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**Automorphisms and quasiconformal mappings of Heisenberg-type groups**

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City University of New York, 1995

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# Automorphisms and quasiconformal mappings of Heisenberg-type groups

by

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# Automorphisms and quasiconformal mappings of Heisenberg-type groups

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Adviser: Professor Martin Moskowitz

The Lie algebras of derivations of trace-zero of Heisenberg-type groups are explicitely computed, along with the connected component of the group of isometries of an H-type Lie group with the metric invariant under left translations and certain automorphisms defined by A. Korányi [18]. Using this we prove a result on stabilizers of lattices and give a necessary and sufficient condition for the existence of non-conformal quasi-conformal mappings. The latter shows that except for the abelian and Heisenberg cases, all quasi-conformal mappings must actually be conformal. A characterization of the isometry group of solvable extensions of H-type groups is also given. Finally we show an application of these techniques to prove that in certain rank-one Lie groups all non-uniform lattices are arithmetic. The last chapter is dedicated to the study of the $L^1$-algebras and representation theory of quaternionic Lie groups of Heisenberg-type.

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# Contents

# Chapter 1

# Introduction and Background

## 1.1 Introduction

In this thesis we study properties of automorphisms of so called groups *of Heisenberg type* which were first introduced by A. Kaplan ([13]), as a generalization of the Heisenberg group itself.

A. Korányi in [18] has characterized the homogeneous left invariant metrics on these manifolds. These generalized Heisenberg groups play an important role in various fields of modern analysis: A. Kaplan and F. Ricci ([14]) have studied the harmonic analysis some of Gelfand pairs associated to $H$-type groups; aspects of complex analysis connected to the structure of Carnot metrics on Heisenberg groups were studied by Korànyi and Reimann in [19] (for the classical Heisenberg group) and by Cowling, Dooley, Korànyi and Ricci in [2] (for the general case). In the first case as groups of transformations of the boundary of a noncompact rank-one symmetric space, in the second as an important tool in the study of quasi-conformal mappings. It is important to note also that $H$-type groups, together with their solvable extensions $A \ltimes N$ by their one-dimensional groups of dilations $A \simeq \mathbb{R}^+$, have provided counterexamples to several important conjectures in differential and spectral geometry. The noncompact Lichnerowicz conjecture was settled negatively by F. Ricci and E. Damek in [4], Z. Szabò found isospectral non isometric harmonic manifolds ([31]) and C. Gordon in [7] gave examples of *closed* nilmanifolds with that same property, obtained as quotients of certain quaternionic $H$-type groups which are isospectral but non-isometric.

In their 1994 paper [24] R. Mosak and M. Moskowitz were interested in the following problem: let $G$ be a Lie group and $\Gamma$ a so called log-lattice in $G$. Consider the group of measure preserving automorphisms $M(N)$ and its identity connected component $M_0(N)$ and let:

$$Stab_{M_0(N)}(\Gamma) = \{\phi \in M_0(N) \mid \phi(\Gamma) = \Gamma\}$$

be those elements that stabilize $\Gamma$. The natural question to ask is whether it is a lattice in $M_0(N)$ and if so, when is it uniform (=cocompact). In the case of $G$ nilpotent and $\Gamma$ a *log*-lattice, Mosak and Moskowitz developed a criterion that makes use of a property of the *nilradical* of the Lie algebra $Der_0(\mathfrak{N})$. They suggested that this criterion could be applied to Lie groups of Heisenberg type. In the second chapter we show that this is the case and settle the question for any $H$-type group. The approach that we give here to these problems is fairly general. It allows us to study all irreducible $H$-type groups simultanously.

The last chapter contains a study of some properties of quaternionic $H$-type groups from a strictly analytical point of view; namely those associated with isospectral non isometric closed Riemannian manifolds: the kind of $H$-type groups studied in [7] turn out to provide a class of non isomorphic non abelian as well as noncompact Lie groups which have $*$-isomorphic $L^1$-algebras.

## 1.2 Lie Groups and Lie Algebras

A Lie group over the field $\mathbf{K}$ (=$\mathbb{R}$ or $\mathbb{C}$) is a group $G$ equipped with the structure of a differentiable manifold over $\mathbf{K}$ in such a way that the map

$$\begin{aligned} \mu : G \times G &\rightarrow G \\ (g_1, g_2) &\mapsto g_1 \cdot g_2 \end{aligned}$$

is differentiable. In other words, the coordinates of the product of two elements have to be differentiable functions of the coordinates of the factors.

It follows as an application of the implicit function theorem that under this assumption also the inverse map $i(g) = g^{-1}\ \forall g \in G$ is a differentiable map ([6], pg.6). A Lie group over $\mathbb{R}$ is called a real Lie group; Lie groups over $\mathbb{C}$ are also called complex Lie groups. Any complex Lie group can be viewed as a real Lie group of twice the (complex) dimension.

Examples of Lie groups are:

1. The additive group of the field $\mathbf{K}$;

2. The *circle* $\mathbb{T} = \{z \in \mathbb{C}^* | z| = 1\}$ (it is a real group);

3. the group $GL(n, \mathbf{K})$ of invertible $n \times n$ matrices over the field $\mathbf{K}$;

4. any finite or countable group equipped with the discrete topology vand the structure of a 0-dimensional differentiable manifold;

5. the group $G$ of orthogonal linear transformations of $\mathbf{K}^n$; if $\mathbf{K} = \mathbb{R}$, $G = O(n)$, for $\mathbf{K} = \mathbb{C}$, $G = U(n)$.

A subgroup $H$ of a Lie group $G$ is called a *Lie subgroup* if it is a submanifold of the underlying manifold of $G$. The basic method of the theory of Lie groups, which makes it possible to obtain powerful results, consists in reducing questions concerning Lie groups to certain problems in linear algebra. This is done by assigning to every Lie group $G$ its "tangent algebra": the *Lie algebra* $Lie(G) = \mathfrak{G}$, which to a large extent determines $G$.

An abstract *Lie algebra* $\mathfrak{L}$ is a finite dimensional vector space over the field $\mathbf{K}$ equipped with a bilinear operation $[\cdot,\cdot] : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$, called the Lie bracket, such that, $\forall X, Y \in \mathfrak{L}$:

1. $[X, X] = 0$.

2. $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$.

From the first it follows $[X, Y] = -[Y, X]$, while the second is known with the name of *Jacobi Identity*.

There are many ways one can define the Lie algebra of a Lie group $G$. It is possible to give one in which the Lie bracket arises from the commutator of vector fields.

With the help of left or right translation one can construct natural isomorphisms between the tangent spaces of a Lie group $G$ at different points. Let $l(g)$ denote left translation by $g$ and $r(g)$ right translation by $g$. Then for any $g, h \in G$ and $\xi \in T_h(G)$ let

$$g\xi \;=\; (dl(g))(\xi) \in T_{gh}(G)$$

and

$$\xi g \;=\; (dr(g))(\xi) \in T_{hg}(G).$$

From the associativity of group multiplication we derive the identities:

$$(gh)\xi = g(h\xi),\ (g\xi)h = g(\xi h),\ (\xi g)h = \xi(gh).$$

If, in particular, $G = A^*$ is the group of invertible elements of an associative algebra $A$, then the "products" $g\xi$ and $\xi g$ coincide with the products in the sense of the algebra $A$.

If we define, for every $\xi \in T_e(G)$, the *right invariant* vector field $\xi_*$:

$$\xi_*(g) \;=\; \xi g,$$

it can be proven that the map $\xi \rightarrow \xi_*$ is an isomorphism from $T_e(G)$ to the vector space $T_*(G)$ of all right invariant vector fields on $G$. As the commutator of two vector fields is invariant under arbitrary diffeomorphisms, the commutator of right invariant vector fields is again right invariant. Thus $T_*(G)$ is an algebra with respect to the operation of taking the commutator of two vector fields. Given two right invariant vector fields $X, Y$ on $G$ we write:

$$[X,Y] = X \circ Y - Y \circ X;$$

this operation turns out to satisfy the relations 1. and 2. of a Lie algebra.

For all the details of the theory of Lie groups and their Lie Algebras we will refer to Helgason's book ([8]).

A *one-parameter subgroup* $g(t)$ of a Lie group $G$ is a smooth homomorphism from the Lie group $\mathbb{R}$ to $G$: $g(t+s) = g(t)g(s)$. It can be proven that a generic path [1] $g(t)$ in $G$ is a one-parameter subgroup if and only if its velocity $\xi(t)$ is constant and $g(0) = e$. We denote with $g_\xi(t)$ the one-parameter subgroup with velocity $\xi(t) = \xi$.
A central object for the study of Lie groups and their Lie algebras is the *exponential map*. For any Lie group $G$ and any $\xi \in \mathfrak{G}$, we set by definition:

$$exp(\xi) = g_\xi(1).$$

The map $exp : \mathfrak{G} \rightarrow G$ is called the *exponential map.*
In the case when $G$ is the group of invertible elements of an associative algebra, the exponential map is given by:

$$exp(X) = \sum_0^\infty \frac{X^n}{n!}.$$

Given any Lie group $G$ there is always a neighbourhood $V_0$ of zero in the Lie algebra $\mathfrak{G}$ that the exponential maps diffeomorphically onto a neighbourhood $U_e$ of the identity in $G$.

[1] A *path* is a differentiable map of a connected subset of the real line into a given manifold.

A Lie group $G$ always admits a uniquely (up to scalar multiple) defined left-invariant ($\mu_l$) as well as a right-invariant ($\mu_r$) positive regular Borel measure[2]: the so called *Haar measure*.
The existence of a left-invariant measure allows the definition of an integral on $G$ For any continuous function $f$ with compact support we have:

$$\int_G f(g_1 \cdot g) d\mu_l = \int_G f(g) d\mu_l \qquad \forall g_1 \in G.$$

Since a *right* translate of the left-invariant measure is again left-invariant and all such measures are multiples of each other we can define the so called *modular function* $\Delta_G$ of $G$ as, for any integrable function $f$ and any $x \in G$ by:

$$\Delta_G(x) \cdot \int_G f(g) d\mu_l = \int_G f(gx) d\mu_l.$$

That defines a homomorphism $\Delta_G : G \to \mathbb{R}^+$. It may happen that $\mu_l$ and $\mu_r$ coincide up to a scalar multiple; in that case our $\Delta_G$ will be constantly equal to *one* and $G$ is said to be *unimodular*. It can be proven that any compact group is unimodular.

A discrete subgroup $\Gamma$ of a Lie group $G$ is called a *lattice* if the quotient space $G/\Gamma$ carries a $G$-invariant *finite* measure. It is very easy to prove that a *necessary* condition for a Lie group to admit a lattice subgroup is unimodularity.

In the case of a connected and simply connected nilpotent Lie group it turns out that the exponential map is actually globally invertible. We set

[2]This is actually true for any locally compact *topological* group: a group equipped with a topology that makes the group operations (multiplication, inversion) continuous (see e.g. [21])

$exp^{-1} = log$. In case the image under the *log*-map of a lattice $\Gamma \subset G$ is a lattice in the Lie algebra $\mathfrak{G}$ we call it a *log-lattice*. Malcev proved in [22] that a connected simply connected nilpotent Lie group $N$ has a lattice if and only if its Lie algebra has a *rational structure*, that means there is a basis of $Lie(N) = \mathfrak{N}$ with rational structure constants. In such a group every lattice $\Gamma$ is always a sublattice and a superlattice of two log-lattices $\Gamma'$ and $\Gamma''$:

$$\Gamma' \subset \Gamma \subset \Gamma''.$$

Two lattices $\Gamma$ and $\Lambda$ in a Lie group $G$ are said to be commensurable if their intersection has finite index in both of them; in this case we write $\Gamma \simeq \Lambda$.

