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THEORY OF SOLVMANIFOLDS

by

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SECTION A. ALGEBRAIC GROUPS

1. Structured Spaces

Let X be a topological space. A functional structure on X is a function F which assigns to each open subset U of X a subalgebra F_U of the algebra of real valued continuous functions on U satisfying the following.

1. If $U \neq \phi$ is an open subset of X then F_U contains all the constant functions on U. By convention we let F_ϕ be solely the set consisting of the constant function 0.

2. If U and V are open subsets of X where $U \subset V$, then for each $f \in F_V$, the restriction of f to U, $f_U \in F_U$.

3. If (U_α) is a collection of open subsets of X and $U = \bigcup_\alpha U_\alpha$, and if f is a real valued function defined on U such that for each α, $f_{U_\alpha} \in F_{U_\alpha}$, then $f \in F_U$. A structure space is a pair (X, F_X) where X is a topological space, and F_X is a functional structure on X. A morphism of structured spaces $r: (X,F_X) \to (Y,F_Y)$ is a continuous map $r: X \to Y$ such that for each open subset V of X and each element f of $F_Y(V)$ we have $f \circ r \in F_X(r^{-1}(V))$. It is clear that the collection of functional structures and morphisms of structured spaces is a category.

2. Functional Structures on R^n

Let R denote the reals and R^n denote the set of all ordered n-tuples of R. We shall define two functional structures on R^n. Let R^n be considered as a topological space with the Euclidean topology. The collection of all analytic functions defined on open subsets of R^n defines a functional structure on R^n and we shall call the resulting structured space the analytic manifold R^n. Let $R[X_1,\dots,X_n]$ be the integral domain of all polynomials with real coefficients in n variables and let $R(X_1,\dots,X_n)$ be the quotient field of $R[X_1,\dots,X_n]$. We can regard $R[X_1,\dots,X_n]$ as a ring of functions on R^n, and $R(X_1,\dots,X_n)$ as a collection of functions defined on subsets of R^n. An algebraic subset of R^n is the set of all common zeros of a finite number or equivalently an ideal of polynomials of $R[X_1,\dots,X_n]$. The algebraic subsets of R^n determine the closed subsets for a topology on R^n which we call the Zareski topology. Relative to the topology we can regard $R(X_1,\dots,X_n)$ as a collection of functions defined on open subsets of R^n. Let A be an open subset of R^n. A rational function f of A is a map $f\colon A \to R$ which in some neighborhood of each point of A can be given by an element of $R(X_1,\dots,X_n)$. The topological space R^n with the Zareski topology along with the map which assigns to each open subset A of R^n the collection of all rational functions of A is a structure space which we call the affine variety R^n. The morphisms of the affine variety A where A is an open subset of R^n having the induced functional structure from the affine variety R^n into R^m include the collection of all functions $f\colon A \to R^m$ given by $f(X_1,\dots,X_n) = (f_1(X_1,\dots,X_n), \dots , f_m(X_1,\dots,X_n))$ where each $f_i \in R(X_1,\dots,X_n)$ and is defined on A, and can locally be represented by such functions.

3. <u>Functional Structures on</u> M(n)

Let M(n) denote the collection of all real nXn matrices. Let $i: M(n) \to R^{n^2}$ be the bijection given by

$$i((\alpha_{i_j})) = (\alpha_{1_1}, \ldots, \alpha_{1_n}, \ldots, \alpha_{n_1}, \ldots, \alpha_{n_m}) .$$

We make M(n) into a vector space by requiring that i be a vector space isomorphism. Defining a multiplication on M(n), $[X,Y] = XY - YX$ we make M(n) into a real Lie algebra. By requiring that i be a structured space isomorphism of M(n) onto the analytic manifold R^{n^2} we make M(n) into a structured space called the analytic manifold M(n). By requiring i be a structure space isomorphism of M(n) onto the affine variety R^{n^2} we make M(n) into a structured space called the affine variety M(n) .

Let GL(n) be the group of all nonsingular matrices on M(n) . GL(n) is an open subset of the analytic manifold M(n) and inherits a functional structure from M(n) . GL(n) considered both as a group and as the above structured space will be called the Lie group GL(n) . We shall assume that the reader is familiar with the general theory of Lie groups (see Chevelley [7]), although we shall list some of the theorems we shall need. GL(n) is an open subset of the affine variety M(n) and inherits a functional structure from M(n) . GL(n) considered both as a group and as the above structure space will be called the affine group GL(n) . We are now finally in a position to define the concept of an algebraic group. An algebraic subgroup of GL(n) is a subset of GL(n) which is both a subgroup of GL(n) and which is a closed subset of the affine variety GL(n) . A rational representation r of an algebraic subgroup A of GL(m) on GL(n) is a rational function of an open subset of the affine variety M(m) containing A into GL(n) whose restriction to A is a group homomorphism. We shall list all the theorems which we will need from the general theory of algebraic groups. For further reference (see Chevelley [7], Mostow [9]).

We collect the algebraic group facts needed in this section. Let X be a matrix subgroup of GL(n) . Since GL(n) is an algebraic group and since the intersection of a collection of algebraic groups is an algebraic group there is a smallest algebraic group containing X , called the algebraic hull of X and denoted by $\mathfrak{a}(X)$. Another relevant concept is the canonical decomposition of an element of GL(n) . Let X be an element of GL(n), then there is an element u of GL(n) all of whose eigen values are one and an element t of GL(n) which is diagonalizable over some extension field, uniquely determined by X, such that X = UT = TU . In general an element u of GL(n) all of whose eigen values are one is called unipotent, and an element t of GL(n) which can be diagonalized over $\mathbb{C}$ is called semi-simple.

1. Let X and Y be matrix subgroups of GL(n) .
   (a) if X normalizes Y, then $\mathfrak{a}$ (X) normalizes $\mathfrak{a}$ (Y) .
   (b) $\mathfrak{a}(\mathfrak{a}(X), \mathfrak{a}(Y)) = \mathfrak{a}(X,Y)$.
   (c) if X is abelian (solvable) then $\mathfrak{a}$ (X) is abelian (solvable).
   (d) if X is an algebraic group then the unipotent and semi-simple parts of every element on X are again in X .
   (e) if each element of X is unipotent then there is a p in GL(n) such that $p X p^{-1}$ consists of upper triangular matrices.

2. Let r be a rational representation of an algebraic subgroup G of $GL(n)$ onto an algebraic subgroup $\bar{G}$ of $GL(m)$.

(a) the image under r of a unipotent element (semi-simple) of G is again unipotent (semi-simple).

(b) every unipotent (semi-simple) element of $\bar{G}$ is the image under r of some unipotent (semi-simple) element of G .

3. Let K be a normal algebraic subgroup of an algebraic group G of $GL(n)$. Then there is a rational representation of G into $GL(m)$ for some m whose kernel is K .

4. Let L be a Lie algebra.

(a) the collection of all Lie algebra automorphisms of L is an algebraic group.

(b) if L is the Lie algebra of an analytic simply connected Lie group G, then there is a topological group isomorphism of $Aut(G)$, the analytic group automorphisms of G, and $Aut(L)$, the Lie algebra automorphisms of L . We make the following conventions.

An automorphism α of G shall be called unipotent when the corresponding automorphism α_L of L is unipotent. An automorphism α of G is called semi-simple when α_L is semi-simple. A group α of automorphisms of G will be called completely reducible when α_L is completely reducible. Let α be a group of automorphisms of G, then by $\mathcal{G}(\alpha)$ we mean the group of automorphisms of G corresponding to $\mathcal{G}(\alpha_L)$. By (a) this does make sense as $\mathcal{G}(\alpha_L)$ is a group of Lie algebra automorphisms of L .

Let α be an automorphism of G . Then α_L is uniquely determined as the Lie algebra automorphism of L such that $\exp \cdot \alpha_L = \alpha \exp$.

SECTION B. THE STRUCTURE OF SIMPLY CONNECTED NILPOTENT ANALYTIC GROUPS

1. Introduction

We shall assume we have identified M(n) with the Lie algebra of the Lie group GL(n) . Where $\alpha(t) = (\alpha_{ij}(t)): R \to GL(n)$ is a one parameter subgroup of GL(n), the corresponding element of the Lie algebra of the Lie group of GL(n) is given by $\alpha'(0) = (\alpha_{ij}'(0))$ in M(n), and under this identification the exponential map from the Lie algebra of the Lie group GL(n) to the Lie group GL(n) is given by the converging power series $\exp X = 1 + X + {}^1/_{2!} X^2 + \ldots$ for X on M(n) . The exponential is an analytic manifold homomorphism of the analytic manifold M(n) into the analytic manifold GL(n) which takes a connected open neighborhood of zero in M(n) analytic manifold isomorphically onto some open connected neighborhood of GL(n) and takes lines through zero in M(n) onto one parameter subgroups of GL(n) .

2. The Algebraic Group U(n)

Let T(n) be the collection of all nil upper triangular matrices on M(n) . T(n) is a Lie subalgebra of the Lie algebra M(n) and a closed subset of the affine variety M(n) . Let U(n) be the collection of all unipotent upper triangular matrices. U(n) is a closed Lie subgroup of GL(n) and an algebraic subgroup of the affine group GL(n) . We shall assume the Hausdorf-Campbell formula for X and Y in M(n) lying in T(n): $\exp X \exp Y = \exp(X + Y + F(X,Y))$ where F(X,Y) is a finite sum of rational multiples of Lie brackets in X and Y . We shall now prove a sequence of lemmas which along with the Birkhoff embedding theorem will allow us to consider the problems of real nilpotent groups as problems in algebraic subgroups of U(n) .

Lemma 2.1. (a) $T(n)$ is the Lie algebra of $U(n)$.

(b) $\exp: T(n) \to U(n)$ is a Zariski homeomorphism of $T(n)$ onto $U(n)$.

Proof: (a) $U(n)$ being homeomorphic to $R^{n(n-1)/2}$ is an analytic subgroup of $GL(n)$ of dimension $n(n-1)/2$. To find the Lie algebra of $U(n)$ it is sufficient to find $(n-1)n/2$ one parameter subgroups of $U(n)$ whose corresponding elements in $M(n)$ are linearly independent in the real Lie algebra $M(n)$. Let us consider the one parameter subgroup of $U(n)$ given as follows. Let $1 \le i < j \le n$. $\alpha(i,j)(t)$ is the matrix of $U(n)$ having zeroes below the diagonal, one on the diagonal, zeroes above the diagonal except on the $(i,j)^{th}$ place where we have t . There are $(n-1)n/2$ such one parameter groups. The corresponding element of the Lie algebra of $U(n)$ in $M(n)$ is given by the matrix $E(i,j)$ which has zeroes everywhere except in the $(i,j)^{th}$ place where it has a one. For $1 \le i < j \le n$, the $E(i,j)$'s are linearly independent in the real Lie algebra $M(n)$ and span $T(n)$.

(b) Let X be in $T(n)$. Since $X^{n+1} = 0$, we have $\exp X = 1 + X + \ldots + \frac{1}{n!}X^n$ which is in $U(n)$. For $Y \in U(n)$ let

$$\log Y = (Y-1) - \frac{(Y-1)^2}{2} + \ldots (-1)^{n+1} \frac{(Y-1)^n}{n} ,$$

which is in $T(n)$. Direct computation tells us that $\exp(\log Y) = Y$ for Y in $U(n)$ and $\log(\exp X) = X$ for $X \in T(n)$. Therefore $\exp: T(n) \to U(n)$ and its inverse $\log: U(n) \to T(n)$ are both polynomial.

Lemma 2.2. Let C be a subalgebra of the Lie algebra $T(n)$, then $\exp C$ is an algebraic subgroup in $U(n)$.

Proof: For X and Y in C, $\exp X \exp Y = \exp Z$ where $Z = X + Y + F(X,Y)$, and $F(X,Y)$ is a finite sum of rational multiples of Lie brackets in X and Y. Therefore Z is in C. Since $-Y$ is in C and $\exp(-Y) = (\exp Y)^{-1}$ we have that $\exp C$ is a subgroup of $U(n)$. Since the exponential map is a Zariski homeomorphism, $\exp C$ is an algebraic group since subspaces of $M(n)$ are clearly algebraic.

Corollary 2.2. Every analytic subgroup of $U(n)$ is an algebraic subgroup of $U(n)$.

Proof: Every analytic subgroup of $U(n)$ is the exponential of a subalgebra of $T(n)$.

Lemma 2.3. Let $X \neq 1$ be in $U(n)$. Then $\mathfrak{a}(X) = \exp(t \log X: t \in R)$.

Proof: There is a nonsingular matrix $P \in GL(n)$ such that $Y = P X P^{-1}$ is a matrix of $U(n)$ having zeroes everywhere except on the super diagonal where its entries are zero or one. Since $X \neq 1$, there is an $1 \leq \alpha \leq n-1$ such that $Y_{\alpha,\alpha+1} = 1$. Then $(Y^m)_{\alpha,\alpha+1} = m$. Therefore X has infinite order. Let $A = \{t \log X: t \in R\}$ which is a subalgebra of $T(n)$. Therefore since X is in $\exp A$ we have $\mathfrak{a}(X) \leq \exp A$. $\log \mathfrak{a}(X)$ is an algebraic subset of A. Moreover it is infinite. Let $\alpha: R \to A$ be given by $\alpha(t) = t \log X$. α is a Zariski homeomorphism of R onto A, therefore $\alpha^{-1}(\log \mathfrak{a}(X))$ is an infinite algebraic subset of R. But R is the only infinite algebraic subset of R. Therefore $R = \alpha^{-1} \log \mathfrak{a}(X)$, $A = \log \mathfrak{a}(X)$ and $\exp A = \mathfrak{a}(X)$.

Corollary 2.3. Every algebraic subgroup of $U(n)$ is an analytic subgroup of $U(n)$.

Proof: Let C be an algebraic subgroup of $U(n)$. Since

$$C = \bigcup_{X \in C} \mathfrak{a}(X) , \quad \bigcap_{X \in C} \mathfrak{a}(X) \text{ contains } 1$$

and $\mathfrak{G}(X)$ is connected we have that C is an analytic subgroup of $U(n)$.

Lemma 2.4. Let C be an algebraic subgroup of $U(n)$. Let r be a rational representation of C in $GL(m)$. Then $r(C)$ is an algebraic group.

Proof: Since r is an analytic group homomorphism we have that $r(C)$ is an analytic subgroup of $GL(m)$. Each element of $r(C)$ is a unipotent matrix. Therefore there is a $p \in GL(m)$ such that $pr(C)p^{-1} \subseteq U(m)$. But then $pr(C)p^{-1}$ is an algebraic subgroup of $U(m)$ and therefore $r(C)$ is an algebraic subgroup of $GL(m)$.

3. The Birkhoff Imbedding Theorem

Theorem A. Let L be a nilpotent Lie algebra.

Then there is a faithful Lie algebra representation $\beta: L \to T(n)$ for some n of L into $T(n)$. To each automorphism A of L, there corresponds a p in $GL(n)$ such that $\beta(A(\ell)) = p\beta(\ell)p^{-1}$ for all $\ell \in L$. To each group A of automorphisms of L there is an isomorphism of A onto $GL(n)$ which we denote by $i: A \to GL(n)$ such that for each $a \in A$ and $\ell \in L$ we have $\beta(A(\ell)) = i(a)\, \beta(\ell)\, i(a)^{-1}$.

Theorem A'. Let N be a simply connected nilpotent analytic group.

Then there is a faithful analytic group representation $\beta: N \to U(n)$ for some n of N into $U(n)$. To each automorphism A of N, there corresponds a matrix $i(A)$ of $GL(n)$ such that $i(A)\, \beta(n)\, i(A)^{-1} = \beta(A(n))$. Moreover $\beta(N)$ is an algebraic group.

Let $L = L(N)$ be the Lie algebra of the real nilpotent group N . Let $\bar{\beta}: L \to T(n)$ be the faithful Lie algebra representation of L into $T(n)$. Since N is a simply connected analytic group there is an analytic group isomorphism of N onto U(n), β: $N \to U(n)$ such that $\exp \circ \bar{\beta} = \beta \circ \exp$. $\beta(N)$ is an analytic subgroup of U(n) whose Lie algebra is $\bar{\beta}(L)$. Let A be an automorphism of the analytic group N. Then A_L is a Lie algebra automorphism of L. There is a p in GL(n) such that $\bar{\beta}(A_L(\ell)) = p\bar{\beta}(\ell)p^{-1}$. But $\exp(p\bar{\beta}(\ell)p^{-1}) = p \exp \bar{\beta}(\ell)p^{-1}$, and we have $\beta(A(n)) = p\beta(n)p^{-1}$ for all $n \in N$.

It is useful to note that if N is an algebraic subgroup of U(n) and α is an automorphism of N given by $\alpha(n) = \bar{\alpha}\, n\, \bar{\alpha}^{-1}$ where $\bar{\alpha}$ is in GL(n). Then $\alpha_{L(n)}(\ell) = \bar{\alpha}\, \ell\, \bar{\alpha}^{-1}$.

4. Central Theorem

Theorem 4.1. Let N be a simply connected nilpotent analytic group which we consider as an algebraic subgroup of U(n).

Let C be a closed subgroup of N. Then $N = \mathfrak{a}(C)$ if and only if N/C is compact.

In this case C_0 is a normal subgroup of N.

Proof: We may assume that N is an algebraic subgroup of U(n) and that C is a closed subgroup of N.

Assume $N = \mathfrak{a}(C)$. We shall use induction on the length of the lower central series of N. If N were abelian then N is a vector group. $\mathfrak{a}(C)$ is contained in the subspace of N generated by C and therefore the subspace generated by C is N. In such a case it is well-known that N/C is compact. Let us assume that the theorem is true for all N of length $\ell-1$ and let N have length ℓ. (N,N) is an algebraic subgroup of U(n) and $(N,N) = \mathfrak{a}(C,C)$. Since $(C,C) \subseteq C \cap (N,N)$ which is a closed subgroup of (N,N), and since $\mathfrak{a}(C \cap (N,N)) = (N,N)$ we have that $(N,N)/C \cap (N,N)$ is compact. Let $f\colon N \to N/C$ be the natural map of N onto N/C. Since (N,N) is a topological subgroup of N, f restricted to (N,N) is a continuous, open map of (N,N) onto $(N,N)C/C$. The map $\eta\colon (N,N) \to (N,N)/C \cap (N,N)$ is a continuous open map and f is constant on the cosets of (N,N) relative to $C \cap (N,N)$. Therefore f factors into a continuous map $\bar{f}\colon (N,N)/C \cap (N,N) \to N/C$. Since $\bar{f}$ is continuous and $(N,N)/C \cap (N,N)$ is compact we have that $(N,N)C/C$ is a compact subspace of N/C. Therefore $(N,N)C/C$ is a closed subspace of N/C and $(N,N)C$ is a closed subspace of N. The natural map of N onto $N/C \cdot (N,N)$ is a continuous open map which is constant on N modulo C, therefore we have a continuous open map of N/C onto $N/(N,N)\cdot C$. Assume $N/(N,N)C$ is compact. Since the fiber above each point of the base space of the fiber space $N/C \to N/(N,N)C$ is homeomorphic to the compact space $(N,N)C/C$, we have N/C compact. We must therefore prove $(N,N)C$ is a uniform subgroup of N. $N/(N,N)C$ is the continuous image of $N/(N,N) \,/\, (N,N)C/(N,N)$ so we must prove the latter compact.

Since (N,N) is a normal algebraic subgroup N, there is a rational representation r of N into $GL(m)$ for some m whose kernel is (N,N). We may take $r(N)$ to be an algebraic subgroup of $U(m)$. Since r is an analytic group homomorphism of N onto $r(N)$ we have that $N/(N,N)$ is homeomorphic to $r(N)$ and that $N/(N,N) \,/\, (N,N)C/(N,N)$ is homeomorphic to $r(N)/r(C)$. Since $r(C) = r((N,N)C)$ and since $(N,N)C$ is a closed subgroup of N we have that $r(C)$ is a closed subgroup of $r(N)$. Since $N = \mathfrak{a}(C)$ and since r is rational and defined on N, we have $r(N) = \mathfrak{a}(r(C))$. Therefore since $r(N)$ is abelian we have $r(N)/r(C)$ is compact.

(b) Assume that N/C is compact. Since N is algebraic, $\mathcal{G}(C) \subseteq N$, and $\mathcal{G}(C)$ is an analytic subgroup of N. Therefore $N/\mathcal{G}(C)$ is homeomorphic to Euclidean space and since it is also the continuous image of N/C which is compact we have that $N = \mathcal{G}(C)$.

(c) Let N/C be compact, then $N = \mathcal{G}(C)$. Since C_0 is a normal subgroup of C, $\mathcal{G}(C_0)$ is a normal subgroup of $N = \mathcal{G}(C)$. But C_0 being an analytic subgroup of $U(n)$ is algebraic. Therefore C_0 is a normal subgroup of N.

Corollary 4.2. Let N be a connected, simply connected nilpotent Lie group. Let C be a closed cocompact subgroup of N. Then

(a) $C \cap N^i$ is a closed cocompact subgroup of N^i.

(b) CN^i is a closed subgroup of N.

Proof: (a) Again we shall assume that N is an algebraic subgroup of $U(n)$ for some n. $\mathcal{G}(C,C) = (\mathcal{G}(C),\mathcal{G}(C)) = (N,N)$, and by continuing on this way $\mathcal{G}(C^i) = \mathcal{G}(C)^i = N^i$. But $C \cap N^i$ is a closed subgroup of N^i, and since $C^i \subset C \cap N^i \subset N^i$ we have that $N^i = \mathcal{G}(C \cap N^i)$ and therefore $N^i/C \cap N^i$ is compact.

(b) Since N^i is a topological subgroup of N, the restriction of the natural map of N onto N/C, to N^i, is continuous. The natural map of N^i onto $N^i/C \cap N^i$ is open and continuous. Therefore we have a continuous map of $N^i/C \cap N^i$ onto N/C whose image is N^iC/C. Since $N^i/C \cap N^i$ is compact we have N^iC/C is a compact subset of N/C and is therefore closed in N/C. Therefore N^iC is a closed subset of N.

5. Structure Theorem

We are now in a position to prove a structure theorem for real nilpotent groups containing discrete cocompact subgroups. We shall, as a part of our results, characterize discrete cocompact subgroups of real nilpotent groups as torsion free finitely generated nilpotent groups. We shall assume that for such groups Γ there exists a sequence $0 = \Gamma_{r+1} \subset \Gamma_r \subset \ldots \subset \Gamma_1 = \Gamma_0 = \Gamma$ such that Γ_i is a normal subgroup of Γ and such that $\Gamma_i/\Gamma_{i+1} = Z$. Moreover we shall assume that any two such sequences has the same length $r+1$ which we call the length of Γ.

We first prove the following lemma.

<u>Lemma 5.1</u>. Let $c_1, \ldots, c_r$ be in $U(n)$.

Assume c_i is in the normalizer of $(c_{i+1}, \ldots, c_r)$ = the group generated by $c_{i+1}, \ldots, c_r$. Then

(1) c_i is in the normalizer of $\mathfrak{G}(c_{i+1}, \ldots, c_r)$, and $\mathfrak{G}(c_i)$ normalizes $\mathfrak{G}(c_{i+1}, \ldots, c_r)$.

(2) $\mathfrak{G}(c_i, \ldots, c_r) = \mathfrak{G}(c_i) \cdot \mathfrak{G}(c_{i+1}, \ldots, c_r)$.

