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ON THE HOMOTOPY THEORY OF MONOIDS

by

Carol M. Hurwitz

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

1984

This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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ACKNOWLEDGMENTS

I would like to thank my advisor, Alex Heller, for suggesting this problem and providing the encouragement and guidance necessary to see it to completion.

In addition, there were many students and faculty members, in particular A. T. Vasquez, to whom I am indebted for their help and support.

Finally, I wish to thank my typist, Jennifer Lord, for her extraordinary effort and attention to detail in typing this dissertation.

CONTENTS

Section One

HISTORICAL BACKGROUND

Homotopy theory has been studied within the context of many categories. Among the better known examples are the categories of CW-complexes introduced by J.H.C. Whitehead in 1949 [21] and that of simplicial sets (originally called semi-simplicial complexes) introduced by S. Eilenberg and J.A. Zilber in 1950 [2]. While CW-complexes are special kinds of topological spaces, simplicial sets are algebraic in nature.

Much of the early work in the homotopy theory of simplicial complexes was accomplished by D.H. Kan [6, 7, 8]. He worked out the details of defining homotopy groups on a sub-category of simplicial sets that satisfied a certain extension condition. These simplicial sets have come to be known as Kan complexes. He eventually developed a combinatorial definition of homotopy groups intended to focus on the structure of a simplicial set rather than on the associated topological space.

Motivated by the fact that the proofs of homotopy theoretic results in various categories were formally the same as those found in algebraic topology, Quillen [16] decided to define an axiomatic notion of homotopy theory in sufficient generality as to encompass the theories that had been studied. He introduced the notion of a model category, and the somewhat stronger notion of a closed model category, endowed with three distinguished families of morphisms: cofibrations, fibrations, and weak equivalences. He showed that a category, N, endowed with such a structure can be localized with respect to its weak equivalences to yield Ho N, and that many of the constructions within algebraic topology, such as mapping cylinders, homotopy fibres, loop and suspension functors, etc., carry over for the category N. The category Ho N, together with whatever structure is inherited from N, was termed the homotopy theory of N.

Different model categories can give rise to the same homotopy theories, and indeed there are many results along these lines. For instance, for both the category of simplicial sets, $\text{Sets}^{\Delta^{op}}$, and the category of topological spaces, Top, a closed model structure has been specified [16]. The geometric realization functor of J. Milnor, $| \, |$: $\text{Sets}^{\Delta^{op}} \longrightarrow \text{Top}$, [15], and the singular functor, Sing: $\text{Top} \longrightarrow \text{Sets}^{\Delta^{op}}$ form an adjoint pair which conserve these structures. Furthermore, the unit and counit of adjunction are homotopy equivalences. Thus, the functors $\text{Ho Sets}^{\Delta^{op}} \underset{\text{Sing}}{\overset{| \, |}{\rightleftarrows}} \text{Ho Top}$, induced by the original adjoint pair, yield an equivalence of homotopy categories.

For the category of small categories, Cat, D.M. Latch [11] showed that Ho Cat, a suitable category of fractions, is equivalent to Ho Sets$^{\Delta^{op}}$, and later, together with R. Fritsch [12], that this equivalence can be realized by adjoint functors. R.W. Thomason [19] used their results to prove that Cat has a closed model structure in the sense of Quillen.

Prior to Thomason's description of the closed model structure for Cat, the notion of a homotopy theory for Cat had been studied within the following framework. By composing the nerve functor, which goes from the category of small categories to the category of simplicial sets, with the geometric realization functor, one gets the classifying functor $B:\text{Cat} \longrightarrow \text{Top}$, which is the usual classifying space functor for a small category which happens also the be a group. By the homotopy type, or indeed the homology, of a category, N, we mean that of its classifying space, BN.

Some of the important results for the homotopy theory of Cat were presented by Quillen [17]. For instance, a natural transformation $T:F \longrightarrow G$ induces a homotopy from BF to BG. Thus the notion of two functors being homotopic to each other can be defined internally within Cat.

Quillen noted that both the homology with local coefficients and the fundamental group of a category can be defined algebraically; that is, directly from the category without resorting to its classifying space. In this paper, Quillen introduced the construction of a homotopy fibre, which then yields the standard long exact homotopy sequence for fibrations.

In 1976, D.M. Kan and W.P. Thurston [9] proved that for any path-connected space X, there is a homology equivalence $BG \longrightarrow X$, for some discrete group G. The kernel of the surjection $G \cong \Pi_1 BG \longrightarrow X$ is a perfect, normal subgroup P, such that the Quillen $(\ )^+$ construction applied to the pair (G,P) yields a space having the same weak homotopy type as X. One of the central ideas in their proof is the notion of embedding a group in a homological cone, that is, a group whose homology is trivial. Their methods produce an uncountable cone for any group, thus in their construction G is always uncountable even when X is a finite CW-complex.

They posed the question as to whether given a finite CW-complex, X, one might produce a finite CW-complex that is a K(G,1) homologically equivalent to X. This amounts to producing a group G as above whose classifying space has the homotopy type of a finite CW-complex. Such a group is said to be geometrically finite.

Following this paper, G. Baumslag, E. Dyer, and A. Heller [1], showed that the category of pointed, connected CW-complexes with homotopy classes of maps is equivalent to a category of fractions of the category of pairs (G,P) where

G is a group, P a perfect, normal subgroup. This equivalence provides yet another category in which to do homotopy theory.

In response to the question posed by Kan and Thurston, they showed that coning a group can be done so that the cone of the group is geometrically finite if G is, finitely generated if G is, or locally finite if G is. They then constructed a functor to do the same thing as the one of Kan and Thurston but effectively, that is, providing a presentation for the groups associated with the spaces and taking finite CW-complexes to geometrically finite groups.

In a later paper, Heller [4] named the category of fractions whose objects are pairs (G,P), as above, the category of topogenic groups. He also defined a category for unpointed theory, that of topogenic groupoids. Within this context, he pursued the notion of doing homotopy along the lines of Quillen's model category theory, although it is clear that there is only a fragment of a model category structure in the algebraic sense.

Although every homology type of a path-connected space can be realized as a K(G,1), clearly this is not true about the homotopy type. However, along the lines of Kan and Thurston, D. McDuff [14] showed that every path-connected space has the weak homotopy type of the classifying space of a monoid and, in fact, that the monoid can be chosen so that its maximal subgroup is homologically equivalent to the monoid and hence, of course, to the original space.

With these various equivalences of categories, one is tempted to ask in which of these it might be most fruitful to do homotopy theory. The interest in doing homotopy theory in Cat, so far from the very geometric setting of Top, stems from the definition of higher K-groups, $K_n(N) = \Pi_{n+1}(BQN)$, where QN is a category constructed from N, an exact category in the sense of algebraic K-theory.

A category in which there is well-developed theory might be useful when trying to do homotopy theory. Thus one might attempt to do homotopy theory of Cat by "going to" moniods, since every homotopy type of path-connected spaces occurs as the classifying space of a monoid. In theory, one can always do this by going through Top, i.e., by applying B, finding a simplicial complex with the same homotopy type and then using McDuff's construction. However, the question was posed: Given a small category, can one find an effective construction within the category of small categories which yields a monoid of the same homotopy type?

In this paper, this question is answered affirmatively. The construction is totally algegraic and effective. Given a category presented by generators and relations, we get a monoid of the same homotopy type presented in generators and relations. Furthermore, for a finitely presented category, one gets a finitely presented monoid, and this construction is essentially algorithmic.

Section Two

PRELIMINARIES

The first two theorems in this paper are generalizations of results in McDuff's paper, "On the Classifying Space of Discrete Monoids" [14]. In her paper, McDuff proves that for a push-out in the category of monoids

$$\begin{array}{ccc} L & \longrightarrow & N^1 \\ \downarrow & & \downarrow \\ N^2 & \longrightarrow & N^1 *_L N^2 \end{array}$$

where the initial legs satisfy a certain freeness condition, that the classifying space functor preserves homotopy push-outs; i.e., the induced map $BN^1 \cup_{BL} BN^2 \longrightarrow B\left(N^1 *_L N^2\right)$ is a homotopy equivalence.

In order to do this, the first thing that she does is to prove that in the presence of right-freeness, there is a word theorem for monoids analogous to the one in group theory for free products with amalgamation [10]. Since groups and monoids can be thought of as small categories with one object, their elements being morphisms from that object to itself, these word theorems can be thought of as special cases of a more general theorem about categories.

We generalize McDuff's result for monoids to the category, Cat_I, of small categories all having a fixed object set, I, with morphisms that are the identity on objects. We consider the same push-out diagram as above, with L, N^1, N^2 now being objects of Cat_I, where i_1, i_2 are right-free inclusions, as defined below, and the "words" are morphisms from one object to another in the push-out category $N^1 *_L N^2$.

After this we show this result implies that under the above hypotheses, we again get $BN^1 \cup_{BL} BN^2 \xrightarrow{\simeq} B\left(N^1 *_L N^2\right)$.

These results provide the tools necessary to make the desired construction. This is done by first constructing a mapping cylinder for any morphism $F:L \longrightarrow N$ in Cat_I. Using this we get the final result by embedding a category in a semi-groupoid of the same homotopy type. (Following the usage of McDuff, a semi-groupoid is a small category for which there is at least one isomorphism between any two objects.) Once this is done, we have essentially accomplished what we have set out to do. There is a monoid at each vertex (object), each one isomorphic to any other, and each one having the same homotopy type as the semi-groupoid. If the semi-groupoid is presented in terms of generators and relations, we may select a set of isomorphisms, one between a fixed object and any other object, and pull back the generators and relations via these selected morphisms, yielding a monoid presented in terms of generators and relations.

The nature of the following proofs, particularly the first one, is technical and detailed. In an attempt to make them easier to follow, some background material is presented here.

As the reader may have noticed, a push-out will be represented by a commuting square with a little "corner" added to the lower right-hand position.

Let $N^1 \xrightarrow{i_1} L \xrightarrow{i_2} N^2$ be as above. For convenience, we introduce the category, $Sets^{L,N^1,N^2}$. An object $\mathcal{F} \in Sets^{L,N^1,N^2}$ consists of functors $F \in Sets^L$, $F_1 \in Sets^{N^1}$, $F_2 \in Sets^{N^2}$, such that $F = i_1^* F_1 = i_2^* F_2$. For $\mathcal{F},\mathcal{G} \in Sets^{L,N^1,N^2}$, a morphism $\mathcal{F}$ consists of natural transformations $T:F \longrightarrow G$, $T_1:F_1 \longrightarrow G_1$, and $T_2:F_2 \longrightarrow G_2$, where $T = i_1^* T_1 = i_2^* T_2$. For simplicity, we say $F \in Sets^{L,N^1,N^2}$ and T is a natural transformation from F to G.

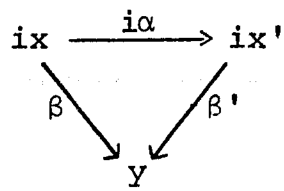
Given two categories, M and N, with functors $F:M \longrightarrow N$ and $G:N \longrightarrow M$, we say F and G are an adjoint pair of there is a function, ϕ, which assigns to each pair of objects, $x \in Ob\ M$, $y \in Ob\ N$, a bijection $\phi_{x,y}:N(Fx,y) \longrightarrow M(x,Gy)$, which is natural in x and y. We say F is left adjoint to G or G is the right adjoint to F. The naturality of this bijection means that for $h:x' \longrightarrow x$, $k:y \longrightarrow y'$, both of the following diagrams commute.

$$\begin{array}{ccc} N(Fx,y) & \xrightarrow{\phi} & M(x,Gy) \\ {\scriptstyle (Fh)^*}\downarrow & & \downarrow{\scriptstyle h^*} \\ N(Fx',y) & \xrightarrow{\phi} & M(x',Gy) \end{array} \quad \text{and} \quad \begin{array}{ccc} N(Fx,y) & \xrightarrow{\phi} & M(x,Gy) \\ {\scriptstyle k_*}\downarrow & & \downarrow{\scriptstyle (Gk)_*} \\ N(Fx,y') & \xrightarrow{\phi} & M(x,Gy') \end{array}$$

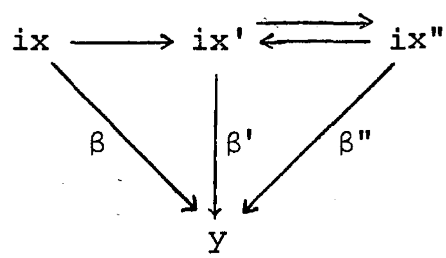
We exploit this definition extensively. In particular, for $f \in N(Fx,y)$, then $h^* \circ \phi(f) = \phi f \circ h$ and $\phi \circ (Fh)^* \circ f = \phi \circ (f \circ Fh)$ so, by the commutativity of the left-hand square, we get $\phi(f \circ Fh) = \phi f \circ h$. Similarly, the right-hand square yields $\phi(k \circ f) = Gk \circ \phi f$.