Given a lattice $\Gamma \subset G$ the *commensurator* of $\Gamma$ in $G$ is the subgroup of $G$ defined by:

$$Comm_G(\Gamma) = \{g \in G \,|\, g\Gamma g^{-1} \simeq \Gamma\}.$$

Let $\mathfrak{G}$ be the Lie algebra of a Lie group $G$. If there is a number $s$ such that, given any $X_1, \ldots, X_s \in \mathfrak{G}$ it holds:

$$[X_1, [X_2, [\cdots, X_s] \cdots] = 0,$$

we say that $\mathfrak{G}$ is $(s-1)$-step *nilpotent*; a nilpotent Lie group is a Lie group whose Lie algebra is nilpotent. Also nilpotent groups are unimodular.
A 1-step nilpotent Lie algebra is called abelian.

There is a more general type of Lie algebra that includes the nilpotent case: if we define $\mathfrak{G}^{(1)} = [\mathfrak{G}, \mathfrak{G}]$, and:

$$\mathfrak{G}^{(n)} = [\mathfrak{G}^{(n-1)}, \mathfrak{G}^{(n-1)}],$$

we say that $\mathfrak{G}$ is solvable if $\mathfrak{G}^{(s)} = 0$ for some positive number $s$.[3]
Obviously an abelian or nilpotent Lie algebra is also solvable.
Any vector subspace $\mathfrak{H} \subset \mathfrak{G}$ is called a *Lie subalgebra* of $\mathfrak{G}$ provided it is closed under the Lie bracket operation:

$$[\mathfrak{H}, \mathfrak{H}] \subseteq \mathfrak{H}.$$

A subalgebra $\mathfrak{H}$ of $\mathfrak{G}$ with the property that, given any $h \in \mathfrak{H}$:

$$[\mathfrak{G}, h] \subseteq \mathfrak{H}$$

is called an *ideal* of $\mathfrak{G}$.

An important normal subgroup of $G$ is its *center* $Z(G)$ (the group of elements in $G$ commuting with any other element). Its Lie algebra $\mathfrak{Z} = Z(\mathfrak{G})$ consists of all elements $X \in \mathfrak{G}$ such that $[X, Y] = 0$ for any $Y$ in $\mathfrak{G}$.

Given a Lie subgroup $H$ of a Lie group $G$, it happens that $\mathfrak{H} = Lie(H)$ is a Lie subalgebra of $\mathfrak{G} = Lie(G)$. If in particular $H$ is a normal subgroup, its Lie algebra will be an ideal of $\mathfrak{G}$.

A *simple* Lie group is a Lie group whose Lie algebra is *simple*; that means non-abelian and with no non-trivial ideals. If a Lie algebra is decomposed as direct sum of simple ideals is called *semisimple*: the same will be said of the Lie groups it is associated to.
The sum of two solvable ideals of a Lie algebra is again a solvable ideal; therefore it is natural to introduce the concept of a maximal solvable ideal. The maximal (with respect to inclusion) solvable ideal of a Lie algebra is its *radical*

[3] A Lie group with a solvable Lie algebra is called *solvable*.

and is denoted by: $\mathcal{R}(\mathfrak{G})$, whereas its maximal nilpotent ideal is the *nilradical*, $\mathcal{R}_n(\mathfrak{G})$ of the Lie algebra.

## 1.3 Automorphisms and Derivations

A smooth $1-1$ homomorphism of a Lie group $G$ onto itself is called an *automorphism*. Its analogue in the Lie algebra context is also defined: the Lie group $Aut(\mathfrak{G})$ consists of *Lie* automorphisms of $\mathfrak{G}$; any of its elements, say $\phi$, is a linear map satisfying:

$$\phi([X,Y]) = [\phi(X), \phi(Y)]\ \forall X, Y \in \mathfrak{G}.$$

The groups $Aut(G)$ and $Aut(\mathfrak{G})$ are in a natural way Lie groups.
If $G$ is connected they are related in the following sense: the map

$$d:\ Aut(G) \longrightarrow Aut(\mathfrak{G})$$

which assigns to each automorphism of $G$ its differential is injective. If $G$ is also simply connected it will actually be an isomorphism ([6], pg. 49).
The Lie algebra of $Aut(\mathfrak{G})$ -denoted with $Der(\mathfrak{G})$- consists of so called *derivations* of $\mathfrak{G}$; a derivation of the Lie algebra $\mathfrak{G}$ is a linear map $D : \mathfrak{G} \rightarrow \mathfrak{G}$, satisfying the property:

$$D([X,Y]) \ = \ [D(X), Y] + [X, D(Y)].$$

Given any group of automorphisms $A$ of $G$, we define the *semidirect product* $A \ltimes G$ to be the Lie group with the product manifold structure and group multiplication:

$$(a,g) \cdot (a',g') = (aa', g \cdot a(g')),$$

and similarly, given a Lie algebra $\mathfrak{D}$ of derivations of $\mathfrak{G}$, the *semidirect sum* $\mathfrak{D} \ltimes \mathfrak{G}$ will be the Lie algebra with bracketing:

$$[(D,X),(D',X')] = ([D,D'],[X,X'] + [X,D(X')] + [X',D'(X)],$$

with obvious notation. If $\mathfrak{D} = Lie(A)$ and $\mathfrak{G} = Lie(G)$, then: $Lie(A \ltimes G) = \mathfrak{D} \ltimes \mathfrak{G}$.

A well known fact is that if $A$ and $G$ are unimodular, then $A \ltimes G$ is unimodular if and only if $A$ consists of measure preserving automorphisms.

Given a lattice $\Gamma \subset G$, we can consider the group of automorphisms of $G$ with the property:

$$Stab_{Aut(G)}(\Gamma) = \{\phi \in Aut(G) | \phi(\Gamma) = \Gamma\},$$

called the *stabilizer* of $\Gamma$.

It turns out that, since $G/\Gamma$ carries a finite $G$-invariant measure, $Stab_{Aut(G)}(\Gamma)$ consists of *measure-preserving* automorphisms; identity component of the group of measure preserving automorphisms of $G$ is denoted by $M_0(G)$ and its Lie algebra -consisting of *trace zero derivations* - will be $Der_0(\mathfrak{G})$ ([24]).

## 1.4 Generalities of Representation Theory

Let $G$ be a Lie group, and let $B$ be a Banach space. A *representation* $\pi$ of $G$ on $B$ is a family of bounded operators such that:

1.

$$\pi(g) : B \longrightarrow B, \qquad g \in G;$$

2.

$$\pi(g_1 g_2) = \pi(g_1)\pi(g_2), \qquad \pi(e) = Id_B;$$

3.

$$g_k \rightarrow g \; in \; G \Longrightarrow \forall v \in B \; \pi(g_k)v \rightarrow \pi(g)v \; in \; B$$

In the case when $B = \mathcal{H}$ is a Hilbert space and each of the $\pi(g)$ is a unitary operator ($\pi(g)^* = \pi(g)^{-1} = \pi(g^{-1})$), we say that $(\pi, \mathcal{H})$ is a *unitary* representation of $G$. In this thesis we will always assume all Hilbert spaces and groups to be *separable*.

Some examples of unitary representations of a general Lie group $G$ -all realized on the Hilbert space $\mathcal{H} = L^2(G)$ of square integrable functions w.r.t. the measure $\mu_l$- are:

1. The *left regular* representation $\lambda$: $\lambda(g)(f(x)) = f(g^{-1}x)$; this is an example of a *faithful* (=injective) representation
2. The *identity* representation $I$: $I(g)(f(x)) = f(x)$;

It is very useful to consider the action a representation $(\pi, \mathcal{H})$ induces on the Banach algebra $L^1(G)$; for any $f \in L^1(G)$ and $v \in \mathcal{H}$ we set:

$$\pi(f)v = \int_G f(g)\pi(g)v d\mu;$$

This defines a bounded operator on $\mathcal{H}$ ([30], pg. 11) with the properties:

1. $\pi(f_1 * f_2) = \pi(f_1) \cdot \pi(f_2) \qquad \pi(f^*) = \pi(f)^*$;

2. $\|\pi(f)\| \leq \|f\|_{L^1}$;

3. the linear span of all $\pi(f)v$, for $v \in \mathcal{H}$ is dense in $\mathcal{H}$.

In other words the induced representation on $L^1(G)$ is a non degenerate $*$-representation. Furthermore it can be proven that there is a 1-1 correspondence between unitary representations of $G$ on a Hilbert space $\mathcal{H}$ and the $*$-representations of $L^1(G)$ (cf. [5], pg. 4) satisfying the properties 1.-3.. Moreover the representations $\pi$ and $\pi(f)$, $f \in L^1(G)$ have the same invariant subspaces (see e.g. [17], pgs. 12-13).

A fundamental concept is the one of irreducible representation. An *irreducible* representation $(\pi, \mathcal{H}_\pi)$ of a Lie group $G$ is one allowing no proper, closed, $\pi(G)$-invariant subspaces in $\mathcal{H}_\pi$. For the kinds of groups we work with, arbitrary representations can be uniquely decomposed into "direct integrals" of irreducibles ([16],pg. 146) and hence the unitary representations of $G$ and the $*$-representations of $L^1(G)$ correspond.

Two unitary representations $\pi$ and $\pi'$ are alled *unitarily equivalent* ($\pi \simeq \pi'$), if there is an invertible isometry $A : \mathcal{H}_{\pi'} \rightarrow \mathcal{H}_\pi$ such that for all $g \in G$:

$$A\pi(g) = \pi'(g)A.$$

We denote by $\check{G}$ (resp. $\hat{G}$) the set of equivalence classes of unitary (resp. irreducible unitary) representations of the Lie group $G$ on separable Hilbert spaces. The set $\hat{G}$ is said to be the *dual space* of $G$.

One can define a topology on $\tilde{G}$ as follows. Let $\pi$ a unitary represenatation of $G$ on a Hilbert space $\mathcal{H}_\pi$. For any compact set $K \subset G$ consider a finite collection of $n$ vectors $\{v_k\} \subset \mathcal{H}_\pi$ and a number $\epsilon > 0$ and define the set $U(K, \{v_k\}, \epsilon)$ consisting of those classes of representations $[\pi']$ in the space of which there exist $n$ vectors $\{w_k\}$ such that:

$$| < \pi(g)v_i, v_j > - < \pi'(g)w_i, w_j > | \; < \; \epsilon,$$

for all $g \in K$, $1 \leq i, j \leq n$. This way we obtain that the family of the sets $U(K, \{v_k\}, \epsilon)$ is a neighbourhood base at the point $[\pi] \in \tilde{G}$ (see e.g. [23], pg. 72).

Basically, a Hilbert space $\mathcal{H}$ is a *direct integral* of Hilbert spaces (of equal dimension) if it has the form $L^2(X, \mu, \mathcal{K})$, all measurable functions $f : X \to \mathcal{K}$ defined on a measure space $(X, \mu)$ with values in a Hilbert space $\mathcal{K}$ such that

$$\|f\| \; = \; \int_X \|f(x)\|^2 \mu(dx) \; < \infty.$$

For our purposes we will assume $X$ to be a standard Borel space, $\mu$ $\sigma$-finite and $\mathcal{K}$ separable. This splitting of $\mathcal{H}$ into fibers will be indicated by:

$$\mathcal{H} \; = \; \int_X^{\oplus} \mathcal{H}_x \mu(dx),$$

where for each $x$: $\mathcal{H}_x = \mathcal{K}$.

Now let $G$ be a Lie group and $\{\pi_x | x \in X\}$ a measurable field of unitary representations each one modeled in $\mathcal{K} = \mathcal{H}_x$ (measurability means that $x \to < \pi_x(g)f(x), h(x) >$ is measurable for all $g \in G$ and all $f$ and $h$ in $L^2(X, \mathcal{K})$. We define the *direct integral* representation $\pi = \int^{\oplus} \pi_x \mu(dx)$ modeled in $\mathcal{H} = L^2(X,\mathcal{K})$ by:

$$\pi(g)f(x) = \pi_x(g)(f(x)) \qquad \forall g \in G,\ \forall f \in L^2(X,\mathcal{K}).$$

It can be proven that $\pi$ is a (SO)-continuous unitary representation of $G$ ([3] chapter 2,[16] pgg. 59 ff.).