(3) Every $x \in \mathfrak{G}(c_i, \ldots, c_r)$ can be written as $x = \Pi_i^r \exp(t_j \log c_j)$.

(4) If $c_i \notin \mathfrak{G}(c_{i+1}, \ldots, c_r)$ then $\mathfrak{G}(c_i) \cap \mathfrak{G}(c_{i+1}, \ldots, c_r) = (1)$, and if $c_i \notin \mathfrak{G}(c_{i+1}, \ldots, c_r)$ for $1 \le i \le r-1$ then the representation in (3) is unique.

<u>Proof</u>: Consider the separately Zariski continuous map $U \times U \to U$ given by $(x,y) \to xyx^{-1}$. Since c_i takes $(c_{i+1}, \ldots, c_r)$ into $\mathfrak{G}(c_{i+1}, \ldots, c_r)$ we have that c_i takes $\mathfrak{G}(c_{i+1}, \ldots, c_r)$ onto itself and that $\mathfrak{G}(c_i)$ normalizes $\mathfrak{G}(c_{i+1}, \ldots, c_r)$. We have that $\mathfrak{G}(c_i) \cdot \mathfrak{G}(c_{i+1}, \ldots, c_r)$ is an algebraic group which since it contains $c_i, \ldots, c_r$ contains $\mathfrak{G}(c_i, \ldots, c_r)$. But $\mathfrak{G}(c_i)$ and $\mathfrak{G}(c_{i+1}, \ldots, c_r)$ are in $\mathfrak{G}(c_i, \ldots, c_r)$ therefore we get (2). We therefore have $\mathfrak{G}(c_i, \ldots, c_r) = \mathfrak{G}(c_i) \ldots \mathfrak{G}(c_r)$

and from $G(c_j) = \exp(t_j \log c_j)$ we get (3). Suppose $c_i \notin G(c_{i+1},\ldots,c_r)$. If $y \in G(c_i) \cap G(c_{i+1},\ldots,c_r)$, and $y \neq 1$, from $G(y) = \exp(t \log y) \subseteq G(c_i) = \exp(t \log c_i)$ we get $\{t \log y\} \subseteq \{t \log c_i\}$ for real t. Therefore $\log y = t \log c_i$, $t \neq 0$ since $y \neq 1$, therefore $\log c_i = 1/t \log y \in \{t \log y\}$. Therefore $G(c_i) \subseteq G(y) \subseteq G(c_{i+1},\ldots,c_r)$, which is a contradiction.

Theorem 5.2. Let N be a real nilpotent group and let Γ be a discrete cocompact subgroup of N. Then

(a) there exists elements $\gamma_1,\ldots,\gamma_r$ in Γ where $r = \dim N$ such that (1) each element γ in Γ can be uniquely written as $\gamma = \Pi_1^r \gamma_i^{e_i}$ where the e_i are integers. (2) where $\Gamma_i = \{\Pi_i^r \gamma_j^{e_j} : e_j \text{ integers}\}$ we have that each Γ_i is a normal subgroup of Γ and that $\Gamma_i/\Gamma_{i+1} = Z$.

(b) Let $\gamma_1',\ldots,\gamma_{r'}'$ be any elements on Γ satisfying (1) and (2). Let $\gamma_i'(t_i) = \exp(t_i \log \gamma_i')$ where t_i is real. Then $r' = \dim N$, and (1) each element n in N can be uniquely written as $n = \Pi_1^{r'} \gamma_i'(t_i)$ where the t_i are real. (2) where $N_i = \{\Pi_i^{r'} \gamma_j'(t_j) : t_j \text{ real}\}$ we have that each N_i is a normal subgroup of N and that $N_i/N_{i+1} = R$.

(c) The map $R^r \to N$ given by $(t_1,\ldots,t_r) \to \Pi_1^{r'} \gamma_i'(t_i)$ is an analytic manifold isomorphism of R^r onto N.

Proof: Let N^c be the last term of the lower central series for N. When $c = 1$, N is a vector group and the theorem is known. Assume that N has length c and that the theorem is known for all real nilpotent groups of length $< c$. $N^c\Gamma/N^c = \Gamma/N^c \cap \Gamma$ is a discrete cocompact subgroup of N/N^c. We can find elements $\gamma_1,\ldots,\gamma_n$ in Γ whose images $\bar{\gamma}_1,\ldots,\bar{\gamma}_n$ in $\Gamma/N^c \cap \Gamma$ under the natural map of Γ onto $\Gamma/N^c \cap \Gamma = N^c\Gamma/N^c$ satisfy (1) and (2) of the theorem. $\Gamma \cap N^c$ is a discrete cocompact subgroup of N^c, therefore there are elements $\gamma_{n+1},\ldots,\gamma_r$ in $\Gamma \cap N^c$ satisfying (1) and (2) of the theorem. Consider the exact sequence $1 \to \Gamma \cap N^c \to \Gamma \xrightarrow{\eta} \Gamma/\Gamma \cap N^c \to 1$ and let $\bar{\gamma} = \eta(\gamma)$ for γ in Γ. Let γ be in Γ. Then we can write $\bar{\gamma} = \Pi_1^n \bar{\gamma}_i^{e_i}$ where e_i are integer. Therefore $\gamma = \Pi_1^n \gamma_i^{e_i} \cdot x$ where x is in $\Gamma \cap N^c$. Then we can write $x = \Pi_{n+1}^r \gamma_i^{e_i}$ where e_i are integer. Therefore $\gamma = \Pi_1^r \gamma_i^{e_i}$. The uniqueness of the representation in $\Gamma/\Gamma \cap N^c$ givesthe uniqueness of the integers e_i with $1 \le i \le n$, and the uniqueness of the representation in $\Gamma \cap N^c$ give the uniqueness of the integers e_i with $n+1 \le i \le r$. Let $\Gamma_i = \{\Pi_i^r \gamma_j^{e_j} : e_j \text{ integer}\}$ and $\bar{\Gamma}_i = \eta(\Gamma_i) = \{\Pi_i^r \bar{\gamma}_j^{e_j} : e_j \text{ integer}\}$. For $i \ge n+1$ we have Γ_i is a central and therefore normal subgroup and Γ and we have $\Gamma_i/\Gamma_{i+1} = Z$. For $i \le n$ we use the induction step to give us $\bar{\Gamma}_i$ is a normal subgroup of $\eta(\Gamma)$ and $\bar{\Gamma}_i/\bar{\Gamma}_{i+1} = Z$, which gives us Γ_i is a normal subgroup of Γ and $\Gamma_i/\Gamma_{i+1} = \bar{\Gamma}_i/\bar{\Gamma}_{i+1} = Z$. Since as is known $r - (n+1) = \dim N^c$ and by induction $n = \dim N/N^c$ we have $r = \dim N$.

Let $\gamma_1',\ldots,\gamma_{r'}'$ be elements in Γ satisfying (1) and (2). Then r' is the length of the torsion free finitely generated nilpotent group. Therefore $r = r' = \dim N$. Assume $\gamma_i' \notin G(\gamma_{i+1}',\ldots,\gamma_r')$ for $\ell+1 \le L \le r$. Then $\dim G(\gamma_{\ell+1}',\ldots,\gamma_r') = r - (\ell+1)$. Suppose $\gamma_\ell' \in G(\gamma_{\ell+1}',\ldots,\gamma_r')$. Then Γ_ℓ' is a discrete cocompact subgroup of $G(\gamma_{\ell+1}',\ldots,\gamma_r')$. Since the length of Γ_ℓ' is $r - \ell$ by the first part of the theorem $r - \ell = \dim G(\gamma_{\ell+1}',\ldots,\gamma_r') = r - (\ell+1)$, a contradiction. Therefore $\gamma_i' \notin G(\gamma_{i+1}',\ldots,\gamma_r')$ for $1 \le i \le r$. The rest of theorem follows from the lemma.

<u>Theorem 5.2.1</u>. Let N be a real nilpotent group and let C be a closed cocompact subgroup of N. Let C_0 be the identity component of C. Then

(a) There are elements $c_1,\ldots,c_r$ in C where $c_{s+1},\ldots,c_r$ are in C_0 such that

(1) each element c in C_0 can be uniquely written as $c = \Pi_{s+1}^r \exp(t_j \log c_j)$ where the t_j are real. Each element c in C can be written uniquely as $c = \Pi_1^s c_i^{e_i} \cdot \Pi_{s+1}^r \exp(t_j \log c_j)$ where e_i are integers and t_j are reals.

(2) where $$C_i = \begin{cases} \Pi_i^s c_j^{e_j} \cdot C_0 & e_j \text{ integer } i \le s \\ \Pi_i^r \exp(t_j \log c_j) & t_j \text{ real } i > s \end{cases}$$

we have that C_i is a normal subgroup of C and that $C_i/C_{i+1} = \begin{cases} Z & i \le s \\ R & i > s \end{cases}$.

(b) Let $c_1',\ldots,c_{r'}'$ in C where $c_{s+1}',\ldots,c_r'$ are in C_0 satisfy (1) and (2), then $r' = r = \dim N$ and $r' - (s'+1) = r - (0+1) = \dim C_0$ and where $c_i'(t_i) = \exp(t_i \log c_i')$ we have

(3) each element n of N can be written uniquely as $n = \Pi_1^r c_i'(t_i)$ where t_i are real.

(4) where $N_i = \{\Pi_i^r c_j'(t_j) : t_j \text{ real}\}$ we have N_i is a normal subgroup of N and that $N_i/N_{i+1} = R$.

(c) The map $R^r \longrightarrow N$ given by $(t_1,\ldots,t_r) \longrightarrow \Pi_1^r c_i'(t_i)$ is an analytic manifold isomorphism.

Consider the case where C is closed. C_0 the identity component of C is a normal analytic subgroup of N, and C/C_0 is a discrete cocompact subgroup of N/C_0. We can find elements $c_1,\ldots,c_s$ whose images $\bar{c}_1,\ldots,\bar{c}_s$ in C/C_0 satisfy the conclusions of the theorem.

Therefore each element c of C can uniquely be written as $c = \Pi_1^s c_i^{e_i} \cdot x$ where e_i are integers and x is in C_0, $C_i = \{\Pi_i^s c_i^{e_i} \cdot C_0 : e_i \text{ integers}\}$ is a normal subgroup of C and $C_i/C_{i+1} = Z$. Also each element n of N can uniquely be written as $n = \Pi_1^s \exp(t_i \log c_i) \cdot x$ with x in C_0 and t_i real, $N_i = \{\Pi_i^s \exp(t_i \log c_i) \cdot C_0 : t_i \text{ real}\}$ closed analytic subgroup of N which is normal in N and $N_i/N_{i+1} = R$. Consider the sequence $C_0, C_0 \cap (N,N), \ldots, C_0 \cap (N^{\ell})$ of normal closed analytic subgroups of C_0. $C_0/C_0 \cap (N,N)$ is a vector group, therefore there are elements $c_{s+1}, \ldots, c_{s+r_1}$ whose images in $C_0/C_0 \cap (N,N)$, $\bar{c}_{s+1}, \ldots, \bar{c}_{x+r_1}$ under the natural map satisfy the following. Each $\bar{c}_i \in C_0/C_0 \cap (N,N)$ can be written uniquely as $\bar{c}_i = \prod_{s+1}^{s+r_1} \exp(t_j \log \bar{c}_j)$ where t_j are real. Therefore each c_i in C_0 can be written uniquely as $c_i = \prod_{s+1}^{s+r_1} \exp(t_j \log c_j) \cdot x$ with $x \in C_0 \cap (N,N)$ and t_j real. Moreover, since N acts by the identity on $C_0/C_0 \cap (N,N)$ we have $C_i = \{ \Pi_i^{s+r_1} \exp(t_j \log c_j) : t_j \text{ real}\} \cdot C_0 \cap (N,N)$ is a normal analytic subgroup of N for $L \geq s+1$. Also $C_i/C_{i+1} = R$. Continuing in this way we complete the theorem.

<u>Corollary 5.2.1</u>. Every closed cocompact subgroup of a real nilpotent group is a torsion free nilpotent Lie group whose component group is finitely generated.

<u>Corollary 5.2.2</u>. The fundamental group of a compact nil manifold is a torsion free finitely generated nilpotent group.

6. Coordinates Systems

<u>Lemma 6.1</u>. Let $X_1, \ldots, X_n$ be a basis of $L = L(N)$. Then the map

$R^n \longrightarrow N$, $(t_1,\ldots,t_n) \longrightarrow \exp(\sum_1^n t_i X_i)$ is an analytic manifold isomorphism of R^n on N. Such a coordinate system will be said to be of type 1.

Lemma 6.2. Let $X_1,\ldots,X_n$ be a basis of $L = L(N)$ such that

$$L_i = \left\{\sum_i^n t_j X_j : t_j \text{ real}\right\} \text{ is an ideal in } L, \text{ and}$$

$[L_j L_i] \subseteq L_{i+1}$.

Such a basis is called a canonical basis of L. Then the map $R^n \longrightarrow N$, $(t_1,\ldots t_n) \longrightarrow \Pi_1^n \exp(t_i X_i)$ is an analytic manifold isomorphism of R^n onto N. Such a coordinate system will be said to be of type 2.

Proof: Let $N_i = \exp L_i$. Then N_i is an algebraic subgroup of N, N_i is a normal subgroup of N, and $(N,N_i) \subseteq N_{i+1}$. Since $X_i \notin L_{i+1}$ we have that $\exp X_i \notin N_{i+1}$ and $\mathcal{G}(\exp X_i) \notin N_{i+1}$. Therefore $N_i = (\exp t_i X_i) \cdot N_{i+1}$ and $\exp t_i X_i \cap N_{i+1} = (1)$. Assume that the map $(t_{i+1},\ldots,t_n) \longrightarrow \Pi_{i+1}^n \exp(t_j X_j)$ is an analytic manifold isomorphism of $R^{n-(i+1)}$ onto N_{i+1}. Since the map $t_i \longrightarrow \exp t_i X_i$ is an analytic manifold isomorphism of R onto $\exp t_i X_i$ we have that the map of $(t_i,\ldots,t_n) \longrightarrow \Pi_i^r \exp(t_j X_j)$ is an analytic manifold isomorphism.

We shall use the Hausdorf-Campbell formula to investigate relationships between these two coordinate systems.

Theorem 6.3. Let $X_1,\ldots,X_n$ be a canonical basis of $L = L(N)$. Then

(1) $\Pi_1^n \exp(t_i X_i) = \exp(\sum_1^n u_i X_i)$ implies that $u_i = t_i + q_i(t_1,\ldots,t_{i-1})$ where q_i is a real polynomial each of whose terms have degree ≥ 2 and where the coefficients of q_i are polynomials with

rational coefficients in the structure constants of the basis.

(2) $\Pi_1^n \exp(t_i X_i) \Pi_1^n \exp(s_i X_i) = \exp(\sum_1^n u_i X_i)$ implies that $u_i = t_i + s_i + p_i(t_1,\ldots,t_{i-1}; s,\ldots,s_{i-1})$ where p_i is a real polynomial each of whose terms has degree ≥ 2 and where the coefficients of p_i are polynomials with rational coefficients in the structure constants of $X_1,\ldots,X_n$.

(3) $\Pi_1^n \exp(t_i X_i) \Pi_1^n \exp(s_i X_i) = \Pi_1^n \exp(u_i X_i)$ implies $u_i = t_i + s_i + \bar{q}_i(t_1,\ldots,t_{i-1}; s_1,\ldots,s_{i-1})$ where $\bar{q}_i$ is a real polynomial each of whose terms has degree ≥ 2 and where the coefficients of $\bar{q}_i$ are polynomials with rational coefficients in the structure constants of $X_1,\ldots,X_n$.

<u>Proof</u>: Let $[X_i,X_j] = \sum_k \gamma_{ij}^k X_k$. Since $[X_i,X_j] \in L_{j+1}$ we have $\gamma_{ij}^k = 0$ for $k \leq j$. Also $\gamma_{ij}^k = 0$ for $k \leq i$. $\Pi_1^n \exp(t_i X_i) = \exp(\sum t_i X_i + \ldots)$ where ... is a finite sum of rational multiples of brackets in $t_i X_i$. Fix $1 \leq i_1,\ldots,i_p \leq n$ then $[\ldots[t_{i_1} X_{i_1}, t_{i_2} X_{i_2}]\ldots t_{i_p} X_{i_p}]$ is equal to $t_{i_1}\ldots t_{i_p} \sum_{j_1,\ldots,j_p=1}^{n} \gamma_{i_1 i_2}^{j_1} \gamma_{j_1 i_3}^{j_2} \ldots \gamma_{j_{p-1} i_p}^{j_p} X_{j_p}$.

Fix j_p . If $j_p \leq i_p$ then for all $j_1,\ldots,j_{p-1}$ we have $\gamma_{j_{p-1} i_p}^{j_p} = 0$ and we have each term of the sum equal to 0 . If $j_p \leq i_{p-1}$ then for $j_p \leq j_{p-1}$ for some j_{p-1} the corresponding term of the sum for all $j_1,\ldots,j_{p-1}$ is 0, and for $j_p > j_{p-1}$ for some j_{p-1} we have $j_{p-1} < i_{p-1}$ in which case the corresponding term of the sum for all $j_{p-2},\ldots,j_1$ is 0 . In this way if $j_p \leq i_1$ or $\leq i_2,\ldots$ or $\leq i_p$ then the X_{j_p} contribution from the bracket is 0 . For

$j_p > i_1,\dots,i_p$ the X_{j_p} contribution of this bracket is

$$t_{i_1} \dots t_{i_p} \sum_{j_1=1}^{n} \dots \sum_{j_{p-1}=1}^{n} \gamma_{i_1 i_2}^{j_1} \dots \gamma_{j_{p-1} i_p}^{j_p} \quad X_{j_p} \quad .$$

Therefore $u_i = t_i + q_i(t_1,\dots,t_{i-1})$ where q_i has the properties stated for it in (1).

The proof of (2) is the same.

$\Pi_1^n \exp(t_i X_i)\, \Pi_1^n \exp(s_i X_i) = \exp(\sum_1^n (t_i+s_i+p_i(t_1,\dots,t_{i-1};s_1,\dots,s_{i-1}))X_i)$ and $\Pi_1^n \exp(u_i X_i) = \exp(\sum_1^n (u_i+q_i(u_1,\dots,u_{i-1}))X_i)$ where q_i and p_i have the properties stated for them in (1) and (2). Therefore $u_1 = s_1+t_1$, $u_2+q_2(u_1) = t_2+s_2+p_2(t_1;s_1)$. Therefore $u_2 = t_2+s_2+p_2(t_1;s_1) + q_2(s_1+t_1)$. Continuing in this way we get (3). In the same way we get

<u>Theorem 6.4</u>. Let $X_1,\dots,X_n$ be a canonical basis of $L = L(N)$. Then

(a) $\Pi_1^n \exp(t_i X_i) = \exp(\sum_1^n t_i' X_i)$ implies

$t_i = t_i' + q_i(t_1',\dots,t_{i-1}')$ where q_i is a real polynomial each of whose terms has degree ≥ 2 and where the coefficients of q_i are rational polynomials in the structure constants of $X_1,\dots,X_n$.

(b) $\exp(\sum_1^n s_i X_i) \exp(\sum_1^n t_i X_i) = \exp(\sum_1^n u_i X_i)$ implies

$u_i = s_i + t_i + p_i(s_1,\dots,s_{i-1};\ t_1,\dots,t_{i-1})$ where p_i is a polynomial whose coefficients are rational polynomials in the structure constants of $X_1,\dots,X_n$.

7. <u>Extensions</u>

<u>Theorem 7.1</u>. Let C be a closed cocompact subgroup of a real nilpotent group N and let $\bar{C}$ be a closed cocompact subgroup of a real nilpotent

group $\bar{N}$. Let $\alpha: C \to \bar{C}$ be a continuous topological group isomorphism of C onto $\bar{C}$.

Then there is a unique extension α_N of α to a topological group isomorphism of N onto $\bar{N}$.

Corollary 7.2. A nil manifold is the quotient space of a real nilpotent group by a closed subgroup.

Then two compact nil manifolds having isomorphic fundamental groups are homeomorphic.

Proof: Let $c_1,\ldots,c_r$ be elements of C with $c_{s+1},\ldots,c_r$ in C_s satisfying the conclusions of theorem . Let $\bar{c}_i = \alpha(c_i) \cdot \alpha(C_o) = \bar{C}_o$, and since α is an analytic group isomorphism of C_0 onto $\bar{C}_0$ we have $\alpha(\exp(t_j \log c_j)) = \exp(t_j \log \bar{c}_j)$. Therefore $\bar{c}_1,\ldots,\bar{c}_r$ are elements of $\bar{C}$ with $\bar{c}_{s+1},\ldots,\bar{c}_r$ in $\bar{C}_0$ satisfying the conclusions of the theorem. Let $\alpha_N: N \to \bar{N}$ be given by $\Pi_1^r \exp(t_i \log c_i) \to \Pi_1^r \exp(t_i \log \bar{c}_i)$. The theorem tells us this is a well-defined analytic manifold isomorphism of N onto $\bar{N}$, and is an extension of α . We need to prove α_N is a homomorphism. Let X and Y be elements of N and write $X = \Pi_1^r \exp(t_i \log c_i)$, $Y = \Pi_1^r \exp(s_i \log c_i)$ and $XY = \Pi_1^r \exp(u_i \log c_i)$, where $u_i = t_i+s_i+q_i(t_1,\ldots,t_{i-1};s_1,\ldots,s_{i-1})$. We have $\alpha_N(X) = \Pi_1^r \exp(t_i \log \bar{c}_i)$, $\alpha_N(Y) = \Pi_1^r \exp(s_i \log \bar{c}_i)$ and $\alpha_N(XY) = \Pi_1^r \exp(u_i \log \bar{c}_i)$. But then we have $\alpha_N(X)\,\alpha_N(Y) = \Pi_1^r \exp(v_i \log \bar{c}_i)$ where $v_i = t_i+s_i+\bar{q}_i(t_1,\ldots,t_{i-1};\, s_1,\ldots,s_{i-1})$. We must prove that $q_i(X_1,\ldots,X_{i-1};\, Y_1,\ldots,Y_{i-1}) = \bar{q}_i(X_1,\ldots,X_{i-1};\, Y_1,\ldots,Y_{i-1})$. However α is a homomorphism of C onto $\bar{C}$, therefore we have $q_i(n_1,\ldots,n_{i-1};m_1,\ldots,m_{i-1}) = \bar{q}_i(n_1,\ldots,n_{i-1};m_1,\ldots,m_{i-1})$ for all integers n_j and m_j $1 \le j \le i-1$. But since both q_i and $\bar{q}_i$ are real polynomials we have that this is sufficient for $q_i = \bar{q}_i$.

Proof of the Corollary: Let $M = N/C$ and $\bar{M} = \bar{N}/\bar{C}$ where N and $\bar{N}$ are real nilpotent groups and C and $\bar{C}$ are closed cocompact subgroups of N and $\bar{N}$ respectively. We then have that C/C_s is a discrete cocompact subgroup of N/C_s and is the fundamental group of N/C and that $\bar{C}/\bar{C}_s$ is a discrete cocompact subgroup of $\bar{N}/\bar{C}_s$ and is the fundamental group of $\bar{N}/\bar{C}$. But if C/C_s is isomorphic to $\bar{C}/\bar{C}_s$ then N/C_s is isomorphic to $\bar{N}/\bar{C}_s$. But $N/C = N/C_s \;/C/C_s = \bar{N}/\bar{C}_s \;/\bar{C}/\bar{C}_s = \bar{N}/\bar{C}$, where by equality we mean homeomorphic.

