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A

PRICING THE INTEREST RATE DERIVATIVES WITH THE HEATH-JARROW-MORTON MODELS

by

JINTAO ZHU

A dissertation submitted to the Graduate Faculty in Economics in partial fulfillment of the requirement for the degree of Doctor of Philosophy,
The City University of New York

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Abstract

PRICING THE INTEREST RATE DERIVATIVES WITH THE HEATH-JARROW-MORTON MODELS

by

Jintao Zhu

Advisor: Professor Nusret Cakici

The Heath-Jarrow-Morton models are the theoretically most satisfied models in pricing the interest rate derivatives. However, the non-Markovian property of the models makes the lattice explode exponentially, which makes the implementation of the models to practical calculations impossible. Recently, Ritchken and Sankarsubramanian put forward a method to deal with this problem, which restricts the volatility structure to a certain type so that the non-Markovian property could be partly eliminated.

Lots of tests have been done on the HJM models modified by Ritchken and Sankarsubramanian in pricing the interest rate derivatives such as options on Eurodollar futures, on discount bonds, on coupon bonds, and on spot rates. The RS algorithm is quite powerful in pricing the short-term interest rate options, but it fails in many cases in pricing the long-term options unless the parameters for the models are selected purposely. Mathematical analysis is given to show the range, in which the RS algorithm works or does not work.

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This dissertation is dedicated to my loving wife, Xinxin Jiang.

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I. Introduction

The Heath-Jarrow-Morton model is theoretically most satisfied model in pricing the interest rate derivatives. First, the initial term structure is an input of the model, not an endogenous result. Secondly, the evolution of the term structure and contingent claim prices are independent of the investor's preferences. Finally, estimates of the drifts are not needed. However, the non-Markovian property of the model makes the lattice explode exponentially, which makes the implementation of the model to practical calculation impossible. Therefore lots of research done on the HJM model focus on how to conquer the non-Markovian property. Recently, Ritchken and Sankarsubramanian put forward a method to deal with this problem, which restricts the volatility structure to certain type so that the non-Markovian property could be partly eliminated.

The purpose of this dissertation is to test the HJM model modified by Ritchken and Sankarsubramanian in pricing the interest rate derivatives. Through lots of research on the HJM model done in this dissertation, I find out that the RS algorithm is quite powerful in pricing the short-term interest rate options, but it fails in many cases in pricing the long-term options unless the parameters for the model are selected purposely. Therefore the research about how to eliminate the non-Markovian property of the HJM model is far from over.

This dissertation includes two independent papers. The first paper prices the options on Eurodollar futures with the HJM model. The second prices the options on discount bonds, coupon bonds, and spot rates.

II. Pricing Eurodollar Futures Options with the Heath-Jarrow-Morton Models

Abstract

The HJM model has generalized the arbitrage method pioneered by Ho and Lee in pricing the interest rate contingent claims. It has many advantages over traditional equilibrium models, for it can reconcile the initial term structures. However, the empirical tests on the HJM model has so far been quite limited due to the path dependence of the term structure, which makes the number of lattices increase exponentially. Recently, Ritchken and Sankarsubramanian has put forward a method that does not erase non-Markov property completely, but rather capture the term structure by two state variables. RS algorithm thus provides a prospect to implement the HJM model in practice.

This paper uses the RS algorithm to test the HJM model in pricing the Eurodollar futures options. The differences among different HJM models with various term structures are analyzed through simulations. The results show that when the volatility of forward rate is not so high, different term structures are going to provide very similar results. Therefore, the constant volatility HJM model, which is the continuous time limit of the Ho and Lee model, should work well in pricing options on Eurodollar futures. The Ho and Lee model will fail to yield similar results to those obtained from the non-constant HJM model only when the term structure is very volatile.

This paper also compares the HJM model with Black's model in pricing the Eurodollar futures options. As long as the implied volatility parameter for each maturity time is calculated separately, and provided that the volatility of spot rate is proportional

to the spot rate, Black's model gives us option prices that are very similar to those obtained from the HJM model. Black's model is really much stronger than it might be supposed. But when the volatility structure is in the same class considered by CIR model, the difference between the HJM model and Black's model is very significant.

A comparison between the American options and European options is also made in this paper using the HJM models. The results show that when the options are deeply in the money, the difference between American style and European style is significant. For the options at the money or out of the money within 50 basis points, the difference between American and European style is within about half a basis point even when the time to maturity is as long as one year. For options in the money within 50 basis points and the maturity is not beyond 1 year, the difference between American style and European style is less than 1 basis point, which is not significant.

Moreover, our findings show that the difference between the forward price and futures price and its effects on pricing the options on Eurodollar futures options are negligible when the time to maturity is less than or equal to half a year. The difference between the futures price and forward price could be as large as 1.7 basis point for maturity equal to one year, but its effects on the prices of options are still negligible.

1. Introduction

Interest rate contingent claims are instruments whose payoff depends in some way on the term structure of interest rates. It is quite natural then that the evolution of the term structure has become the focal point in pricing interest rate contingent claims. Equilibrium models like Rendleman/Bartter model, Vasicek model, Cox/Ingersoll/Ross

(CIR) model, and Longstaff /Schwartz model all begin with specific assumptions about the economy and derive the term structure of interest rate endogenously. A shortcoming of this methodology was that the initial term structure could not be matched. In contrast, the Heath-Jarrow-Morton (HJM) model, which generalized the methodology pioneered by Ho and Lee model, makes assumptions about the evolution of the term structure and imposes the arbitrage free conditions to price the contingent claims. The HJM model has several advantages over the general equilibrium models. First, the initial term structure is an input of the model, not an endogenous result. Secondly, the evolution of the term structure and contingent claim prices are independent of the investor's preferences. Finally, estimates of the drifts are not needed. Therefore, HJM model has found increasing favor in both theoretical research and practical trading.

Unfortunately, the computation of prices is complex because without severely restricting the term structure, the evolution of the term structure under the martingale measure is usually not Markovian with respect to a finite-dimensioned state space. HJM (1990) use the binomial method to price the contingent claims. Due to the path dependence, the lattice does not recombine, and grows exponentially. To our knowledge, there was no powerful algorithm for implementing the HJM model in pricing interest rate derivatives before Ritchken and Sankarasubramnian (1995) published their paper, "Volatility Structures of Forward Rates and The Dynamics of the Term Structure".

Ritchken and Sankarasubramnian have identified the conditions on the volatility structure which do not completely remove the path dependence, but rather capture the term structure by two state variables instead of the entire term structure. Interestingly, the restrictions are imposed on the volatility structure of the forward rate, not the spot rate.

Therefore, RS algorithm still permits a broad set of volatility structures. The volatility structures considered by the traditional equilibrium models, such as Vasicek model or Cox/Ingersoll/Ross model, are just some specific cases.

The goal of this paper is to use the HJM model combined with RS algorithm to price the Eurodollar futures options and test the differences among the HJM models with different term structures in pricing Eurodollar futures options. Flesaker (1993a) has studied the HJM model with constant volatility of forward rate. His conclusion is negative about the constant volatility HJM model. This paper examines the difference between the constant volatility HJM model and the non-constant HJM model. The results of this paper show that under certain conditions, Ho and Lee model could provide very close results to the HJM model with non-constant volatility.

The simplest way to price the Eurodollar futures option is to use Black's model. When the latter is used to price the interest rate derivatives, there are two approximations. First, the futures price and forward price are assumed equal. In reality, however, they are not equal. Second, interest rates are assumed to be constant for discounting purpose even though they are assumed to be stochastic when the payoff from the option is calculated. Since the two effects tend to cancel each other out, Black's model is more powerful in pricing the interest rate derivatives than it appears. This paper tests the difference between the HJM model and Black's model through simulations. The results show that when the volatility of the spot rate is proportional to the spot rate, the difference in pricing Eurodollar futures options is insignificant. However, when the volatility structure of the spot rate is in the same class considered by CIR, then the difference between Black's model and the HJM model becomes significant.

Recently, Flasaker (1993b) used the HJM model with constant volatility and provided closed form formulas for both the Eurodollar futures price and forward price. His research shows that for contract maturities up to around 6 months, Eurodollar futures prices are not significantly different from the corresponding forward price. For longer maturities, the magnitude of the difference may be quite significant and increases with the maturity and the level of the interest rate volatility. This paper examines the difference between the forward price and futures price and its effects on the Eurodollar futures options. The results show that the difference between the forward price and futures price could be significant but its impacts on the options price are trivial.

The paper proceeds as follows. Sections 2 and 3 provide a brief description of the HJM model and RS algorithm. The theories of pricing the Eurodollar futures are discussed in section 4. Section 5 tests the convergence of the model. The sensitivity of option prices to the parameters is provided in section 6. Section 7 compares the HJM model with Black's model. The differences among the HJM models with different term structures are examined in section 8. Section 9 tests the differences between the European options and American options. The difference between the forward price and futures price and its effects on the option prices are provided in section 10. Finally, section 11 concludes the paper.

2. HJM Model

The following equations construct the one factor HJM model:

$$df(t,T) = \mu_f(t,T)dt + \sigma_f(t,T)dw(t) \quad . \tag{1}$$

$$P(t,T) = e^{-\int_t^T f(t,s)ds} \quad . \tag{2}$$

$$\mu_f(t,T) = \sigma_f(t,T)\int_t^T \sigma_f(t,s)ds\,. \tag{3}$$

$$g(0) = E_0[e^{-\int_0^s r(t)dt} g(s)] \;\; . \tag{4}$$

In this model, $f(t,T)$ is the forward rate at date t for instantaneous and riskless borrowing or lending at date T. Equation (1) is the stochastic process of forward rate of every maturity T. $\mu_f(t,T)$ and $\sigma_f(t,T)$ are the drift and volatility of this stochastic process and could be functions of the term structure itself. $dw(t)$ is a Brownian motion. The spot rate $r(t)$ is given by $f(t,t)$. Equation (2) is the definition for the price of a pure discount bond with maturity T at time t. HJM prove that given the no-arbitrage condition, the drift term $\mu_f(t,T)$ and volatility parameter $\sigma_f(t,T)$ are not independent and their relationship is described by equation (3). In equation (4), $g(0)$ represents the value of a European claim at date 0 having a terminal payout at date s that is fully determined by the yield curve at that time. The expectation is taken in a risk neutral world.

The volatility function $\sigma_f(t,T)$ is selected arbitrarily. As long as it is given, we can get the drift term $\mu_f(t,T)$ directly from (3). Then the whole stochastic process of the term structure is determined as well as the value of the European claim. Such an arbitrary property leads to a generality of the model and can be used to consistently price all contingent claims (both American and European) theoretically. But the arbitrary property also makes the term structure not follow a Markovian process with respect to finite state variables, which leads to difficulties in implementing the model in practice.

Recently, Ritchken and Sankarsubramanian (1995) put a restriction on the volatility function. Although this restriction does sacrifice some part of the generality of the HJM model, it permits the term structure to be represented by a two-state Markovian model. It is described by

$$\sigma_f(t,T) = \sigma_f(t,t)k(t.T) \tag{5}$$

with

$$k(t,T) = e^{-\int_t^T \kappa(x)dx} .$$

Here, $\sigma_f(t,t)$ is the volatility of the spot rate, and $\kappa(x)$ is a deterministic function given exogeneously. Very interestingly, the spot rate volatility $\sigma_f(t,t)$ is still selected arbitrarily. This allows the class of term structure characterized by (5) to be wide enough to include models like Vasicek's model or CIR model as its some of its special cases. The implication is that its practical application could be quite broad.

Under the above condition (5), the term structure could be captured by two state variables. One is the spot rate $r(t)$ and another $\phi(t)$, which represents the accumulated variance for forward rate up to date t. The stochastic process of these two state variables are described by the following two equations:

$$dr(t) = \mu(r,\phi,t)dt + \sigma_f(t,T)dw(t) \tag{6}$$

$$d\phi(t) = (\sigma_f^2(t,t) - 2\kappa(t)\phi(t))dt \tag{7}$$

with

$$\mu(r,\phi,t) = \kappa(t)[f(0,t) - r(t)] + \phi(t) + \frac{d}{dt} f(0,t) .$$

Note that in the class of volatility structure of (5), the price of a pure discount bond is given by

$$P(t,T) = (\frac{P(0,T)}{P(0,t)})e^{-\beta(t,T)(r(t)-f(0,t))-\frac{1}{2}\beta^2(t,T)\phi(t)}, \tag{8}$$

where

$$\beta(t,T) = \int_t^T k(t,u)du \ .$$

3. Algorithm

The spot rate volatility structure $\sigma_f(t,t)$ is selected arbitrarily and could depend on both state variables, $r(t)$ and $\phi(t)$. Since it is generally not constant, in order to use the lattice approach to generate simulations, Li/Ritchken/Sankrasubramanian (1995) follow Nelson/Ramaswamy (1990) in making the following transformation:

$$Y(t) = \int \frac{1}{\sigma[r(t),\phi(t),t]}dr(t) \ . \tag{9}$$

Let $r(t) = h(Y(t))$ be the inverse function. Then we have

$dY(t) = m(Y,\phi,t)dt + dw(t)$, and (10)

$d\phi(t) = (\sigma^2[r(t),\phi(t),t)] - 2\kappa(t)\phi(t))dt$, (11)

where

$$m(Y,\phi,t) = \frac{\partial Y(t)}{\partial t} + \mu(r,\phi,t)\frac{\partial Y(t)}{\partial r(t)} + \frac{1}{2}\sigma^2[r(t),\phi(t),t]\frac{\partial^2 Y(t)}{\partial r(t)^2} \ .$$

As can be seen, the transformed process $dY(t)$ and $d\phi(t)$ have constant volatility, and a lattice approximation can be established. A very interesting feature of the above algorithm is that the lattice is built on the lifetime of the contingent claim, not on the lifetime of the underlying assets. Suppose the maturity of the option is T. Then we divide T into n intervals, $\Delta t = T/n$. Let the approximating variables be y^a and ϕ^a at the beginning of some time increment. Then in the next time increment, the variables move to either (y^{a+}, ϕ^{a+}) or (y^{a-}, ϕ^{a-}), where

$$y^{a+} = y^a + J\sqrt{\Delta t} \tag{12a}$$

$$y^{a-} = y^a - J\sqrt{\Delta t}\,. \tag{12b}$$

Now, J is an integer and could be different at each node. It is chosen specifically to make the probability of up jump or down jump to be positive. The up jump probability is given by the following formula:

$$p = \frac{m(y^a, \Phi^a, t)\Delta t + (y^a - y^{a-})}{(y^{a+} - y^{a-})} \tag{13}$$

From the relationship set up by (9), we can get easily the corresponding tree for the spot rate $r(t)$. The process of $\phi(t)$ has two different properties from that of spot rate $r(t)$ or $y(t)$. First, $\phi(t)$ is locally deterministic, which implies

$$\phi^{a+} = \phi^{a-} = \phi^a + [\sigma^2(r(t), \phi^a t) - 2\kappa(t)\phi^a]\Delta t\,. \tag{14}$$

Second, the number of distinct Φ values at each node will equal the number of unique paths leading to that node. Rather than keeping track of all distinct paths, we just identify two paths at each node, which give the maximum value $\bar{\phi}^a$ and minimum value $\underline{\phi}^a$. We then partition the interval $[\underline{\phi}^a, \bar{\phi}^a]$ into m equidistant points with $\phi^a(k)$, $k = 1 \ldots m$, representing *kth* points and

$$\phi^a(k) = \underline{\phi}^a + \frac{k-1}{m-1}(\bar{\phi}^a - \underline{\phi}^a) \quad . \tag{15}$$

When we finish the lattices of the spot rate $r(t)$ and $\phi(t)$, the lattice for the underlying assets can be established using formula (8). The terminal value of the contingent claim can then be captured. With backward recursion, we can get the contingent value at each node.