A crucial property of the direct integral representation is that if $\nu$ is another $\sigma$-finite measure on $X$ with the same null sets as $\mu$ it holds that:

$$\int_X^{\oplus} \pi_x \mu(dx) \simeq \int_X^{\oplus} \pi_x \nu(dx),$$

which shows that only the equivalence class (=up to null sets) of $\mu$ is important in direct integrals.

In the case of abelian an Lie group, each of its irreducible unitary representations is one-dimensional. In this case the tensor product operation defines a commutative group structure on $\hat{G}$ and this group is naturally identified with the group of characters of $G$ ([23], pgg. 72 ff.).

The direct integral decomposition of a representation is given by their characters: a *character* $\chi$ of an abelian Lie group $A$ is a continuous homomorphism of the group into the group of complex numbers of modulus one: $\forall g, g_1, g_2 \in A$, $\chi(g_1 g_2) = \chi(g_1)\chi(g_2)$, $|\chi(g)| = 1$

## 1.5 The Linear Algebra of Quadratic Forms

A (real) quadratic space is the data $(V, Q)$ of a finite dimensional vector space $V \simeq \mathbb{R}^k$, equipped with a quadratic form $Q$.

Given two quadratic spaces $(R, Q_R)$ and $(S, Q_S)$ a *composition of quadratic forms* on $R$ and $S$ is a bilinear map $\mu$

$$\mu : R \times S \longrightarrow S,$$

satisfying $Q_S(\mu(r,s)) = Q_R(r) \cdot Q_S(s)$. A composition of quadratic forms $\mu$ is said to be *normalized*, provided there is a $r \in R$ such that, for all $s \in S$: $\mu(r,s) = s$.

The sets of normalized compositions of quadratic forms $\mu : \mathbb{R}^k \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are in 1-1 correspondence with the sets of $\{u_i\}_{1 \leq i \leq k-1} \in O(n)$ such that:

1. $u_i^2 = -1$;

2. $u_i u_j + u_j u_i = 0, \ (i \neq j)$.

This is achieved ([10], pg. 141) by defining $\forall x \in \mathbb{R}^n$: $u_i(x) = \mu(e_i, x)$, where $\{e_k\}$ is an orthonormal basis of $\mathbb{R}^k$.

Given a quadratic space $(V, Q)$, the *Clifford Algebra* $C(V, Q)$ is the data of an $\mathbb{R}$-algebra and a linear map $\theta : V \rightarrow C(V, Q)$ such that: $\theta(x)^2 = Q(x) \cdot 1$, satisfying the following universal property: given any $\mathbb{R}$-algebra $A$ with a linear function $u : V \rightarrow A$ such that $u(x)^2 = Q(x) \cdot 1$, there is an algebra morphism $u' : C(V, Q) \rightarrow A$ such that:

$$u' \circ \theta = u.$$

The algebra morphism $-\theta : V \rightarrow C(V, Q)$ determines an involution $\beta : C(V, Q) \rightarrow C(V, Q)$ with: $\beta \circ \theta = -\theta$; so that a so-called $\mathbb{Z}_2$-grading is induced:

$$C(V,Q) = C^0 \oplus C^1, \qquad C^iC^j \subset C^{i+j} \ \ i,j \in \mathbb{Z}_2.$$

If $\beta(x) = x$ we say that $x \in C^0$, if $\beta(x) = -x$, then $x \in C^1$.

A *Clifford module* $M$ over $C = C(V,Q)$ is a $\mathbb{Z}_2$-graded module over $C$: $M = M^0 \oplus M^1$, $C^iM^j \subset M^{i+j}$, with $i,j \in \mathbb{Z}_2$.

In this thesis we will discuss Clifford algebras and Clifford modules associated to the quadratic spaces $(\mathbb{R}^n, -Q)$, where $Q$ is the square of the euclidean norm; those Clifford algebras will be denoted by $C(n)$.

It can be proven ([10], pg. 151) that all Clifford modules are completely reducible. It also turns out from the theory that there is a very close relation between Clifford modules and compositions of quadratic forms; in fact ([10], pg. 156) there exists a composition of quadratic forms $\mu : \mathbb{R}^{m+1} \times \mathbb{R}^n \to \mathbb{R}^n$ if and only if $\mathbb{R}^n$ admits a structure of a $C(m)$ module. Moreover, it can be proven that every module over $C(m)$ arises from a composition of quadratic forms.

In this thesis we will use properties of the algebras: real, complex, quaternionic and ocotionic numbers (denoted resp. with $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$) equipped with the standard norm $\|X\|^2 = Q(X) = \bar{X} \cdot X = X \cdot \bar{X}$. The real scalar product $< \cdot, \cdot >$ *induced by* $Q$ is defined for any pair of quaternions or octonions $X, Y$ as:

$$< X, Y > = Re(\bar{X}Y).$$

A simple calculation shows that $Re(\bar{X}Y) = Re(X\bar{Y})$. , and is obtained by polarisation of $Q$ as follows; it holds that:

$$Q(X+Y) = (\overline{X+Y})(X+Y) = \bar{X}X + \bar{Y}Y + \bar{X}Y + \bar{Y}X.$$

From that we get:

$$\|X+Y\|^2 - \|X\|^2 - \|Y\|^2 \; = \; \bar{X}Y + \bar{Y}X \; = \; 2Re(\bar{X}Y),$$

which proves our formula.

Despite the lack of commutativity on $\mathbb{H}$ we can still define a sort of linear structure on any vector space $V \simeq \mathbb{H}^n$ as follows: given any two elements $e = (e_1, \ldots, e_n)$ and $f = (f_1, \ldots, f_n)$ in $V$ we write:

$$< e, f > = \sum \bar{e}_i \cdot f_i.$$

That produces a so called *right* linear structure over $\mathbb{H}$, with the property: for any $h \in \mathbb{H}$: $< e, fh > = < e, f > h$ where $eh = (e_1, \ldots, e_n) \cdot h = (e_1 h, \ldots, e_n h)$. If we set the *bar* on the right instead the left hand side in the definition of the product $< \cdot, \cdot >$ we obtain a *left* linear quaternionic structure. For these two structures it holds:

$$< f, f > = \; \|f_1\|^2 + \ldots + \|f_n\|^2 \qquad\qquad \forall f \in \mathbb{H}^n.$$

A right-linear (resp. left linear) quaternionic transformation $T$ will be one that has the property, with the same notation as above: $T(eh) = T(e) \cdot h$ (resp. $T(he) = hT(e)$) for all $e \in V$ and all $h \in \mathbb{H}$.

With respect to either of these two products we can define the *quaternionic adjoint* of $T$: $< Te, f > = < e, T^* f >$.

The (left or right) linear transformations satisfying

$$< Te, Tf > = < e, f >$$

are exactly the ones in $Sp(n)$, while those for which $T + T^* = 0$ are the ones in the Lie algebra $sp(n)$ (see [17], pgg. 27 ff.) These provide the exact analogue to the examples of pg. 4.

# Chapter 2

# Lie Groups of Heisenberg-type

A Heisenberg-type Lie group $N$ (or H-type group) is a connected and simply connected two-step nilpotent Lie group, on whose Lie algebra $\mathfrak{N}$ there is a positive definite real quadratic form $<,>$ which is *compatible* with the *natural decomposition*,

$$\mathfrak{N} = \mathfrak{Z} \oplus \mathfrak{V} \tag{2.1}$$

where $\mathfrak{Z}$ is the center of $\mathfrak{N}$ and $\mathfrak{V}$ is its orthocomplement with respect to $< \cdot, \cdot >$. Here compatibility refers to Kaplan's basic assumption that the family of operators

$$\{ad_X : X \in \mathfrak{V} | < X, X >= 1\} \tag{2.2}$$

consists of partial isometries onto $\mathfrak{N}$. Kaplan refers to this as *property H.*

The existence of the quadratic form gives rise to two other equivalent algebraic descriptions of H-type groups.

### 2.0.1 Clifford Modules

Given the usual decomposition $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ with $m = dim(\mathfrak{z})$ of a Lie algebra of Heisenberg type with its quadratic form $Q(\cdot) = < \cdot, \cdot >$, consider the map $J : \mathfrak{z} \rightarrow End(\mathfrak{v})$ defined, with obvious notation, by:

$$< J_z(v), v' > = < z, [v, v'] > \quad z \in \mathfrak{z}. \tag{2.3}$$

By straightforward computation one can prove that, given an orthonormal basis $\{e_i\}$ of $\mathfrak{z}$, the family $\{J_{e_i}\}$ satisfies the relations of the generators of the Clifford algebra $Cl(-Q, \mathfrak{z})$. In other words $J : \mathfrak{z} \rightarrow End(\mathfrak{v})$ extends to a unitary representation of $Cl(-Q, \mathfrak{z})$ on the orthocomplement of the center that therefore carries a structure of so called Clifford module over $C(m)$, where $C(m)$ is the *uniquely defined* Clifford algebra $Cl(-Q, \mathfrak{z})$ with $m$ generators. (see [14], pg. 418). We then have that $\mathfrak{z}$ acts on $\mathfrak{v}$ as a set of linear transformations satisfying, for any two orthogonal $e_i, e_j \in \mathfrak{z}$ of unit norm:

$$J_{e_i} J_{e_j} + J_{e_j} J_{e_i} = 0 \; and \; J_{e_i}^2 = -Id$$

### 2.0.2 Composition of Quadratic Forms

An alternative description of $H$-type algebras can be given by using *compositions of quadratic forms.*

Given a normalized composition of quadratic forms $\mu : R \times S \rightarrow S$ we can define a Lie bracket as follows. Consider the dual map $\phi : S \times S \rightarrow R$

defined by the linear system

$$< r, \phi(s, s') > = < \mu(r, s), s' >, \ r \in R. \tag{2.4}$$

Now consider now the element $r_0$ that satisfies $\mu(r_0, s) = s$ for all $s \in S$ and let $\pi$ be the orthogonal projection onto $\mathbb{R}r_0^{\perp}$. Then

$$< \mu(r, s), \mu(r', s) > = < r, r' > q_S(s),$$

if we set $r' = r_0$ we get that $< \mu(r, s), s > = < r, \phi(s, s) > = 0$. Therefore the new map $\pi \circ \phi$ is skew symmetric. Given any $(r_1, s_1), (r_2, s_2) \in \pi(R) \oplus S$, by writing:

$$[(r_1, s_1), (r_2, s_2)] = (\pi \circ \phi(s_1, s_2), 0) \tag{2.5}$$

we obtain a two-step nilpotent Lie algebra structure on $\mathfrak{N} = \pi(R) \oplus S$, in which $\pi(R)$ becomes the center and $S$ its orthocomplement. An easy computation shows that such an $\mathfrak{N}$ satisfies property $(H)$ and thus is a Lie algebra of Heisenberg type with the map $J$ in (2.3) given by: $J_r(s) = \mu(r, s)$ (see [13]).

## 2.1 Irreducible $H$-type Algebras

An H-type algebra $\mathfrak{N}$ is said to be *irreducible* if its associated Clifford module is.

Two H-type algebras $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are said to be *isomorphic* if there is a linear isomorphism $\Phi : \mathfrak{L}_1 \to \mathfrak{L}_2$ such that, with a slight abuse of notation:

$$J_{\Phi(z)}\Phi(v) \ = \ \Phi(J_z v), \qquad v \in \mathfrak{V}_1, \ z \in \mathfrak{z}_1.$$

In some sense the isomorphism map intertwines the two representations $J(\mathfrak{z}_1)$ and $J(\mathfrak{z}_2)$ of the Clifford algebra. This allows inequivalent Clifford modules to give rise to isomorphic $H$-type Lie algebras: for each $m = dim(\mathfrak{z})$ there is exactly one irreducible $H$-type Lie algebra ([14], pg. 419).