8. Embedding

Theorem 8.1. Let Γ be a torsion free finitely generated nilpotent group. Then Γ can be embedded in a unique way as a discrete cocompact subgroup of a real nilpotent group $N(\Gamma)$.

Proof: There is a sequence $(e) = \Gamma_0 \subset \Gamma_1 \subset \ldots \subset \Gamma_r = \Gamma$ where each Γ_i is a normal subgroup of Γ and $\Gamma_i/\Gamma_{i+1} = Z$. The number r is an invariant called the length of Γ . We shall induct on r . By induction there is a unique real nilpotent group $N(\Gamma_{r-1})$ containing Γ_{r-1} as a discrete cocompact subgroup. Let γ in Γ be a pre-image in Γ of a generator of $\Gamma/\Gamma_{r-1} = Z$. We have that γ induces an automorphism $\mathrm{ad}\,\Gamma_{r-1}\,\gamma$ of Γ_{r-1} which extends uniquely to a topological group automorphism $\gamma^{\#}$ of $N(\Gamma_{r-1})$. By the Birkhoff Embedding Theorem we may regard $N(\Gamma_{r-1})$ as an algebraic group of unipotent upper triangular matrices in $GL(n)$ for some n . Moreover there is a $\bar{\gamma}$ in $GL(n)$ such that $\gamma^{\#}(n) = \bar{\gamma}\, n\, \bar{\gamma}^{-1}$ for all $n \in N(\Gamma_{r-1})$. Since Γ is nilpotent there is an integer k such that $\overset{k}{(\gamma,\ldots(\gamma,x)\ldots)} = (1)$ for all $x \in \Gamma_{r-1}$. But then $\overset{k}{(\bar{\gamma},\ldots,(\bar{\gamma},x)\ldots)} = (1)$ for all $x \in \Gamma_{r-1}$.

The map of $U(n)$ which takes $x \to (\bar{\gamma},\ldots,(\bar{\gamma},x)\ldots)$ is Zariski continuous. In this way we have that

$$(G(\bar{\gamma}),\ldots\overset{k}{(}G(\bar{\gamma}),N(\Gamma_{r-1})\ldots) = (1) .$$

Let $\bar{\gamma} = u\ t$ where u in $GL(n)$ is unipotent, t in $GL(n)$ is semi-simple and $(u,t) = 1$. Then since $t \in G(\bar{\gamma})$ we have $t\ N(\Gamma_{r-1})\ t^{-1} = N(\Gamma_{r-1})$ and $(t,\ldots\overset{k}{(}t,N(\Gamma_{r-1})\ldots) = (1)$. Let $\tilde{t}$ be the automorphism of $L(N(\Gamma_{r-1}))$ given by $x \to t\ x\ t^{-1}$, and let $t^{\#}$ be the automorphism of $N(\Gamma_{r-1})$ given by $n \to t\ n\ t^{-1}$. We have that $\exp \circ \tilde{t} = t^{\#} \exp$ and $\tilde{t}$ is the induced automorphism of $L(N(\Gamma))$ given by $t^{\#}$. $t^{\#}$ is clearly semi simple and therefore there is over the complexification of $L(N(\Gamma_{r-1}))$ a bases consisting of eigen vectors of $\tilde{t}$. Let X be an eigen vector in $L(N(\Gamma_{r-1}))$ of $\tilde{t}$. Since $[\tilde{t}(X),X] = 0$ we have $(t, \exp X) = \exp((\tilde{t} - I)X)$ and $(1) = (t,\ldots(t,\exp X)\ldots) = \exp((\tilde{t} - I)^k X)$. Therefore $\tilde{t}(X) = X$. Therefore $\tilde{t} = I$ and $\gamma^{\#}$ is unipotent. Let $N(\Gamma) = N(\Gamma_{r-1}) \cdot G(\gamma^{\#})$. $N(\Gamma)$ is a real nilpotent group and contains $\Gamma = \Gamma_{r-1} \cdot \hat{\gamma}^{\#}$ (semi direct) as a discrete cocompact subgroup.

<u>Corollary 8.2</u>. Let C be a torsion free nilpotent Lie group whose component group C/C_0 is finitely generated.

Then C can be imbedded as a closed cocompact subgroup of a real nilpotent group.

<u>Proof</u>: C_0 is a real nilpotent group.

SECTION C. RATIONALITY

1. Introduction

Let N be a real nilpotent group. Let Γ be a discrete cocompact subgroup of N . Then we can find a canonical basis $X_1,\ldots,X_n$ of L(N) such that the analytic manifold isomorphism $R^n \to N$, $(t_1,\ldots,t_n) \to \Pi_1^n \exp(t_i X_i)$ takes Z^n onto Γ . By Theorem 6.2,

$$\Pi_1^n \exp(s_i X_i) \Pi_1^n \exp(t_i X_i) = \Pi_1^n \exp(u_i X_i) \quad \text{where}$$

$u_i = s_i + t_i + q_i(s_1,\ldots,s_{i-1}; t_1,\ldots,t_{i-1})$ with q_i a real polynomial. Since Γ is a subgroup we have that for integer values of $s_1,\ldots,s_{i-1}$; $t_1,\ldots,t_{i-1}$ we get that $q_i(s_1,\ldots,s_{i-1}; t_1,\ldots,t_{i-1})$ has integer values. Therefore using the fact that a real polynomial which takes on integer values for all integer variables must have purely rational coefficients, we have that the image of Q^n in N is a subgroup of N which we call $N_Q(\Gamma)$, and consider as a rational Lie group.

Let N be a real nilpotent group. Let $X_1,\ldots,X_n$ be a rational canonical basis of L(N) . We then have coordinates of type one for N given by $R^n \to N$, $(t_1,\ldots,t_n) \to \exp(\sum_1^n t_i X_i)$ and coordinates of type two for N given by $R^n \to N$, $(t_1,\ldots,t_n) \to \Pi_1^n \exp(t_i X_i)$. By Theorem 6.4, $\exp(\sum_1^n s_i X_i) \exp(\sum_1^n t_i X_i) = \exp(\sum_1^n u_i X_i)$ where

$u_i = s_i + t_i + p_i(s_1,\ldots,s_{i-1}; t_1,\ldots,t_{i-1})$ with p_i a polynomial whose coefficients are rational polynomials in the structure constants of $X_1,\ldots,X_n$. Therefore $\exp(\sum_1^n s_i X_i)$ where s_i ranges over Q is a subgroup of N . Therefore the subset of N whose coordinates of type one are rational as a subgroup of N . In the same way the subset of N whose coordinates of type two are rational is a subgroup of N . Moreover by Theorem 6.2, $\Pi_1^n \exp(t_i X_i) = \exp(\sum_1^n u_i X_i)$ implies $u_i = t_i + p_i(t_1,\ldots,t_{i-1})$ where p_i is a polynomial whose coefficients are rational polynomials in the structure constants of $X_1,\ldots,X_n$. Therefore the subset of N whose coordinates of type one are rational and whose coordinates of type two are rational coincide. Denote by $N_Q(X_1,\ldots,X_n)$.

We have shown that under two different assumptions we have been able to define a rational Lie group. We shall now proceed to show how these two assumptions are related, and to investigate this rational Lie group.

2. Rational Lie Groups

Let F be a field of characteristic 0. Assume that we have a group multiplication on F^n which satisfies

(1) 0 is the identity of the group

(2) $(a_1,\ldots,a_n)(b_1,\ldots,b_n) = (c_1,\ldots,c_n)$ where $c_i = a_i + b_i + q_i(a_1,\ldots,a_{i-1};b_1,\ldots,b_{i-1})$ and $q_i(X_1,\ldots,X_{i-1};Y_1,\ldots,Y_{i-1}) \in F[X_1,\ldots,X_{i-1};Y_1,\ldots,Y_{i-1}]$.

Note:

$$\frac{d}{dt} q_i(X_1,\ldots,X_{i-1};Y_1t,\ldots,Y_{i-1}t)_{t=0} =$$

$$= \sum_{k=1}^{i-1} Y_k \left(\sum \gamma(e_1,\ldots,e_{i-1};\delta_{k1},\ldots,\delta_{k,i-1}) \quad \Pi_{j=1}^{i-1} X_j^{e_j}\right.$$

where

$$q_i(X_1,\ldots,X_{i-1};Y_1,\ldots,Y_{i-1}) =$$

$$\sum \gamma(e_1,\ldots,e_{i-1};f_1,\ldots,f_{i-1}) \, \Pi_{j=1}^{i-1} X_j^{e_j} \, \Pi_{j=1}^{i-1} Y_j^{f_j}$$

If F were the reals then giving R^n its usual manifold structure this multiplication would define a real analytic group. We shall establish a differential geometry on F^n which when $F = R$ would be the usual differential geometry of a real analytic group.

(a) The tangent space at a point

Let $a = (a_1,\ldots,a_n)$ be a point in F^n. A curve through "a" is an n-tuple of polynomials $(p_1(t),\ldots,p_n(t))$ where $p_i(t) \in F[t]$ and $p_i(0) = a_i$ for $1 \le i \le n$. We shall say two curves $(p_i(t))$ and $(\bar{p}_i(t))$ through "a" are equivalent whenever $p_i'(0) = \bar{p}_i'(0)$ for $1 \le i \le n$. By the tangent space at "a" which we denote by $T(a)$ we shall mean the collection of all equivalence class of curves through "a". Let α in $T(a)$ be given by the curve through "a", $(p_i(t))$. Make correspond to α the n-tuple in F^n, $(p_i'(0))$. This defines a bijection of $T(a)$ onto F^n and we make $T(a)$ into an F vector space by requiring that this bijection be an F isomorphism.

(b) Translation

Let $b = (b_1,\ldots,b_n)$ be a point of F^n. We want to define an F isomorphism $b^{\#} : T(0) \to T(b)$. Let α in $T(0)$ be given by the curve through 0, $(a_1t,\ldots,a_nt)$ where $a_i \in F$ for $1 \le i \le n$. Then $b^{\#}(\alpha)$ is the element in $T(b)$ defined by the curve through "b", $b \cdot (a_it) = (b_1+a_1t),\ldots,b_n+a_nt+q_n(b_1,\ldots,b_{n-1}; a_1t,\ldots,a_{i-1}t))$.
Direct computation will show us that this defines an F isomorphism of $T(0)$ onto $T(b)$.

Computation: Let $(a_it + p_i(t))$ be another curve through 0 defining α. Then $p_i(t)$ has each of its terms with degree ≥ 2 in t. The curve through "b" given by $b \cdot (a_it + p_i(t)) = (b_i + a_it + p_i(t) + q_i(b_1,\ldots,b_{i-1}; a_1t + p_1(t),\ldots,a_{i-1}t + p_{i-1}(t)))$ defines the same element of $T(b)$ as does $b \cdot (a_it)$ since we have that the linear term of $q_i(b_1,\ldots,b_{i-1};\ a_1t + p_1(t),\ldots,a_{i-1}t + p_{i-1}(t))$ considered as a polynomial in t is the same as the linear term of $q_i(b_1,\ldots,b_{i-1};\ a_1t,\ldots,a_{i-1}t)$ considered as a polynomial in t. Therefore $b^{\#}$ is a well defined map of $T(0)$ into $T(b)$. Since F^n is a group it is clear that $b^{\#}: T(0) \to T(b)$ is a bijection.

Let β be an element of $T(0)$ defined by $(\beta_1 t,\ldots,\beta_n t)$. Then we have $b^{\#}(\beta)$ is given by $b \cdot (\beta_i t) = (b_i+\beta_i t+q_i(b_1,\ldots,b_{i-1};\beta_i t,\ldots\beta_{i-1}t))$. $\alpha + \beta$ is given by $((a_i + \beta_i)t)$ and $b^{\#}(\alpha + \beta)$ is given by $b \cdot ((a_i+\beta_i)t) = (b_i+(a_i+\beta_i)t + q_i(b_1,\ldots,b_{i-1};(a_1+\beta_1)t,\ldots,(a_{i-1}+\beta_{i-1})t))$. Moreover, $b^{\#}(\alpha)+b^{\#}(\beta)$ is given by $(b_i+(a_i+\beta_i)t + q_i(b_1,\ldots,b_{i-1};\ a_1t,\ldots,a_{i-1}t) + q_i(b_1,\ldots,b_{i-1};\beta_1 t,\ldots,\beta_{i-1}t))$.

Let $q_i(X_1,\ldots,X_{i-1};Y_1,\ldots,Y_{i-1}) = \sum \gamma(e_1,\ldots,e_{i-1};f_1,\ldots,f_{i-1}) \Pi_1^{i-1} X_j^{e_j}\ \Pi_1^{i-1} Y_j^{f_j}$ where e_j, f_j are integers. The linear term in t of $q_i(b_1,\ldots,b_{i-1};(a_1+\beta_1)t,\ldots,(a_{i-1}+\beta_{i-1})t)$ is given by

$$\sum_{j=1}^{i-1} \left(\sum \gamma(e_1,\ldots,e_{i-1};\delta_{1j},\ldots,\ \delta_{i-1,j})\ \Pi_1^{i-1}\ b_j^{e_j}\right)(a_j+\beta_j)$$

and the linear term in T of $q_i(b_1,\ldots,b_{i-1};a_1t,\ldots,a_{i-1}t) + q_i(b_1,\ldots,b_{i-1};\ \beta_1 t,\ldots,\beta_{i-1}t)$ is given by

$$\sum_{j=1}^{i-1}\left(\sum \gamma(e_1,\ldots,e_{i-1};\delta_{1j},\ldots,\delta_{i-1,j})\ \Pi_1^{i-1}\ b_j^{e_j}\right)a_j + \sum_{j=1}^{i-1}\left(\sum \gamma(e_1,\ldots e_{i-1};\delta_{1j},\ldots,\delta_{i-1,j})\ \Pi_1^{i-1}\ b_j^{e_j}\right)\beta_j .$$

Therefore $b^{\#}(\alpha+\beta) = b^{\#}(\alpha) + b^{\#}(\beta)$. In the same way $b^{\#}(x\alpha) = x\ b^{\#}(\alpha)$ for $x \in F$. Therefore $b^{\#}: T(0) \to T(b)$ is an F homomorphism.

Let $b^{\#}(\alpha)$ given by $(b_i+a_it+q_i(b_1,\ldots,b_{i-1};a_1t,\ldots,a_{i-1}t))$ be zero. Then $a_i + \sum_{j=1}^{i-1}\left(\sum \gamma(e_1,\ldots,e_{i-1};\delta_{1j},\ldots,\delta_{i-1,j})\ \Pi_1^{i-1}\ b_j^{e_j}\right)a_j = 0.$

But then clearly $a_1 = 0,\ a_2 + c_{11}\ a_1 = 0, \ldots, a_i + \sum_{j=1}^{i-1} c_{ij}\ a_j = 0, \ldots,$ and therefore $a_i = 0 \quad 1 \le i \le n$. Therefore $b^{\#}$ is an F isomorphism.

(c) Vector Fields

A vector field on F^n is an n-tuple $(\alpha_1(Y_1,\ldots,Y_n),\ldots,\alpha_n(Y_1,\ldots,Y_n))$ where $\alpha_i(Y_1,\ldots,Y_n)$ is a polynomial in $F[Y_1,\ldots,Y_n]$. We shall let $V(F^n)$ be the set of all vector fields on F^n. Let $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$ be two vector fields on F^n and let $p(Y_1,\ldots,Y_n)$ be an element of $F[Y_1,\ldots,Y_n]$. Defining $\alpha+\beta = (\alpha_i+\beta_i)$ and $p\cdot\alpha = (p\ \alpha_i)$ we define $V(F^n)$ as an $F[Y_1,\ldots,Y_n]$ module. A derivation of $F[Y_1,\ldots,Y_n]$ we mean a map α of $F[Y_1,\ldots,Y_n]$ into itself such that α is an F linear map and $\alpha(pq) = \alpha(p)\ q + p\ \alpha(q)$ for p and q on $F[Y_1,\ldots,Y_n]$. Let α and β be derivations of $F[Y_1,\ldots,Y_n]$ and let p be an element of $F[Y_1,\ldots,Y_n]$. Defining $(\alpha+\beta)(\bar{p}) = \alpha(\bar{p}) + \beta(\bar{p})$ and $(p\alpha)(\bar{p}) = p \cdot \alpha(\bar{p})$ for $\vec{p} \in F[Y_1,\ldots,Y_n]$ we define $\mathrm{Der}\ F[Y_1,\ldots,Y_n]$ as a $F[Y_1,\ldots,Y_n]$ module. Let $\frac{\partial}{\partial Y_i}$ denote the derivation of $F[Y_1,\ldots,Y_n]$ which takes Y_j into δ_{i_j}. This defines $\frac{\partial}{\partial Y_j}$ for we can, using the multiplication property, extend $\frac{\partial}{\partial Y_j}$ uniquely to the monomials of $Y_1,\ldots,Y_n$ and by F linearity to F linear combinations of the monomials in $Y_1,\ldots,Y_n$ which is all of $F[Y_1,\ldots,Y_n]$. Consider the map of $V(F^n)$ into $\mathrm{Der}\ F[Y_1,\ldots,Y_n]$ which takes $(\alpha_i(Y_1,\ldots,Y_n)) \to \sum_1^n \alpha_i(Y_1,\ldots,Y_n) \frac{\partial}{\partial Y_i}$. This is an $F[Y_1,\ldots,Y_n]$ isomorphism of $V(F^n)$ onto $\mathrm{Der}(F[Y_1,\ldots,Y_n])$ which can be determined by direct computation.

(d) Lie Bracket

Let α and β be in $\mathrm{Der}(F[Y_1,\ldots,Y_n])$ then we define

$[\alpha,\beta] = \alpha\beta - \beta\alpha$ which is again in $\mathrm{Der}(F[Y_1,\ldots,Y_n])$ by computation. In this way $\mathrm{Der}(F[Y_1,\ldots,Y_n])$ becomes a Lie algebra. We can make $V(F^n)$ into a Lie algebra by requiring that the $F[Y_1,\ldots,Y_n]$ isomorphism onto $\mathrm{Der}(F[X_1,\ldots,X_n]$ be an F Lie algebra isomorphism. We can directly compute that for $\alpha = (\alpha_i)$ and $\beta = (\beta_j)$ in $V(F^n)$ we have

$$[\alpha,\beta] = \left(\sum_{j=1}^{n} \left(\beta_j \frac{\partial \alpha_i}{\partial Y_j} - \alpha_j \frac{\partial \beta_i}{\partial Y_j}\right)\right) .$$

(d) Left Invariant Vector Fields

We shall use (b) to define an F isomorphism of $T(0)$ onto a Lie subalgebra of $V(F^n)$. The image which we denote by $L(F^n)$ and call the left invariant vector fields of F^n is then the analagous Lie algebra of a Lie group. Let α in $T(0)$ be given by $(a_i t)$. We make correspond to α, $\alpha(Y_1,\ldots,Y_n) = (\alpha_i(Y_1,\ldots,Y_n))$ an element of $V(F^n)$ as follows. We define

$$\alpha_i(Y_1,\ldots,Y_n) = \frac{d}{dt}\left(a_i t + q_i(Y_1,\ldots,Y_{i-1};\ a_1 t,\ldots,a_{i-1}t)\right)_{t=0} .$$

As in (b) the map $T(0) \to V(F^n)$, $\alpha \to \alpha(Y_1,\ldots,Y_n)$ is a well-defined F isomorphism of $T(0)$ into $V(F^n)$. $L(F^n)$ is therefore an n-dimensional F vector space. Let $L(\mathrm{Der}(F[Y_1,\ldots,Y_n]))$ be the image of $L(F^n)$ in $\mathrm{Der}(F[Y_1,\ldots,Y_n])$. We shall now characterize $L(F^n)$ and $L(\mathrm{Der}(F[Y_1,\ldots,Y_n]))$. Clearly $\alpha(Y_1,\ldots,Y_n) = (\alpha_i(Y_1,\ldots,Y_n))$ is an element of $L(F^n)$ if and only if we have for $b = (b_i)$ in F^n

$$\alpha_j(b_1,\ldots,b_n) = \alpha_j(0,\ldots,0) + \frac{d}{dt} q_j(b_1,\ldots,b_{j-1}\ ;\ \alpha_1(0)t,.,\alpha_{j-1}(0)t)_{t=0}$$

for all j . But $\frac{d}{dt} q_j(b_1,\ldots,b_{j-1};\alpha_1(0)t,\ldots,\alpha_{j-1}(0)t)_{t=0} =$

$$\sum_{e=1}^{j-1} \alpha_k(0)\left(\sum \gamma(e_1,\ldots,e_{j-1};\ \delta_{1k},\ldots,\delta_{j-1,k})\ \Pi_{i=1}^{j-1} b_i^{e_i}\right)$$

where $g_L(X_1,\ldots,X_{i-1};Y_1,\ldots,Y_{j-1}) = \sum \gamma(e_1,\ldots,e_{i-1};f_1,\ldots,f_{i-1})$

$$\Pi_1^{i-1} X_j^{e_j} \Pi_1^{i-1} Y_j^{f_j} .$$

Therefore $\alpha(Y_1,\ldots,Y_n) = (\alpha_i(Y_1,\ldots,Y_n))$ is in $L(F^n)$ if and only if for $b = (b_1,\ldots,b_n)$ in F^n we have

$$\alpha_j(b) = \alpha_j(0) + \sum_{k=1}^{j-1} \alpha_k(0)(\sum [\gamma(e_1,\ldots,e_{j-1};\delta_{1k},\ldots,\delta_{j-1,k})) \Pi_{\ell=1}^{j-1} b_\ell^{e_\ell}]),$$

for $1 \leq j \leq n$.

Fix b in F^n. Let $tb : F^n \to F^n$, $t_b(a) = ba$. Let $(\alpha_j(Y_1,\ldots,Y_n))$ be an element of $V(F^n)$ and let α be the corresponding element in $\operatorname{Der} F[Y_1,\ldots,Y_n]$. Where f is an element of $F[Y_1,\ldots,Y_n]$ we have $\alpha(f)b = \alpha(f \circ t_b)(0)$ if and only if

$$\sum_{j=1}^{n} \alpha_j(b) \frac{\partial f}{\partial Y_j} (b) = \sum_{j=1}^{n} \alpha_j(0) \frac{\partial}{\partial Y_j} (f \circ t_b)(0) .$$

Let $X_i = b_i + Y_i + q_i(b_1,\ldots,b_{i-1};Y_1,\ldots,Y_{i-1})$. We have $\alpha(f)b = \alpha(f \circ t_b)(0)$ if and only if $\alpha_j(b) = \sum_{i=1}^{n} \alpha_i(0) \frac{\partial X_j}{\partial Y_i}(0)$.

However, $\frac{\partial X_j}{\partial Y_i}(0) = \frac{\partial}{\partial Y_i} g_j(b_1,\ldots,b_{j-1};Y_1,\ldots,Y_{j-1})_{Y_i=0} =$

$$\sum \gamma(e_1,\ldots,e_{j-1};\delta_{1i},\ldots,\delta_{j-1,i}) \Pi_{\ell=1}^{j-1} b_\ell^{e_\ell} ,$$

for $i < j$ and for $i > j$. $\frac{\partial X_j}{\partial Y_i}(0) = 1$ if $i = j$.

