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On the finite Hilbert transform

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ON THE FINITE HILBERT TRANSFORM

by

DONA V. BOCCIO

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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Abstract

ON THE FINITE HILBERT TRANSFORM

by

Dona V. Boccio

Adviser: Professor Richard Sacksteder.

Several aspects of the finite Hilbert transform are investigated. Among these are its relevance to the boundary value problems in potential theory, its actions on various function spaces, specific formulas for its effects on Chebychev polynomials, and solvability of the airfoil equation. Some of the results obtained are re-derivations of classical formulas and discussions of their full range of validity. The formulas for the finite Hilbert transform involving Chebychev polynomials lead to explicit solutions of certain boundary value problems.

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Chapter One

Introduction

1.1 History.

The finite Hilbert transform has been studied because of its connections to potential theory and hence to aerodynamics, fluid dynamics, acoustics, and electrostatics. It is also of interest from a purely mathematical standpoint, in the fields of complex analysis and partial differential equations. The transform provides an example of the type of singular integral operator that arises in connection with such applications as boundary value problems in partial differential equations and solutions of integral equations. (cf. [5], [6], and [10]).

The theory arose from the study of incompressible flow around a surface such as an airplane wing (cf. [8] for references to earlier works). From a mathematical viewpoint, this work placed emphasis on formulas such as (1) and (2) below (section 2.1). Later work was concerned with understanding reasonable domains and ranges for the finite Hilbert transform. Söhngen [8], [9] worked with Hölder continuous functions; Tricomi [11], [12] dealt with functions in $\mathcal{L}^p$ spaces for certain values of p. Widom

[13], [14] and Jörgens [4] studied the transform on $\mathscr{L}^p$ for $1 \le p < \infty$. Jörgens' contributions included a complete analysis of the finite Hilbert transform on $\mathscr{L}^p$.

1.2 Summary

In this paper, new derivations of formulas will be given using generating functions. These formulas suggest that $\mathscr{L}_2^+$ and $\mathscr{L}_2^-$ (defined in section 2.2 below) are more natural spaces on which to investigate the finite Hilbert transform. In particular, we consider the airfoil equation $\mathscr{F}g = h$ and discuss its solvability on both of these spaces

In chapter 3, we turn to the implications of these results for potential theory. We use complex analysis to obtain new derivations of the formulas mentioned above. We then find the boundary data for which certain boundary value problems are solvable in the almost everywhere sense.

1.3 Further applications.

Techniques from Chapter 3 can be used to investigate the Dirichlet and impedance problems in the exterior of the line segment [-1,1] in the complex plane. Similar methods might be used to solve boundary value problems in the exterior of an arc.

Another area to be researched might be surfaces in higher dimensions, for example, the disk in the xy-plane considered in 3-space.

Chapter Two

The Airfoil Equation

2.1 Formulas

In this section we will give new proofs of classical results (Theorems 1 and 2) regarding the finite Hilbert transform of Chebychev polynomials (cf. [12, pp. 180-181]).

The finite Hilbert transform of a suitable function ϕ is defined by

$$\mathscr{F}[\phi(y)](x)=\pi^{-1}\int_{-1}^{1}(y-x)^{-1}\phi(y)dy.$$

Here the integral is to be interpreted as a Cauchy principal value.

We use the following definitions of the Chebychev polynomials, $T_n(x)$, $U_n(x)$, $x = \cos\theta$;

$T_n(\cos\theta) = \cos(n\theta) \quad n = 0,1,...$

$U_n(\cos\theta) = [\sin(n+1)\theta]/\sin\theta \quad n = 0,1,...$

Lemma 1: For any a with $|a| < 1$,

$$\int_{-1}^{1} f(y)(1-y^2)^{-1}dy = \int_{-1}^{1} f[(z+a)(az+1)^{-1}](1-z^2)^{-1}dz.$$

Proof: Change variables with $y = (z+a)(az+1)^{-1}$.

Lemma 1 is easily seen to be valid if the integral is a Cauchy principal value.

Lemma 2: $\mathscr{F}[(1-y^2)^{-1/2}](x) = 0$ for $|x| < 1$.

Proof: First note that for $f(y) = (1-y^2)^{1/2}(y-x)^{-1}$,

$$\pi^{-1}\int_{-1}^{1} (1-y^2)^{-1}f(y)dy = \mathscr{F}[(1-y^2)^{-1/2}](x).$$

Applying Lemma 1 with a = x, we obtain

$$\int_{-1}^{1}(1-y^2)^{-1}f(y)dy = (1-x^2)^{-1/2}\int_{-1}^{1} z^{-1}(1-z^2)^{-1/2}dz .$$

But the integrand is an odd function of z; therefore

$\mathscr{F}[(1-y^2)^{-1/2}](x) = 0$.