Let $g_{i,j+1}(k), k = 1,\dots,m$ be the value of the claim at node $(i, j+1)$ and the corresponding state variables are $r_{i,j+1}$ and $\phi_{i,j+1}$. Similarly, let $g_{i-1,j+1}(k), k = 1,\dots,m$ be the value of the claim at node $(i-1, j+1)$ and the corresponding state variables be $r_{i-1,j+1}$ and $\phi_{i-1,j+1}(k)$. Assume that $g_{i,j}(k)$ has been computed. Knowing $\phi_{i,j}(k)$, we can calculate the corresponding value ϕ^* after a time interval Δt,

$$\phi^* = \phi_{i,j}(k) + [\sigma_f^2 - 2\kappa\phi_{i,j}(k)]\Delta t \quad . \tag{16}$$

We then go to the up node $(i, j+1)$ to compare ϕ^* with $\phi_{i,j+1}(k), k = 1,\dots,m$. If ϕ^* lies in the interval $[\phi_{i,j+1}(l), \phi_{i,j+1}(l+1)], 1 \le l \le m$, then using simple interpolation the value of claim g^{up} at node $(i, j+1)$ given ϕ^* can be calculated as follows:

$$g^{up} = g_{i,j+1}(l) + \frac{g_{i,j+1}(l+1) - g_{i,j+1}(l)}{\phi_{i,j+1}(l+1) - \phi_{i,j+1}(l)}[\phi^* - \phi_{i,j+1}(l)] \quad . \tag{17}$$

Similarly, we go to down node $(i-1, j+1)$ and compare ϕ^* with $\phi_{i-1,j+1}(k)$. Using the same method as above, the value of claim g_{down} at node $(i-1, j+1)$ given ϕ^* is calculated as

$$g_{down} = g_{i-1,j+1} + \frac{g_{i-1,j+1}(l+1) - g_{i-1,j+1}(l)}{\phi_{i-1,j+1}(l+1) - \phi_{i-1,j+1}(l)}[\phi^* - \phi_{i-1,j+1}(l)] \, , \tag{18}$$

where $\phi_{i-1,j+1}(l) \le \phi^* \le \phi_{i-1,j+1}(l+1)$.

Given g^{up} and g_{down} , the claim value $g_{i,j}(k)$ is calculated using backward recursion. Specifically,

$$g_{i,j}(k) = [pg^{up} + (1-p)g_{down}]e^{-r_i \Delta t}. \tag{19}$$

If the claim is American style, then the price given by the above formula is compared with the exercise value of the claim and the higher value is selected. An American call option is then calculated by

$$g_{i,j} = \max[P_{i,j} - X, (pg^{up} + (1-p)g_{down})e^{-r_i \Delta t})], \tag{20}$$

where P is the bond price.

4. The Valuation of Options on Eurodollar Futures

To price the Eurodollar futures options, we use the forward price as a substitute for the futures price. We also assume that the Eurodollar rates are default free. Let τ be the maturity of the option on Eurodollar futures and $T = \tau + 0.25$. Then the forward price for the time period from τ to T at time t is given by the following formula:

$$G(t,\tau,T) = \frac{P(t,T)}{P(t,\tau)} , \tag{21}$$

where $G(t,\tau,T)$ is forward price at time t starting from time τ, $P(t,T)$ is the price of a pure discount bond at time t with a payoff \$1 at maturity date T, and $P(t,\tau)$ is the price of a pure discount bond with a payoff \$1 at maturity date τ. The Eurodollar futures contract is quoted on the base of a 90-day quarterly compounding rate and the quoted Eurodollar futures price equals 100 minus the 90-day quarterly compounding rate. Thus we have

$$efp = 100[1 - 4(\frac{P(t,\tau) - P(t,T)}{P(t,T)})] \,. \tag{22}$$

where efp is the Eurodollar futures price.

A substantial amount of academic research has been carried out on the differences between the forward price and futures price when the interest rates are stochastic. Recently, Bjorn Flesaker (1993b) has used the HJM model with constant volatility and got closed form formulas for both the Eurodollar futures price and forward price. His research has shown that for contract maturity up to around 6 months, Eurodollar futures prices are not significantly different from the corresponding forward price. For longer maturity, however, magnitude of the difference may be quite significant and increases with the maturity and the level of interest rate volatility. But Flesaker's conclusions may well depend on the specific restrictions on volatility structure of the forward rate.

The futures in a risk neutral world are treated as an asset paying continuously compounding dividend at the same rate as the risk free rate. Therefore, the expected rate of return on the futures equals zero in the risk neutral world. Recently, K. I. Amin and A. J. Morton (1994) have observed that the futures price for a continuously marked-to-market futures contract follows a martingale, since opening a futures position requires no investment. If the futures price at date t for a contract that matures at date T is $efp_T(t)$, then

$$efp_T(t) = E_t[efp_T(T)] \,. \tag{23}$$

Note that $efp_T(T)$ is the futures price at maturity, which equals the forward price and also the spot price at maturity date T. It follows that we can use the lattice method to calculate the futures price by going backward without the discount factor.

In this paper, we price the Eurodollar futures options based on the formulas (21) and (22) to utilize the simplicity of the forward price. The implication is that the forward price is used as a substitute for the futures price. Later in section 9, we examine the difference between the forward price and futures price and its effect on the options.

With regard to the spot rate volatility $\sigma_f(t,t)$ in formula (5), we follow the academic convention and assume constant elasticity,

$$\sigma_f(t,t) = \sigma[r(t)]^{\gamma}\,;\gamma \geq 0\,. \tag{24}$$

Different γ values mean different volatility structure of the spot rate, therefore different term structures, and the transformation derived from (9) will be different. In this study, we consider three different cases to test the sensitivity of Eurodollar futures option to γ. When γ is fixed, there are only two parameters. One is σ and the other κ.

Case 1. γ=0, which implies a constant spot rate volatility. In this case, the appropriate transform would be

$$Y(t) = r(t)/\sigma \text{ and} \tag{25a}$$

$$m(Y,\phi,t) = \frac{1}{\sigma}[\kappa(t)(f(0,t) - r(t)) + \phi(t) + \frac{df(0,t)}{dt}]\,. \tag{25b}$$

Case 2. $\gamma = 0.5$. The appropriate transform would now be

$$Y(t) = 2\frac{\sqrt{r(t)}}{\sigma} \text{and} \tag{26a}$$

$$m(Y,\phi,t) = \frac{1}{Y(t)}[\frac{\kappa}{2}(\frac{4v(\phi,t)}{\sigma^2} - [Y(t)^2]) - \frac{1}{2}], \tag{26b}$$

where

$$v(\phi,t) = f(0,t) + \frac{1}{\kappa}\frac{d}{dt}f(0,t) + \frac{\phi(t)}{\kappa}\,.$$

Case 3. $\gamma = 1$. Here the transform is

$$Y(t) = \frac{\log(r(t))}{\sigma} \text{ and} \tag{27a}$$

$$m(Y,\phi,t) = \frac{1}{\sigma}[v(Y,\phi,t) - \frac{1}{2}\sigma^2], \tag{27b}$$

where

$$v(Y,\phi,t) = \{\kappa[f(0,t) - r(t)] + \phi(t) + \frac{d}{dt}f(0,t)\}e^{-r(t)} .$$

5. Test of Convergence

The convergence behavior depends on the number of ϕ values and the number of time partitions n. Li/Ritchken/Sankrasubramanian (1995) conclude that 25 partitions for ϕ and 50 partitions for time to maturity is enough to price the options on the long term bonds accurately. The convergence test on pricing the Eurodollar futures options show that 50 time partitions are necessary and enough to provide accurate results. But with regard to the partitions on the sate variable ϕ for pricing the options on the Eurodollar futures, 25 partitions are not necessary. Below, we denote the exercise price as X, the current Eurodollar futures price as efp, time to maturity as τ, time partition as n, and the number of partition for state variable ϕ as m.

Assume the initial term structure is flat at 5% and volatility structure of the forward rate follows formula (24). Let the current Eurodollar futures price be (efp)=95, and $\kappa = 0.01$. Also assume the options are American call options.

Table 1 shows the convergence results for options with different exercise prices and different time to maturity at various volatility parameter levels when $\gamma = 1$. When

$\sigma = 0.1$, 3 partitions for ϕ give us the same option prices as 25 partitions for ϕ. When the volatility parameter is as large as 0.5 and the time to maturity equals one year, the deviation of 3 partitions for ϕ from 25 partitions is within 0.03 basis point. For the whole table, 5 partitions for ϕ provide results whose deviation from 25 partitions is within 0.01 basis point. Therefore, 3 or 5 partitions for ϕ provide sufficient accuracy.

Notice that the convergence behavior could be quite different for different volatility structures. When $\kappa = 0$ and $\gamma = 0$, the forward rate term structure follows a constant volatility process, which happens to be the simplest case of the HJM model, as well as the continuous time limit of the Ho and Lee model. Then the parameter ϕ is deterministic and the convergence behavior depends only on the time partitions. Table 2 shows the convergence results for options with different exercise prices and different time to maturity. It can easily be seen that 300 time partitions give us very accurate results.

We also test the convergence behavior of the model when $\gamma = 0.5$. In this case, the convergence behavior is quite similar to the case in which $\gamma = 1$. Therefore, 50 time partitions and 3 or 5 partitions for state variable ϕ are going to provide results accurate enough in pricing the Eurodollar futures options.

6. Sensitivity of Option Prices to Model Parameters

Let the volatility structure follow formula (24) and $\gamma = 1$. Assume the initial term structure is flat at the interest rate implied by the current Eurodollar futures price, which is 95. The price sensitivity of American call options to parameters is shown in Table 3.

When σ is less or equal to 10%, all the prices of options with exercise price 94 are completely determined by the intrinsic value, and all the prices of options with exercise price 96 is worthless except the option with 1 year to maturity and σ is at 10% level. Those option prices do not bear any information of the model parameters, therefore it is meaningless to talk about the parameter sensitivity for those options.

For all the options that are not completely determined by the intrinsic value or are not worthless, their prices are relatively sensitive to the volatility parameter σ, and not quite sensitive to the parameter κ. For example, when $X = 95$, $\tau = 0.5$, and $\kappa = 0.01$, the option price increase about 200% from 0.1378 to 0.4115 when σ goes up about 200% from 0.1 to 0.3. For the same exercise price and time to maturity, if we fix the parameter σ at 10% and let the parameter κ increases about 10 times from 0.01 to 0.1, the option price changes about 33% from 0.1378 to 0.1332.

We also tested the sensitivity of the option price to parameters when $\gamma = 0.5$. The conclusions are very similar to those of the HJM model when $\gamma = 1$. The option prices are not so sensitive to the parameter κ, but relatively quite sensitive to the volatility parameter σ.

7. Comparison of the HJM Model with Black's Model

In 1976, Black extended the well-known Black-Scholes model to value options on commodity futures. The application of the model is far beyond Black's original intention and it is frequently used to value interest rate options. When Black's model is applied to the interest rate derivatives such as the options on Eurodollar futures, we really assume the interest rates are constant for discounting purposes, even though they are assumed to be stochastic when the payoff from the options is calculated. Black has a closed form equation for the European options. Denote τ as the maturity date of the option, *efp* futures price with maturity τ, X strike price, r zero coupon yield for maturity τ, and σ volatility of futures price, then we have

$$c = e^{-r\tau}[efpN(d_1) - XN(d_2)], \text{ and} \tag{28a}$$

$$p = e^{-r\tau}[XN(-d_2) - efpN(-d_1)], \tag{28b}$$

where $d_1 = \dfrac{\log(efp / X) + \sigma^2\tau/2}{\sigma\sqrt{\tau}}$ and

$$d_2 = \frac{\log(efp / X) - \sigma^2\tau/2}{\sigma\sqrt{\tau}}$$

In the above expressions, c and p are the call option price and put option price, respectively.

When Black's model is used to price the Eurodollar futures options, we really use the put option formula to calculate the call option price and vice versa. Suppose the Eurodollar futures price is *efp* and the exercise price X. Then substituting 100-*efp* and 100-X for the futures price and exercise price, respectively, into the put option formula, we can get the call option price directly in basis points with multiplying the result by 100. The American option could then be valued using lattice method.

Unlike the equilibrium models and arbitrage models, which focus on the evolution of the term structure, Black's model concentrates on the distribution of the futures prices directly. This property makes Black's model simple and clear. If Black's model and the HJM model do not provide significantly different results in pricing the Eurodollar futures options, then Black's model would make the pricing procedure simpler and convenient.

First, we compare Black's model with the HJM model when $\sigma_f(t,t) = \sigma r$, which is considered by Rendelman and Bartter's model. Assume that the current Eurodollar futures price equals 95. κ is fixed at 0.01 for the HJM model. The results of comparison between the HJM model and Black's model with different time to maturity and different volatility parameters are listed in Table 4, where Δ is difference between the option prices from Black's model and those from the HJM model.

The call option with exercise price of 95 is at the money. We first calculate the option prices with different exercise prices for the HJM model, then use the price of the option at the money from the HJM model as input for Black's model to obtain the implied volatility. This implied volatility is then used to calculate the options prices with different exercise prices. When $\tau = 0.25$ and $\sigma = 0.1$ for the HJM model, the implied volatility for Black's model equals 0.09988. When $\tau = 0.25$ and $\sigma = 0.3$ for the HJM model, the implied volatility for Black's model is 0.29919. When $\tau = 0.5$ and $\sigma = 0.1$ for the HJM model, the implied volatility for Black's model equals 0.09971. When $\tau = 0.5$ and $\sigma = 0.3$ for the HJM model, the implied volatility for Black's model equals 0.29833. When $\tau = 1$ and $\sigma = 0.1$ for the HJM model, the implied volatility for Black's model is 0.09934. When $\tau = 1$ and $\sigma = 0.3$ for the HJM model, the implied volatility for Black's model is 0.29637. Table 4 shows that when the volatility parameter for the HJM model is 0.1, the HJM model and Black's model give us almost the same results. When the volatility parameter for the HJM model is as high as 0.3, the difference between the HJM model and Black's model is within 0.4 basis point, which is not significant.

These findings do not necessarily lead us to conclude that the difference between Black's model and the HJM model is negligible in pricing the options on Eurodollar futures when $\sigma_f(t,t) = \sigma r$, since we assume the initial term structure is flat. So we also test the difference between these two models when the initial term structure is not flat. Assume the three-month Eurodollar rate equals 3%, six-month rate equals 4%, and current six-month Eurodollar futures price equals 95. The initial term structure is derived with the method of linear interpolation. For the HJM model, we have $\kappa = 0.01$ and $\sigma = 0.1$. Suppose the option prices for Black's model are discounted by the six-month Eurodollar rate. The volatility for Black's model is still calculated by at the money option price from the HJM model. For options with maturity within one year, the difference between the HJM model and Black's model is within half a basis point. Therefore, we conclude that even when the initial term structure is not flat, the difference between two models is insignificant.

The conclusions about the difference between Black's model and the HJM model are different from the HJM model when $\sigma_f(t,t) = \sigma\sqrt{r}$ Assume the initial term structure is flat at the interest rate implied by the current Eurodollar futures price of 95. We first use Black's model to calculate the option prices with different exercise prices and then calculate the implied volatility for the HJM model with the option price at the money. The results are given in Table 5, where we find that when the volatility for Black's model is 0.1, the difference between Black's model and the HJM model is within 0.4 basis points even when the maturity is as long as one year. When the volatility for Black's model goes up 0.3, then the difference between the HJM model and Black's model is still negligible for the options whose maturity is 0.25 year. But for the options whose maturity are larger or equal to half a year, the difference between Black's model and the HJM model could be larger than 1 basis point, which is very significant.

Clearly, when the volatility of the spot rate is proportional to the spot rate, the difference between the HJM model and Black's model is insignificant. In this sense, Black's model is much stronger than it seems to be in pricing the Eurodollar futures options. But when the spot rate structure is in the same class considered by the CIR model, Black's model fails to get the results similar to the HJM model in pricing the Eurodollar futures options, when the maturity of the option is larger or equal to half a year and the volatility for Black's model is about 30%.

8. The HJM Models with Non-constant Volatility and with Constant Volatility

Without some specific restrictions on the volatility structure of the forward rate, it is difficult to apply the lattice method in pricing the interest rate derivatives because the non-Markovian property makes lattice procedures to increase exponentially. Therefore, the empirical test on the HJM model is quite difficult. The non-Markovian property could be completely removed if we restrict the forward rate to a constant volatility process. Recently, Bjorn Flesaker used the General Method of Moment to test the validity of the HJM model in pricing the Eurodollar futures option. His results show that the constant volatility HJM model is unable to explain the cross-sectional pricing pattern of the Eurodollar futures options. His conclusion depends on the specific volatility structure and might also depend on the volatility parameter he uses. Simulation tests are carried out in the following to examine the difference among the HJM models with different volatility structures.