Therefore $\alpha_j(b) = \sum_{i=1}^{n} \alpha_i(0) \frac{\partial X_j}{\partial Y_i}(0)$ if and only if

$$\alpha_j(b) = \alpha_j(0) + \sum_{k=1}^{j-1} \alpha_k(0)(\sum [\gamma(e_1,\ldots,e_{j-1};\delta_{k1},\ldots,\delta_{ki-1}) \Pi_{\ell=1}^{j-1} b_\ell^{e_\ell}]).$$

Therefore α is in $L(\operatorname{Der}(F[Y_1,\ldots,Y_n]))$ if and only if $\alpha(f)b = \alpha(f \circ t_b)(0)$ which is clearly equivalent to $\alpha(f \circ tb) \circ t_b^{-1} = \alpha(f)$.

We are now in a position to prove $L(F^n) = L(\mathrm{Der}(F[Y_1,\ldots,Y_n])$ is a Lie subalgebra of $V(F^n)$. Let α and β be in $L(\mathrm{Der}(F[Y_1,\ldots,Y_n]))$.

$[\alpha,\beta](f \circ t_b) \circ t_b^{-1} = (\alpha\beta)((f \circ t_b)) \circ t_b^{-1} - (\beta\alpha)(f \circ tb) \circ t_b^{-1}$.

$(\alpha\beta)(f \circ t_b) \circ t_b^{-1} = \alpha(\beta(f \circ t_b)) \circ t_b^{-1} = \alpha(\beta(f) \circ t_b) \circ t_b^{-1} = \alpha(\beta(f))$.

Therefore $[\alpha,\beta](f \circ t_b) \circ t_b^{-1} = [\alpha,\beta](f)$.

We have the following

Theorem B. E is an extension of F

Suppose we have a group multiplication on E^n satisfying

(1) 0 is the identity.

(2) $(a_i)(b_i) = (c_i)$ where $c_i = a_i+b_i+q_i(a_1,\ldots,a_{i-1};b_1,\ldots,b_{i-1})$ with $q_i(X_jY) \in F[X_jY]$.

Then

(a) F^n is a subgroup of E^n.

(b) The injection of F^n in E^n defines an injection of $T_F(0) = L(F^n)$ into $T_E(0) = L(E^n)$ which defines an F isomorphism of $L(F^n)$ into $L(E^n)$ which preserves the Lie bracket.

(c) A basis of $L(F^n)$ as an F vector space is a basis of $L(E^n)$ as an E vector space.

3. Structure Theorem

Theorem 9.1. Let N be a real nilpotent group. Then N has a discrete cocompact subgroup if and only if there is a canonical basis for $L(N)$ whose structure constants are rational.

Proof: Only if: this follows from the preceeding discussion.

if: We shall prove the following lemma.

Lemma 9.2. Let $X_1,\ldots,X_M$ be a canonical rational basis of $L(N)$. Then there exists integers $k_1,\ldots,k_n$ such that $\exp(\sum_1^m \frac{m_i}{k_i} X_i)$ is a subgroup of N as the m_i run over the integers.

Proof: X_n is central in $L = L(N)$, and $\exp X_n$ is central on N. Let $N' = N/\exp t_n X_n$ which is a real nilpotent group and $L(N') = L(N)/R X_n$. Let $p : N \to N'$ be the natural map and let $p^{\#}: L(N) \to L(N')$ be the differential of p. Clearly $p^{\#} X_1,\ldots,p^{\#} X_{n-1}$ is a rational canonical basis of $L(N')$. By induction we can find integers $k_1,\ldots,k_{n-1}$ such that $\exp(\sum_1^{n-1} \frac{m_i}{k_i} p^{\#} X_i)$ is a subgroup of N' where the m_i's run over the integers. Now

$$\exp(\sum_1^{n-1} \frac{m_i}{k_i} X_i) \exp(\sum_1^{n-1} \frac{m_i'}{k_i} X_i) = \exp(\sum_1^{n-1} \frac{n_i}{k_i} X_i) \cdot Z$$

where m_i, m_i' and n_i are integers and Z is in $\exp t_n X_n$. But then

$$\exp(\sum_1^{n-1} \frac{m_i}{k_i} X_i) = \exp(\sum_1^{n-1} \frac{m_i'}{k_i} X_i) = \exp(\sum_1^{n-1} \frac{n_i}{k_i} X_i + T_n X_n)$$

since Z is central. The Hausdorf-Campbell formula tells us that $T_n = q_n(\frac{m_1}{k_1},\ldots,\frac{m_{n-1}}{k_{n-1}} ; \frac{m_1'}{k_1},\ldots,\frac{m_{n-1}'}{k_{n-1}})$ where each coefficient of q_n is a rational polynomial in the structure constants of $X_1,\ldots,X_n$ which have been assumed to be rational. Therefore there is an integer k_n depending solely on $k_1,\ldots,k_{n-1}$ and the denominators of the structure constants such that for arbitrary m_i, m_j' $1 \le i, j \le n-1$ there is an integer n_n such that $Z = \exp \frac{n_n}{k_n}$. Clearly $\exp(\sum_1^n \frac{m_i}{k_i} X_i)$ is a subgroup of N.

Let Γ be a finitely generated torsion free nilpotent group for the remainder of this chapter. Let C be a torsion free nilpotent Lie group whose component group C/C_0 is torsion free and finitely generated. Let $N(\Gamma)$ and $N(C)$ be the unique simply connected nilpotent analytic groups containing Γ and C as discrete and closed cocompact subgroups respectively. We shall say that Γ is a discrete lattice group whenever $\log \Gamma$ is an additive subgroup of the vector space $L(N(\Gamma))$. We shall say that C is a closed lattice group whenever C/C_0 is a discrete lattice group. The following lemma follows directly from the proof of lemma. Let N_C be the last term of the lower central series of N a real nilpotent group. We shall denote the image of an element n of N by $\bar{n}$ under the natural map of N onto N/N_C . We shall denote the image of an element X of $L(N)$ by $\bar{X}$ under the natural map of $L(N)$ onto $L(N/N_C)$.

<u>Lemma 9.3</u>. Let $X_1,\ldots,X_r$ be a rational canonical basis of the nilpotent Lie algebra $L(N)$, where $X_{t+1},\ldots,X_r$ is a basis of $L(N_C)$. Assume $\exp(\sum_1^t n_i \bar{X}_i)$ is a subgroup of N/N_C where the n_i run over the integers.

Then there are integers $m_{t+1},\ldots,m_r$ such that

$\exp(\sum_1^t n_i X_i + \sum_{t+1}^r n_i \frac{X_i}{m_i})$ is a subgroup of N where the n_i run over the integers. Moreover we can choose the $m_{t+1},\ldots,m_r$ such that $\exp(\sum_1^t n_i X_i + \sum_{t+1}^r n_i \frac{X_i}{m_i})$ is the subgroup of N generated by $\exp(\sum_1^t n_i X_i)$ and $\exp X_j$ for $t+1 \leq j \leq r$.

Theorem 9.4. There is a discrete lattice group $\Gamma_{\#}$ contained in $N(\Gamma)$ and containing Γ up to finite index. We shall call any such $\Gamma_{\#}$ a lattice envelope of Γ.

Proof: Let $\gamma_1,\ldots,\gamma_r$ be elements of Γ satisfying

1. Each element of Γ can be written uniquely as $\Pi_1^r \gamma_j^{e_j}$ where the e_j are integers.
2. $\Gamma_i = \Pi_i^r \gamma_j^{e_j}$ where the e_j run over the integers, then Γ_i is a normal subgroup of Γ and $\Gamma_i/\Gamma_{i+1} = Z$.
3. Each element of $N(\Gamma)$ can be uniquely written as $\Pi_1^r \exp(t_j \log \gamma_j)$ where the t_j are real.
4. $N_i = G(\Gamma_i) = \Pi_i^r \exp(t_j \log \gamma_j)$ where the t_j run over the reals. Then N_i is a normal subgroup of N and $N_i/N_{i+1} = R$.

Such elements are called a canonical basis of Γ. Let $X_i = \log \gamma_i$. Then $X_1,\ldots,X_r$ is a canonical rational basis of $L(N(\Gamma))$. There are integers $m_1,\ldots,m_r$ such that $\Gamma_{\#} = \exp(\sum_1^r n_i \frac{X_i}{m_i})$ is a subgroup of N as the n_i run over the integers. Since Γ is generated by $\exp X_i$ $1 \leq i \leq r$ which are contained in $\Gamma_{\#}$ we have $\Gamma \subset \Gamma_{\#}$. Moreover $\Gamma_{\#}$ is a discrete cocompact subgroup of $N(\Gamma)$. Therefore $\Gamma_{\#}/\Gamma$ is finite since both $\Gamma, \Gamma^{\#}$ are discrete and have the same algebraic hull.

Theorem 9.5. Let $x_1^1,\ldots,x_{i1}^{},\ldots,x_1^c,\ldots,x_{ic}^c$ be a rational canonical basis of $L(N(\Gamma))$ where

(a) $x_1^k,\ldots,x_{ik}^k,\ldots,x_1^c,\ldots,x_{ic}^c$ is a basis of $L(N^k)$.

(b) The elements $\exp x_1^1,\ldots \exp x_{i1}^1,\ldots, \exp x_1^c,\ldots, \exp x_{ic}^c$ is a canonical basis of Γ.

Let A be an automorphism of Γ, $A_N(\Gamma)$ its unique extension to $N(\Gamma)$. Then there exist integers $m_1 = 1, m_2, \ldots, m_c$ such that

$$\Gamma^{\#} = \exp\left\{ \left(\sum_{\ell=1}^{c} \sum_{j=1}^{i_\ell} n_j^\ell \frac{x_j^\ell}{m_\ell} \right) : n_j^\ell \text{ integers} \right\}$$ is a subgroup of $N(\Gamma)$

and $A_{N(\Gamma)}(\Gamma^{\#}) = \Gamma^{\#}$.

<u>Proof</u>: We use induction on the steps of nilpotency of Γ. If Γ is abelian we are clearly done. Let Γ be c step nilpotent and assume the theorem true for $\ell < c$. Let N^c be the last term in the lower central series for $N = N(\Gamma)$ and let $\Gamma_c = N^c \cap \Gamma$. Then $N(\Gamma/\Gamma_c) = N(\Gamma)/N_c$. Let us denote the image of an element $n \in N$, by $\bar{n}$ under the natural map of N onto N/N^c and the image of an element X on $L(N)$ by $\bar{X}$ under the natural map of $L(N)$ onto $L(N)/L(N^c)$. Let $\bar{A}$ be the automorphism of Γ/Γ_c induced by A. By induction there are integers $m_1 = 1, m_2, \ldots, m_{c-1}$ such that where

$$\bar{\gamma} = \left\{ \sum_{\ell=1}^{c-1} \sum_{j=1}^{i_\ell} n_j^\ell \frac{\bar{X}_j^\ell}{m_\ell} : n_j^\ell \text{ integers} \right\}$$ we have that

$\exp \bar{\gamma}$ is a subgroup of $N(\Gamma)/N^c$ and that $\bar{A}_{N(\Gamma)}/N^c\,(\exp \bar{\gamma}) = \exp \bar{\gamma}$.

There are integers $m_1^c, \ldots, m_{ic}^c$ such that where

$$\gamma' = \left\{ \sum_{\ell=1}^{c-1} \sum_{j=1}^{i_\ell} n_j^\ell \frac{X_j^\ell}{m_\ell} + \sum_{j=1}^{i_c} n_j \frac{X_j^c}{m_j^c} : n_j^\ell,\ n_j \text{ integers} \right\}$$ we have

$\exp \gamma'$ is a subgroup of $N(\Gamma)$ and is generated by

$$\exp\left\{ \left(\sum_{\ell=1}^{c-1} \sum_{j=1}^{i_\ell} n_j^\ell \frac{X_j^\ell}{m_\ell} : n_j^\ell \text{ integers} \right\} \right.$$ and $\exp X_j^c$ $1 \le j \le i_c$.

Clearly there is an integer $m_c = \Pi_{j=1}^{i_c} n_j^c$ such that where

$$\gamma = \left\{ \sum_{\ell=1}^{c} \sum_{j=1}^{i_\ell} n_j^\ell \frac{X_j^\ell}{m_\ell} : n_j^\ell \text{ integers} \right\}$$ we have $\exp \gamma$ a subgroup of

$N(\Gamma)$. We have that $\exp \gamma/\Gamma$ is finite say of order m . Let X be in γ, then $A_{N(\Gamma)}(\exp X) = \exp Y \cdot Z$ where Y is in γ and Z is in N^c . Since $(\exp X)^m = \exp mX$ is in Γ, we have that Z^m is in $\Gamma_{\#} = \exp \gamma$. Therefore if we let m_c now be $m \cdot m_c$ the integers $m_1 = 1,\ldots,m_c$ will satisfy the conclusions of the theorem.

CHAPTER II

THEORY OF SOLV-MANIFOLDS

SECTION 1.

1. Introduction

At this point we shall generalize our concept of an algebraic group. Our old algebraic groups become real algebraic groups. Let k be any field of characteristic zero, and let k^n denote the set of all ordered n-tuples of k. Let $k[X_1,\ldots,X_n]$ be the integral domain of all polynomials with k-coefficients in n variables and let $k(X_1,\ldots,X_n)$ be its quotient field. We can regard $k[X_1,\ldots,X_n]$ as a ring of functions on k^n and $k(X_1,\ldots,X_n)$ as a collection of functions defined on subsets of k^n. An algebraic subset of k^n is the set of all common zeroes of a finite number or equivalently an ideal of polynomials of $k[X_1,\ldots,X_n]$. The algebraic subsets of k^n determine the closed subsets of a topology for k^n which we call its Zariski topology. Relative to this topology we can regard $k(X_1,\ldots,X_n)$ as a collection of functions defined on open subsets of k^n. Let A be an open subset of k^n. Then we can give to A the induced Zariski topology. A rational function f of A is a map $f\colon A \to k$ which locally can be represented by an element of $k(X_1,\ldots,X_n)$. The space k^n with the Zariski topology along with the map which assigns to each open subset of k^n, A, the collection of all rational functions of A is a k structured space which we call the affine variety k^n.

Let $M_k(n)$ denote the collection of all $n \times n$ matrices whose coefficients are in k. Let $i\colon M_k(n) \to k^{n^2}$ be the bijection given by $i((\alpha_{ij})) = (\alpha_{11},\ldots,\alpha_{1n},\ldots,\alpha_{n1},\ldots,\alpha_{nn})$. We make $M_k(n)$ into a vector space over k by requiring that i be a vector space isomorphism. Defining a multiplication on $M_k(n)$, $[X,Y] = XY - YX$ we make $M_k(n)$ into a k Lie algebra. By requiring that i be a structured space isomorphism of $M_k(n)$ onto the structure affine space k^n we make $M_k(n)$ into a k structured space called the affine variety $M_k(n)$.

Let $GL_k(n)$ be the group of all nonsingular matrices in $M_k(n)$. Clearly $GL_k(n)$ is open in the affine variety $M_k(n)$ and therefore inherits in a natural way a functional structure from $M_k(n)$. $GL_k(n)$ as a group and as a structured space will be called the affine group $GL_k(n)$. An algebraic subgroup of $GL_k(n)$ is a closed subgroup of the affine group $GL_k(n)$. A rational representation r of an algebraic subgroup A of $GL_k(m)$ is a rational function of an open subset of the affine variety $M_k(n)$ containing A into $GL_k(n)$ whose restriction to A is a group homomorphism. It is a fact that all the algebraic group theorems of Chapter 1, Section 1 hold for algebraic subgroups of $GL_k(n)$. Also since k contains the rationals and since both exponential and log are defined over the rationals all the algebraic statements of Chapter 1, Section 2 hold for algebraic subgroups of $GL_k(n)$.

To study solvable Lie groups we will need two further theorems which we now state.

1. Let S be a connected solvable algebraic subgroup of $GL_k(n)$. Then

(a) U = the collection of all unipotent matrices in S is a connected algebraic group which is a normal subgroup of S .

(b) Where T is any maximal completely reducible subgroup of S we have $S = U \cdot T$ (semi-direct product).

Moreover where T, $\bar{T}$ are two maximal completely reducible subgroups of S, there is an h in $\mathfrak{G}((S,S))$ such that $h T h^{-1} = \bar{T}$.
This theorem allows us to define a semi-simple splitting of a simply connected solvable analytic group.

The following generalization due to Mostow will allow us to define a semi-simple splitting for a more general type of group.

2. Let Γ be a solvable algebraic group of $GL_k(n)$.
Then

(a) U = the collection of all unipotent matrices of Γ is a connected algebraic normal subgroup of Γ.

(b) Where T is a maximal completely reducible subgroup of Γ we have $\Gamma = U \cdot T$ (semi-direct product).

Moreover for any two maximal completely reducible subgroup of Γ, T and $\bar{T}$ there is a u in U such that $u T u^{-1} = \bar{T}$.

SECTION 2. The Semi-Simple Splitting

Let S be a simple connected solvable analytic group. We can write $\mathfrak{G}(ad_S\ S) = U^{\#} \cdot T^{\#}$ where $U^{\#}$ is the group of a unipotent automorphisms of $\mathfrak{G}(ad_S\ S)$ and $T^{\#}$ is a maximal completely reducible subgroup of $\mathfrak{G}(ad_S\ S)$. Let $\zeta_1: \mathfrak{G}(ad_S\ S) \to T^{\#}$ be the homomorphism of $\mathfrak{G}(ad_S\ S)$ onto $T^{\#}$ with kernel $U^{\#}$. Then $\zeta: S \to T^{\#}$ given by $\zeta = \zeta_1 \circ ad$ is called the semi-simple homomorphism of S relative to the decomposition $\mathfrak{G}(ad_S\ S) = U^{\#} \cdot T^{\#}$. Let $T_S = \zeta(S)$.

Now $ad_S\ S$ is a group of automorphisms of S and therefore the Lie algebra of S. Since the group of automorphisms of a Lie algebra is algebraic, $\mathfrak{G}(ad_S\ S)$ is a group of automorphisms of S. Therefore we can form $S \cdot T_S$ (semi direct product). We call this the semi-simple splitting of S relative to the decomposition $\mathcal{G}(ad_S\ S) = U^{\#} \cdot T^{\#}$. We now state the following theorem found in [ ].

<u>Theorem 2.1</u>. Let N_S be the nil radical of $S \cdot T_S$, then

(a) $S \cdot T_S = N_S \cdot T_S$ (semi direct product).

(b) The nil radical N of S is normal in $S \cdot T_S$ and $S \cdot T_S/N$ is abelian.

(c) The projection $p: S \to N_S$, $p(s) = n$ where $s = nt$ with $n \in N_S$, $t \in T_S$ is a homeomorphism of S onto N_S.

(d) S, N_S generate $S \cdot T_S$.

<u>Theorem 2.2</u>. Let S be a solvable simply connected analytic group. Assume $S \subset N_S \cdot T_S$ (semi direct product) such that N_S is a real nilpotent group and T_S is an abelian group of semi-simple automorphisms of N_S. Assume, moreover, that

(a) If: $p : N_S \cdot T_S \to N_S$ is the projection onto the first factor then p restricted to S is a homeomorphism of S onto N_S.

(b) S and N_S generated $N_S \cdot T_S$

then

there exists a representation of $\mathcal{G}(ad_S\ S) = U^{\#} \cdot T^{\#}$ such that $N_S\ T_S = S \cdot T_S$ is the semi-simple splitting relative to this representation.

SECTION 3. The Discrete Semi-Simple Splitting

(a) Introduction

We will generalize the semi-simple splitting discussed in Section 2. Let Γ_R be a group satisfying the exact sequence $1 \to N \to \Gamma_R \overset{\pi}{\to} H \to 1$ where N is a real nilpotent group and H is a discrete finitely generated abelian group. We can write $H = H_1 \oplus H_2$ where H_1 is a torsion free finitely generated abelian group and H_2 is the torsion subgroup of H. Let $h_1,\ldots,h_s$ be a basis of H_1 and let $h_{s+1},\ldots,h_n$ be the elements of H_2. By 2. there is a semi-direct product representation $\mathfrak{a}(ad_N \Gamma_R) = U^{\#} \cdot T^{\#}$ where $U^{\#}$ is the group of all unipotent automorphisms of $\mathfrak{a}(ad_N \Gamma_R)$ and $T^{\#}$ is a maximal completely reducible subgroup of $\mathfrak{a}(ad_N \Gamma_R)$. Let $\zeta_1 : \mathfrak{a}(ad_N \Gamma_R) \to T^{\#}$ be the homomorphism of $\mathfrak{a}(ad_N \Gamma_R)$ onto $T^{\#}$ with kernel $U^{\#}$. Then $\zeta : \Gamma_R \to T^{\#}$ given by $\zeta = \zeta_1 \circ ad_N$ is called the semi-simple homomorphism of Γ_R relative to the representation $\mathfrak{a}(ad_N \Gamma_R) = U^{\#} \cdot T^{\#}$. We shall define for each representation a semi-simple splitting of Γ_R and we shall show that any two semi-simple splittings of Γ_R relative to the same representation are isomorphic. We can then quite easily show that any two semi-simple splittings of Γ_R are isomorphic. In what follows we fix a representation $\mathfrak{a}(ad_N \Gamma_R) = U^{\#} \cdot T^{\#}$ and let $\zeta : \Gamma_R \to T^{\#}$ be the corresponding semi-simple homomorphism of Γ_R.

Lemma 1.1. Γ is nilpotent if and only if $ad_{N(\Delta)}\Gamma$ consists of unipotent transformations.

Proof: Assume Γ is nilpotent, and take $\gamma \in \Gamma$. Since Γ is nilpotent we have that there is an integer k such that $\overbrace{(\gamma,\ldots(\gamma,\Delta)\ldots)}^{k} = (e)$. Let $\gamma^{\#} = ad_{N(\Delta)}\gamma$. By the Birkhoff Embedding theorem we may consider $N(\Delta)$ to be an algebraic subgroup of $U(n)$ for some n and we may find a $\bar{\gamma}$ in $GL(n)$ such that $\gamma^{\#}(n) = \bar{\gamma}\, n\, \bar{\gamma}^{-1}$ for all $n \in N(\Delta)$. Therefore $(\bar{\gamma},\ldots(\bar{\gamma},\Delta)\ldots) = (e)$ and since the map $U(n) \to U(n)$ given by $n \to (\bar{\gamma},\ldots(\bar{\gamma},n)\ldots)$ is Zariski continuous we have $(G(\bar{\gamma}),\ldots,(G(\bar{\gamma}), N(\Delta)\ldots) = (e)$. Write $\bar{\gamma} = t\, u$ where t is a semi-simple element and u is a unipotent element of $GL(n)$ and $tu = ut$. We shall show that $t = 1$. Let t_L be the automorphism of $L(N(\Delta))$ given by $X \to t\, X\, t^{-1}$ and let t_N be the automorphism of $N(\Delta)$ given by $n \to t\, n\, t^{-1}$. We then have that t_N is the semi-simple part of $ad_{N(\Delta)}\gamma$ and that t_L is its differential. There is over the complexification of $L(N(\Delta))$ a basis consisting of eigen vectors of t_L . Let $X = \log n$ for $n \in N(\Delta)$ be an eigen vector of t_L . Then $[X,t_L(X)] = 0$ and $(t,n) = \exp[(t_L - I)X]$. Therefore $\exp[(t_L - I)^k X] = \overset{k}{(t,\ldots(t,n)\ldots)} = (e)$. Therefore $(t_L - I)^k X = 0$ and since X is an eigen vector $t_L(X) = X$. Therefore $t_L = 1_{L(N(\Delta))}, (t,n) = 1$ for all n in $N(\Delta)$ and $\gamma^{\#}$ is unipotent.