Lemma 3: $\mathscr{F}[(1-y^2)^{-1/2}](x) = -(x^2-1)^{-1/2}$ for $|x| > 1$.

Proof: We proceed as in Lemma 2 but with $a = x^{-1}$.

Then

$$\pi^{-1}\int_{-1}^{1}(1-y^2)^{-1}f(y)dy = -(x^2-1)^{-1/2}\pi^{-1}\int_{-1}^{1}(1-w^2)^{-1/2}dw$$

$$= -(x^2-1)^{-1/2}\pi^{-1}\pi ,$$

so $\mathscr{F}[(1-y^2)^{-1/2}](x) = -(x^2-1)^{-1/2}$.

Theorem 1: $\mathscr{F}[(1-y^2)^{-1/2} T_n(y)](x) = U_{n-1}(x)$ $n=1,2,...$

Proof: Using the generating function (cf. [1, p. 783]),

$$\sum_{n=0}^{\infty} 2T_n(y)z^n=(1-z^2)(1+z^2-2zy)^{-1}+1$$

for $-1 < y < 1$, $|z| < 1$,

we obtain $2\sum_{n=1}^{\infty} T_n(y)\ z^n = (1-z^2)(1+z^2-2zy)^{-1} - 1$

and

$$\mathscr{F}[(1-y^2)^{-1/2}\sum_{n=1}^{\infty} T_n(y)z^n](x)$$

$$= -(2\pi)^{-1}\int_{-1}^{1}(1-y^2)^{-1/2}(y-x)^{-1}dy$$
$$+ (2\pi)^{-1}(1-z^2)\int_{-1}^{1}(1-y^2)^{-1/2}(y-x)^{-1}(1+z^2-2zy)^{-1}dy.$$

Note that the first integral above is 0 by Lemma 2. Using partial fractions, the second integral becomes

$$(2\pi)^{-1}(1-z^2)\int_{-1}^{1}(1-y^2)^{-1/2}\Big[(y-x)^{-1}(1+z^2-2zx)^{-1}$$
$$+ 2z(1+z^2-2zx)^{-1}(1+z^2-2zy)^{-1}\Big]dy$$

$$=(1-z^2)[2\pi(1+z^2-2zx)]^{-1}\Big\{\int_{-1}^{1}(1-y^2)^{-1/2}(y-x)^{-1}dy$$
$$+\int_{-1}^{1}2z(1-y^2)^{-1/2}(1+z^2-2zy)^{-1}dy\Big\}.$$

Again, the first integral above is 0 by Lemma 2; and since $|(1+z^2)(2z)^{-1}| > 1$, we obtain from Lemma 3

$$-(1-z^2)[2\pi(1+z^2-2zx)]^{-1}\int_{-1}^{1}(1-y^2)^{-1/2}[y-(1+z^2)(2z)^{-1}]^{-1}dy$$

$$=-(1-z^2)[2\pi(1+z^2-2zx)]^{-1}\Big\{-\pi\{[(1+z^2)(2z)^{-1}]^2-1\}^{-1/2}\Big\}$$

$$=z(1+z^2-2zx)^{-1}.$$

Also, $\sum_{n=1}^{\infty}U_n(x)z^n = (1+z^2-2zx)^{-1}$ (cf. [1, p. 783]),

and $\sum_{n=1}^{\infty}U_{n-1}(x)z^n = z\sum_{n=1}^{\infty}U_{n-1}(x)z^{n-1} = z\sum_{n=0}^{\infty}U_n(x)z^n$

$$= z(1+z^2-2zx)^{-1}.$$

We have shown

$$\mathcal{F}[(1-y^2)^{-1/2}\sum_{n=1}^{\infty} T_n(y)\ z^n](x) = \sum_{n=1}^{\infty} U_{n-1}(x)\ z^n \quad n=1,2,\ldots$$

