, we will consider the graph-realizability problem of 3-connected matroids.

To begin, we shall define a reduction sequence of a matroid. Let $\underline{M} = (\underline{C}, E)$ be a 3-connected matroid, where $|E| \geqslant 6$. Then the sequence of matroids, $\underline{S} = < \underline{M}_0, \underline{M}_1, \ldots \underline{M}_n >$, is called a reduction sequence of $\underline{M}$ if it satisfies the following conditions: Let $\underline{M}_i = (\underline{C}_i, E_i)$, $1 \leqslant i \leqslant n$.

(i) $\underline{M}_0 = \underline{M}$

(ii) either $\underline{M}_i = \underline{M}_{i-1} \times E_i$ or $\underline{M}_i = \underline{M}_{i-1} \cdot E_i$ for $1 \leqslant i \leqslant n$, where $E_i = E_{i-1} - \{e_{i-1}\}$ and $e_{i-1} \in E_{i-1}$

(iii) $\underline{M}_i$ is 3-connected for $1 \leqslant i \leqslant n$, and $|E_n| \geqslant 6$

(iv) neither $\underline{M}_n \times (E_n - \{e_n\})$ nor $\underline{M}_n \cdot (E_n - \{e_n\})$ is 3-connected for any $e_n \in E_n$.

In a reduction sequence $\underline{M}_n$ is called the irreducible matroid of the sequence. A reduction sequence terminates when $\underline{M}_n$ is a wheel or a whirl, or $|E_n| = 6$. If $|E_n| = 6$ and $\underline{M}_n$ contains a non-essential cell, then $\underline{M}_n$ is one of the following matroids: ${}_6\underline{M}_3$, ${}_6\underline{M}_4$, ${}_6\underline{M}_{4(1)}$, ${}_6\underline{M}_{4(2)}$, or ${}_6\underline{M}_5$. These matroids will be referred to as the non-essential matroids. A wheel matroid is obviously realizable as a wheel graph. However, a whirl and non-essential matroids are neither graphic nor cographic. We state these characteristics of a reduction sequence in the following theorem:

Theorem 6.1 If $\underline{M} = (\underline{C}, E)$ is a 3-connected matroid, where $|E| \geq 6$, then the irreducible matroid $\underline{M}_n$ of a reduction sequence of $\underline{M}$ is a wheel, a whirl, or a non-essential matroid. In addition, if $\underline{M}$ is graph-realizable, then $\underline{M}_n$ is a wheel matroid.

According to this theorem, if $\underline{M}$ is graphic or cographic, then $\underline{M}_n$ is a wheel; in other words, if $\underline{M}_n$ is not a wheel, then $\underline{M}$ is neither graphic nor cographic. However, though the requirement that $\underline{M}_n$ be a wheel is a necessary condition for $\underline{M}$ to be realizable as a graph, it is not sufficient. Consider the binary matroid of a heptahedron. This matroid $\underline{M} = (\underline{C}, E)$ is 3-connected and consists of seven cells and fourteen circuits: $E = \{1, 2, \ldots, 7\}$ and $\underline{C} = \{123, 146, 157, 245, 267, 347, 356, 1247, 1256, 1345, 1367, 2346, 2357, 4567\}$, where, for convenience, the sequence of numbers is used to represent the corresponding set, e.g. 123 represents $\{1, 2, 3\}$. Let $\underline{M}_1 = (\underline{C}_1, E_1)$ $\underline{M} \times (E - \{7\})$. Then $E_1 = \{1, 2, 3, 4, 5, 6\}$ and $\underline{C}_1 = \{123, 146, 245, 356, 1256, 1345, 2346\}$. It is clear that $\underline{M}_1$ is a wheel matroid, namely the matroid of W_3, and that cell 7 is not essential. According to Tutte [Tu 3, 6], a heptahedron is one of the smallest binary matroids which are neither graphic nor cographic.

The graph-realizability of $\underline{M}$ depends on the realizability of $\underline{M}_i$ for $1 \leq i \leq n$. We will formulate a condition for $\underline{M}_i$ to be graphic or cographic given $\underline{M}_n$, a wheel matroid.

In the following discussion, we will use notations $\underline{M}_i$ and e_i to refer to a member of a reduction sequence and a non-essential cell of $\underline{M}_i$ which is used in the reduction operation. Throughout this chapter we assume $E_i = E_{i-1} - \{e_{i-1}\}$ for $1 \leq i \leq n$.

6.3 REALIZABILITY CONDITIONS

The realizability conditions established in this section are based on Theorems 2.2 and 6.1.

Let $\underline{S} = \langle \underline{M}_0, \underline{M}_1, \ldots, \underline{M}_n \rangle$ be a reduction sequence of $\underline{M}$. The inverse operations of $\underline{M}_{i-1} \times E_i$ and $\underline{M}_{i-1} \bullet E_i$ are defined by $\underline{M}_i(x)^{-1} E_{i-1} = \underline{M}_{i-1}$, and $\underline{M}_i(\bullet)^{-1} E_{i-1} = \underline{M}_{i-1}$, respectively. A sequence $\underline{S}^{-1} = \langle \underline{M}_0, \underline{M}_1, \ldots, \underline{M}_n \rangle^{-1} = \langle \underline{M}_n, \underline{M}_{n-1}, \ldots \underline{M}_0 \rangle$ is called the inverse reduction sequence of $\underline{M}$.

Suppose $\underline{M}_i = \underline{M}_{i-1} \times E_i$ and $\underline{M}_i$ is cographic. By definition, $\underline{M}_i$ is the polygon matroid of some graph, G_i, which, by Theorem 2.2, is unique. Then the inverse operation $\underline{M}_i(x)^{-1} E_{i-1}$ is called admissible if the following condition is satisfied:

Condition 1 There exists a graph G_{i-1} which is obtained by joining two distinct vertices of G_i with edge e_{i-1}, and every circuit of $\underline{M}_{i-1}$ containing e_{i-1} is a polygon of G_{i-1}.

How these two vertices are determined will be explained in the next section.

Now, we state the following lemma, which will be applied to devise an inverse operation algorithm.

Lemma 6.1 Let $\underline{M}_i = \underline{M}_{i-1} \times E_i$ be a cographic matroid. Then $\underline{M}_{i-1}$ is cographic if and only if the inverse operation is admissible.

Proof. Suppose $\underline{M}_{i-1}$ is a cographic matroid and G_{i-1} a corresponding 3-connected graph. Then $\underline{M}_i = \underline{M}_{i-1} \times E_i = \underline{P}(G_{i-1}) \times E_i = \underline{P}(G_{i-1} \bullet E_i)$

by Theorem 2.11(a), and $G_i = G_{i-1} \cdot E_i$ is 3-connected by Theorem 2.15. G_{i-1} is obtained by joining two distinct vertices of G_i with e_{i-1}, since, otherwise, G_{i-1} is separable. Clearly, the inverse operation $(x)^{-1}$ is admissible.

Suppose Condition 1 is satisfied. Let $\underline{P}(G_{i-1}) = (\underline{C}_P, E_{i-1})$. We will prove $\underline{M}_{i-1} = \underline{P}(G_{i-1})$. Since $\underline{C}_{i-1} \subseteq \underline{C}_P$, we only have to show $\underline{C}_P \subseteq \underline{C}_{i-1}$. Let C be a circuit of $\underline{P}(G_{i-1})$ containing e_{i-1}. We choose a circuit C' of $\underline{M}_{i-1}$ so that $e_{i-1} \in C'$ and $C \cup C'$ is minimal, consistent with this condition. Since C and C' are polygons of G_{i-1}, the symmetric difference $C \oplus C'$ is a disjoint union of polygons of G_{i-1}. Let C_1 $(\subseteq C \oplus C')$ be a polygon of G_{i-1} and e be any member of $C_1 \cap C'$ $(e \notin C)$. $e_{i-1} \notin C_1 \in \underline{C}_P$ and hence, $C_1 \in \underline{C}_{i-1}$ since $\underline{C}_i = \underline{C}_{i-1} \times E_i = \underline{C}_P \times E_i$. Then there exists a circuit C_2 of $\underline{M}_{i-1}$ such that $e_{i-1} \in C_2 \subseteq C_1 \cup C' - \{e\}$. However, $C_2 \cup C \subseteq (C_1 \cup C' - \{e\}) \cup C = C_1 \cup C' \cup C - \{e\} = C' \cup C - \{e\}$. This contradicts the minimality of $C' \cup C$. Thus, $C \oplus C' = \Phi$, and we have $C = C' \in \underline{C}_{i-1}$. If $e_{i-1} \notin C \in \underline{C}_P$, then $C \in \underline{C}_i = \underline{C}_{i-1} \times E_i \subset \underline{C}_{i-1}$ by hypothesis. Accordingly, $\underline{C}_P \subseteq \underline{C}_{i-1}$, and the proof is complete. ∎

Now, let $\underline{M}_i = \underline{M}_{i-1} \cdot E_i$ be cographic and G_i a corresponding graph such that $\underline{M}_i = \underline{P}(G_i)$. Then the inverse operation $\underline{M}_i\,(\cdot)^{-1}\,E_{i-1}$ is <u>admissible</u> if the following condition is satisfied:

<u>Condition 2</u> There exists a graph G_{i-1} which is obtained from G_i by splitting a vertex v of G_i into vertices v_1 and

v_2 and adjoining edge e_{i-1} between v_1 and v_2 so that every edge incident to v in G_i is incident to either v_1 or v_2 in G_{i-1}. Every circuit $\underline{M}_{i-1}$ is a polygon of G_{i-1}.

<u>Lemma 6.2</u> Let $\underline{M}_i = \underline{M}_{i-1} \ \ E_i$ be a cographic matroid. Then $\underline{M}_{i-1}$ is cographic if and only if the inverse operation is admissible.

<u>Proof.</u> Since the necessary part of the lemma is obvious, we prove the sufficiency part.

Suppose Condition 2 is fulfilled. By hypothesis, every circuit of $\underline{M}_{i-1}$ containing e_{i-1} is a polygon of G_{i-1}. We show that the converse of this statement is also true, that is, every polygon of G_{i-1} containing e_{i-1} is a member of $\underline{C}_{i-1}$. Let $\underline{P}(G_{i-1}) = (\underline{C}_P, E_{i-1})$, and let C be a polygon of G_{i-1} such that $e_{i-1} \in C \notin \underline{C}_{i-1}$. Since $C_1 = C - \{e_{i-1}\}$ is a polygon of G_i, we have $C_1 \in \underline{C}_i$. By definition, there exists a circuit C' of $\underline{M}_{i-1}$ such that $C_1 = C' \cap E_i = C' - \{e_{i-1}\}$. Thus, $C' = C_1$ or $C' = C_1 \cup \{e_{i-1}\} = C$. By the assumption, the second case is impossible and hence, $C_1 \in \underline{C}_{i-1}$. By hypothesis, C_1 is a polygon of G_{i-1}, which is contrary to $C \ \ C_1 \cup \{e_{i-1}\}$ being a polygon of G_{i-1}. Therefore, every polygon of G_{i-1} containing e_{i-1} is a circuit of $\underline{M}_{i-1}$. By Theorem 2.18, $\underline{M}_{i-1} = \underline{P}(G_{i-1})$, which proves the lemma. ∎

Lemma 6.2 remains true after replacing "every circuit of $\underline{M}_{i-1}$" in Condition 2 by "every circuit of $\underline{M}_{i-1}$ with an edge incident to v_1 or v_2".

We have defined the admissible inverse operation on cographic matroids and formulated necessary and sufficient conditions for graph-realizable matroids. We now define the admissible inverse operations on graphic matroids.

Let $\underline{M}_i = \underline{M}_{i-1} \times E_i$, where $\underline{M}_i$ is graphic. Then the inverse operation, $\underline{M}_i (x)^{-1} E_{i-1}$, is admissible if it satisfies the following condition:

> Condition 3 Let $\underline{M}_i = \underline{B}(G_i)$. There exists a graph G_{i-1} which is obtained from G_i by splitting a vertex v of G_i into two vertices v_1 and v_2 and adjoining edge e_{i-1} between v_1 and v_2 so that every edge incident to v in G_i is incident to either v_1 or v_2 in G_{i-1}. Every circuit of $\underline{M}_{i-1}$ containing e_{i-1} is a cut-set of G_{i-1}.

Lemma 6.3 Let $\underline{M}_i = \underline{M}_{i-1} \times E_i$ be a graphic matroid. Then $\underline{M}_{i-1}$ is graphic if and only if the inverse operation is admissible.

Proof. The necessary condition may easily be proved. A proof of the sufficiency part is similar to that of Lemma 6.1.

Let $\underline{B}(G_{i-1}) = (\underline{C}_B, E_{i-1})$. If C is a circuit of $\underline{M}_{i-1}$ which does not contain e_{i-1}, then $C \in \underline{C}_i = \underline{C}_{i-1} \times E_i$ is a cut-set of G_i. By definition, $G_i \bullet (E_i - C)$ consists of two components, and the vertex v belongs to one of the components. Since graph G_{i-1} is obtained by splitting v into two vertices v_1 and v_2, C is also a cut-set of G_{i-1}. Thus, $\underline{C}_{i-1} \subseteq \underline{C}_B$.

Now. let $e_{i-1} \in C \in \underline{C}_B$ and $C \notin \underline{C}_{i-1}$. Choose $C' \in \underline{C}_{i-1}$ so that $e_{i-1} \in C'$ and $C \cup C'$ is minimal with respect to this condition. The symmetric difference $C \oplus C'$ consists of non-null disjoint cut-sets of G_{i-1}, for C and C' are distinct cut-sets of G_{i-1}. Let $C_1 \subseteq C \oplus C'$ be a cut-set of G_{i-1}. Since $e_{i-1} \notin C_1$, we have $C_1 \in \underline{C}_i$ and hence, $C_1 \in \underline{C}_{i-1}$. Suppose e is a member of $C_1 \cap C'$. Of course $e \notin C$. By Axiom II, there exists a circuit C_2 of $\underline{M}_{i-1}$ such that $e_{i-1} \in C_2 \subseteq C_1 \cup C' - \{e\}$. Then, $C \cup C_2 \subsetneq C \cup (C_1 \cup C' - \{e\}) = C \cup C' - \{e\}$, which is contrary to the minimality of $C \cup C'$. Accordingly, $C \in \underline{C}_{i-1}$ and $\underline{C}_B \subseteq \underline{C}_{i-1}$. Therefore, we have shown $\underline{M}_{i-1} = \underline{B}(G_{i-1})$. ∎

Let us consider the remaining case. Suppose $\underline{M}_i = \underline{M}_{i-1} \bullet E_i$ is a graphic matroid, where $\underline{M}_i = \underline{B}(G_i)$. Then the inverse operation, $\underline{M}_i (\bullet)^{-1} E_{i-1}$, is <u>admissible</u> if Condition 4 is satisfied.

<u>Condition 4</u> There exists a graph G_{i-1} which is obtained by joining two distinct vertices v_1 and v_2 of G_i with edge e_{i-1}, and every circuit of $\underline{M}_{i-1}$ is a cut-set of G_{i-1}.

<u>Lemma 6.4</u> Let $\underline{M}_i = \underline{M}_{i-1} \bullet E_i$ be a graphic matroid. Then, $\underline{M}_{i-1}$ is graphic if and only if the inverse operation is admissible.

<u>Proof.</u> A proof is obtained simply by replacing "polygon" by "cut-set" in the proof of Lemma 6.2. ∎

The combination of Lemmas 6.1, 6.2, 6.3, and 6.4 provides a necessary and sufficient condition for $\underline{M}$ to be graph-realizable. If

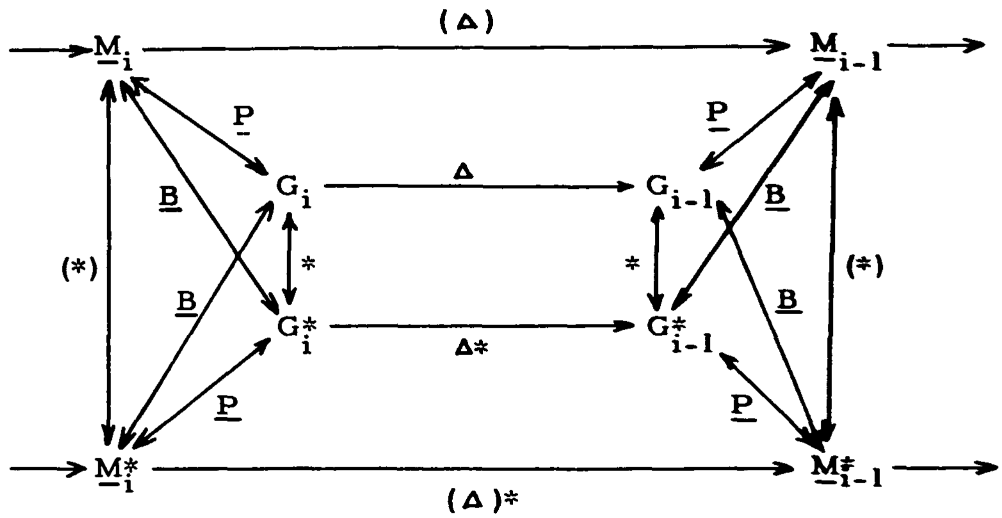
$\underline{M}$ is realizable, it is necessary that the irreducible matroid be a wheel. When we perform the inverse operations, there are two possible cases to start the algorithm: (1) The irreducible matroid is identified as the polygon matroid of a wheel graph; or (2) It is identified as the bond matroid of a wheel graph. Suppose the irreducible matroid M_n is the polygon (bond) matroid of a wheel graph, that is, $\underline{M}_n = \underline{P}(W_k)$ $(\underline{B}(W_k))$. Starting with $\underline{M}_n = \underline{P}(W_k)$ $(\underline{B}(W_k))$, if all the inverse operations are admissible in reduction sequence $\underline{S}$, then $\underline{S}^{-1}$ is called P-admissible (B-admissible) and $\underline{M}$ is a cographic (graphic) matroid. It is evident that $\underline{M}$ is planar if $\underline{S}^{-1}$ is P- and B-admissible.

Theorem 6.2 A 3-connected matroid $\underline{M}$ is graphic (cographic) if and only if its inverse reduction sequence $\underline{S}^{-1}$ is B-admissible (P-admissible), and $\underline{M}$ is planar if $\underline{S}^{-1}$ is B- and P-admissible.

We have stated a realizability condition in terms of bond and polygon matroids. However, by defining the dual inverse reduction sequence, it is possible to state an equivalent condition in terms of bond or polygon matroids alone.