Assume the initial term structure is flat at interest rate implied by the current futures price. Let $\kappa = 0.01$ for the non-constant HJM models. We first calculate the option prices with different exercise prices when $\gamma = 1$, $\sigma = 0.1$. Then we use the option price at the money to calculate the implied volatilities for the constant volatility HJM model and the HJM model with γ equal 0.5. The comparison among different HJM models in pricing the Eurodollar futures options with different time to maturity and exercise prices for both the American and European style are shown in Table 6. Δ_1 is the difference between column (4) and column (2), and Δ_2 is the difference between column (4) and column (3). The difference between the constant HJM models and non-constant HJM models increases with the increase of time to maturity. But even when the time to maturity is as long as one year, the difference between the constant volatility HJM model and the non-constant HJM model when $\gamma = 1$ is within 0.6 basis point for both American and European options. Moreover, the difference between the constant HJM model and non-constant HJM model when $\gamma = 0.5$ is within 0.4 basis point. Of course, for the deeply out of the money options, the difference in percentage is much higher, but those options prices rarely bear parameter information. Therefore, based on the option prices, which are not deeply out of the money or not deeply in the money, we conclude that there is no significant difference for different volatility structures in pricing the Eurodollar futures options.

Flesaker uses very high volatility relative to the number used above for the HJM model with constant volatility. Thus, we use the same volatility number used by Flesaker below to repeat the testing process above, which is 0.02 for the constant volatility HJM model. This time we first calculate the option prices with different exercise prices for the constant volatility HJM model, and then use at-the-money price to calculate the implied volatilities for the HJM models when $\gamma = 1$ and $\gamma = 0.5$, respectively. The results are shown in Table 7, where Δ_1 is the difference between fourth column and second, Δ_2 between fourth and third, and Δ_3 between third and second. As can be seen, the difference among different HJM models increases with the increase of time to maturity. Even for the short-term options, which has 0.25 year to maturity, the difference among these three HJM models could be larger than 1 basis points. Therefore, we conclude that the differences among these three different models are very significant. To sum, when the forward rate is quite volatile and depends on the interest rate, the constant volatility HJM model is significantly different from the non-constant HJM model in pricing the Eurodollar futures options.

The deviations of the option prices of the constant HJM model from the non-constant HJM model depend on the exercise price. Table 7 shows that both Δ_1 and Δ_2 are negative for the options in the money and positive for the options out of the money. Therefore the constant HJM model undervalues the options in the money and overvalues the options out of the money relative to the non-constant HJM models. This also can be seen very clearly in Table 8, where σ_1 is the implied volatility for the constant HJM model with the option prices of HJM model when $\gamma = 0.5$ as inputs. Similarly, σ_2 is the implied volatility for the constant HJM model with the option prices of the HJM model when $\gamma = 1$ as inputs. σ_1 and σ_2 are calculated using the put option prices for the options in the money and using the call option prices for the options out of the money. The relationship of the implied volatility for options with half a year to maturity with respect to the exercise price is shown in Figure 1.

The strong relationship between the implied volatility and the exercise price really highlights the importance of correctly estimating the parameter γ. When the true volatility structure is such that $\gamma = 1$ or $\gamma = 0.5$, then the constant HJM model will leads to incorrect results when the forward rate is very volatile. That is because the constant HJM model uses just a single volatility number to represent the volatilities at different exercise prices, which in fact are significantly different. Flesaker shows that the deviation of the theoretical value of the constant volatility HJM model from the market price is a function of the moneyness. When the non-constant HJM model matches the market prices well, then the volatility smile explains quite well the deviations of the theoretical value of the constant HJM model from the market prices.

In Figure 1, we also see that σ_1 and σ_2 intersect at the exercise price of the option at the money, then diverge from each other with the deviation of the exercise price from that at the money. The difference between σ_1 and σ_2 reflects the difference between the non-constant HJM models when $\gamma = 1$ and when $\gamma = 0.5$. The difference between the two HJM models increases with the increase of the deviations of the exercise price from that at the money.

The results in Flesaker (1993a) show that the constant HJM model overvalues short-term options relative to long-term options. The results in Table 8 show that such a relationship does not exist. Flesaker uses the options in the money or out of the money within 50 basis points. So we also consider the options with 50 basis points in the money or out of the money. Consider the options with exercise price 95.5. The implied volatilities for 0.25 year to maturity and 0.5 year to maturity are higher than the volatility for 1 year to maturity, which means that the constant HJM model overvalues the options with 0.25 and 0.5 year to maturity relative to the option with 1 year to maturity. But the implied volatility for 0.5 year to maturity is higher than the implied volatility for 0.25

year to maturity, which means that the constant HJM model overvalues the option with 0.5 year to maturity relative to the option with 0.25 year to maturity.

9. European Option vs. American Option

Due to its simplicity, European option models are frequently used for theoretical research. Unfortunately, the options traded on Eurodollar futures in Chicago Mercantile Exchange are American style. If the difference between American options and European options are trivial under certain conditions, then we really can use the European option models to price American options. It is very obvious that for deeply in the money options, American style and European style are quite different, because the early exercise property makes the American options mainly determined by the intrinsic value, whereas the European options are mainly determined by the whole stochastic process of its lifetime. For options deeply out of the money, the probability of early exercise is very small, and the difference between the European and American is trivial. Flesaker claims that the differences for options in the money within 50 basis points are within 1 basis points, which is the tick size in the market.

The HJM models could be used to examine the difference of European options from the American option. Assume the spot rate volatility structure follows (24) and the initial term structure is flat at the interest rate implied by the current Eurodollar futures price. Let $\kappa = 0.01$ when γ equals 0.5 or 1 and $\kappa = 0$ when γ equals 0. Since the differences between the European and American for options deeply in the money or out of the money are already clear, here we focus on the options in the money or out of the money within 50 basis points. The call option prices for both American and European with different time to maturity and different exercise prices at various volatility parameter levels are listed in Table 9.

The volatility parameter σ for different HJM models is selected so as to make the value of σr^{γ} at the initial point comparable for different HJM models. The conclusions about the difference between the American and European style from the three different HJM model are the same. Consider the options with exercise price 94.5. From table 9, we can see that the difference between the American options and European options increases with the increase of the time to maturity. When the maturity is less or equal to half a year, the difference between American and European is within 0.65 basis point for $\gamma = 1$, 0.75 for $\gamma = 0.5$, and 0.65 for $\gamma = 0$. Even the highest one 0.75 is within 1 basis point, which is not significant. When the maturity is one year, the difference is about one basis point, which is significant.

From Table 9, we can see that the difference between the American and European for options at the money or out of the money with 50 basis points is much smaller than that of the options with 50 basis points in the money. Table 9 shows that the difference between the American and European is less than 0.21 basis point when the maturity is less or equal to 0.5 year. When the maturity goes up to 1 year, the difference between the two increases, but is still less than 0.53 basis point, which is insignificant.

On the whole Table 9, except the options, which is one year to maturity and has 50 basis points in the money, the difference between the European and American style is insignificant for all other options. To sum, the difference between the European and American style for options with a maturity less or equal to 0.5 year is within 1 basis point, which is the tick size of the market.

10. The Difference between Futures Price and Forward Price and its Effects on Options Prices

Our findings so far follow from formula (21) and (22), which means that we have been using the forward price to substitute for the futures price. We are aware of the fact that such substitutions may lead to some errors. In this section, we are going to examine the differences between the futures price and the forward price and its effects on the Eurodollar futures options.

First, we calculate the futures price based on formula (23), then calculate the options price. Suppose the spot rate volatility follows the process described by (24) with $\gamma = 1$. Assume also that the initial term structure is flat at the interest rate implied by the forward price. Because the futures price equals the forward price on the maturity date, as long as we know the initial term structure, we know the futures price on the maturity date. Thus, using formula (23) and going backward, we can get the lattice of the futures price. The initial futures price on the lattice should be completely consistent with the current futures price. In practice, we only know the current futures price, not the forward rate. Therefore, we can go inversely. We use the current futures price to calculate the implied forward rate. Then we use this initial term structure as inputs to calculate the option prices, which are based on the futures prices.

Second, we use the initial term structure obtained from above as inputs to calculate the forward price and options prices based on the formula (21) and (22), which is what we have done in previous sections.

First let $\sigma = 0.1$. Using the methods described above, when the current forward price is fixed at 95, the current Eurodollar futures prices is 94.9998 for 0.25 year to maturity, 94.9994 for 0.5 year to maturity and 94.9982 for 1 year to maturity. The difference between the forward price and the futures price is 0.02 basis point for 0.25 year to maturity, 0.06 basis point for 0.5 year to maturity, and 0.18 basis point for 1 year to maturity. Thus, the difference increases with the increase of the time to maturity.

The options prices based on both forward prices and futures prices with different exercise prices and different time to maturity are listed in table 10. Column (2) lists the option prices based on the forward price, while column (1) prices are based on the futures price. For the deeply in the money options, the prices of options are determined by the intrinsic value. Therefore, the difference is just the difference of the forward price and futures price. The differences for all other options, which are not deeply in the money, are within 0.04 basis points for options even with 1 year to maturity.

Next, let σ increase from 0.1 to 0.3 and the forward price still fixed at 95. The futures price is 94.9979 for 0.25 year to maturity, 94.9944 for 0.5 year to maturity, and 94.9830 for 1 year to maturity. The difference between the forward price and the futures price increases with the increase of the time to maturity. For the options with one year to maturity, the difference between the forward price and futures price goes up to 1.70 basis points. But the difference between the forward price and futures price has very limited effect on pricing the options. For options with 1 year to maturity, the biggest value of Δ is 0.54 basis point, which is not significant. So even for the volatility parameter as high as 0.3, the difference between the option price based on formula (21) and (22) and that based on formula (23) is insignificant, which verifies the methodology we used in the previous sections.

11. Conclusion

Without severe restrictions on the term structure, the evolution of the term structure is not Markovian under the martingale with respect to finite state variables. The restrictions put by RS algorithm has partly removed the non-Markovian property and allowed the evolution of term structure be described by two state variables. Quite interestingly, the restrictions are imposed on the volatility structure of the forward rate, not on the volatility structure of the spot rate, which keeps most of the generality of the original HJM model. Term structures considered by Vasick, Rendelman/Bartter, and those considered by Cox/Ingosoll/Ross, which are widely used in practice are just some specific cases here. Furthermore, Vasick's model, Rendelman/Bartter model, and CIR model are equilibrium models, which could not reconcile the model with the initial term structure, a shortcoming that was overcome by the HJM model. The HJM model combined with the RS algorithm could be very powerful in pricing interest rate derivatives in practice. The simulation tests in this paper on the Eurodollar futures options with the HJM model indeed provide evidence for its capacity.

As long as we get the implied volatility for Black's model with respect to each maturity time, the results from Black's model do not deviate significantly from the HJM model, when the volatility of spot rate is proportional to the spot rate. In this sense, Black's model is much stronger than one might think in pricing Eurodollar futures options. But when the volatility structure of the spot rate follows the process considered by CIR, the difference between the HJM model and Black's model is significant. Therefore, the HJM model is more powerful than Black's model in pricing the interest rate derivatives like the options on Eurodollar futures.

The simulation tests in this paper show that the constant HJM model gives very close results to the non-constant HJM model when the term structure is not quite volatile. If the volatility of the forward rate is very high, then it fails to provide results close to the non-constant HJM model. This highlights the importance of the estimates of the γ parameter when the term structure is very volatile.

The simulation tests show that the difference between the European and American style options at the money or out of the money within 50 basis points is less than 0.21 basis point when the maturity is less or equal to 0.5 year. When the maturity goes up to 1 year, the difference between the two increases, but is still less than 0.53 basis point. With regard to the options with 50 basis points in the money, the difference between the American and European style is less than 0.75 basis point when the time to maturity is less or equal to 0.5 year. When the maturity goes up to 1 year, the difference between the two is about 1 basis point.

When the volatility of the spot rate is proportional to the spot rate, the difference between the forward price and futures price is negligible if the volatility parameter is at the level of 10%. In this case, the difference between the option price based on formula (21) and (22) and that on formula (23) is also negligible. When the volatility parameter goes up to 30%, the difference between the forward price and futures price could be larger than 1 basis point, but its effects on pricing the options are still trivial.

Table1 Results of Convergence Tests

		$m\ (\tau = 0.5)$				$m\ (\tau = 1)$			
σ	X	3	5	10	25	3	5	10	25
0.1	94	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	95	0.1378	0.1378	0.1378	0.1378	0.1905	0.1905	0.1905	0.1905
	96	0.0001	0.0001	0.0001	0.0001	0.0018	0.0018	0.0018	0.0018
0.3	94	1.1040	1.1040	1.1040	1.1040	1.2235	1.2234	1.2234	1.2234
	95	0.4115	0.4115	0.4115	0.4115	0.5665	0.5665	0.5665	0.5665
	96	0.0701	0.0700	0.0700	0.0700	0.1686	0.1686	0.1686	0.1686
0.5	94	1.3421	1.3420	1.3420	1.3420	1.5828	1.5825	1.5825	1.5825
	95	0.6818	0.6818	0.6818	0.6818	0.9332	0.9330	0.9329	0.9329
	96	0.2452	0.2451	0.2451	0.2451	0.4494	0.4493	0.4493	0.4492

$\gamma = 1$, $n = 50$, and $efp = 95$. The initial term structure is assumed be flat at the interest rate implied by the current futures price.

Table 2 Results of Convergence

n	$\tau = 0.25$			$\tau = 0.5$			$\tau = 1$		
	94.00	95.00	96.00	94.00	95.00	96.00	94.00	95.00	96.00
50	1.0771	0.3971	0.0857	1.1842	0.5560	0.2016	1.3540	0.7718	0.3885
100	1.0758	0.3980	0.0843	1.1833	0.5573	0.2007	1.3538	0.7736	0.3886
200	1.0759	0.3985	0.0844	1.1821	0.5579	0.1993	1.3530	0.7745	0.3879
300	1.0764	0.3986	0.0850	1.1820	0.5581	0.1995	1.3525	0.7748	0.3874
400	1.0762	0.3987	0.0848	1.1821	0.5582	0.1995	1.3521	0.7749	0.3867

The options are American call. $\gamma = 0$, $\kappa = 0$ $\sigma = 0.02$, and $efp = 95$. The initial term structure is assumed flat at the rate implied by the current Eurodollar futures price.

Table 3 Sensitivity of Option Prices to Model Parameters

σ	X	$\tau = 0.25$		$\tau = 0.5$		$\tau = 1$	
		$\kappa = 0.01$	$\kappa = 0.1$	$\kappa = 0.01$	$\kappa = 0.1$	$\kappa = 0.01$	$\kappa = 0.1$
0.05	94	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	95	0.0493	0.0482	0.0689	0.0667	0.0954	0.0902
	96	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	94	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	95	0.0985	0.0963	0.1378	0.1332	0.1905	0.1801
	96	0.0000	0.0000	0.0000	0.0000	0.0018	0.0011
0.3	94	1.0365	1.0335	1.1040	1.0949	1.2234	1.1978
	95	0.2949	0.2884	0.4115	0.3979	0.5665	0.5358
	96	0.0199	0.0180	0.0700	0.0624	0.1686	0.1467

The options are American call. $\gamma = 1$, $n = 50$, and $efp = 95$. The initial term structure is flat at the interest rate implied by the current futures price. The call options are American style.