Since $|T_n(y)z^n| \le |z|^n$, the Weierstrass M-Test shows that $\sum_{n=1}^{\infty} T_n(y)\ z^n$ converges uniformly in y for $|z| \le c < 1$. Thus for any $\epsilon > 0$, term by term integration over the set $Y(\epsilon) = \{|y| < 1 : |x-y| > \epsilon\}$ is justified. The resulting series converges uniformly in ϵ for fixed x and z. This fact follows easily from the mean value estimate $|[T_n(x+\delta) - T_n(x-\delta)](2\delta)^{-1}z^n| \le c^n \sup|T_n'(y)| \le c^n n^2$ (cf. [1, p. 786] for the second inequality). The limit as $\epsilon \to 0$ can therefore be taken to give:

$$\mathcal{F}[(1-y^2)^{-1/2}\sum_{n=1}^{\infty} T_n(y)z^n](x) = \sum_{n=1}^{\infty} \mathcal{F}[(1-y^2)^{-1/2}T_n(y)](x)z^n$$

and

(1) $\mathcal{F}[(1-y^2)^{-1/2}T_n(y)](x) = U_{n-1}(x) \qquad n = 1,2,\ldots$

<u>Lemma 4</u>: For $|x| > 1$, $\mathcal{F}[(1-y^2)^{1/2}](x) = -x + (x^2-1)^{1/2}$.

<u>Proof</u>: First note that

$$(1-y^2)(y-x)^{-1} = -(y+x)[1+(x^2-1)(y^2-x^2)^{-1}].$$

Then

$$\begin{aligned}\mathcal{F}[(1-y^2)^{1/2}](x) &= \pi^{-1}\int_{-1}^{1}(1-y^2)^{1/2}(y-x)^{-1}dy \\ &= \pi^{-1}\int_{-1}^{1}(1-y^2)^{-1/2}[(1-y^2)(y-x)^{-1}]dy \\ &= -\pi^{-1}\int_{-1}^{1}(1-y^2)^{-1/2}(y+x)[1+(x^2-1)(y^2-x^2)^{-1}]dy\end{aligned}$$

$$= \pi^{-1}\int_{-1}^{1}(1-y^2)^{-1/2}(y+x)dy \quad - \quad \pi^{-1}(x^2-1)\int_{-1}^{1}(1-y^2)^{-1/2}(y-x)^{-1}dy$$

$$= -\pi^{-1}\int_{-1}^{1}(1-y^2)^{-1/2}\ y\ dy \quad - \pi^{-1}x\int_{-1}^{1}(1-y^2)^{-1/2}dy$$

$$- \pi^{-1}(x^2-1)\int_{-1}^{1}(1-y^2)^{-1/2}(y-x)^{-1}dy$$

$= 0 - x + (x^2-1)^{1/2}$ where the value of the third integral is obtained from Lemma 3.

<u>Lemma 5</u>: $\mathcal{F}[(1-y^2)^{1/2}](x) = -x \quad$ for $|x| < 1$.

<u>Proof</u>: The proof is identical to that of Lemma 4, except that in the last integral, $-\pi^{-1}(x^2-1)\int_{-1}^{1}(1-y^2)^{-1/2}(y-x)^{-1}dy$ $= 0$ by Lemma 2 since $|x| < 1$ (cf. [12, p. 175]).

<u>Theorem 2</u>: $\mathcal{F}[(1-y)^{1/2}U_{n-1}(y)](x) = - T_n(x) \quad n = 1,2,...$

<u>Proof</u>: Using the generating function (cf. [1, p. 783])

$$\sum_{n=0}^{\infty}U_n(y)z^n = (1+z^2-2zy)^{-1} \quad \text{for } -1 < y < 1,\ |z| < 1,$$

we obtain

$$\mathcal{F}[(1-y^2)^{1/2}\sum_{n=1}^{\infty}U_{n-1}(y)z^{n-1}](x) =$$

$$\pi^{-1}\int_{-1}^{1}(1-y^2)^{1/2}(y-x)^{-1}(1+z^2-2zy)^{-1}dy \ .$$

By partial fractions, this becomes

$$= [\pi(1+z^2-2zx)]^{-1}\int_{-1}^{1}(y-x)^{-1}(1-y^2)^{1/2}dy \ +$$

$$2z[\pi(1+z^2-2zx)]^{-1}\int_{-1}^{1}(1-y^2)^{1/2}(1+z^2-2zy)^{-1}dy \ .$$

By Lemma 5, $\pi^{-1}\int_{-1}^{1}(y-x)^{-1}(1-y^2)^{1/2}dy = -x$.