Let $\underline{S} = \langle \underline{M}_0, \underline{M}_1, \ldots, \underline{M}_n \rangle$ be a reduction sequence of $\underline{M}$. By Theorem 2.14, $\underline{M}_0^*, \underline{M}_1^*, \ldots, \underline{M}_n^*$ are 3-connected, and $\underline{S}^* = \langle \underline{M}_0^*, \underline{M}_1^*, \ldots, \underline{M}_n^* \rangle$ is a reduction sequence of $\underline{M}^*$ called the dual reduction sequence of $\underline{M}$. The dual inverse reduction sequence of $\underline{M}$ is defined by $(\underline{S}^*)^{-1} = \langle \underline{M}_n^*, \underline{M}_{n-1}^*, \ldots, \underline{M}_0^* \rangle$. Let $\underline{M}_n$ be the bond matroid of a wheel graph. Suppose $\underline{M}_{i-1} = \underline{M}_i(x)^{-1} E_{i-1}$ and

$\underline{M}_i = \underline{B}(G_i)$. If the inverse operation does not satisfy Condition 3, then $\underline{M}_{i-1}$ is not graphic, however, it may be cographic. $\underline{M}_{i-1}$ being cographic depends on whether G_i is planar or not. If G_i is not planar, then $\underline{M}_i$ is not cographic. Suppose G_i is planar and G_i^* is the dual graph of G_i. Then $\underline{M}^*_{i-1} = \underline{M}_i^* (\bullet)^{-1} E_{i-1} = \underline{B}(G_i^*)(\bullet)^{-1} E_{i-1}$, and $\underline{M}^*_{i-1}$ is graphic if the inverse operation satisfies Condition 4, where in Condition 4 G_i, $\underline{M}_i$, and $\underline{M}_{i-1}$ are replaced by G_i^*, $\underline{M}_i^*$, and $\underline{M}^*_{i-1}$, respectively. Thus the graph-realizability of $\underline{M}$ is determined by use of bond matroids alone. A similar argument may be applied to cographic matroids. See Fig. 6.1 in connection with this discussion.



(Δ)	matroid inverse operation
(Δ)*	matroid dual inverse operation
(*)	matroid dual operation
Δ	graph inverse operation
Δ*	graph dual inverse operation
*	graph dual operation

Note. The operation * is defined when a graph is planar.

Figure 6.1 Dual Inverse Operations and the Corresponding Graph Operations

6.4 ALGORITHMS

Even though we have found no complete efficient algorithm for determining graph-realizability of matroids, we will devise some of the algorithms in this section.

A reduction sequence of a 3-connected matroid is obtained by examining 3-connectedness of a matroid at each step of the reduction process. An efficient algorithm for determing connectivity is not known at present. However, a connected matroid with series or parallel cells is easily identified and has connectivity two. Therefore, if a reduction and the corresponding contraction of a 3-connected matroid contain series or parallel cells, then the cell used in the operation is an essential cell. If a reduced matroid $\underline{M}_i$ contains neither series nor parallel cells, then in general it requires $2^{r(\underline{M}_i)}$ or $2^{\mu(\underline{M}_i)}$ computations of the connectivity function to determine the connectivity.

After we find a reduction sequence, we have to determine whether the irreducible matroid $\underline{M}_n = (\underline{C}_n, E_n)$ is a wheel matroid. This can be done very easily. By Theorem 6.1, the cell set of $\underline{M}_n$ has an even cardinality, since a wheel graph consists of an even number of edges and a wheel and whirl matroids are obtained from a wheel graph. If $\|E_n\| = 6$, then the number of circuits consisting of four cells is $3 = \|E_n\|/2$ for the wheel matroid, $6 = |E_n|$ for the whirl matroid, and neither three nor six for a non-essential matroid. For $\|E_n\| > 6$, $\underline{M}_n$ is either a wheel or whirl matroid, and the number of circuits consisting of $|E_n|/2+1$ cells is $|E_n|/2$ for a wheel matroid and $|E_n|$ for a whirl matroid. Therefore, the number of circuits consisting of $|E_n|/2+1$ cells indicates whether the irreducible matroid is a wheel, a whirl, or a non-essential matroid.

Algorithm W The irreducible matroid $\underline{M}_n$ in a reduction sequence is a wheel, a whirl, or a non-essential matroid according to whether the number of circuits of $\underline{M}_n$, consisting of $|E_n|/2+1$ cells, is $|E_n|/2$, $|E_n|$, or neither.

The algorithm for testing the admissibility of inverse operations is based on the four conditions described in the previous section. In the following discussion the notation $S \longleftarrow T$ is used to mean "replace T by S".

Let $\underline{M}_i = \underline{M}_{i-1} \times E_i$ be cographic. The next algorithm is devised to test whether $\underline{M}_{i-1}$ is cographic.

Algorithm A $\underline{M}_i = \underline{P}(G_i)$ and $\underline{M}_{i-1} = \underline{M}_i(x)^{-1} E_{i-1}$

Step 1. Choose any $C \in \underline{C}_{i-1}$ containing e_{i-1}. If $C - \{e_{i-1}\}$ is a path of G_i, construct G_{i-1} by joining the ends of the path with e_{i-1}. Go to Step 2. If $C - \{e_{i-1}\}$ does not form a path in G_i, $\underline{M}_{i-1}$ is not cographic.

Step 2. Every circuit of $\underline{M}_{i-1}$ containing e_{i-1} is a polygon of G_{i-1}. If this condition is satisfied, $\underline{M}_{i-1} = \underline{P}(G_{i-1})$; otherwise $\underline{M}_{i-1}$ is not cographic.

Validity of Algorithm A It is easy to see the equivalence of this algorithm and Condition 1, and by Lemma 6.1 Algorithm A is valid. ∎

Suppose $\underline{M}_i = \underline{M}_{i-1} \cdot E_i$ is cographic. Then Algorithm B provides a condition for $\underline{M}_{i-1}$ to be cographic.

Algorithm B $\underline{M}_i = \underline{P}(G_i)$ and $\underline{M}_{i-1} = \underline{M}_i(\cdot)^{-1} E_{i-1}$

Step 1. Choose any circuit of $\underline{M}_{i-1}$ in $\underline{C}_{i-1} - \underline{C}_i$ which does not contain e_{i-1}. This circuit consists of two edge-disjoint polygons of G_i with only one vertex v in common. Go to Step 2. If this condition is not satisfied, $\underline{M}_{i-1}$ is not cographic.

Step 2. Choose a circuit C of $\underline{M}_{i-1}$ containing e_{i-1}. $C - \{e_{i-1}\}$ is a polygon of G_i and contains exactly two edges e_α, e_β which are incident to v. Set $E_{(1)} = \{e_\alpha\}$ and $E_{(2)} = \{e_\beta\}$. Go to Step 3. If this condition is not satisfied, $\underline{M}_{i-1}$ is not cographic.

Step 3. Choose an unexamined circuit C in $\underline{C}_{i-1}$ containing e_{i-1} and a member of $E_{(1)} \cup E_{(2)}$, say e'_α. $C - \{e_{i-1}\}$ is a polygon of G_i and contains exactly two edges e'_α, e'_β incident to v. If $e'_\alpha \in E_{(1)}$ ($e'_\alpha \in E_{(2)}$), then let $E_{(2)} \leftarrow E_{(2)} \cup \{e'_\beta\}$ ($E_{(1)} \leftarrow E_{(1)} \cup \{e'_\beta\}$). Thus the updated set $E_{(2)}$ ($E_{(1)}$) is the union of $\{e'_\beta\}$ and the outdated set $E_{(2)}$($E_{(1)}$). Go to Step 4. If C does not satisfy the above condition, $\underline{M}_{i-1}$ is not cographic.

Step 4. Is $E_{(1)} \cap E_{(2)} = \phi$? If not, then $\underline{M}_{i-1}$ is not cographic.

If so, check whether all the edges incident to v in G_i are in $E_{(1)} \cup E_{(2)}$. If the answer is yes, go to Step 5; otherwise, go to Step 3.

Step 5. Construct graph G_{i-1} from G_i by splitting vertex v into two vertices v_1 and v_2 so that the edges in $E_{(1)}$ $(E_{(2)})$ are incident to v_1 (v_2) and joining v_1 and v_2 with e_{i-1}. Now test whether the unexamined circuits in $\underline{C}_{i-1}$ are polygons of G_{i-1}.

If the answer is yes, then $\underline{M}_{i-1} = \underline{P}(G_{i-1})$; otherwise, $\underline{M}_{i-1}$ is not cographic.

<u>Validity of Algorithm B</u> We shall show the equivalence of Condition 2 and Algorithm B. Suppose Condition 2 is satisfied. By Lemma 6.2, $\underline{M}_{i-1}$ is a cographic matroid and $\underline{M}_i = \underline{P}(G_{i-1}) \cdot E_i = \underline{P}(G_{i-1} \times E_i)$. Since G_{i-1} is 3-connected, by Menger's theorem there exist two paths not containing e_{i-1} whose only common vertices are the ends of e_{i-1}. The union of these paths forms a polygon in G_{i-1}, and the contraction of e_{i-1} yields edge-disjoint polygons having one vertex v in common.

Let C be a polygon of G_{i-1} containing e_{i-1}. Since G_{i-1} is 3-connected, exactly two edges of C - $\{e_{i-1}\}$ are incident to the ends of e_{i-1}. If we shrink e_{i-1} to vertex v, path C - $\{e_{i-1}\}$ becomes a polygon in G_i and those two edges are incident to v.

Let v_1 and v_2 be the ends of e_{i-1} in G_{i-1}, and let E_1 and E_2 be the incident edges with v_1 and v_2, respectively. Then $E_1 \cap E_2 = \{e_{i-1}\}$, and $E_1 \cup E_2 - \{e_{i-1}\}$ are the incident edges with v in G_i. Consequently, the edges incident to v are divided into two disjoint sets $E_{(1)} = E_1 - \{e_{i-1}\}$ and $E_{(2)} = E_2 - \{e_{i-1}\}$. The edges in $E_{(1)}(E_{(2)})$ are incident to $v_1(v_2)$, and v_1, v_2 are joined with e_{i-1}. The other part of the algorithm follows from this fact.

Suppose every condition in Algorithm B is satisfied. Let C be a circuit of $\underline{M}_{i-1}$ containing e_{i-1}. $C - \{e_{i-1}\}$ is a polygon of G_i and has exactly two edges e'_α and e'_β incident to v by Step 3. Each of these edges is incident to different end vertices of e_{i-1} in G_{i-1} by Steps 4 and 5. Thus, C is a polygon of G_{i-1}. If $e_{i-1} \notin C \in \underline{C}_{i-1}$, then C is a polygon of G_{i-1} by Step 5. Thus, Algorithm B is valid. ∎

In the above algorithm it is sufficient, as mentioned in the previous section, to examine the circuits of $\underline{M}_{i-1}$ with an edge incident to v_1 or v_2.

Algorithms for Conditions 3 and 4 are implemented as Algorithms C and D.

<u>Algorithm C</u> $\underline{M}_i = \underline{B}(G_i)$ and $\underline{M}_{i-1} = \underline{M}_i(x)^{-1} E_{i-1}$

Step 1. Find a circuit C of $\underline{M}_{i-1}$ such that $e_{i-1} \in C_{i-1}$ and the members of $C - \{e_{i-1}\}$ have a common vertex v in G_i. Split v into two vertices v_1 and v_2, and join v_1 and v_2 with e_{i-1} to form G_{i-1} so that C is a star cut-set of G_{i-1}. Go to Step 2. If there is no circuit satisfying this condition, then $\underline{M}_{i-1}$ is not graphic.

Step 2. Every member of $\underline{C}_{i-1}$ containing e_{i-1} is a cut-set of G_{i-1}.

Then $\underline{M}_{i-1}$ is graphic; otherwise, $\underline{M}_{i-1}$ is not graphic.

<u>Validity of Algorithm C</u> Suppose Condition 3 is satisfied. By Lemma 6.3, there exists a graph G_{i-1} such that $\underline{M}_{i-1} = \underline{B}(G_{i-1})$ and $\underline{M}_i = \underline{B}(G_i) = \underline{B}(G_{i-1} \times E_i)$, where G_{i-1} is obtained from G_i by splitting a vertex v and adjoining edge e_{i-1}. The star cut-sets of G_{i-1} at v_1 and v_2 are clearly members of $\underline{C}_{i-1}$, that is, one of those star cut-sets satisfies the condition in Step 1. These cut-sets are the only cut-sets of G_{i-1} which contain e_{i-1} and $C - \{e_{i-1}\}$ has a common vertex in G_i, where C is a star cut-set at v_1 or v_2. Therefore, the condition in Step 2 is also satisfied.

If all the conditions in Algorithm C are fulfilled, $\underline{M}_{i-1}$ is a graphic matroid by Lemma 6.3. $\blacksquare$

<u>Algorithm D</u> $\underline{M}_i = \underline{B}(G_i)$ and $\underline{M}_{i-1} = \underline{M}_i(\bullet)^{-1} E_{i-1}$

Step 1. Find two circuits C_1, C_2 of $\underline{M}_{i-1}$ containing e_{i-1} such that $C_1 - \{e_{i-1}\}$ and $C_2 - \{e_{i-1}\}$ are star cut-sets of G_i. Construct G_{i-1} by adding e_{i-1} to the vertices defined by the star cut-sets of G_i. Go to Step 2. If there are no such circuits, then $\underline{M}_{i-1}$ is not graphic.

Step 2. Every circuit of $\underline{M}_{i-1}$ containing e_{i-1} is a cut-set of G_{i-1}. If this is the case, $\underline{M}_{i-1}$ is graphic; otherwise $\underline{M}_{i-1}$ is not.

<u>Validity of Algorithm D</u> Since Algorithm D implies Condition 4, we show the converse. If Condition 4 is satisfied, then, by Lemma 6.4,

$\underline{M}_{i-1}$ is graphic and $\underline{M}_{i-1} = \underline{B}(G_{i-1})$ for some graph G_{i-1}. Let $G_i = G_{i-1} \cdot E_i$. Then $\underline{M}_i = \underline{B}(G_i)$. The star cut-sets C_1 and C_2 of G_{i-1} at vertices v_1 and v_2 contain e_{i-1}, and $C_1 - \{e_{i-1}\}$, $C_2 - \{e_{i-1}\}$ are star cut-sets of G_i. Since C_1 and C_2 are the only cut-sets at v_1 and v_2, G_{i-1} is uniquely constructed from C_1 and C_2 as shown in Step 1. Step 2 immediately follows from this fact. Thus, Algorithm D provides a means to construct a graph G_{i-1}. $\blacksquare$

An algorithm determining graph realizability of matroids may be devised according to Algorithms A through D and Algorithm W. This can be done also by making use of the dual inverse sequence. However, use of dual graphs avoids the complexity of finding dual matroids and the double computations of mutually dual reduction sequences as noted in Section 6.3.

<u>Algorithm for Determining Realizability</u> Let $\underline{S} = \langle \underline{M}_0, \underline{M}_1, \ldots, \underline{M}_n \rangle$ be a reduction sequence of $\underline{M}$.

Step 1. Use Algorithm W to determine whether $\underline{M}_n$ is a wheel matroid.

If $\underline{M}_n$ is not a wheel matroid, then $\underline{M}$ is not realizable.

If $\underline{M}_n$ is a wheel matroid, construct a graph G_n so that $\underline{M}_n = \underline{B}(G_n)$. If $n = 0$, $\underline{M}$ is planar. If $n \neq 0$, let $k \longleftarrow n$ and go to Step 2.

Step 2. If $\underline{M}_{k-1} = \underline{M}_k (x)^{-1} E_{k-1}$ $(\underline{M}_{k-1} (\cdot)^{-1} E_{k-1})$, use Algorithm C (D) to determine whether $\underline{M}_{k-1}$ is graphic. If $\underline{M}_{k-1}$ is graphic, construct G_{k-1} by Algorithm C (D) and go to Step 3. If $\underline{M}_{k-1}$

is not graphic, then $\underline{M}$ is not graphic; go to Step 4.

Step 3. Is $k=1$? If $k=1$, $\underline{M}$ is graphic. In addition, it is planar if G_{k-1} is planar. If $k\neq 1$, let $k \leftarrow k-1$, and go to Step 2.

Step 4. Is G_k planar? If not, $\underline{M}$ is not realizable. If it is planar, take a dual $G_k{}^*$ of G_k. Let $G_k \leftarrow G_k{}^*$, and go to Step 5.

Step 5. If $\underline{M}_{k-1} = \underline{M}_k(x)^{-1}\, E_{k-1}$ $(\underline{M}_{k-1}\, (\bullet)^{-1}\, E_{k-1})$, use Algorithm A (B) to test if $\underline{M}_{k-1}$ is cographic. If $\underline{M}_{k-1}$ is cographic, construct G_{k-1} and go to Step 6. If $\underline{M}_{k-1}$ is not cographic, $\underline{M}$ is not realizable.

Step 6. Is $k=1$? If $k=1$, $\underline{M}$ is cographic and not graphic. If $k\neq 1$, let $k \leftarrow k-1$, and go to Step 5.

<u>Validity of Algorithm</u> By Theorem 6.1, $\underline{M}_n$ is a wheel matroid if $\underline{M}$ is realizable, and a wheel graph whose bond matroid is $\underline{M}_n$ is uniquely determined within isomorphism.

Suppose $\underline{M}_k = \underline{B}(G_k)$, and an inverse operation is not admissible. In this case, the original matroid $\underline{M}$ is not graphic. However, if G_k is planar, then $\underline{M}_k$ is planar and $\underline{M}_k$ is a polygon matroid of a dual graph $G_k{}^*$. That is, $\underline{P}(G_k{}^*) = \underline{P}^*(G_k) = \underline{B}(G_k)$. Dual graph $G_k{}^*$ can be uniquely determined since G_k is 3-connected. The above algorithm is valid by Theorem 6.2. ∎

A flowchart of the algorithm is given in Fig. 6.2. In the process of reconstruction we might have to find a dual graph, or, in other words, we have to examine the planarity of a graph. However, our method used here is a generalization of the planarity testing procedure given by Bruno, Steiglitz, and Weinberg [Br 2] . At each step of the algorithm we have a 3-connedted graph G_k. The process of constructing G_{k-1} is automatically an application of the planarity testing algorithm, and by inspection, we can see if G_{k-1} is planar.