For two maps, $f \in N(Fx,y)$ and $g \in M(x,Gy)$, which correspond to each other by the adjunction ϕ, i.e., $g = \phi f$, we will use Mac Lane's terminology and refer to g as the right adjunct of f, and f as the left adjunct of g. For notational convenience, we will denote the left adjunct of g by $\hat{g}$. We will be using Mac Lane's notation for the unit of adjunction, $\eta : I \longrightarrow Gf$, and the co-unit of adjunction, $\varepsilon : FG \longrightarrow I$. Whenever necessary, these maps will be subscripted to indicate, in an obvious way, the adjoint pair to which they belong.

Given a category A and a functor $i:L \longrightarrow N$ in Cat then i induces a map between the functor categories, $i^*:A^N \longrightarrow A^L$, A^N and A^L denoting the categories of functors from N,L respectively into A. If A is an abelian category admitting all colimits (co-complete), there exists a left adjoint to i^*, $Lan_i: A^L \longrightarrow A^N$, called the left Kan extension along i. For $T: L \longrightarrow Ab$, the left Kan extension of T, Lan T, is constructed as follows. Let $y \in Ob\ N$, then form the comma category, $(i\downarrow y)$, consisting of objects, (x,β), where $ix \xrightarrow{\beta} y$, $x \in Ob\ L$. Morphisms in $(i\downarrow y)$ consist of maps $x \xrightarrow{\alpha} x'$ such that the following commutes:



We now define a functor $T_y: (i\downarrow y) \longrightarrow A$, by $T_y(x,\beta) = T(x)$, $T_y(\alpha) = T(\alpha)$. (Pictorially we have



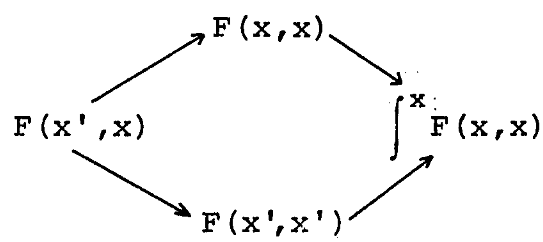
that is, various objects, ix, ix', ix", etc., with maps hitting y and compatible maps between them. We apply T to the x, x', x"'s, etc., and to the maps of the sort $x \xrightarrow{\alpha} x'$.) Then we take the colimit in A.

Thus, we can define $(Lan_i T)(y) = \text{colim } T(y)$ in A, which has all colimits. For $y \xrightarrow{\gamma} y'$, γ induces a morphism $Lan_i\alpha: (Lan_i T)(y) \longrightarrow (Lan_i T)(y')$.

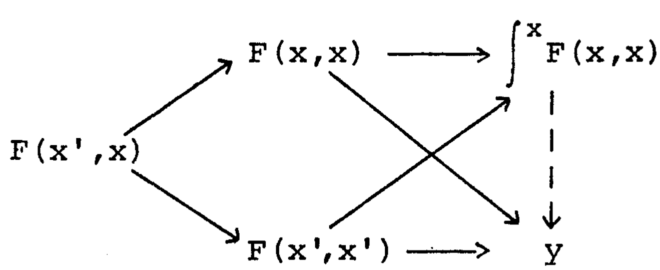
We make extensive use of the left Kan extension and the adjointness properties of the pair Lan_i, i^*.

Whenever we use the Kan extension, the target category of the functor in question is either Sets, or Ab, the category of abelian groups. Both of these categories are cocomplete so that the Kan extensions discussed do, in fact, exist.

For a functor $F: L^{op} \times L \longrightarrow N$, we describe the notion of the coend of F, denoted $\int^x F(x,x)$. The coend is an object of N such that whenever $x \xrightarrow{\alpha} x'$ in L, we have the commuting diagram



The coend is universal in the sense that given the commuting diagram



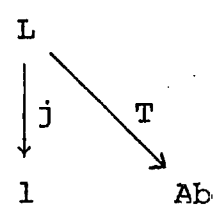
for y in N, there is a unique morphism $\int^x f(x,x) \longrightarrow y$, such that the diagram together with this morphism still commutes.

The notion of coends generalizes such familiar ideas as the tensor product of modules over a ring or the usual geometric realization.

Under mild restrictions [13], we can use the theory of coends to obtain a useful and revealing expression for $\text{Lan}_i T$, when evaluated at an object x in N. Taking as our functor $N(z,\ ) \times T : L^{op} \times L \longrightarrow N$, we have $(\text{Lan}_i T)(x) = \int^x N(z,x) \times T(z)$.

As mentioned in the previous section of this paper, Π_1 and H_n of a category can be calculated algebraically. To obtain $\Pi_1(L,x) = \Pi_1(BL,x)$ directly form the category L, form the category of fractions $G = L\left[\text{ArL}^{-1}\right]$, the groupoid obtained from L by formally adjoining the inverses of all the morphisms, and making the appropriate identifications [3]. Then $\Pi_1(L,x) = G_x$, the vertex group of G at x.

To calculate the homology of L, algebraically [5], consider the following diagram



where 1 is the trivial category, T a functor from L to Ab. Starting with T, take a projective resolution; PT:

$$\cdots\cdot\ P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow T$$

Apply Lan_j to get $Lan_j PT$:

$$\cdots \longrightarrow Lan_j P_2 \longrightarrow Lan_j P_1 \longrightarrow Lan_j P_0 \longrightarrow Lan_j T$$

Take the homology of $Lan_j PT$, and set $H_n(L,T) = H_n(Lan_j PT)$. If T is a morphism-inverting functor, it constitutes a system of local coefficients corresponding to $\Pi_1 : BL \xrightarrow{A_T} Ab$, and we note without proof that $H_n(BL, A_T) = H_n(L,T)$ [18].

Section Three

A WORD THEOREM FOR PUSH-OUTS IN Cat_I

The notion of "right-free" inclusion we introduce below insures that for two small categories in Cat_I, one included in the other, the subcategory induces a decomposition of the larger category into disjoint "cosets" in a manner analogous to that for groups. Each "coset" is to be in bijective correspondence with any other, and there must exist a fixed set of "coset representatives" such that any element of the larger category can be written in a canonical way, as the composition of a "coset representative" with an element from the smaller category. (Note: The inclusion of a sub-monoid into a monoid is not a right-free inclusion, in general.) [20]

In the case where the categories are groups, the definition of right-free inclusion reduces to the statement that one group is a subgroup of another. The word theorem for the free product with amalgamation, which our theorem generalizes, fails unless the group over which the amalgamation takes place lies as a subgroup or is injected into the other groups.

If L is a subgroup of N, then $\mathbb{Z}(N)$ is a right-free module over $\mathbb{Z}(L)$, that is, $\mathbb{Z}(N) \otimes_{\mathbb{Z}(L)} \_\_$ is left exact (preserves injections). Analogously, if $L \longrightarrow N$ is right-free, then the left Kan extension along i, which can be thought of as a generalization of the tensor product above, is left exact. We use this left exactness first in the proof of the word theorem and later in a "change of rings" type argument.

<u>Definition</u>: Given two categories, L, N, in Cat_I, $i:L \longrightarrow N$ an inclusion, we say i is a *right-free inclusion* if for all $x,y \in I$,

$$N(x,y) \xleftarrow{\sim} L(x,y) \sqcup \coprod_{z \in I} \bar{N}(z,y) \times L(x,z)$$

where $\bar{N}(z,y)$ $N(z,y)$ is a fixed set for each $z,y \in I$, and $1 \notin N(x,x)$ for all $x \in I$. The arrow maps $L(x,y)$ into $N(x,y)$ by inclusion and $\bar{N}(z,y) \times L(x,z)$ into $N(x,y)$ by composition. We will let $\bar{N} = \coprod_{x,y \in I} N(x,y)$ and refer to $\bar{N}$ as the set of *generators* of N over L.

The composition of two right-free inclusions is right-free. Given $L \xrightarrow{i} M \xrightarrow{j} N$, i, j, right-free inclusions, $\bar{M}$ and $\bar{N}$, the set of generators of M over L, and N over M, respectively, then

$$N(x,y) \simeq M(x,y) \sqcup \left[\coprod_z \overline{N}(z,y) \times M(x,z)\right]$$

$$\simeq L(x,y) \sqcup \left[\coprod_z \overline{M}(z,y) \times L(x,z)\right]$$

$$\sqcup \left[\overline{N}(z,y) \times \left(L(x,z) \sqcup \coprod_{z'} \overline{M}(z',z) \times L(x,z')\right)\right]$$

$$\simeq L(x,y) \sqcup \left[\coprod_z M(z,y) \times L(x,z)\right] \sqcup \left[\coprod_z N(z,y) \times L(x,z)\right]$$

$$\sqcup \left[\coprod_z N(z,y) \times \coprod_{z'} M(z',z) \times L(x,z')\right]$$

By interchanging the dummy variables z and z' in the third term, we see that we can factor out the set L(x,z) on the right yielding

$$N(x,y) \simeq L(x,y) \sqcup \left[\coprod_z \overline{M}(z,y) \sqcup \coprod_z \overline{N}(z,y) \sqcup \left(\coprod_{z'} \overline{N}(z',y) \times \coprod_z \overline{M}(z,z')\right)\right] \times L(x,z),$$

the set in brackets being the set of generators of N over L. Every element of M is, of course, in N by inclusion and $(\overline{m},\overline{n}) \in \coprod_{z'} N(z',y) \times \coprod_z M(z,z')$ actually represents $\overline{m}\overline{n} \in N$, a generator in N over L.

We can now state the first theorem.

<u>Theorem 1</u>: Given the following push-out diagram,

$$\begin{array}{ccc} L & \xrightarrow{i_1} & N^1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ N^2 & \longrightarrow & N^1 *_L N^2 \end{array}$$

where i_1, i_2 are both right-free inclusions, then every morphism of the push-out can be represented uniquely by an (alternating) sequence $(\bar{n}_\lambda, \bar{n}_{\lambda-1}, \ldots, \bar{n}_1, \ell)$, $\lambda \geqslant 0$, where ℓ is a morphism in L, and $\bar{n}_1, \bar{n}_2, \ldots, \bar{n}_\lambda$ come alternately from $\bar{N}^1$ and $\bar{N}^2$, in such a way as to be composable in $N^1 *_L N^2$, i.e., $x \xrightarrow{\ell} z_1 \xrightarrow{\bar{n}_1} z_2 \xrightarrow{\bar{n}_2} \cdots \xrightarrow{\bar{n}_{\lambda-1}} z_\lambda \xrightarrow{\bar{n}_\lambda} y$.

<u>Proof</u>: Due to the length and technical nature of this proof, it has been separated into five steps.

<u>Step 1</u>: We construct a functor $W_x \in \text{Sets}^{L,N^1,N^2}$, which can be thought of as "maps out of x"; when evaluated at y, it can be thought of as a "set of words" representing morphisms from x to y.

Starting with $N^1 \xleftarrow{i_1} L \xrightarrow{i_2} N^2$ as in the hypothesis, we construct the succession of functors indicated in the following diagram:

$$\begin{array}{ccccccccccccccc} & & A_x^1 & & & & & & A_x^3 & & & & & & \\ W_x^0 & \nearrow & & \searrow & W_x^1 & & & W_x^2 & & \searrow & W_x^3 & & & & \cdots \\ & & & & & \searrow & B_x^2 \nearrow & & & & & \searrow & B_x^4 & \nearrow & \end{array}$$

with the functors $W_x \in \mathrm{Sets}^L$, $A_x^{2\mu-1} \in \mathrm{Sets}^{N^1}$, $B_x^{2\mu} \in \mathrm{Sets}^{N^2}$. For $x \in I$, we define the functors:

$W_x^0 = L(x,\_)$, the representable functor.

$A_x^1 = \mathrm{Lan}_{i_1} W_x^0$.

$W_x^1 = i_1^* A_x^1 = i_1^* \mathrm{Lan}_{i_1} W_x^0$.

From the unit of the adjunction for the adjoint pair Lan_{i_1}, i_1^*, we get $\eta_{i_1} : W_x^0 \longrightarrow i_1^* \mathrm{Lan}_{i_1} W_x^0 = W_x^1$. Set $\gamma^0 = \eta_{i_1} : W_x^0 \longrightarrow W_x^1$.

$B_x^2 = \mathrm{Lan}_{i_2} W_x^1$.