Given any Clifford algebra with $m$ generators there is up to equivalence one or possibly two (when $m = 3, 7 \mod 8$) irreducible Clifford modules associated to it. The theory of quadratic forms provides a complete classification of all Clifford modules for $m = 0$ to 7 mod 8. A detailed account on the construction of these modules is given in Husemoller's book ([10] Chapter 11, sec. 8). The fact that $dim(\mathfrak{z})$ determines the dimension of $\mathfrak{v}$ and that two inequivalent Clifford modules of the same dimension produce isomorphic $H$-type algebras allows one to classify all of them. A complete list of the dimension of $\mathfrak{v}$ for each value of $m = dim(\mathfrak{z})$ can be found in the paper of Kaplan and Ricci [14].

A new explicit realization of irreducible H-type Lie algebras for $0 \leq dim(\mathfrak{z}) \leq 7$ is given in the next paragraph. Since Clifford modules are completely reducible, any $H$-type Lie algebra $\mathfrak{n}$ is decomposable as: $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}_1 \oplus \cdots \mathfrak{v}_k$ where all the irreducible $H$-type subalgebras $\mathfrak{z} \oplus \mathfrak{v}_i$ are isomorphic. $H$-type Lie algebras arise in a very natural way: consider a simple Lie group $G$ of real-rank one. Classification tells us that it has to belong to either one of the families $SO_0(n,1)$, $SU(n,1)_0$ and $Sp(n,1)$ or the exeptional group $F_4$. Such a group admits an Iwasawa decomposition: $G = K \cdot A \cdot N$; Korànyi ([18]) has proved that the nilpotent group $N$ is always of Heisenberg type: $N$ can be resp. $\mathbb{R}^{n-1}$, the classical $2(n-1)+1$ dimensional Heisenberg group $N_1^{n-1}$ or its quaternionic and octonionic analogues $N_3^{n-1}$ and $N_7$.

### 2.1.1 Real, Complex, Quaternionic, and Cayley algebras

By making use of the quadratic forms characterization of H-type algebras we will prove a new basic result that makes explicit computations considerably easier. Let $\mathbf{K}_i^*$ be an $i$-dimensional $\mathbb{R}$-subspace of the *imaginary* elements in $\mathbf{K}(=\mathbb{R},\ \mathbb{C},\ \mathbb{H}$ or $\mathbb{O})$, we can define the composition of quadratic forms: $\mu_i : \mathbf{K}_i^* \times \mathbf{K} \to \mathbf{K}$ by:

$$\mu_i(X,Y) \;=\; X \cdot Y, \qquad \forall X,Y \in \mathbf{K}.$$

By putting $\pi(R) = K_i^*$ and $S = \mathbf{K}$ and performing the construction discussed in the previous section we can equip the vector spaces $\mathbf{K}_i^* \oplus \mathbf{K}$ with the structure of an $H$-type algebra.

The following then holds: since, with obvous notation, $< \mu_i(Z,X), X' >$ $= < Z, \phi(X,X') >$ we get: $Re((ZX)\bar{X}') = Re(Z\bar{\phi}(X,X'))$; by letting $Z = 1$ we get:

$$Re(X\bar{X}') \;=\; Re(\bar{\phi}(X,X'))$$

and thus:

$$\phi(X,X') \;=\; X' \cdot \bar{X} \;\; \forall X,X' \in \mathbf{K}.$$

PROPOSITION 2.1.1 *An irreducible H-type Lie algebra $\mathfrak{n}$ with $dim(\mathfrak{z}) \leq 7$ is isomorphic to one of the algebras $\mathfrak{n}_i \simeq \mathbf{K}_i^* \oplus \mathbf{K}$, where $dim(Z(\mathfrak{n})) = i$.*

PROOF. In the case when $i = 0$, $\mathfrak{N} \simeq \mathbb{R}^n$, so there is nothing to prove. In the other cases we note that the mapping $\pi \circ \phi(\cdot,\cdot)$ can be taken as the standard hermitian product defined on $\mathbf{K} \times \mathbf{K}$. If $i = 1$ we obtain $\mathfrak{N} = i\mathbb{R} \oplus \mathbb{C}$ and so $\mathfrak{N} \simeq \mathfrak{N}_1$, the classical Heisenberg Lie algebra.

If $i = 2, 3$ we choose $\mathbf{K} = \mathbb{H}$; this way we can construct two different quaternionic compositions of quadratic forms: $\mu_2 : \mathbb{H}_2^* \times \mathbb{H} \to \mathbb{H}$ and $\mu_3 : \mathbb{H}_3^* \times \mathbb{H} \to \mathbb{H}$. Each is defined by the equation, for $X \in \mathbf{K}_i^*$ and $Y \in \mathbf{K}$

$$\mu_i(X, Y) = X \cdot Y \ (i = 2, 3).$$

We also note that, in the case of $\mu_3$, there is an inequivalent Clifford module corresponding to the composition defined by:

$$\mu_r(X,Y) \ = \ Y \cdot X \qquad X \in \mathbb{H}^*, Y \in \mathbb{H}$$

The Lie algebras with these two (left) compositions will be denoted by $\mathfrak{N}_2 \simeq \mathbb{H}_2^* \oplus \mathbb{H}$ and $\mathfrak{N}_3 \simeq \mathbb{H}^* \oplus \mathbb{H}$[1].

For $i = 4, 5, 6$ or $7$ we choose $\mathbf{K} = \mathbb{O}$; the realizations are exactly the same as for the quaternionic cases. The algebras we get this way are $\mathfrak{N}_4$, $\mathfrak{N}_5$, $\mathfrak{N}_6$ and $\mathfrak{N}_i \simeq \mathbb{O}_i^* \oplus \mathbb{O}$, $4 \leq i \leq 7$ [2].

For all the Lie algebras described above the Lie bracket is defined as follows: given two elements $(Z, X)$ and $(Z', X')$ in $\mathfrak{N}_i$, we have:

$$\phi(X, X') = X' \cdot \bar{X}$$

[1]We will refer to them as resp. the quasi-quaternionic and quaternionic H-type algebras

[2]These are the quasi-octonionic and octonionic algebras

and therefore:

$$[(Z,X),(Z',X')] = \left(Im_i(X'\cdot\bar{X}),0)\right),$$

where $Im_i$ is the projection of the imaginary part onto the space $\mathbf{K}_i^*$. In doing so, we get the identity

$$<\mu(Z,X),X'>= Re(\bar{Z}XX') = Re(\bar{Z}X'\bar{X}) =<\bar{Z},\pi\circ\phi(X,X')>$$

as required by our definition (2.4).[3] To prove irreducibility one has to observe that, since $J_Z(X) = Z\cdot X$ there is no invariant subspace. For $i = 3$ or $7$ this is obvious. Consider $i = 2$. Then there are two orthonormal vectors in $\mathbb{H}_2^*$, say, $X_1$ and $X_2$. Since ([12], pg.149) $J_{X_1}V \perp J_{X_2}V$ we get that $1, X_1, X_2, X_1X_2$ are an orthonormal basis of $\mathbb{H}$ and the module is irreducible.

For $i = 4$ the same argument runs by taking as an orthonormal basis a subset of $\{1, X_1, \ldots, X_4, X_iX_j\}$, $1 \le i, j \le 4$.

The above constructed H-type Lie algebras with centers $\mathfrak{z}_i$ of real dimensions ranging from zero to seven exhaust all possibilities. ◇

An $H$-type Lie algebra $\mathfrak{N}$ is said to satisfy the $J^2$-condition if ([2]), for any $X\in\mathfrak{V}$:

$$(J(\mathfrak{z})\oplus\mathbb{R})(J(\mathfrak{z})\oplus\mathbb{R})X = (J(\mathfrak{z})\oplus\mathbb{R})X.$$

A characterization of such algebras was given by Koràny and his co-authors; here we give another proof of their result:

[3]In case we choose the *right* action described before ($\mu(Z,X) = X\cdot Z$), the Lie bracket becomes: $[(Z,X),(Z',X')] = (Im_i(\bar{X}X'),0)$.

**Proposition** 2.1.2 *If an H-type Lie algebra $\mathfrak{N}$ satisfies the $J^2$-condition, it is isometrically isomorphic to $\mathbb{R}^n$, $\mathfrak{N}_1^n$, $\mathfrak{N}_3^n$ or $\mathfrak{N}_7$.*

**Proof.** First we need the observation that the $J^2$-condition is equivalent to saying that, for any two mutually orthogonal $Z, Z' \in \mathfrak{Z}$ and any fixed $X \in \mathfrak{V}$, there is always a third element $Z'' \in \mathfrak{Z}$ such that ([2], pg.5)

$$J_Z \cdot J_{Z'} X = J_{Z''} X \tag{2.6}$$

This is trivially verified when $dim(\mathfrak{Z})$ is either 0 or 1. In case $dim(\mathfrak{Z}) = 2, 4, 5, 6$ or $> 7$, we observe ([10], pg. 150) that the dimension of the Clifford module is larger than $dim(\mathfrak{Z}) + 1$, so we can find two orthogonal elements of $\mathfrak{Z}$ such that their product lies in the complement of $J(Z)$ in $C(m)$. To see this we have to observe that the action $J_Z X = Z \cdot X$ is (in the way stated in the introduction) a rotation in $O(n)$ ($n = dim(\mathfrak{V})$). If $\{e_i\}$ is an orthonormal basis of $\mathfrak{Z}$ the action of the $J_{e_i}$'s is by rotations by $\frac{\pi}{2}$ of $\mathfrak{V}$, with the property, for different $i$ and $j$ $J_{e_i} X \perp J_{e_j} X$ ([2], pg. 6). If $\mathfrak{N}$ is irreducible and isomorphic to $\mathbf{K}_i^* \oplus \mathbf{K}$ with $i = 2, 4, 5$ or 6, the space $\mathbf{K}_i^*$ being just a subset of the imaginary elements of $\mathbf{K}$ equation (2.6) is not verified for all $Z, Z' \in \mathfrak{Z}$: any two unit $Z$ and $Z'$ in an orthonormal basis of $\mathbf{K}_i^*$ such that $Z \cdot Z' \notin \mathbf{K}_i^*$ will do. For the quaternionic and octonionic algebras $\mathfrak{N}_3$ and $\mathfrak{N}_7$ we get that such a $Z''$ can always be found. For $i = 3$ we can take $Z'' = Z \cdot Z'$, whereas in the octonionic case the lack of associativity makes the choice depend on $X$, in which case a simple calculation shows that the equation $Z(Z'X) = Z'' \cdot X$ with orthogonal $Z$ and $Z'$ in $\mathbb{O}^*$ always has a solution $Z''$ for any $X$ in $\mathbb{O}$. The calculation can be done as follows: if we define $\{J_{e_i}\}$ to be the rotations induced by an orthonormal basis of $\mathfrak{z} = \mathbb{O}^*$ we get that

$$Z \cdot X = \sum a_i J_{e_i} X \tag{2.7}$$

for some real numbers $a_i$ such that w.r.t. the chosen basis: $(a_1, \ldots, a_7) = Z$. Since $n = dim(\mathfrak{v}) = dim(\mathfrak{z}) + 1$ and $< Z, Z' >= Re(\bar{Z}Z') = 0$ we know that $Z(Z' \cdot X)$ will still be a linear combination of the type $\sum b_i J_{e_i} X$. So if we take $Z'' = (b_1, b_2, \ldots, b_7)$ we have the solution we wanted.