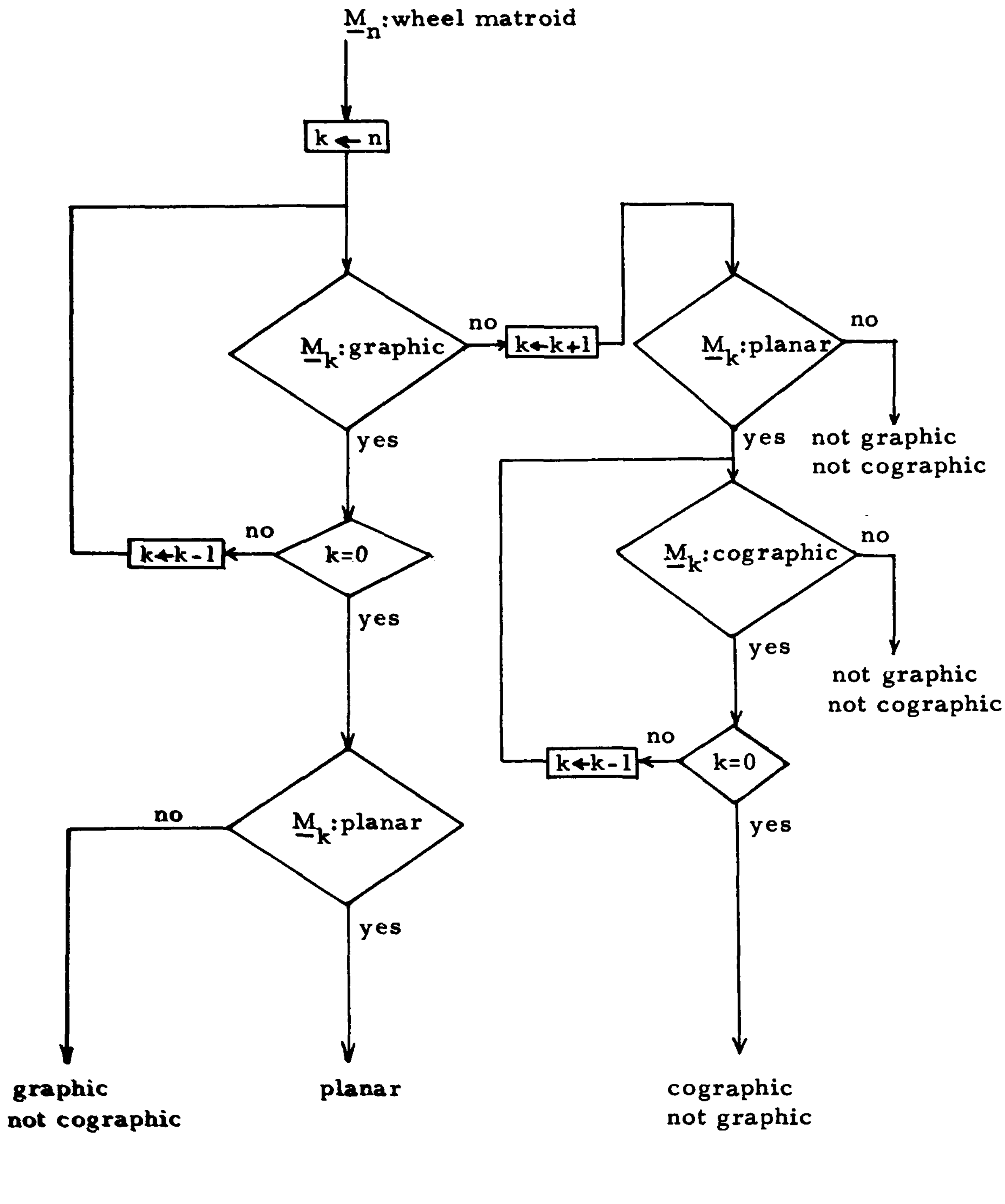


Figure 6.2 Flowchart of Realizability Algorithm of Matroids

6.5 EXAMPLES

Example 6.1 Let $\underline{M}=\underline{M}_0=(\underline{C}_0, E_0)$ be the 3-connected matroid given by

$E_0 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 0\}$

$\underline{C} = \{$125, 160, 170, 236, 247, 348, 350, 459, 678, 890, 1230, 1249, 1356, 1457, 1689, 1780, 2378, 2468, 2560, 2579, 3467, 3490, 3589, 4580, 6790, 12389, 12480, 13469, 1347013578, 14568, 23790, 24690, 25689, 25780, 35679, 45670$\}$.

Let $\underline{M}_1=(\underline{C}_1, E_1)=\underline{M}_0\cdot(E_0-\{1\})$. Then $\underline{M}_1$ contains circuit $\{1, 2, 5\}\cap E_1=\{2, 5\}$ and hence, $\underline{M}_1$ is not 3-connected. Now let $\underline{M}_1=\underline{M}_0 \times (E_0-\{1\})$, where E_1 and $\underline{C}_1$ are given by

$E_1 = \{2, 3, 4, 5, 6, 7, 8, 9, 0\}$

$\underline{C}_1 = \{$236, 247, 348, 350, 459, 678, 890, 2378, 2468, 2560, 2579, 3467, 3490, 3589, 4580, 6790, 23790, 24690, 25689, 25780, 35679, 45670$\}$

If $S=\{2, 3, 6\}$, then $\xi(\underline{M}_1;S,\overline{S})=3$. Therefore, $\lambda(\underline{M}_1)\leq 3$ and a little calculation shows $\lambda(\underline{M}_1)=3$, and hence $\underline{M}_1$ is 3-connected.

Similarly, we can show $\underline{M}_2=(\underline{C}_2, E_2)=\underline{M}_1 \times (E_1-\{4\})$ is a 3-connected matroid, where

$E_2=\{2, 3, 5, 6, 7, 8, 9, 0\}$

$\underline{C}_2=\{$236, 350, 678, 890, 2378, 2560, 2579, 3589, 6790, 23790, 25689, 25780, 35679$\}$.

Since any reduction and contraction of $\underline{M}_2$ contains parallel or series cells, $\underline{M}_2$ is an irreducible matroid. The number of circuits of $\underline{M}_2$ consisting of $|E_2|/2 + 1=5$ cells is four; $\underline{M}_2$ is a wheel matroid by

Algorithm W.

The reduction sequence is $\underline{S} = \langle \underline{M}_0, \underline{M}_1, \underline{M}_2 \rangle$. We apply the algorithm described in the previous section to test if $\underline{M}$ is realizable as a graph.

Step 1. Construct a wheel graph G_2 so that $\underline{M}_2 = \underline{B}(G_2)$ (Fig. 6.3).

Step 2. $\underline{M}_1 = \underline{M}_2(x)^{-1} (E_2 \cup \{4\})$. Use Algorithm C.

Choose any circuit C_1 of $\underline{M}_1$ such that $4 \in C_1$ and the members of $C_1 - \{4\}$ have a common vertex. Let $C_1 = \{4, 5, 9\}$. $C_1 - \{4\} = \{5, 9\}$ has common vertex v_5. Split v_5 into two vertices v_5 and v_6 so that only edges 5 and 9 are incident to v_5, and then join v_5 and v_6 with edge 4 (Fig. 6.4). Now test if every member of $\underline{M}_1$ containing edge 4 is a cut-set of G_1. We see this is the case, and $\underline{M}_1$ is cographic.

Step 3. $\underline{M}_0 = \underline{M}_1(x)^{-1} (E_1 \cup \{1\})$. Use Algorithm C. Choose any circuit C_0 of $\underline{M}_0$ containing cell 1 and such that $C_0 - \{1\}$ has a common vertex. However, this is impossible, and, hence, $\underline{M}_0$ is not cographic.

Step 4. Since G_1 is planar, take the dual $G_1{*}$ of G_1 (Fig. 6.5). Apply Algorithm A for $\underline{M}_0 = \underline{P}(G_1{*})(x)^{-1} (E_1 \cup \{1\})$. Choose any $C_0 \in \underline{C}_0$ containing cell 1. Let $C_0 = \{1, 2, 5\}$. Since $C_0 - \{1\}$ forms a path in $G_1{*}$, join v_2 and v_4 with edge 1 so that C_0 is a polygon (Fig. 6.5). Now examine every
$C_0 - \{1\}$ forms a path in $G_1{*}$, join v_2 and v_4 with edge 1
circuit of $\underline{M}_0$ containing edge 1 to see if it is a polygon of G_0.

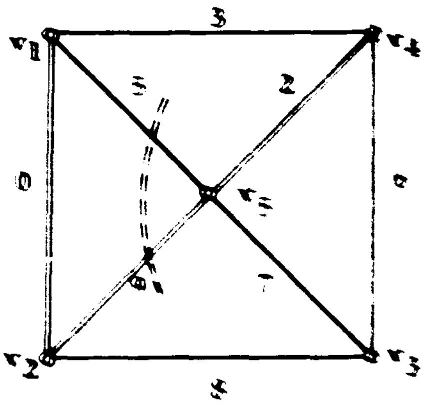


Figure 6.3 Graph G_2 of Example 6.1

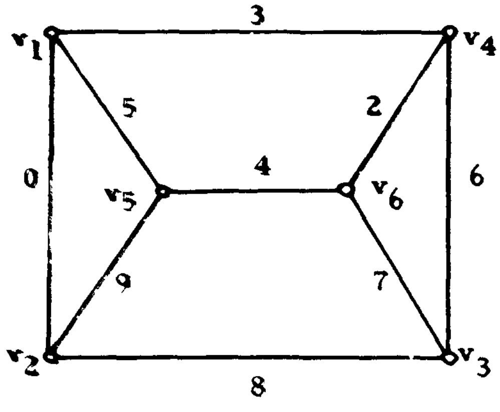


Figure 6.4 Graph G_1 of Example 6.1

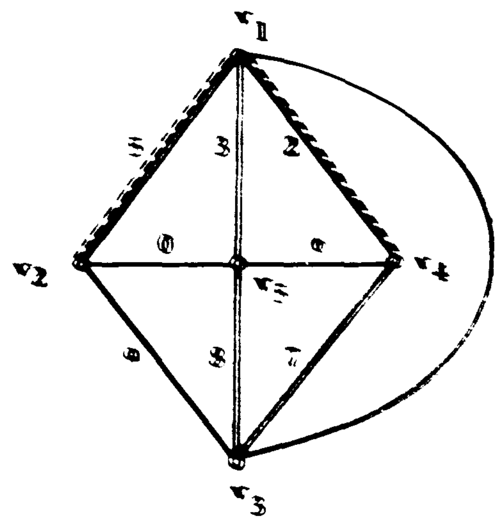


Figure 6.5 Graph G_1^* of Example 6.1

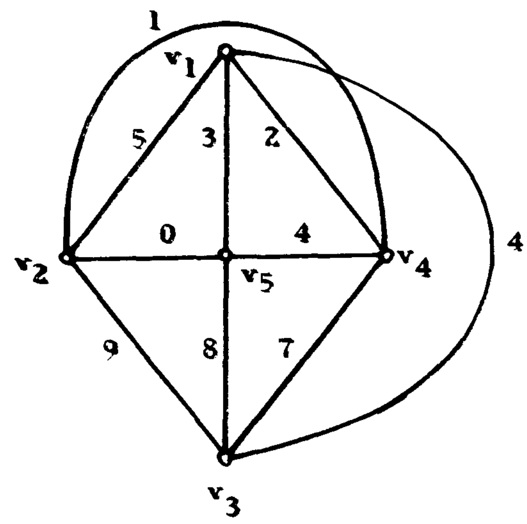


Figure 6.6 Graph G_0 of Example 6.1

Since this is satisfied $\underline{M}_0=\underline{M}$ is cographic, and its corresponding graph is given as G_0 in Fig. 6.6.

Example 6.2 Let $\underline{M}=\underline{M}_0=(\underline{C}_0, E_0)$ be the 3-connected matroid given by

$E_0 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 0, \alpha\}$

$\underline{C}_0 = \{123, 24\alpha, 280, 460, 567, 68\alpha, 134\alpha, 1380, 1789, 2468, 260\alpha, 3459,$
$3790, 4570, 480\alpha, 578\alpha, 12459, 12790, 13468, 1360\alpha, 15689,$
$1679\alpha, 2359\alpha, 23789, 24578, 2570\alpha, 34679, 35690, 124679,$
$125690, 134578, 13570\alpha, 145890, 14790\alpha, 235689, 23679\alpha,$
$34789\alpha, 35890\alpha\}.$

We form the following reduction sequence:

$\underline{M}_1=(\underline{C}_1, E_1)=\underline{M}_0\cdot(E_0-\{9\})$, where

$E_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 0, \alpha\}$

$\underline{C}_1 = \{123, 178, 24\alpha, 280, 345, 370, 460, 567, 68\alpha, 1245, 1270, 134\alpha,$
$1380, 1568, 167\alpha, 235\alpha, 2378, 2468, 260\alpha, 3467, 3560, 4570,$
$480\alpha, 578\alpha, 12467, 12560, 13468, 1360\alpha, 14580, 1470\alpha, 23568,$
$2367\alpha, 24578, 2570\alpha, 3478\alpha, 3580\alpha\}.$

$\underline{M}_2=(\underline{C}_2, E_2)=\underline{M}_1 \times (E_1-\{5\})$, where

$E_2 = \{1, 2, 3, 4, 6, 7, 8, 0, \alpha\}$

$\underline{C}_2 = \{123, 178, 24\alpha, 280, 370, 460, 68\alpha, 1270, 134\alpha, 1380, 167\alpha, 2378,$
$2468, 260\alpha, 3467, 480\alpha, 12467, 13468, 1360\alpha, 1470\alpha, 2367\alpha,$
$3478\alpha\}.$

$\underline{M}_3=(\underline{C}_3, E_3)=\underline{M}_2 \times (E_2-\{0\})$, where

$E_3 = \{1, 2, 3, 4, 6, 7, 8, \alpha\}$

$\underline{C}_3 = \{123, 178, 24\alpha, 68\alpha, 134\alpha, 167\alpha, 2378, 2468, 3467, 12467, 13468,$
$2367\alpha. 3478\alpha\}$.

The inverse reduction sequence is $\underline{S}^{-1} = \langle \underline{M}_3, \underline{M}_2, \underline{M}_1, \underline{M}_0 \rangle$ and the inverse operations are performed as follows:

Step 1. $\underline{M}_3$ is a wheel matroid, and let $\underline{M}_3 = \underline{B}(G_3)$ (Fig. 6.7).

Step 2. $\underline{M}_2 = \underline{M}_3(x)^{-1}(E_3 \cup \{0\})$. Use Algorithm C. Choose any circuit C_2 of $\underline{M}_2$ such that $0 \in C_2$ and the members of $C_2 - \{0\}$ have a common vertex of G_3. Let $C_2 = \{3, 7, 0\}$. The unique vertex is v_5. Split v_5 into two vertices, v_5 and v_6, so that C_2 is a star cut-set of G_2 (Fig. 6.8). Every circuit of $\underline{M}_2$ containing edge 0 is a cut-set of G_2.

Step 3. $\underline{M}_1 = \underline{M}_2(x)^{-1}(E_2 \cup \{5\})$. Use Algorithm C.
Since there is no circuit of $\underline{M}_1$ which satisfies the condition in Algorithm C, $\underline{M}_1$ is not graphic.

Step 4. Since G_2 is planar, take the dual $G_2{}^*$ of G_2 (Fig. 6.9). Apply Algorithm A for $\underline{M}_1 = \underline{M}_2(x)^{-1}(E_2 \cup \{5\}) = \underline{P}(G_2{}^*)(x)^{-1}(E_2 \cup \{5\})$. Choose any $C_1 \in \underline{C}_1$ containing cell 5. Let $C_1 = \{1, 2, 5, 6, 0\}$. Since $C_1 - \{5\} = \{1, 2, 6, 0\}$ forms a path in $G_2{}^*$, join v_2 and v_4 with edge 5 to construct G_1 (Fig. 6.10). Now test every circuit of $\underline{M}_1$ containing cell 5 to see if it is a polygon of G_1. We find this to be the case.

Step 5. $\underline{M}_0 = \underline{M}_1(\cdot)^{-1}(E_1 \cup \{9\})$. Use Algorithm B.
Choose $C_0 = \{1, 3, 4, 5, 7, 8\} \in \underline{C}_0$, which is not in $\underline{C}_1$ and does not contain cell 9. C_0 consists of two polygons of G_1, $\{1, 7, 8\}$

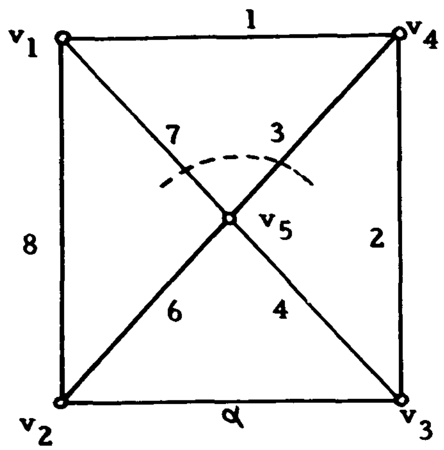


Figure 6.7 Graph G_3 of Example 6.2

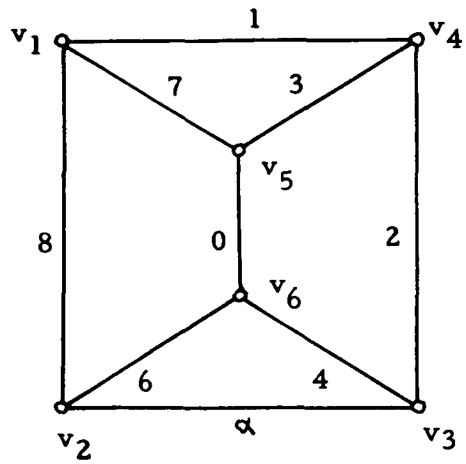


Figure 6.8 Graph G_2 of Example 6.2

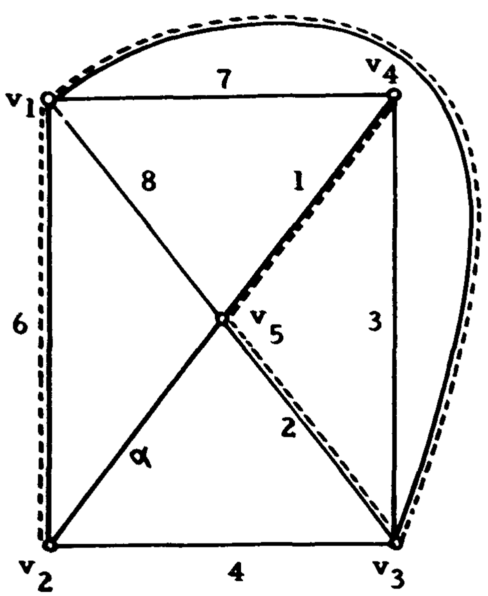


Figure 6.9 Graph G_2^* of Example 6.2

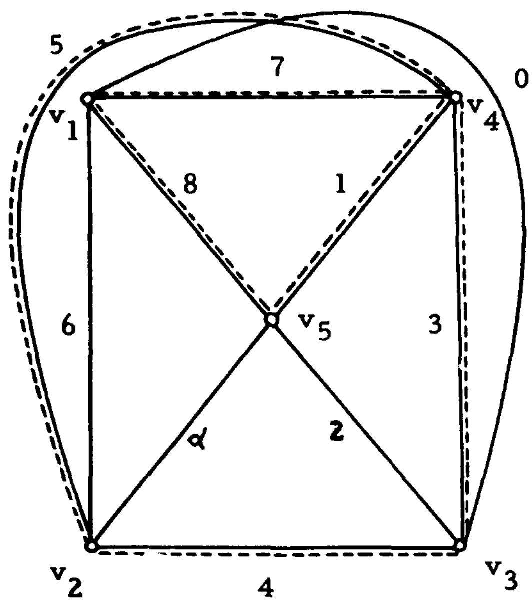


Figure 6.10 Graph G_1 of Example 6.2

and $\{3, 4, 5\}$, which have one vertex, v_4, in common. If $\underline{M}_0$ is graph realizable, then v_4 should be the splitting vertex.