$W_x^2 = i_2^* B_x^2 = i_2^* \mathrm{Lan}_{i_2} W_x^1$

Set $\gamma^1 = \eta_{i_2} : W_x^1 \longrightarrow i_2^* \mathrm{Lan}_{i_2} W_x^1 = W_x^2$

A_x^3 is formed from the push-out diagram in Sets^{N^1}:

$$\begin{array}{ccc} Lan_{i_1} i_1{}^* A_x{}^1 & \xrightarrow{\;Lan_{i_1}\gamma^1\;} & Lan_{i_1} i_2{}^* B_x{}^2 \\ {\scriptstyle \varepsilon_{i_1}}\downarrow & & \downarrow{\scriptstyle \hat{\gamma}^2} \\ A_x{}^1 & \xrightarrow{\;\alpha^1\;} & A_x{}^3 \end{array}$$

$W_x{}^3 = i_1{}^* A_x{}^3$.

We take the left adjunct of γ^2 to get

$\gamma^2 : W_x{}^2 \longrightarrow i_1{}^* A_x{}^3 = W_x{}^3$.

$B_x{}^4$ is formed analogously in $Sets^{N^2}$:

$$\begin{array}{ccc} Lan_{i_2} i_2{}^* B_x{}^2 & \xrightarrow{\;Lan_{i_2}\gamma^2\;} & Lan_{i_2} i_1{}^* A_x{}^3 \\ {\scriptstyle \varepsilon_{i_2}}\downarrow & & \downarrow{\scriptstyle \hat{\gamma}^3} \\ B_x{}^2 & \xrightarrow{\;\beta^2\;} & B_x{}^4 \end{array}$$

$W_x{}^4 = i_2{}^* B_x{}^4$.

As before, we get $\gamma^3 : W_x{}^3 \longrightarrow i_2{}^* B_x{}^4 = W_x{}^4$, the left adjunction to $\hat{\gamma}^3$.

In general, $A^{2\mu+1}$, $\mu \geqslant 1$ comes from the push-out in $Sets^{N^1}$ shown in Diagram 3.1 below:

Diagram 3.1

$$\begin{array}{ccc} \mathrm{Lan}_{i_1} W_x^{2\mu-1} & \xrightarrow{\mathrm{Lan}_{i_1}\gamma^{2\mu-1}} & \mathrm{Lan}_{i_1} W_x^{2\mu} \\ \Big\downarrow \varepsilon_{i_1} & & \Big\downarrow \hat{\gamma}^{2\mu} \\ A_x^{2\mu-1} & \xrightarrow{\alpha^{2\mu-1}} & A_x^{2\mu+1} \end{array}$$

$$W_x^{2\mu+1} = i_1{*}A_x^{2\mu+1},\ \gamma^{2\mu}: W_x^{2\mu} \longrightarrow W_x^{2\mu+1}.$$

In the same manner, $B_x^{2\mu}, \mu > 1$, comes from the push-out in Sets^{N^2} shown in Diagram 3.2:

Diagram 3.2

$$\begin{array}{ccc} \mathrm{Lan}_{i_2} W_x^{2\mu-2} & \xrightarrow{\mathrm{Lan}_{i_1}\gamma^{2\mu-1}} & \mathrm{Lan}_{i_2} W_x^{2\mu-1} \\ \Big\downarrow \varepsilon_{i_2} & & \Big\downarrow \hat{\gamma}^{2\mu-1} \\ B_x^{2\mu-2} & \xrightarrow{\beta^{2\mu-2}} & B_x^{2\mu} \end{array}$$

$$W_x^{2\mu} = i_2{*}B_x^{2\mu},\ \gamma^{2\mu-1}: W_x^{2\mu-1} \longrightarrow W_x^{2\mu}.$$

Note that the bottom map of Diagram 3.1 gives a map $A_x^{2\mu-1} \xrightarrow{\alpha^{2\mu-1}} A_x^{2\mu+1}$, so that we have $A_x^1 \xrightarrow{\alpha^1} A_x^3 \xrightarrow{\alpha^3} A_x^5 \longrightarrow \cdots$ in Sets^{N^1}. Similarly, Diagram 3.2 gives $B_x^{2\mu} \xrightarrow{\beta^{2\mu}} B_x^{2\mu+2}$ to yield $B_x^{\,2} \xrightarrow{\beta^2} B_x^{\,4} \xrightarrow{\beta^4} B_x^{\,6} \xrightarrow{\beta^6} \cdots$ in Sets^{N^2}. From our construction of W_x^{λ}'s we have $W_x^{\,0} \xrightarrow{\gamma^0} W_x^{\,1} \xrightarrow{\gamma^1} W_x^{\,2} \xrightarrow{\gamma^2} \cdots$ in Sets^{L}.

We define the following functors:

$$A_x = \operatorname*{colim}_{\mu \to \infty} A_x^{\,2\mu+1} \quad \text{in } \mathrm{Sets}^{N^1}$$

$$B_x = \operatorname*{colim}_{\mu \to \infty} B_x^{\,2\mu} \quad \text{in } \mathrm{Sets}^{N^2}$$

$$W_x = \operatorname*{colim}_{\mu \to \infty} W_x^{\,\lambda} \quad \text{in } \mathrm{Sets}^{L}$$

Observe that the commutativity of Diagram 3.1 implies, by the naturality properties of adjoint functors, the commutativity of the following diagram:

$$\begin{array}{ccccccc}
 & & W_x^{\,2\mu-1} & \xrightarrow{\gamma^{2\mu-1}} & W_x^{\,2\mu} & & \\
 & & \Big\downarrow \mathrm{Id} & & \Big\downarrow \gamma^{2\mu} & & \\
i_1{}^{*}A^{2\mu-1} & = & W_x^{\,2\mu-1} & \xrightarrow{i_1\alpha^{2\ -1}} & W_x^{\,2\mu+1} & = & i_1{}^{*}A_x^{\,2\mu+1}
\end{array}$$

Therefore $i_1{}^{*}\alpha^{2\mu-1} = \gamma^{2\mu} \circ \gamma^{2\mu-1} : W_x^{\,2\mu-1} \longrightarrow W_x^{\,2\mu+1}$. Thus $W_x = \operatorname*{colim}_{\mu \to \infty} W_x^{\,2\ +1} = \operatorname*{colim}_{\mu \to \infty} i_1^{*} A^{2\ +1} = i_1^{*} A_x$. (The functor i_1^{*} is left adjoint to the right Kan extension, and therefore commutes with colimits.)

An entirely analogous argument yields $W_x = \operatorname{colim}_{\mu \to \infty} W_x{}^2$ $= \operatorname{colim}_{\mu \to \infty} i_2{}^* B_x{}^2 = i_2{}^* B_x{}^2 = i_x{}^* B_x$, so W_x is in Sets^{L,N^1,N^2}.

Step 2: We prove the following Yoneda-type lemma.

Lemma: Let $F \in \text{Sets}^{L,N^1,N^2}$, $x \in I$, then given $w_0 \in F(x)$, there exists a unique natural transformation $T: W_x \longrightarrow F$ in Sets^{L,N^1,N^2} such that $[1] \in W_x(x)$ is sent to $w_0 \in F(x)$ by T. In other words, $\text{n.t.}(W_x, F) \longrightarrow F(x)$. (n.t. = natural transformations.)

Proof: We have $1 \in W_x{}^0(x)$. Since $W_x{}^0$ is a representable functor, the Yoneda lemma tells us there exists a unique natural transformation $T^0: W_x{}^0 \longrightarrow F$ such that $1 \longrightarrow w_0$.

We now show that $T^0: W_x{}^0 \longrightarrow F$ in Sets^L determines a unique natural transformation $T: W_x \longrightarrow F$ in Sets^{L,N^1,N^2}. Since $T^0: W_x{}^0 \longrightarrow F = i_1{}^* F_1$, we take the adjunction of T^0 to get $S^1: A_x{}^1 = \text{Lan}_{i_1} W_x{}^0 \longrightarrow F_1$, where $S^1 = \hat{T}^0$. Applying $i_1{}^*$, we form $T^1: W_x{}^1 = i_1{}^* \text{Lan}_{i_1} W_x{}^0 \longrightarrow i_1{}^* F_1 = F$, where $T^1 = i_1{}^* S^1$ $= i_1{}^* T^0$. Because $T^1 \circ \gamma^0 = i_1{}^* \hat{T}^0 \circ \eta_1 = T^0$, the following diagram commutes.

$$\begin{array}{ccc} W_x{}^0 & \xrightarrow{\quad 0 \quad} & W_x{}^1 \\ & {\scriptstyle T^0}\searrow \quad \swarrow{\scriptstyle T^1} & \\ & F & \end{array}$$

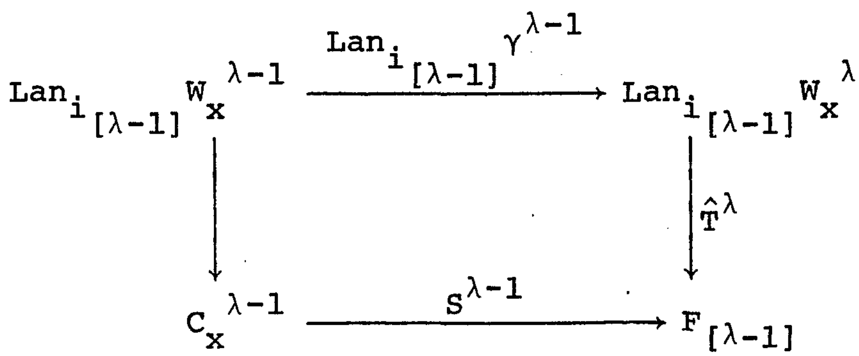
We have $\gamma^0 = \eta_1$ by definition. We repeat this series of steps using i_2. Take the left adjunct of $T^1 \colon W_x{}^1 \longrightarrow i_2^* F_2 = F$, to get $S^2 \colon B_x{}^2 = \mathrm{Lan}_{i_2} W_x{}^1 \longrightarrow F_2$, where $S^2 = \hat{T}^1$. Applying i_2^* gives $T^2 \colon W_x{}^2 = i_2{}^* B_x{}^2 \longrightarrow i_2{}^* F = F$, where $T^2 = i_2{}^* S^2 = i_2{}^* T^1$. Since $T^2 \circ \gamma^1 = i_2{}^* \hat{T}^1 \circ \eta_2 = T^1$, the following diagram commutes.

$$\begin{array}{ccc} W_x{}^1 & \xrightarrow{\quad \gamma^1 \quad} & W_x{}^2 \\ & {\scriptstyle T^1}\searrow \quad \swarrow{\scriptstyle T^2} & \\ & F & \end{array}$$

At this point the construction must change to accomodate the fact that the successive $W_x{}^\lambda$'s are the result of a push-out construction. If we assume $T^\lambda \circ \gamma^{\lambda-1} = T^{\lambda-1}$, we have the following commuting square.

$$\begin{array}{ccc} W_x{}^{\lambda-1} & \xrightarrow{\quad \gamma^{\lambda-1} \quad} & W_x{}^\lambda \\ \Big\downarrow{\scriptstyle \mathrm{Id}} & & \Big\downarrow{\scriptstyle T^\lambda} \\ W_x{}^{\lambda-1} & \xrightarrow{\quad T^{\lambda-1} \quad} & F \end{array}$$

The naturality properties immediately imply that the following square commutes.



where $C_x = \begin{cases} A_x^{\lambda} & \lambda \text{ odd} \\ B_x^{\lambda} & \lambda \text{ even} \end{cases}$ and $[\lambda] = \begin{cases} 1 & \lambda \text{ odd} \\ 2 & \lambda \text{ even} \end{cases}$

We use this as the outer square of the following diagram to get the map $S^{\lambda+1}$:

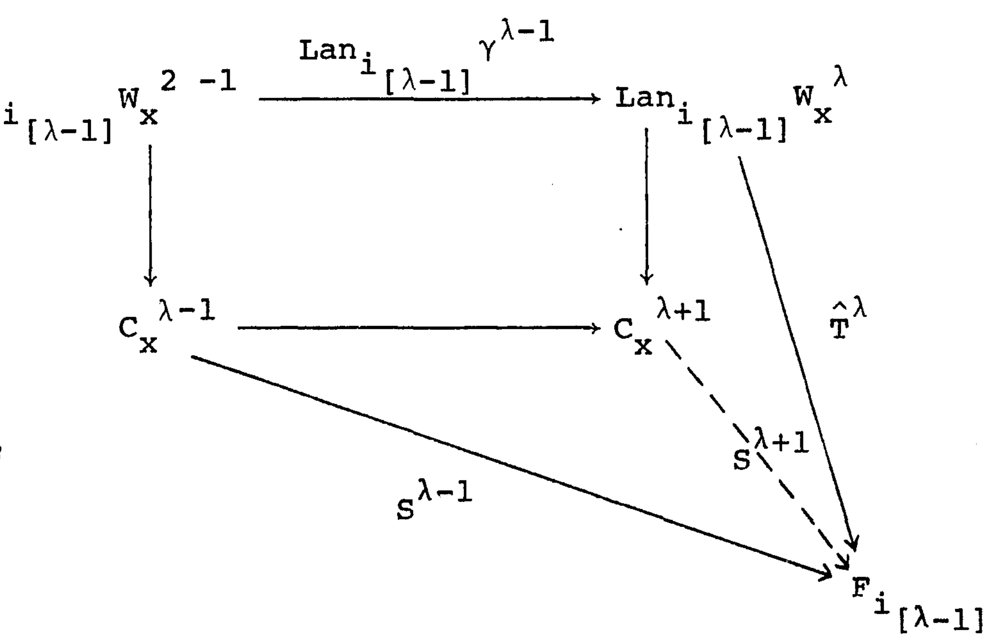
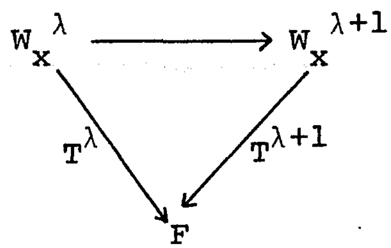
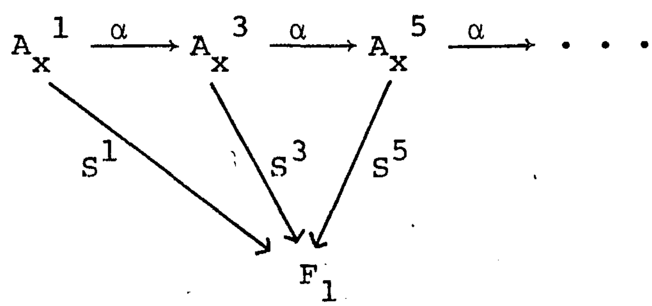


Diagram 3.3

As before, we let $T^{\lambda+1} = {i_i}^{*}_{[\lambda-1]} S^{\lambda+1}$, then the commuting triangle on the right of the above diagram gives $\hat{T}^{\lambda} = S^{\lambda+1} \circ \hat{\gamma}^{\lambda}$, which implies $T^{\lambda} = {i_{i_\lambda}}^{*} S^{\lambda+1} \circ \gamma^{\lambda} = T^{\lambda+1} \circ \gamma^{\lambda}$, by the naturality properties of adjoints. We now have a sequence of natural transformations, $T^{\lambda}: W_x \longrightarrow F$, such that the following diagram commutes, and we set $T = \operatorname{colim}_{\lambda \to \infty} T^{\lambda}$.

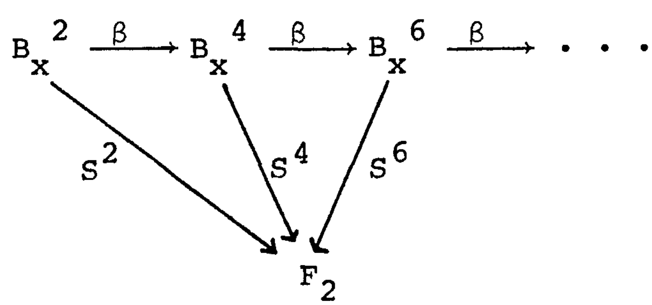


Also, note that from Diagram 3.3, we get the following commuting diagrams:



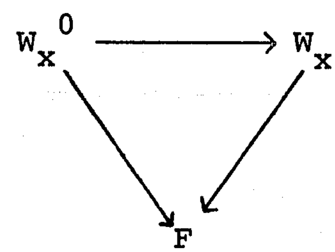
(λ even)

and

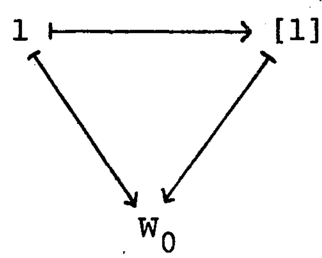


Set $S_1 = \operatorname*{colim}_{\mu \geqslant 0} S^{2\mu+1}$ and $S_2 = \operatorname*{colim}_{\mu \geqslant 1} S^{2\mu}$. Then $i_1^* S_1 = i_2^* S_2$ $= T$, and $T: W_x \longrightarrow F$ in Sets^{L,N^1,N^2}.

Thus we have shown that $T^0: W_x{}^0 \longrightarrow F$ determines a unique natural transformation $T: W_x \longrightarrow F$ where the following diagram commutes.



Thus we observe that



This concludes the proof of the lemma.

Step 3: For $x,y \in I$, we have W_x and W_y in Sets^{L,N^1,N^2}. If w_0 is any element in $W_x(y)$, then by the above lemma, there is a unique transformation $T_{w_0}: W_y \longrightarrow W_x$ such that $[1] \longmapsto w_0$. We now construct a category, $\mathbb{D}$, in Cat_I. For each $x,y \in I$, we set $\mathbb{D}(x,y)$ equal to the set of natural transformations from W_x to W_y. Thus by the lemma, we have $\mathbb{D}(x,y) = \text{n.t.}(W_x,W_y) = W_x(y)$.

Step 4: At this point, we show that in the presence of right freeness, $W_x(y) = \{(\bar{n}_\lambda, \bar{n}_{\lambda-1}, \ldots, \bar{n}_1, \ell) \mid x \xrightarrow{\ell} z_1 \xrightarrow{\bar{n}_1} z_2 \longrightarrow \cdots \longrightarrow z \xrightarrow{\bar{n}_\lambda} y,\ \lambda \geqslant 0\}$ where the n's come alternately from $\bar{N}^1$ and $\bar{N}^2$. In other words, each element in $W_x(y)$ has a unique canonical representative. We do this explicitly for $W_x{}^0(y)$, $W_x{}^1(y)$, $W_x{}^2(y)$, and $W_x{}^3(y)$, then proceed by induction. Finally, $W_x(y) = \operatorname*{colim}_{\lambda \to \infty} W_x{}^\lambda(y)$.

First we observe that given $i: L \longrightarrow N$, i a right-free inclusion, $\bar{N}$, the set of generators over L, and $F \in \mathrm{Sets}^L$, we may exploit the bijection

$$N(x,y) \xleftarrow{\sim} L(x,y) \sqcup \coprod_{z'} N(z',y) \times L(x,z')$$

for $x,y \in I$, given by the definition of right-free inclusion. This enables us to write $(\mathrm{Lan}_i\ F)(y)$ in a particularly simple simple way. Note that $\bar{N}(x,\_)$ is in $\mathrm{Sets}^{L^{op}}$ and proceed:

$$\begin{aligned}(\mathrm{Lan}_i\ F)(y) &= \int^z N(z,y) \times F(z) \\ &= \int^z \Big[L(z,y) \sqcup \coprod_{z'} \bar{N}(z',y) \times L(z,z')\Big] \times F(z) \\ &= \int^z L(z,y) \times F(z) \sqcup \Big[\coprod_{z'} \bar{N}(z',y) \times L(z,z') \times F(z)\Big],\end{aligned}$$

by the definition of right-free inclusion and the fact that colimits commute.

The left Kan extension along an identity map leaves the functor unchanged so $\int^{z} L(z,y) \times F(z)$ is merely F(y), and since $\overline{N}(z',y)$ contains no variable of integration, we pull it out from under the coend (integral) sign. Thus,

$$(\mathrm{Lan}_i\, F)(y) = F(y) \sqcup \left[\coprod_{z'} \overline{N}(z',y) \times \int^{z} L(z,z') \times F(z) \right]$$

$$= F(y) \sqcup \left[\coprod_{z'} \overline{N}(z',y) \times F(z') \right]$$

Thus we have

$$(\mathrm{Lan}_i\, F)(y) = F(y) \sqcup \left[\coprod_{z'} \overline{N}(z',y) \times F(z') \right].$$

It is easy to see this implies Lan_i preserves injections whenever i is a right-free inclusion. We will use this simplified expression for $(\mathrm{Lan}_i\, F)(y)$ to get the canonical "word" form for $W_x^{\lambda}(y)$, $\lambda = 0,1,2, \ldots$. We proceed:

$$W_x^{0}(y) = L(x,y)$$

$$W_x^{1}(y) = A_x^{1}(y) = (\mathrm{Lan}_{i_1} W_x^{0})(y)$$

$$= W_x^{0}(y) \sqcup \coprod_{z_1} \overline{N}^{1}(z_1,y) \times W_x^{0}(z_1)$$

$$W_x^{2}(y) = B_x^{2}(y) = (\mathrm{Lan}_{i_2} W_x^{1})(y)$$

$$= W_x^{1}(y) \sqcup \coprod_{z_2} \overline{N}^{2}(z_2,y) \times W_x^{1}(z_2)$$

(The z's are being labelled thusly to correspond to strings of morphisms $x \longrightarrow z_1 \longrightarrow z_2 \longrightarrow \cdots \longrightarrow z_\lambda \longrightarrow y$ in $W_x^{\lambda}(y)$, but are in fact, dummy variables.)

To simplify notation in the remaining part of Step 4, we let

$$U_x^{\;0} = W_x^{\;0}$$

$$U_x^{\;\lambda} = W_x^{\;\lambda} - W_x^{\;\lambda-1}, \quad \lambda \geqslant 1$$

Note that $U_x^{\;\lambda} \; \varepsilon \; \mathrm{Sets}^L$, $\lambda \geqslant 0$. From this point on, we proceed by induction.

The idea in both the initial step of the induction and the inductive step will be to reduce the push-out diagram in question to the following form.

$$\begin{array}{ccc} X & \longrightarrow & Y \sqcup Z \\ \big\downarrow & \ulcorner & \big\downarrow \\ Y & \longrightarrow & Y \sqcup Z \end{array}$$

In other words, by rewriting the upper right hand corner in the form $X \sqcup Z$, we immediately get an explicit expression for the push-out as $Y \sqcup Z$.

The first step in the inductive process is to obtain $W_x^{\;3}(y)$. We consider the following diagram:

$$\begin{array}{ccc}
(\mathrm{Lan}_{i_1} W_x{}^1)(y) & \xrightarrow{\ \mathrm{Lan}_{i_1}\gamma^1\ } & (\mathrm{Lan}_{i_1} W_x{}^2)(y) \\
\Big\downarrow{\scriptstyle \varepsilon_{i_1}} & & \Big\downarrow{\scriptstyle \hat{\gamma}^2} \\
W_x{}^1\ (y) & \xrightarrow{\ \gamma^1\ } & W_x{}^3\ (y)
\end{array}$$

Using the scheme mentioned above, we have $X = (\mathrm{Lan}_{i_1}\ W_x{}^1)(y)$, $Y = W_x{}^1(y)$. We use our expression for $W_x{}^2(y)$ to rewrite $(\mathrm{Lan}_{i_1}\ W_x{}^2)(y)$ as follows:

$$(\mathrm{Lan}_{i_1} W_x{}^2)(y) = \left(\mathrm{Lan}_{i_1}\left[W_x{}^1 \sqcup \coprod_{z_2} \bar{N}^2(z_2,\_) \times W_x{}^1(z_2)\right]\right)(y)$$

$$= (\mathrm{Lan}_{i_1} W_x{}^1)(y) \sqcup \left(\mathrm{Lan}_{i_1}\left[\coprod_{z_2} \bar{N}^2(z_2,\_) \times W_x{}^1(z_2)\right]\right)(y)$$

since left adjoints commute with coproducts. We now have $(\mathrm{Lan}_{i_1} W_x{}^2)(y)$ in the form $X \sqcup Z$, giving

$$Z = \left(\mathrm{Lan}_{i_1}\left[\coprod_{z_2} \bar{N}^2(z_2,\_\_) \times W_x{}^1(z_2)\right]\right)(y).$$

From this we get the following:

$$W_x{}^3(y) = Y \sqcup Z = W_x{}^1(y) \sqcup \left(\mathrm{Lan}_{i_1}\left[\coprod_{z_2} \bar{N}(z_2,\_) \times W_x{}^1(z_2)\right](y)\right)$$

$$= W_x{}^1(y) \sqcup \coprod_{z_2} \bar{N}^2(z_2,y) \times W_x{}^1(z_2)$$

$$\sqcup \coprod_{z_3} \bar{N}^1(z_3,y) \times \left[\coprod_{z_2} \bar{N}^2(z_2,z_3) \times W_x{}^1(z_2)\right]$$

$$= W_x{}^2 \sqcup \coprod_{z_3} \bar{N}^1(z_3,y) \times \left(W_x{}^2(z_3) - W_x{}^1(z_3)\right).$$

Using the notation, $U_x{}^2(y) = W_x{}^2(y) - W_x{}^1(y)$, we get

$$W_x{}^3(y) = W_x{}^2(y) \sqcup \left[\coprod_{z_3} \bar{N}^1(z_3,y) \times U_x{}^2(z_3)\right].$$

We now use this expression for $W_x{}^3(y)$ as the model for the inductive hypothesis:

$$W_x{}^{\lambda+1}(y) = W_x{}^{\lambda}(y) \sqcup \coprod_{z_{\lambda+1}} \bar{N}^{[\lambda+1]}(z_{\lambda+1},y) \times U_x{}^{\lambda}(z_{\lambda+1}).$$

We combine the arguments for the $A_x{}^{\lambda}$'s, and the $B_x{}^{\lambda}$'s by making the following simplifications. By construction $W_x{}^{2\mu-1} = i_1{*}A_x{}^{2\mu-1}$, and $W_x{}^{2\mu} = i_2{*}B_x{}^{2\mu}$, $\mu = 1, 2, \ldots$; this implies $W_x{}^{2\mu-1}(y) = A_x{}^{2\mu-1}(y)$ and $W_x{}^{2\mu}(y) = B_x{}^{2\mu}(y)$, since i_1 and i_2 are inclusions. Using this fact, the push-out diagrams (see Diagrams 3.1 and 3.2) used to construct $A_x{}^3$, $B_x{}^4$, $A_x{}^5$, etc. can be written identically when they are evaluated at y.