We rewrite the second integral as

$$-[\pi(1+z^2-2zx)]^{-1}\int_{-1}^{1}(1-y^2)^{1/2}[y-(1+z^2)(2z)^{-1}]^{-1}dy$$

and, since $|(1+z^2)(2z)^{-1}| > 1$, we apply Lemma 4 with $x = (1+z^2)(2z)^{-1}$ to obtain

$$-[(1+z^2-2zx)]^{-1}\left\{-(1+z^2)(2z)^{-1} + \{[(1+z^2)(2z)^{-1}]^2-1\}^{1/2}\right\}$$

$$= (1+z^2-2zx)^{-1}\left\{(1+z^2)(2z)^{-1} - (1-z^2)(2z)^{-1}\right\} .$$

So $\mathscr{F}[(1-y^2)^{1/2}\sum_{n=1}^{\infty}U_{n-1}(y)z^{n-1}](x) =$

$$(1+z^2-2zx)^{-1}(-x)+(1+z^2-2zx)^{-1}\left\{(1+z^2)(2z)^{-1}-(1-z^2)(2z)^{-1}\right\}$$

$$= (1+z^2-2zx)^{-1}(-x+z).$$

Also,

$$\sum_{n=1}^{\infty}-T_n(x)z^{n-1} = -(2z)^{-1}\sum_{n=1}^{\infty}2T_n(x)z^n$$

$$= -(2z)^{-1}\left\{\sum_{n=0}^{\infty}2T_n(x)z^n -2\right\}$$

$$= -(2z)^{-1}\left\{[(1-z^2)/(1+z^2-2zx)^{-1}+1]-2\right\}$$

$$=(z-x)(1+z^2-2zx)^{-1}.$$

We have shown

$$\mathscr{F}[(1-y^2)^{1/2}\sum_{n-1}^{\infty}U_{n-1}(y)z^{n-1}](x) = \sum_{n-1}^{\infty}-T_n(x)z^{n-1}.$$

By the same argument as in the proof of Theorem 1, we obtain

(2) $\mathscr{F}[(1-y)^{1/2}U_{n-1}(y)](x) = - T_n(x) \quad n = 1,2,...$

2.2 Solvability of the airfoil equation.

Here, we will investigate the finite Hilbert transform on a "weighted " $\mathscr{L}_2$ space in which it will be seen to have a unique inverse. This inverse will be used to re-prove several known theorems in a more concise way and to put them into an appropriate context.

Define $$\mathscr{L}_2^- = \{f: 2\pi^{-1}\int_{-1}^{1}(1-y^2)^{-1/2}|f(y)|^2\,dy < \infty\}$$

$$\mathscr{L}_2^+ = \{f: 2\pi^{-1}\int_{-1}^{1}(1-y^2)^{1/2}\,|f(y)|^2\,dy < \infty\}$$

When $f \in \mathscr{L}_2^-$ denote the finite Hilbert transform of f(x) as $\mathscr{F}_-f$ (resp., $\mathscr{F}_+f$ for $f \in \mathscr{L}_2^+$).

Clearly, $\mathscr{L}_2^+ \supseteq \mathscr{L}_2^-$ since for $|y| < 1$, $(1-y^2)^{1/2} < 1$ and

$$||f||_+^2 = 2\pi^{-1}\int_{-1}^{1}(1-y^2)^{1/2}|f(y)|^2dy$$

$$< 2\pi^{-1}\int_{-1}^{1}(1-y^2)^{-1/2}|f(y)|^2dy = ||f||_-^2 .$$

Proposition 1: $\mathscr{F}_-$ is an isometry of $\mathscr{L}_2^-$ into $\mathscr{L}_2^-$.

Proof: It is easily verified that $\{(1-y^2)^{1/2}U_{n-1}\}$ and $\{2T_0, T_1, T_2, ...\}$ are orthonormal sets in $\mathscr{L}_2^-$ (cf. [7, p. 30]). Now, we show that $\{(1-y^2)^{1/2}U_{n-1}\}$ is complete and therefore forms a basis for $\mathscr{L}_2^-$. (Note that an analogous argument can be used to show that $\{(1-y^2)^{-1/2}T_n\}$ forms a basis for $\mathscr{L}_2^+$).