Assume now that $ad_{N(\Delta)}\Gamma$ consists of unipotent automorphisms. Let C be the center of $N(\Delta)$ and let $C_1 = \{x \in C\colon (f,x) = e$ for all $f \in \Gamma\}$. C_1 is an algebraic subgroup of $N(\Delta)$ and since $ad_{N(\Delta)}\Gamma$ consists of unipotent automorphisms, $\dim C_1 > 0$. Let $\Delta_1 = C_1 \cap \Delta$. Then Δ_1 is a normal subgroup of Γ and $\Delta/\Delta_1 = \Delta C_1/C_1$ being a closed subgroup of $N(\Delta)/C_1$ is a torsion free nilpotent Lie group whose component group is finitely generated. Consider the exact sequence

$1 \to \frac{\Delta}{\Delta_1} \to \frac{\Gamma}{\Delta_1} \to H \to 1$. Since $\dim N(\Delta)/C_1 < \dim N(\Delta)$ we have that Γ/Δ_1 is nilpotent. However, since $(\Gamma,\Delta_1) = (e)$ we have that Γ is nilpotent.

Lemma 2. $G(ad_{N(\Delta)}\Gamma_N)/\ ad_{N(\Delta)}N(\Delta)$ is abelian.

Proof: $(\Gamma_N,\Gamma_N) \subseteq N(\Delta)$. Therefore $(ad_{N(\Delta)}\Gamma_N,\ ad_{N(\Delta)}\Gamma_N) \subseteq ad_{N(\Delta)}N(\Delta)$.

Since $ad_{N(\Delta)}N(\Delta)$ is algebraic being an analytic group of unipotent automorphism we have $(G(ad_{N(\Delta)}\Gamma_N),\ G(ad_{N(\Delta)}\Gamma_N)) \subseteq ad_{N(\Delta)}N(\Delta)$.

Lemma 3. Let t and t' be semi-simple automorphisms of $G(ad_{N(\Delta)}\Gamma_N)$. Then there is an $X \in ad_{N(\Delta)}N(\Delta)$ such that $t\,t'\,X$ is a semi-simple automorphism.

Proof: Since $ad_{N(\Delta)}N(\Delta)$ is a normal algebraic subgroup of $G(ad_{N(\Delta)}\Gamma_N)$ there is a rational representation of $G(ad_{N(\Delta)}\Gamma_N)$ $r\colon G(ad_{N(\Delta)}\Gamma_N) \to Q \subseteq GL(n)$ for some n whose kernel is $ad_{N(\Delta)}N(\Delta)$. By Lemma 2, Q is abelian. Since r is a rational representation and t and t' are semi-simple, $r(t)\,r(t') = r(t\,t')$ is semi-simple. But then there is a semi-simple automorphism of $G(ad_{N(\Delta)}\Gamma_N)$, $\bar{t}$ such that $r(\bar{t}) = r(t\,t')$.

Lemma 4. Let t be a semi-simple element of $G(ad_{N(\Delta)}\Gamma_N)$ and let u be a unipotent element of $G(ad_{N(\Delta)}\Gamma_N)$.

Then the semi-simple part of tu is equal to $t \bmod (ad_{N(\Delta)}N(\Delta))$. The unipotent part of tu is equal to $u \bmod (ad_{N(\Delta)}N(\Delta))$.

Proof: Let $tu = \bar{t}\bar{u}$ where $\bar{t}(\bar{u})$ is a semi-simple (unipotent) automorphism and $(\bar{t},\bar{u}) = 1$. Then $\bar{t}^{-1}t = \bar{u}\,u^{-1}$. There is an x in $ad_{N(\Delta)}N(\Delta)$ such that $\bar{t}^{-1}tx$ is semi-simple. But since $x,u,\bar{u}$ are all in $U^{\#}$ we have $\bar{u}\,\bar{u}^{-1}x$ is unipotent. Therefore $\bar{t}^{-1}tx = \bar{u}\,u^{-1}x = 1$.

Lemma 5. $T^{\#}$ acts completely reducibly on $U^{\#}$ under the adjoint representation.

<u>Proof</u>: $T^{\#}$ acts completely reducibly on N and is therefore a completely reducible subgroup of the algebraic group $\mathrm{Aut}(N)$. The map φ: $\mathrm{Aut}(N) \longrightarrow \mathrm{Aut}(\mathrm{Aut}(N))$ which takes $x \to \varphi_x(y) = x\, y\, x^{-1}$ is a rational representation. Therefore $\varphi(T^{\#})$ is a completely reducible subgroup of $\mathrm{Aut}(\mathrm{Aut}(N))$. Since $U^{\#}$ is a normal connected subgroup of $U^{\#} \cdot T^{\#}$ our theorem follows.

Consider the exact sequence $1 \to N \to \Gamma_R \xrightarrow{\eta} H \to 1$ where N is a real nilpotent group, and where H is a discrete finitely generated abelian group. Let $H = H_1 \oplus H_2$ be as before a decomposition of H into a torsion free part H_1 with basis $h_1,\ldots,h_s$ and the torsion part H_2 with elements $h_{s+1},\ldots,h_n$. Choose $\gamma_1,\ldots,\gamma_n$ in Γ_R such that $\eta(\gamma_i) = h_i$. Write $\mathrm{ad}_N\gamma_i = u_i \cdot \zeta(\gamma_i)$ where u_i is in $U^{\#}$ and $\zeta(\gamma_i) \in T^{\#}$, where $\mathfrak{a}(\mathrm{ad}_N\Gamma_R) = U^{\#} \cdot T^{\#}$ and where $\zeta\colon \Gamma_R \to T^{\#}$ is the corresponding semi-simple homomorphism of Γ_R.

<u>Theorem</u>. (a) $U^{\#} = \mathrm{ad}_N N \cdot \mathfrak{a}(u_n) \cdots\cdot \mathfrak{a}(u_1)$

$T^{\#} = \mathfrak{a}(\zeta(\gamma_1),\ldots,\zeta(\gamma_n))$.

(b) $U' = \mathrm{ad}_N\, N \cdot \hat{u}_n \cdots\cdot \hat{u}_1$ is the group generated by the unipotent parts of $\mathrm{ad}_N\Gamma_R$.

$T' = \zeta(\hat{\gamma}_n) \cdots\cdot \zeta(\hat{\gamma}_1)$ is a group and contains modulo $\mathrm{ad}_N\, N$ every semi-simple part of elements from $\mathrm{ad}_N\,\Gamma_R$.

(c) $U' \cdot T'$ is a group and is the group generated by the semi-simple and unipotent parts of $\mathrm{ad}_N\,\Gamma_R$.

<u>Proof</u>: $\mathfrak{a}(\mathrm{ad}_N\,\Gamma_R)\,/\,\mathrm{ad}_N\, N$ is abelian, therefore since u_n is in $\mathfrak{a}(\mathrm{ad}_N\,\Gamma_R)$ we have $u_n\,\mathrm{ad}_N\, N\, u_n^{-1} = \mathrm{ad}_N\, N$. Therefore $\mathfrak{a}(u_n)$ normalizes $\mathrm{ad}_N\, N$ and $\mathrm{ad}_N\, N \cdot \hat{u}_n$ and $\mathrm{ad}_N\, N \cdot \mathfrak{a}(u_n)$ are groups, the latter an algebraic group.

Suppose $ad_N N \cdot \hat{u}_n \ldots \hat{u}_2$ and $ad_N N \cdot \mathfrak{G}(u_n) \ldots \mathfrak{G}(u_2)$ are groups, the latter an algebraic group. Since u_1 is in $\mathfrak{G}(ad_N \Gamma_R)$ we have that u_1 and therefore $\mathfrak{G}(u_1)$ normalizes both groups. Therefore $ad_N N \hat{u}_n \ldots \hat{u}_1$ and $ad_N N \cdot \mathfrak{G}(u_n) \ldots \mathfrak{G}(u_1)$ are groups, the latter an algebraic group. Let $\bar{U} = ad_N N \cdot \mathfrak{G}(u_n) \ldots \mathfrak{G}(u_1)$. Then $\bar{U}$ is an algebraic subgroup of $U^{\#}$ as well as a normal algebraic subgroup of $\mathfrak{G}(ad_N \Gamma_R)$.

There is a rational representation $r: \mathfrak{G}(ad_N \Gamma_R) \to GL(m)$ for some m whose kernel is $\bar{U}$. Let $Q = r(\mathfrak{G}(ad_N \Gamma_R))$, then both Q and $\mathfrak{G}(Q)$ are abelian. $\mathfrak{G}(ad_N \Gamma_R) = \mathfrak{G}(ad_N N, u_1, \ldots, u_n, \zeta(\gamma_1), \ldots, \zeta(\gamma_n))$, and $r^{-1}(\mathfrak{G}(r(\zeta(\gamma_1), \ldots \zeta(\gamma_n)))$ is an algebraic subgroup of $\mathfrak{G}(ad_N \Gamma_R)$ which contains $\bar{U}$ and $\zeta(\gamma_1), \ldots, \zeta(\gamma_n)$. It must be $\mathfrak{G}(ad_N \Gamma_R)$. Therefore $\mathfrak{G}(Q) = \mathfrak{G}(r(\zeta(\gamma_1)), \ldots, r(\zeta(\gamma_n)))$. Since each $r(\zeta(\gamma_i))$ $1 \le i \le n$ is semi-simple, and since $\mathfrak{G}(Q)$ is abelian we have that $\mathfrak{G}(Q)$ consists of all semi-simple automorphisms. Therefore $r(u) = 1$ for all u in $U^{\#}$ since $r(u)$ is both unipotent and semi-simple and $\bar{U} = U^{\#}$. Moreover $U^{\#} \cdot \mathfrak{G}(\zeta(\gamma_n), \ldots, \zeta(\gamma_1))$ is an algebraic subgroup of $U^{\#} T^{\#}$ containing $ad_N \Gamma_R$ and must therefore be $U^{\#} T^{\#}$. Therefore $T^{\#} = \mathfrak{G}(\zeta(\gamma_n), \ldots, \zeta(\gamma_1))$.

Let $U' = ad_N N \cdot \hat{u}_n \ldots \hat{u}_1$. Take γ in Γ_R and write $ad_N \gamma = ad_N x \cdot \Pi_1^n u_i^{e_i} t_i^{e_i}$ where $\gamma = x \cdot \Pi_1^n \gamma_i^{e_i}$ with $x \in ad_N N$ and e_i integer $1 \le i \le n$ and u_i and t_i the unipotent and semi-simple part of $ad_N \gamma$. Since $\mathfrak{G}(ad_N \Gamma_R) / ad_N N$ is abelian we can write $ad_N \gamma = \Pi_1^n u_i^{e_i} \cdot \Pi_1^n t_i^{e_i} \bmod (ad_N N)$, and therefore the unipotent part of $ad_N \gamma$ is equal to $\Pi_1^n u_i^{e_i} \bmod (ad_N N)$ and is in U' and the semi-simple part of $ad_N \gamma$ is equal to $\Pi_1^n t_i^{e_i} \bmod (ad_N N)$.

(b) Construction

We shall construct at this time a semi-simple splitting of the group Γ_R relative to the representation $\mathfrak{G}(ad_N \Gamma_R) = U^{\#} \cdot T^{\#}$.

Theorem 3.1. There are elements $\gamma_1, \ldots, \gamma_n$ in Γ_R with $\eta(\gamma_i) = h_i$ for $1 \leq i \leq n$ satisfying

(a) $(T^{\#}, ad_N \gamma_i) = 1_N$ for $1 \leq i \leq n$.

(b) $T^{\#}((\gamma_i, \gamma_j)) = (\gamma_i, \gamma_j)$ for $1 \leq i, j \leq n$.

Moreover corresponding to any set of elements $\gamma_1, \ldots, \gamma_n$ in Γ_R with $\eta(\gamma_i) = h_i$ satisfying (a) and (b) there exists a unique extension of $T^{\#}$ to a group of automorphisms of Γ_R, $T^{\#} = T^{\#}_{(\gamma_i)}$ such that $T^{\#}_{(\gamma_i)}(\gamma_j) = \gamma_j$ for $1 \leq j \leq n$.

Proof: Let $\gamma_1, \ldots, \gamma_n$ be arbitrary elements of Γ_R such that $\eta(\gamma_i) = h_i$. Since $T^{\#}$ acts completely reducibly and since the adjoint map is rational we have that $ad_{U^{\#}} T^{\#}$ is completely reducible. Since $\mathfrak{G}(ad_N \Gamma_R)/ ad_N N$ is abelian we have that $ad_{U^{\#}/ ad_N N} T^{\#} = 1_{U^{\#}/ ad_N N}$. Let $L(U^{\#}) = W \oplus L(ad_N N)$ as real vector spaces where we have $ad_{L(U^{\#})} T^{\#}(W) = W$. For x in $U^{\#}$ and t in $T^{\#}$ we have $ad\, t\, (\log x) = t \log x\, t^{-1}$. Since $\exp(t \log x\, t^{-1}) = t\, x\, t^{-1} = xy$ with y in $ad_N N$ and since $xy = \exp(\log x + \bar{z})$ where $\bar{z}$ is in $L(ad_N N)$ by the Hausdorf-Campbell formula, we have that $ad\, t\, (\log x) = \log x + \bar{\bar{z}}$ with $\bar{\bar{z}}$ in $L(ad_N N)$. Therefore $ad_W T^{\#} = 1_W$. Let $ad_N \gamma_i = u_i \cdot \zeta(\gamma_i)$ with u_i in $U^{\#}$ for $1 \leq i \leq n$. Let $\log u_i = \log v_i + \bar{y}_i$ where $\log v_i$ is in W and $\bar{y}_i$ is in $L(ad_N N)$. Then $(T^{\#}, v_i) = 1_N$. Moreover $u_i = x_i v_i$ with $x_i = ad_N n_i$, n_i in N again by the Hausdorff-Campbell formula.

Let $\gamma_i' = n_i^{-1}\gamma_i$. We have that γ_i' is in Γ_R, $\eta(\gamma_i') = \eta(\gamma_i) = h_i$ and $ad_N \gamma_i' = v_i \zeta(\gamma_i)$, and since v_i is in $\exp W$, $(T^{\#}, v_i) = 1_N$. Let us fix an integer k, $1 \le k \le n$. We shall assume that we can find elements $\gamma_1,\ldots,\gamma_n \in \Gamma_R$ such that $(ad_N \gamma_i, \zeta(\gamma_j)) = 1_N$ for $1 \le i, j \le n$ and such that $\zeta(\gamma_i)((\gamma_j,\gamma_\ell)) = (\gamma_j,\gamma_\ell)$ for $0 \le i \le k-1$ and $1 \le j, k \le n$. Let $t_i = \zeta(\gamma_i)$. Let $N_1 = \cap_0^{k-1} \ker_C(t_j - I)$. Then N_1 is a connected subgroup of C which is kept invariant by $ad_N \gamma_i$ for $1 \le i \le n$ since $(ad_N \gamma_i, t_j) = 1_N$. Let us write $t_k((\gamma_k,\gamma_i)) = (\gamma_k,\gamma_i) z_i$ with z_i in N. Since $(ad_N(\gamma_k,\gamma_i), t_k) = 1_N$ we have that z_i is in C for $1 \le i \le n$. Since $(t_i,t_k) = 1_N$ and since $t_j((\gamma_k,\gamma_i)) = (\gamma_k,\gamma_i)$ for $0 \le j \le k-1$ and $1 \le i \le n$ we have that $t_j(t_k((\gamma_k,\gamma_i))) = (\gamma_k,\gamma_i)\, t_j(z_i) = (\gamma_k,\gamma_i)\, z_i$. Therefore $t_j(z_i) = z_i$ for $0 \le j \le k-1$ and $1 \le i \le n$. Therefore z_i and $\gamma_i^{-1} z_i \gamma_i$ are contained in N_1. Let $z_i' = \gamma_i^{-1} z_i \gamma_i$. Let $\bar{N}_1 = \ker_{N_1}(t_k - I)$ and $\bar{\bar{N}}_1 = (t_k - I) N_1$. Since t_k is semi-simple and since $(ad_N \gamma_k, t_k) = 1_N$ we have that $N_1 = \bar{N}_1 \oplus \bar{\bar{N}}_1$ and that $\bar{\bar{N}}_1 = (t_k - I)(ad\,\gamma_k - I)\,\bar{\bar{N}}_1$. Write $z_i' = \bar{z}_i' + \bar{\bar{z}}_i'$ where $\bar{z}_i' \in \bar{N}_1$ and $\bar{\bar{z}}_i' \in \bar{\bar{N}}_1$. Then $-\bar{\bar{z}}_i' = (t_k - I)(ad\,\gamma_k - I)\, y_i$ where y_i is in $\bar{\bar{N}}_1$. Let $\bar{\gamma}_i = \gamma_i y_i$. Since y_i is in C we have $ad_N \gamma_i = ad_N \bar{\gamma}_i$ and therefore we have $(ad_N \bar{\gamma}_i, t_j) = 1_N$ for $1 \le i$, $j \le n$. Also $(\bar{\gamma}_i,\bar{\gamma}_j) = (\gamma_i,\gamma_j)\, ad\,\gamma_j((ad\,\gamma_i(ad\,\gamma_j^{-1}(y_i)y_j\, y_i^{-1})\, y_j^{-1})_1$, and since for $0 \le \ell \le k-1$, $t_\ell((\gamma_i,\gamma_j) = \gamma_i,\gamma_j)$ and since $(t_\ell, ad_N \gamma_i) = 1_N$ we have since $t_\ell(y_i) = y_i$ $1 \le i \le n$ that $t_\ell(\bar{\gamma}_i,\bar{\gamma}_j) = (\bar{\gamma}_i,\bar{\gamma}_j)$ for $0 \le \ell \le k-1$ and $1 \le i, j \le n$. Now $(\bar{\gamma}_k,\bar{\gamma}_i) = (\gamma_k,\bar{\gamma}_i) = (\gamma_k,\gamma_i)\, ad\,\gamma_i(ad\,\gamma_k(y_i)y_i^{-1})$. Therefore

$t_k(\bar{\gamma}_k,\bar{\gamma}_i) = (\gamma_k,\gamma_i)\ z_i$ ad $\gamma_i(t_k(\text{ad}\ \gamma_k(y_i)\ y_i^{-1}))$ since $t_k(\gamma_k,\gamma_i) = (\gamma_k,\gamma_i)\ z_i$ and $(t_k,\text{ad}_N\ \gamma_i) = 1_N$. But then $t_k(\bar{\gamma}_k,\bar{\gamma}_i) = (\gamma_k,\gamma_i)$ ad $\gamma_i(\gamma_i^{-1}\ z_i\ \gamma_i\ t_k(\text{ad}\ \gamma_k(y_i)\ y_i^{-1})$. Consider $\gamma_i^{-1}\ z_i\ \gamma_i\ t_k(\text{ad}\ \gamma_k(y_i)\ y_i^{-1}) = \text{ad}\ \gamma_i(z_i) + t_k(\text{ad}\ \gamma_k(y_i) - y_i) = \bar{z}_i' + \bar{\bar{z}}_i' + t_k(\text{ad}\ \gamma_k - I)\ y_i = \bar{z}_i' + (\text{ad}\ \gamma_k - I)\ y_i$. Therefore $t_k(\bar{\gamma}_k,\bar{\gamma}_i) = (\gamma_k,\gamma_i) \cdot \gamma_i\ \bar{z}_i'\ \gamma_i^{-1} \cdot \gamma_i\ \gamma_k\ y_i\ \gamma_k^{-1}\ y_i^{-1}\ \gamma_i^{-1} = (\gamma_k,\gamma_i)$ ad $\gamma_i(\gamma_k\ y_i\ \gamma_k^{-1}\ y_i^{-1}) \cdot \gamma_i\ \bar{z}_i'\ \gamma_i^{-1} = (\gamma_k,\gamma_i)$ ad $\gamma_i(\gamma_k\ y_i\ \gamma_k^{-1}\ y_i^{-1}) \cdot \gamma_i\ \bar{z}_i'\ \gamma_i^{-1} = (\bar{\gamma}_k,\bar{\gamma}_i)\gamma_i\ \bar{z}_i'\ \gamma_i^{-1}$ But $\gamma_i\ \bar{z}_i'\ \gamma_i^{-1}$ is in $\bar{N}_1$ therefore $(t_k - I)^2(\bar{\gamma}_k,\bar{\gamma}_i) = 0$ and since t_k is semi-simple we have that $t_k(\bar{\gamma}_k,\bar{\gamma}_i) = (\bar{\gamma}_k,\bar{\gamma}_i)$ for $1 \le i \le n$. Now we have ad $\gamma_k(\bar{\gamma}_j,\bar{\gamma}_i)$ ad $\gamma_k(\bar{\gamma}_i\ \bar{\gamma}_j)\ \bar{\gamma}_j^{-1}\ \bar{\gamma}_i^{-1}\ (\bar{\gamma}_i,\bar{\gamma}_j) = \text{ad}\ \gamma_k(\bar{\gamma}_j\ \bar{\gamma}_i)\bar{\gamma}_i^{-1}\ \bar{\gamma}_j^{-1}$ and ad $\gamma_k(\bar{\gamma}_i\ \bar{\gamma}_j)\ \bar{\gamma}_j\ \bar{\gamma}_i^{-1} = (\gamma_k,\bar{\gamma}_i)$ ad $\bar{\gamma}_i(\gamma_k,\bar{\gamma}_j)$. Both are invariant by t_k . Let $t_k((\bar{\gamma}_i,\bar{\gamma}_j)) = (\bar{\gamma}_i,\bar{\gamma}_j)c$ where c is in C . Then ad $\gamma_k(\bar{\gamma}_j,\bar{\gamma}_i) \cdot$ ad $\gamma_k(\bar{\gamma}_i\ \bar{\gamma}_j)\ \bar{\gamma}_j^{-1}\ \bar{\gamma}_i^{-1}\ (\bar{\gamma}_i,\bar{\gamma}_j) = t_k(\text{ad}\ \gamma_k(\bar{\gamma}_j,\bar{\gamma}_i) \cdot$ ad $\gamma_k(\bar{\gamma}_i\ \bar{\gamma}_j)\ \bar{\gamma}_j^{-1}\ \bar{\gamma}_i^{-1}\ (\bar{\gamma}_i,\bar{\gamma}_j)) = \text{ad}\ \gamma_k(t_k((\bar{\gamma}_j,\bar{\gamma}_i)))$ ad $\gamma_k(\bar{\gamma}_i\ \bar{\gamma}_j)\bar{\gamma}_j^{-1}\ \bar{\gamma}_i^{-1}$ $t_k((\bar{\gamma}_i,\bar{\gamma}_j)) = \text{ad}\ \gamma_k((\bar{\gamma}_j,\bar{\gamma}_i))$ ad $\gamma_k(c)$ ad $\gamma_k(\bar{\gamma}_i\ \bar{\gamma}_j)\ \bar{\gamma}_j^{-1}\ \bar{\gamma}_i^{-1}(\bar{\gamma}_i,\bar{\gamma}_j) \cdot c^{-1} =$ ad $\gamma_k((\bar{\gamma}_j,\bar{\gamma}_i))$ ad $\gamma_k(\bar{\gamma}_i\ \bar{\gamma}_j)\ \bar{\gamma}_j^{-1}\ \bar{\gamma}_i^{-1}\ (\bar{\gamma}_i,\bar{\gamma}_j)$ ad $\gamma_k(c)\ c^{-1}$ since c is central. Therefore ad $\gamma_k(c) = c$, and $t_k(c) = c$. We then have $(t_k - I)^2(\bar{\gamma}_i,\bar{\gamma}_j) = 0$ and since t_k is semi-simple we have $t_k(\bar{\gamma}_i,\bar{\gamma}_j) = (\bar{\gamma}_i,\bar{\gamma}_j)$ for $1 \le i \le n$ and $1 \le j \le n$. Each element γ of Γ_R can be uniquely written $\gamma = x\ \Pi_1^n\ \gamma_i^{e_i}$. Let t be in $T^{\#}$. Let $t_\Gamma(\gamma) = t(x)\ \Pi_1^n\ \gamma_i^{e_i}$, where $\gamma = x\ \Pi_1^n\ \gamma_i^{e_i}$. Let $\gamma' = x'\ \Pi_1^n\ \gamma_i^{e_i'}$ be another element of Γ_R. Then $t_\Gamma(\gamma') = t(x')\ \Pi_1^n\ \gamma_i^{e_i'}$, and

$$\gamma\,\gamma' = x\,\Pi_1^n \gamma_i^{e_i}\, x'\, \Pi_n^1 \gamma_i^{-e_i}\, \Pi_1^n \gamma_i^{e_i}\, \Pi_1^n \gamma_i^{e_i'} = x\,\Pi_1^n \gamma_i^{e_i}\, x' \Pi_n^1 \gamma_i^{-e_i} \cdot z \cdot \Pi_1^n \gamma_i^{(e_i + e_i')}$$

where z is a product of commutators of γ_i, γ_j. Then $t_\Gamma(\gamma\,\gamma') =$ $t(x)\, \Pi_1^n \gamma_i^{e_i}\, t(x')\, \Pi_n^1 \gamma_i^{-e_i} \cdot z \cdot \Pi_1^n \gamma_i^{(e_i+e_i')}$ since $(T^{\#}, \mathrm{ad}_N\, \gamma_i) = 1_N$ and $T^{\#}(z) = z$. But this is $t_\Gamma(\gamma) \cdot t_\Gamma(\gamma')$.