In general, given W_x^{λ}, $W_x^{\lambda+1}$, we obtain $W_x^{\lambda+2}$ from the following diagram:

$$\begin{array}{ccc} (\mathrm{Lan}_{i_{[\lambda]}} W_x^{\lambda})(y) & \longrightarrow & (\mathrm{Lan}_{i_{[\lambda]}} W_x^{\lambda+1})(y) \\ \downarrow & & \downarrow \\ W_x^{\lambda}(y) & \longrightarrow & W_x^{\lambda+2}(y) \end{array}$$

$$\text{where } [\lambda] = \begin{cases} 1 & \text{if } \lambda \text{ odd} \\ 2 & \text{if } \lambda \text{ even} \end{cases}$$

We use the same scheme as before:

$$X = (\mathrm{Lan}_{i_{[\lambda]}} W_x^{\lambda})(y)$$

$$Y = W_x^{\lambda}(y)$$

$$\begin{aligned} &(\mathrm{Lan}_{i_{[\lambda]}} W_x^{\lambda+1})(y) \\ &\quad = \mathrm{Lan}_{i_{[\lambda]}} \left[W_x^{\lambda} \sqcup \coprod_{z_{\lambda+1}} \overline{N}^{[\lambda+1]}(z_{\lambda+1}, \_\_) \times U_x^{\lambda}(z_{\lambda+1}) \right](y) \\ &\quad = (\mathrm{Lan}_{i_{[\lambda]}} W_x^{\lambda})(y) \\ &\qquad \left\{ \mathrm{Lan}_{i_{[\lambda]}} \left[\coprod_{z_{\lambda+1}} \overline{N}^{[\lambda+1]}(z_{\lambda+1}, \_\_) \times U_x^{\lambda}(z_{\lambda+1}) \right] \right\}(y) \\ &\quad = X \sqcup Z. \end{aligned}$$

Thus, $W_x^{\lambda+2}(y) = Y \sqcup Z$

$$= W_x^{\lambda}(y) \sqcup \left(\mathrm{Lan}_{i_{[\lambda]}} \left[\coprod_{z_{\lambda+1}} \bar{N}^{[\lambda+1]}(z_{\lambda+1}, \_\_) \times U_x^{\lambda}(z_{\lambda+1}) \right] \right)(y)$$

$$= W_x^{\lambda}(y) \sqcup \coprod_{z_{\lambda+1}} \bar{N}^{[\lambda+1]}(z_{\lambda+1}, y) \times U_x^{\lambda}(z_{\lambda+1})$$

$$\sqcup \coprod_{z_{\lambda+2}} \bar{N}^{[\lambda]}(z_{\lambda+2}, y) \times \coprod_{z_{\lambda+1}} \bar{N}^{[\lambda+1]}(z_{\lambda+1}, z_{\lambda+2}) \times U_x^{\lambda}(z_{\lambda+1})$$

Therefore $W_x^{\lambda+2}(y) = W_x^{\lambda+1}(y) \sqcup \coprod_{z_{\lambda+2}} \bar{N}^{[\lambda+2]}(z_{\lambda+2}, y) \times U_x^{\lambda+1}(z_{\lambda+2})$

and the induction is complete.

Since $W_x(y) = \operatorname*{colim}_{\lambda \geqslant 0} W_x(y) = {}_{\lambda \geqslant 0} W_x(y) = \coprod_{\lambda \geqslant 0} U_x(y)$, we we can see exactly what is in $W_x(y)$ by inspecting $U_x^{\lambda}(y)$, $\lambda \geqslant 0$.

$$U_x^{0}(y) = L(x,y) = \{ \ell \mid \ell : x \longrightarrow y;\ x,y \in I \}$$

$$U_x^{1}(y) = \left\{ (\bar{n}, \ell) \,\middle|\, \bar{n}_1 \in \bar{N}^1,\ \ell \in L,\ x \xrightarrow{\ell} z_1 \xrightarrow{\bar{n}_1} y \right\}$$

Assume $U_x^{1}(y) = \left\{ (\bar{n}_{\lambda-1}, \ldots, \bar{n}_1, \ell),\ \bar{n}_1 \in \bar{N}^2,\ \bar{n}_{\lambda-1} \in \bar{N}^{i_\lambda} \right\}$

$$\cup \left\{ (\bar{n}_{\lambda}, \ldots, \bar{n}_1, \ell),\ \bar{n}_1 \in \bar{N}^1,\ \bar{n}_{\lambda} \in \bar{N}^{i_\lambda} \right\}$$

Then, by definition, $U_x^{\lambda+1}(y) = W_x^{\lambda+1}(y) - W_x^{\lambda}(y)$

$$= \coprod_{z_{\lambda+1}} \bar{N}^{i[\lambda+1]}(z_{\lambda+1}, y) \times \left[W_x^{\lambda}(z_{\lambda+1}) - W_x^{\lambda-1}(z_{\lambda+1}) \right]$$

$$= \coprod_{z_{\lambda+1}} \overline{N}^{i}[\lambda](z_{\lambda+1},y) \times \Big[\Big\{(\overline{n}_\lambda,\ldots,\overline{n}_1,\ell),\overline{n}_1 \varepsilon \overline{N}, \overline{n}_\lambda \varepsilon \overline{N}^{i}[\lambda+2]$$

$$\cup\ (\overline{n}_{\lambda+1},\ldots,\overline{n}_1,\ell),\ \overline{n}_1 \ \varepsilon\ \overline{N}^1,\ \overline{n}_{\lambda+1}\ \varepsilon\ \overline{N}^{i}[\lambda+2]\Big\}\Big]$$

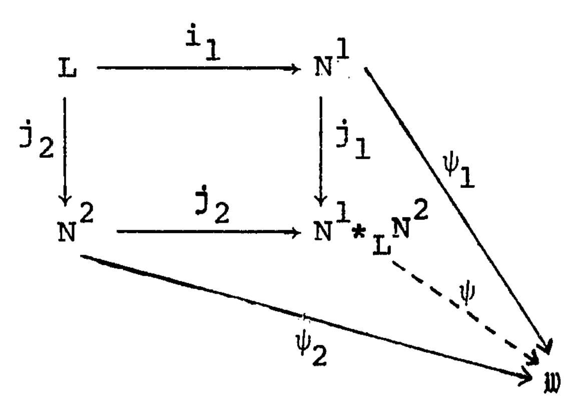
These are clearly alternating sequences increasing in length by one element as we go from U_x^{λ} to $U_x^{\lambda+1}$. Thus,

$$W_x(y) = \coprod_{\lambda \geq 0} U_x^{\lambda} = \{(n, \ldots, n_1, \ell) \mid \lambda \geq 0\}$$

Step 5: $\mathfrak{W} \simeq N^1 *_L N^2$ in Cat_I.

Proof: We define two maps $\mathfrak{W} \xrightarrow{\pi} N^1 *_L N^2 \xrightarrow{\psi} \mathfrak{W}$. The first map, $\pi : \mathfrak{W} \longrightarrow N^1 *_L N^2$ is defined by $\pi(n, \ldots n_1, \ell) = n_\lambda \circ \ldots \circ n_1 \circ \ell$. Every element of $N^1 *_L N^2$ can be written as the alternating product of elements from N^1 and N^2. Since i_1, i_2 are right-free, we can put every element into the "canonical" form. Thus π is onto.

To define ψ, consider the following push-out diagram.



Since we are in Cat_I, $N^i \xrightarrow{\psi_i} \mathbb{D}$, $j = 1,2$, is the identity on objects. If $x \xrightarrow{i} y$ in N^j, $\psi_j(n) = T_n$ which sends $1 \longmapsto [n] = [n,1] = [\bar{n},\ell]$ where $\bar{n}$ and ℓ are unique since i_j is a right-free inclusion. The outer square commutes, thus giving a unique map, ψ. By defining

$$\psi(n_\lambda \ldots n_1) = T_{n_\lambda} \circ T_{n_{\lambda-1}} \circ \ldots \circ T_{n_1} : [1] \longmapsto [n_\lambda, \ldots, n_1, \ell]$$

we make the entire diagram commute. Since ψ is unique, this must be ψ.

To show π is one-to-one, it suffices to show that $\psi \circ \pi$ is one-to-one. We observe that

$$\psi \circ \pi(\bar{n}_\lambda, \ldots, \bar{n}_1, \ell) = T_{(\bar{n}_\lambda, \ldots \bar{n}_1, \ell)} : [1] \longmapsto [\bar{n}_\lambda, \ldots, \bar{n}_1, \ell]$$

Since both elements are in canonical form we may justify eliminating the equivalence class notation, i.e., $1 \longmapsto (\bar{n}_\lambda, \ldots, \bar{n}_1, \ell)$. Thus $\psi \circ \pi$ must be one-to-one since each element of $\mathbb{D}$ determines a different natural transformation.

π is one-to-one and onto. Therefore $= N^1 *_L N^2$, i.e., every element of $N^1 *_L N^2$ can be written uniquely as $(\bar{n}_\lambda, \ldots \bar{n}_1, \ell)$, where i_1, i_2 are right-free inclusions. This concludes the proof.

<u>Corollary</u>: The terminal legs, $N^1 \xrightarrow{j_1} N^1 *_L N^2$ and $N^2 \xrightarrow{j_2} N^1 *_L N^2$, and hence, the composition map $L \xrightarrow{k} N^1 *_L N^2$, $k = j_1 i_1 = j_2 i_2$ are right-free inclusions.

<u>Proof</u>: Since i_1 is a right-free inclusion, $n \varepsilon N^1$ can be written as $\bar{n}\ell$ in a unique fashion, with $\bar{n} \varepsilon \bar{N}^1, \ell \varepsilon L$. The map j_1 merely sends n to $\bar{n}\ell$, its canonical representation in $N^1 *_L N^2$. So j_1 is clearly one-to-one. By setting $\tilde{N}^1 = \left\{ (\bar{n}_\lambda, \ . \ . \ . \ , \bar{n}_1) \mid \lambda \geqslant 1, \bar{n}_1 \varepsilon \bar{N}^2 \right\}$, we obtain a set of generators for $N^1 *_L N^2$ over N^1. If $n \varepsilon N^1 *_L N^2$, then we have just shown $n = \bar{n}_\lambda \bar{n}_{\lambda-1} \ . \ . \ . \ \bar{n}_1 \ \ell$ in a unique way. If $\bar{n}_1 \varepsilon \bar{N}^2$ then $(\bar{n}_\lambda, \ . \ . \ . \ , \bar{n}_1) \varepsilon \tilde{N}^1$ and n has been written uniquely as the product of an element of $\tilde{N}^1$ and an element of N^1. If $\bar{n}_1 \varepsilon \bar{N}^1$, then $(\bar{n}_\lambda, \ . \ . \ . \ , \bar{n}_2) \varepsilon \tilde{N}^2$ and $n = (\bar{n}_\lambda, \ . \ . \ . \ , \bar{n}_2)\bar{n}_1\ell$, again written uniquely as the product of an element of $\tilde{N}^1$ and an element of N^1.

The same argument, of course, serves to show j_2 is a right-free inclusion. Finally, set

$$\bar{N} = \tilde{N}^1 \quad \tilde{N}^2 = \left\{ (\bar{n}_\lambda, \ . \ . \ . \ , \bar{n}_1), \ \lambda \geqslant 1 \right\}$$

and we have a set of generators for $N^1 *_L N^2$ over L with respect to $k : L \longrightarrow N^1 *_L N^2$.