To this end, we show that if f is in $\mathscr{L}_2^-$ s.t. for all n, $(f,(1-y^2)^{1/2}U_{n-1})_- = 0$ then $f = 0$, where $(\ \ )_-$ denotes the inner product in $\mathcal{L}_2^-$.

$$(f,(1-y^2)^{1/2}U_{n-1})_- =$$

$$2\pi^{-1}\int_{-1}^{1}(1-y^2)^{-1/2}f(y)\ (1-y^2)^{1/2}U_{n-1}(y)\ dy$$

$$= \ 2\pi^{-1}\int_{\pi}^{0} f(\cos\theta)\ (\sin\theta)^{-1}\sin n\theta\ (-\sin\theta)\ d\theta$$

$$= 2\pi^{-1}\int_{0}^{\pi} f(\cos\theta)\sin n\theta\ d\theta = 0 \quad \text{for all } n \text{ implies } f = 0$$

as this is the n^{th} Fourier sine coefficient of $f(\cos\theta)$.

Now, since $\mathcal{F}_-[(1-y^2)^{1/2}U_{n-1}(y)](x) = -T_n(x)$ we see that $\mathcal{F}_-$ maps an orthonormal basis injectively into an orthonormal set, and therefore is an isometry of $\mathcal{L}_2^-$ into $\mathcal{L}_2^-$.

Since $\mathcal{F}_-$ is an isometry,

$$\mathcal{F}_-^*\mathcal{F}_- = I \quad \text{in } \mathcal{L}_2^-$$

where $\mathcal{F}_-^*$ is the adjoint map.

Theorem 3: $\mathcal{F}_-^*[g(z)](y) = -(1-y^2)^{1/2}\mathcal{F}_+[(1-z^2)^{-1/2}g(z)](y)$.

Proof: First, note that if $g(z)$ is in $\mathcal{L}_2^-$ then

$(1-z^2)^{-1/2}g(z)$ is in $\mathcal{L}_2^+$.

Now, since $(\mathcal{F}_-f,g)_- = (f,\mathcal{F}_-^*g)_-$ for any f,g in $\mathcal{L}_2^-$ is the defining relation for $\mathcal{F}_-^*$, it suffices to prove

$$(\mathcal{F}_-f,g)_- = (f,\ -(1-y^2)^{1/2}\mathcal{F}_+[(1-z^2)^{-1/2}g(z)])_-$$

for the case $f = (1-y^2)^{1/2}U_{n-1}$ and $g = T_m$, because the $(1-y^2)^{1/2}U_n$'s and the T_n's form bases for $\mathcal{L}_2^-$.

$$(\mathcal{F}_- f, g)_- = (\mathcal{F}_-[(1-y^2)^{1/2}U_{n-1}], T_m)_-$$

$$= (-T_n, T_m)_- = \begin{cases} -1 & n=m \\ 0 & n\neq m \end{cases},$$

and

$$(f, -(1-y^2)^{1/2}\mathcal{F}_+[(1-z^2)^{-1/2}g(z)])_-$$

$$= ((1-y^2)^{1/2}U_{n-1}, -(1-y^2)^{1/2}\mathcal{F}_+[(1-z^2)^{-1/2}T_m])_-$$

$$= ((1-y^2)^{1/2}U_{n-1}, -(1-y^2)^{1/2}(U_{m-1}))_-$$

$$= \begin{cases} -1 & n=m \\ 0 & n\neq m \end{cases}.$$

An analogous result can be shown for $\mathcal{L}_2^+$:

<u>Theorem 4</u>: $\mathcal{F}_+^*[g(z)](y) = -(1-y^2)^{-1/2}\mathcal{F}_-[(1-z^2)^{1/2}g(z)](y)$.

In section 2.1, we established Theorems 1 and 2 by means of generating functions. These results can be written in our new notation as

(1') $\mathcal{F}_+[(1-y^2)^{-1/2}\ T_n(y)](x) = U_{n-1}(x) \qquad n = 1,2,...$

(2') $\mathcal{F}_-[(1-y^2)^{1/2}\ U_{n-1}(y)](x) = -T_n(x) \qquad n = 1,2,...$

We remark that (1') and (2') are not independent; (1') can be derived from (2') by applying $\mathcal{F}_-^*$ to both sides of (2'), as follows:

$$\mathcal{F}_-^*\ \mathcal{F}_-\ [(1-y^2)^{1/2}U_{n-1}(y)] = -\mathcal{F}_-^*\ T_n(x),$$

hence, by Theorem 3,

$$(1-x^2)^{1/2}U_{n-1}(x) = (1-x^2)^{1/2}\mathcal{F}_+[(1-x^2)^{-1/2}T_n(x)],$$

and $\mathcal{F}_+[(1-x^2)^{-1/2}T_n(x)] = U_{n-1}(x)$.