In the case of $dim(\mathfrak{v}) > dim(\mathfrak{z}) + 1$ the condition $< Z, Z' >= 0$ only makes sure that the elements in the linear combination (2.7) are not just the multiplication by a real number. This implies that there are two elements in an orthonormal base of $\mathfrak{z}$, say $e_1$ and $e_2$ such that their product defines a orthogonal rotation of $\mathfrak{v}$ which is not given by any of the $J_{e_i}$'s. So that: $J_{e_1} J_{e_2} \notin \{J_{e_i}\}$ and the claim is proven. The same fact can be seen as a consequence of the following

LEMMA 2.1.3 *For $X, Y \in \mathbb{O}$ and $Z \in \mathbb{O}^*$ it holds: $Re(Z(XY)) = Re((ZX)Y)$*

PROOF. From our construction it follows that ($X, X' \in \mathbb{O}$): $< Z, [X, X'] >=< J_Z X, X' >$; that is:

$$Re(Z \cdot Im(\bar{X}X')) = Re((ZX) \cdot \bar{X}'),$$

which, since $Z$ is purely imaginary, is equivalent to:

$$Re(Z \cdot (X\bar{X}')) = Re((ZX) \cdot \bar{X}');$$

And thus, by setting $Y = X'$, we get our lemma. ◇

If $\mathfrak{N}$ is not irreducible, it is isomorphic to $\mathbf{K}_i^* \oplus \mathbf{K}^n$ and therefore the proven statement will be true for each of the irreducible components $\mathbf{K}_i^* \oplus \mathbf{K}$ of $\mathfrak{N}$. If $m = 3$ or $m = 7$ the only possibility not already considered is that $\mathfrak{N} = \mathfrak{Z} \oplus \mathfrak{V}_1 \oplus \ldots \mathfrak{V}^k$ is obtained using non equivalent actions of $\mathfrak{Z}$ on $\mathfrak{V}$. These are explicitely realized in our model by left- and right-multiplication. But since these operations are not commutative ($Z \cdot Z' \neq Z' \cdot Z$). So the $J^2$-condition cannot be satisfied. $\diamondsuit$

# Chapter 3

# Automorphisms and Derivations

In this section we want to explicitely compute the identity component of the group of Haar measure preserving automorphisms $M_0(N)$ of an $H$-type group $N$ and its Lie algebra, $Der_0(\mathfrak{N})$, the derivations of $\mathfrak{N}$ of trace zero. These were first studied by R. Mosak and M. Moskowitz in [24]. There they assumed the quite general connected simply connected nilpotent group had a *log-lattice* $\Gamma$ (which by Malcev's results is always the case for $H$-type groups) and they studied its stabilizer in $M_0(N)$ defined by:

$$Stab_{M_0(N)}(\Gamma) = \{\phi \in M_0(N) \mid \phi(\Gamma) = \Gamma\}$$

In [24] (Theorem 2.2.), a criterion was developed which shows when $Stab_{M_0(N)}(\Gamma)$ is a lattice or a uniform lattice in $M_0(N)$. This criterion, established on the Lie algebra $Der_0(\mathfrak{N}) = Lie(M_0(N))$, deals with the radical $\mathcal{R} = Rad(Der_0(\mathfrak{N}))$ and its maximal nilpotent ideal $\mathcal{R}_n = Rad(Der_0(\mathfrak{N}))_n$:

1. If $\mathcal{R} = \mathcal{R}_n$, then $Stab_{M_0(N)}(\Gamma)$ is a lattice in $M_0(N)$.

2. If $Der_0(\mathfrak{N})/\mathcal{R}_n$ is in addition of compact type, then $Stab_{M_0(N)}(\Gamma)$ is uniform.

Furthermore, this criterion remains valid also in the case we replaced $M_0(N)$ by any of its closed subgroups. As we will see this result finds a straightforward application to the full family of Heisenberg type Lie algebras. Using the notation introduced for the composition of quadratic forms, will facilitate this computation.

If we let $D_3$ be the restriction of the derivation to the center and write $D_{\mathfrak{v}}$ for the restriction to its orthocomplement, equation we can write:

$$D_3([X,Y]) = ad(D_{\mathfrak{v}}X)(Y) + ad(X)(D_{\mathfrak{v}}Y) \tag{3.1}$$

which, because of 2-step nilpotency, simply implies that such a derivation can be written as an upper triangular matrix of the form:

$$D = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where $A = D_3$ is a square matrix acting on the center, $C = D_{\mathfrak{v}}$ a square matrix acting on its orthocomlement and $B$ a rectangular block with free coefficients.

Using this we compute the derivations of trace zero for each irreducible case. If we write down a generic $D \in Der_0(\mathfrak{N}_i)$ as an upper block triangular matrix, as above, we realize that in order to compute explicitely the action of $D$ on the elements of $\mathfrak{N}_i$ $(2 \leq i \leq 7)$ we can use (3.1) along with the definition of derivation:

$$A(Im_i(Y \cdot \bar{X})) = Im_i(Y \cdot \bar{CX}) + Im_i(CY \cdot \bar{X}).$$

In the associative cases the latter is equivalent to the equation:

$$A(Im_i(< X, Y >)) = Im_i(< X, (C + C^*)Y >)^1 \qquad (3.2)$$

Where $A$ is a linear transformation of the center of the lie algebra and $C + C^*$ a self adjoint transformation of the orthocomplement.

We will need a property that immediately follows from (3.2):

LEMMA 3.0.4 *For $A$, $B$ and $C$ as above, $A + A^* = 0$.*

PROOF. First note that for $i = 1$ the assertion is obvious, so consider $3 \geq i > 1$; if we rewrite (3.2) by choosing $X = 1$ and $Y = e_j$, where $e_j$ is any non-real element of the canonical base of $\mathbb{H}$ or $\mathbb{O}$ such that $A(Im((e_j)) \neq 0$, by posing $\tilde{C} = C + C^*$ we have:

$$A(Im(e_j)) = Im\left(\tilde{C}(e_j)\right).$$

Hence, for any $Y = X$ the equation gives:

$$0 = < (C + C^*)X, X >;$$

therefore, either $C(X) = \lambda \cdot X$ and in that case $A(X) = 2 \cdot \lambda$ and $D$ is not of trace zero, or $C + C^* = 0$ which implies $A + A^* = 0$ That proves the lemma.◇

[1] Here $< \cdot, \cdot >$ stands for either the usual Hermitian or the quaternionic scalar product; the $^*$ is, depending on the context, the conjugate transpose or the quaternionic transpose. For details see e.g. Adams' book on compact groups

It should be noted here that a general derivation the $A$ can be written as $A = \lambda \cdot Id + A'$ with $A'$ of trace zero. Using this lemma we can prove the following result for $\mathfrak{N}_3^n \simeq \mathbb{H}^n \oplus \mathbb{H}^n$. We note here that these algebras are obtained by using the same composition of quadratic forms as for the irreducible ones $\mathfrak{N}_i$ and taking a higher dimensional $\mathbf{K}$-vector space for the orthocomplement of the center: $\mathfrak{N}_i^n = \mathbf{K}_i^* \oplus \mathbf{K}^n$. Among these H-type Lie algebras, the ones where $i = 0, 1, 3$ or $7$, turn out to be the $\mathfrak{N}$-part of the $\mathfrak{K} \oplus \mathfrak{A} \oplus \mathfrak{N}$ Iwasawa decomposition of all simple Lie algebras of rank one.

PROPOSITION 3.0.5 *With the above notation:*

$$Der_0(\mathfrak{N}_3^n) \simeq sp(1) \oplus sp(n) \ltimes \mathbb{R}^{12n}$$

PROOF. Using the previous lemma we may write: $A + A^* = 0$ and so $A \in sp(1)$. At the same time, if $A \neq 0$, by (3.2) the elements of $C$ will be completely determined by $A$.

If $A = 0$, (3.2) becomes:

$$Im(< (C + C^*)X, Y >) = 0,$$

and thus

$$< (C + C^*)X, Y >= 0 \qquad X, Y \in \mathbb{H}.$$

This means: $C \in sp(n)$.

Up to now we have proven that our derivation algebra consists of the union of two sets ($sp(1)$ and $sp(n)$) with the nilradical. These three sets are

Lie subalgebras. For suppose we have two derivations $D_1$ and $D_2$ of the type discussed in the first case of our proof; that means they will be of the form:

$$D_1 = \begin{pmatrix} A_1 & B_1 \\ 0 & C_1(A_1) \end{pmatrix}, \quad D_2 = \begin{pmatrix} A_2 & B_2 \\ 0 & C_2(A_2) \end{pmatrix}$$

computing their Lie bracket we get:

$$[D_1, D_2] = \begin{pmatrix} [A_1, A_2] & * \\ 0 & [C_1(A_1), C_2(A_2)] \end{pmatrix};$$

In the case when $[A_1, A_2] \neq 0$ there is nothing to prove, so we have to show that if $[A_1, A_2] = 0$ we get also: $[C_1, C_2] = 0$.
But since we have $A_i \in sp(1) \cong so(3)$, the two matrices will commute only if one is a multiple of the other. Since the $C_i$ are determined from the $A_i$ by solving a linear system of equations (3.2) and thus:

$$C_1 = C_2 \cdot \lambda Id,$$

which implies that $[C_1, C_2] = 0$. Thus the first set of derivations is a subalgebra of $Der_0(\mathfrak{N}_3^n)$. If we consider two matrices $D_1$, $D_2$, in the second set, we have:

$$[D_1, D_2] = \begin{pmatrix} 0 & * \\ 0 & [C_1, C_2] \end{pmatrix};$$

showing that this set is actually an ideal of $Der_0(\mathfrak{N}_3)$.

At this stage we need to point out that the condition $tr(D) = 0$ is automatically satisfied for any derivation of $\mathfrak{N}_3^n$ (we will shortly that this is a general fact). Taking the last observation into account it follows that the algebra we are looking for is:

$$Der_0(\mathfrak{N}_3^n) \simeq sp(1) \oplus sp(n) \ltimes \mathbb{R}^{12n}.$$

and the proof of the proposition is complete. ◇

The case of $\mathfrak{N}_7 \simeq \mathbb{O}^* \oplus \mathbb{O}$ was studied before in [24]; the non-associativity of the Cayley numbers forces the second set to vanish:

$$Der_0(\mathfrak{N}_7) \simeq so(7) \ltimes \mathbb{R}^{7\cdot 8}.$$

To complete our list we have to produce an explicit realization of the derivations of trace zero also in the cases when $dim(Z(\mathfrak{N})) = 2, 4, 5,$ or 6. The argument will run like the ones discussed above. In the case of $\mathfrak{N}_2 \simeq \mathbb{H}_2^* \oplus \mathbb{H}$ the $A$'s will be elements of $so(2)$, while the $C$'s are the same as in the quaternionic case.

The quasi-octonionic cases have been computed directly following [24]: the $C$ is trivial for $i = 6, 7$, whereas for $i = 5$ and $i = 4$ the rotation group of $2-$ and $3-$dimensional spaces acts trivially on the center [2].

The complete list of derivation algebras is:

[2]the computation follows the direct one done by Mosak and Moskowitz.

| Table 1 - Derivations | | |
|---|---|---|
| $dim\,(Z(N))$ | H-type Algebra | $Der_0(\mathfrak{N})$ |
| 0 | $N_0^n$ | $sl(n,\mathbb{R})$ |
| 1 | $N_1^n$ | $sp(n,\mathbb{R}) \ltimes \mathbb{R}^{2n}$ |
| 2 | $N_2^n$ | $so(2) \oplus sp(n) \ltimes \mathbb{R}^{8n}$ |
| 3 | $N_3^n$ | $sp(1) \oplus sp(n) \ltimes \mathbb{R}^{12n}$ |
| 4 | $N_4^n$ | $so(4) \oplus so(3) \ltimes \mathbb{R}^{32}$ |
| 5 | $N_5^n$ | $so(5) \oplus so(2) \ltimes \mathbb{R}^{40}$ |
| 6 | $N_6^n$ | $so(6) \ltimes \mathbb{R}^{48}$ |
| 7 | $N_7^n$ | $so(7) \ltimes \mathbb{R}^{56}$ |

COROLLARY 3.0.6 *Given simple Lie algebra of Heisenberg type* $\mathfrak{N}$ *with* $8 > dim_{\mathbb{R}}(Z(\mathfrak{N})) > 1$, *its Lie algebra of derivations,* $Der_0(\mathfrak{N})/\mathcal{R}$ *is a Lie algebra of compact type.*

We are now able to state the main result of this section.