Section Four

PRESERVATION OF HOMOTOPY TYPE IN THE PRESENCE OF RIGHT-FREE INCLUSIONS: $BN^1 \cup_{BL} BN^2 \xrightarrow{\sim} B\left(N^1 *_L N^2\right)$

<u>Theorem 2</u>: Given the following diagram

$$\begin{array}{ccc} L & \xrightarrow{i_1} & N^1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ N^2 & \xrightarrow{j_2} & N^1 *_L N^2 = N \end{array}$$

with i_1, i_2 right-free inclusions of categories, then the induced map $\beta : BN^1 \cup_{BL} BN^2 \longrightarrow B\left(N^1 *_L N^2\right)$ is an equivalence.

<u>Proof</u>: The proof will proceed by showing

(I): $\Pi_1 B\left(N^1 *_L N^2, x\right) \xrightarrow{\sim} \Pi_1\left(BN^1 \cup_{BL} BN^2, x\right)$, and

(II): $BN^1 \cup_{BL} BN^2 \xrightarrow{\beta} B\left(N^1 *_L N^2\right)$ induces an isomorphism in homology with all local coefficients.

<u>Proof of I</u>: Let Γ be the groupoid reflection, then $\Gamma N \longrightarrow \Gamma N^1 *_{\Gamma L} \Gamma N^2$ because Γ is a left-adjoint and thus preserves pushouts. To get Π_1, we consider the vertex groups of these categories at x. Doing this yields the following diagram which in turn gives the desired result.

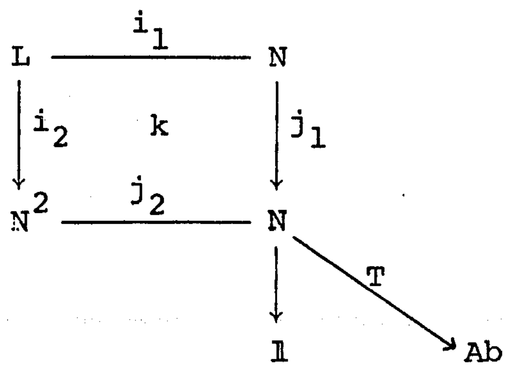
$$\begin{array}{ccc}
\Gamma N(x) & \xrightarrow{\quad \sim \quad} & \left(\Gamma N^1 *_{\Gamma L} \Gamma N^2\right)(x) \\
\downarrow\wr & & \wr| \\
\Gamma N(x) & \xrightarrow{\quad \sim \quad} & \Gamma N^1(x) *_{\Gamma L(x)} \Gamma N^2(x) \\
\downarrow\wr & & \wr| \\
\Pi_1(BN,x) & \xrightarrow{\quad \sim \quad} & \Pi_1(BN^1, x) *_{\Pi_1(BL,x)} \Pi_1(BN^2,x) \\
 & & \downarrow\wr \\
 & & \Pi_1\left(BN^1 \cup_{BL} BN^2, x\right)
\end{array}$$

Since i_1 and i_2 are inclusions, we know that $\Gamma(i_1)$ and $\Gamma(i_2)$ are inclusions also, which implies that $\left(\Gamma N^1 *_{\Gamma L} \Gamma N^2\right)(x) \xrightarrow{\simeq} \Gamma N^1(x) *_{\Gamma L(x)} \Gamma N^2(x)$. For any category N, $\Gamma N(x) \xrightarrow{\simeq} \Pi_1(BN,x)$ [18]. We use this to get both vertical isomorphisms for the bottom square. Finally, from the van Kampen theorem, we have $\Pi_1\left(BN^1 \cup_{BL} BN^2\right)(x)$ $= \Pi_1(BN^1,x) *_{\Pi_1(BL,x)} \Pi_1(BN^2,x)$. Thus we conclude that $\Pi_1\left(BN^1 *_L N^2, x\right) \xrightarrow{\simeq} \Pi_1\left(BN^1 \cup_{BL} BN^2, x\right)$.

<u>Proof of II</u>: Starting with the following diagram

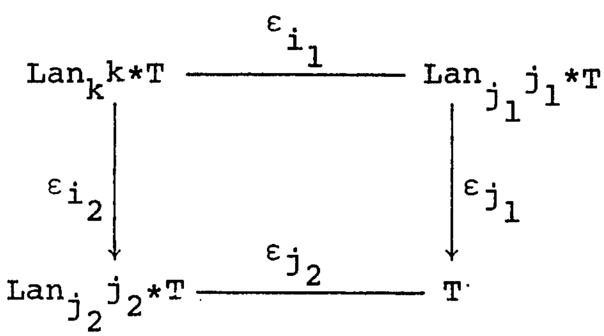
$$\begin{array}{ccc}
L & \xrightarrow{\quad i_1 \quad} & N^1 \\
{\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\
N^2 & \xrightarrow{\quad j_2 \quad} & N^1 *_L N^2 = N
\end{array}$$

we let T be a morphism-inverting functor from N to Ab, and 1 be the category with one object and one morphism. Then consider



By first pulling T back to each of the other categories, then taking the appropriate Kan extensions, we get the following commutative diagram in Ab^N:

Diagram 4.1



The maps in this diagram come from adjoint pairs of the form Lan_j, j^* where $j = i_1, i_2, j_1, j_2$ or k. For instance, $\mathrm{Lan}_k k^*T =$ $\mathrm{Lan}_{j_1 i_1}(j_1 i_1)^*T = \mathrm{Lan}_{j_1}(\mathrm{Lan}_{i_1} i_1^*) j_1^* T \xrightarrow{\varepsilon_{i_1}} \mathrm{Lan}_{j_1} j_1^*T$. This diagram commutes from straightforward application of properties of adjoints, Lan, etc.

This diagram is in Ab^N, an abelian category. It will be shown that it is a push-out and that $(\varepsilon_{i_1}, \varepsilon_{i_2})$: $Lan_k k^* T \longrightarrow Lan_{i_1} i_1^* T \oplus Lan_{j_2} i_2^* T$ is an injection. Thus, the following sequence is exact.

$$0 \longrightarrow Lan_k k^* T \longrightarrow Lan_{i_1} i_1^* T \oplus Lan_{i_2} i_2 T \longrightarrow T \longrightarrow 0.$$

This, in turn, yields the long exact sequence

$$\cdots \longrightarrow H_n\left(N, Lan_k k^* T\right) \longrightarrow H_n\left(N, Lan_{j_1} j_1^* T\right) \oplus H_n\left(N, Lan_{j_2} j_2^* T\right) \longrightarrow H_n(N,T) \longrightarrow \cdots$$

using the commutativity of H_n with $\oplus$.

For any right-free inclusion, $M \xrightarrow{i} N$, Lan_i preserves injections, in which case we have $H_n(N, Lan_i i^*T) = H_n(M, i^*T)$. Thus we may rewrite the long exact sequence to yield the top row of the following diagram:

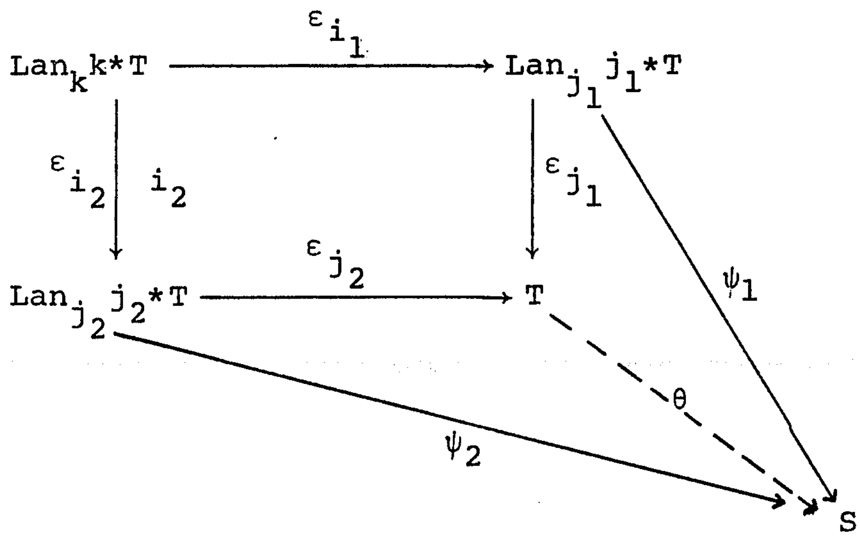
$$\begin{array}{ccccccc}
\cdots \longrightarrow H_N(L, k^*T) & \longrightarrow & H_n\left(N^1, j_1^* T\right) \oplus H_n\left(N^2, j_2^* T\right) & \longrightarrow & H_n(N,T) \longrightarrow \cdots \\
\uparrow \wr & & \uparrow \wr & & \uparrow \wr \\
 & & & & H_n(BN;T) \\
 & & & & \uparrow \\
\cdots \longrightarrow H_n\left(BL, k^* A_T\right) & \longrightarrow & H_n\left(BN^1, j_1^* A_T\right) \oplus H_n\left(BN^2, j_2^* A_T\right) & \longrightarrow & H_n\left(BN^1 \;\; {}_{BL}BN^2; A_T\right) \longrightarrow \cdots
\end{array}$$

where A_T is the local coefficient system associated with T. Using the 5-Lemma we get $H_n\left(BN^1 \cup_{BL} BN^2, A_T\right) \longrightarrow H_n(B(N^1 *_L N^2), A_T)$.

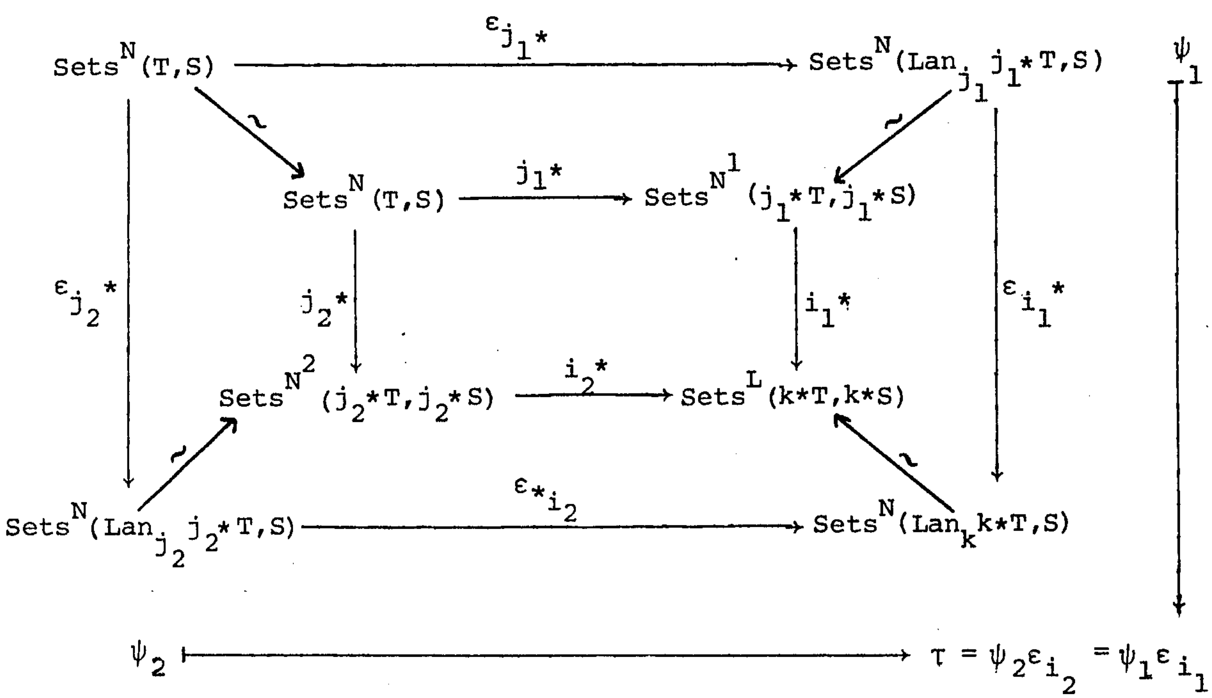
Thus, to complete the proof, we must show that Diagram 4.1 is a push-out and that $(\varepsilon_{i_1}, \varepsilon_{i_2})$ is an injection.

To show that Diagram 4.1 is a push-out, consider the following diagram in $Sets^N$:

Diagram 4.2



Given $\psi_2 \varepsilon_{i_2} = \psi_1 \varepsilon_{i_1}$, we need $\theta : T \longrightarrow S$. We now make the following commutative diagram:



We take the adjunctions of ψ_1, ψ_2, and $\tau = \psi_2 \varepsilon_{i_2} = \psi_1 \varepsilon_{i_1}$, thereby moving from the outer square to the inner square. These adjunctions agree on objects, so we can define θ to equal any of these and it will satisfy the necessary commutativity conditions for naturality.