Corollary 1: $\ker(\mathcal{F}_+ \circ (1-y^2)^{-1/2}) = \{c\}$.

Proof: $(\text{Range } \mathcal{F}_-)^\perp = \ker \mathcal{F}_-^*$.

From Theorem 3, $\ker \mathcal{F}_-^* = \ker(\mathcal{F}_+ \circ (1-y^2)^{-1/2})$.

But from (2'), we have $(\text{Range } \mathcal{F}_-)^\perp = \{cT_0\} = \{c\}$.

Corollary 2: $\ker(\mathcal{F}_- \circ (1-y^2)^{1/2}) = \{0\}$.

Proof: Analogous to Corollary 1, using Theorem 4 and the fact that, by (1'), $\mathcal{F}_+$ is onto.

Corollary 1 and (1') imply:

Theorem 5: Given $g \in \mathcal{L}_2^+$, $\mathcal{F}_+ f = g$ has a solution $f_0 \in \mathcal{L}_2^+$. The most general solution in $\mathcal{L}_2^+$ is $f_0 + c(1-y^2)^{-1/2}$.

Corollary 2 and (2') imply:

Theorem 6: For $g \in \mathcal{L}_2^-$, $\mathcal{F}_- f = g$ has a solution $f = \mathcal{F}_-^* g$ iff $(g,1)_- = 0$. Then that solution is unique.

Chapter Three

Applications to Potential Theory

3.1 The Neumann problem.

We will now show how the finite Hilbert transform can be used to solve the Neumann problem for Laplace's equation in the exterior of the interval [-1,1] in the complex plane. It will be shown that if limiting values of the normal derivatives from above and below, u_N^+ and u_N^-, are specified, then there is a function expressible as the difference of a single and a double layer potential satisfying the given requirements. It will be seen that since $\mathcal{F}_+ f = g$ is solvable with solution $f = \mathcal{F}_+^* g$, the Neumann problem is solvable.

Define the operator $\mathcal{K}h(x,y)$ for suitable h by

$$(4\pi)^{-1}\int_{-1}^{1}\{\partial/\partial s \ \ln\,[(x-t)^2 + (y-s)^2]\}\big|_{s=0} h(t)dt$$

for $(x,y) \notin [-1,1] \subseteq \mathbb{C}$, $h(1) = h(-1) = 0$.

Note that this defines a double layer potential with kernel

$$K(z,z_0) = (4\pi)^{-1}\partial/\partial s \ \ln\,|z-z_0|^2\big|_{s=0}$$

where $z = (x,y)$, $z_0 = (t,s)$, $|t| \le 1$.

Then

$$\mathcal{K}h(x,y) = -(2\pi)^{-1}\int_{-1}^{1} y[(x-t)^2 + y^2]^{-1}h(t)dt.$$

Integrating by parts, we obtain (assuming that h' exists and is in $\mathscr{L}^1[-1,1]$)

$$\mathcal{K}h(x,y) = -(2\pi)^{-1}\int_{-1}^{1}\tan^{-1}[(x-t)y^{-1}]h'(t)dt.$$

For $|x| < 1$, let

$$Ph(x) = \lim_{y\to 0} \partial/\partial y\ [\mathcal{K}h(x,y)]$$

$$= (2\pi)^{-1}\lim_{y\to 0}\int_{-1}^{1}(x-t)[(x-t)^2 + y^2]^{-1}h'(t)dt.$$

The limit can be taken under the integral sign (cf. [10, p. 218]). So for almost every x,

$$Ph(x) = (2\pi)^{-1}\int_{-1}^{1}(x-t)^{-1}h'(t)\,dt.$$

This is the finite Hilbert transform of h'; thus,

(3) $\quad Ph(x) = -2^{-1}\mathcal{F}[h'(t)](x).$

Now, define the single layer potential $Qg(x,y)$ by

$$(2\pi)^{-1}\int_{-1}^{1}\ln\ [(x-t)^2 + y^2]^{1/2}\ g(t)dt$$

and let

(4) $\quad u(x,y) = Qg(x,y) - \mathcal{K}h(x,y)$

for $(x,y) \notin [-1,1] \subseteq \mathbb{C}$.

We attempt to solve the following Neumann problem: Given $u_N^{\pm}(x)$, find h(x), g(x) such that

$\Delta u = 0 \qquad$ on $\mathbb{C} - [-1,1]$,

(5) $\quad u_N^{+}(x) = \lim_{y\to 0^+} \partial/\partial y\ u(x,y),$

(6) $\quad u_N^{-}(x) = \lim_{y\to 0^-} \partial/\partial y\ u(x,y)$ for $x \in (-1,1)$.