Table 4 Comparison of the HJM Model with Black's Model

τ		$\sigma = 0.1$ for HJM model			$\sigma = 0.3$ for HJM model		
	X	$\gamma = 1$	*Black*	Δ	$\gamma = 1$	*Black*	Δ
0.25	94.00	1.0000	1.0000	0.0000	1.0365	1.0366	0.0001
	94.50	0.5008	0.5008	0.0000	0.6197	0.6188	-0.0009
	95.00	0.0985	0.0985	0.0000	0.2949	0.2949	0.0000
	95.50	0.0015	0.0015	0.0000	0.1013	0.0993	-0.0020
	96.00	0.0000	0.0000	0.0000	0.0199	0.0197	-0.0002
0.5	94.00	1.0000	1.0000	0.0000	1.1040	1.1061	0.0021
	94.50	0.5090	0.5090	0.0000	0.7242	0.7221	-0.0021
	95.00	0.1378	0.1378	0.0000	0.4115	0.4115	0.0000
	95.50	0.0098	0.0097	-0.0001	0.1958	0.1931	-0.0027
	96.00	0.0001	0.0001	0.0000	0.0700	0.0686	-0.0014
1.0	94.00	1.0000	1.0000	0.0000	1.2234	1.2256	0.0022
	94.50	0.5323	0.5323	0.0000	0.8685	0.8686	0.0001
	95.00	0.1905	0.1905	0.0000	0.5665	0.5665	0.0000
	95.50	0.0341	0.0334	-0.0007	0.3342	0.3307	-0.0035
	96.00	0.0018	0.0018	0.0000	0.1686	0.1648	-0.0038

The options are American call. For the HJM model, $\kappa = 0.01$, $n = 50$, and $m = 5$. For Black's model, $n = 400$, $r = 0.0497$. $efp = 95$. The initial term structure is assumed be flat at the interest rate implied by the current Eurodollar futures price. When $\tau = 0.25$ and $\sigma = 0.1$ for the HJM model, the implied volatility for the Black's model=0.09988. When $\tau = 0.25$ and $\sigma = 0.3$ for the HJM model, the implied volatility for the Black's model=0.29919. When $\tau = 0.5$ and $\sigma = 0.1$ for the HJM model, the implied volatility for the Black's model=0.09971. When $\tau = 0.5$ and $\sigma = 0.3$ for the HJM model, the implied volatility for the Black's model=0.29833. When $\tau = 1$ and $\sigma = 0.1$ for the HJM model, the implied volatility for Black's model is 0.09934. When $\tau = 1$ and $\sigma = 0.3$ for the HJM model, the implied volatility for Black's model is 0.29637.

Table 5 Comparison of the HJM model with Black's model

τ		$\sigma = 0.1$ for the Black's model			$\sigma = 0.3$ for the Black's model		
	X	$\gamma = 0.5$	*Black*	Δ	$\gamma = 0.5$	*Black*	Δ
0.25	94.00	1.0000	1.0000	0.0000	1.0306	1.0370	0.0064
	94.50	0.5005	0.5008	0.0003	0.6147	0.6195	0.0048
	95.00	0.0986	0.0986	0.0000	0.2957	0.2957	0.0000
	95.50	0.0018	0.0015	-0.0003	0.1077	0.0998	-0.0079
	96.00	0.0000	0.0000	0.0000	0.0250	0.0199	--0.0051
0.5	94.00	1.0000	1.0000	0.0000	1.0932	1.1078	0.0146
	94.50	0.5078	0.5092	0.0014	0.7172	0.7243	0.0071
	95.00	0.1381	0.1381	0.0000	0.4138	0.4138	0.0000
	95.50	0.0110	0.0098	-0.0012	0.2074	0.1951	-0.0123
	96.00	0.0001	0.0001	0.0000	0.0841	0.0698	-0.0143
1.0	94.00	1.0000	1.0000	0.0000	1.2073	1.2317	0.0244
	94.50	0.5303	0.5331	0.0028	0.8621	0.8754	0.0133
	95.00	0.1917	0.1917	0.0000	0.5733	0.5733	0.0000
	95.50	0.0378	0.0341	-0.0037	0.3536	0.3369	-0.0167
	96.00	0.0029	0.0019	-0.0010	0.1961	0.1695	-0.0266

The options are American call. For the HJM model, $\kappa = 0.01$, $n = 50$, $m = 5$. For Black's model, $n = 400$. $efp = 95$. The initial term structure is assumed flat at the interest rate implied by the current Eurodollar futures price. When $\tau = 0.25$ and the volatility=0.1for the Black's model, $\sigma = 0.06705$ for the HJM model. When $\tau = 0.25$ and the volatility =0.3 for the Black's model, $\sigma = 0.02232$ for the HJM model. When $\tau = 0.5$ and the volatility =0.1for the Black's model, $\sigma = 0.02236$ for the HJM model. When $\tau = 0.5$ and the volatility =0.3 for the Black's model, $\sigma = 0.06705$ for the HJM model. When $\tau = 1$ and the volatility =0.1for the Black's model, $\sigma = 0.02244$ for the HJM model. When $\tau = 1$ and the volatility =0.3 for the Black's model, $\sigma = 0.06765$ for the HJM model.

Table 6 Comparison of the Different HJM Models

τ		X	$\gamma=1$	$\gamma=0.5$	$\gamma=0$	Δ_1	Δ_2
0.25	A	94.00	1.0000	1.0000	1.0000	0.0000	0.0000
		94.50	0.5008	0.5005	0.5003	-0.0005	-0.0002
		95.00	0.0985	0.0985	0.0985	0.0000	0.0000
		95.50	0.0015	0.0017	0.0021	0.0006	0.0004
		96.00	0.0000	0.0000	0.0000	0.0000	0.0000
	E	94.00	0.9875	0.9875	0.9876	0.0001	0.0001
		94.50	0.4965	0.4961	0.4958	-0.0007	-0.0003
		95.00	0.0983	0.0983	0.0983	0.0000	0.0000
		95.50	0.0015	0.0017	0.0021	0.0006	0.0004
		96.00	0.0000	0.0000	0.0000	0.0000	0.0000
0.5	A	94.00	1.0000	1.0000	1.0000	0.0000	0.0000
		94.50	0.5090	0.5077	0.5065	-0.0025	-0.0012
		95.00	0.1378	0.1378	0.1378	0.0000	0.0000
		95.50	0.0098	0.0108	0.0121	0.0023	0.0003
		96.00	0.0001	0.0001	0.0002	0.0001	0.0001
	E	94.00	0.9757	0.9757	0.9757	0.0000	0.0000
		94.50	0.5020	0.5010	0.4999	-0.0026	-0.0011
		95.00	0.1371	0.1371	0.1371	0.0000	0.0000
		95.50	0.0098	0.0108	0.0121	0.0023	0.0003
		96.00	0.0001	0.0001	0.0002	0.0001	0.0001
1.0	A	94.00	1.0000	1.0000	1.0000	0.0000	0.0000
		94.50	0.5323	0.5296	0.5263	-0.0060	-0.0033
		95.00	0.1905	0.1905	0.1905	0.0000	0.0000
		95.50	0.0341	0.0370	0.0391	0.0050	0.0021
		96.00	0.0018	0.0027	0.0039	0.0021	0.0012
	E	94.00	0.9580	0.9563	0.9560	-0.0020	-0.0003
		94.50	0.5203	0.5182	0.5151	-0.0052	-0.0031
		95.00	0.1885	0.1885	0.1885	0.0000	0.0000
		95.50	0.0339	0.0369	0.0388	0.0049	0.0019
		96.00	0.0018	0.0027	0.0038	0.0020	0.0011

The initial term structure is assumed flat at the interest rate implied by the current Eurodollar futures price, which is 95. For $\gamma=1$, $\sigma=0.1$, $\kappa=0.01$, $n=50$, and $m=5$. For $\gamma=0.5$, $\kappa=0.01, n=50$, and $m=5$. For $\gamma=0$, $\kappa=0$, and $n=300$. When $\tau=0.25$, the implied volatility is 0.02229 when $\gamma=0.5$ and 0.00494 when $\gamma=0$. When $\tau=0.5$, the implied volatility is 0.02229 when $\gamma=0.5$ and 0.00493 when $\gamma=0$. When $\tau=1$, the implied volatility is 0.02229 when $\gamma=0.5$ and 0.00490 when $\gamma=0$. The options are American call options.

Table 7 Comparison of the different HJM models

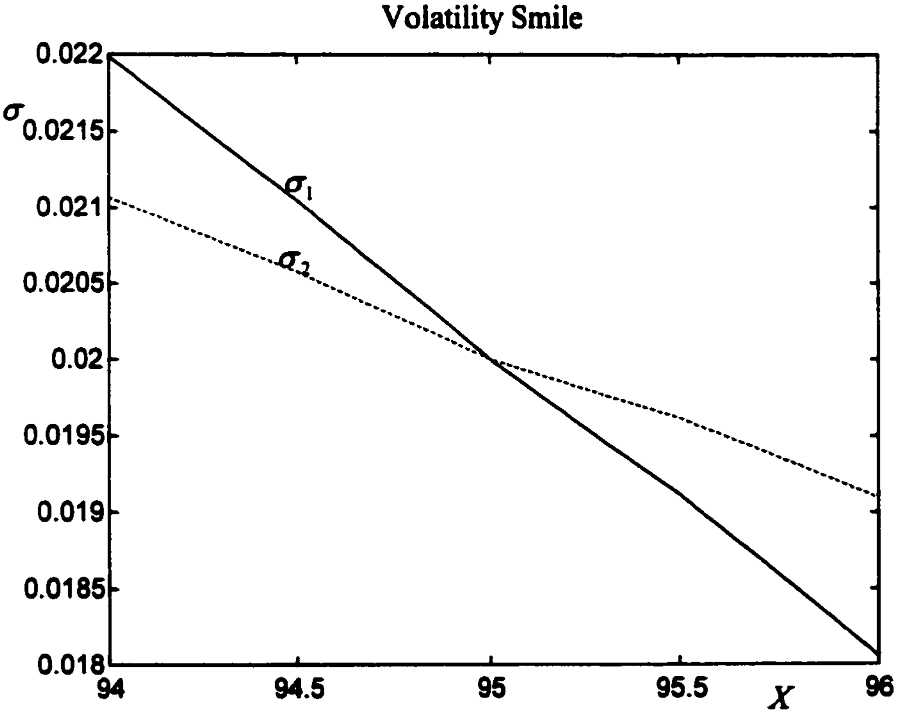
τ		X	$\gamma=1$	$\gamma=0.5$	$\gamma=0$	Δ_1	Δ_2	Δ_3
0.25	Call	94.00	1.1014	1.0894	1.0764	-0.0250	-0.0130	-0.0120
		94.50	0.7148	0.7057	0.6947	-0.0201	-0.0110	-0.0091
		95.00	0.3986	0.3986	0.3986	0.0000	0.0000	0.0000
		95.50	0.1830	0.1923	0.1995	0.0165	0.0072	0.0093
		96.00	0.0614	0.0738	0.0850	0.0236	0.0112	0.0124
	Put	94.00	0.1114	0.0993	0.0861	-0.0253	-0.0132	-0.0121
		94.50	0.2210	0.2119	0.2009	-0.0201	-0.0110	-0.0091
		95.00	0.4004	0.4004	0.4004	0.0000	0.0000	0.0000
		95.50	0.6806	0.6898	0.6970	0.0164	0.0072	0.0092
		96.00	1.0561	1.0683	1.0792	0.0231	0.0109	0.0122
0.5	Call	94.00	1.2273	1.2060	1.1820	-0.0453	-0.0240	-0.0213
		94.50	0.8658	0.8531	0.8376	-0.0282	-0.0155	-0.0127
		95.00	0.5581	0.5581	0.5581	0.0000	0.0000	0.0000
		95.50	0.3241	0.3370	0.3469	0.0228	0.0099	0.0129
		96.00	0.1584	0.1804	0.1995	0.0411	0.0191	0.0220
	Put	94.00	0.2470	0.2256	0.2014	-0.0456	-0.0242	-0.0214
		94.50	0.3773	0.3646	0.3491	-0.0282	-0.0155	-0.0127
		95.00	0.5611	0.5611	0.5611	0.0000	0.0000	0.0000
		95.50	0.8193	0.8322	0.8418	0.0225	0.0096	0.0129
		96.00	1.1472	1.1691	1.1878	0.0406	0.0187	0.0129
1.0	Call	94.00	1.4244	1.3897	1.3525	-0.0719	-0.0372	-0.0347
		94.50	1.0805	1.0614	1.0401	-0.0404	-0.0213	-0.0191
		95.00	0.7748	0.7748	0.7748	0.0000	0.0000	0.0000
		95.50	0.5221	0.5414	0.5582	0.0361	0.0168	0.0193
		96.00	0.3189	0.3547	0.3874	0.0685	0.0327	0.0358
	Put	94.00	0.4608	0.4263	0.3893	0.0715	-0.0370	-0.0345
		94.50	0.6009	0.5820	0.5610	0.0399	-0.0210	-0.0189
		95.00	0.7796	0.7796	0.7796	0.0000	0.0000	0.0000
		95.50	1.0119	1.0310	1.0480	0.0361	0.0170	0.0191
		96.00	1.2969	1.3320	1.3645	0.0676	0.0325	0.0351

The initial term structure is assumed flat at the interest rate implied by the current Eurodollar futures price. For $\gamma = 1$ and $\gamma = 0.5$, $\kappa = 0.01$, $n = 50$, and $m = 5$. For $\gamma = 0$, $\sigma = 0.02$, $\kappa = 0$, and $n = 300$. When $\tau = 0.25$, the implied volatility for call options equals 0.40616 and 0.40512 for put options, when $\gamma = 1$ and the implied volatility for call options equals 0.09051 and for put options 0.090283 when $\gamma = 0.5$. When $\tau = 0.5$, the implied volatility equals 0.40813 for call options and 0.40525 for put options, when $\gamma = 1$, and the implied volatility for call options is 0.09091 and 0.09029 for put options when $\gamma = 0.5$. When $\tau = 1$, the implied volatility equals 0.41290 for call options and 0.40495 for put options when $\gamma = 1$, and the implied volatility equals 0.09190 for call options and 0.09022 for put options when $\gamma = 0.5$. The options are American style.

Table 8 Implied Volatility

	$\tau = 0.25$		$\tau = 0.5$		$\tau = 1$	
X	σ_1	σ_2	σ_1	σ_2	σ_1	σ_2
94.00	0.02197	0.02107	0.02199	0.02106	0.02205	0.02109
94.50	0.02115	0.02063	0.02105	0.02058	0.02106	0.02056
95.00	0.02000	0.02000	0.02000	0.02000	0.02000	0.02000
95.50	0.01906	0.01959	0.01912	0.01962	0.01903	0.01955
96.00	0.01796	0.01909	0.01805	0.01909	0.01798	0.01904

σ_1 is the implied volatility for the constant volatility HJM model with the theoretical price from the HJM model when $\gamma = 1$ as input, and σ_2 from the HJM model when $\gamma = 0.5$.



σ_1 is the implied volatility for the constant volatility HJM model with the theoretical price from the HJM model when $\gamma = 0.5$ as input, and σ_2 from the HJM model when $\gamma = 1$.

Table 9 European vs. American Options

X										
	$\gamma=1$									
		$\tau=0.25$			$\tau=0.5$			$\tau=1$		
	σ	A	E	Δ	A	E	Δ	A	E	Δ
94.5	0.1	0.5008	0.4965	0.0043	0.5090	0.5025	0.0065	0.5323	0.5210	0.0113
	0.3	0.6197	0.6177	0.0020	0.7244	0.7203	0.0041	0.8685	0.8592	0.0093
	$\gamma=0.5$									
		$\tau=0.25$			$\tau=0.5$			$\tau=1$		
	σ	A	E	Δ	A	E	Δ	A	E	Δ
	0.02	0.5000	0.4949	0.0051	0.5036	0.4961	0.0075	0.5184	0.5060	0.0124
	0.08	0.6624	0.6604	0.0020	0.7906	0.7865	0.0041	0.9612	0.9525	0.0087
	$\gamma=0$									
		$\tau=0.25$			$\tau=0.5$			$\tau=1$		
	σ	A	E	Δ	A	E	Δ	A	E	Δ
	0.005	0.5004	0.4960	0.0044	0.5071	0.5006	0.0065	0.5284	0.5175	0.0109
	0.02	0.6947	0.6930	0.0017	0.8376	0.8346	0.0030	1.0401	1.0360	0.0041
95	$\gamma=1$									
		$\tau=0.25$			$\tau=0.5$			$\tau=1$		
	σ	A	E	Δ	A	E	Δ	A	E	Δ
	0.1	0.0985	0.0983	0.0002	0.1378	0.1371	0.0007	0.1905	0.1885	0.0020
	0.3	0.2949	0.2942	0.0007	0.4115	0.4097	0.0018	0.5665	0.5619	0.0046
	$\gamma=0.5$									
		$\tau=0.25$			$\tau=0.5$			$\tau=1$		
	σ	A	E	Δ	A	E	Δ	A	E	Δ
	0.02	0.0884	0.0882	0.0002	0.1236	0.1230	0.0006	0.1710	0.1692	0.0018
	0.08	0.3526	0.3518	0.0008	0.4917	0.4898	0.0019	0.6763	0.6715	0.0048
	$\gamma=0$									
		$\tau=0.25$			$\tau=0.5$			$\tau=1$		
	σ	A	E	Δ	A	E	Δ	A	E	Δ
	0.005	0.0998	0.0996	0.0002	0.1398	0.1392	0.0006	0.1940	0.1923	0.0017
	0.02	0.3986	0.3980	0.0006	0.5581	0.5568	0.0013	0.7748	0.7730	0.0018
95.5	$\gamma=1$									
		$\tau=0.25$			$\tau=0.5$			$\tau=1$		
	σ	A	E	Δ	A	E	Δ	A	E	Δ
	0.1	0.0015	0.0015	0.0000	0.0098	0.0098	0.0000	0.0341	0.0339	0.0002
	0.3	0.1013	0.1012	0.0001	0.1958	0.1953	0.0005	0.3342	0.3325	0.0017
	$\gamma=0.5$									
		$\tau=0.25$			$\tau=0.5$			$\tau=1$		
	σ	A	E	Δ	A	E	Δ	A	E	Δ
	0.02	0.0008	0.0008	0.0000	0.0068	0.0068	0.0000	0.0256	0.0255	0.0001
	0.08	0.1513	0.1510	0.0003	0.2774	0.2767	0.0007	0.4462	0.4439	0.0023
	$\gamma=0$									

	$\tau = 0.25$			$\tau = 0.5$			$\tau = 1$		
σ	A	E	Δ	A	E	Δ	A	E	Δ
0.005	0.0022	0.0022	0.0000	0.0128	0.0128	0.0000	0.0413	0.0412	0.0001
0.02	0.1995	0.1993	0.0002	0.3469	0.3463	0.0006	0.5582	0.5576	0.0006

$efp = 95$. The initial term structure is assumed to be flat at the interest rate implied by the current Eurodollar futures price. When $\gamma = 1$ and $\gamma = 0.5$, $m = 5$, $\kappa = 0.01$, and $n = 50$. When $\gamma = 0$, $\kappa = 0$ and $n = 300$. A represents American call options and E denotes European options.