(c) Uniqueness

We can now form the group $\Gamma_R \cdot T^{\#}$ where we consider $T^{\#}$ as a group of automorphisms of Γ_R. Let $T = \zeta(\Gamma_R)$ and form the group $\Gamma_R \cdot T$ where T is considered as a group of automorphisms of Γ_R. We note that there are two variables in this construction, the representation of $\mathfrak{a}(\mathrm{ad}_N\, \Gamma_R) = U^{\#} \cdot T^{\#}$ and the extension of $T^{\#}$ to Γ_R. We make now some general definitions.

Let Γ_R be a group satisfying the exact sequence $1 \to N \to \Gamma_R \xrightarrow{\eta} H \to 1$ where N is a real nilpotent group and H is a finitely generated discrete abelian group. Let $\mathfrak{a}(\mathrm{ad}_N\, \Gamma_R) = U^{\#} \cdot T^{\#}$ be a Mostow decomposition of $\mathfrak{a}(\mathrm{ad}_N\, \Gamma_R)$ and let $\zeta\colon \Gamma_R \to T^{\#}$ be the corresponding semi-simple homomorphism. Let T be an abelian group of automorphisms of Γ_R. We say that Γ_R is a semi-simple splitting of Γ_R relative to the representation $\mathfrak{a}(\mathrm{ad}_N\, \Gamma_R) = U^{\#} \cdot T^{\#}$ if and only if $T|N = \zeta(\Gamma_R)$ and there exist elements $\gamma_1, \ldots, \gamma_n$ in Γ_R with $\eta(\gamma_i) = h_i$ satisfying $T(\gamma_i) = \gamma_i$ for $1 \le i \le n$. The previous theorem gives us the existence of a semi-simple splitting relative to any representation of $\mathfrak{a}(\mathrm{ad}_N\, \Gamma_R)$. Let $\Gamma_R \cdot T$ and $\Gamma_R \cdot \bar{T}$ be two semi-simple splittings of Γ_R relative to the same representation $\mathfrak{a}(\mathrm{ad}_N\, \Gamma_R) = U^{\#} \cdot T^{\#}$. Let $\zeta\colon \Gamma_R \to T^{\#}$ be the corresponding semi-simple homomorphism. Let $\gamma_1,\dots,\gamma_n$ $(\bar{\gamma}_1,\dots,\bar{\gamma}_n)$ be chosen in Γ_R such that $\eta(\gamma_i) = h_i$ $(\eta(\bar{\gamma}_i) = h_i)$ and $T(\gamma_i) = \gamma_i$ $(\bar{T}(\bar{\gamma}_i) = \bar{\gamma}_i)$.

Choose t_i $(\bar{t}_i)$ in $T(\bar{T})$ such that $t_i|N = \zeta(\gamma_i) = \zeta(\bar{\gamma}_i) = \bar{t}_i|N$. We then have the following theorem.

<u>Theorem 3.2</u>. A. There exists an h in the center of N such that

$$\mathrm{ad}_{\Gamma_R} h \circ T \circ \mathrm{ad}_{\Gamma_R} h^{-1} = \bar{T} \text{ , and in particular}$$

$$\mathrm{ad}_{\Gamma_R} h \circ t_i \circ \mathrm{ad}_{\Gamma_R} h^{-1} = \bar{t}_i \quad \text{for } 1 \le i \le n \text{ .}$$

B. $$: \Gamma_R \cdot T \longrightarrow \Gamma_R \cdot \bar{T}; \quad (\gamma, t) = (h\,\gamma\, h^{-1}, \mathrm{ad}_{\Gamma_R} h \circ t \circ \mathrm{ad}_{\Gamma_R} h^{-1})$$

establishes an isomorphism of the two semi-simple splittings of Γ_R .

<u>Note</u>: We have not proven that any two arbitrary semi-simple splittings of Γ_R are isomorphic, as yet.

We shall first show that there is an h in C the center of N such that

$$\mathrm{ad}_{\Gamma_R}(h\, t_i\, h^{-1}) = \bar{t}_i \quad .$$

<u>Proof</u>: Fix k, $1 \le k \le n$, and assume there is an h in C where C is the center of N such that $\mathrm{ad}\ h \circ t_i \circ \mathrm{ad}\ h^{-1} = \bar{t}_i$ on Γ_R for $0 \le i \le k-1$. Let $t_i' = \mathrm{ad}\ h \circ t_i \circ \mathrm{ad}\ h^{-1}$ and let $\gamma_i' = h\,\gamma_i\,h^{-1}$ for $1 \le i \le n$. It follows that $t_i'(\gamma_j') = \gamma_j'$ for $1 \le i, j \le n$ and that $t_i' = \bar{t}_i$ on Γ_R for $0 \le i \le k-1$, Let $\gamma_i' = \ell_i\,\bar{\gamma}_i$ where ℓ_i is in N . $t_j'(\gamma_i') = \gamma_i' = t_j'(\ell_i)\ t_j'(\bar{\gamma}_i)$. For $0 \le j \le k-1$ since $t_j' = \bar{t}_j$ on Γ_R we have $t_j'(\gamma_i') = \gamma_i' = \ell_i\,\bar{\gamma}_i = t_j'(\ell_i)\bar{t}_j(\bar{\gamma}_i) = t_j'(\ell_i)\ \bar{\gamma}_i$. Therefore $t_j'(\ell_i) = \ell_i$ for $0 \le j \le k-1$ and $1 \le i \le n$.

Therefore $\ell_i \in \cap_0^{k-1} \ker_N (t_j - I)$ for $1 \le i \le n$. $\bar{t}_k(\gamma_i') = \bar{t}_k(\ell_i)\bar{\gamma}_i = \bar{t}_k(\ell_i)\ell_i^{-1} \gamma_i'$, therefore $\bar{t}_k(\gamma_i')\gamma_i'^{-1} = \bar{t}_k(\ell_i)\ell_i^{-1}$. Since $t_k' = \bar{t}_k$ on N and since $(\mathrm{ad}_N \gamma_i', t_k') = 1_N$ we have that $(\mathrm{ad}_N \gamma_i', \bar{t}_k) = 1_N$. Therefore for $n \in N$ we have $\bar{t}_k(\gamma_i' n \gamma_i'^{-1}) = \gamma_i' \bar{t}_k(n) \gamma_i'^{-1} = \bar{t}_k(\gamma_i')\bar{t}_k(n)\bar{t}_k(\gamma_i'^{-1})$ and we can conclude that $\bar{t}_k(\gamma_i')\gamma_i'^{-1} = \bar{t}_k(\ell_i)\ell_i^{-1}$ is contained in C. But since $\ell_i \in \cap_0^{k-1} \ker_N (t_j - I)$ and since $\bar{t}_k$ commutes with t_j on N for all $j = 1,\dots, k-1$ we have that $\bar{t}_k(\ell_i)\ell_i^{-1} \in \cap_0^{k-1} \ker_C(t_j - I)$. But $\cap_0^{k-1} \ker_C(t_j - I)$ is a connected subgroup of C, which is invariant under $\bar{t}_k$, therefore there is a c_i in $\cap_0^{k-1} \ker_C(t_j - I)$ such that $\bar{t}_k(\ell_i)\ell_i^{-1} = \bar{t}_k(c_i)c_i^{-1}$. Let $\gamma_i'' = c_i^{-1} \gamma_i'$. For $0 \le i \le k-1$, $\bar{t}_i(c_j^{-1}\gamma_j') = t_i'(c_j^{-1}\gamma_j') = t_i'(c_j^{-1})t_i'(\gamma_j') = c_j^{-1}\gamma_j'$. Moreover $\bar{t}_k(c_j^{-1}\gamma_j') = \bar{t}_k(c_j^{-1})\bar{t}_k(\gamma_j') = \bar{t}_k(c_j^{-1})\bar{t}_k(c_j)c_j^{-1}\gamma_j' = c_j^{-1}\gamma_j'$. Therefore $\bar{t}_i(\gamma_j'') = \gamma_j''$ for $1 \le i \le k$. Let $C_{k-1} = \cap_0^{k-1} \ker_C(t_j - I)$. Then $c_j^{-1} \in C_{k-1}$ for $1 \le j \le n$. Let $\bar{C}_{k-1} = \cap_0^{k} \ker_C(t_j - I)$ and let $\bar{\bar{C}}_{k-1} = (t_k - I)\, C_{k-1}$. Since $t_k|C_{k-1}$ is semi-simple and since $(\mathrm{ad}_N \gamma_k', t_k) = 1_N$ we have $C_{k-1} = \bar{C}_{k-1} \oplus \bar{\bar{C}}_{k-1}$ and $\bar{\bar{C}}_{k-1} = (I - \mathrm{ad}\, \gamma_k')\, \bar{\bar{C}}_{k-1} = (\mathrm{ad}\, \gamma_k' - I)\, \bar{\bar{C}}_{k-1}$. Since $c_k^{-1} \in C_{k-1}$ we can write $c_k^{-1} = \bar{d}_k \bar{\bar{d}}_k$ where $\bar{d}_k$ is in $\bar{C}_{k-1}$ and $\bar{\bar{d}}_k$ is in $\bar{\bar{C}}_{k-1}$. But then there is an h' in $\bar{\bar{C}}_{k-1}$ such that $\bar{\bar{d}}_k = \gamma_k' h'^{-1} \gamma_k'^{-1} h'$ and since h' is in the center $\bar{\bar{d}}_k = h' \gamma_k' h'^{-1} \gamma_k'^{-1}$. Let $t_i'' = \mathrm{ad}\, h'\, t_i'\, \mathrm{ad}\, h'^{-1}$, But then $t_i''(\gamma_j'') = \mathrm{ad}\, h' t_i'\, \mathrm{ad}\, h'^{-1}(\gamma_j'') = \gamma_j''$ for $0 \le i \le k-1$ and $1 \le j \le n$ since $t_i'(h') = h'$ for

$0 \le i \le k\text{-}1$, and since $t_i'(\gamma_j'') = \gamma_j''$ for $0 \le i \le k\text{-}1$ and $1 \le j \le n$. Also $t_k''(c_k^{-1}\gamma_k') = \text{ad } h'\ t_k'\ \text{ad } h'^{-1}(\bar{d}_k\ \bar{\bar{d}}_k\ \gamma_k') =$

$$= \text{ad } h'\ t_k'(h'^{-1}\ \bar{d}_k\ h'\ \gamma_k'\ h'^{-1}\ \gamma_k'^{-1}\ \gamma_k'\ h')$$

$$= \text{ad } h'\ t_k'(\bar{d}_k\ \gamma_k') = \text{ad } h'(t_k'(\bar{d}_k)t_k'(\gamma_k')) .$$

Since $\bar{d}_k \in \bar{C}_{k-1} = \cap_0^k \ker_C(t_j - I)$ we have

$t_k''(c_k^{-1}\gamma_k') = \text{ad } h'(\bar{d}_k\ \gamma_k') = \bar{d}_k\ h'\ \gamma_k'\ h'^{-1} = \bar{d}_k\ \bar{\bar{d}}_k\ \gamma_k'$.

Therefore $t_k''(c_k^{-1}\gamma_k') = c_k^{-1}\gamma_k'$. It follows from $t_i'(h') = h'$, that $t_i'' = \bar{t}_i$ on Γ_R for $0 \le i \le k\text{-}1$ and that $\bar{t}_k(c_k^{-1}\gamma_k') = c_k^{-1}\gamma_k' = t_k''(c_k^{-1}\gamma_k')$. Therefore we have $t_k'' = \bar{t}_k$ on $N \cdot \hat{\gamma}_k'$. Since $\gamma_i''\ \gamma_k''\ \gamma_i''^{-1} \in N \cdot \hat{\gamma}_k'$ we have $\bar{t}_k(\gamma_i''\ \gamma_k''\ \gamma_i''^{-1}) = \gamma_i''\ \gamma_k''\ \gamma_i''^{-1} = t_k''(\gamma_i''\ \gamma_k''\ \gamma_i''^{-1}) = t_k''(\gamma_i'')\ \gamma_k''\ t_k''(\gamma_i''^{-1})$. Therefore we have that $\text{ad } \gamma_k''(t_k''(\gamma_i'')\ \gamma_i''^{-1}) = t_k''(\gamma_i'')\ \gamma_i''^{-1}$. Since $\text{ad}_N\ \gamma_i'' = \text{ad}_N\ \gamma_i'$ we have that t_k'' is the semi-simple part of $\text{ad}_N\ \gamma_k''$. Therefore $t_k''(\gamma_i'')\ \gamma_i''^{-1} \in \ker_N(t_k'' - I)$.

Let $\gamma_i''' = h'\ \gamma_i'\ h'^{-1}$ for $1 \le i \le n$. We have $t_k''(\gamma_i''') = \gamma_i'''$. Let $\gamma_i'' = v_i\ \gamma_i'''$ with v_i in N . Then $\gamma_i'' = c_i^{-1}\ \gamma_i' = v_i\ h'\ \gamma_i'\ h'^{-1}$. Therefore $v_i = c_i^{-1}\ \gamma_i'\ h'^{-1}\ \gamma_i'^{-1}\ h'$ which is contained in C . Since $t_k'(\gamma_i'') = t_k''(v_i)\gamma_i''' = t_k''(v_i)v_i^{-1}\ \gamma_i''$, we have $t_k''(\gamma_i'')\gamma_i''^{-1} = t_k''(v_i)v_i^{-1}$. Since v_i is in C we have that $(t_k'' - I)v_i$ is in $\ker_C(t_k'' - I)$. But then $(t_k'' - I)^2 v_i = 0$ and since t_k'' is semi-simple, $t_k''(v_i) = v_i$, $t_k''(\gamma_i'') = \gamma_i''$ for $1 \le i \le n$. Since $t_k'' = \bar{t}_k$ on N we have since

$t_k''(\gamma_i'') = \bar{t}_k(\gamma_i'') = \gamma_i''$ for $1 \leq i \leq n$ that $t_k'' = \bar{t}_k$ on Γ_R. Therefore we have found an $\bar{h}$ in C such that $ad_{\Gamma_R} \bar{h} \circ t_i \ ad_{\Gamma_R} \bar{h}^{-1} = \bar{t}_i$ for $1 \leq i \leq k$ by letting $\bar{h} = h\, h'$.

Suppose now we have two representations $\mathcal{G}(ad_N \Gamma_R) = U^{\#} \cdot T_1^{\#}, = U^{\#} \cdot T_2^{\#}$. Let $\zeta_1 : \Gamma_R \longrightarrow T_1^{\#}$, $\zeta_2 : \Gamma_R \longrightarrow T_2^{\#}$ be the corresponding semi-simple homomorphisms of Γ_R. Let $\Gamma_R \cdot T_i$ $i = 1,2$ be a semi-simple splitting of Γ_R relative to the representation $\mathcal{G}(ad_N \Gamma_R) = U^{\#} \cdot T_i^{\#}$.

<u>Theorem 3.3</u>. A. There is an h in N such that $ad_{\Gamma_R} h \circ T_1 \ ad_{\Gamma_R} h^{-1} = T_2$.

B. $\varphi : \Gamma_R \cdot T_1 \longrightarrow \Gamma_R \cdot T_2$, $\varphi(\gamma, t) = (h\,\gamma\, h^{-1}, ad\ h\ t\ ad\ h^{-1})$ is an isomorphism of $\Gamma_R \cdot T_1$ onto $\Gamma_R \cdot T_2$.

<u>Proof</u>: All we need prove is that there is an h in N such that $ad_N h \circ T_1^{\#} \circ ad_N h^{-1} = T_2^{\#}$. For we then have from the commutativity of the diagram

$$\begin{array}{ccc} \Gamma_R & \xrightarrow{\ \zeta_1\ } & T_1^{\#} \\ \Big\downarrow {\scriptstyle ad_{\Gamma_R} h} & & \Big\downarrow {\scriptstyle \psi} \\ \Gamma_R & \xrightarrow[\ \zeta_2\ ]{} & T_2^{\#} \end{array}$$

where $\psi(t) = ad_N h \circ t \circ ad_N h^{-1}$ that $T_2 | N = \zeta_2(\Gamma_R) = ad_N h \circ \zeta_1(\Gamma_R)$ $ad_N h^{-1} = ad_N h \circ T_1 | N \circ ad_N h^{-1}$ and therefore $\Gamma_R \cdot T_1'$ with $T_1' =$ $ad_{\Gamma_R} h \circ T_1 \circ ad_{\Gamma_R} h^{-1}$ is a semi-simple splitting of Γ_R relative to $\mathcal{G}(ad_N \Gamma_R) = U^{\#} \cdot T_2^{\#}$. But by Theorem 3.2 there is an h' in the center of N for which $ad_{\Gamma_R} h' \circ T_1' \circ ad_{\Gamma_R} h'^{-1} = T_2$.

Therefore $ad_{\Gamma_R} \bar{h} \circ T_1 \circ ad_{\Gamma_R} \bar{h}^{-1} = T_2$. By 2. there is a u in $U^{\#}$ such that $u^{-1} T_1^{\#} u^{+1} = T_2^{\#}$. Since $ad_{U^{\#}} T_1^{\#}$ is completely reducible we can write $U^{\#} = W \oplus ad_N N$ as vector spaces where $ad\, T_1^{\#}(W) = W$. But since $(T_1^{\#}, U^{\#}) \subseteq ad_N N$ as before we have that $ad_W T_1^{\#} = 1_W$. Write $u = w \cdot ad_N h$ where w is in W and h is in N. Then $T_1^{\#}(w) = w$. We claim $ad_N h^{-1} \circ T_1^{\#} \circ ad_N h = T_2^{\#}$. Take t in $T_1^{\#}$ then $u t u^{-1}$ is in $T_2^{\#}$. But $u^{-1} t u^{+1} = ad_N h^{-1} \cdot w^{-1} t w\, ad_N h = ad_N h^{-1} \circ t \circ ad_N h$. Therefore there is an h in N such that $ad_N h \circ T_1^{\#} \circ ad_N h^{-1} = T_2^{\#}$.

(d) Properties

Let us assume that $\Gamma_R \cdot T$ is a semi-simple splitting of Γ_R. We shall give T the discrete topology. Choose $\gamma_1, \ldots, \gamma_n$ in Γ_R such that $\eta(\gamma_i) = h_i$, $T(\gamma_i) = \gamma_i$ and $T|N = \hat{\zeta}(\gamma_1) \ldots \hat{\zeta}(\gamma_n)$. Choose t_i in T such that $T|N = \hat{\zeta}(\gamma_1) \ldots \hat{\zeta}(\gamma_n) = \hat{t}_1 \ldots \hat{t}_n|N$ and $t_i|N = \zeta(\gamma_i)$.

Fix a j $s+1 \leq j \leq n$. Then h_j is in H_2 which is the finite torsion subgroup of H. Let e be equal to the number of elements on H_2. Then $h_j^e = 1$. Therefore $\gamma_j^e \in N$ and since N is divisible there is an $n \in N$ such that $n_j^e = \gamma_j^e$, and $(n_j, \gamma_j) = 1$, and $T(n_j) = n_j$. This may be done as follows. Let $N_j = \{x \in N;\ ad_N \gamma_j(x) = x\}$ let $N_2 = \cap_1^n \ker_N(t_i - I)$ and let $\bar{N}_j = N_j \cap N_2$ which is a connected and algebraic subgroup of N. We have clearly γ_j^e in $\bar{N}_j$. Therefore since $\bar{N}_j$ is divisible there is a unique n_j in $\bar{N}_j$ such that $n_j^e = \gamma_j^e$. Do this for j, $s+1 \leq j \leq n$.

Let $\bar{\gamma}_i = \gamma_i$ $1 \le i \le s$ and let $\bar{\gamma}_i = \gamma_i n_i^{-1}$ for $s + 1 \le i \le n$. Clearly $\eta(\bar{\gamma}_i) = \eta(\gamma_i) = h_i$, $T(\bar{\gamma}_i) = T(\gamma_i) \cdot T(n_i^{-1}) = \gamma_i n_i^{-1} = \bar{\gamma}_i$, and $\zeta(\gamma_i) = \zeta(\bar{\gamma}_i)$ from which it follows that $T|N = \zeta(\hat{\bar{\gamma}}_1)\ldots\zeta(\hat{\bar{\gamma}}_n)$. Since $(n_j, \gamma_j) = 1$ we have $\bar{\gamma}_j{}^e = 1$ for $s + 1 \le j \le n$.

<u>Theorem 3.4</u>. Let $\Gamma_R \cdot T$ be a semi-simple splitting of Γ_R. Give T the discrete topology. Choose $\gamma_1, \ldots, \gamma_n$ on Γ_R such that $\eta(\gamma_i) = h_i$, $T(\gamma_i) = \gamma_i$ and $T|N = \zeta(\hat{\gamma}_1) \ldots \zeta(\hat{\gamma}_n)$. Moreover we can assume that γ_j has finite order for $s + 1 \le j \le n$. Let t_i in T be chosen so that $t_i|N = \zeta(\gamma_i)$. Let $n_i = \gamma_i t_i^{-1}$. Then

(a) $F = N \cdot \hat{n}_1 \ldots \hat{n}_s$ is a closed normal nilpotent subgroup of $\Gamma_R \cdot T$ such that F/N is a discrete torsion free finitely generated abelian group.

(b) $K = \hat{n}_{s+1} \ldots \hat{n}_n$ is a finite central subgroup of $\Gamma_R \cdot T$, and the group $K \oplus T$ is a discrete finitely generated abelian group.

(c) $\Gamma_R \cdot T = F \cdot (K \oplus T)$ semi direct product.