By the jump relations (cf. [2 pp. 169, 174]), the fact that Q and P are continuous on $\mathbb{C}$, and assuming for now that h and g are continuous on (-1,1), we obtain

$$u^{\pm}(x) \quad = \quad Qg(x) - (\mathcal{K} \mp I/2)h(x)$$

where $u^{+}(x) = \lim_{y \to 0^{+}} u(x,y)$, $\quad u^{-}(x) = \lim_{y \to 0^{-}} u(x,y)$,

and

(7) $\quad u_N^{\pm}(x) = (\mathcal{K}^t \pm I/2)g(x) - Ph(x) \quad$ for $x \in (-1,1)$.

Note that since $y = 0$, $\mathcal{K}^t = \mathcal{K} = 0$ here.

It follows easily that

(8) $\quad h(x) = u^{+}(x) - u^{-}(x)$

and

(9) $\quad g(x) = u_N^{+}(x) - u_N^{-}(x)$.

It can be shown by using Theorem 1.25 of [10, p. 13] that (8) and (9) hold almost everywhere even if g and h are only in $\mathcal{L}^1$.

Using (7) and (9), the Neumann problem is reduced to solving the following for h:

$Ph(x) = 2^{-1}g(x) - u_N^{+}(x) = -2^{-1}g(x) - u_N^{-}(x) = -v(x)$,

where $v(x) = 2^{-1}[u_N^{+}(x) + u_N^{-}(x)]$.

By (3) above,

$$v(x) = 2^{-1}\mathcal{F}[h'(t)](x).$$

If $v(x) \in \mathcal{L}_2^{+}$, then by Theorem 5 we can find solutions of the form $h_0'(t) + c(1-t^2)^{-1/2}$. But we require $h(1) = h(-1) = 0$. This can be arranged by choosing c such that $\int_{-1}^{1} h'(t)dt = 0$.

We have proved the following theorem:

Theorem 7: Suppose u_N^+ and u_N^- are real-valued functions in $\mathcal{L}_2^+$. Then there is a harmonic function u(z) defined in $\mathbb{C} - [-1,1]$ by $u(z) = Qg(z) - \mathcal{K}h(z)$ such that conditions (5) and (6) hold almost everywhere.

3.2 Another solution of the Neumann problem.

In this section we will present another method of solving the Neumann problem. More explicit representations of solutions will be given, and new proofs of previous results will be furnished.

We will require the following lemma.

Lemma 6: $d/dx\ [(1-x^2)^{1/2}U_{n-1}(x)] = -n(1-x^2)^{-1/2}T_n(x)$.

Proof: $d/dx\ [(1-x^2)^{1/2}U_{n-1}(x)] = d/dx\ [\sin(n\cos^{-1}x)]$

$= [\cos(n\cos^{-1}x)](-n)(1-x^2)^{-1/2} = -n(1-x^2)^{-1/2}\ T_n(x)$.

Let w be the branch of $\cos^{-1}z$, $z \in \mathbb{C} - [-1,1]$, for which $\cos^{-1}(1) = 0$, $\cos^{-1}(0) = \pi/2$, and $\cos^{-1}(-1) = \pi$ are the limiting values as z approaches [-1,1] from the upper half plane, i.e., the "usual" branch.

Lemma 7: w is globally defined on $\mathbb{C} - [-1,1]$.

Proof: Write $z = \cos(u + iv)$. Then the identities

$$z = \cos u \cos iv - \sin u \sin iv$$

and

$$z = 2^{-1}[e^{i(u+iv)} + e^{-i(u+iv)}]$$

give

(10) $z = \cos u \cosh v - i \sin u \sinh v$

and

$$x = \cos u \cosh v, \quad y = -\sin u \sinh v.$$

Define the domains $\mathcal{D}_i$,$i = 1,\ldots,4$ of the w-plane by

$\mathcal{D}_1 = \{(u,v): \ \pi/2 \le u \le \pi, \ v < 0 \}$

$\mathcal{D}_2 = \{(u,v): \ 0 \le u \le \pi/2, \ v < 0 \}$

$\mathcal{D}_3 = \{(u,v): \ -\pi/2 \le u \le 0, \ v < 0 \}$

$\mathcal{D}_4 = \{(u,v): \ -\pi \le u \le -\pi/2, \ v < 0 \}$.