THEOREM 3.0.7 *If* $\Gamma$ *is a log-lattice in a Lie group* $N$ *of Heisenberg type, the lattice* $Stab_{M_0(N)}(\Gamma)$ *is uniform if and only if* $7 \geq dim(Z(N)) > 1$.

PROOF. In the Lie algebras $Lie(M_0(N_i))$ with $i = 3,4,6,7$ the nilradical coincides with the radical; the quotient $Der_0(\mathfrak{N})/\mathcal{R}_n$ is always of compact type. The above montioned criterion applies directly. For the remaning two cases we should apply the result of Mosak and Moskowitz to the subgroup of

$M_0(N)$ obtained by taking the quotient by the compact abelian factor $SO(2)$; then we extend it to the full group of measure preserving automorphisms. ◇

A very important example of *non-irreducible* H-type Lie algebra can be constructed by considering the composition of quadratic forms arising by taking left and right multiplications by imaginary quaternions, say $\mu_r$ and $\mu_l$: $\mathfrak{N}_3^{a,b} = \mathbb{H}' \oplus \mathbb{H}^a \oplus \mathbb{H}^b$; where the action of $\mathbb{H}'$ is by left multiplication on $\mathbb{H}^a$ and right multiplication on $\mathbb{H}^b$ (cf. [31]).

PROPOSITION 3.0.8 *Any trace zero derivation $D$ of a Lie algebra of Heisenberg-type with center of dimension greater then one and less than eight can be decomposed as $D = D_K \cdot D_N$, where $D_N$ is nilpotent and $D_K$ is in a Lie algebra of compact type.*

PROOF. Just consider the derivation restricted to each irreducible component and apply the result proven in the irreducible case (Corollary 3.3.). ◇

In the case of $\mathfrak{N}_3^{a,b}$ this yields: $Der_0(\mathfrak{N}_3^{a,b}) = sp(1) \oplus sp(a) \oplus sp(b)$ which implies the corresponding $M_0(N_3^{a,b})$ to be considerably smaller than the group of automorphisms that is obtained with the irreducible quaternionic group.

## 3.1 Isometries of H-type groups

In this section we study the isometries of H-type groups and show how their structure gives a necessary and sufficient condition for the existence of non-conformal quasi-conformal mappings. From this will follow that quasi-conformal mappings of certain H-type groups must be conformal. Finally we will study lattices of isometries of a class of solvmanifolds.

Consider an H-type group, $N$, equipped with the left-invariant metric as in [2]. Let $Iso(N)$ be its group of isometries. Then by ([12], sec. 3):

$$Iso(N) = A(N) \ltimes N \tag{3.3}$$

The group $A(N)$ consists of those automorphisms of $N$ whose differentials are isometries of the Lie algebra $\mathfrak{N}$:

$$A(N) = \{\phi \in Aut(N) \mid \phi_* \in Iso(\mathfrak{N})\}.$$

In our situation we can see that in the subalgebra $Der_0(\mathfrak{N})$ of $Der(\mathfrak{N})$, all the elements of the nilradical act on a generic element $X = (Z, V) \in \mathfrak{N}$ by:

$$\phi(X) = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Z \\ V \end{pmatrix} = \begin{pmatrix} B \cdot V \\ 0 \end{pmatrix} \tag{3.4}$$

This implies that they are *not* isometries. We conclude that in the case of a Heisenberg-type group the Lie algebra of $A(N)$ or $A(N)_0$, the connected component of $A(N)$, satisfies:

$$Lie(A(N)) \subseteq \frac{Der_0(\mathfrak{N})}{\mathcal{R}_n}. \tag{3.5}$$

Our computations will be based on this latter fact. As a result we are able to deal with groups of automorphisms *locally* and thus avoid covering space arguments which make their appearence in previous work on the subject (see for example Pansu [27] and Riehm [29]). Since we are interested in the compactness of $A(N)$, and since there are a finite number of connected components, we can restrict our attention to the identity component of the automorphism group.

A straightforward computation gives us the complete list of the identity components of the isometry groups:

| Table 2 - Isometries | | |
|---|---|---|
| $dim(Z(N))$ | H-type Group | $Iso(N)_0$ |
| 0 | $N_0^n$ | $SO(n) \ltimes \mathbb{R}^n$ |
| 1 | $N_1^n$ | $U(n) \ltimes N_1^n$ |
| 2 | $N_2^n$ | $SO(2) \cdot Sp(n) \ltimes N_2^n$ |
| 3 | $N_3^n$ | $Sp(1) \cdot Sp(n) \ltimes N_3^n$ |
| 4 | $N_4^n$ | $SO(4) \cdot Sp(1) \ltimes N_4$ |
| 5 | $N_5^n$ | $SO(5) \cdot U(1) \ltimes N_5$ |
| 6 | $N_6^n$ | $SO(6) \ltimes N_6$ |
| 7 | $N_7^n$ | $SO(7) \ltimes N_7$ |

We see that all the lattices of isometries will be uniform (all groups here clearly are ameanable). At the same time, using the results of the preceeding sections we can prove an analogous statement for $M_0(N)$:

THEOREM 3.1.1 *An irreducible Heisenberg-type group $N$ with center of dimension less than eight has both uniform and non-uniform lattices of measure-preserving automorphisms if and only if* $7 \geq dim(Z(N)) \geq 2$.

PROOF. Equation (3.3) combined with equation (3.5) gives everything we need to know about lattices in the group $M_0(N)$. Given a log-lattice $\Gamma \subset N$ we can set:

$$\tilde{\Gamma} = Stab_{M_0(N)}(\Gamma) \cap A(N)_0$$

we obtain lattice in $A(N)_0$.

LEMMA 3.1.2 *The group* $\Lambda = \tilde{\Gamma} \times \Gamma$ *is a lattice in* $Iso(N)_0$.

PROOF. The set $\Lambda$ is discrete in $Iso(N)_0$.
At the same time for every $\tilde{\gamma} \in \tilde{\Gamma}$:

$$\tilde{\gamma}(\Gamma) = \Gamma,$$

by the definition of stabilizer. Thus $\tilde{\Gamma}$ is a group and therefore a lattice. ◇

In case $dim\,(Z(N)) > 1$, the group $M_0(N)$ is the semidirect product of the a compact $A(N)$ acting on $N$ as a group of automorphisms. Such a semidirect product gives an amenable group and thus all of its lattices are uniform.

Consider now the situation: $dim\,(Z(N)) < 2$.
This implies, using the same notation as before: $A(N)_0 = U(n)$ for $N$ equal to the Heisenberg group

$$M_0(N_1) = Sp(n, \mathbb{R}) \ltimes \mathbb{R}^{2n}. \tag{3.6}$$

The only other possibility is the trivial abelian H-type group $N_0 \simeq \mathbb{R}^n$. Here the isometry group will be:

$$M_0(N_0^n) = SL(n, \mathbb{R}) \tag{3.7}$$

In these two cases $Stab_{M_0(N)}(\Gamma)$ will never be uniform ([24]) and the by the above lemma our theorem follows.◇

The last theorem can be reformulated for general $H$-type groups:

THEOREM 3.1.3 *An Heisenberg type group $N$ with $7 \geq dim(Z) \geq 2$ admits both uniform and non-uniform lattices of measure preserving automorphisms if and only if its Lie algebra $\mathfrak{n} \simeq \mathfrak{z} \oplus \mathfrak{v}^{(1)} \oplus \cdots \oplus \mathfrak{v}^{(n)}$ is either abelian or its irreducible factors are isomorphic to $\mathfrak{n}_1$.*

PROOF. The only thing to note is that the abelian factor needs to be at least two-dimensional in order to ensure the existence of non uniform lattices in the group of measure preserving automorphisms. The rest of the arguments is just the restriction of the previous result to the single irreducible components of $\mathfrak{n}$. ◇

### 3.1.1 Conformal mappings

In his paper ([27]) P. Pansu establishes a result on conformal mappings for the groups $N_3^n$ and $N_7^n$. A homeomorphism $T : U \to U'$ between open subsets of an H-type group is called *$\lambda$-quasiconformal* if there exists a real number $\lambda \in [1, \infty)$ such that for all $x \in U$, $\epsilon > 0$ and all sufficiently small $r$ there is an $R > 0$ such that:

$$B(Tx, R) \subseteq T\left(B(x, r)\right) \subseteq B(Tx, (\lambda + \epsilon)R).$$

A quasiconformal map $\phi$ is said to be *conformal* when $\lambda = 1$.

This is equivalent to saying that $\phi$ is quasiconformal and $\phi_*$ is an isometry of the Lie Algebra of $N$ times a dilation ([27], pg. 44).

Pansu proves the following result ([27], Corollary 11.2.):

*Theorem - A quasiconformal homeomorphism of $N_3^n$ (resp. of $N_7^n$), acting as maximal unipotent group of isometries on the hyperbolic quaternionic (resp. octonionic) symmetric space, is conformal.*

Combining our results with those of Pansu we can prove a slightly more general statement.

THEOREM 3.1.4 *A quasiconformal homeomorphism of a simple H-type group with center of dimension greater than two and less than eight must be conformal.*

PROOF. Let $\phi$ be the homeomorphism, $N$ our H-type group and $\mathfrak{N} \simeq \mathfrak{Z} \oplus V$ its Lie Algebra satisfying $dim(Z(\mathfrak{N})) > 2$. By Pansu's differentiability theorem ([27], sec.VII) the differential exists almost everywhere. We first observe that ([2], pg.12) that its differential is a grading-preserving automorphism[3] of $\mathfrak{N}$:

$$\phi_*(\mathfrak{V}) \subset \mathfrak{V}$$

By equation (3.4) it is clear that the component with respect to the radical of any grading-preserving automorphism is zero. This in turn implies, by the hypothesis and equation (3.5), that $\phi_* = \phi'_* + \phi''_*$, where $\phi'_* \in Der_0(\mathfrak{N})/\mathcal{R}_n$

[3]In fact for the Heisenberg group this follows immediately from the properties of *contact transformations* (cf. [19])

and $\phi''_*$ is a matrix of the type $\phi''_* = \begin{pmatrix} \lambda \cdot Id & 0 \\ 0 & \lambda/2 \cdot Id \end{pmatrix}$ and therefore $\phi = exp(\phi_*)$ is a dilation times an isometry, which equivalent to saying that the map is conformal.◇

### 3.1.2 Solvable extensions

Consider the Riemannian solvmanifold obtained as extension $A \ltimes N$ $(AN)$ of an H-type group $N$ by its one-dimensional group $A$ of dilations, equipped with the $Iso(N)$-invariant metric as in [2]. The connected component of its isometry group certainly contains all left-translations by elements of the group itself as well as the compact group $A(N)_0$, the rotations of $N$. Now take $N \simeq N_3^{a,b}$, with $a, b \neq 0$. We will then have: $A(N)_0 = Sp(1) \cdot Sp(a) \cdot Sp(b)$. The groups $A \ltimes N_3^{a,b}$ have interesting properties and (when $a + b$ is constant) provide a class of isospectral non-isometric harmonic manifolds (cf.[31], [4]). We now prove the following result:

THEOREM 3.1.5 *The group of isometries of $AN_3^{a,b}$ is not unimodular unless $a$ or $b$ is zero and conversely.*

PROOF. In case one of the two indices is zero the isometry group is the simple Lie group $Sp(n, 1)$ and $AN$ is a symmetric space ([2], section 6) and hence its isometry group is unimodular.

Otherwise, we claim that the elements in $A(N)$ are the only isometries that fix the origin. To see this consider for semplicity the case when $a = b = 1$ and a geodesic $\gamma(t)$ such that $\gamma(0) = e$ and $\gamma'(0) = (X_a, X_b, Z, s) \in \mathfrak{N}_3^{a,b} \oplus \mathfrak{A}$ of unit norm. These geodesics will have the form ([2], prop. 2.2)

$$\gamma(t) = (c_1 X + c_2 J_Z X, c_3 Z, c_4 s), \tag{3.8}$$

where the $c_i$'s are functions of $t$ and $\|Z\|$.