From the word theorem, any $w \in N = N^1 *_L N^2$ can be written uniquely as $w = \bar{n}_\lambda \; \bar{n}_{\lambda-1} \; \bar{n}_3 \cdots \bar{n}_1 \ell$ where the $\bar{n}_1$'s come alternately from N^1 and N^2, $\ell \in L$.

If $x \xrightarrow{w} y$, for $x, y \in N$, then the following diagram

$$\begin{array}{ccc} T(x) & \xrightarrow{\theta(x)} & S(x) \\ {\scriptstyle T(w)}\downarrow & & \downarrow{\scriptstyle S(w)} \\ T(y) & \xrightarrow{\theta(y)} & S(y) \end{array}$$

can be rewritten in the following form

$$\begin{array}{ccc} T(x) & \xrightarrow{\theta_x} & S(x) \\ {\scriptstyle T(\ell)}\downarrow & & \downarrow{\scriptstyle S(\ell)} \\ T(x_1) & \xrightarrow{\theta_{x_1}} & S(x_1) \\ {\scriptstyle T(\bar{n}_1)}\downarrow & & \downarrow{\scriptstyle S(\bar{n}_1)} \\ T(x_2) & \xrightarrow{\theta_{x_2}} & S(x_2) \\ \downarrow & & \downarrow \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ T(x_\lambda) & \xrightarrow{\theta_{x_\lambda}} & S(x_\lambda) \\ {\scriptstyle T(\bar{n}_\lambda)}\downarrow & & \downarrow{\scriptstyle S(\bar{n}_\lambda)} \\ T(y) & \xrightarrow{\theta_y} & S(y) \end{array}$$

where w is the composition of $x \xrightarrow{\ell} x_1 \xrightarrow{\bar{n}_1} x_2 \longrightarrow \cdots \longrightarrow x_\lambda \xrightarrow{\bar{n}_\lambda} y$. Each square in this ladder commutes because θ is equal to the appropriate natural transformation in L, N^1 or N^2, and gives a square that commutes in L, N^1 or N^2, depending on where the $\bar{n}_i$ or ℓ lie. To clarify this a bit, look at the first square:

$$\begin{array}{ccc} T(x) & \xrightarrow{\theta_x} & S(x) \\ {\scriptstyle T(\ell)}\downarrow & & \downarrow{\scriptstyle S(\ell)} \\ T(x_1) & \xrightarrow{\theta_{x_1}} & S(x_1) \end{array}$$

The commutativity of this square comes precisely from the commutativity of the following diagram.

$$\begin{array}{ccc} k^*T(x) & \xrightarrow{\phi\tau_x} & k^*S(x) \\ {\scriptstyle k^*T(\ell)}\downarrow & & \downarrow{\scriptstyle k^*S(\ell)} \\ k^*T(x_1) & \xrightarrow{\phi\tau_{x_1}} & k^*S(x_1) \end{array}$$

We have $k^*T = T$ and $k^*S = S$ on objects since k is an inclusion, and $\theta_x = \phi\tau_x$ on objects, by definition of θ.

The commutativity of each little square gives the commutativity of Diagram 4.3, of course. Thus, θ is a natural transformation from T to S. T(w) being defined on a unique representation of w, it follows that T is well-defined.

We now show that $(\varepsilon_{i_1}, \varepsilon_{i_2}): \mathrm{Lan}_k k\ T \longrightarrow \mathrm{Lan}_{j_1} j_1^* T \oplus \mathrm{Lan}_{j_2} j_2\ T$ is an injection. We begin with the following push-out diagram.

$$\begin{array}{ccc} L & \xrightarrow{\;i_1\;} & N^1 \\ {\scriptstyle i_2}\downarrow & \searrow^{k} & \downarrow{\scriptstyle j_1} \\ N^2 & \xrightarrow{\;j_2\;} & N \end{array} \qquad\qquad k = j_1 i_1 = j_2 i_2$$

All maps are assumed to be right-free inclusions, and we introduce the following notation: for $L \longrightarrow N^i$ the generators of N^i over L will be denoted by $\bar{N}^i$, for $N^i \longrightarrow N$ the generators of N over N^i will be denoted by $\tilde{N}^i$, for $L \longrightarrow N$ the generators of N over L will be denoted by $\bar{N}$, $i = 1,2$. The elements of $\bar{N}$ consist of all alternating words from $\bar{N}^1$ and $\bar{N}^2$. The elements of $\tilde{N}^1$ are all alternating words from $\bar{N}$ that end in an element from $\bar{N}^2$. Thus $\bar{N}(x,y) = \coprod_z \tilde{N}^1(z,y) \times \bar{N}^1(x,z)$. Analogously, $\bar{N}(x,y) = \coprod_z \tilde{N}^2(z,y) \times N^2(x,z)$

Diagram 4.1 involving the left Kan extensions, can be rewritten when evaluated at y.

$$\begin{array}{ccc} T(y) \sqcup \coprod_v \bar{N}(v,y) \times T(v) & \xrightarrow{\;\varepsilon_{i_1}\;} & T(y) \sqcup \coprod_z \bar{N}(z,y) \times T(z) \\ {\scriptstyle \varepsilon_{i_2}}\downarrow & & \downarrow \\ T(y) \sqcup \coprod_u \tilde{N}^2(u,y) \times T(u) & \longrightarrow & T(y) \end{array}$$

The T_v's, T_z's, T_u's and T_y's are all abelian groups. The statement is clear for the $T(y)$ part of the diagram. The other elements move as in the diagram below:

$$\begin{array}{ccc} (\bar{n},t) & \longrightarrow & (n^1, T(\bar{n}^1)t) \\ \downarrow & & \downarrow \\ (n^2, T(\bar{n}^2)t) & \longrightarrow & (T\bar{n},t) \end{array} \qquad \bar{n} = \tilde{n}^2\bar{n}^2 = \tilde{n}^1\bar{n}^1 \qquad \begin{array}{c} \tilde{n}^i \in \tilde{N}^i \\ \bar{n}^i \in \bar{N}^i \end{array}$$

Note that, in fact, $\bar{n}$ ends in an element from $\bar{N}^1$ or $\bar{N}^2$. If it ends in an element from $\bar{\bar{N}}^1$, then $\tilde{n}^2 = \bar{n}$ and $\bar{n}^2 = 1$. A similar statement holds for $\bar{n}$ ending in an element from $\bar{N}^2$.

To show that $(\varepsilon_{i_1}, \varepsilon_{i_2})$ is an injection, it is sufficient to show that $\ker(\varepsilon_{i_1}, \varepsilon_{i_2}) = \ker \varepsilon_{i_1} \cap \ker \varepsilon_{i_2} = \{0\}$.

Let $(\bar{n},t) \in \ker \varepsilon_{i_1}$, $t \neq 0$. Suppose $\bar{n}$ ends in an element from $\bar{N}^2$, then $(\tilde{n}^2, T(\bar{n}^2)t) \longleftarrow (\bar{n},t) \xrightarrow{\varepsilon_{i_1}} (\bar{n},t) = (\bar{n},0)$, which would imply $t = 0$. Therefore $(\bar{n},t) \in \ker \varepsilon_{i_1}$ and since we have assumed $t \neq 0$, $\bar{n}$ ends in an element from $\bar{N}^1$. Then $(\bar{n},t) \longleftarrow (\bar{n},t) \longrightarrow (\tilde{n}^1, T(\bar{n}^1)(t)) = (\tilde{n}^1, 0)$, but $t = 0$ certainly means $(\bar{n},t) \in \ker \varepsilon_{i_2}$.

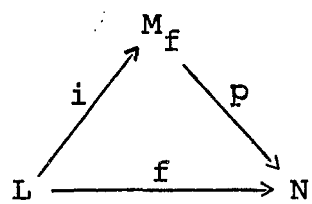
Clearly the argument works when i_1 and i_2 are reversed. thus $\ker(\varepsilon_{i_1}, \varepsilon_{i_2}) = \{0\}$ and the proof is done.

Section Five

CONSTRUCTION OF A MAPPING CYLINDER

Let L,N be two categories in Cat_I and $f:L \longrightarrow N$ be a functor which is the identity on objects. The following theorem allows us to construct a mapping cylinder, M_f, for the functor f.

<u>Theorem 3</u>: There exists a factorization of f



with the following properties:

i) $pi = f$

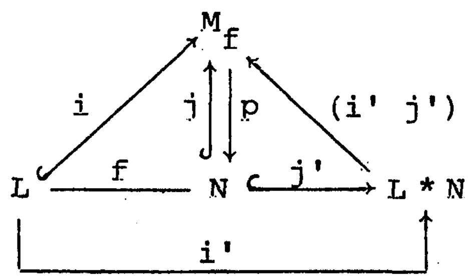
ii) i is a right-free inclusion

iii) p is a homotopy equivalence.

<u>Proof</u>: We describe the mapping cylinder by presenting it in terms of generators and relations, that is,

$$M_f = \langle L, N, t_x \mid x \in I, \text{ and } \alpha t_x = f(\alpha) t_x, \; t_y \beta t_x = \beta t_x, \text{ for } \alpha, \beta : x \longrightarrow y, \alpha \in L, \beta \in N \rangle$$

Consider the following diagram in Cat_I:



The functors i, j, i', j' are the obvious inclusions, and p is defined on generators of M_f by

$$p(\alpha) = f(\alpha) \qquad \alpha \in L$$

$$p(\beta) = \beta \qquad \beta \in N$$

$$p(t_x) = 1_x$$

Property i) is satisfied by definition of p. To show property ii), that the inclusion $i:L \longrightarrow M$ is right-free, we note that i' is a right-free inclusion and show that (i' j') is also. Then $i = (i'\ j') \cdot i'$ will be a right-free inclusion, since composition preserves this property. We now show that $i':L\ \ N \longrightarrow M_f$ is a right-free inclusion with generators βt_x and 1_x, $x \in I$. By looking at the generators of M_f, we see that every morphism in M_f is one of two types. Either it is the product of α's, β's, and t_x's, or there are no t_x's. In the latter case, we have an element, w, of $L * N$.

<u>Lemma</u>: For $w \in L * N$, $w : x \longrightarrow y$, $t_x \in M_f$, then $wt_x = p(w)t_x$.

<u>Proof of Lemma</u>: For arbitrary $w \in L$, we write $w = \alpha_1^{i_1}\beta_1 \cdots \alpha_{\lambda-1}\beta_{\lambda-1}\alpha_\lambda^{i_\lambda}$, where $i_1 = 0$ or 1, and proceed by induction on λ.

For $\lambda = 1$, $\alpha_1^{i_1}t_x = f\left(\alpha_1^{i_1}\right)t_x = p\left(\alpha_1^{i_1}\right)t_x$, using the definition of f and p, respectively.

Assuming the statement holds for $\lambda - 1$, we consider

$$\begin{aligned} wt_x &= \alpha_1^{i_1}\beta_1 \cdots \alpha_{\lambda-1}\beta_{\lambda-1}\alpha_\lambda^{i_\lambda}t_{x_\lambda} \\ &= \alpha_1^{i_1}\beta_1 \cdots \alpha_{\lambda-1}\beta_{\lambda-1}f\left(\alpha_\lambda^{i_\lambda}t_{x_\lambda}\right) \\ &= \alpha_1^{i_1}\beta_1 \cdots \alpha_{\lambda-1}t_{x_{\lambda-1}}\beta_{\lambda-1}f\left(\alpha_\lambda^{i_\lambda}\right)t_{x_\lambda} \\ &= p\left(\alpha_1^{i_1}\beta_1 \cdots \alpha_{\lambda-1}\right)t_{x_{\lambda-1}}p\left(\beta_{\lambda-1}\alpha_\lambda^{i_\lambda}\right)t_{x_\lambda} \\ &= p(w)t_{x_\lambda}. \end{aligned}$$

We use this lemma to show that any element of M_f with t_x's in the product can be written in the form $\beta t_x w$. We start with such an element written in terms of generators:

$t_{x_1}^{i_1} w_1 t_{x_2} w_2 \ldots w_{\lambda-1} t_{x_\lambda} w_\lambda^{i_\lambda}$, $i_1 = 1$ or 0, $i = 1$ or 0. By the lemma, this equals $t_{x_1}^{i_1} p(w_1) t_{x_2} p(w_2) \ldots p(w_{\lambda-1}) t_{x_\lambda} w^{i_\lambda}$ which reduces to $p(w_1)\ p(w_2) \ldots p(w_{\lambda-1}) t_{x_\lambda} w^{i_\lambda}$ because of the relation $t_y \beta t_x = \beta t_x$ if $\beta : x \longrightarrow y$, given in the definition of M_f. This equals $p(w_1 \ldots w_{\lambda-1}) t_{x_\lambda} w_\lambda^{i_\lambda} = \beta t_x w^{i_\lambda}$, setting $\beta = p(w_1 \ldots w_{\lambda-1})$, which is in N. This expression is unique since p is a functor, and there is no ambiguity.