Table 10 Effects on the option prices from the differences between the forward price and futures price

σ		$\tau = 0.25$			$\tau = 0.5$			$\tau = 1$		
	X	(1)	(2)	Δ	(1)	(2)	Δ	(1)	(2)	Δ
0.1	93.00	1.9998	2.0000	0.0002	1.9994	2.0000	0.0006	2.0000	1.9982	0.0018
	93.50	1.4998	1.5000	0.0002	1.4994	1.5000	0.0006	1.5000	1.4982	0.0018
	94.00	0.9998	1.0000	0.0002	0.9994	1.0000	0.0006	1.0000	0.9982	0.0018
	94.50	0.5006	0.5008	0.0002	0.5087	0.5090	0.0003	0.5323	0.5319	0.0004
	95.00	0.0985	0.0985	0.0000	0.1377	0.1378	0.0001	0.1905	0.1904	0.0001
	95.50	0.0015	0.0015	0.0000	0.0098	0.0098	0.0000	0.0341	0.0341	0.0000
	96.00	0.0000	0.0000	0.0000	0.0001	0.0001	0.0000	0.0018	0.0018	0.0000
0.3	93.00	1.9979	2.0000	0.0021	2.0056	2.0088	0.0024	2.0494	2.0548	0.0054
	93.50	1.5043	1.5055	0.0012	1.5386	1.5403	0.0017	1.6181	1.6213	0.0032
	94.00	1.0359	1.0365	0.0006	1.1031	1.1040	0.0009	1.2216	1.2234	0.0018
	94.50	0.6195	0.6197	0.0002	0.7240	0.7244	0.0004	0.8676	0.8685	0.0009
	95.00	0.2948	0.2949	0.0001	0.4113	0.4115	0.0002	0.5660	0.5665	0.0005
	95.50	0.1013	0.1013	0.0000	0.1958	0.1958	0.0000	0.3341	0.3342	0.0001
	96.00	0.0199	0.0199	0.0000	0.0700	0.0700	0.0000	0.1685	0.1686	0.0001

$\gamma = 1$, $\kappa = 0.01$, $m = 5$, and $n = 50$. The forward price is fixed at 95. When $\sigma = 0.1$, the futures price is 94.9998 for 0.25 year to maturity, 94.9994 for 0.5 year to maturity, and 94.9982 for 1year to maturity. When $\sigma = 0.3$, the futures price is 94.9979 for 0.25 year to maturity, 94.9944 for 0.5 year to maturity, and 94.9830 for 1 year to maturity.

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III. Pricing Options on Discount Bonds, Coupon Bonds, and Spot Rates with the Heath-Jarrow-Morton models

1. Introduction

The research in pricing the interest rate derivatives has followed two different tracks. One is the equilibrium models, which make assumptions about the stochastic process of the interest rate in the risk-neutral world, then price the derivatives in this risk-neutral world. The disadvantage of the equilibrium models is that the initial term structure can not be fit automatically. Another is the no-arbitrage models, which is pioneered by Ho and Lee (1986) and generalized by Heath, Jarrow and Morton (1992). Rather than as a result in the equilibrium models, the initial term structure is input in the no-arbitrage models.

The HJM model is the one of the most popular and theoretically satisfying models. But without strict restrictions on the volatility structures, the evolution of the term structures may not be Markovian with respect to a finite dimensioned state space. When the lattice procedures are used to simulate the HJM model to price the interest rate derivatives, the paths may not combine and explode exponentially, which makes the computation almost impossible. Recently, Ritchken and Sankarasubramanian (RS) (1995a) has modified the HJM model by restricting the volatility structure to a certain type, which does not completely remove the path dependence, but makes the lattice method efficient.

Ritchken and Sankarasubramnian (1995b) has done lots of simulation test with their method, which shows that their method could be widely used in pricing the interest rate derivatives such as the options on the bonds, options on the spot rate, and options on stocks with changing term structures. Cakici and Zhu (1999) research in pricing the options on Eurodollar futures also shows that RS algorithm is quite efficient. However the algorithm does not get success in any situations. The research in the paper shows that for options less than two years the algorithm is quite efficient, but for options larger than two years it may fail in many cases unless we put lots of restrictions on parameters.

The research in this paper follows RS (1995b) to compare the HJM models under different volatility structures in pricing the options on the discounted bond, coupon bonds, and spot rate. Rather than fixing volatility of the bond price, we fix at-the-money option price at the same level for HJM models with different volatility structure, then use the implied volatility parameter to calculate other option prices to make comparison among them.

The results in this paper shows that HJM models under different volatility structures are going to provide us quite different option prices. Therefore any incorrect selection of the volatility structures may lead to great errors in pricing the options on discounted bonds, coupon bonds, and spot rate. The difference among the HJM models with different volatility structures also depends on the exercise prices. When we fix at-the-money option at the same price level for all different HJM models, the difference among the HJM models will increase if the exercise price is more deeply out of the money or in the money.

This paper also compares the difference between the American style options and European style options on discount bonds, spot rate, and coupon bonds. It is quite interesting that American style and European style are indifferent for options on bonds, which do not pay coupons during the life of the options. Therefore, the American and European options on discount bonds are priced the same. This property is very similar to the options on stocks. When the stocks have no dividends payment during the life of the option, the early exercise right is worthless.

The RS algorithm depends on a transformation it takes to create lattice. Without the transformation, the variance of the spot rate is not constant, which makes the lattice impossible. Through the transformation, a new variable $y(t)$ with constant volatility is created. But this transformation has a shortcoming that it puts additional mathematical and financial restrictions on the spot rate. For example, when volatility elasticity is 1.5, the transformation requires negative values for $y(t)$. For a short-term option, this condition can be satisfied easily. But for a long term options, the lattice constructed may provide negative $y(t)$ values. In such a case, this algorithm lost may not be efficient.

2. The Model

Let $f(t,T)$ be the forward rate at date t for the small time increment that begins at date T and let $r(t,t)$ represent the spot rate. Assumes the volatility structure of the forward rate follows the following specific form:

$$\sigma_f(t,T) = \sigma[r(t)]^{\gamma} e^{-\kappa(T-t)} \qquad (1)$$

RS prove that the dynamics of the forward rates can be linked to two state variables. One is the spot rate $r(t)$. Another is $\phi(t)$, which acts as a sufficient statistics for capturing all path information necessary for pricing. Then the one factor HJM model could be described by the following equations:

$$P(t,T)=\frac{P(0,T)}{P(0,t)}e^{-\beta(t,T)[r(t)-f(0,t)]-\beta^2(t,T)\phi(t,T)/2} \qquad (2)$$

$$df(t,T)=\mu_f(t,T)dt+\sigma[r(t)]^{\gamma}e^{-\kappa(T-t)}dw(t) \qquad (3)$$

$$g(0)=E_0[e^{-\int_0^s r(t)dt}g(s)] \ . \qquad (4)$$

$P(t,T)$ is the price at date t of a pure discount bond that matures at time T. Equation (2) is the definition of a pure discount bond price, which is related to the forward rate. $f(t,T)$ is the forward rate at date t for instantaneous and riskless borrowing or lending at date T. Equation (3) is the stochastic process of forward rate of every maturity T.

$g(0)$ represents the value of an European claim at date 0 having a terminal payout at date s that is fully determined by the yield curve at that time. The expectation is taken in a risk neutral world.

The restriction to the forward rate volatility structure given by (1) is important. Without this restriction, the information needed to capture the evolution process of the forward rate is going to explode, which makes the popular lattice method obsolete. Interestingly, though the restriction excludes many other different types of the volatility structure, the model still includes the most popular volatility structures. For example, when $\gamma=0$, this is Vasicek model (1977). When $\gamma=0.5$, this is Cox, Ingersoll, and Ross model (1985).

The dynamics of the two state variables are given by:

$$dr(t)=\mu(r,\phi,t)dt+\sigma[r(t)]^{\gamma}dw(t) \qquad (5)$$

and:

$$d\phi(t) = [\sigma^2[r(t)^{2\gamma}] - 2\kappa\phi(t)]dt \tag{6}$$

where:

$$\mu(r,\phi,t) = \kappa[f(0,t) - r(t)] + \frac{\partial}{\partial t} f(0,t) + \phi(t). \tag{7}$$

Because the volatility of the spot rate is not a constant, the implementation of a lattice approach is impossible. Li/Ritchken/Sankrasubramanian (1995) follow Nelson/Ramaswamy (1990) in making the following transformation:

$$y(t) = \int \frac{1}{\sigma[r(t)]^{\gamma}} dr(t) \tag{8}$$

Then the dynamics of the two state variables is changed to the following the two equations:

$$dy(t) = m(y,\phi,t)dt + dw(t) \tag{9}$$

$$d\phi(t) = [\sigma^2[r(t)]^{2\gamma} - 2\kappa(t)\phi(t)]dt \tag{10}$$

where

$$m(y,\phi,t) = \frac{\partial}{\partial t} y(t) + \mu(r,\phi,t)\frac{\partial}{\partial t} y(t) + \frac{1}{2}\sigma^2[r(t)]^{2\gamma}\frac{\partial^2 y(t)}{\partial r(t)^2} \tag{11}$$

The transformed process $dy(t)$ and $d\phi(t)$ have constant volatility, a lattice approximation can be established. A very interesting feature of above algorithm is that the lattice is built on the lifetime of the contingent claim, not on the lifetime of the underlying assets. Suppose the maturity of the option is T. Then we divide T into n intervals. $\Delta t = T/n$. Let the approximating variables be y^a and ϕ^a at the beginning of some time increment. Then in the next time increment, the variables move to either (y^{a+}, ϕ^{a+}) or (y^{a-}, ϕ^{a-}) where,

$$y^{a+} = y^a + J\sqrt{\Delta t} \tag{12a}$$

$$y^{a-} = y^a - J\sqrt{\Delta t} \tag{12b}$$

J is an integer and could be different at each node. It is chosen specifically to make the probability of up jump or down jump to be positive. The up jump probability is given by the following formula.

$$p = \frac{m(y^a, \Phi^a, t)\Delta t + (y^a - y^{a-})}{(y^{a+} - y^{a-})} \tag{13}$$

From the relationship set up by (8), we can get easily the corresponding tree for the spot rate $r(t)$. The process of $\phi(t)$ has two different properties from that of spot rate $r(t)$ or $y(t)$. First $\phi(t)$ is locally deterministic which means

$$\phi^{a+} = \phi^{a-} = \phi^a + [\sigma^2(r(t), \phi^a t) - 2k(t)\phi^a]\Delta t. \tag{14}$$

Second the number of distinct Φ values at each node will equal the number of unique paths leading to that node. Rather keep track of all different distinct paths, we just identify two paths at each node which will give the maximum value $\bar{\phi}^a$ and minimum value $\underline{\phi}^a$. Then partition the interval $[\underline{\phi}^a, \bar{\phi}^a]$ into m equidistant points with $\phi^a(k)$, $k = 1 ... m$, representing *kth* points and

$$\phi^a(k) = \underline{\phi}^a + \frac{k-1}{m-1}(\bar{\phi}^a - \underline{\phi}^a) \quad . \tag{15}$$

When we finished the lattices of the spot rate $r(t)$ and $\phi(t)$, the lattice for the underlying assets can be established using formula (2), then terminal value of the contingent claim can be captured. With the backward recursion, we can get the contingent value at each node.

3. An unsolved problem with RS algorithm

A big problem with the HJM model is that it fails to provide an efficient lattice method to price the options. Therefore a successful modification to the HJM model must conquer this shortcoming so that the lattice method could be applied efficiently. To create the efficient lattice method for the HJM model with restriction (1), transformation (8) is necessary. Without the transformation, the volatility of the spot rate is non-constant, which prevents to set up the lattice method. But the transformation (8) put a strict relationship between the spot rate $r(t)$ and variable $y(t)$. The relationships between $r(t)$ and variable $y(t)$ under different γ values are shown in the following:

$$y(t) = \frac{r(t)}{\sigma}, \ \gamma = 0; \tag{16.1}$$

$$y(t) = \frac{2\sqrt{r(t)}}{\sigma}, \ \gamma = 0.5; \tag{16.2}$$

$$y(t) = \frac{\ln[r(t)]}{\sigma}, \ \gamma = 1; \tag{16.3}$$

$$y(t) = -\frac{2}{\sigma\sqrt{r(t)}}, \ \gamma = 1.5; \tag{16.4}$$

Equation (16.1) shows the restriction on $y(t)$ when $\gamma = 0$. Because interest rate is positive, $y(t)$ values should be positive. Unfortunately the lattice of $y(t)$ created by (12) does not guarantee positive values for $y(t)$. Specifically, when the number of time partition n satisfies the following condition, then the $y(t)$ values on some of the lattice nodes will be negative, which makes the spot rates $r(t)$ on the corresponding nodes negative. This of course is not reasonable.

$$n > \mathrm{int}(\frac{r(0)^2}{\sigma^2 \tau}), \tag{17.1}$$

where int is the integer function and τ is the time to maturity of the option;

For example, assume the initial term structure is flat at 10%. Considering a 5-year option on a bond with 15-year to maturity, we are going to have negative $y(t)$ values when n is larger than 80 if $\sigma = 0.005$. With the increasing of the volatility parameter σ, the n value in (17.1) is going to be smaller. If $\sigma = 0.01$, this negative interest rate problem is going to happen when n is larger than 20. When σ reaches to 0.015, Even 8 time partitions will lead to this problem. Of course, mathematically, we still can calculate the option's value even if the interest rate is negative, but financially it does not make senses. Therefore when $\gamma = 0$, though the volatility form of the spot rate is similar to Vasicek's model, RS's lattice algorithm is inferior to Vacik's model in the senses that it does not incorporate the mean reversion characteristics.

The restriction on $y(t)$ when $\gamma = 0.5$ is presented by equation (16.2). Because $\sqrt{r(t)}$ must be positive, then $y(t)$ must be strictly positive. This restriction is more severe than when $\gamma = 0$, because this is not only a financial restriction but also a mathematical restriction. As long as $y(t)$ goes negative, the lattice method fails to provide any results. But the lattice for $y(t)$ created by (12) does not guarantee the positive values for $y(t)$. Specifically,

When

$$n > \mathrm{int}(\frac{4r(0)}{\sigma^2 \tau}), \tag{17.2}$$

the negative $y(t)$ values will happen, which makes the calculation impossible. For example, assume that initial term structure is flat at 10%. For a 5-year option on a discount bond, when $\sigma = 0.015$, 356 time partitions would result in negative $y(t)$ values. Of course, 356 time partitions are not necessary for convergence in this case. Therefore when σ is 0.015, the negative $y(t)$ values will not happen in fact. When $\sigma = 0.03$, 89 time partitions would result in negative $y(t)$ values. But 89 time partitions are not enough for pricing 5-year options. Therefore in this case the lattice method does not work. If σ goes as high as 0.045, only 40 time partitions would have this problem. Then the lattice method fails to provide results.