Circuits containing cell 9	$E_{(1)}$	$E_{(2)}$
1789	1	7
3790	3	7
3459	3	5

We have $E_{(1)} = \{1, 3\}$ and $E_{(2)} = \{5, 7\}$, and hence, $E_{(1)} \cap E_{(2)} = \phi$. Construct graph G_0 (Fig. 11). Now, we test to see if the unexamined circuits in $\underline{C}_0 - \underline{C}_1$ are polygons of G_0. We see that the condition of Algorithm B is satisfied. Accordingly, $\underline{M}_0$ is cographic, but not graphic.

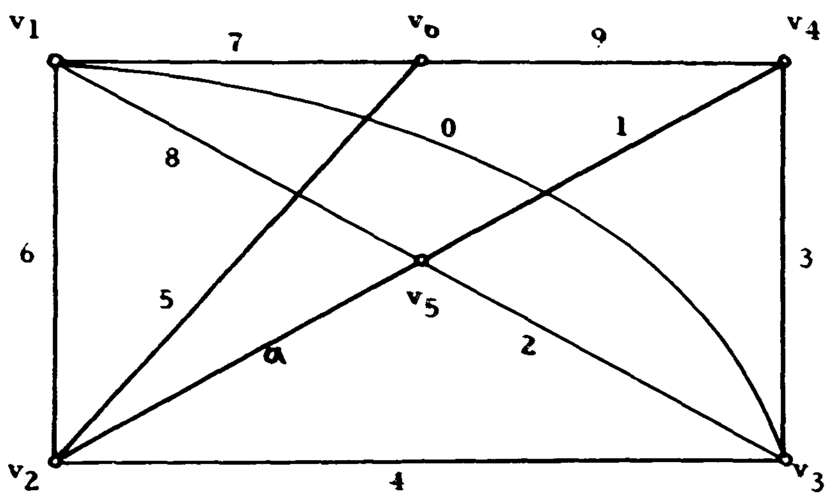


Figure 6.11 Graph G_0 of Example 6.2

CHAPTER 7 BIPARTITE AND EULER MINORS

In this chapter, we present another decomposition of a matroid into a unique family of minors, specifically, maximal bipartite and Euler minors. It is shown that a binary matroid may be reconstructed from these minors, which gives a theorem on matroid isomorphism.

7.1 MAXIMAL BIPARTITE CONTRACTION

Welsh obtained an interesting result for a class of binary matroids [We 5]. In defining bipartite and Euler matroids, he showed that these matroids embody dual concepts; that is, the dual of an Euler matroid is a bipartite matroid, and the converse is also true. In this section we will show the existence of bipartite minors in terms of so-called even sets.

Let $\underline{M}$ be a matroid. A circuit of $\underline{M}$ is called <u>even</u> or <u>odd</u> according to whether its cardinality is even or odd. If every circuit of $\underline{M}$ is even, then $\underline{M}$ is called <u>bipartite</u>.

<u>Theorem 7.1</u> Let $\underline{M} = (\underline{C}, E)$ be a connected binary matroid and $|\underline{C}| \geqslant 2$. Then there always exists an even circuit of $\underline{M}$.

<u>Proof.</u> Let C_1 and C_2 be two distinct circuits of $\underline{M}$ which have a non-null intersection. Since $\underline{M}$ is connected, these circuits always exist. We choose C_1 and C_2 so that $C_1 \cup C_2$ is minimal with respect

to the above condition. By Theorem 2.20, the symmetric difference $C_1 \oplus C_2$ is a disjoint union of circuits. Since C_1 and C_2 are distinct, $C_1 \oplus C_2$ contains at least one circuit of $\underline{M}$. Suppose $C_1 \oplus C_2$ contains two distinct circuits C_1' and C_2' of $\underline{M}$. Let $e_1 \in C_1 \cap C_1'$ and $e_2 \in (C_1 - C_1') \cap C_2' = C_1 \cap C_2'$. By Axiom II there exists a circuit C_3 of $\underline{M}$ such that $e_2 \in C_3 \subseteq C_1 \cup C_1' - \{e_1\}$. Then we have $C_2 \cup C_3 \subseteq C_2 \cup (C_1 \cup C_1' - \{e_1\}) = C_1 \cup C_2 - \{e_1\}$. Thus, $C_2 \cup C_3 \subset C_1 \cup C_2$, and, by Axiom I, $C_2 \cap C_3 \neq \phi$. However, this is contrary to $C_1 \cup C_2$ being minimal. Therefore, $C_1 \oplus C_2$ is a circuit of $\underline{M}$.

Suppose every member of $\underline{C}$ has odd cardinality. Choose two distinct circuits C_1 and C_2 so that $C_1 \cap C_2 \neq \phi$ and $C_1 \cup C_2$ is minimal, consistent with this condition. As shown above, $C = C_1 \oplus C_2$ is a circuit of $\underline{M}$. By hypothesis

$$\text{odd} = |C| = |C_1 \oplus C_2| = |C_1| + |C_2| - 2\,|C_1 \cap C_2| = \textbf{even}.$$

Thus, we have a contradiction. Accordingly, $\underline{M}$ has a circuit of even cardinality. ∎

Corollary 7.1 Let $\underline{M} = (\underline{C}, E)$ be a binary matroid. If there exists no even circuit of $\underline{M}$, then $\underline{C}$ is null, or it consists of disjoint odd circuits.

Proof. Let $\underline{M}_i = (\underline{C}_i, E_i)$, $1 \leq i \leq k$, be the components of $\underline{M}$. By definition, each $\underline{M}_i$ is connected. According to Theorem 7.1, $\underline{C}_i$ is null, or it consists of an odd circuit. Consequently, the corollary follows. ■

Theorem 7.1 implies that every connected binary matroid with at least two circuits contains a non-trivial bipartite matroid as a minor, where a trivial bipartite matroid is a matroid containing no circuits. In the same theorem, the binary condition on $\underline{M}$ is a necessary one, because, for instance, non-binary matroid ${}_4\underline{M}_3$ is connected and its only circuits are odd.

Let $\underline{M} = (\underline{C}, E)$ be a matroid and $S \subseteq E$. Then $\underline{M} \times S$ is called a <u>bipartite contraction</u> of $\underline{M}$ to S if it is bipartite, and it is <u>maximal</u> if there does not exist a subset T of E such that S is a proper subset of T and $\underline{M} \times T$ is bipartite. To show that every maximal bipartite contraction (max. bpt. contraction) is determined by fundamental circuits, we need to prove the following lemma:

<u>Lemma 7.1</u> Let $\underline{M} = (\underline{C}, E)$ be binary and B be a base of $\underline{M}$. If every fundamental circuit of $\underline{M}$ with respect to B is even, then $\underline{M}$ is bipartite.

<u>Proof.</u> Let C be any circuit of $\underline{M}$. If C is a fundamental circuit with respect to B, then, by hypothesis, C is even.

Suppose C is not a fundamental circuit and $C - B = \{e_1, e_2, \ldots, e_n\}$. Then, by Theorem 2.21, C is expressed as the symmetric difference of the fundamental circuits determined by B and $e_1, e_2, \ldots, e_n$. Since a symmetric difference of even cardinality subsets contains an even number of elements, C is an even circuit of

$\underline{M}$. Accordingly, the lemma follows. ∎

Let B be a base of a binary matroid $\underline{M} = (\underline{C}, E)$. We define the <u>even set</u> E_B of $\underline{M}$ spanned by base B as follows:

$$E_B = \{ e \mid e \in B \text{ or } B \cup \{e\} \text{ contains an even circuit of } \underline{M} \}.$$

By Lemma 7.1, $\underline{M} \times E_B$ is a bipartite matroid. Furthermore, $\underline{M} \times E_B$ is, as shown in the next lemma, a max. bpt. contraction of $\underline{M}$.

<u>Lemma 7.2</u> Let E_B be an even set of a binary matroid $\underline{M}$. Then $\underline{M} \times E_B$ is a max. bpt. contraction.

<u>Proof.</u> Let $\underline{M} \times E'$ be bipartite, where E' contains E_B properly. Let $e \in E' - E_B$. Then, by definition, $B \cup \{e\}$ $(\subseteq E_B \cup \{e\})$ contains a circuit C of $\underline{M}$. Since C is a circuit of $\underline{M} \times E'$, C is an even circuit of $\underline{M}$. Thus $e \in E_B$, which is contrary to the assumption. Therefore, $\underline{M} \times E_B$ is a max. bpt. contraction of $\underline{M}$. ∎

The converse statement of Lemma 7.2 is also true.

<u>Lemma 7.3</u> Let $\underline{M} \times E'$ be a max. bpt. contraction of a binary matroid $\underline{M}$. Then E' is an even set of $\underline{M}$.

<u>Proof.</u> First, we show E' contains a base of $\underline{M}$. Suppose E' does not contain a base of $\underline{M}$. Let B' be a base of $\underline{M} \times E'$. Choose a base B of $\underline{M}$ so that $B \supset B'$, and let E_B be the even set of $\underline{M}$ spanned by B. If C is the fundamental circuit of $\underline{M} \times E'$ determined by B' and $e \in E' - B'$, then $C \in \underline{C} \times E_B$, since $C \subseteq B' \cup \{e\} \subset B \cup \{e\}$ $\subseteq E_B$. Thus E' is a proper subset of E_B. Since $\underline{M} \times E_B$ is

bipartite, this contradicts $\underline{M} \times E'$ being maximal. Consequently, E' contains a base B of $\underline{M}$.

Let E_B be an even set of $\underline{M}$ spanned by B and let $e \in E' - B$. Then $B \cup \{e\}$ contains an even circuit of $\underline{M}$ and hence, $e \in E_B$. Accordingly, $E' \subseteq E_B$. Since $\underline{M} \times E'$ is maximal, $E' \subseteq E_B$ and we have $E' = E_B$. $\blacksquare$

Combining Lemmas 7.2 and 7.3 we state the main theorem of this section.

<u>Theorem 7.2</u> Let $\underline{M} = (\underline{C}, E)$ be a binary matroid and $E' \subseteq E$. Then $\underline{M} \times E'$ is a max. bpt. contraction of $\underline{M}$ if and only if E' is an even set of $\underline{M}$.

<u>Example 7.1</u> Let $\underline{M} = (\underline{C}, E)$ he the polygon matroid of the graph G shown in Fig. 7.1. Let $B = \{1, 2, 3, 5, 9\}$ be a base of $\underline{M}$. The even set E_B spanned by B is $E_B = \{1, 2, 3, 4, 5, 7, 8, 9, 11\}$. The max. bpt. contraction of $\underline{M}$ is $\underline{M} \times E_B = \underline{P}(G) \times E_B = \underline{P}(G \cdot E_B)$, which is the polygon matroid of $G \cdot E_B$ (Fig. 7.2).

Now let $\underline{M} = (\underline{C}, E)$ be the bond matroid of the same graph. $B = \{4, 5, 7, 8, 10, 11\}$ is a base of $\underline{B}(G)$. The corresponding even set is $E_B = \{1, 2, 4, 5, 6, 7, 8, 9, 10, 11\}$ and the max. bpt. contraction $\underline{M} \times E_B$ is the bond matroid of the graph $G \times E_B$ (Fig. 7.3).

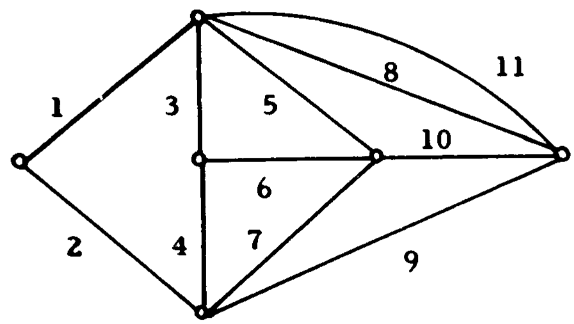


Figure 7.1 Graph G of Example 7.1

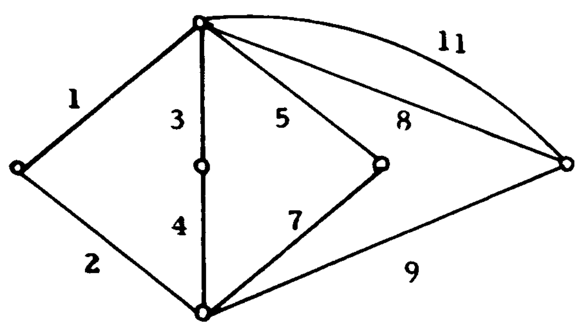


Figure 7.2 Graph $G \cdot E_B$ of Example 7.1

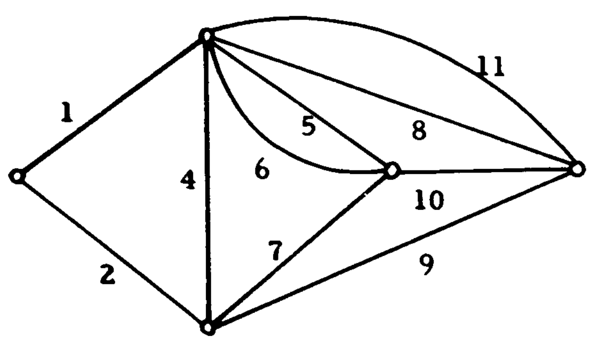


Figure 7.3 Graph $G \times E_B$

7.2 MAXIMAL BIPARTITE REDUCTION

Let $\underline{M} = (\underline{C}, E)$ be a binary matroid and S be a subset of E. A minor of the form $\underline{M}$ S is called a <u>bipartite reduction</u> of $\underline{M}$ to S if it is bipartite, and it is <u>maximal</u> if there does not exist T such that T contains S properly and $\underline{M}$ T is bipartite.

<u>Lemma 7.4</u> Let $\underline{M} = (\underline{C}, E)$ be a binary matroid. If $\underline{M} \times E'$, where $E' \subseteq E$, is a max. bpt. contraction of $\underline{M}$, then $\underline{M} \cdot E'$ is bipartite.

<u>Proof.</u> By Theorem 7.2, E' is an even set and there exists a base B of $\underline{M}$ by which E' is spanned. Let C' be a member of $\underline{C} \cdot E'$. By definition, we can find $C \in \underline{C}$ such that $C' = C \cap E'$. Since C is a circuit of $\underline{M}$ and $\underline{M}$ is binary, C may be written as a symmetric difference of fundamental circuits of $\underline{M}$.

Let $C - B = \{e_1, e_2, \ldots, e_m, e_1', e_2', \ldots, e_n'\}$, where $e_1, e_2, \ldots, e_m$ are members of E' and $e_1', e_2', \ldots, e_n'$ are in $E - E'$. By Theorem 2.21, we can write:

$$C = C_1 \oplus C_2 \oplus \cdots \oplus C_m \oplus C_1' \oplus C_2' \oplus \cdots \oplus C_n',$$

where C_i, $1 \leq i \leq m$, is the fundamental circuit of $\underline{M}$ determined by B and e_i, and C_j', $1 \leq i \leq n$, is the fundamental circuit determined by B and e_j'. Since E' is the even set of $\underline{M}$ spanned by B, C_i is even for $1 \leq i \leq m$ and $|C_j'|$ is odd for $1 \leq j \leq n$. Let $S = C_1 \oplus C_2 \oplus \cdots \oplus C_m$.

Then,
$$\begin{aligned} |C'| &= |C \cap E'| = |(S \oplus C_1' \oplus C_2' \oplus \cdots \oplus C_n') \cap E'| \\ &= |(S \cap E') \oplus (C_1' \cap E') \oplus (C_2' \cap E') \oplus \cdots \oplus (C_n' \cap E')| \\ &= |S \oplus (C_1' - \{e_1'\}) \oplus (C_2' - \{e_2'\}) \oplus \cdots \oplus (C_n' - \{e_n'\})|. \end{aligned}$$

Since $|C_1' - \{e_1'\}|$, $|C_2' - \{e_2'\}|, \ldots, |C_n' - \{e_n'\}|$ are even and the symmetric difference of subsets with even cardinalities consists of an even number of elements, $|C'|$ is even. This is true for every circuit of $\underline{C}$ E' and the lemma follows. ∎

In the above lemma, if $\underline{M}$ x E' is not maximal, then $\underline{M}$•E' is not necessarily bipartite. Consider the polygon matroid $\underline{P}$(G) of the graph G given in Fig. 7.4. Choose E' = $\{1, 2, \ldots, 6\}$. Then $\underline{P}$(G) x E' = $\underline{P}$(G•E') is bipartite, but not maximal. However, $\underline{P}$(G)•E' = $\underline{P}$(G x E') is not a bipartite matroid (Fig. 7.5).

<u>Corollary 7.2</u> Let $\underline{M}$ = ($\underline{C}$, E) be binary. If E' is an even set of $\underline{M}$, then $\underline{M}$•E' is bipartite.

<u>Proof.</u> By Theorem 7.2 and Lemma 7.4. ∎

To show the converse of Lemma 7.4, we need to prove two lemmas.

<u>Lemma 7.5</u> Let $\underline{M}$ = ($\underline{C}$, E) be binary. If e $\in$ E and $\underline{M}'$ = ($\underline{C}'$, E') = $\underline{M}$•(E - $\{e\}$), then every circuit of $\underline{M}$ not containing e is a disjoint union of circuits of $\underline{M}'$.