<u>Proof</u>: Let $v_j = \gamma_j t_j^{-1}$. For $j \ge s + 1$ we have that γ_j has finite order, and that $t_k(\gamma_j) = \gamma_j$ for $1 \le k \le n$. Therefore v_j has finite order and since $ad_N v_j$ is unipotent we have $ad_N v_j = 1_N$. Write for x in $\Gamma_R \cdot T$, $ad\ v_j(x) = x \cdot n$ for some n in N. For some integer $f > 0$ we have $v_j^f = 1$ and therefore $(ad\ v_j)^f x = x = x \cdot n^f$ since $ad_N v_j = 1N$. From the divisibility of N and $n^f = 1$ we have $n = 1$.

Therefore $K = \hat{v}_{s+1} \ldots \hat{v}_r$ is a finite central subgroup of $\Gamma_R \cdot T$. Let $F = N \cdot \hat{v}_1 \ldots \hat{v}_s$. Since Γ_R / N is discrete and T is discrete we have $\Gamma_R \cdot T / N$ discrete therefore since $N \subset F$ we have that F is a closed subgroup of $\Gamma_R \cdot T$. Moreover $\Gamma_R \cdot T / N$ is abelian therefore F is a normal subgroup of $\Gamma_R \cdot T$ and F/N is abelian. Since $ad_N F$ consists of unipotent automorphisms and F/N is abelian we have that F is nilpotent. Clearly F/N is finitely generated. If v_j^{ℓ} were in N for some integer ℓ then γ_j^{ℓ} would also be in N which would contradict the torsion freeness of h_j for $1 \leq j \leq s$. Therefore F/N is torsion free. Form $K \oplus T$ and $F \cdot (K \oplus T)$. Let x be in $F \cap (K \oplus T)$. Write $x = k \cdot t$ for k in K and t in T. Then $ad_N x = ad_N t$ which is both semi-simple and unipotent and therefore $ad_N t = 1_N$. But then $t = 1$ and $x = 1$ since $F \cap K = (1)$. Therefore this is a semi-direct product and clearly $\Gamma_R \cdot T = F \cdot (K \oplus T)$.

<u>Corollary 3.3.1</u>. Let $H = Z^s$.

Then $F = \{x \in \Gamma_R \cdot T\colon ad_N\, x \text{ is unipotent}\}$

<u>Corollary 3.3.2</u>. Let $H = Z^s$. Let $\Gamma_R \cdot T_1$ and $\Gamma_R \cdot T_2$ be two semi-simple splitting of Γ_R. Let h in the center of N be chosen so that $ad\, h\, T_1\, ad\, h^{-1} = T_2$. Then

(a) the map $\Gamma_R T_1 \xrightarrow{\varphi} \Gamma_R T_2$ $\varphi(\gamma, t) = (h \gamma h^{-1}, ad\, h\, t\, ad\, h^{-1})$ is an isomorphism.

(b) Let $\Gamma_R T_i = F_i T_i \quad i = 1,2$ as above, then $\varphi(F_1) = F_2$.

SECTION 4. A Refinement

Let Γ be a group satisfying the exact sequence $1 \to \Delta \to \Gamma \xrightarrow{\pi} Z^s \to 1$ where Δ is a torsion free nilpotent Lie group whose component group Δ / Δ_0 is finitely generated torsion free. By the results of Chapter one there exists a subgroup $\bar{\Delta}$ of $N(\Delta) = \Delta_R$ containing Δ as a subgroup of finite index satisfying

(a) $\bar{\Delta}$ is a closed lattice nilpotent polycyclic group.

(b) $(\Delta,\bar{\Delta})$ has the automorphism extension property.

Therefore we can form the group $\bar{\Gamma}$ containing Γ as a subgroup of finite index satisfying the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta & \longrightarrow & \Gamma & \longrightarrow & Z^s & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 1 & \longrightarrow & \bar{\Delta} & \longrightarrow & \bar{\Gamma} & \longrightarrow & Z^s & \longrightarrow & 1 \end{array}$$

Since we are only interested in groups commenserable with Γ we shall assume that Δ itself is a closed lattice nilpotent polycyclic group. We can form then the rational hull Δ_Q of Δ as in Chapter one and we have that both (Δ,Δ_Q) and (Δ,Δ_R) have the automorphism extension property. This allows us to form the groups Γ_Q, Γ_R satisfying the

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta & \longrightarrow & \Gamma & \xrightarrow{\pi} & Z^s & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 1 & \longrightarrow & \Delta_Q & \longrightarrow & \Gamma_Q & \xrightarrow{\pi} & Z^s & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 1 & \longrightarrow & \Delta_R & \longrightarrow & \Gamma_R & \xrightarrow{\pi} & Z^s & \longrightarrow & 1 \end{array}$$

Let $\mathcal{G}(\mathrm{ad}_{\Delta_R}\ \Gamma_R) = U^{\#} \cdot \bar{T}^{\#}$ be a representation of $\mathcal{G}(\mathrm{ad}_{\Delta_R}\ \Gamma_R)$ and let $\bar{\zeta}: \Gamma_R \to \bar{T}^{\#}$ be the corresponding semi-simple homomorphism. Let $\Gamma_R \cdot \bar{T}$ be a semi-simple splitting of Γ_R relative to this representation.

<u>Theorem 4.1</u>.A. There exists a semi-simple splitting $\Gamma_R \cdot T$ of Γ_R for which there are elements $\gamma_1,\ldots,\gamma_s$ with $\eta(\gamma_i) = z_i$ for $1 \le i \le s$ satisfying

(a) $\gamma_1,\ldots,\gamma_s$ are in Γ_Q .

(b) $T(\gamma_i) = \gamma_i$ for $1 \le i \le s$.

B. Moreover there is a closed lattice nilpotent polycyclic group $\bar{\Delta}$ satisfying

(c) $\bar{\Delta}$ is a subgroup of Δ_Q containing Δ as a subgroup of finite index.

(d) Γ normalizes $\bar{\Delta}$ so that we can form the group $\bar{\Gamma} = \bar{\Delta} \cdot \Gamma$.

(e) $\bar{\Gamma}$ contains $\gamma_1,\ldots,\gamma_s$ and is a subgroup of Γ_Q containing Γ as a subgroup of finite index satisfying the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta & \longrightarrow & \Gamma & \longrightarrow & Z^s & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 1 & \longrightarrow & \bar{\Delta} & \longrightarrow & \bar{\Gamma} & \longrightarrow & Z^s & \longrightarrow & 1 \end{array}$$

<u>Proof</u>: Let us consider first the case where Δ is a discrete lattice nilpotent polycyclic group. Then $\log \Delta_Q = L(\Delta_Q)$ is a rational Lie algebra and $\mathrm{ad}\ \Delta_Q\ \Gamma_Q$ is a group of automorphisms of the rational Lie algebra $L(\Delta_Q)$. Let $\mathcal{G}_Q(\mathrm{ad}\ \Delta_Q\ \Gamma_Q) = U_Q^{\#} \cdot T_Q^{\#}$ be a Mostow decomposition of $\mathcal{G}_Q(\mathrm{ad}\ \Delta_Q\ \Gamma_Q)$, and let $\zeta_Q: \Gamma_Q \to T_Q^{\#}$ be the corresponding semi-simple

homomorphism. We can proceed exactly as before to conclude that there are elements $\gamma_1,\ldots,\gamma_s$ in Γ_Q with $\eta(\gamma_i) = z_i$ satisfying $(T_Q^{\#}, \text{ad}\,\Delta_Q\,\gamma_i) = 1\,\Delta_Q$ and $T_Q^{\#}((\gamma_i,\gamma_j)) = (\gamma_i,\gamma_j)$ for $1 \le i, j \le s$. Let $U^{\#} = \mathcal{G}(U_Q^{\#})$ and let $T^{\#} = \mathcal{G}(\zeta_Q(\Gamma_Q))$. Then $\mathcal{G}(\text{ad}\,\Delta_R\,\Gamma_R) = U^{\#} \cdot T^{\#}$ is a Mostow decomposition with $T_Q^{\#} \subseteq T^{\#}$. Let $\zeta\colon \Gamma_R \to T^{\#}$ be the corresponding semi-simple homomorphism. Clearly $\zeta_Q(\Gamma_Q) = \zeta(\Gamma_R)$. Therefore there are elements $\gamma_1,\ldots\gamma_s$ in Γ_Q with $\eta(\gamma_i) = z_i$ satisfying $(T^{\#},\text{ad}\,\Delta_R\,\gamma_i) = 1\,\Delta_Q$ and $T^{\#}((\gamma_i,\gamma_j)) = (\gamma_i,\gamma_j)$ for $1 \le i, j \le s$.

Let us proceed to the general case. Δ_0 is normal in Δ. Therefore being characteristic in Δ, Δ_0 is normal in Γ. From before Δ_0 is normal in Δ_R therefore Δ_0 is normal in Γ_R. Therefore $\text{ad}_{\Delta_R}\Gamma_R(\Delta_0) = \Delta_0$ and $\mathcal{G}(\text{ad}_{\Delta_R}\Gamma_R)(\Delta_0) = \Delta_0$. Therefore Δ_0 is normal in any semi-simple splitting $\Gamma_R \cdot \bar{T}$ of Γ_R. Clearly $\Gamma_R/_{\Delta_0} \cdot \bar{T}'$ is a semi-simple splitting of Γ_R / Δ_0 where $\bar{T}'$ is the group of automorphisms of Γ_R / Δ_0 induced from $\bar{T}$. However there is a semi-simple splitting $\Gamma_R / \Delta_0 \cdot T''$ of Γ_R / Δ_0 such that there are elements $\tilde{\gamma}_1,\ldots,\tilde{\gamma}_s$ of Γ_Q with $\eta(\tilde{\gamma}_i) = z_i$ whose images $\tilde{\gamma}_1',\ldots,\tilde{\gamma}_s'$ in Γ_R / Δ_0 satisfy $T''(\tilde{\gamma}_i') = (\tilde{\gamma}_i')$ for $1 \le i \le s$. There is an h in Δ_R whose image h' in Δ_R / Δ_0 satisfies $T'' = \text{ad}_{\Gamma_R / \Delta_0} h' \circ \bar{T}' \circ \text{ad}_{\Gamma_R / \Delta_0} h'^{-1}$. Let $T = \text{ad}_{\Gamma_R} h \circ \bar{T} \circ \text{ad}_{\Gamma_R} h^{-1}$ and form the group $\Gamma_R \cdot T$ which is clearly a semi-simple splitting of Γ_R. We claim there are elements $z_1,\ldots,z_s$ in Δ_0 such that $T(\gamma_i) = \gamma_i$ for $1 \le i \le s$ with

$\gamma_i = z_i \tilde{\gamma}_i$. Let t_i in T be chosen so that $t_i|\Delta_R = \zeta(\tilde{\gamma}_i)$ where $\zeta(\gamma) = ad_{\Delta_R} h \circ \bar{\zeta}(\gamma) \circ ad_{\Delta_R} h^{-1}$. Assume $t_i(\gamma_j) = \gamma_j$ for $0 \leq i \leq k \leq s$ and $1 \leq j \leq s$. Let $\Delta_0^{(k)} = \cap_0^k \ker_{\Delta_0}(t_i - I)$. Since $T|\Gamma_R / \Delta_0 = T''$ we have $t_{k+1}(\gamma_j) = x_j \gamma_j$ with x_j in Δ_0 . From T being abelian we have x_j in $\Delta_0^{(k)}$. Therefore there is a z_i in $\Delta_0^{(k)}$, such that $t_{k+1}(\gamma_j)\gamma_j^{-1} = t_{k+1}(z_j) z_j^{-1}$. Let $\gamma_i' = z_i^{-1}\gamma_i$, then $t_j(\gamma_i') = \gamma_i'$ for $0 \leq j \leq k+1$ and $1 \leq i \leq s$.

We need the following lemma for part B.

Lemma 4.1.1. Let x be in Δ_Q . Then there is a subgroup $\bar{\Delta}$ of Δ_Q containing Δ as a subgroup of finite index satisfying

(a) $\bar{\Delta}$ is normalized by Γ .

(b) x is in $\bar{\Delta}$.

Part B then clearly follows for where $\bar{\gamma}_1,\ldots,\bar{\gamma}_s$ are arbitrary elements of Γ such that $\eta(\bar{\gamma}_i) = \eta(\gamma_i)$ then $\gamma_i = x_i \bar{\gamma}_i$ where x_i is in Δ_Q .

Proof: Again assume Δ is a discrete lattice nilpotent polycyclic group.

Let $e_1,\ldots,e_n$ be a canonical basis of $L(\Delta_R)$ such that $\exp(\sum_1^n n_i e_i : n_i \text{ integer}) = \Delta$. Since x is in Δ_Q there is an integer m such that m log x is in $\log \Delta$. If Δ_R is abelian then $\bar{\Delta} = \exp(\sum_1^n n_i \frac{e_i}{m} : n_i \text{ in } Z)$ is a group normalized by Γ containing x . We may see this as follows. Clearly $\bar{\Delta}$ being the isomorphic image of a group is a group. Since $m \log x = \sum_1^n n_i e_i$ for some integers n_i , $\log x = \sum_1^n n_i \frac{e_i}{m}$, and x is in $\bar{\Delta}$. Let A be in $ad_{\Delta_R} \Gamma$. Then for any integers n_i

$$A\left(\sum_1^n n_i \frac{e_i}{m}\right) = {}^1/_m A\left(\sum_1^n n_i e_i\right) = {}^1/_m \sum_1^n \bar{n}_i e_i = \sum_1^n \bar{n}_i \frac{e_i}{m}.$$

where since Δ is normalized by Γ, the $\bar{n}_i$ are integers. Therefore $\bar{\Delta}$ is invariant under A, and Γ normalizes $\bar{\Delta}$.

Let us assume the theorem to be true when Δ_R is $c-1$ step nilpotent and let Δ_R be c step nilpotent. Let $(\Delta_R)_c$ be the last non-trivial term in the lower central series of Δ_R and let t be the integer such that $e_{t+1},\ldots,e_n$ span $(\Delta_R)_c$. We denote the image of an element e in $L(\Delta_R)$ by $\bar{e}$ in $L(\Delta_R / (\Delta_R)_c)$. Therefore there are integers $m_1,\ldots,m_t$ such that where $\bar{\gamma}$ is the lattice generated by $\frac{\bar{e}_i}{m_i}$ $1 \le i \le t$ we have that $\exp \bar{\gamma}$ is a subgroup of $\Delta_R / (\Delta_R)_c$, $\bar{x}$ is in $\exp \bar{\gamma}$ and $\exp \bar{\gamma}$ is normalized by $\bar{\Gamma}$, it is invariant under the action of $\Delta_R / (\Delta_R)_c$ induced by the action of Γ on Δ_R. As before there is some integer $\bar{m}$ such that $\exp\left(\sum_1^n n_i \frac{e_i}{m_i} : n_i \text{ in } Z\right)$ is a group where $m_i = \bar{m}$ for $t+1 \le i \le n$. Let $\bar{\bar{m}} = \Pi_1^t m_i$ and let $p_i = m_i$ for $1 \le i \le t$, $p_i = m\bar{m}\bar{\bar{m}}$. Clearly where γ is the lattice generated by $\frac{e_1}{p_1},\ldots,\frac{e_t}{p_t},\ldots,\frac{e_n}{p_n}$ we have $\exp \gamma$ is a subgroup of Δ_R, containing Δ as a subgroup of finite index. Since $m \log x$ is in $\log \Delta$ and since $\log x = \sum_1^t n_i \frac{e_i}{p_i} + Z$ where n_i are integers and Z is in $L((\Delta_R)_c)$ we have $m\,Z$ in $\log \Delta \cap L((\Delta_R)_c)$. But then $Z = \sum_{t+1}^n n_i \frac{e_i}{m} = \sum_{t+1}^n n_i \bar{m}\bar{\bar{m}} \frac{e_i}{p_i}$ for some integer n_i. Therefore $\log x$ is in γ and x is in $\exp \gamma$.

Let A be in $\mathrm{ad}_{\Delta_R}\Gamma$, and let $y = \sum_1^t n_i \frac{e_i}{p_i}$ for any integer n_i. $A(y) = \sum_1^t n_i' \frac{e_i}{p_i} + z$ where n_i' are integers and z is in $L((\Delta_R)_c)$. $\bar{\bar{m}}\, y$ is in $\log \Delta$ therefore $A(\bar{\bar{m}}\, y)$ is in $\log \Delta$. Therefore $\bar{\bar{m}}\, z$ is in $\log \Delta \cap L((\Delta_R)_c)$. Therefore as before $z = \sum_{t+1}^n \bar{n}_i \frac{e_i}{p_i}$ where the $\bar{n}_i$ are integers. Therefore since $1 / m\, \bar{m}\, \bar{\bar{m}} \log \Delta \cap L((\Delta_R)_c)$ is invariant under A we have γ is invariant under A. Therefore $\exp \gamma$ is normalized by Γ.

<u>Theorem 4.2</u>. Let $\Gamma_R \cdot T$ be a semi-simple splitting of Γ_R for which there exists elements $\gamma_1,\ldots,\gamma_n$ in Γ such that $\hat{\eta}(\gamma_i) = z_i$ for $1 \le i \le n$, satisfying $T(\gamma_i) = \gamma_i$ and $T|\Delta_R = \Pi_1^s\, \zeta(\gamma_i)$. Choose t_i in T such that $t_i|\Delta_R = \zeta(\gamma_i)$ for $1 \le i \le n$.

Write $\Gamma_R \cdot T = F \cdot T$ where $F = \Delta_R\, \hat{v}_1 \ldots \hat{v}_n$ where $v_i = \gamma_i\, t_i^{-1}$.

Then A.

(a) (v_i, v_j) is in Δ.

(b) $\mathrm{ad}_{\Delta_R / \Delta_0} v_i$ is rational relative to a canonical basis of $L(\Delta_R / \Delta_0)$, $e_1,\ldots,e_s$ for which $\log \Delta / \Delta_0 = \{\sum_1^e n_j e_j : n_j \text{ in } Z\}$

B. There is a closed lattice nilpotent polycyclic group $\bar{\Delta}$ contained in Δ_Q containing Δ as a subgroup of finite index such that

(a) $\bar{\Delta} \cdot \hat{v}_1,\ldots,\hat{v}_n$ is a closed cocompact lattice nilpotent polycyclic subgroup of F_R.

(b) $\bar{\Delta} \cdot \hat{v}_1,\ldots,\hat{v}_n$ is invariant under T.

(c) $\bar{\Delta}$ is normalized by Γ and so we can form the group $\bar{\Gamma} = \bar{\Delta} \cdot \Gamma$. Then $\bar{\Gamma} \cdot T = \bar{\Delta} \cdot \hat{v}_1 \ldots \hat{v}_n \cdot T$.

Proof of A.

(a) Since $(t_j, \gamma_i) = 1$ we have $(\gamma_i, \gamma_j) = (v_i, v_j)$ in Δ .

(b) $ad_{\Delta_R / \Delta_0} \gamma_i$ preserves Δ / Δ_0 and therefore Δ_Q / Δ_0 . But v_i and t_i acting on Δ_R / Δ_0 are in the algebraic hull of $ad_{\Delta_R / \Delta_0} \gamma_i$ over the rationals.

Proof of B.

Let us assume Δ is a discrete lattice nilpotent polycyclic group. Let $M = F_R$. Let $\tilde{e}_1, \ldots, \tilde{e}_n$ be a basis of $L(\Delta_R)$ such that $\log \Delta = \left\{ \sum_{j=1}^{n} n_i \tilde{e}_i : n_i \text{ in } Z \right\}$, and chose $V_1, \ldots, V_s$ in $L(M)$ so that $\exp V_i = v_i$ for $1 \leq i \leq s$. Since (v_i, v_j) is in Δ and since $ad_{\Delta_R} v_i(\Delta_Q) = \Delta_Q$ the collection of all rational linear combinations of $\tilde{e}_i$, V_j for $1 \leq i \leq n$, $1 \leq j \leq s$ is a rational Lie algebra. Moreover, $L(M) / L(\Delta_R)$ is abelian. Therefore we can find a rational canonical basis $e_1, \ldots, e_n$; $V_1, \ldots, V_s$ of $L(M)$ such that the lattice generated by the e_i's is of finite index in $\log \Delta$. Moreover where t is the integer such that $e_{t+1}, \ldots, e_n$ span $L(M_c)$ where M_c is the last non-trivial term in the lower central series of M we have that the lattice spanned by $e_{t+1}, \ldots, e_n$ is equal to $\log \Delta \cap L(M_c)$ which is invariant under Γ . We claim that there are integers $m_1, \ldots, m_n$ such that where γ is the lattice spanned by $\frac{e_i}{m_i}$, V_j for $1 \leq i \leq n$ and $1 \leq j \leq s$ we have that $\exp \gamma$ is a subgroup of M containing Δ and invariant under $ad_M \gamma_i$ for $1 \leq i \leq s$.

We proceed by induction. If M is abelian we are done. Let us assume the theorem to be true for $M(c-1)$-step nilpotent and let M be c-step nilpotent. We shall denote the image of an element X of $L(M)$ in $L(M/M_c)$ by $\bar{X}$. By induction there are integers $m_1,\ldots,m_t$ such that the lattice $\bar{\gamma}$ in $L(M/M_c)$ generated by $\frac{e_i}{m_i}$, V_j for $1 \le i \le t$ and $1 \le j \le s$ has the property that $\exp \bar{\gamma}$ is a subgroup of M/M_c containing $\bar{\Delta}$ and invariant under $\mathrm{ad}_{M/M_c} \gamma_i$ $1 \le i \le s$. As in Chapter one, there is an integer m such that where γ' is the lattice generated by $\frac{e_i}{m_i}$, V_j for $1 \le i \le n$, and $1 \le j \le s$ and $m_i = m$ for $t+1 \le i \le n$ we have that $\exp \gamma'$ is a subgroup of M.