In this case, let's go inversely. Assume that 100 time partitions are necessary and enough to get the convergent results when the initial term structure is flat at 10%. If the volatility parameter is relatively high around 0.045, then the negative $y(t)$ problem does not exist for a option with time to maturity less than 1.97 year. When the volatility is around 0.03, the time to maturity of the option that can be priced could reach to 4.44 years.

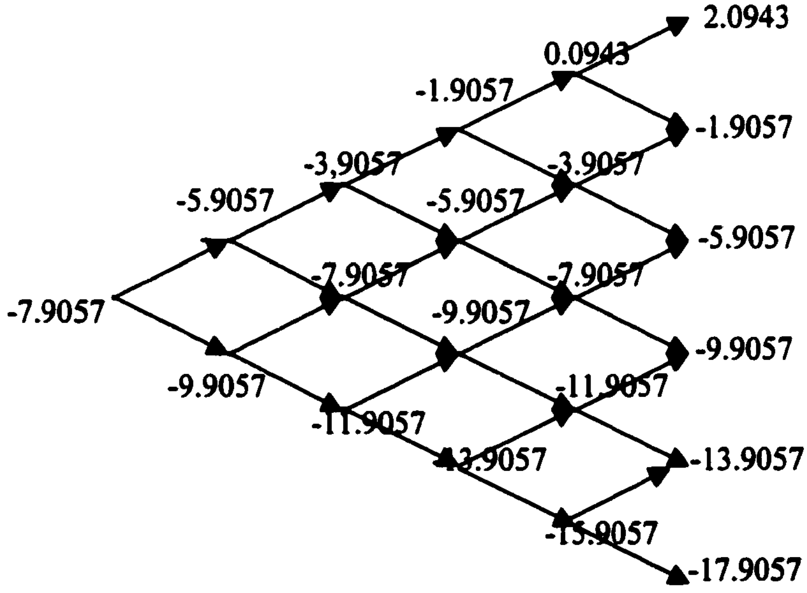
When $\gamma = 1.5$, the relationship between spot rate $r(t)$ and variable $y(t)$ is determined by (16.4), which requires $y(t)$ being negative. This restriction is as strong as in the case of $\gamma = 0.5$. Specifically,

When

$$n > \text{int}(\frac{4}{\sigma^2 r(0)\tau}), \tag{17.3}$$

the lattice of $y(t)$ is going to have positive values. For example, assume that the initial term structure is flat at 10%. For a 5-year option, 356 time partitions are going to have positive $y(t)$ values when $\sigma = 0.15$. If σ goes up to 0.3, 89 time partitions will give us the trouble positive $y(t)$ values. Similarly, even 40 time partitions will lead to positive $y(t)$ values if σ reaches 0.45. Figure1 provides us an example of such a case, in which the volatility parameter is specifically selected as large as 0.8 so that this problem could be observed in 5 steps. The time to maturity of the option is 20 years and the initial term structure is assumed flat at 10%. In Figure 1, beginning with the fourth time partition, we have positive $y(t)$. When such a case happen, the algorithm fails to provide us any results.

Figure 1



Let's go inversely. If 100 time partitions are necessary for us to get the convergent option prices, then less than 1.97 year to maturity option could be priced when the volatility of the forward rate is around 0.45. If volatility is around 0.3, less than 4.44 year to maturity options could be priced.

To summarize, the largest number of time partitions within which the lattice of $y(t)$ will be consistent with the restrictions given by (16) is negatively correlated with the volatility parameter in every HJM model. For short-term options, even when the volatility parameter is very high for each HJM model, the largest number is big enough to provide us convergence results and does not meet the trouble problem. But for long term options, with high volatility parameter, the RS lattice method could not give us a convergent result without violating the restrictions of (16). Therefore RS algorithm is very efficient in pricing the short-term options, but it may lose its efficiency for the long-term options unless we select the parameters purposely. The failure of RS algorithm in many cases is due to the transformation (8), rather than the volatility structure (1).

4. Comparison of HJM Models with Different Volatility Structures

According to the above discussion, the comparison of the different HJM models is restricted to relatively short-term options so that the restrictions put by equation (16) are strictly abided by. The selection of the parameter values follows RS (1995). γ takes four different values: 0, 0.5, 1, and 1.5. $\gamma = 0$ corresponds to the generalized Vasciek model. $\gamma = 0.5$ corresponds to square root model of volatility, similar to the Cox, Ingersoll, and Ross model, and $\gamma = 1$ is similar to the lognormal volatility considered by Dorthan (1978). The value of $\gamma = 1.5$ is motivated by Chan, Karolyi, Longstaff, and Sanders (1992) in their unrestricted models. Their research suggests that γ may even be as high as 1.5.

To compare the different HJM models under different volatility structures, the initial term structure must be the same. The only difference is the volatility parameter. RS have selected the volatility parameters for different volatility structures so as to make the initial volatility of the forward rate be the same. They also selected the HJM model under a specific volatility structure as the benchmark, then used the implied method to select the different volatility parameters for other HJM models with the least squares from the benchmark as the criterion. Their conclusion is quite clear that the HJM models under different volatility structures are significantly different, therefore a misspecified model is going to result in big errors in pricing the interest rate derivatives.

In this paper, we use a different method to compare different HJM models in pricing the interest derivatives. We set the HJM model when $\gamma = 0$ as the benchmark model. We also set the at the money option as the benchmark option. We first use the benchmark model to calculate the benchmark option price. Then we use the benchmark option price from the benchmark model to calculate the implied volatility parameter for different volatility structures. The logic is that if the different HJM models are similar in pricing the interest rate derivatives, then all other option prices from different HJM models should not deviate from each other too much when the benchmark option's price is the same for all the different HJM models.

The volatility parameter for the benchmark model takes three different values: 0.005, 0.01, and 0.015. k takes two different values:0.01 and 0.05. Such selection for volatility parameter and k is reasonable and consistent with empirical experience (RS 1995b)

Assuming the initial term structure is flat at 10%. Table 1 shows 6-month call option prices from different HJM models under different parameter values and different exercise prices. The options are American style and written on 15-year to maturity discount bond. Table 1.1 presents the percentage difference from the benchmark $\gamma = 0$.

From Table 1 and 1.1 we find that when the at the money option price is fixed at the same level for all different HJM models, the deviations of other HJM models from the benchmark is increasing with γ values. And the difference also depends on the exercise prices. In most the cases, the farther of the exercise price away from the at the money exercise price, the larger the difference of other HJM models from the benchmark model. The deviations for in-the-money options are positive and negative for the out-of-the-money options. The deviations decrease when the parameter κ increases from 0.01 to 0.05. The percentage changes of the out-of-the-money options are relatively larger than in-the-money options.

Table 2 presents the prices of 6-month call options on bonds with 10% coupon rate. The time to maturity of the bond is 15 years. The coupon payments are made semiannually. Table 2.1 provides the percentage change of deviations from the benchmark model. Table 3 collects the 6-month American call option prices on spot rates. Table 3.1 provides the percentage change of the deviations from the benchmark model. Similar conclusions can be drawn from Table 3 and 3.1. Comparing with the options on discount bonds, the percentage change of deviations for the deeply out-of-the-money options on both the coupon bonds and spot rates are much larger.

Table 4, 5, and 6 provides the 2- year American option prices on a discount bond, a 10% coupon bond, and spot rates respectively. Table 4.1, 5.1, and 6.1 are the corresponding percentages of deviations respectively. The conclusions about the different HJM models in pricing the options are very similar here to the above. Making comparison between the 6-month options and the 2-year options, we find that the longer the time to maturity, the larger the absolute value of deviations from the benchmark model. But the percentage changes may not be bigger.

To sum, the difference among the different HJM models in pricing the options on discount bonds, coupon bonds, and spot rates are quite significant, especially when the volatility of the forward rate is relatively high. Incorrect selection of a HJM model in pricing the options will lead to big errors, especially for the deeply out-of-the-money options. This is very similar to the conclusions drawn by RS.

5. American Option vs. European Options

Since Black-Scholes's Model for the European options, lots of research has done to price the American options. The early exercise property of the American options makes it worth more than the European options. Interestingly, Robert Merton (1973) proved theoretically that the American option and European option are indifferent if they are written on stocks without dividend payments under the assumption that the interest rate is non-stochastic. Because interest rate derivatives are strongly related to interest rates, which follow a stochastic process, it seems that the American options and European options are quite different. This might be true in most cases. But it is also possible that the difference between European and American options is trivial in some cases. The results in the following show that the American option and European option on the discount bonds are priced the same. Therefore the early exercise right is worthless if the option is written on a discount bond. This is really surprising for the bond price is very sensitive to the interest rate, which here follows a stochastic process.

Assume the initial term structure is flat at 10%. Table 7 presents the comparison between 6-month American and European options on 15 –year-to-maturity discount bond with different HJM models under a wide range of volatility parameter. Table 8 provides comparison of 2-year American and European option. From Table 7 and 8, we note the American and European options on discount bonds are priced the same. The early exercise right is worthless for discount bonds. We also provide 6-month American and European option prices on bonds with coupon rate 10% on Table 9. Assuming that the coupon payments are made semiannually, there is no coupon payment during the life of the option. Table 9 shows that the 6-month American option and European option have the same price. This property is very similar to the options written on stocks. If there is no dividend payments during the life of the option, the option price is the same for both American style and European style.

Table 10 presents 2-year American and European option prices on bonds with 10% coupon rate. The difference between American and European style is quite significant. For example, when γ=0.5 and σ=0.015, at-the–money American call option is \$50.01 and European \$44.66. The American option is 11.98% higher than the European option.

To sum, the options on discount bonds are priced the same for both American style and European style. With regard to the options on coupon bonds, the American and European style are priced significantly different from each other when there are coupon payments during the life of the option. If during the life of option there is no coupon payments, the American and European are also the same.

Table 11 and 12 provides comparisons between 6-month and 2-year American and European options respectively on spot rates with different HJM models under different volatility parameter values. The difference between American and European is significant. For example, for the 6-moth options, when $\gamma=1$ and $\sigma=0.05$, at-the-option is priced $5.03 for American style and $4.86 for European style. The American style is 3.50% higher than the European style.

6. Conclusion

This paper makes research on the HJM model modified by RS, then makes comparisons among different HJM models with the different forward rate volatility structures. The restrictions put on the HJM model by RS has successfully removed part of the non-Markovian property, which makes the application of the lattice method possible. The lattice method can price the short-term interest rate derivatives like the options on discount bonds, coupon bonds, and spot rates. But the lattice method created by RS still has many shortcomings. The major problem is that the lattice method depends on a specific transformation (8), and transformation in fact put additional strong restrictions on the new variable $y(t)$. However the lattice created by (12) for $y(t)$ does not guarantee the restriction be followed. Therefore the lattice method loses its power in pricing the long-term options unless the volatility of the forward rate is very low. But the failure of the lattice method in some cases is due to the specific transformation, rather than to the specific volatility structure put on the HJM model by RS.

The specific comparison method used here gives us the same conclusion as RS about the different HJM models. The deviations from the benchmark model increase with the elasticity parameter γ and the volatility of the forward rate. Incorrect selection of the HJM models could lead to big errors in pricing the interest rate derivatives like the options on discount bonds, coupon bonds, and spot rates. The differences among the HJM models are especially big in pricing the out-of-the-money options.

Through comparison between the American style and European style options on discount bonds, and coupon bonds, we find that the early exercise right on the discount bond is worthless. Therefore the American and European option are priced the same on discount bonds. With regard to the options on the coupon bonds, the American and European option are priced the same if there are no coupon payments during the life of the option, otherwise the difference is quite significant. This property is very similar to the options on stock price under the assumption that interest rate is not stochastic. The American style is worth more than the European style for the options on spot rates.

Table 1 Comparison of Prices of 6-Month Options on 15-year to Maturity Discount Bonds

σ	K	γ	0.950X	0.975X	1.000X	1.025X	1.050X
0.005	0.01	0.0	11.90	7.55	4.24	2.07	0.86
		0.5	11.93	7.58	4.24	2.05	0.85
		1.0	11.96	7.60	4.24	2.03	0.83
		1.5	11.98	7.62	4.24	2.01	0.80
	0.05	0.0	11.43	6.71	3.21	1.19	0.33
		0.5	11.44	6.74	3.21	1.19	0.32
		1.0	11.46	6.75	3.21	1.18	0.30
		1.5	11.48	6.77	3.21	1.16	0.28
0.010	0.01	0.0	15.01	11.45	8.48	6.08	4.22
		0.5	15.10	11.51	8.48	6.02	4.11
		1.0	15.17	11.55	8.48	5.98	4.05
		1.5	15.24	11.58	8.48	5.95	3.99
	0.05	0.0	13.35	9.51	6.41	4.08	2.44
		0.5	13.43	9.54	6.41	4.07	2.41
		1.0	13.49	9.57	6.41	4.03	2.35
		1.5	13.55	9.61	6.41	4.00	2.28
0.015	0.01	0.0	18.76	15.52	12.71	10.26	8.20
		0.5	18.79	15.62	12.71	10.18	8.11
		1.0	18.92	15.67	12.71	10.12	8.00
		1.5	19.02	15.73	12.71	10.06	7.89
	0.05	0.0	15.99	12.55	9.62	7.20	5.27
		0.5	16.13	12.64	9.62	7.09	5.16
		1.0	16.23	12.69	9.62	7.05	5.06
		1.5	16.34	12.75	9.62	6.98	4.95

X is the six-month forward price on the underlying bond. The notional principal for all options is set at $1,000. The volatilities of 0.005, 0.01, and 0.015 refers to the instantaneous volatility parameter, in the $\gamma = 0$ model, which is the benchmark model. The time to maturity of the bond is 15 years. The volatility parameter σ for all other γ is selected so as to make at-the-money option prices equal to the benchmark at-the-money option prices.

Table 1.1 Comparison of Percentage Differences of 6-Month Call Options on 15-Year to Maturity Discount Bond relative to The $\gamma = 0$ Benchmark

σ	K	γ	0.950X	0.975X	1.000X	1.025X	1.050X
0.005	0.01	0.5	0.25	0.40	0	-0.97	-1.16
		1.0	0.50	0.66	0	-1.93	-3.49
		1.5	0.67	0.93	0	-2.90	-6.98
	0.05	0.5	0.09	0.45	0	0.00	-3.03
		1.0	0.26	0.60	0	-0.84	-9.09
		1.5	0.44	0.89	0	-2.52	-15.15
0.01	0.01	0.5	0.60	0.52	0	-0.99	-2.61
		1.0	1.07	0.87	0	-1.64	-4.03
		1.5	1.53	1.14	0	-2.14	-5.45
	0.05	0.5	0.60	0.32	0	-0.25	-1.23
		1.0	1.05	0.63	0	-1.23	-3.69
		1.5	1.50	1.05	0	-1.96	-6.56
0.015	0.01	0.5	0.16	0.64	0	-0.78	-1.10
		1.0	0.85	0.97	0	-1.36	-2.44
		1.5	1.39	1.35	0	-1.95	-3.78
	0.05	0.5	0.88	0.72	0	-1.53	-2.09
		1.0	1.50	1.12	0	-2.08	-3.98
		1.5	2.19	1.59	0	-3.06	-6.07

X is the six-month forward price on the underlying bond. The notional principal for all options is set at $1,000. The volatilities of 0.005, 0.01, and 0.015 refers to the instantaneous volatility parameter, in the $\gamma = 0$ model, which is the benchmark model. The time to maturity of the bond is 15 years. The volatility parameter σ for all other γ is selected so as to make at-the-money option prices equal to the benchmark at-the-money option prices.