<u>Proof.</u> If $\{e\}$ is a circuit of $\underline{M}$, then $\underline{M}'$ = $\underline{M}$•(E - $\{e\}$) = $\underline{M}$ x (E - $\{e\}$). Clearly, every circuit of $\underline{M}$ not containing e is a circuit of $\underline{M}'$, and the lemma follows.

Suppose $\{e\} \notin \underline{C}$ and (e $\notin$) C is any circuit of $\underline{M}$. If C is a circuit of $\underline{M}'$, obviously C is a disjoint union of itself. Therefore, we

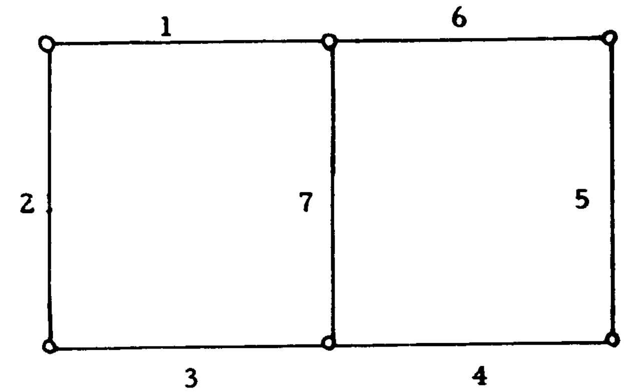


Figure 7.4 Graph G

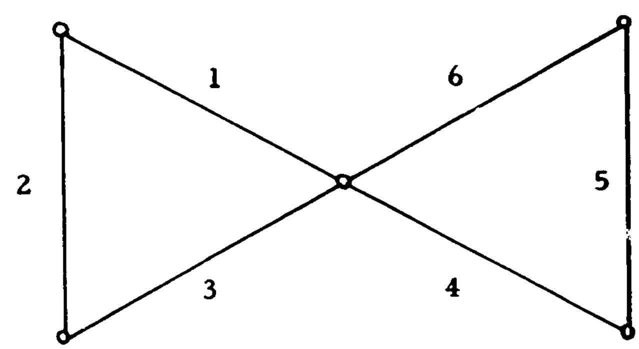


Figure 7.5 Graph G x E'

assume $e \notin C \notin \underline{C}'$. By definition, there exists $C_1 \in \underline{C}$ such that $C_1 \cap E' = C_1 - \{e\} \in \underline{C}'$ and $C_1 - \{e\} \subset C$, where $e \in C_1$. Let $S = C \oplus C_1$. Since $\underline{M}$ is binary, by Theorem 2.20, S is a disjoint union of circuits of $\underline{M}$. Let C_2 be a circuit of $\underline{M}$ contained in S and not containing e. Then

$$C_2 \subseteq S - \{e\} = (C \cup C_1 - C \cap C_1) - \{e\}$$
$$\subseteq C \cup C_1 - \{e\} = C.$$

By Axiom I, $C_2 = C$ and $S - C_2 = \{e\}$ contains a circuit of $\underline{M}$, contrary to the hypothesis. Consequently, $C_2 = S$ is a circuit of $\underline{M}$. Let $C_1' = C_1 \cap E' = C_1 - \{e\}$ and $C_2' = C_2 \cap E' = C_2 - \{e\}$. Then C_1' and C_2' are disjoint, and $C = C_1' \oplus C_2' = C_1' \cup C_2'$. If C_2' is not a circuit of $\underline{M}'$, then there is a circuit $(e \in)$ $C_3 \in \underline{C}$ such that

$$C_3 \cap E' = C_3 - \{e\} \subset C_2' = C_2 - \{e\}.$$

Thus, $C_3 \subset C_2$, contrary to Axiom I. Accordingly, C_2' is a circuit of $\underline{M}'$, and C is a disjoint union of circuits C_1' and C_2' of $\underline{M}'$. ∎

In Lemma 7.5, if we remove the condition "binary", the lemma is not true. A counterexample is $\underline{M} = {}_4\underline{M}_3 = (\underline{C}, E)$, where $E = \{e_1, e_2, e_3, e_4\}$. $\underline{M} \cdot (E - \{e_4\})$ consists of three circuits which are the subsets of $\{e_1, e_2, e_3\}$ containing exactly two elements. $\{e_1, e_2, e_3\}$ is a circuit of $\underline{M}$; however, it cannot be expressed as a disjoint union of circuits of $\underline{M} \cdot (E - \{e_4\})$.

Lemma 7.6 Let $\underline{M} = (\underline{C}, E)$ be binary, and $\underline{M}' = \underline{M} \cdot E'$, where $E' \subseteq E$. Then every circuit of $\underline{M} \times E'$ is a disjoint union of circuits of $\underline{M}'$.

Proof. A proof is obtained by induction on $|E - E'|$. If $|E'| = |E| - 1$, the lemma follows from Lemma 7.5.

Suppose the lemma is true for any E' which has more than n elements. Let $E' = \{e_1, e_2, \ldots, e_n\}$ and $E'' = E \cup \{e\}$ where $e \in E - E'$. By assumption, every circuit of $\underline{M} \times E''$ is a disjoint union of circuits of $\underline{M} \bullet E''$. Let C be a circuit of $\underline{M} \times E'$. C is also a circuit of $\underline{M} \times E''$ and $e \notin C$. Then

$$C = C_1 \cup C_2 \cup \ldots \cup C_k,$$

where $C_i \in \underline{C}\ E''$, $1 \leq i \leq k$, and $C_i \cap C_j = \phi$ for $i \neq j$. Each C_i, $1 \leq i \leq k$, is a circuit of $\underline{M} \bullet E''$ not containing e. Therefore, by Lemma 7.5, C_i is a disjoint union of circuits of $\underline{M} \bullet (E'' - \{e\}) = \underline{M} \bullet E'$.

$$C_i = C_{i1} \cup C_{i2} \cup \ldots \cup C_{ik_i},$$

where $C_{ij} \in \underline{C} \bullet E'$, $1 \leq j \leq k_i$, and $C_{ij} \cap C_{im} = \phi$ for $j \neq m$. Accordingly, we ca^ express C as a disjoint union of circuits of $\underline{M} \bullet E'$. By induction the lemma fo^lows. ∎

The converse of Lemma 7.4 is obtained from the above lemma.

Lemma 7.7 Let $\underline{M} = (\underline{C}, E)$ be binary and $E' \subsetneq E$. If $\underline{M} \bullet E'$ is bipartite, then $\underline{M} \times E'$ is also bipartite.

Proof. If $\underline{M} \bullet E'$ is bipartite, every circuit of $\underline{M} \bullet E'$ has even cardinality. Since, by Lemma 7.6, any circuit of $\underline{M} \times E'$ is a disjoint union of circuits of $\underline{M} \bullet E'$, every circuit of $\underline{M} \times E'$ has even cardinality. Thus, $\underline{M} \times E'$ is bipartite. ∎

<u>Theorem 7.3</u> Let $\underline{M} = (\underline{C}, E)$ be a binary matroid and $E' \subsetneq E$. Then $\underline{M} \times E'$ is a max. bpt. contraction if and only if $\underline{M} \cdot E'$ is a max. bpt. reduction.

<u>Proof.</u> Let $\underline{M} \times E'$ be a max. bpt. contraction of $\underline{M}$. By Lemma 7.4, $\underline{M} \cdot E'$ is bipartite. If $\underline{M} \cdot E'$ is not maximal, then, by Lemma 7.7, $\underline{M} \times E'$ is not maximal, which is contrary to the hypothesis. Therefore, $\underline{M} \cdot E'$ is a max. bpt. reduction.

Similarly, the sufficiency part follows from Lemmas 7.7 and 7.4. ∎

7.3 MAXIMAL EULER MINORS

For planar graphs the dual of a bipartite graph is an Euler graph, and vice versa. A matroid generalization of this theorem is proved by Welsh [We 5].

A matroid $\underline{M} = (\underline{C}, E)$ is called Euler if there exist mutually disjoint circuits $C_1, C_2, \ldots, C_n$ such that $E = C_1 \cup C_2 \cup \ldots \cup C_n$.

Theorem 7.4 [We 5] Let $\underline{M}$ be a binary matroid. Then $\underline{M}$ is bipartite if and only if its dual $\underline{M}^*$ is Euler.

Let $\underline{M} = (\underline{C}, E)$ be a matroid. A minor of the form $\underline{M} \times E'$ ($\underline{M} \cdot E'$), where $E' \subseteq E$, is called an Euler contraction (Euler reduction) of $\underline{M}$ to E' if $\underline{M} \times E'$ ($\underline{M} \cdot E'$) is an Euler matroid. If $\underline{M} \times E'$ ($\underline{M} \cdot E'$) is Euler and there is no subset E'' of E such that $E' \subset E''$ and $\underline{M} \times E''$ ($\underline{M} \cdot E''$) is Euler, the $\underline{M} \times E'$ ($\underline{M} \cdot E'$) is maximal.

Lemma 7.8 Let $\underline{M} = (\underline{C}, E)$ be binary and $E' \subseteq E$. Then

(a) if $\underline{M} \cdot E'$ is a max. Euler reduction, then $\underline{M} \times E'$ is an Euler contraction.

(b) If $\underline{M} \times E'$ is an Euler contraction, then $\underline{M} \cdot E'$ is an Euler reduction.

Proof. The lemma follows from Lemmas 7.4, 7.7, and Theorem 7.4. ∎

Using Lemma 7.8, we restate Theorem 7.3.

<u>Lemma 7.9</u> Let $\underline{M}$ be a binary matroid and $E' \subseteq E$. Then $\underline{M} \times E'$ is a max. Euler contraction if and only if $\underline{M} \cdot E'$ is a max. Euler reduction.

The combination of Theorems 7.2 and 7.3 and Lemma 7.7 yields the next theorem.

<u>Theorem 7.5</u> Let $\underline{M} = (\underline{C}, E)$ be a binary matroid and $E' \subseteq E$. Then the following statements are equivalent:

(a) E' is an even set of $\underline{M}$.

(b) $\underline{M} \times E'$ is a max. bpt. contraction of $\underline{M}$.

(c) $\underline{M} \cdot E'$ is a max. bpt. reduction of $\underline{M}$.

(d) $\underline{M}^* \times E'$ is a max. Euler contraction of $\underline{M}^*$.

(e) $\underline{M}^* \cdot E'$ is a max. Euler reduction of $\underline{M}^*$.

An even set of a matroid determines max. bpt. minors. Now we shall find sets which determine Euler minors. Let $\underline{M} = (\underline{C}, E)$ be a binary matroid and B be a base of $\underline{M}$. A subset E' of E is called a <u>coeven set</u> of $\underline{M}$ spanned by B if E' consists of $\overline{B}$ and all the members $e_1, e_2, \ldots, e_n$ of B such that $\underline{M}(\overline{B} \cup \{e_i\})$ is Euler for each e_i, $1 \leqslant i \leqslant n$.

<u>Theorem 7.6</u> Let $\underline{M} = (\underline{C}, E)$ be a binary matroid and $E' \subseteq E$. Then E' is a coeven set of $\underline{M}$ if and only if $\underline{M}$ E' is a max. Euler reduction of $\underline{M}$.

<u>Proof</u>. Let B be a base and E' be a coeven set of $\underline{M}$ spanned by B. Let $e \in E' - \overline{B}$. By definition, $\underline{M} \cdot (\overline{B} \cup \{e\})$ is Euler, and, by Theorem 7.4, $\underline{M}^* \times (\overline{B} \cup \{e\})$ is bipartite. Since $\overline{B}$ is a base of $\underline{M}^*$, $\overline{B} \cup \{e\}$ contains one circuit C* of $\underline{M}^*$ and C* is even. Therefore, E' is an even set of $\underline{M}^*$ and $\underline{M}^* \times E'$ is a max. bpt. contraction. Accordingly, by Theorem 7.5, $\underline{M} \cdot E'$ is a max. Euler reduction of $\underline{M}$.

Conversely, if $\underline{M} \cdot E'$ is a max. Euler reduction, then, by Theorem 7.5, E' is an even set of $\underline{M}^*$. E' contains a base $\overline{B}$ of $\underline{M}^*$, where B is a base of $\underline{M}$. For each $e \in E' - \overline{B}$, $\underline{M}^* \times (\overline{B} \cup \{e\})$ contains only one circuit which is even. Thus, $\underline{M}^* \times (\overline{B} \cup \{e\})$ is bipartite for each $e \in E' - \overline{B}$. By Theorem 7.4, $\underline{M} \cdot (\overline{B} \cup \{e\})$ is Euler for each $e \in E' - \overline{B}$. Since $\underline{M} \cdot E'$ is maximal, E' is a coeven set of $\underline{M}$. ∎

Let B be a base of a binary matroid $\underline{M}$. If $e \in E - \overline{B}$, then $\underline{M}^* \times (\overline{B} \cup \{e\})$ has only one circuit which is even, and the components of $\underline{M} \cdot (\overline{B} \cup \{e\})$ consist of loop matroids and one binomial matroid of the form ${}_{n}\underline{M}_{2}$. Clearly, cell e belongs to the coeven set of $\underline{M}$ spanned by B if and only if n is even. Thus, whether $e \in E - \overline{B}$ belongs to a coeven set is easily determined by counting the number of even circuits of $\underline{M} \cdot (\overline{B} \cup \{e\})$.

We may rewrite Theorem 7.5 in terms of coeven sets.

<u>Theorem 7.7</u> Let $\underline{M} = (\underline{C}, E)$ be a binary matroid and $E' \subsetneq E$.

Then the following statements are equivalent:

(a) E' is a coeven set of $\underline{M}$.

(b) $\underline{M}$ x E' is a max. Euler contraction of $\underline{M}$.

(c) $\underline{M}$•E' is a max. Euler reduction of $\underline{M}$.

(d) $\underline{M}$* x E' is a max. bpt. contraction of $\underline{M}$*.

(e) $\underline{M}$*•E' is a max. bpt. reduction of $\underline{M}$*.

We shall illustrate Theorems 7.5 and 7.7 in the next example.

<u>Example 7.2</u> Let $\underline{P}$(G) be the polygon matroid of the graph G given in Fig. 7.6. We choose B = $\{1, 5, 8, 9, 10, 12\}$ as a base of $\underline{P}$(G). Then, the coeven set of $\underline{P}$(G) spanned by B is:

$$E_B = \bar{B} \cup \{1, 9, 10\} = \{1, 2, 3, 4, 6, 7, 9, 10, 11\}.$$

The max. Euler contraction and max. Euler reduction are the polygon matroids of the graphs G•E_B and G x E_B, respectively, given in Fig. 7.7. If we take the bond matroids of G•E_B and G x E_B, then those bond matroids are a max. bpt. reduction and max. bpt. contraction of $\underline{B}$(G) = $\underline{P}$*(G).

The even set of $\underline{P}$(G) spanned by the same base B = $\{1, 5, 8, 9, 10, 12\}$ is

$$E_B' = \{1, 3, 5, 6, 8, 9, 10, 11, 12\}.$$

$\underline{P}$(G•E_B') and $\underline{P}$(G x E_B') are the max. bpt. contraction and max. bpt. reduction of $\underline{P}$(G) spanned by B, and $\underline{B}$(G•E_B') and $\underline{B}$(G x E_B')

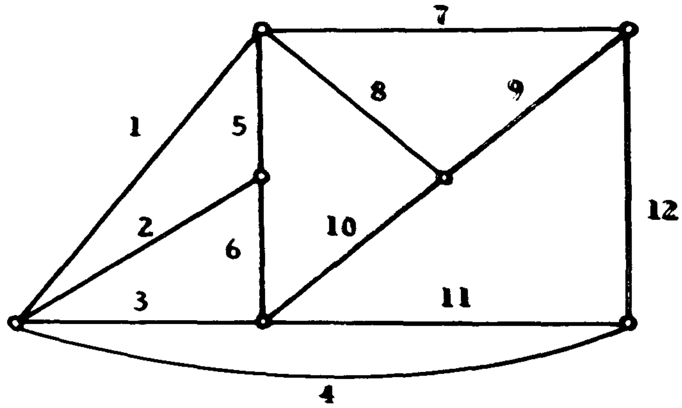


Figure 7.6 Graph G of Example 7.2

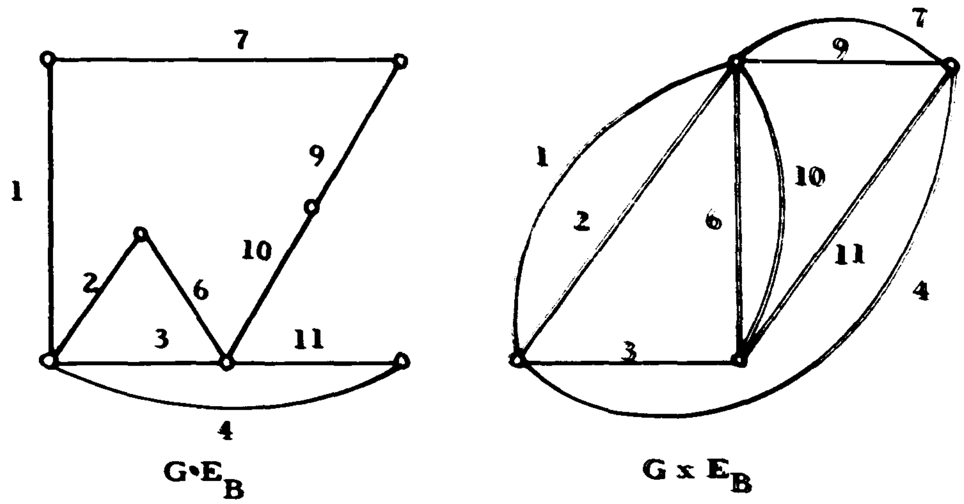


Figure 7.7 Graphs $G \cdot E_B$ and $G \times E_B$

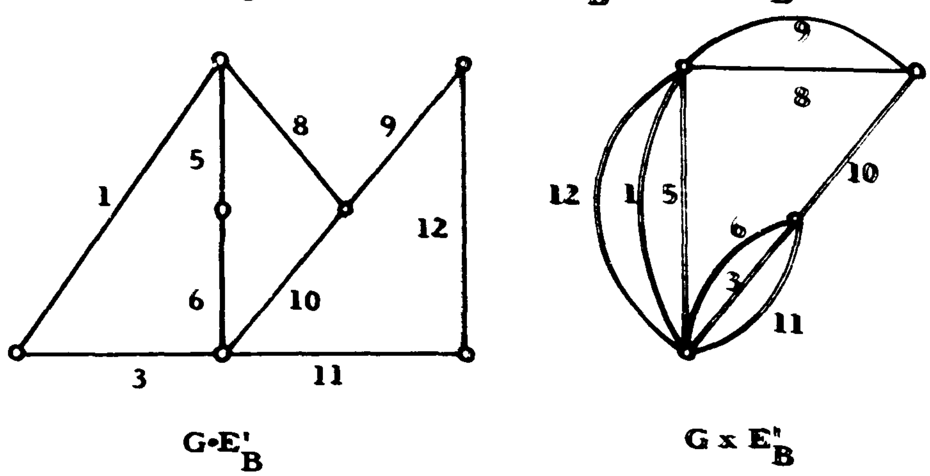


Figure 7.8 Graphs $G \cdot E'_B$ and $G \times E''_B$

are the max. Euler reduction and max. Euler contraction of $\underline{B}(G)$. (See Fig. 7.8.)