Suppose there is an isometry $\Phi$, fixing the origin, which "mixes" the components of $N_3^{a,b}$ belonging to $\mathbb{H}^a$ and $\mathbb{H}^b$; in other words since $\Phi$ permutes such geodesics, it will induce a map $\phi$ which acts as an orthogonal transformation on the tangent space

$$T_e(NA) \;=\; \mathfrak{N} \oplus \mathfrak{A} \;\simeq\; \mathbb{H}^a \oplus \mathbb{H}^b \oplus \mathbb{H}.$$

Now let, with obvious notation, $\gamma(t) = g = (X, Z, s) = (X_a, X_b, Z, s)$; we get:

$$\begin{aligned}
\Phi(\gamma(t)) \;&= \\
&= \;\Phi(c_1(X_a, X_b) + c_2(ZX_a, X_b Z), c_3 Z, c_4 s) \\
&= \;(c_1(\Phi_a(g), \Phi_b(g)) + c_2(\Phi_Z(g)\Phi_a(g), \Phi_b(g)\Phi_Z(g)), c_4\Phi_A(g))
\end{aligned}$$

where the subscripts denote the projections of $\Phi$ onto $A$, the center $Z$ and its complement $V$; this amounts to a norm-preserving map satisfying both identities: $\Phi(g \cdot g_1) = \Phi(g) g_1$ and $\Phi(g_1 \cdot g) = g_1 \Phi(g)$, which is impossible since by non-commutativity of the quaternions the left and right multiplications are inequivalent. For the other values of $a$ and $b$ the argument is completely analogous. Since the group of dilations commutes with $Iso(N)$ the isometry group is: $(A \times Sp(1) \cdot Sp(a) \cdot Sp(b)) \ltimes N_3^{a,b}$. Since $A$ acts by measure-distorting automorphisms this group is non-unimodular. $\diamondsuit$

By using a variant of the previous argument we can prove *directly* the following fact:

PROPOSITION 3.1.6 *The full group of isometries of $A \ltimes N_3^{n-1}$ is $Sp(n,1)$.*

PROOF. We know by looking at the Iwasawa decomposition of $Sp(n,1)$ that it is generated by $AN_3^{n-1}$ together with $K \simeq Sp(1) \cdot Sp(n)$. By hypothesis the group $AN$ acts isometrically by left-translations. The action group of isometries that preserve $e$ on $AN_3^{n-1}$ permutes all geodesics in (3.8). Let $\Phi$ be such an isometry. If we set $g = (X, Z, s)$

$$\|g\| = \|(X, Z, s)\| = \|X\| + \|Z\| + |s| = 1,$$

and $\gamma$ a geodesic such that $\gamma'(0) = g$. We can consider the map $\phi : \mathbb{H}^n \to \mathbb{H}^n$ satisfying $\Phi(\gamma(0))' = \phi(g)$; we can write:

$$\Phi(\gamma(t)) = (c_1\phi_1(g) + c_2 J_{\phi_2(g)}\phi_1(g), c_3\phi_2(g), c_4\phi_3(g)),$$

where the subscripts indicate the projections onto the first, second and third component. The map $\phi$ has to be *left-linear* in the sense that:

$$\phi_1(g_1 \cdot g) = g_1\phi_1(g) \qquad \forall g \in \mathbb{H}^n, g_1 \in \mathbb{H}$$

in order to preserve structure of the geodesics, in addition it has to be norm-preserving: $\|g\| = \|\phi(g)\|$. If we choose $\phi \in Sp(n)$, it is obvious that $\Phi$ is an isometry that fixes the identity. The set of all such $\Phi$'s forms a group isomorphic to $Sp(n)$.

The only other maps that send the unit quaternionic sphere in $\mathbb{H}^n$ into itself satisfying the described properties are the ones acting as left multiplications by a unit quaternion on the complement of the center and by conjugation on the center of the Lie algebra of $AN_3^{a+b}$. [4] In this way we get a set of maps $\{\Phi_h\}$, for any unit quaternion $h$, given by:

$$\begin{aligned} \phi_h : \mathbb{H}^n &\longrightarrow \mathbb{H}^n \\ (X, Z, s) &\mapsto (hX, h(Z)h^{-1}, s))) \end{aligned}$$

A simple calculation shows that the induced isometries are:

$$\Phi_h(c_1 X + c_2 Z \cdot X, c_3 Z, c_4 s) \mapsto (c_1 hX + c_2 J_{Z'} hX, c_3 Z', c_4 s'), \qquad (3.9)$$

where $Z' = hZh^{-1}$ is imaginary (conjugations preserve real and imaginary parts) and $s' = s$. This new set of isometries forms a group isomorphic to $Sp(1)$.

We then get that $Iso(A \ltimes N_3^{n-1})$ is generated by the subgroups $A \ltimes N_3^{n-1}$ and $Sp(n) \cdot Sp(1)$; therefore $Iso(A \ltimes N_3^{n-1})$ is isomorphic to $Sp(n, 1)$ $\diamond$

## 3.2 Arithmetic Lattices in Rank 1 Groups

In our last section we will obtain an existence theorem on arithmeticity of lattices in the groups $Sp(n, 1)$, $n > 1$ and the exeptional group. This will be done by use of techniques based on our present results together with results of Borel (on maximal subgroups) as well as those of Kazdan-Margulis on the

[4] Here everything depends on the choice that we have to make to give to $\mathbb{H}^n$ the structure of a left- or right module over $\mathbb{H}$.

intersection of a non-uniform lattice with the $N$ part of the Iwasawa decomposition, together with the Margulis criterion for arithmeticity envolving the commensurator of a lattice: a lattice in a semisimple Lie group is arithmetic if and only it has infinite order in its commensurator.

Let $\Gamma \subset G$ be a non-uniform lattice. By the results of Kazdhan-Margulis ([28], XI Cor.11.13) ) we have that a suitable conjugate $\Gamma'$ of $\Gamma$ exists such that $\Lambda = \Gamma' \cap N$ is a lattice in $N$. We recall that a lattice $\Gamma$ in real Lie group $H$ is said to be *arithmetic* if there exists an algebraic group $G$ defined over $\mathbb{Q}$, such that its integer points $G_{\mathbb{Z}}$ are mapped by a continuous homomorphism with compact kernel $\phi : G_{\mathbb{Z}} \to H$ onto another lattice $\Gamma' = \phi(\Gamma)$ such that $\Gamma$ and $\Gamma'$ are commensurable.

In the case when $G = Sp(n,1)$ equipped with a $\mathbb{Q}$-structure we observe that the structure of the measure preserving automorphisms can be used to prove the result below. A similar argument also works for the exceptional group, $F_4$.

Before we state our result we remind the reader of a simple result concerning simply connected nilpotent Lie groups. Given any lattice $\Lambda$ in such a group $N$ with a $\mathbb{Q}$-structure, it holds that $\Lambda \cap N_{\mathbb{Z}}$ has finite index in $\Lambda$ and $N_{\mathbb{Z}}$, the elements of $N$ with integer coordinates. That means that $\Lambda$ is commensurable to $N_{\mathbb{Z}}$ (see e.g. [6], pg. 162).

THEOREM 3.2.1 *Any non-uniform lattice $\Gamma$ in $Sp(n,1)$ must be arithmetic.*

PROOF. Since $N$ is simply connected nilpotent we can assume $\Lambda = \Gamma \cap N \simeq N_{\mathbb{Z}}$.

Let $K_\Lambda$ be defined as the subgroup of the group of elements of $Sp(n)$ with integer entries that stabilize $\Lambda \subset \mathbb{H}^* + \mathbb{H}^{n-1}$.[5] Since $\Lambda \subset \Gamma$ we have that: $\Lambda \cdot \Gamma = \Gamma$. Now take $g \in K_\Lambda$, and consider the action $\phi_g(\gamma) = g\gamma g^{-1}$ for $\gamma \in \Gamma$. By writing $\gamma = \lambda_1 \bar{\gamma} \lambda_2$ for some $\lambda_i \in \Lambda$, we have: $\phi_g(\gamma) = g\lambda_1\bar{\gamma}\lambda_2 g^{-1}$. But since $K_\Lambda$ fixes $\Gamma$, we get $\phi_g(\gamma) \in \Gamma$. This in turn implies $K_\Lambda \subset Comm_G(\Gamma)$. Obviuosly $\Lambda \subset Comm_G(\Gamma)$. Moreover, since the elements of $Sp(n)_{\mathbb{Z}}$ not contained in $K_\Lambda$ are *central*, they also belong to the commensurator of $\Gamma$. Since the maximal compact subgroup $K$ of $Sp(n,1)$ is $Sp(n) \cdot Sp(1)$, with the last factor acting simply by left translations and conjugations as described in Prop. 3.1.6., we conclude that the entire group $Sp(n,1)_{\mathbb{Z}}$, of elements with integer entries - which is generated by $Sp(n)_{\mathbb{Z}} \cdot Sp(1)_{\mathbb{Z}}$ and $\Lambda$- is contained in the commensurator. Now there are only two possibilities: either $Sp(n,1)_{\mathbb{Z}} \subset \Gamma$ and then, by Borel's maximality theorem, $\Gamma = Sp(n,1)_{\mathbb{Z}}$ ([1], Theorem 7), or there is an element of infinite order in $Sp(n,1)_{\mathbb{Z}}$ not contained in $\Gamma$; that implies $[\Gamma : Comm_G(\Gamma)] = \infty$ and that in turn, by Margulis' criterion forces it to be arithmetic ([23], chap. IX). $\diamond$

[5]This is because a quotient group of $Sp(1)$, the orthogonal group $SO(3) \simeq Sp(1)/\{\pm I\}$, acts on $\mathbb{H}^*$.

# Chapter 4

# Harmonic Analysis on H-type Groups

## 4.1 Motivation and Background

It is a well known fact that, given any two finite abelian groups $G_1$ and $G_2$ with the same number of elements $|G_1| = |G_2| = n$ have isomorphic group algebras. We will denote with $F(G_i)$ $(i = 1, 2)$ the convolution algebras of complex-valued functions in the two groups equipped with the involution $f^*(g) = f(\bar{g}^{-1})$ and the usual product:

$$f_1 * f_2(s) = \frac{1}{n} \sum f_1(s \cdot g) f_2(g^{-1}) \qquad \forall s \in G_i, \;\; f_1, f_2 \in F(G_i). \tag{4.1}$$

Any $f \in F(G_i)$ can be written as a linear combination of the $n$ orthonormal characters $\{\chi_k^{(i)}\}$ of $G_i$: $f_1(g) = \sum_1^k a_k \chi_k^{(i)}(g)$, $f_2(g) = \sum_1^k a_k \chi_k^{(i)}(g)$. So (4.1) becomes:

$$f_1 * f_2(s) = \frac{1}{n}\sum_1^k a_k \bar{b}_k \chi_k^{(i)}(g),$$

implying that both algebras $F(G_1)$ and $F(G_2)$ are $*$-isomorphic to $D_n(\mathbb{C})$, the algebra of diagonal $n \times n$ complex matrices with the product $A \cdot B = A\bar{B}$ (for detailes see e.g. the discussion in [26], pgg. 185 ff.).

In this last chapter we present a short study of this phenomenon in the context of Lie groups.