Choose an integer ℓ such that $\ell \cdot \log \Delta$ is contained in the lattice generated by e_i for $1 \le i \le n$ and choose $\ell' = \Pi_1^t m_i$. Let $p_i = m_i$ for $1 \le i \le t$ and $p_i = \ell \ell' m$ for $t+1 \le i \le n$, and let γ be the lattice generated by $\frac{e_i}{p_i}$, V_j for $1 \le i \le n$ and $1 \le j \le s$. Clearly $\exp \gamma$ is a subgroup of M. Let x be in Δ. Then

$$x = \sum_1^t n_i \frac{e_i}{p_i} + \sum_1^s k_i V_i + z$$

where n_i, k_i are integers and z is in $L(M_c)$. But ℓx is in the lattice generated by $e_1,\ldots,e_n$, therefore $\ell z = \sum_{t+1}^n n_i e_i$ where n_i integers, and $z = \sum_{t+1}^n n_i \frac{e_i}{\ell}$. Therefore x is in γ, and $\Delta \subseteq \exp \gamma$. Let y be in γ. Then $y = \sum_1^t n_i \frac{e_i}{p_i} + \sum_{t+1}^n n_i \frac{e_i}{p_i} + \sum_1^\ell k_j V_j$ where n_i, k_j are integers. Let $A_i = \mathrm{ad}_M \gamma_i$. Since $p_i = m \ell \ell'$ for $t+1 \le i \le n$ and since $\{\sum_{t+1}^n \bar{n}_i e_i : \bar{n}_i \text{ in } Z\} = \log \Delta \cap L(M_c)$ is invariant under Γ, it is invariant under A_i. Therefore $A_i \left(\sum_{t+1}^n n_i \frac{e_i}{p_i}\right)$ is equal to $\sum_{t+1}^n n_i' \frac{e_i}{p_i}$ where the n_i' are integers. Moreover $A_i \left(\sum_1^s k_j V_j\right) = \sum_1^s k_j V_j$ since $(\gamma_i, V_j) = 1$. Therefore to prove that $A_i(y)$ is in γ

it is sufficient to prove that $A_i\left(\sum_1^t n_i \frac{e_i}{p_i}\right)$ is in γ. But $A_i\left(\sum_1^t n_i \frac{e_i}{p_i}\right) = \sum_1^t n_i'' \frac{e_i}{p_i} + z$ where n_i'' are integers and z is in $L(M_c)$. Let $y' = \sum_1^t n_i \frac{e_i}{p_i}$. $\ell' y'$ is in the lattice generated by $e_1,\ldots,e_n$, and therefore in $\log \Delta$. But then $A_i(\ell' y')$ is in $\log \Delta$, and $\ell A_i(\ell' y')$ is in the lattice generated by $e_1,\ldots,e_n$. Therefore $\ell \ell' z$ is in the lattice generated by $e_1,\ldots,e_n$ and is in $L(M_c)$. Therefore $\ell \ell' z = \sum_{t+1}^n n_i'' e_i$ and $z = \sum_{t+1}^n n_i'' \frac{e_i}{\ell\ell'}$. Therefore γ is invariant under $\mathrm{ad}_M \gamma_i$ for $1 \le i \le s$. Since $t_i = \gamma_i v_i^{-1}$ and since $\mathrm{ad}\, \gamma_i v_i^{-1}(\exp \gamma) = t_i(\exp \gamma) = \exp \gamma$ we have that $\exp \gamma$ is invariant under T.

Let $\bar{\Delta} = \exp\left(\sum_1^n n_i \frac{e_i}{p_i} : n_i \text{ in } Z\right)$. We now proceed to the general case. Clearly Δ_0 is normal in $\Gamma_R \cdot T$. We may consider the short exact sequence $1 \to \Delta/\Delta_0 \to \Gamma_R / \Delta_0 \to Z^s \to 1$. Then Δ/Δ_0 is a discrete lattice nilpotent polycyclic group. Moreover $\Gamma_R \cdot T / \Delta_0 = F/\Delta_0 \cdot T$ is a semi-simple splitting of Γ_R / Δ_0 as above. Therefore we can find a lattice nilpotent polycyclic subgroup $\bar{\Delta}/\Delta_0$ in Δ_Q / Δ_0 containing Δ/Δ_0 as a subgroup of finite index such that denoting the image of an element x of $\Gamma_R \cdot T$ by $\bar{x}$ in $\Gamma_R \cdot T/\Delta_0$ we have $\bar{\Delta}/\Delta_0 \cdot \hat{v}_1 \ldots \hat{v}_s$ is a discrete lattice nilpotent polycyclic group invariant under the action of F_R/Δ_0 induced by T. But then $\bar{\Delta} \cdot \hat{v}_1 \ldots \hat{v}_s$ is a closed lattice nilpotent polycyclic group with $\bar{\Delta}/\Delta$ finite and $\bar{\Delta} \cdot \hat{v}_1 \ldots \hat{v}_s$ invariant under T.

<u>Theorem 4.3</u>. Let Γ be a group satisfying the short exact sequence $1 \to \Delta \to \Gamma \xrightarrow{\pi} Z^s \to 1$. Where Δ is a torsion free nilpotent Lie group

whose component group is finitely generated.

Then there is a semi-simple splitting of Γ_R, $\Gamma_R \cdot T$ and a subgroup $\bar{\Delta}$ of Δ_R containing Δ as a subgroup of finite index and normalized by Γ such that the group $\bar{\Gamma} = \bar{\Delta}\,\Gamma$ satisfies

(a) $\bar{\Gamma}$ is a subgroup of Γ_R containing Γ as a subgroup of finite index satisfying the diagram

$$1 \longrightarrow \bar{\Delta} \longrightarrow \bar{\Gamma} \xrightarrow{\eta} Z^s \longrightarrow 1$$

(b) $\bar{\Gamma}$ is invariant under T and $\bar{\Delta}$ is invariant under T.

(c) There are elements $\gamma_1,\ldots,\gamma_s$ in $\bar{\Gamma}$ such that $\eta(\gamma_i) = z_i$ and $T(\gamma_i) = \gamma_i$ for $1 \leq i \leq s$.

(d) Moreover $\bar{\Delta} \cdot \hat{v}_1,\ldots,\hat{v}_s$ is a closed lattice nilpotent polycyclic group with $v_i = \gamma_i \circ t_i^{-1}$. We call the group $\bar{\Gamma} \cdot T$ a semi-simple splitting of $\bar{\Gamma}$.

<u>Proof</u>: Put together previous theorems.

SECTION 5. THE NIL SHADOW

Let Γ be a closed subgroup of a simply connected solvable analytic group S. The finer properties of solvable Lie theory can be seen by comparing the semi-simple splittings of Γ and of S.

We begin by choosing an arbitrary semi-simple splitting $S \cdot T_S = N_S \cdot T_S$ of S. Using the Birkhoff imbedding theorem we can consider $N_S \cdot T_S$ as a matrix group where N_S is an algebraic group of unipotent matrices and T_S is an abelian group of semi-simple matrices. Let $T = \mathfrak{a}(T_S)$. Then $\mathfrak{a}(S) = N_S \cdot T$ where N_S is the group of all unipotent matrices of $\mathfrak{a}(S)$ and T is a maximal completely reducible subgroup of $\mathfrak{a}(S)$. Let $\mathfrak{a}(\Gamma) = U_\Gamma \cdot T_\Gamma$ be a Mostow decomposition of $\mathfrak{a}(\Gamma)$. Let $\tilde{T}$ be a maximal completely reducible subgroup of $\mathfrak{a}(S)$ containing T_Γ and choose h in H so that $h\, T\, h^{-1} = \bar{T}$. Clearly $\mathfrak{a}(S) = N_S \cdot \bar{T}$. Let $\bar{T}_S$ be the projection of S into $\bar{T}$ relative to the representation $\mathfrak{a}(S) = N_S\bar{T}$. Clearly $S \cdot \bar{T}_S = N_S \cdot \bar{T}_S$ is a semi-simple splitting of S. We shall assume therefore that $S \cdot T_S = N_S \cdot T_S$ is a semi-simple splitting of S in such a way that for some representation $\mathfrak{a}(\Gamma) = U_\Gamma \cdot T_\Gamma$ we have $U_\Gamma \subseteq N_S$ and $T_\Gamma \subset \mathfrak{a}(T_S)$.

Let H be the nil radical of S We shall assume that if $\Gamma_0 \subset H$ we then have the exact sequence $1 \to \Gamma \cap H \to \Gamma \overset{\pi}{\longrightarrow} z^s \to 1$.

<u>Theorem 5.1</u>. Let Γ be a closed cocompact subgroup of S. Assume that Γ has no connected normal subgroups of S. Then

(a) $\Gamma_0 \subset H$ where H is the nil radical of S. We have then that the group Γ satisfies the exact sequence $1 \to \Gamma \cap H \to \Gamma \overset{\pi}{\longrightarrow} z^s \to 1$. Let $\Delta = \Gamma \cap H$ and let Δ_R be the Lie hull of Δ.

(b) There exists a semi-simple splitting $S \cdot T_S = N_S \cdot T_S$ of S

and a subgroup $\bar{\Delta}$ of Δ_R containing Δ as a subgroup of finite index satisfying

(1) $\bar{\Delta}$ is normalized by Γ and $t(\Gamma)$. Let $\bar{\Gamma} = \Delta\Gamma$.

(2) The group $\bar{\Gamma}$ satisfies the exact sequence $1 \to \bar{\Delta} \to \bar{\Gamma} \xrightarrow{\eta} z^s \to 1$, contains Γ as a subgroup of finite index, is normalized by $t(\bar{\Gamma})$ and contains elements $\gamma_1, \ldots, \gamma_s$ with $\eta(\gamma_i) = z_i$ such that $t(\bar{\Gamma})(\gamma_i) = \gamma_i$ for $1 \leq i \leq s$,.

(3) $n(\bar{\Gamma})$ is a closed cocompact lattice nilpotent polycyclic subgroup of N_S, is given by $n(\bar{\Gamma}) = \bar{\Delta} \cdot \hat{n(\gamma_1)} \ldots \hat{n(\gamma_s)}$ and is normalized by $t(\bar{\Gamma})$.

(4) $\bar{\Gamma} \cdot t(\bar{\Gamma}) = n(\bar{\Gamma}) \cdot t(\bar{\Gamma})$. Where $t: S \to T_S$ and $n: S \to N_S$ are the projections onto T_S and N_S in $N_S \cdot T_S$.

Proof: Let $\Gamma_R = (\Gamma \cap H)_R \Gamma$. We shall show that Γ_R is a closed subgroup of S and that $n(\Gamma_R)$ is a subgroup of N_S . Moreover that $N_S / n(\Gamma_R)$ is compact.

Γ_R clearly satisfies the property that $\mathfrak{a}(\Gamma_R) = \mathfrak{a}(\Gamma)$. For γ in Γ_R write $\gamma = n\, t$ with n in N_S and t in T_S , then n,t are in $\mathfrak{a}(\Gamma)$. Our first claim is that Γ_R is closed in S . $(\Gamma \cap H)_R$ is closed in S , therefore the restriction of the rational map $f: S \to S / \Gamma$ to $(\Gamma \cap H)_R$ is continuous. $\Gamma \cap H$ is a closed cocompact subgroup of $(\Gamma \cap H)_R$ therefore the natural map $g: (\Gamma \cap H)_R \longrightarrow (\Gamma \cap H)_R / \Gamma \cap H$ is open and continuous. Therefore since $\Gamma \cap H \subset \Gamma$, there is a continuous map $h: (\Gamma \cap H)_R / \Gamma \cap H \longrightarrow S/\Gamma$ such that $h \circ g = f$. The image of h is Γ_R / Γ and is a compact subset of the Hausdorf space S/Γ . Therefore Γ_R/Γ is closed in S/Γ and Γ_R is closed in S . Let

$\mathfrak{G}(\Gamma) = U_\Gamma \cdot T_\Gamma$ be a Mostow decomposition of $\mathfrak{G}(\Gamma)$ with $U_\Gamma \subset N_S$ and $T_\Gamma \subset T_S$. Since $(\mathfrak{G}(\Gamma),\mathfrak{G}(\Gamma)) \subseteq \mathfrak{G}(\Gamma,\Gamma) \subseteq (\Gamma \cap H)_R$ we have $(U_\Gamma,T_\Gamma) \subseteq (\Gamma \cap H)_R$.

We now prove that $n(\Gamma_R)$ is a subgroup of N_S. Take x,y in Γ_R and write $x = nt$, $y = \bar{n}\,\bar{t}$ with n, $\bar{n}$ in N_S, t, $\bar{t}$ in T_S. $xy = n\,\bar{n}(\bar{n}^{-1},t)\,t\,\bar{t}$ is in Γ_R. But $(\bar{n}^{-1},t) \in (\Gamma \cap H)_R \subseteq N_S$, therefore $(\bar{n},t^{-1})\,xy = n\,\bar{n}\,t\,\bar{t}$ is in Γ_R. But then $n\,\bar{n}$ is in $n(\Gamma_R)$. Therefore $n(\Gamma_R)$ is a subgroup of N_S. Since $n\colon S \to N_S$ is a homeomorphism $n(\Gamma_R)$ is a closed subgroup of N_S. We claim $n(\Gamma_R)$ is cocompact in N_s i.e., $N_S = \mathfrak{G}(n(\Gamma_R))$. $t(\Gamma_R) \subseteq T_\Gamma$ and T_Γ normalizes Γ_R since $\Gamma_R \supset (\Gamma \cap H)_R$. Therefore $t(\Gamma_R)$ normalizes Γ_R. Let $\overline{t(\Gamma_R)}$ be the closure of $t(\Gamma_R)$ in T_S. Then $\Gamma_R \cdot \overline{t(\Gamma_R)}$ is a closed cocompact subgroup of $S \cdot T_S$. Therefore $n(\Gamma_R) \cdot \overline{t(\Gamma_R)}$ is a closed cocompact subgroup of $N_S \cdot T_S$. Since $N_S\,\overline{t(\Gamma_R)}$ is closed in $N_S \cdot T_S$ we have that $n(\Gamma_R)$ is a closed cocompact subgroup of $N_S\,t(\Gamma_R)$.

Let $\bar{p}\colon N_S\,\overline{t(\Gamma_R)} \to N_S / n(\Gamma_R)$ be the composition of the projection onto N_S followed by the natural map. It is open and continuous. Let $\bar{g}\colon N_S\,\overline{t(\Gamma_R)} \to N_S\,\overline{t(\Gamma_R)} / n(\Gamma_R)\,\overline{t(\Gamma_R)}$ be the natural map. It is open and continuous. Since $n(\Gamma_R)$ is normalized by $\overline{t(\Gamma_R)}$ there is a continuous map $\bar{h}\colon N_S\,t\overline{(\Gamma_R)} / n(\Gamma_R)\,\overline{t(\Gamma_R)} \longrightarrow N_S / n(\Gamma_R)$ such that $\bar{h} \circ \bar{g} = \bar{p}$. But then the image of $\bar{h}$ which is $N_S / n(\Gamma_R)$ is compact.

We are now in a position to prove our theorem. Γ_0 is normalized by Γ and therefore by $\mathfrak{G}(\Gamma)$. But $N_S \subset \mathfrak{G}(\Gamma)$ therefore Γ_0 is normalized by N_S. Our next claim is that $\Gamma_0 \cap H$ is connected. Since Γ_0, H, $H\,\Gamma_0$ are analytic subgroups of the solvable simply connected analytic group S, they are also simply connected. Therefore we have that

$\Gamma_0 / \Gamma_0 \cap H = H\Gamma_0 / H$ is simply connected and an elementary fact in topology gives us $\Gamma_0 \cap H$ is connected. From S/H abelian and since $H\Gamma_0 / \Gamma_0 = H / \Gamma_0 \cap H$ is nilpotent we have $H\Gamma_0 / \Gamma_0 \cap H$ is nilpotent and therefore acts on itself by unipotent inner automorphisms. Take y in $H\Gamma_0$ and write $y = n\,t$ with n in N_S, t in T_S.

Let $K = (t - I)H$. Since t induces the identity map on $H\Gamma_0 / \Gamma_0 \cap H$ we have $K \subseteq \Gamma_0 \cap H$. Since T_S is abelian, $T_S(K) = K$. Let $I(K)$ be the ideal in N_S generated by K. $I(K)$ is T_S invariant and since $C_0 \cap H$ is normalized by N_S we have $I(K) \subseteq C_0 \cap H$. By hypothesis we must have $I(K) = K = 0$ and $H\Gamma_0$ acts on H by unipotent automorphisms. Therefore $H\Gamma_0$ is nilpotent. Since it is also connected $H\Gamma_0 = H$. We now prove the bulk of the theorem.

Since $\mathfrak{a}(\Gamma_R) = U_\Gamma \cdot T_\Gamma$ with $U_\Gamma \subset N_S$, $T_\Gamma \subset T$, and since $(\mathfrak{a}(\Gamma_R), \mathfrak{a}(\Gamma_R)) \subseteq (\Gamma \cap H)_R$ arguing as in Theorem 3.1 there are elements $\gamma_1,\ldots,\gamma_n$ in Γ_R with $\eta(\gamma_i) = z_i$ with the property that writing $\gamma_i = n_i\, t_i$ with n_i in N_S, t_i in T_S we have $t_i(n_j) = n_j$ for $1 \le i, j \le n$. Let $\Delta_R = (\Gamma \cap H)_R$. Since $\mathrm{ad}_{N_S}\, \gamma_i(\Delta_R) = \Delta_R$ we have $\mathrm{ad}\, n_i(\Delta_R) = \Delta_R$, $t_i(\Delta_R) = \Delta_R$. Moreover (n_i, n_j) is in Δ_R. Therefore $\eta(\Gamma_R) = \Delta_R \cdot \hat{n}_1 \ldots \hat{n}_n$ and $t(\Gamma_R) = \Pi_1^n\, \hat{t}_i$. Let $T = t(\Gamma_R)$. Since $n(\Gamma_R)$ is a closed cocompact subgroup of N_S, T is isomorphic to $A(T) = \mathrm{ad}_{\Gamma_R} T$ and $\Gamma_R \cdot T$ is isomorphic to $\Gamma_R \cdot A(T)$. Clearly $\Gamma_R \cdot A(T)$ is a semi-simple splitting of Γ_R and $n(\Gamma_R)$ is the collection of all elements of $\Gamma_R \cdot T$ which act by unipotent automorphisms on Δ_R. Since we are solely interested in groups containing Γ up to finite index, we may assume that there exists a semi-simple splitting of Γ_R, $\Gamma_R \cdot \bar{T}$ such that Γ is invariant under $\bar{T}$, $\Delta = \Gamma \cap H$ is invariant under $\bar{T}$, and for which there exist elements $\bar{\gamma}_1, \ldots, \bar{\gamma}_n$ in Γ such that $\eta(\bar{\gamma}_i) = \bar{\gamma}_i$ and $\bar{T}(\bar{\gamma}_i) = \bar{\gamma}_i$ for $1 \leq i \leq n$. Choose h in Δ_R such that $\bar{T} = ad_{\Gamma_R}(h\, T\, h^{-1})$ and let $T' = h\, T\, h^{-1}$, $T'_S = h\, T_S\, h^{-1}$. We have then $ST_S = N_S\, T_S = ST'_S = N_S\, T'_S$, and $\Gamma_R \cdot T = n(\Gamma_R) \cdot T = \Gamma_R \cdot T' = n(\Gamma_R) \cdot T'$. Moreover Γ is normalized by T' and $ad\, T'(\bar{\gamma}_i) = \bar{\gamma}_i$. Let $\bar{n}_i = \bar{\gamma}_i \cdot t'_i$. Then $ad_{\Delta_R} \bar{n}_i$ is unipotent, $(\bar{\gamma}_i, t'_i) = 1$. Therefore $\bar{\gamma}_i = \bar{n}_i\, t'_i$ with $\bar{n}_i$ in N_S and t'_i in T'_S and $t'_i(\bar{n}_j) = \bar{n}_j$. Therefore letting $n: S \to N_S$, $t: S \to T'_S$ relative to the representation $S \cdot T'_S = N_S \cdot T'_S$ we have that $n(\Gamma) = \Delta \cdot \hat{\bar{n}}_1 \ldots \hat{\bar{n}}_n$ is a closed cocompact subgroup of N_S, invariant under $t(\Gamma)$.

<u>Theorem 5.2.</u> Let Γ be a closed cocompact subgroup of S.

Let us assume that Γ contains no connected normal subgroups of S. Then $H\Gamma$ is closed in S where H is the nil radical of S.

<u>Proof</u>: We shall assume that $S \cdot T_S = N_S \cdot T_S$ is a semi-simple splitting of S and that Γ satisfies the conclusions for $\bar{\Gamma}$ relative to $N_S \cdot T_S$ in Theorem 5.1. Let G be the normalizer of Γ_0 in S. Then G is a closed subgroup of S and since H normalizes Γ_0, Γ normalizes Γ_0 bsth H and Γ are in G. In particular G is normal in S and the nil radical of the identity component of G is H. Let

$$N^{\#} = N_S / \Gamma_0 / (N_{S/\Gamma_0}, N_{S/\Gamma_0})$$

Since N_S normalizes Γ_0, $N^{\#}$ is a well defined vector group. $n(\Gamma)$ is a closed cocompact lattice nilpotent polycyclic subgroup of N_S and since $n: S \to N_S$ is a homeomorphism Γ_0 is the identity component of $n(\Gamma)$. Therefore $n(\Gamma) / \Gamma_0$ is a discrete lattice nilpotent polycyclic subgroup of N_S / Γ_0. Let $(n(\Gamma) / \Gamma_0)^{\#}$ be the image of $n(\Gamma) / \Gamma_0$ in $N^{\#}$. By Chapter one, $(n(\Gamma) / \Gamma_0)^{\#}$ is a discrete cocompact lattice nilpotent polycyclic subgroup of $N^{\#}$.

Let $\varphi: \mathrm{aut}(N_S;\Gamma_0) \longrightarrow \mathrm{aut}(N^{\#})$ be the map which takes an X in $\mathrm{aut}(N_S;\Gamma_0)$ the group of automorphisms of N_S keeping Γ_0 invariant into its induced map on $N^{\#}$. Let $\ell: G \to \mathrm{ad}_{N^{\#}} G$ be given by $\ell = \varphi \circ \mathrm{ad}\, N_S$. Let $X: G \to G/H$ be the natural map. Since $\ell(H) = 1$, there is a map $y: G/H \to \mathrm{ad}_{N^{\#}} G$ such that $\ell = y \circ X$. Let $K = \ker \ell$. Let us assume for the time being that $K_0 = H$. Then $\ker y = K/H = K/K_0$ is a discrete subgroup of G/H. $\ell(\Gamma)$ keeps $(n(\Gamma) / \Gamma_0)$ invariant therefore $\ell(\Gamma)$ is discrete. But $H\,\Gamma/H = X(\Gamma) \subset y^{-1}(\ell(\Gamma))$. But since $\ker y$ and $\ell(\Gamma)$ are discrete, $y^{-1}(\ell(\Gamma))$ and therefore $X(\Gamma) = H\,\Gamma / H$ is a discrete subgroup of G/H. But then $H\,\Gamma$ is closed in G which is closed in S. Therefore $H\,\Gamma$ is closed in S. All that remains to be proved is that $H = K_0$. Since $H \subset K_0$ and K_0 is connected in S we will be done if we can prove K_0 is nilpotent. Let x be in K_0 and write $x = n \cdot t$ with n in N_S and t in T_S. $\ell(x) = \ell(n)\ell(t) = 1$ but since $\ell(n) = 1$ we have $\ell(t) = 1$. Therefore $\overline{\mathrm{ad}}_{N^{\#}} t = 1_{N^{\#}}$. This implies that t is a semi-simple automorphism of N_S which keeps Γ_0 invariant and acts by the identity on N_S / Γ_0. This will be enough to conclude that $t = 1$. Let $K = (t - I)N_S$. Then $K \subseteq \Gamma_0$ and K is invariant under T_S since T_S is abelian. Let $I(K)$ be the ideal generated by K in N_S. Then $I(K)$ is invariant under the action of T_S and since Γ_0 is normalized by N_S, $I(K) \subset \Gamma_0$. But then $I(K)$ is a connected normal subgroup of S and since ~~it~~ is contained in Γ_0 it is zero. Therefore $K = 0$ and $t = 1_{N_S}$. Therefore K_0 acts on H by unipotent automorphisms and since K_0 / H is abelian we have that K_0 is nilpotent.

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AUTOBIOGRAPHICAL STATEMENT

Richard Tolimieri was born in New York City, on November 19, 1940. He entered The City College of New York in September 1957. He spent the next seven years traveling in the United States, working in various jobs, involving himself in the Civil Rights marches in the South in the late 50's, and studying until he received his Bachelor of Science degree in Mathematics at The City College of New York. He entered The City University of New York in 1965 and has supported himself by working at City College, Pace College and receiving National Science Foundation grants. In his spare time, he taught at the University of the Street in Manhattan, visited the Pentagon and has been active in some form of the radical movement.