Previously, we described in Theorem 7.2 how to generate all the max. bpt. contractions of a binary matroid. If a binary matroid $\underline{M}$ is given, then all the max. bpt. contractions of $\underline{M}$ are uniquely determined. Now, we shall consider the reconstruction problem of the original matroid from max. bpt. contractions.

Theorem 7.8 Let $\underline{M} = (\underline{C}, E)$ be a binary matroid containing no loops, and $\underline{B}$ be the family of all the distinct max. bpt. contractions of $\underline{M}$. Then $\underline{M}$ can be uniquely reconstructed from $\underline{B}$.

Proof. Let E' be the union of the cell sets of the members in $\underline{B}$. Suppose e is any cell of $\underline{M}$ and B is a base of $\underline{M}$ containing e; such a base always exists since $\underline{M}$ contains no loops. If E_B is the even set of $\underline{M}$ spanned by B, then $\underline{M} \times E_B \in \underline{B}$, by Theorem 7.2. Thus, $e \in E'$. Since $E' \subseteq E$, we have $E' = E$.

By the definition of a max. bpt. contraction, every base of $\underline{M}$ is a base of some max. bpt. contraction of $\underline{M}$. Conversely, if B is a base of a max. bpt. contraction, then, clearly, B is a base of $\underline{M}$. Therefore, all the bases of $\underline{M}$ are uniquely determined from the members of $\underline{B}$, and $\underline{M}$ can be reconstructed from the family of max. bpt. contractions of $\underline{M}$. ∎

Corollary 7.3 A binary matroid $\underline{M}$ without isthmuses is uniquely reconstructed from the family of all the distinct max. Euler reductions of M.

Proof. Since M contains no isthmus, M*, the dual of M, contains no loops. By Theorems 7.4 and 7.8, M* is reconstructed from the max. bpt. contractions of M*, and hence, M = (M*)* is reconstructed from the max. Euler reductions of M. ∎

CHAPTER 8 WHITNEY CONNECTIVITY OF MATROIDS

In this chapter we generalize Whitney connectivity of graphs to matroids.

8.1 WHITNEY CONNECTIVITY OF GRAPHS

Let $G = (V, E)$ be a graph of connectivity n, where n is a positive finite integer. By the definition of Whitney connectivity, we can find n vertices of G whose deletion from G results in a disconnected graph. These n vertices are called join vertices of G. The join graph $G_0 = (V_0, E_0)$ is the induced graph on join vertex set V_0, where obviously E_0 consists of the edges of G having both of their ends in V_0. Let $G_i' = (V_i', E_i')$, $1 \leq i \leq k$, be the components of the disconnected graph obtained by deleting the vertices V_0 from G, these components referred to as the join components. Each member of the edge set $(E - E_0 - \bigcup_{i=1}^{k} E_i')$ is incident to a vertex of G_0 and a vertex of exactly one G_i', since $V_1', V_2', \ldots, V_k'$ are mutually disjoint. The set $(E - E_0)$ may be partitioned into k subsets $E_1, E_2, \ldots, E_k$ so that each E_i is the union of E_i' and the edges incident to the vertices of G_i'. Then $G_i = G \cdot E_i$, $1 \leq i \leq k$, are connected and called the n-palms of G associated with the join vertex set V_0.

Lemma 8.1 Let G be a graph of connectivity n, $1 \leq n < \infty$, and V_0 be a join vertex set of G. Then the vertex set of every n-palm associated with V_0 contains V_0, and the intersection of the vertex sets of any two n-palms is V_0.

Proof. Suppose $G_i = G \cdot E_i$ is an n-palm of G associated with V_0 such that the vertex set of G_i does not contain V_0. If G_i' is the join component of G corresponding to G_i, G_i' has no common vertices with other join components of G. Therefore, G_i is connected to the rest of the graph, $G \cdot (E - E_i)$, at less than $|V_0| = n$ vertices. However, this contradicts the fact that G is n-connected. Consequently, every n-palm associated with V_0 contains V_0.

The second part of the lemma is deduced from the first part and the definition of join components. ∎

Lemma 8.2 Let $G = (V, E)$ be a connected graph. If the connectivity of G is n, $1 \leq n < \infty$, then there exists a non-null proper subset S of E such that $\eta(G;S,\bar{S}) = n$ and $\min(r(G \cdot S), r(G \cdot \bar{S})) \geq n$.

Proof. Let V_0 be a join vertex set of G. Since the connectivity of G is finite, there are at least two join components associated with V_0. Let G_1' and G_2' be distinct join components of G. Since each of these join components contains at least one vertex, the corresponding n-palms $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ both contain at least $(n+1)$ vertices of G. Choose $S = E_1$ and $\bar{S} = E - E_1 \supseteq E_2$. Clearly, $r(G \cdot S) = r(G \cdot E_1) \geq n$ and $r(G \cdot \bar{S}) \geq r(G \cdot E_2) \geq n$.

By Lemma 8.1, $G \cdot S$ and $G \cdot \bar{S}$ have the common vertices V_0 and, hence, $\eta(G;S,\bar{S}) = n$. Accordingly, the lemma follows. ∎

Let $G = (V, E)$ be connected. Then $\lambda_W(G)$ denotes the least integer n which satisfies the following conditions:

$$\eta(G;S,\bar{S}) = n \quad \text{and} \quad \min(r(G \cdot S),\ r(G \cdot \bar{S})) \geqslant n,$$

where S is a non-null proper subset of E. If there does not exist such an integer, then we write $\lambda_W(G) = \infty$.

<u>Lemma 8.3</u> If the connectivity of a graph G is n, where $n \geqslant 1$, then $\lambda_W(G) \leqslant n$.

<u>Proof.</u> If n is infinity, the lemma is trivial. For a finite n, it follows from Lemma 8.2. ∎

<u>Lemma 8.4</u> Let $G = (V, E)$ be a connected graph such that $\lambda_W(G) = n$. If a non-null proper subset S of E satisfies the conditions $\eta(G;S,\bar{S}) = n$ and $\min(r(G \cdot S),\ r(G \cdot \bar{S})) \geqslant n$, then $G \cdot S$ and $G \cdot \bar{S}$ are both connected.

<u>Proof.</u> Suppose $G \cdot S$ is not connected. Since $r(G \cdot S) \geqslant n$, $G \cdot S$ contains at least $(n+1)$ vertices, and at least one of the vertices, say v, is not a vertex of $G \cdot \bar{S}$, because the number of common vertices of $G \cdot S$ and $G \cdot \bar{S}$ is n. Let $G_1 = (V_1, E_1)$ be the component of $G \cdot S$ which contains v. Since $G \cdot S$ does not contain isolated vertices, G_1 contains at least two vertices and v is not a vertex of $G \cdot (E - E_1)$. Therefore, $r(G_1) = r(G \cdot E_1) \geqslant |V_1| - 1$ and $r(G \cdot (E - E_1)) \geqslant n > \eta(G;E_1, E - E_1)$. Furthermore, $\eta(G;E_1, E - E_1) \leqslant |V_1| - 1$. Hence, $\lambda_W(G) \leqslant \eta(G;E_1, E - E_1) < n$. This is contrary to the hypothesis; therefore $G \cdot S$ is connected. Similarly $G \cdot \bar{S}$ is also connected. ∎

<u>Lemma 8.5</u> If G is a graph of connectivity n, where $n \geqslant 1$, then

$\lambda_W(G) \geqslant n$.

Proof. If $\lambda_W(G) = \infty$, it is obvious that the lemma is true. To prove the lemma for a finite n we assume $\lambda_W(G) < n$. By definition, we can find a non-null proper subset S of E such that $\eta(G;S,\bar{S}) = \lambda_W(G)$ and min $(r(G \cdot S),\ r(G \cdot S)) \geqslant \lambda_W(G)$. By Lemma 8.4 $G \cdot S$ and $G \cdot \bar{S}$ are connected and both contain at least $\lambda_W(G) + 1$ vertices. Since the number of common vertices of $G \cdot S$ and $G \cdot \bar{S}$ is $\lambda_W(G)$, the deletion of the common vertices results in a disconnected graph. However, this is impossible since G is n-connected. Therefore, $\lambda_W(G) \geqslant n$. ∎

The combination of Lemmas 8.3 and 8.5 yields the next theorem, which gives another equivalent definition of Whitney connectivity.

Theorem 8.1 If G is a connected graph with connectivity n, where $n \geqslant 1$, then $\lambda_W(G) = n$.

According to Theorem 8.1, Whitney connectivity of a connected graph G is the least positive integer $\lambda_W(G)$ such that $\eta(G;S,\bar{S}) = \lambda_W(G)$ and min $(r(G \cdot S),\ r(G \cdot \bar{S})) \geqslant \lambda_W(G)$, where S is a non-null proper subset of E. If there is no such integer, then $\lambda_W(G) = \infty$.

8.2 WHITNEY CONNECTIVITY OF MATROIDS

In this section we generalise Whitney connectivity of graphs to matroids. Let $\underline{M}=(\underline{C}, E)$ be a matroid defined on E. Whitney connectivity of $\underline{M}$, denoted by $\lambda_W(\underline{M})$, is the least integer n which satisfies the following two conditions:

$$\eta(\underline{M};S, \bar{S}) = n \text{ and } \min(r(\underline{M} \times S), r(\underline{M} \times \bar{S})) \geq n,$$

where S is a non-null proper subset of E. The aim of this section is to show $\lambda_W(\underline{P}(G))= \lambda_W(G)$ for a connected graph G.

Let G=(V, E) be a graph. r(G) and c(G) denote the rank and the number of components of G, where the rank of G is the number of elements in a spanning forest of G. The nullity of G, denoted by $\mu(G)$, is the number E - r(G). The following relationship is basic in graph theory:

$$r(G)= |E| - \mu(G)= |V| - c(G).$$

There will arise no confusion in the use of the same symbols "r", "μ", and "λ_W" for matroid and graph invariants.

Lemma 8.6 If G is a graph, then $r(G)=r(\underline{P}(G))$ and $\mu(G)=\mu(\underline{P}(G))$.

Proof. A base of $\underline{P}(G)$ is a maximal set of edges which does not contain circuits of $\underline{P}(G)$. Since the circuits of $\underline{P}(G)$ are the polygons of G, a base of $\underline{P}(G)$ is a spanning forest of G. Therefore $r(G)=r(\underline{P}(G))$. The second part easily follows from the first part and the matroid definition of nullity:

$$\mu(G)= |E| - r(G)= |E| - r(\underline{P}(G))=\mu(\underline{P}(G)). \blacksquare$$

<u>Lemma 8.7</u> If $G = (V, E)$ is a connected graph and S is a subset of E, then:

$$\xi(\underline{P}(G);S,\bar{S}) = \eta(G;S,\bar{S}) - c(G \cdot S) - c(G \cdot \bar{S}) + 2.$$

<u>Proof.</u> By definition

$$\xi(\underline{P}(G);S,\bar{S}) = -r(\underline{P}(G)) + r(\underline{P}(G) \times S) \quad r(\underline{P}(G) \times \bar{S})) + 1$$

$$= -r(\underline{P}(G)) + r(\underline{P}(G \cdot S)) + r(\underline{P}(G \cdot \bar{S})) + 1.$$

Using Lemma 8.6

$$\xi(\underline{P}(G);S,\bar{S}) = -r(G) + r(G \cdot S) + r(G \cdot \bar{S}) + 1$$

$$= -|V| + c(G) + |V_S| - c(G \cdot \bar{S}) + |V_{\bar{S}}| - c(G \cdot \bar{S}) + 1$$

$$= -|V| + |V_S| + |V_{\bar{S}}| - c(G \cdot S) - c(G \cdot \bar{S}) + 2,$$

where V_S and $V_{\bar{S}}$ are the vertex sets of $G \cdot S$ and $G \cdot \bar{S}$, respectively. Since $|V| = |V_S| + |V_{\bar{S}}| - \eta(G;S,\bar{S})$, we can reduce the above equation to

$$\xi(\underline{P}(G);S,\bar{S}) = \eta(G;S,\bar{S}) - c(G \cdot S) - c(G \cdot \bar{S}) + 2. \blacksquare$$

<u>Lemma 8.8</u> Let $G = (V, E)$ be a connected graph. Then $\lambda_W(\underline{P}(G)) \leqslant \lambda_W(G)$.

<u>Proof.</u> Since the lemma is trivial for $\lambda_W(G) = \infty$, we assume $\lambda_W(G) = n$ is a finite positive integer. For some non-null proper subset S of E $\eta(G;S,\bar{S}) = n$ and $\min(r(G \cdot S), r(G \cdot \bar{S})) \geqslant n$. By Lemmas 8.6 and 8.7

$$\xi(\underline{P}(G);S,\bar{S}) = n - c(G \cdot S) - c(G \cdot \bar{S}) + 2 \leqslant n,$$

and $\min(r(\underline{P}(G) \times S), r(\underline{P}(G) \times \bar{S})) \geqslant n$. Therefore

$$\lambda_W(\underline{P}(G)) \leqslant n = \lambda_W(G). \blacksquare$$

<u>Lemma 8.9</u> Let $G = (V, E)$ be a connected graph. Then $\lambda_W(\underline{P}(G)) \geqslant \lambda_W(G)$.

<u>Proof.</u> If $\lambda_W(\underline{P}(G)) = \infty$, the lemma is obvious.

Suppose $\lambda_W(\underline{P}(G)) = n$ is finite and $\lambda_W(G) > n$. Then, by Lemma 8.7, there exists a non-null proper subset S of E such that

$$\eta(G;S,\bar{S}) \leqslant n + c(G \cdot S) + c(G \cdot \bar{S}) - 2$$

and $r(G \cdot S)$, $r(G \cdot \bar{S}) \geqslant n$. We choose S so that the above conditions are satisfied and $\eta(G;S,\bar{S})$ is minimum, consistent with those conditions.

If $c(G \cdot S) + c(G \cdot \bar{S}) = 2$, then $\lambda_W(G) \leqslant n$, which is contrary to the hypothesis. Therefore, $G \cdot S$ or $G \cdot \bar{S}$ has at least two components. In the following the notation $\eta(S_1, S_2)$ denotes the number of common vertices of the two subgraphs $G \cdot S_1$ and $G \cdot S_2$, where S_1 and S_2 are subsets of E. Using this notation we can write $\eta(G;S,\bar{S}) = \eta(S,\bar{S})$. For any subset T of E, $\bar{T}$ is the complement of T in E, that is, $\bar{T} = E - T$.

Case 1. One of the components, say $G \cdot S_0$, of $G \cdot S$ or $G \cdot \bar{S}$ satisfies the following condition: $r(G \cdot S_0) \geqslant \eta(S_0, \bar{S}_0)$.

Without loss of generality we may suppose that $G \cdot S_0$ is a component of $G \cdot S$. If $\eta(S_0, \bar{S}_0) \leqslant n$, then, clearly, $\lambda_W(G) \leqslant \eta(S_0, \bar{S}_0) \leqslant n$ since $r(G \cdot \bar{S}_0) \geqslant r(G \cdot \bar{S}) \geqslant n$. This contradicts the hypothesis. If $\eta(S_0, \bar{S}_0) > n$, then let $G \cdot S_0'$ be a component of $G \cdot (S - S_0)$. Let $S' = S - S_0'$ and $\bar{S'} = E - S = \bar{S} \cup S_0'$. Then

$$\eta(S', \bar{S'}) = \eta(S, \bar{S}) - \eta(S_0', \bar{S_0}') < \eta(S, \bar{S}),$$

$$c(G \cdot S') = c(G \cdot S) - 1,$$

$$c(G \cdot \bar{S'}) \geqslant c(G \cdot \bar{S}) - \eta(S_0', \bar{S_0}') + 1.$$

Therefore,

$$\eta(S'',\overline{S''}) \leq m + c(G\cdot S) + c(G\cdot\overline{S}) - 2 - \eta(S_0',\overline{S_0}')$$

$$\leq m + [c(G\cdot S'') + 1] + [c(G\cdot\overline{S'}) + \eta(S_0',\overline{S_0}') - 1] - 2 - \eta(S_0',\overline{S_0}')$$

$$= m + c(G\cdot S'') + c(G\cdot\overline{S'}) - 2, \text{ and}$$

$$r(G\cdot\overline{S''}) \geq r(G\cdot\overline{S}) \geq n,$$

$$r(G\cdot S'') = r(G\cdot S_0) + r(G\cdot(S - (S_0' \cup S_0))) \geq r(G\cdot S_0) \geq \eta(S_0,\overline{S_0}) - 1 \geq m.$$

However, this contradicts $\eta(S,\overline{S})$ being minimum.

Case 2. For each component $G\cdot S_0$ of $G\cdot S$ and $G\cdot\overline{S}$, $r(G\cdot S_0) <$ $\eta(S_0,\overline{S_0})$ and $r(G\cdot S)$ or $r(G\cdot\overline{S}) \geq n+1$.

