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**Knowing How and Knowing Why:
Expert and In Transition Teacher Understandings of Representations,
Translations, and Connections for Dividing With Fractions**

by

Yasemin Jones

**A dissertation submitted to the Graduate Faculty in
Educational Psychology in partial fulfillment of the requirements
for the Degree of Doctor of Philosophy,
The City University of New York
2005**

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Approval Page

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Abstract

Knowing How and Knowing Why: Expert and In Transition Teacher Understandings of Representations, Translations, and Connections for Dividing With Fractions

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The study is an examination of two connected forms of mathematical content knowledge: subject matter (SMK) and pedagogical content (PCK) knowledge for fractions and division with fractions. Fourteen middle school mathematics teachers “thought out loud” on division with fractions and concept map tasks. From the think alouds, eight teachers in transition (IT) and six teachers with deep knowledge (DK) were identified. Cognitive representations (i.e., models, translations, and connections) for understanding and teaching the content matter were analyzed.

The deep knowledge teacher responses were analyzed to develop a model of content knowledge for understanding dividing with fractions. The proposed model includes the following content knowledge features of subject matter (SMK) and pedagogical content (PCK) knowledge: 1) multiple representations; 2) varied representations; 3) multiple translations; 4) emphasis on context; 5) emphasis on reference unit; and 6) fractions as sets of connected representations. The six teachers with deep knowledge evidenced well developed, organized, and connected knowledge through multiple representations, translations, and connections between and to related “big ideas” (Fosnot & Dolk, 2002) in the mathematics curriculum.

Examining differences in the qualities of “thought in action” between the two groups of teachers, deep knowledge and in transition, provided opportunities to enhance research and practical understandings of mathematics teachers’ knowledge.

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I am grateful to the teachers I interviewed. Their curiosity for and commitment to learning and teaching mathematics shine through their responses. They were true examples of the teachers Lee Shulman (1986) referred to in: "Those who can, do; those who understand, teach" (p. 14) Through this experience, I now know more about the subject I have written about than when I started. In the process, I have realized there is so much more to know. I hope to commit my professional journey to learning about teaching and being fortunate enough to spend more time in classrooms, observing teachers practicing their craft.

My family has been a source of continual encouragement. Seth Jones, my smart and dear husband, who tried many different tactics to help me face the music: tough coaching, sympathetic coaching and sometimes just walking the dog. Seth's support and care carried me through my graduate school experience. Friends: Arija, Cynthia, Dana, Kristin, Krystal, Leah, Marikje, Maureen, Niesha, Sara, Susan, and Wendy, all believed in my capacity when I didn't.

Writing these chapters was humbling and hard work. I wasn't always confident that I could finish. Now, standing close to completion, I am very appreciative of all the intellectual and emotional support I have received from my advisor, committee members, readers, friends, and family.

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CHAPTER 1

Introduction to the Study

In teaching, teachers weave together many different kinds of knowledge and beliefs: of students, of how students learn, of the teacher's role, of pedagogy, and of the subjects they teach. What they do is shaped by considerations of the context and their disposition to do particular things under particular circumstances. Their knowledge, skills, and dispositions are, in varying measures, the product of what they experienced as pupils, whatever professional training they have received, and their experience as teachers. Of the panoply of things that contribute to teaching, the one most frequently taken for granted and overlooked is teachers' own knowledge of the subject matter. Most assume that teachers know the "stuff" they are supposed to teach (Ball & Mosenthal, 1990, p. 1).

Research on teaching no longer takes for granted the "stuff" of teaching: what teachers know and how they teach what they know. These are two connected and interdependent forms of content knowledge: subject matter and pedagogical content knowledge. These two forms of content knowledge are examined in this study to understand qualities of effective mathematics teaching.

The study contributes to current research models that describe the integrated qualities of subject matter (SMK) and pedagogical content (PCK)

knowledge of mathematics teachers. Interviews with fourteen middle school mathematics teachers focused on division with fractions, considered the most difficult operation with the most difficult type of number in arithmetic (Ma, 1999). This topic is typically introduced in the middle grades.

Division with fractions was chosen as the content focus for three main reasons: 1) previous examinations of teachers SMK and PCK in the area provide a foundation for further study (Ball, 1988; Ma, 1999; Schifter, 1995; Sowder, Phillip, Armstrong, & Schappelle, 1998); 2) current NCTM standards (2000) for number and operations standards state “ instructional programs can enable students to understand the meaning and effects of arithmetic operations with fractions”; and 3) the National Research Council (2001) recommends that K-8 teachers have a deep understanding of the mathematics of the school curriculum and the principles behind it.