Since every element in M_f can be written in the form w or $\beta t_x w$ in a unique way, (i' j') is a right-free inclusion with the generators $\overline{M}_f = \{1_x \text{ and } \beta t_x w \mid x \in I, \beta \in N, w \in L * N\}$.

We now prove our last assertion, iii), that p is a homotopy equivalence. It is sufficient to show that $pj = 1_N$ and that there is either a natural transformation $jp \longrightarrow 1$ or $1 \longrightarrow jp$ since two functors connected by a natural transformation become homotopic when we apply the classifying functor, B [17].

We have $pj = 1_N$, but it is not straightforward to show either of the other statements. We introduce $\bar{j}: N \longrightarrow M_f$, defined by $\bar{j}(\beta) = \beta t_x$, where $\beta : x \longrightarrow y$. Then T, defined by $T_x = t_x$, gives a natural transformation $\bar{j} \longrightarrow j$. We need only observe that the following diagram in M_f commutes.

$$\begin{array}{ccc} x & \xrightarrow{T_x} & x \\ {\scriptstyle \bar{j}(\beta)}\downarrow & & \downarrow{\scriptstyle j(\beta)} \\ y & \xrightarrow{T_y} & y \end{array} \qquad\qquad x \xrightarrow{t_x} x \xrightarrow{\beta} y$$

Using the defining relation $\beta t_x = t_y \beta t_x$, we have $j(\beta)T_x = j(\beta)t_x = t_y\beta t_x = t_y\bar{j}(\beta) = T_y j(\beta)$. The existence of the natural transformation $\bar{j} \xrightarrow{\sim} j$ implies that of both $p\bar{j} \longrightarrow pj = 1_N$ and $\bar{j}p \longrightarrow jp$. The same argument shows that there is a natural transformation $\bar{j}p \xrightarrow{\cdot} 1_{M_f}$.

Consider $x \xrightarrow{\gamma} y$ in M_f. Define T as above. Then we need to show that the following diagram commutes in M_f.

$$\begin{array}{ccc} x & \xrightarrow{t_x} & x \\ \downarrow{\scriptstyle \bar{j}p(\gamma)} & & \downarrow{\scriptstyle \gamma} \\ y & \xrightarrow{t_y} & y \end{array}$$

Recall we have three types of generators for M_f: $\alpha \in L$, $\beta \in N$, and $t_x, x \in I$. We use the relations on M_f to see $t_y\bar{j}p(\alpha) = t_y\bar{j}f(\alpha) = t_y f(\alpha)t_x = f(\alpha)t_x = \alpha t_x$, $t_y\bar{j}p(\beta) = t_y\beta t_x = \beta t_x$, and $t_x\bar{j}p(t_x) = t_x\bar{j}(1_x) = t_x t_x = t_x$. Thus, $p\bar{j} \longrightarrow 1_N$ and $jp \longleftarrow \bar{j}p \longrightarrow 1_{M_f}$, which shows p is a homotopy equivalence.

Section Six

EMBEDDING A CATEGORY INTO A SEMI-GROUPOID OF THE SAME HOMOTOPY TYPE

Lemma: Every connected category, L, has a contractible subcategory, L, with at most one morphism between any two objects (i.e., L_0 is a preorder), and Ob L = Ob L_0.

Proof: Consider the set, S, of all contractible, partially ordered, subcategories of L. If L has at most one object, the conclusion is immediate. If a connected category, L, has at least two objects, there exist x,y in Ob L with $x \neq y$, with a morphism $\alpha : x \longrightarrow y$. The subcategories $\{x\}$ and $\{x \xrightarrow{\alpha} y\}$ form a linearly ordered subset of S. By the Hausdorff maximal principle, they are contained in a maximal, linearly ordered subset, S_1, of S. Take the union of the elements of S_1 to be L_0. L is clearly contractible since each element of S_1 is, and S_1 is linearly ordered by inclusion maps.

Furthermore, we claim every object in L is an object in L_0. Suppose this is not the case. Since L, is connected, every object in L is "connected" to every other object in L by a finite sequence of morphisms. Thus if $x \in$ Ob L_0,

$y \notin \mathrm{Ob}\ L_0$, they must be connected by such a sequence, and somewhere in the sequence is a morphism that goes from an object in L_0 to one that is not, or vice versa.

We shall assume that $\alpha : x_0 \longrightarrow y_0$ is such a morphism, and that $x_0 \in \mathrm{Ob}\ L$, $Y_0 \in \mathrm{Ob}\ L_0$. Let $L_1 = L_0 \cup \{\alpha : x_0 \longrightarrow y_0\}$. We now show that L_1 is contractible. $i : L_0 \longrightarrow L_1$, and we define $r : L_1 \longrightarrow L_0$ by $r(y_0) = x_0$, $r(\alpha) = 1_{x_0}$, and $r = \mathrm{Id}$ elsewhere. Then $ri = 1_{L_0}$ and $ir \xrightarrow{\simeq} 1_{L_1}$. We define a natural transformation $T : ir \longrightarrow i_{L_1}$, by

$$T(x) = \begin{cases} 1_x & \text{for } x \in \mathrm{Ob}\ L_0 \\ \alpha & \text{for } x = y_0 \end{cases}$$

It is sufficient to consider $\beta : x \longrightarrow y_0$, $x \in \mathrm{Ob}\ L_0$, and show that the diagram below commutes.

$$\begin{array}{ccc} ir(x) & \xrightarrow{T(x)} & x \\ {\scriptstyle ir(\beta)}\downarrow & & \downarrow{\scriptstyle \beta} \\ ir(y_0) & \xrightarrow{T(y_0)} & y_0 \end{array}$$

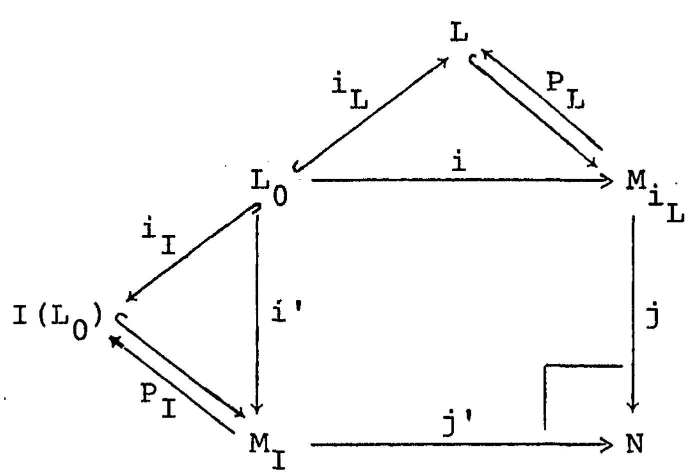
By construction of L_1, $x \xrightarrow{\beta} y_0$ factors as $x \xrightarrow{\gamma} x_0 \xrightarrow{\alpha} y$, in $\mathrm{Mor}(L_0)$. Then going around the bottom of the square we have $T(y_0) \circ ir(\beta) = \alpha \circ ir(\alpha \circ \gamma) = \alpha \circ i(r(\alpha) \circ r(\beta))$ $= \alpha \bullet i(1_x \circ \gamma) = \alpha \circ \gamma = \beta$. Going around the top of the square we have $\beta \circ T(x) = \beta$, giving the desired commutativity.

If on the other hand, $y_0 \xrightarrow{\alpha} x$, then we get a natural transformation $1_{C_1} \longrightarrow ir$ by the same argument. Thus i is a homotopy equivalence of categories. Therefore, L_1 is a contractible, paritally ordered category that strictly contains L_0. Now $S_1 \cup \{L_0\} \neq S_1$ is again a linearly ordered set on contractible, partially ordered categories, thus contradicting the maximality of S_1. We conclude there can be no objects in L, not in L_0, i.e., Ob L_0 = Ob L.

We are now ready to prove our main result.

Theorem 4: Given a connected category L, we can embed it by a right-free inclusion in a semi-groupoid, N, where the inclusion induces a homotopy equivalence.

Proof: Let L_0 be a contractible subcategory of L, with Ob L_0 = Ob L. Let $I(L_0)$ be the indiscrete category containing L_0. Using the mapping cylinders of $i_L : L_0 \longrightarrow L$ and $i_I : L_0 \longrightarrow I(L_0)$, respectively, we take the push-out indicated in the diagram below:



The maps i and i' are the right-free inclusions of L_0 into the mapping cylinders M_{i_L} and M_I, respectively. Thus the terminal legs, j and j', of this push-out are right-free inclusions, by the corollary to Theorem 1. The map $I(L_0) \longrightarrow M_I$ is a right-free inclusion by Theorem 3, therefore $I(L_0)$ is contained in N as a subcategory, making N a semi-groupoid.

By Theorem 2, $BN \simeq BM_{i_L} \cup_{BL_0} BM_{i_I} \simeq BL \cup_{BL_0} BL_0 \simeq BL$.

As mentioned in the preliminaries, the homotopy type of a monoid at any vertex of a semi-groupoid is the same as that of the entire semi-groupoid. Thus, for any L in Cat_I, we can find a monoid with the same homotopy type.

At this point, we would like to point out that the mapping cylinder is constructed so that, if a category, L, is presented in terms of generators and relations, then we have the mapping cylinder presented in terms of generators and relations, and, of course, this is true of the semi-groupoid constructed in Theorem 4. If we denote by M the monoid at a fixed vertex of N, then $\phi : N \longrightarrow I(L_0)$: $n \longmapsto (\ell, \ell^{-1}n)$, where $\ell \in I(L_0)$, $n \in N$, and $\ell : \alpha \longrightarrow \beta$ if $n : \alpha \longrightarrow \beta$. By composing with the canonical projection onto M, we have a surjection of N onto M. Using this surjection, we can easily show that the monoid at any fixed vertex is presented by the generators and relations of N pulled back to that fixed vertex by the isomorphisms provided by $I(L_0)$.

Section Seven

EXAMPLES

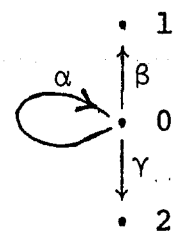
We calculated the monoid for the category $\dot{0} \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \dot{1}$, which has the homotopy type of a circle. We were curious to see if there was anything intuitively obvious about the result. That is asking a lot of a construction that is constrained by right-freeness, a notion that is more technical than intuitive. Nonetheless, the results seem reasonable.

We get the following sets of generators and relations for the associated monoid:

Generators	Relations
$\alpha^{-1}\beta$	$t_0^2 = t_0$
$\alpha^{-1}t_1\alpha$	$t_1^2 = t_1$
$\alpha^{-1}s_1\alpha$	$s_0^2 = s_0$
t_0	$s_1^2 = s_1$
s_0	$\alpha^{-1}t_1\alpha t_0 = t_0$
t_1	$\alpha^{-1}t_1\alpha t_0 = \alpha^{-1}\beta t_0$
s_1	$\alpha^{-1}s_1\alpha s_0 = s_0$
	$s_0\alpha^{-1}s_1 = \alpha^{-1}s_1\alpha$

All of the generators except $\alpha^{-1}\beta$ are idempotent and generate no homotopy, and $\alpha^{-1}\beta$ generates the homotopy of a circle. This reveals no intuitive inconsistency but does not strike one as intuitively obvious, especially since the complex effects of the relations are difficult to account for.

We have also calculated the monoid associated with the category



where $\beta\alpha = \beta$ and $\gamma\alpha = \gamma$. This category has the homotopy type of S^2. The set of generators and relations for the monoid associated with this category are as follows:

Generators	Relations
α	$t_0 \alpha t_0 = \alpha t_0$
t_0	$t_0 \beta^{-1} t_1 \beta = \beta^{-1} t_1 \beta$
s_0	$t_0 \gamma^{-1} t_2 \gamma = \gamma^{-1} t_2 \gamma$
$\beta^{-1} t_1 \beta$	$s_0 \beta^{-1} s_1 \beta = \beta^{-1} s_1 \beta$
$\beta^{-1} s_1 \beta$	$s_0 \gamma^{-1} s_2 \gamma = \gamma^{-1} s_2 \gamma$
$\gamma^{-1} t_2 \gamma$	$t_0^{\,2} = t_0$
$\gamma^{-1} s_2 \gamma$	$s_0^{\,2} = s_0$

This result is similar to the first in that there is only one generator that is not idempotent, but the relations are considerably more complex.

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