Let $G\cdot S_0 = (V_0, S_0)$ be any component of $G\cdot S$ or $G\cdot\overline{S}$. Then

$$|V_0| = r(G\cdot S_0) + 1 \leq \eta(S_0,\overline{S_0}) \leq |V_0| .$$

Therefore, $\eta(S_0,\overline{S_0}) = |V_0|$ and hence, $\eta(S,\overline{S}) = |V|$. Suppose $r(G\cdot S) \geq m+1$, without loss of generality. We shall show that S can be chosen so that $G\cdot S$ contains no polygons. Let S' be a spanning forest of $G\cdot S$. Since the vertex set of $G\cdot S$ is V and $G\cdot S$ contains no isolated vertices, the vertex set of $G\cdot S'$ is V, that is, $\eta(S',\overline{S'}) =$ $\eta(S,\overline{S})$. We also have

$$r(G\cdot S'') = |S''| = r(G\cdot S) \geq n+1,$$

$$r(G\cdot\overline{S''}) \geq r(G\cdot S) \geq n.$$

Thus we can assume that $G\cdot S$ contains no polygons. Let e be an edge of $G\cdot S$ which has valence one at its one end in $G\cdot S$. Let $S' = S - \{e\}$ and $\overline{S'} = \overline{S} \cup \{e\}$. Then

$$\eta(S'',\overline{S''}) \leq \eta(S,\overline{S}) - 1,$$

$$r(G\cdot S') = r(G\cdot S) - 1 \geq n.$$

$$r(G\cdot\overline{S'}) \geq r(G\cdot\overline{S}) \geq n.$$

Hence,

$$\eta(S',\overline{S'}) \leq n + c(G\cdot S) + c(G\cdot\overline{S}) - 3$$

$$\leq n + c(G\cdot S') + \left[c(G\cdot\overline{S'}) + 1\right] - 3$$

$$= n + c(G\cdot S') + c(G\cdot\overline{S'}) - 2.$$

This is contrary to $\eta(S,\overline{S})$ being minimum.

Case 3. For each component $G\cdot S_0$ of $G\cdot S$ and $G\cdot\overline{S}$, $r(G\cdot S_0) < \eta(S_0,\overline{S}_0)$ and $r(G\cdot S) = r(G\cdot\overline{S}) = n$.

By assumption

$$c(G\cdot S) = c(G\cdot\overline{S}) = |V| - n \geq 2.$$

Suppose the vertex set of a component $G\cdot S_1$ of $G\cdot S$ properly contains the vertex set of a component of $G\cdot\overline{S}$. Let $G\cdot T_i$, $1 \leq i \leq k$, be the components of $G\cdot\overline{S}$ of which the vertex sets are properly contained in $G\cdot S_1$. Let $S_1' = \bigcup_{i=1}^{k} T_i$, $S_2 = S - S_1$, and $S_2' = \overline{S} - S_1'$. Let $S' = S_1 \cup S_1'$ and $\overline{S'} = S_2 \cup S_2'$. Then

$$r(G\cdot S') \geq \eta(S',\overline{S'}),$$

$$r(G\cdot\overline{S'}) = \eta(S_1,\overline{S_2}') + \eta(S_2,\overline{S_2}') - c(G\cdot\overline{S'}).$$

Since every component of $G\cdot\overline{S'}$ contains vertices of $G\cdot S_2$, we have

$$r(G\cdot\overline{S'}) \geq \eta(S_1,\overline{S_2}') = \eta(S',\overline{S'}),$$

$$\eta(S',\overline{S'}) \leq r(G\cdot S_1) \leq r(G\cdot S) = n.$$

Accordingly, $\lambda_W(G) \leq n$, contrary to the hypothesis.

If $G_1 = (V_1, S_1)$ and $G_2 = (V_2, S_2)$ are components of $G\cdot S$ and $G\cdot\overline{S}$, respectively, and $V_1 = V_2$, then G is not connected, which is a contradiction.

Lastly, we consider the following case: No component of $G \cdot S$ contains the vertices of a component of $G \cdot \bar{S}$, and no component of $G \cdot \bar{S}$ contains the vertices of a component of $G \cdot S$.

Let $G \cdot S_1$ and $G \cdot S_1'$ be components of $G \cdot S$ and $G \cdot \bar{S}$ which have common vertices. Let $S_2 = S - S_1$ and $S_2' = \bar{S} - S_1'$. If we set $S' = S_1 \cup S_1'$ and $\bar{S'} = S_2 \cup S_2'$, then

$$r(G \cdot S') \geqslant \eta(S', \bar{S'}),$$
$$r(G \cdot \bar{S'}) = \eta(S', \bar{S'}) + \eta(S_2, S_2') - c(G \cdot \bar{S'}).$$

Every component of $G \cdot \bar{S'}$ contains vertices of $G \cdot S_2$, and every common vertex of $G \cdot S_2$ and $G \cdot S_1'$ is contained in a component of $G \cdot \bar{S'}$. For, otherwise the condition of the first part of Case 3 is satisfied.

Therefore,

$$\eta(S_2, S_2') \geqslant c(G \cdot S_2) \geqslant c(G \cdot \bar{S'}),$$
$$r(G \cdot \bar{S'}) \geqslant \eta(S', \bar{S'}).$$

Since

$$r(G \cdot S) = n = \eta(S', \bar{S'}) + \eta(S_1, \bar{S}_1') + \eta(S_2, S_2') - c(G \cdot S_2) - 1,$$

we have

$$\eta(S', \bar{S'}) \leqslant n - \eta(S_2, S_2') + c(G \cdot S_2) \leqslant n.$$

Accordingly, $\lambda_W(G) \leqslant n$, contrary to the hypothesis.

Since all the cases have been examined, we conclude that $\lambda_W(G) \leqslant \lambda_W(\underline{P}(G))$ for a connected graph G. ∎

<u>Theorem 8.2</u> If G is a connected graph, then $\lambda_W(G) = \lambda_W(\underline{P}(G))$.

<u>Proof</u>. By Lemmas 8.8 and 8.9. $\blacksquare$

In Theorem 8.2 we established a generalization of Whitney connectivity of graphs to matroids.

8.3 COMPARISON OF TUTTE AND WHITNEY CONNECTIVITY

We write T- for the terminology of Tutte connectivity, and W- for Whitney connectivity. For instance, "T-n-connected", "T-separable", and "T-n-separator" stand for n-connected, separable, and n-separator in the Tutte sense.

Let $\underline{M}=(\underline{C}, E)$ be a matroid on E. We say that $\underline{M}$ is W-n-connected if $2 \leqslant n \leqslant \lambda_W(\underline{M})$, and that $\underline{M}$ is W-n-connected if $\lambda_W(\underline{M}) \geqslant 2$. A matroid $\underline{M}$ satisfying condition $\lambda_W(\underline{M})=1$ is said to be W-separable. A non-null proper subset S of E is a W-n-separator of $\underline{M}$ if $\lambda_W(\underline{M}) = n$ and $\xi(\underline{M};S,\overline{S})=n$, $\min(r(\underline{M} \times S), r(\underline{M} \times \overline{S})) \geqslant n$. W-1 separators are simply called W-separators.

Theorem 8.3 $\lambda_T(\underline{M}) \leqslant \lambda_W(\underline{M})$.

Proof. If $\lambda_W(\underline{M}) = \infty$, the theorem is obvious. Suppose $\underline{M}=(\underline{C}, E)$ and $\lambda_W(\underline{M})$ is finite. By definition, there exists a non-null proper subset S of E such that

$$\xi(\underline{M};S,\overline{S})=\lambda_W(\underline{M}),$$

$$r(\underline{M} \times S),\ r(\underline{M} \times \overline{S}) \geqslant \lambda_W(\underline{M}).$$

Since $r(\underline{M} \times S) \leqslant |S|$ and $r(\underline{M} \times \overline{S}) \leqslant |\overline{S}|$, we have

$$\min(|S|, |\overline{S}|) \geqslant \lambda_W(\underline{M}).$$

Therefore, $\lambda_T(\underline{M}) \leqslant \lambda_W(\underline{M})$. ∎

In general, T-connectivity is much smaller than W-connectivity. Consider the polygon matroid of the complete bipartite graph $K_{n,n}$, where $n \geqslant 4$. T-connectivity of this matroid is $\lambda_T(\underline{P}(K_{n,n}))=4$, while

$\lambda_W(\underline{P}(K_{n,n})) = n.$

Let $\underline{M} = (\underline{C}, E)$ be a matroid. A cell of $\underline{M}$ is called an <u>isthmus</u> if it does not belong to any circuit of $\underline{M}$, and a <u>loop</u> if it forms a circuit. The <u>loop matroid</u> is a 1-circuit matroid which consists of one cell and one circuit.

We shall explore additional properties of W-connectivity and further compare the two definitions of connectivity.

<u>Theorem 8.4</u> Let $\underline{M} = (\underline{C}, E)$ be a matroid containing no loops and S be a non-null proper subset of E. Then S is a W-separator of $\underline{M}$ if and only if S is a T-separator of $\underline{M}$.

<u>Proof.</u> If S is a W-separator, then

$$\xi(\underline{M}; S, \bar{S}) = 1 \text{ and } \min(r(\underline{M} \times S),\ r(\underline{M} \times \bar{S})) \geq 1.$$

Since $|S| \geq r(\underline{M} \times S)$ and $|\bar{S}| \geq r(\underline{M} \times \bar{S})$, we have $\min(|S|, |\bar{S}|) \geq 1$. Therefore, S is a T-separator of $\underline{M}$.

Suppose S is a T-separator of M. Then

$$\xi(\underline{M}; S, \bar{S}) = 1 \text{ and } \min(|S|, |\bar{S}|) \geq 1.$$

Since $\underline{M}$ contains no loops, $r(\underline{M} \times T) \geq 1$ for any non-null subset T of E. Thus $r(\underline{M} \times S)$, $r(\underline{M} \times \bar{S}) \geq 1$, and hence, S is a W-separator of $\underline{M}$. ∎

The equivalent statements on T-separability of matroids stated as Theorem 4.1 are valid for W-connectivity under a slightly restricted condition.

<u>Corollary 8.1</u> Let $\underline{M} = (\underline{C}, E)$ be a matroid containing no loops, and S be a non-null proper subset of E. Then the following statements are equivalent:

(a) S is a W-separator of $\underline{M}$.

(b) $r(\underline{M} \times S) + r(\underline{M} \times \bar{S}) = r(\underline{M})$, or

$\mu(\underline{M} \times S) + \mu(\underline{M} \times \bar{S}) = \mu(\underline{M})$.

(c) If B_S and $B_{\bar{S}}$ are bases of $\underline{M} \times S$ and $\underline{M} \times \bar{S}$, then $B_S \cup B_{\bar{S}}$ is a base of $\underline{M}$.

(d) If $C \in \underline{C}$, then $C \subseteq S$ or $C \subseteq \bar{S}$.

<u>Proof.</u> By Theorem 8.4, S is a T-separator of $\underline{M}$ if and only if it is a W-separator of M. The corollary follows from Theorem 4.1. ∎

In T-connectivity the connectivity of a matroid is the same as that of the dual matroid. However, this is not true for W-connectivity. W-connectivity of the polygon matroid of graph G, shown in Fig. 8.1, is two; on the other hand, W-connectivity of the bond matroid of G, which is the polygon matroid of G* in Fig. 8.2, is infinity.

<u>Lemma 8.10</u> Cell e of $\underline{M}$ is a loop of $\underline{M}$ if and only if e is an isthmus of $\underline{M}^*$.

<u>Proof.</u> If e is a loop of $\underline{M}$, then, by the definition of duality, we have $|C^* \cap \{e\}| \neq 1$ for every $C^* \in \underline{C}^*$. Thus no circuits of $\underline{M}^*$ contain e, and, as a consequence, e is an isthmus of $\underline{M}^*$.

Conversely, if e is an isthmus of $\underline{M}^*$, then e does not belong to any circuit of $\underline{M}^*$ and, hence, the intersection of $\{e\}$ and

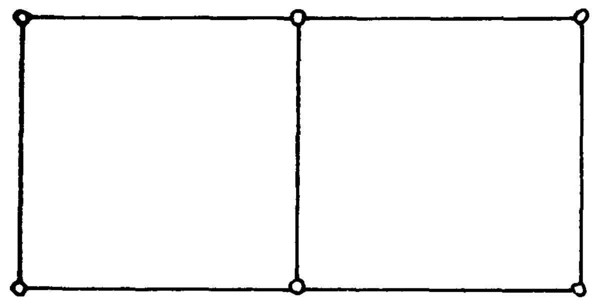


Figure 8. 1 Graph G

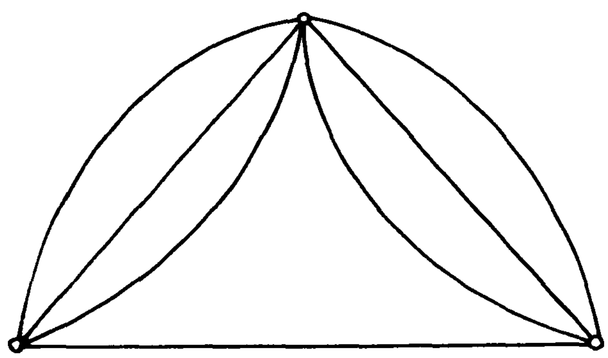


Figure 8. 2 Graph G*

every circuit of $\underline{M}^*$ is null. Therefore, $\{e\}$ is a minimal non-null subset of E such that $|\{e\} \cap C^*| \neq 1$ for every $C^* \in \underline{C}^*$. Since $(\underline{M}^*)^* = \underline{M}$, $\{e\}$ is a circuit of $\underline{M}$ and e is a loop of $\underline{M}$. $\blacksquare$

<u>Theorem 8.5</u> Let $\underline{M} = (\underline{C}, E)$ be a matroid containing neither loops nor isthmuses. Then $\underline{M}$ is connected if and only if $\underline{M}^*$ is connected.

<u>Proof.</u> It suffices to show that $\underline{M}$ is separable if and only if $\underline{M}^*$ is separable. A non-null proper subset S of $\underline{M}$ is a W-separator of $\underline{M}$ if and only if

$$\zeta(\underline{M};S,\overline{S}) = 1 \text{ and } \min(r(\underline{M} \times S),\ r(\underline{M} \times \overline{S})) \geq 1,$$

and these conditions are equivalent to the following conditions:

$$\zeta(\underline{M}^*;S,\overline{S}) = 1 \text{ and } \min(r(\underline{M}^* \times S),\ r(\underline{M}^* \times \overline{S})) \geq 1,$$

since, by Lemma 8.10, $\underline{M}$ and $\underline{M}^*$ both contain neither loops nor isthmuses. Accordingly, $\underline{M}$ is separable if and only if $\underline{M}^*$ is separable. $\blacksquare$

<u>Theorem 8.6</u> The following statements are equivalent:

(a) $\lambda_W(\underline{M}) = \infty$.

(b) For each non-null proper subset S of E, S or $\overline{S}$ contains a base of $\underline{M}$.

(c) $r(\underline{M} \cdot S) = 0$ or $r(\underline{M} \cdot \overline{S}) = 0$ for each non-null proper subset S of E.

(d) For each cocircuit C^* of $\underline{M}$, $E - C^*$ contains no cocircuits of $\underline{M}$.

<u>Proof.</u> (a)$\Leftrightarrow$(b) By assumption, for each non-null proper subset S

of E,

$$\xi(\underline{M};S,\bar{S}) = -r(\underline{M}) + r(\underline{M} \times S) + r(\underline{M} \times \bar{S}) + 1$$

$$\geq r(\underline{M} \times S) + 1 \text{ or } r(\underline{M} \times \bar{S}) + 1.$$

The following condition is equivalent to the one above:

$$r(\underline{M} \times \bar{S}) \geq r(\underline{M}) \text{ or } r(\underline{M} \times S) \geq r(\underline{M}).$$

Since $r(\underline{M} \times S)$, $r(\underline{M} \times \bar{S}) \leq r(\underline{M})$, we have

(1) $$r(\underline{M} \times \bar{S}) = r(\underline{M}) \text{ or } r(\underline{M} \times S) = r(\underline{M}).$$

Accordingly, S or $\bar{S}$ contains a base of $\underline{M}$.

If (b) is true, that is, if condition (1) holds, then, for each non-null proper subset S of E

$$\xi(\underline{M};S,\bar{S}) = r(\underline{M} \times S) + 1 \text{ or } r(\underline{M} \times \bar{S}) + 1,$$

and (a) follows.

(b)$\longleftrightarrow$(c) If $r(\underline{M} \times T) = r(\underline{M})$, where $T \subset E$, then $r(\underline{M} \cdot \bar{T}) = r(\underline{M}) - r(\underline{M} \times T) = 0$. Therefore, condition (1) is equivalent to condition (2):

(2) $$r(\underline{M} \cdot S) = 0 \text{ or } r(\underline{M} \cdot \bar{S}) = 0.$$

(c)$\longleftrightarrow$(d) Since $\mu(\underline{M}^* \times T) = r(\underline{M} \cdot T)$, where $T \subset E$, we have the following equivalent condition to (2):

$$\mu(\underline{M}^* \times S) = 0 \text{ or } \mu(\underline{M}^* \times \bar{S}) = 0.$$

If C^* is a cocircuit of $\underline{M}$, then $\mu(\underline{M}^* \times C^*) = 1$, and hence, $\mu(\underline{M}^* \times \bar{C^*}) = 0$. Thus $\bar{C^*} = E - C^*$ contains no circuits of $\underline{M}^*$, and hence, no cocircuits of $\underline{M}$.

Suppose (d) is satisfied. If a non-null proper subset S of E contains a cocircuit of $\underline{M}$, then $\mu(\underline{M}^* \times \bar{S}) = 0$ by assumption. If S

does not contain cocircuits of $\underline{M}$, then $\mu(\underline{M}^* \times S) = 0$. Thus, for each non-null proper subset S of E, we have:

$$\mu(\underline{M}^* \times \overline{S}) = 0 \quad \text{or} \quad \mu(\underline{M}^* \times S) = 0.$$

Accordingly, (c) follows. ∎

From Theorem 8.6, we can identify all the graphs with infinite connectivity.

<u>Corollary 8.2</u> Let G be a connected graph containing neither loops nor parallel edges. Then $\lambda_W(G) = \infty$ if and only if G is a complete graph.

<u>Proof</u>. If G is a complete graph, clearly $\lambda_W(G) = \infty$. Suppose $\lambda_W(G) = \infty$. By Theorem 8.2, $\lambda_W(\underline{P}(G)) = \lambda_W(G) = \infty$. The dual matroid of $\underline{P}(G)$ consists of all the cut-sets of G. Let v and v' be any distinct vertices of G, and S and S' be the star cut-sets at v and v', respectively. These star cut-sets are uniquely determined for the given vertices since G is not separable. According to Theorem A(d), E - S contains no star cut-sets of G; hence, $S \cap S' \neq \phi$. Consequently, v and v' are adjacent. Since G contains no parallel edges, $S \cap S'$ consists of a single element. Therefore, any two distinct vertices of G are connected by one edge and, consequently, G is a complete graph, for G contains no loops. ∎

<u>Theorem 8.7</u> Let $\underline{M} = (\underline{C}, E)$ be a matroid and $\underline{M}^* = (\underline{C}^*, E)$ be its dual. If $\lambda_W(\underline{M}) = n$ is finite, then $r(\underline{M}) \geq n+1$, and $r(\underline{M} \times C^*) \geq n$

for each $C^* \in \underline{C}^*$.