## 4.2 Abelian Lie groups

The first result concerns compact Lie groups;

PROPOSITION 4.2.1 *Any two compact connected abelian Lie groups have isomorphic $L^1$-algebras.*

PROOF. Let $G_1$ and $G_2$ be the two groups and $\{\chi_k^{(1)}\}$ $\{\chi_k^{(2)}\}$ their characters. Let $f$ be a function $L^1(G_i)$; if we define $a_k = \int_{G_i} f(g)\bar{\chi}_k^{(i)}$, the sequence $(a_1, a_2, \ldots)$ is an element of $S(\mathbb{N})$, the Banach space of sequences $\{a_k\}$ satisfying: $lim_{k\to\infty} a_k = 0$, equipped with the sup-norm. It is a well known fact that not every sequence in $S(\mathbb{N})$ is in the image of the $\Phi_i$ ([15], pg. 22) . Despite of that we can still prove the following

LEMMA 4.2.2 *The images of $\Phi_1$ and $\Phi_2$ in the space $S(\mathbb{N})$ coincide.*

PROOF. Consider any function $f \in L^1(G_1)$ and let $(a_1, a_2, \ldots)$ be its image under the map $\Phi_1$. We can find a sequence $\{f_k\}$ of functions in $L^2(G_1)$ (and since $G_1$ compact also contained in $L^1(G_1)$) such that $\|f - f_k\| \to 0$ in $L^1$-norm. That implies, in the sup-norm of $S(\mathbb{N})$ (see [15], pg. 13):

$$\|\Phi_1(f) - \Phi_1(f_k)\| \longrightarrow 0.$$

Now we have to observe that both $\Phi_1$ and $\Phi_2$ are $1-1$ maps when restricted to $L^2$. Therefore there is a converging sequence $\{g_k\}$ in $L^2(G_2)$ that satisfies: $\Phi_2(g_k) = \Phi_1(f_k)$; if we denote with $g$ the $L^1$ function on $G_2$ satisfying $\lim_k g_k = g$ we get that: $\Phi_1(f) = \Phi_2(g)$; the argument can be proven by switching $G_1$ and $G_2$ and that proves the lemma. $\diamond$

The maps $\Phi_i : f \to (a_1, a_2, \ldots)$ are $*$-morphisms of Banach algebras: they preserve the convolution structure (on $S(\mathbb{N})$ the convolution is the same as defined for the finite group algebras). Thus $L^1(G_1) \simeq \Phi_i(G_i) \simeq L^1(G_2)$ and the theorem is proven. $\diamond$

The assumption of compactness for the Lie groups in the previous proposition is essential.

THEOREM 4.2.3 *Two connected and simply connected abelian Lie groups $A_1$ and $A_2$ are isomorphic if and only if their $L^1$-algebras are.*

PROOF. Suppose that $A_1 \simeq \mathbb{R}^n$ and $A_2 \simeq \mathbb{R}^m$ with $m \neq n$. The irreducible unitary representations are the exponentials

$$u_\nu(x) = exp(i < \nu, x >),$$

where $\nu$ is an element of $\mathbb{R}^n$ or $\mathbb{R}^m$ and $< \cdot, \cdot >$ will be resp. the scalar product in $\mathbb{R}^n$ or $\mathbb{R}^m$. All unitary representations can be decomposed as continuous sums of the characters $u_\nu$ ([16]). Let $\hat{A}_1$ and $\hat{A}_2$ be the unitary duals of the two groups equipped with the described topology. There is a $1-1$ correspondance between the irreducible unitary representations of the groups and the $*$-representations of their $L^1$-algebras; this correspondence is continuous and therefore since $\hat{A}_1 \simeq \mathbb{R}^n$ and $\hat{A}_2 \simeq \mathbb{R}^m$, by dimension such a correspondence cannot exist, thus $L^1(A_1)$ and $L^1(A_2)$ cannot be isomorphic. ◊ By constructiong the finite-dimensional duals, the same argument can be used to prove that any two Heisenberg groups $N_1$ and $N_2$ cannot have isomorphic $L^1$-algebras unless $N_1 \simeq N_2$.

## 4.3 Two-step Nilpotent Lie Groups

Consider any Heisenberg type group $N$ as defined in [2]. For any $g \in N$ we set: $g = exp(Z) + exp(V) = (Z, V)$. By [14] we know that the unitary irreducible representations of $N$ are parametrized by the elements $t \in Z(\mathfrak{N})$ and each one of those is equivalent to one of the form:

$$\pi_s(g)F(W) = \pi_s(exp(Z+V))F)(W) = F(W+V)E(s,g,W),$$

where $E$ is the exponential function defined by:

$$E(s,g,W) = e^{i<s,Z>-\frac{1}{4}\|s\|(\|V\|^2+2(<W,V>+i<t,[V,W]>))}, \tag{4.2}$$

and $s$ and $t = \frac{s}{\|s\|}$ are elements in the center of the Lie algebra $\mathfrak{N}$ of $N$; $F$ is an entire function on the complex space $(\mathfrak{W}, J_t)$ such that

$$\|F\|^2 = \int F(z)e^{-\frac{\|s\|\|w\|^2}{2}}dz < \infty.$$

All irreducible unitary representations of an H-type group that are not one-dimensional are equivalent to one of these. Consider now the two non-isomorphic H-type groups $N_3^{(a,b)}$ and $N_3^{a+b}$, where $a$ and $b$ are any two fixed positive integers. Their structure as $H$-type Lie groups is defined by taking left and right quaternionic muliplication (in the first one) and just left multiplication (for the second). It may be useful to recall here that the difference in the structure of the two groups is given by the fact that the map $J$ is defined, with obvious notation, as $J_t(V_a, V_b)) = (tV_a, tV_b)$ for $N_3^{a+b}$ and as $J_t(V_a, V_b) = (tV_a, V_b t)$ for $N_3^{(a,b)}$. The scalar product on the Lie algebra is obtained by polarisation of the quaternionic norm on the space $\mathbb{H}^* \oplus \mathbb{H}^a \oplus \mathbb{H}^b \simeq \mathbb{H}^* \oplus \mathbb{H}^{a+b}$ and therefore is the same for both groups. In other terms the group structure of $N_3^{(a,b)}$ is obtained by "mixing" left- and right-multiplication by an *imaginary* quaternion $t \in Z(\mathfrak{N}_3^{(a,b)})$. For simplicity consider the case when $a = b = 1$ and the action of $Z(\mathfrak{N}_3^{(a,b)}) \simeq \mathbb{H}^*$ is by multiplication on the left on the first quaternionic component of $\mathfrak{V} \simeq \mathbb{H} \oplus \mathbb{H}$ and by right multiplication on the second.
Consider now the expression in (4.2):

$$< W, V > -i < t, [V, W] > = < W, V > -i < J_t V, W >;$$

in the two groups this is equal to either

$$< (W_a, W_b), (V_a, V_b) > +i < (tV_a, tV_b), (W_a, W_b) >$$

or, case $N = N_3^{(a,b)}$:

$$< (W_a, W_b), (V_a, V_b) > +i < (tV_a, V_b t), (W_a, W_b) > .$$

If we write $V_b^* = V_b t - tV_b$, we get:

$$
\begin{aligned}
& < (W_a, W_b), (V_a, V_b) > +i < (tV_a, V_b t), (W_a, W_b) >= \\
= \quad & < (W_a, W_b), (V_a, V_b) > +i < (tV_a, tV_b), (W_a, W_b) > +i < (0, V_b^*), (0, W_b) > .
\end{aligned}
$$

This means that the operators in $B(\mathcal{H}_\pi)$, produced by the representation $\pi_t^{(a,b)}$ of $N_3^{(a,b)}$ are also obtained from a unitary irreducible representation of $N_3^{a+b}$:

$$
\pi_t^{(a,b)}(Z, V_a, V_b)F(W_a, W_b) = \pi_t^{(a+b)}(Z, V_a, V_b)F(W_a, W_b) \cdot e^{iC<(0,V_b^*),(0,W_b)>},
$$

If we put then, for any $g = (Z, V_a, V_b) \in N_3^{a+b}$:

$$
\bar{\pi}^{a+b}(g)F(W_a, W_b) = \pi^{(a,b)}(g))F(W_a, W_b) \cdot e^{iK<V_b^*,W_b>}
$$

we obtain a unitary *irreducible* representation of $N_3^{a+b}$. We have therefore proven the following result:

**PROPOSITION 4.3.1** *The Lie groups $N_3^{a+b}$ and $N_3^{(a,b)}$ have the same irreducible unitary representations.*

Here *the same* means that given any Bargmann-Fock representation $(\pi, \mathcal{H}_\pi)$ of $N_3^{(a,b)}$, the subalgebra $\pi(N_3^{(a,b)}) \subset B(\mathcal{H}_\pi)$ coincides with the one given by the corresponding representation $\bar{\pi}$ of $N_3^{a+b}$ on the same Hilbert space. It also has to be noticed that this implies that the respective dual spaces of the two groups are isomorphic as topological spaces.

From the last result we get:

**THEOREM 4.3.2** *The groups $N_3^{a+b}$ and $N_3^{(a,b)}$ have isomorphic $L^1$-algebras.*

**PROOF.** From the general theory of unitary representations we know that the left-regular representation of $G = N_3^{(a,b)}$ written, for any $f \in L^2(N_3^{(a,b)})$, as:

$$\lambda(g)(f(x)) = f(g^{-1}x),$$

can be decomposed as a direct integral of irreducible unitary representations:

$$\lambda(g)(f) = \int_{\pi\in\hat{G}}^{\oplus} \pi(g)(f)dg. \tag{4.3}$$

The left-regular representation $\lambda$ induces a *faithful* representation of $L^1(N_3^{(a,b)})$ on $L^2(N_3^{(a,b)}) = \mathcal{H}_\lambda$:

$$\lambda^*(f) = \int_G f(g)\lambda(g)dg.$$

Since the images of the unitary irreducibles of $N_3^{a+b}$ and $N_3^{(a,b)}$in $B(\mathcal{H}_\lambda)$ coincide, the representation $\lambda^*$ can be viewed also as a $*$-representation of $L^1(N_3^{a+b})$ that produces the same $*$-subalgebra of $B(\mathcal{H}_\lambda)$. That implies $L^1(N_3^{a+b})$ and $L^1(N_3^{(a,b)})$ are isomorphic. ◊

Our last result is a simple observation about the radial $L^1$-functions of the two groups considered before. A function $f \in L^1(N)$ is said to be radial if $f(g)$ depends only on the distance of $g$ from the identity. If we write $g = exp(Z + V_a + V_b) = (Z, V_a, V_b)$ this is the same as saying that $f(Z, V_a, V_b)$ depends only on $\|Z\|$ and $\|V_a + V_b\|$

**THEOREM 4.3.3** *The algebras of radial $L^1$-functions of $N_3^{a+b}$ and $N_3^{(a,b)}$ coincide.*

The symbols $*_l$ and $*_{l,r}$ denote resp., the convolution in $L^1(N_3^{a+b})$ and $L^1(N_3^{(a,b)})$.

PROOF. Let $f_1$ and $f_2$ two radial functions in $L^1(N_3^{a+b})$, $g = (Z, V_a, V_b)$ and $g_1 = (Z_1, V_1, W_1)$ an element of $N_3^{a+b}$:

$$f_1 *_l f_2(g_1) = \int_{N_3^{a+b}} f_1(g_1 g) f_2(g^{-1}) d\mu =$$

$$= \int_{N_3^{a+b}} f_1(Z + Z_1 + Im(V_1\bar{V}_a) + Im(W_1\bar{V}_b), V_a + V_1, V_b + W_1) f_2(g^{-1}) d\mu$$

$$\int_{N_3^{a+b}} f_1(Z + Z_1 + Im(V_1\bar{V}_a + Im(W_1 W_1^{-1}\bar{V}_b W_1, V_a + V_1,$$

$$W_1^{-1}(V_b + W_1)W_1) f_2(g^{-1}) d\mu =$$

$$\int_{N_3^{(a,b)}} f_1(Z + Z_1 + Im(V_1\bar{V}_a) + Im(\bar{V}_b W_1), V_a + V_1, V_b + W_1) f_2(g^{-1}) d\mu$$

$$= \int_{N_3^{(a,b)}} f_1(g_1 g) f_2(g^{-1}) d\mu = f_1 *_{l,r} f_2(g_1).$$

In other terms, the convolution structure of the radial functions in the two groups is the same:

$$f_1 *_l f_2(g) \;=\; f_1 *_{l,r} f_2(g). \tag{4.4}$$

◇

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