Then, in view of (10), $z = \cos w$ maps $\mathcal{D}_1$ into Quadrant II of $\mathbb{C} - [-1,1]$, $\mathcal{D}_2$ into Quadrant I, $\mathcal{D}_3$ into Quadrant IV, and $\mathcal{D}_4$ into Quadrant III.

From the results above, it follows easily that $u^+ = -u^-$ and $v^+ = v^- = 0$; hence $w^+ = -w^-$. Also, $w_N^+ = -w_N^-$.

Now, consider the function $f(z) = ie^{-inw(z)}$. Clearly, $f(z)$ is analytic and globally defined on $\mathbb{C} - [-1,1]$.

Using $f(z) = ie^{-in(u+iv)} = ie^{nv}(\cos nu - i\sin nu)$, we obtain

$$R(z) = \text{Re } f(z) = e^{nv}\sin nu$$

and

$$I(z) = \text{Im } f(z) = e^{nv}\cos nu.$$

Then, using the Cauchy-Riemann equation,

$$\partial R/\partial y = -\partial I/\partial x = ne^{nv}(\sin nu \ \partial u/\partial x - \cos nu \ \partial v/\partial x)$$

and

$$R_N^+(x) = \lim_{y \to 0^+} \partial R/\partial y = n \sin nu^+ \ \partial u^+/\partial x$$

$$= -nU_{n-1}(x).$$

We note here that $R_N^+(x) = R_N^-(x)$ since sin is an odd function. Then from (9), $g(x) = 0$.

We have shown that $R(z)$ is the solution of Laplace's equation that is harmonic at infinity (i.e. $o(\ln|z|)$ at infinity) with boundary condition

(11) $$R_N^{\pm}(x) = -nU_{n-1}(x).$$

It is known that a harmonic function with smooth boundary values and harmonic at infinity can be represented as the difference of a single layer and a double layer potential. Then the jump relations (7) are valid for $u(z) = R(z)$ and show that

$$R_N^{\pm}(x) = -Ph(x).$$

Combining this with (3) and (11) gives

$$2^{-1}\mathscr{F}[h'(t)](x) = -nU_{n-1}(x).$$

Using the definition of $R(z)$ and taking limits,

$$R^+(x) = -R^-(x) = (1-x^2)^{1/2}U_{n-1}(x),$$

so from (8), $h(x) = 2R^+(x) = 2(1-x^2)^{1/2}U_{n-1}(x)$.

Using Lemma 6,

$$h'(x) = -2n(1-x^2)^{-1/2}T_n(x)$$

so $2^{-1}\mathscr{F}[-2n(1-x^2)^{-1/2}T_n(t)](x) = -nU_{n-1}(x)$

and we have another proof of Theorem 1.

Now, considering the imaginary part,

$$\partial I/\partial y = \partial R/\partial x = ne^{nv}(\cos nu\ \partial u/\partial x + \sin nu\ \partial v/\partial x)$$

and

$$I_N^+(x) = \lim_{y\to 0^+} \partial I/\partial y = n \cos nu^+\ \partial u^+/\partial x$$
$$= -n(1-x^2)^{-1/2}T_n(x),$$

$I_N^-(x) = -I_N^+(x)$.

Now, taking $u(z) = I(z)$ in (4) and noting that $I^+(x) = I^-(x)$, we obtain from (8) that $h = 0$. Then

$$g(x) = 2I_N^+(x)$$
$$= -2n(1-x^2)^{-1/2}T_n(x)$$

and

$$I(z) = Qg(z)$$
$$= (2\pi)^{-1}\int_{-1}^{1} \ln[(x-t)^2 + y^2]^{1/2}(-2n)(1-t^2)^{-1/2}T_n(t)dt.$$

Applying Lemma 6 and integrating by parts yields

$$I(z) = \pi^{-1}\int_{-1}^{1}(x-t)[(x-t)^2 + y^2]^{-1}(1-t^2)^{1/2}U_{n-1}(t)dt.$$

Taking $\lim_{y \to 0^+}$ under the integral sign (cf.[10, p. 218]),

$$I^+(x) = \pi^{-1}\int_{-1}^{1}(t-x)^{-1}(1-t^2)^{1/2}U_{n-1}(t)dt$$
$$= -\mathcal{F}[(1-t^2)^{1/2}U_{n-1}(t)](x).$$

But $I^{\pm}(x) = T_n(x)$ and we have another proof of Theorem 2.

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