Proof. By hypothesis, there exists a non-null proper subset S of E such that

$$\xi(\underline{M};S,\bar{S}) = -r(\underline{M}) + r(\underline{M} \times S) + r(\underline{M} \times \bar{S}) + 1 = n,$$

$$\min(r(\underline{M} \times S),\ r(\underline{M} \times \bar{S})) \geq n.$$

Then

$$r(\underline{M}) = r(\underline{M} \times S) + r(\underline{M} \times \bar{S}) + 1 - n$$

$$\geq n + n + 1 - n = n + 1.$$

The second part of the theorem is obtained by using the first part. Suppose there exists a cocircuit C^* of $\underline{M}$ such that $r(\underline{M} \times C^*) \leq n - 1$. Since $r(\underline{M} \times \bar{C}^*) = r(\underline{M}) - r(\underline{M} \cdot C^*) = r(\underline{M}) - \mu(\underline{M}^* \times C^*) = r(\underline{M}) - 1$, we have

$$\xi(\underline{M};C^*,\bar{C}^*) = r(\underline{M} \times C^*) \leq n - 1.$$

From the first part of this theorem

$$r(\underline{M} \times \bar{C}^*) = r(\underline{M}) - \mu(\underline{M}^* \times C^*) \geq n + 1 - 1 = n.$$

Therefore, $\lambda_W(\underline{M}) \leq n - 1$, contrary to the hypothesis. Accordingly, $r(\underline{M} \times C^*) \geq n$ for each $C^* \in \underline{C}^*$. ∎

A well-known graph theorem is a natural consequence of Theorems 8.2 and 8.7.

Corollary 8.3 If G is a connected graph of W-connectivity n, where n is finite, then the number of edges of every cut-set of G is at least n, and in particular the valence of every vertex of G is at least n.

Proof. In Theorem 8.7, choose $\underline{M} = \underline{P}(G)$ and $\underline{M}^* = \underline{B}(G)$. If C^* is a cut-set of G, then $|C^*| \geqslant r(\underline{P}(G) \times C^*) \geqslant n$. $\blacksquare$

As shown previously, $\lambda_T(\underline{M}) = \lambda_T(\underline{M}^*)$; however, $\lambda_W(\underline{M}) = \lambda_W(\underline{M}^*)$ is not always true. In the following theorem, we will give a simple sufficient condition for $\lambda_W(\underline{M}) = \lambda_W(\underline{M}^*)$.

Theorem 8.8 Let $\underline{M} = (\underline{C}, E)$ be a matroid and $\underline{M}^* = (\underline{C}^*, E)$ be its dual. If

$$\min_{C \in \underline{C}} |C| = \min_{C^* \in \underline{C}^*} |C^*| = n$$

and $r(\underline{M})$, $\mu(\underline{M}) \geqslant n+1$, then $\lambda_W(\underline{M}) = \lambda_W(\underline{M}^*) \leqslant n$.

Proof. Step 1. Prove $\lambda_W(\underline{M})$, $\lambda_W(\underline{M}^*) \leqslant n$.

Suppose C^* is a circuit of $\underline{M}^*$ such that $|C^*| = n$. Then, by Theorem 2.13,

$$\xi(\underline{M}; C^*, \bar{C}^*) = |C^*| - \mu(\underline{M} \times C^*) - \mu(\underline{M}^* \times C^*) + 1$$

$$= r(\underline{M} \times C^*) \leqslant |C^*| = n.$$

By Theorem 2.12

$$r(\underline{M} \times \bar{C}^*) = r(\underline{M}) - r(\underline{M} \cdot C^*)$$

$$= r(\underline{M}) - \mu(\underline{M}^* \times C^*) = r(\underline{M}) - 1 \geqslant n.$$

Thus, $\lambda_W(\underline{M}) \leqslant n$. Similarly, we can show $\lambda_W(\underline{M}^*) \leqslant n$.

Step 2. Prove $\lambda_W(\underline{M}) = \lambda_W(\underline{M}^*)$.

We assume that $\lambda_W(\underline{M}) = k$ and $\lambda_W(\underline{M}^*) = k^*$, where $k, k^* \leqslant n$, as shown above. Let $\underline{S}$ and $\underline{S}^*$ be the collections of W-k-separators of $\underline{M}$ and W-k*-separators of $\underline{M}^*$, respectively.

$$\underline{S} = \{(S, \bar{S}) \mid \xi(\underline{M}; S, \bar{S}) = k \text{ and } r(\underline{M} \times S),\ r(\underline{M} \times \bar{S}) \geqslant k\}.$$

$$\underline{S}^* = \{(S^*, \bar{S}^*) \mid \xi(\underline{M}^*; S^*, \bar{S}^*) = k^* \text{ and } r(\underline{M}^* \times S),\ r(\underline{M}^* \times \bar{S}^*) \geqslant k^*\}.$$

If $\underline{S} \cap \underline{S}^*$ is not null, then there exists $(S, \overline{S}) \in \underline{S} \cap \underline{S}^*$ and

$$k = \xi(\underline{M}; S, \overline{S}) = \xi(\underline{M}^*; S, \overline{S}) = k^*.$$

Accordingly, $\lambda_W(\underline{M}) = \lambda_W(\underline{M}^*)$.

Suppose $\underline{S} \cap \underline{S}^* = \phi$. Then, if $(S, \overline{S})$ is a member of $\underline{S}$,

$$r(\underline{M}^* \times S) \leq k - 1 \quad \text{or} \quad r(\underline{M}^* \times \overline{S}) \leq k - 1.$$

Without loss of generality, we assume $r(\underline{M}^* \times S) \leq k - 1$. By Theorem 2.13

$$\begin{aligned} \mu(\underline{M} \times S) &= -\xi(\underline{M}; S, \overline{S}) + \mu(\underline{M} \cdot S) + 1 \\ &= -k + r(\underline{M}^* \times S) + 1 \leq -k + (k - 1) + 1 \\ &= 0, \end{aligned}$$

and again, by Theorem 2.13

$$\begin{aligned} \mu(\underline{M}^* \times S) &= -\xi(\underline{M}^*; S, \overline{S}) + \mu(\underline{M}^* \cdot S) + 1 \\ &= -k + r(\underline{M} \times S) + 1 \\ &= 1. \end{aligned}$$

Therefore, S contains no circuits of $\underline{M}$; however, it contains a circuit of $\underline{M}^*$. Let C^* be a circuit of $\underline{M}^*$ contained in S. Since $r(\underline{M}^* \times S) \geq r(\underline{M}^* \times C^*)$, by Theorem 2.13

$$\begin{aligned} \xi(\underline{M}; S, \overline{S}) = k &= \mu(\underline{M} \cdot S) - \mu(\underline{M} \times S) + 1 \\ &= \mu(\underline{M} \cdot S) + 1 = r(\underline{M}^* \times S) + 1 \\ &\geq r(\underline{M}^* \times C^*) + 1 = |C^*| \quad \min_{C^* \in \underline{C}^*} |C^*| \\ &= n. \end{aligned}$$

Consequently, $\lambda_W(\underline{M}) = k = n \geq \lambda_W(\underline{M}^*)$.

Since $\underline{S}^*$ is not null, we now choose $(S^*, \overline{S}^*) \in \underline{S}^*$.

Repeating a similar discussion to the above, we obtain $\lambda_W(\underline{M}^*) = k^* = n$.

Therefore, $\lambda_W(\underline{M}) = \lambda_W(\underline{M}^*) = n$, and the proof is complete. ∎

The condition given in Theorem 8.8 is sufficient for $\lambda_W(\underline{M}) = \lambda_W(\underline{M}^*)$, but not necessary; the reader may easily find a counterexample.

In the next corollary we state the corresponding graph theorem.

Corollary 8.4 Let $G = (V, E)$ be a planar connected graph. If the minimum cardinality of the polygons of G is n, which is also the minimum cardinality of the cut-sets, and $|V| \geq n + 2$ and $|E| \geq |V| + n$, then the connectivity of G coincides with that of a dual graph G^*.

Proof. Since G is connected, $|V| = r(G) + 1 = r(\underline{P}(G)) + 1 \geq n + 2$ by Lemma 8.6 and Theorem 8.8. We also have

$$|E| = r(G) + \mu(G) = |V| - 1 + \mu(G) \geq |V| + n.$$

Accordingly, the corollary follows. ∎

Two sufficient conditions for $\lambda_W(\underline{M}) \geq \lambda_W(\underline{M}^*)$ may be stated in terms of circuits and cocircuits.

Theorem 8.9 Let $\underline{M} = (\underline{C}, E)$ be a matroid of W-connectivity n, where n is finite, and $\underline{M}^* = (\underline{C}^*, E)$ be its dual. Then,

(a) if $\min_{C^* \in \underline{C}^*} |C^*| \geq n + 1$, then $\lambda_W(\underline{M}^*) \leq n$.

(b) if $\min_{C \in \underline{C}} |C| \leq n - 1$ and $\mu(\underline{M}) \geq n$, then $\lambda_W(\underline{M}^*) \leq n - 1$.

Proof. (a) Since $\lambda_W(\underline{M}) = n$, there exists a non-null proper subset of E such that

$$\zeta(\underline{M};S,\bar{S}) = n,$$

$$\min(r(\underline{M} \times S),\ r(\underline{M} \times \bar{S})) \geqslant n.$$

If S contains a circuit of $\underline{M}^*$, then

$$r(\underline{M}^* \times S) \geqslant \min_{C^* \in \underline{C}^*} |C^*| - 1 \geqslant n.$$

Suppose S does not contain circuits of $\underline{M}^*$. Then

$$r(\underline{M}^* \times S) = |S| \geqslant r(\underline{M} \times S) \geqslant n.$$

In both cases we have $r(\underline{M}^* \times S) \geqslant n$. Similarly, $r(\underline{M}^* \times \bar{S}) \geqslant n$ for for every non-null subset S of E. Accordingly, $\lambda_W(\underline{M}^*) \leqslant n$.

(b) Let C be a circuit of $\underline{M}$ satisfying $|C| \leqslant n - 1$.

Then

$$\zeta(\underline{M};C,\bar{C}) = |C| - \mu(\underline{M} \times C) - \mu(\underline{M}^* \times C) + 1$$

$$= |C| - \mu(\underline{M}^* \times C).$$

By Theorem 8.7 $|C^*| \geqslant n$ for every member of $\underline{C}^*$, and C does not contain circuits of $\underline{M}^*$. Hence

$$\zeta(\underline{M};C,\bar{C}) = |C| \text{ and } r(\underline{M}^* \times C) = |C| .$$

We also have

$$r(\underline{M}^* \times \bar{C}) = \mu(\underline{M} \cdot \bar{C}) = \mu(\underline{M}) - \mu(\underline{M} \times C)$$

$$= \mu(\underline{M}) - 1 \geqslant n - 1$$

$$\geqslant |C|.$$

Therefore, $\lambda_W(\underline{M}^*) \leqslant |C| \leqslant n - 1$. ■

<u>Corollary 8.5</u> Let $G = (V, E)$ be a planar connected graph of W-connectivity n, where n is finite, and G^* is its dual. Then,

(a) if the minimum cardinality of the cut-sets of G is greater than n, the connectivity of G* is at most n.

(b) if the minimum cardinality of the polygons of G is less than n and $|E| - |V| + 1 \geq n$, then the connectivity of G* is at most n - 1.

In the last theorem in this section we will give a necessary and sufficient condition for $\lambda_W(\underline{M}) = \lambda_T(\underline{M})$.

Theorem 8.10 Let $\underline{M} = (\underline{C}, E)$ be a matroid of T-connectivity n, where n is finite; then, $\lambda_W(\underline{M}) = \lambda_T(\underline{M})$ if and only if $\underline{M}$ has a T-n-separator S such that neither S nor $\overline{S}$ contains a base of $\underline{M}$.

Proof. Suppose $\lambda_W(\underline{M}) = \lambda_T(\underline{M}) = n$. Let S be a W-n-separator of $\underline{M}$. Then

$$\xi(\underline{M}; S, \overline{S}) = n,$$

$$\min(r(\underline{M} \times S),\ r(\underline{M} \times \overline{S})) \geq n.$$

Since $r(\underline{M} \times S) \leq |S|$ and $r(\underline{M} \times \overline{S}) \leq |\overline{S}|$, S is a T-n-separator of $\underline{M}$. By assumption,

$$-r(\underline{M}) + r(\underline{M} \times S) + r(\underline{M} \times \overline{S}) - 1 \leq r(\underline{M} \times S),\ r(\underline{M} \times \overline{S}),$$

or,

$$r(\underline{M} \times S),\ r(\underline{M} \times \overline{S}) \leq r(\underline{M}) - 1.$$

Therefore, neither S nor $\overline{S}$ contains a base of $\underline{M}$.

Now, suppose S is a T-n-separator of $\underline{M}$ such that neither S nor S contains a base of $\underline{M}$. Then

$$r(\underline{M} \times S),\ r(\underline{M} \times \overline{S}) \leq r(\underline{M}) - 1,$$

and

$$\xi(\underline{M};S,\bar{S}) = n = -r(\underline{M}) + r(\underline{M} \times S) + r(\underline{M} \times \bar{S}) + 1$$

$$\leqslant r(\underline{M} \times S),\ \ r(\underline{M} \times \bar{S}).$$

Thus, $\lambda_W(\underline{M}) \leqslant n$, and $\lambda_W(\underline{M}) = \lambda_T(\underline{M}) = n$, by Theorem 8.3. Accordingly, the theorem follows. ∎

CHAPTER 9 CONCLUSION

In this chapter we summarize the contributions of this dissertation and suggest a number of possible future areas of research.

Tutte defined matroid connectivity by generalizing graph connectivity, and verified the validity of the generalization by applying it to 3-connected matroids. In recent times, however, little research on matroid connectivity has been pursued. In this dissertation new properties of matroid connectivity are uncovered.

In Chapter 3 the maximum value of a connectivity function is evaluated in terms of maximally distant bases. This value fixes, for a matroid of finite connectivity, an upper bound on the connectivity. The central-tree problem in graph theory is extended to matroids, and the evaluation of a connectivity function may give a solution to this problem.

A direct calculation of Tutte connectivity of a matroid requires 2^r and 2^μ computations of a connectivity function, where r and μ are rank and nullity of the matroid. At present there exists no efficient algorithm for determining the connectivity of matroids. An efficient algorithm will provide a practical use of a reduction sequence for graph realizability of matroids. We do show, however, how to determine the connectivity of binomial matroids, which computation leads to an important matroid theorem--the existence of a matroid with a prescribed connectivity.

Tutte's theorem, stated as Theorem 2.16, is applied to the reduction of a 3-connected matroid to a wheel, a whirl, or a matroid containing a non-essential cell. The last class of irreducible matroids is identified as follows:

$$ {}_6\underline{M}_3, {}_6\underline{M}_4, {}_6\underline{M}_{4(1)}, {}_6\underline{M}_{4(2)}, \text{ and } {}_6\underline{M}_5 . $$

A planar matroid cannot have an arbitrarily high connectivity. Actually, in Section 3.5 we give a proof stating that if the connectivity of a matroid is greater than three, the matroid is nonplanar. A related question is: "What is the maximum connectivity of a graph-realizable matroid for matroids of finite connectivity?" or, "Does there exist a graph with a prescribed Tutte connectivity?". For Whitney connectivity the answer is affirmative. No answer has been given to the question of Tutte connectivity.

Chapters 4 and 5 are concerned with the decomposition of matroids into 3-connected minors. MacLane's graph decomposition is generalized to matroids, and we obtain a characterization of matroid structures: A matroid is binary, regular, or graph-realizable if and only if the atoms are binary, regular, or graph-realizable, respectively.

The matroid decomposition in Chapter 4 is extended to P- and S- decompositions in Chapter 5, which includes C-decomposition as a special case. A split decomposition of a graph may be considered as a P-decomposition of the polygon matroid or S-decomposition of the bond matroid. The split decomposition is applied to n-port networks to determine the planarity of n-port networks, a concept which generalizes that of graph planarity.

The graph-realizability problem of matroids has a wide application to the realizability of physical systems. By Theorem 4.15 this problem is reduced to one of 3-connected minors.

In Chapter 6 a new graph-realizability condition is provided by generalizing the wheel theorem of Bruno, Steiglitz, and Weinberg. The realizability condition is stated on the inverse operations of reduction and contraction of matroids. The algorithms A, B, C, and D to test this condition consist of two parts: construction of a graph and examination of whether the matroid circuits are polygons or cut-sets of the graph. Once a graph is constructed, we show that the second part is easily checked. We also provide two examples to demonstrate the algorithms.

The contraction and reduction operations are dual concepts in matroid theory. Therefore, it is natural to assume that a theorem on reduction operation is similar to a corresponding theorem on contraction, and vice-versa. In Lemma 6.1 it is sufficient to test only the circuits of $\underline{M}_{i-1}$ which contain e_{i-1} to determine whether $\underline{M}_{i-1}$ is cographic. However, in Lemma 6.2 it is necessary to examine all the circuits for the graph-realizability of $\underline{M}_{i-1}$. The same thing is true for Lemmas 6.3 and 6.4 Although the author has not been able to simplify Conditions 2 and 4 as Conditions 1 and 3, he believes that these conditions can be stated as simply as the latter conditions. We leave the simplification of these conditions for future research.

In Chapter 7 we define maximal bipartite and maximal Euler minors of a binary matroid and state necessary and sufficient conditions for a matroid to be a max. bpt. minor and a max. Euler minor. One may easily generate all the max. bpt. and max. Euler minors of a binary matroid from the bases of the matroid. We also prove that the original matroid is uniquely reconstructed from the families of distinct max. bpt. contractions or max. Euler reductions of the matroid.

Chapter 8 is devoted to a study of Whitney connectivity of matroids. Whitney connectivity of graphs has been accepted as a standard definition of graph connectivity among most researchers. We show that this connectivity can be expressed using the η- function of Tutte and that, by generalizing it to matroids, Whitney connectivity of a graph coincides with Whitney matroid connectivity of the polygon matroid of the graph. An immediate consequence of this result is that Tutte connectivity of a matroid can not exceed Whitney connectivity. We also present a sufficient condition for a matroid and its dual to have the same Whitney connectivity; this condition is stated in terms of circuits and cocircuits of the matroid.

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