Table 2 Comparison of Prices of 6-Month Options on 15-year to Maturity Coupon Bonds

sigma	K	gamma	0.950X	0.975X	1.000X	1.025X	1.050X
0.005	0.01	0.0	49.20	26.47	9.80	2.17	0.26
		0.5	49.22	26.55	9.80	2.15	0.23
		1.0	49.24	26.60	9.80	2.09	0.20
		1.5	49.27	26.66	9.80	2.04	0.18
	0.05	0.0	49.07	25.49	7.91	1.10	0.06
		0.5	49.07	25.54	7.91	1.07	0.05
		1.0	49.08	25.57	7.91	1.03	0.04
		1.5	49.09	25.61	7.91	0.99	0.03
0.01	0.01	0.0	52.72	33.92	19.61	10.06	4.55
		0.5	52.96	34.17	19.61	9.89	4.38
		1.0	53.16	34.32	19.61	9.73	4.16
		1.5	53.37	34.48	19.61	9.57	3.94
	0.05	0.0	50.85	30.78	15.82	6.72	2.32
		0.5	51.01	30.90	15.82	6.61	2.17
		1.0	51.15	31.03	15.82	6.47	2.02
		1.5	51.30	31.16	15.82	6.34	1.86
0.015	0.01	0.0	59.34	42.76	29.40	19.26	11.98
		0.5	59.73	42.85	29.40	19.09	11.67
		1.0	60.11	43.10	29.40	18.85	11.25
		1.5	60.49	43.36	29.40	18.60	10.83
	0.05	0.0	55.28	37.59	23.73	13.82	7.42
		0.5	55.68	37.90	23.73	13.70	6.95
		1.0	56.07	38.13	23.73	13.45	6.60
		1.5	56.46	38.36	23.73	13.22	6.28

X is the six-month forward price on the underlying coupon bond. The notional principal for all options is set at $1,000. The volatilities of 0.005, 0.01, and 0.015 refers to the instantaneous volatility parameter, in the $\gamma = 0$ model, which is the benchmark model. The time to maturity of the bond is 15 years. Coupon rate is assumed to be 10%. The volatility parameter σ for all other γ is selected so as to make at-the-money option prices equal to the benchmark at-the-money option prices.

Table 2.1 Comparison of Percentage Differences of 6-Month Call Options on 15-Year to Maturity Coupon Bond relative to The $\gamma = 0$ Benchmark

σ	K	γ	0.950X	0.975X	1.000X	1.025X	1.050X
0.005	0.01	0.5	0.04	0.30	0.00	-0.92	-11.54
		1.0	0.08	0.49	0.00	-3.69	-23.08
		1.5	0.14	0.72	0.00	-5.99	-30.77
	0.05	0.5	0.00	0.20	0.00	-2.73	-16.67
		1.0	0.02	0.31	0.00	-6.36	-33.33
		1.5	0.04	0.47	0.00	-10.00	-50
0.01	0.01	0.5	0.46	0.74	0.00	-1.69	-3.47
		1.0	0.83	1.18	0.00	-3.28	-8.57
		1.5	1.23	1.65	0.00	-4.87	-13.41
	0.05	0.5	0.31	0.39	0.00	-1.64	-6.47
		1.0	0.59	0.81	0.00	-3.72	-12.93
		1.5	0.88	1.23	0.00	-5.65	-19.83
0.015	0.01	0.5	0.66	0.21	0.00	-0.88	-2.59
		1.0	1.30	0.80	0.00	-2.17	-6.09
		1.5	1.94	1.40	0.00	-3.43	-9.60
	0.05	0.5	0.72	0.82	0.00	-0.87	-6.33
		1.0	1.42	1.42	0.00	-2.68	-11.05
		1.5	2.13	2.05	0.00	-4.34	-15.36

X is the six-month forward price on the underlying coupon bond. The notional principal for all options is set at $1,000. The volatilities of 0.005, 0.01, and 0.015 refers to the instantaneous volatility parameter, in the $\gamma = 0$ model, which is the benchmark model. The time to maturity of the bond is 15 years. Coupon rate is assumed to be 10%. The volatility parameter σ for all other γ is selected so as to make at-the-money option prices equal to the benchmark at-the-money option prices.

Table 3 Comparison of Prices of 6-Month Options on Spot Rates

σ	K	γ	0.950X	0.975X	1.000X	1.025X	1.050X
0.005	0.01	0.0	5.04	2.91	1.35	0.48	0.12
		0.5	5.03	2.91	1.35	0.49	0.12
		1.0	5.03	2.90	1.35	0.50	0.13
		1.5	5.03	2.90	1.35	0.50	0.14
	0.05	0.0	5.04	2.91	1.35	0.47	0.11
		0.5	5.04	2.91	1.35	0.48	0.12
		1.0	5.04	2.91	1.35	0.49	0.13
		1.5	5.03	2.90	1.35	0.50	0.14
0.01	0.01	0.0	5.82	4.10	2.71	1.67	0.95
		0.5	5.81	4.10	2.71	1.70	1.00
		1.0	5.78	4.08	2.71	1.72	1.02
		1.5	5.76	4.07	2.71	1.73	1.05
	0.05	0.0	5.83	4.09	2.70	1.66	0.94
		0.5	5.82	4.09	2.70	1.69	0.98
		1.0	5.79	4.08	2.70	1.70	1.01
		1.5	5.76	4.06	2.70	1.72	1.03
0.015	0.01	0.0	6.96	5.40	4.07	2.97	2.10
		0.5	6.94	5.39	4.07	3.02	2.17
		1.0	6.89	5.37	4.07	3.04	2.22
		1.5	6.85	5.34	4.07	3.07	2.27
	0.05	0.0	6.96	5.39	4.05	2.95	2.07
		0.5	6.94	5.38	4.05	2.99	2.15
		1.0	6.89	5.36	4.05	3.02	2.19
		1.5	6.84	5.33	4.05	3.04	2.24

X is the six-month forward interest rate. The notional principal for all options is set at \$1,000. The volatilities of 0.005, 0.01, and 0.015 refers to the instantaneous volatility parameter, in the $\gamma = 0$ model, which is the benchmark model. The volatility parameter σ for all other γ is selected so as to make at-the-money option prices equal to the benchmark at-the-money option prices.

Table 3.1 Comparison of Percentage Differences of 6-Month Call Options on Spot Rate Relative to the $\gamma = 0$ Benchmark

σ	K	γ	0.950X	0.975X	1.000X	1.025X	1.050X
0.005	0.01	0.5	-0.20	0.00	0.00	2.08	0.00
		1.0	-0.20	-0.34	0.00	4.17	8.33
		1.5	-0.20	-0.34	0.00	4.17	16.67
	0.05	0.5	0.00	0.00	0.00	2.13	9.09
		1.0	0.00	0.00	0.00	4.26	18.18
		1.5	-0.20	-0.34	0.00	6.38	27.27
0.01	0.01	0.5	-0.17	0.00	0.00	1.80	5.26
		1.0	-0.69	-0.49	0.00	2.99	7.37
		1.5	-1.03	-0.73	0.00	3.59	10.53
	0.05	0.5	-0.17	0.00	0.00	1.81	4.26
		1.0	-0.69	-0.24	0.00	2.41	7.45
		1.5	-1.20	-0.73	0.00	3.61	9.57
0.015	0.01	0.5	-0.29	-0.19	0.00	1.68	3.33
		1.0	-1.01	-0.56	0.00	2.36	5.71
		1.5	-1.58	-1.11	0.00	3.37	8.10
	0.05	0.5	-0.29	-0.19	0.00	1.36	3.86
		1.0	-1.01	-0.56	0.00	2.37	5.80
		1.5	-1.72	-1.11	0.00	3.05	8.21

X is the six-month forward interest rate. The notional principal for all options is set at $1,000. The volatilities of 0.005, 0.01, and 0.015 refers to the instantaneous volatility parameter, in the $\gamma = 0$ model, which is the benchmark model. The volatility parameter σ for all other γ is selected so as to make at-the-money option prices equal to the benchmark at-the-money option prices.

Table 4 Comparison of Prices of 2-Year Options on 15-Year to Maturity Discount bonds

σ	K	γ	0.950X	0.975X	1.000X	1.025X	1.050X
0.005	0.01	0.0	14.28	10.62	7.60	5.22	3.44
		0.5	14.36	10.66	7.60	5.20	3.37
		1.0	14.43	10.69	7.60	5.16	3.30
		1.5	14.49	10.73	7.60	5.12	3.23
	0.05	0.0	12.85	8.87	5.73	3.44	1.91
		0.5	12.92	8.93	5.73	3.39	1.86
		1.0	12.97	8.97	5.73	3.36	1.80
		1.5	13.03	9.00	5.73	3.32	1.74
0.01	0.01	0.0	21.04	17.95	15.18	12.73	10.61
		0.5	21.13	18.01	15.18	12.61	10.44
		1.0	21.30	18.09	15.18	12.55	10.29
		1.5	21.48	18.18	15.18	12.50	10.16
	0.05	0.0	17.62	14.31	11.46	9.01	7.00
		0.5	17.78	14.41	11.46	8.93	6.80
		1.0	17.92	14.48	11.46	8.85	6.68
		1.5	18.06	14.56	11.46	8.78	6.55
0.015	0.01	0.0	28.20	25.35	22.72	20.32	18.15
		0.5	28.39	25.40	22.72	20.13	17.90
		1.0	28.64	25.56	22.72	20.08	17.65
		1.5	28.90	25.73	22.72	20.03	17.42
	0.05	0.0	22.91	19.89	17.16	14.73	12.56
		0.5	23.10	19.95	17.16	14.53	12.34
		1.0	23.12	20.10	17.16	14.48	12.12
		1.5	23.54	20.25	17.16	14.40	11.90

X is the 2-year forward price on the underlying bond. The notional principal for all options is set at $1,000. The volatilities of 0.005, 0.01, and 0.015 refers to the instantaneous volatility parameter, in the $\gamma = 0$ model, which is the benchmark model. The time to maturity of the bond is 15 years. The volatility parameter σ for all other γ is selected so as to make at-the-money option prices equal to the benchmark at-the-money option prices.

Table 4.1 Comparison of Percentage Differences of 2-Year Call Options on 15-Year to Maturity Discount Bonds Relative to the $\gamma = 0$ Benchmark

σ	K	γ	0.950X	0.975X	1.000X	1.025X	1.050X
0.005	0.01	0.5	0.56	0.38	0.00	-0.38	-2.32
		1.0	1.05	0.66	0.00	-1.15	-4.35
		1.5	1.47	1.04	0.00	-1.92	-6.38
	0.05	0.5	0.54	0.68	0.00	-1.45	-2.62
		1.0	0.93	1.13	0.00	-2.33	-5.76
		1.5	1.40	1.47	0.00	-3.49	-8.90
0.01	0.01	0.5	0.43	0.33	0.00	-0.94	-1.61
		1.0	1.19	0.78	0.00	-1.41	-3.02
		1.5	2.09	1.28	0.00	-1.81	-4.24
	0.05	0.5	0.91	0.70	0.00	-0.89	-2.71
		1.0	1.70	1.19	0.00	-1.78	-4.71
		1.5	2.50	1.75	0.00	-2.55	-6.43
0.015	0.01	0.5	0.67	0.20	0.00	-0.94	-1.38
		1.0	1.56	0.83	0.00	-1.18	-2.75
		1.5	2.48	1.50	0.00	-1.43	-4.02
	0.05	0.5	0.83	0.30	0.00	-1.36	-1.75
		1.0	0.92	1.06	0.00	-1.70	-3.50
		1.5	2.75	1.81	0.00	-2.24	-5.25

X is the 2-year forward price on the underlying bond. The notional principal for all options is set at $1,000. The volatilities of 0.005, 0.01, and 0.015 refers to the instantaneous volatility parameter, in the $\gamma = 0$ model, which is the benchmark model. The time to maturity of the bond is 15 years. The volatility parameter σ for all other γ is selected so as to make at-the-money option prices equal to the benchmark at-the-money option prices.

Table 5 Comparison of Prices of 2-Year Options on 15-year to Maturity Coupon bond

σ	K	γ	0.950X	0.975X	1.000X	1.025X	1.050X
0.005	0.01	0.0	50.23	31.02	17.01	8.20	3.44
		0.5	49.86	31.00	17.01	8.13	3.26
		1.0	50.02	31.13	17.01	7.99	3.10
		1.5	50.19	31.26	17.01	7.86	2.94
	0.05	0.0	49.18	28.54	13.71	5.31	1.62
		0.5	48.64	28.41	13.71	5.20	1.49
		1.0	48.75	28.53	13.71	5.08	1.38
		1.5	48.86	28.65	13.71	4.97	1.26
0.01	0.01	0.0	61.78	46.55	34.15	24.38	16.94
		0.5	62.08	46.70	34.15	24.14	16.45
		1.0	62.57	47.01	34.15	23.84	15.89
		1.5	63.35	47.44	34.15	23.49	15.29
	0.05	0.0	57.10	40.63	27.59	17.86	11.04
		0.5	57.28	40.82	27.59	17.62	10.54
		1.0	57.75	41.09	27.59	17.32	10.05
		1.5	58.23	41.38	27.59	17.03	9.59
0.015	0.01	0.0	76.36	62.79	51.14	41.29	33.00
		0.5	77.06	63.21	51.14	40.83	32.02
		1.0	77.92	63.66	51.14	40.38	31.19
		1.5	78.64	64.04	51.14	39.85	30.38
	0.05	0.0	68.22	53.65	41.39	31.40	23.34
		0.5	68.81	53.97	41.39	30.94	22.50
		1.0	69.62	54.40	41.39	30.49	21.67
		1.5	70.46	54.85	41.39	30.04	20.87

X is the 2-year forward price on the underlying coupon bond. The notional principal for all options is set at $1,000. The volatilities of 0.005, 0.01, and 0.015 refers to the instantaneous volatility parameter, in the $\gamma = 0$ model, which is the benchmark model. The time to maturity of the bond is 15 years. Coupon rate is assumed to be 10%. The volatility parameter σ for all other γ is selected so as to make at-the-money option prices equal to the benchmark at-the-money option prices.

Table 5.1 Comparison of Percentage Differences of 2-Year Call Options on 15-Year to Maturity coupon Bonds Relative to the $\gamma = 0$ Benchmark

σ	K	γ	0.950X	0.975X	1.000X	1.025X	1.050X
0.005	0.01	0.5	-0.74	-0.06	0.00	-0.85	-5.23
		1.0	-0.42	0.35	0.00	-2.56	-9.88
		1.5	-0.08	0.77	0.00	-4.15	-14.53
	0.05	0.5	-1.10	-0.46	0.00	-2.07	-8.02
		1.0	-0.87	-0.04	0.00	-4.33	-14.81
		1.5	-0.65	-0.39	0.00	-6.40	-22.22
0.01	0.01	0.5	0.49	0.32	0.00	-0.98	-2.89
		1.0	1.28	0.99	0.00	-2.21	-5.67
		1.5	2.54	1.91	0.00	-3.65	-9.74
	0.05	0.5	0.32	0.47	0.00	-1.34	-4.53
		1.0	1.14	1.13	0.00	-3.02	-8.97
		1.5	1.98	1.85	0.00	-4.65	-13.13
0.015	0.01	0.5	0.92	0.67	0.00	-1.11	-2.97
		1.0	2.04	1.39	0.00	-2.20	-5.48
		1.5	3.00	1.99	0.00	-3.49	-7.94
	0.05	0.5	0.86	0.60	0.00	-1.46	-3.60
		1.0	2.05	1.42	0.00	-2.90	-7.16
		1.5	3.28	2.24	0.00	-4.33	-10.58

X is the 2-year forward price on the underlying coupon bond. The notional principal for all options is set at $1,000. The volatilities of 0.005, 0.01, and 0.015 refers to the instantaneous volatility parameter, in the $\gamma = 0$ model, which is the benchmark model. The time to maturity of the bond is 15 years. Coupon rate is assumed to be 10%. The volatility parameter σ for all other γ is selected so as to make at-the-money option prices equal to the benchmark at-the-money option prices.

Table 6 Comparison of Prices of 2-Year Options on spot rate

σ	K	γ	0.950X	0.975X	1.000X	1.025X	1.050X
0.005	0.01	0.0	5.48	3.78	2.46	1.49	0.84
		0.5	5.47	3.77	2.46	1.51	0.87
		1.0	5.44	3.76	2.46	1.53	0.89
		1.5	5.42	3.74	2.46	1.54	0.91
	0.05	0.0	5.51	3.79	2.43	1.45	0.79
		0.5	5.50	3.78	2.43	1.47	0.82
		1.0	5.48	3.76	2.43	1.49	0.84
		1.5	5.46	3.75	2.43	1.50	0.87
0.01	0.01	0.0	7.58	6.17	4.93	3.88	2.99
		0.5	7.54	6.14	4.93	3.92	3.07
		1.0	7.48	6.11	4.93	3.95	3.12
		1.5	7.42	6.08	4.93	3.98	3.18
	0.05	0.0	7.59	6.15	4.88	3.81	2.91
		0.5	7.55	6.12	4.88	3.85	2.98
		1.0	7.49	6.09	4.88	3.88	3.04
		1.5	7.44	6.06	4.88	3.91	3.10
0.015	0.01	0.0	9.95	8.62	7.42	6.33	5.36
		0.5	9.88	8.61	7.42	6.41	5.46
		1.0	9.79	8.56	7.42	6.45	5.55
		1.5	9.70	8.52	7.42	6.50	5.63
	0.05	0.0	9.93	8.57	7.34	6.23	5.25
		0.5	9.86	8.56	7.34	6.31	5.34
		1.0	9.77	8.51	7.34	6.35	5.43
		1.5	9.69	8.47	7.34	6.40	5.51

X is the 2-year forward interest rate. The notional principal for all options is set at \$1,000. The volatilities of 0.005, 0.01, and 0.015 refers to the instantaneous volatility parameter, in the $\gamma = 0$ model, which is the benchmark model. The volatility parameter σ for all other γ is selected so as to make at-the-money option prices equal to the benchmark at-the-money option prices.