In this introduction, the research areas of interest to the study are outlined. These areas are described in more detail in the literature review section.

Understanding Fractions

In a recent publication by the National Research Council (2001), rational numbers, of which fractions are a subset, are emphasized:

Learning about rational numbers is more complicated and more difficult than learning about whole numbers. Rational numbers are more complex, in part because they are represented in several ways (e.g., common fractions and decimal fractions) and used in many ways (e.g., as parts of

regions and sets, as ratios, as quotients). There are numerous properties for students to learn, including the significant fact that the numerator and denominator are related through multiplication and division, not addition. Further, students are likely to have less out of school experiences with rational numbers than with whole numbers. The result is a number system that presents great challenges to students and teachers (p. 231).

Research studies examining the mathematical structure of and proposed pedagogical approaches for teaching fractions have assessed the types of cognitive models that students have regarding division, multiplication, partitioning, and unitizing fractions (Behr, Havel, Lesh, & Post, 1992; English & Halford, 1999; Goldin & Kaput, 1996; Lamon, 1993, 1999; Mack, 1990, 1995, 2001). Contributing to this research, cognitive models of students' rational number and fractional knowledge have been proposed (Behr et. al., 1992; Kieren, 1988, 1992). These areas of research have examined student rather than teacher thinking, with the understanding that through access to students' strategies for understanding the topic, instruction will benefit.

While such approaches are valuable for understanding how students approach and reason intuitively and formally about the topic, similar efforts at examining teacher knowledge of this topic in detail are missing. Cognitive understandings of teacher's knowledge of and about dividing with fractions are the major focus of the study because this type of number (i.e., fraction) and operation (i.e., division) is a content area often learned procedurally, unrelated to

the different types of representations, solution methods, or to connections between related mathematical concepts.

Research on Teaching

In 1986, Shulman wrote an important critique of research on teacher thinking and decision making. He summarized, “The missing research programs are those which clarify teachers’ understandings of subject matter content and the relationships between such understanding and the instruction teachers provide for students” (p. 25). Early studies under critique lacked specifics for understanding teacher knowledge and practice and focused on “process/product” outcomes. Since that time, research on teachers and teaching (Ball, 1988; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Grossman, 1990; Lampert, 1999, Sherin, 2002; Shulman, 1986; Wilson, 1989; Wineburg, 1990, 1997) connects *teaching for understanding* to teachers’ content knowledge as the organization of the knowledge in the teacher’s mind (i.e., SMK) and ways for teaching the subject matter (i.e., PCK). Current approaches to research on teacher knowledge include: 1) pre service teachers’ subject matter misunderstandings and expert-novice differences in content knowledge (Ball, 1988, 1991; Ma, 1999; Even & Tirosh, 1995; Leinhardt & Smith, 1985; Tirosh, 2000) and 2) content knowledge as SMK and PCK of in service teachers in the work of Carpenter, Fennema, and Franke (1996), Franke, Carpenter, Levi, and Fennema (2001), Ma (1999), Leinhardt (1993, 2001), Schifter and Simon (1992), Schifter and Fosnot, (1993), Schifter (1995, 1997), and Sherin (2002).

Recent publications by The National Council for the Teaching of Mathematics (NCTM) Professional Teaching Standards (1991, 2000) also emphasize the importance of mathematics teachers having depth and breadth in their content knowledge and teaching practice. The standards serve as national guides for the teaching of mathematics in American schools and have implications for educational research on studying teachers and how research on teaching may be conducted.

Subject matter and pedagogical content knowledge

The *Handbook of Research on Teaching* chapter (Munby, Russell & Martin, 2001) on teacher knowledge explains the construct along two main forms of content knowledge: subject matter knowledge (SMK) and pedagogical content knowledge (PCK). These knowledge categories (i.e., SMK and PCK) dominate research on mathematics teacher thinking and knowledge (Ball, 1988, 1991; Koehler & Grouws, 1992; Leinhardt, 2001; Hiebert, Gallimore, & Stigler, 2002; Shulman & Quinlan, 1996).

Shulman (1986) described PCK as, “ The ways of representing the subject which makes it comprehensible to others. It also includes an understanding of what makes the learning of specific topics easy or difficult” (p. 9). This form of content knowledge can include teacher knowledge of student understandings and misunderstandings of a subject area, knowledge of the curriculum, and knowledge of various models for the subject matter (Hiebert et. al., 2002).