Table 6.1 Comparison of Percentage Differences of 2-Year Call Options on Spot Rates Relative to the $\gamma = 0$ Benchmark

σ	K	γ	0.950X	0.975X	1.000X	1.025X	1.050X
0.005	0.01	0.5	-0.18	-0.26	0.00	-1.34	3.57
		1.0	-0.73	-0.53	0.00	-2.68	5.95
		1.5	-1.09	-1.06	0.00	-3.36	9.52
	0.05	0.5	-0.18	-0.26	0.00	1.38	3.80
		1.0	-0.54	-0.79	0.00	2.76	6.33
		1.5	-0.91	-1.06	0.00	3.45	10.13
0.01	0.01	0.5	-0.53	-0.49	0.00	1.03	2.68
		1.0	-1.32	-0.97	0.00	1.80	4.35
		1.5	-2.11	-1.46	0.00	2.58	6.35
	0.05	0.5	-0.53	-0.49	0.00	1.05	2.41
		1.0	-1.32	-0.98	0.00	1.84	4.47
		1.5	-1.98	-1.46	0.00	2.62	6.53
0.015	0.01	0.5	-0.70	-0.12	0.00	1.26	1.87
		1.0	-1.61	-0.70	0.00	1.90	3.54
		1.5	-2.51	-1.16	0.00	2.69	5.04
	0.05	0.5	-0.70	-0.12	0.00	1.28	1.71
		1.0	-1.61	-0.70	0.00	1.93	3.43
		1.5	-2.42	-1.17	0.00	2.73	4.95

X is the 2-year forward interest rate. The notional principal for all options is set at \$1,000. The volatilities of 0.005, 0.01, and 0.015 refers to the instantaneous volatility parameter, in the $\gamma = 0$ model, which is the benchmark model. The volatility parameter σ for all other γ is selected so as to make at-the-money option prices equal to the benchmark at-the-money option prices.

Table 7 Comparison of 6-Month American and European Options on 15-Year to Maturity discount bonds

γ	σ	α	American	European	Difference	%
0.0	0.005	0.95	11.90	11.90	0.00	0.00
		1.00	4.24	4.24	0.00	0.00
		1.05	0.87	0.87	0.00	0.00
	0.01	0.95	15.02	15.02	0.00	0.00
		1.00	8.48	8.48	0.00	0.00
		1.05	4.21	4.21	0.00	0.00
	0.015	0.95	18.74	18.74	0.00	0.00
		1.00	12.72	12.72	0.00	0.00
		1.05	8.19	8.19	0.00	0.00
0.5	0.015	0.95	11.80	11.80	0.00	0.00
		1.00	4.01	4.01	0.00	0.00
		1.05	0.71	0.71	0.00	0.00
	0.03	0.95	14.71	14.71	0.00	0.00
		1.00	8.03	8.03	0.00	0.00
		1.05	3.74	3.74	0.00	0.00
	0.045	0.95	18.21	18.21	0.00	0.00
		1.00	12.05	12.05	0.00	0.00
		1.05	7.49	7.49	0.00	0.00
1.0	0.05	0.95	11.95	11.95	0.00	0.00
		1.00	4.23	4.23	0.00	0.00
		1.05	0.82	0.82	0.00	0.00
	0.1	0.95	15.15	15.15	0.00	0.00
		1.00	8.46	8.46	0.00	0.00
		1.05	4.03	4.03	0.00	0.00
	0.15	0.95	18.87	18.87	0.00	0.00
		1.00	12.68	12.68	0.00	0.00
		1.05	7.97	7.97	0.00	0.00
1.5	0.15	0.95	11.84	11.84	0.00	0.00
		1.00	4.01	4.01	0.00	0.00
		1.05	0.66	0.66	0.00	0.00
	0.3	0.95	14.83	14.83	0.00	0.00
		1.00	8.02	8.02	0.00	0.00
		1.05	3.61	3.61	0.00	0.00
	0.45	0.95	18.37	18.37	0.00	0.00
		1.00	12.01	12.01	0.00	0.00
		1.05	7.24	7.24	0.00	0.00

X is the six-month forward price on the underlying discount bond. The notional principal for all options is set at $1,000. The exercise price is αX. The time to maturity of the bond is 15 years.

Table 8 Comparison of 2-Year American and European Options on 15-year to Maturity discount bonds American and European options

γ	σ	α	American	European	Difference	%
0.0	0.005	0.95	14.28	14.28	0.00	0.00
		1.00	7.60	7.60	0.00	0.00
		1.05	3.45	3.45	0.00	0.00
	0.01	0.95	21.04	21.04	0.00	0.00
		1.00	15.18	15.18	0.00	0.00
		1.05	10.61	10.61	0.00	0.00
	0.015	0.95	28.20	28.20	0.00	0.00
		1.00	22.72	22.72	0.00	0.00
		1.05	18.15	18.15	0.00	0.00
0.5	0.015	0.95	14.02	14.02	0.00	0.00
		1.00	7.20	7.20	0.00	0.00
		1.05	3.04	3.04	0.00	0.00
	0.03	0.95	20.41	20.41	0.00	0.00
		1.00	14.38	14.38	0.00	0.00
		1.05	9.64	9.64	0.00	0.00
	0.045	0.95	27.17	27.17	0.00	0.00
		1.00	21.45	21.45	0.00	0.00
		1.05	16.63	16.63	0.00	0.00
1.0	0.05	0.95	14.41	14.41	0.00	0.00
		1.00	7.59	7.59	0.00	0.00
		1.05	3.29	3.29	0.00	0.00
	0.10	0.95	21.23	21.23	0.00	0.00
		1.00	15.10	15.10	0.00	0.00
		1.05	10.21	10.21	0.00	0.00
	0.15	0.95	28.33	28.33	0.00	0.00
		1.00	22.42	22.42	0.00	0.00
		1.05	17.37	17.37	0.00	0.00
1.5	0.15	0.95	14.14	14.14	0.00	0.00
		1.00	7.20	7.20	0.00	0.00
		1.05	2.92	2.92	0.00	0.00
	0.30	0.95	20.63	20.63	0.00	0.00
		1.00	14.30	14.30	0.00	0.00
		1.05	9.32	9.32	0.00	0.00
	0.45	0.95	27.41	27.41	0.00	0.00
		1.00	21.25	21.25	0.00	0.00
		1.05	16.01	16.01	0.00	0.00

X is the 2-year forward price on the underlying discount bond. The notional principal for all options is set at $1,000. The exercise price is αX. The time to maturity of the bond is 15 years.

Table 9 Comparison of 6-Month American and European Options on 15-Year to Maturity Coupon bonds

γ	σ	α	American	European	Difference	%
0.0	0.005	0.95	49.20	49.20	0.00	0.00
		1.00	9.80	9.80	0.00	0.00
		1.05	0.26	0.26	0.00	0.00
	0.010	0.95	52.71	52.71	0.00	0.00
		1.00	19.61	19.61	0.00	0.00
		1.05	4.54	4.54	0.00	0.00
	0.015	0.95	59.34	59.34	0.00	0.00
		1.00	29.40	29.40	0.00	0.00
		1.05	11.98	11.98	0.00	0.00
0.5	0.015	0.95	49.16	49.16	0.00	0.00
		1.00	9.27	9.27	0.00	0.00
		1.05	0.16	0.16	0.00	0.00
	0.03	0.95	52.36	52.36	0.00	0.00
		1.00	18.56	18.56	0.00	0.00
		1.05	3.71	3.71	0.00	0.00
	0.045	0.95	58.60	58.60	0.00	0.00
		1.00	27.85	27.85	0.00	0.00
		1.05	10.40	10.40	0.00	0.00
1.0	0.05	0.95	49.23	49.23	0.00	0.00
		1.00	9.77	9.77	0.00	0.00
		1.05	0.20	0.20	0.00	0.00
	0.10	0.95	53.13	53.13	0.00	0.00
		1.00	19.55	19.55	0.00	0.00
		1.05	4.12	4.12	0.00	0.00
	0.15	0.95	60.04	60.04	0.00	0.00
		1.00	29.31	29.31	0.00	0.00
		1.05	11.17	11.17	0.00	0.00
1.5	0.15	0.95	49.20	49.20	0.00	0.00
		1.00	9.27	9.27	0.00	0.00
		1.05	0.13	0.13	0.00	0.00
	0.30	0.95	52.73	52.73	0.00	0.00
		1.00	18.54	18.54	0.00	0.00
		1.05	3.28	3.28	0.00	0.00
	0.45	0.95	59.27	59.27	0.00	0.00
		1.00	27.78	27.78	0.00	0.00
		1.05	9.55	9.55	0.00	0.00

X is the six-month forward price on the underlying coupon bond. The notional principal for all options is set at $1,000. The exercise price is αX. The time to maturity of the bond is 15 years. Coupon rate is 10%.

Table 10 Comparison of 2-Year American and European Options on 15-Year to Maturity Coupon Bonds

γ	σ	α	American	European	Difference	%
0	0.005	0.95	50.23	44.93	5.30	11.80
		1.00	17.01	16.00	1.01	6.31
		1.05	3.44	3.33	0.11	3.30
	0.010	0.95	61.78	56.44	5.34	9.46
		1.00	34.15	31.99	2.16	6.75
		1.05	16.94	16.15	9.79	4.89
	0.015	0.95	76.36	70.34	6.02	8.56
		1.00	51.14	47.87	3.27	6.83
		1.05	33.00	31.29	1.71	5.47
0.5	0.015	0.95	50.01	44.66	5.35	11.98
		1.00	16.19	15.17	1.02	6.72
		1.05	2.79	2.70	0.09	3.33
	0.03	0.95	60.90	55.54	5.36	9.65
		1.00	32.43	30.29	2.13	7.03
		1.05	14.78	14.06	0.72	5.12
	0.045	0.95	74.67	68.71	5.96	8.67
		1.00	48.43	45.24	3.19	7.05
		1.05	29.43	27.85	1.58	5.67
1	0.05	0.95	50.54	45.23	5.31	11.74
		1.00	17.05	15.97	1.08	6.76
		1.05	3.09	2.99	0.10	3.34
	0.1	0.95	62.77	57.32	5.45	9.51
		1.00	34.08	31.83	2.25	7.07
		1.05	15.72	14.94	0.78	5.22
	0.15	0.95	77.54	71.42	6.12	8.57
		1.00	50.71	47.34	3.37	7.12
		1.05	30.71	29.03	1.68	5.79
1.5	0.15	0.95	50.23	44.96	5.28	11.74
		1.00	16.17	15.15	1.02	6.73
		1.05	2.46	2.38	0.08	3.36
	0.30	0.95	61.74	56.37	5.37	9.53
		1.00	32.28	30.15	2.13	7.06
		1.05	13.72	13.05	0.67	5.13
	0.45	0.95	75.90	69.93	5.97	8.54
		1.00	48.05	44.86	3.19	7.11
		1.05	27.46	25.95	1.51	5.82

X is the 2-year forward price on the underlying coupon bond. The notional principal for all options is set at $1,000. The exercise price is αX. The time to maturity of the bond is 15 years. Coupon rate is 10%.

Table 11 Comparison of 6-Month American and European Options on Spot Rate

γ	σ	α	American	European	Difference	%
0.0	0.005	0.95	5.04	4.87	0.17	3.49
		1.00	1.35	1.34	0.01	0.75
		1.05	0.12	0.12	0.00	0.00
	0.010	0.95	5.82	5.70	0.12	2.11
		1.00	2.71	2.67	0.04	1.50
		1.05	0.95	6.94	0.01	1.06
	0.015	0.95	6.96	6.84	0.12	1.75
		1.00	4.07	4.01	0.06	1.50
		1.05	2.10	2.08	0.02	0.96
0.5	0.015	0.95	5.02	4.85	0.17	3.51
		1.00	1.28	1.27	0.01	0.79
		1.05	0.10	0.10	0.00	0.00
	0.03	0.95	5.69	5.57	0.12	2.15
		1.00	2.56	2.53	0.04	1.58
		1.05	0.87	0.87	0.00	0.00
	0.045	0.95	6.74	6.61	0.13	1.97
		1.00	3.85	3.80	0.05	1.32
		1.05	1.96	1.94	0.02	1.03
1.0	0.05	0.95	5.03	4.86	0.17	3.50
		1.00	1.35	1.33	0.02	1.50
		1.05	0.13	0.13	0.00	0.00
	0.10	0.95	5.77	5.65	0.12	2.12
		1.00	2.71	2.67	0.04	1.50
		1.05	1.01	1.00	0.01	1.00
	0.15	0.95	6.87	6.75	0.12	1.78
		1.00	4.06	4.00	0.06	1.50
		1.05	2.19	2.17	0.02	0.92
1.5	0.15	0.95	5.01	4.84	0.17	3.51
		1.00	1.28	1.27	0.01	0.79
		1.05	0.11	0.11	0.00	0.00
	0.30	0.95	5.64	5.52	0.12	2.17
		1.00	2.57	2.53	0.04	1.58
		1.05	0.92	0.92	0.00	0.00
	0.45	0.95	6.65	6.53	0.12	1.84
		1.00	3.85	3.80	0.05	1.32
		1.05	2.05	2.03	0.02	0.99

X is the six-month forward interest rate. The notional principal for all options is set at \$1,000. The exercise price is αX. The time to maturity of the bond is 15 years.

Table 12 Comparison of 2-Year American and European Options on Spot Rate

γ	σ	α	American	European	Difference	%
0.0	0.005	0.95	5.48	4.89	0.59	12.07
		1.00	2.46	2.28	0.18	7.89
		1.05	0.84	0.80	0.04	5.00
	0.01	0.95	7.58	6.91	0.67	9.70
		1.00	4.93	4.57	0.36	7.88
		1.05	2.99	2.81	0.18	6.41
	0.015	0.95	9.95	9.10	0.85	9.34
		1.00	7.42	6.85	0.57	8.32
		1.05	5.36	5.00	0.36	7.20
0.5	0.015	0.95	5.38	4.78	0.60	12.55
		1.00	2.33	2.16	0.17	7.87
		1.05	0.77	0.73	0.04	5.48
	0.03	0.95	7.29	0.63	0.66	9.95
		1.00	4.67	4.32	0.35	8.10
		1.05	2.81	2.64	0.17	6.44
	0.045	0.95	9.50	8.67	0.83	9.57
		1.00	7.02	6.48	0.54	8.33
		1.05	5.07	4.74	0.33	6.96
1.0	0.05	0.95	5.44	4.85	0.59	12.16
		1.00	2.45	2.28	0.17	7.46
		1.05	0.89	0.85	0.04	4.71
	0.10	0.95	7.47	6.80	0.67	9.85
		1.00	4.92	4.56	0.36	7.89
		1.05	3.11	2.92	0.19	6.51
	0.15	0.95	9.77	8.91	0.86	9.65
		1.00	7.40	6.82	0.58	8.50
		1.05	5.53	5.15	0.38	7.38
1.5	0.15	0.95	5.34	4.74	0.60	12.66
		1.00	5.33	2.16	0.17	7.87
		1.05	0.81	0.77	0.04	5.19
	0.30	0.95	7.18	6.53	0.65	9.95
		1.00	4.67	4.32	0.35	8.10
		1.05	2.92	2.74	0.18	6.57
	0.45	0.95	9.34	8.51	0.83	9.75
		1.00	7.02	6.47	0.55	8.50
		1.05	5.24	4.88	0.36	7.38

X is the 2-year forward price on the underlying discount bond. The notional principal for all options is set at $1,000. The exercise price is αX. The time to maturity of the bond is 15 years.

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