Subject matter knowledge (SMK) includes the understanding of concept features and connections between mathematical ideas and this form of content knowledge contains two sublevels: substantive knowledge and syntactic knowledge (Ball, 1988, 1991, 1993). Substantive knowledge of subject matter refers to the key facts and concepts and connections among concepts, procedures, and mathematical ideas of the relationships between division with whole numbers and division with fractional numbers and knowledge of various division models. Syntactic knowledge is knowledge within the content area, knowledge that answers the “why” questions of the problem and explains the ways which new knowledge is introduced. Syntactic knowledge can include knowledge and application of different models of division or a conceptual understanding of the meaning of the division with fractions algorithm. Teachers’ content knowledge forms, SMK and PCK, can be examined in teachers’ organizations, interpretations, and representations of subject matter (Ball, 1988; Ball & Lampert, 1999; Fennema & Franke, 1992; Kennedy, Ball, & McDiarmid, 1993). What teachers know about teaching the content (i.e. pedagogical content knowledge) relates to what they know about what they teach (i.e., subject matter knowledge).

Deep knowledge: Teaching mathematics for understanding

Another important area of educational research is the examination of differences between accomplished and beginning thinkers in a domain (Bransford, Brown, & Cocking, 1999). The classic research methodology used for

this type of examination has been comparison of think alouds of expert and novices on teaching and/or understanding the subject matter (Leinhardt, 1983, 2001; Leinhardt & Smith, 1985; Turner-Bisset, 1999; Wilson, 1989; Wineburg, 1990). These comparisons have provided insight into the complex qualities of knowledge necessary for teaching.

The goal of these expert/novice studies was not to turn novices into experts by having novices copy expert behaviors. The examinations of the differences in the qualities of “thought in action” were opportunities to enhance research and practical understandings of teacher knowledge.

In mathematics, organization of content knowledge is considered the foundation for expertise in the domain (Bransford et. al., 1999; Chi & Ceci, 1987; Munby et. al., 2001). Expert domain knowledge is organized around concepts and principles. These are the “big ideas” in mathematics (Leinhardt, Putnam, Stein, & Baxter, 1991; Ma, 1999; Neimi, 1997; Schifter, 1997), not isolated facts and algorithmic procedures. Fosnot & Dolk (2002) consider “big ideas” as “the central, organizing ideas of mathematics- principles that define mathematical order and are characteristics of shifts in learners’ reasoning- shifts in perspective, logic, and the mathematical relationships they set up” (p.17). Teachers must have a deep understanding of these ideas in order to give students access to them.

Teacher change

Research on issues of teacher change and development also informs this study. In a current review of research on teacher change programs, Richardson and Placier (2001) summarize research on studying change in teacher thinking, emphasizing the contextual, constructivist, and cognitive principles in teachers changing their knowledge and practice. Educational research on individual teachers captures the developing changes in teacher thinking within professional development contexts. Research conducted by Franke, Fennema, & Carpenter (1997), Goldsmith and Schifter (1997) and Schifter and Simon (1992). Schifter (1995) and Schifter and Fosnot (1993) define elementary mathematics teachers *in transition* as those becoming more constructivist in their approach to teaching and thinking about the subject. Others (Stein & Brown, 1997) have explored teacher change (i.e., transition) as a “transformation of participation” within a community of teaching practice in school settings.

Two teachers in a pilot study (Schneider, 2001) and the teachers in this study have participated or are participating in professional development (PD) programs to broaden their content knowledge for teaching mathematics. These programs are designed to improve and deepen teacher mathematics knowledge and practice (Even & Tirosh, 1995; Franke, et. al., 2001; Franke, et. al., 1997; Schifter & Fosnot, 1993). They can be formal experiences, such as summer institutes/ workshops, or informal experiences in schools as teachers plan and

shared classes together using reform based curricula such as 1) *Connected Mathematics Project (CMP)*; 2) *TERC: Investigations in Data, Number, and Space*; or 3) *MiC: Mathematics in Context*. Curricula and workshops approach mathematics as investigations, with problems contextualized in story situations, challenging teachers, and examining student thinking. In workshop style projects, teachers often play the role of student, transforming their teaching conceptions of the content and practice.

Overview: Pilot and Current Study

This study's purpose is to describe the continuum of knowledge for teachers who have developed *deep knowledge* for teaching division with fractions as well as those teachers *in transition* towards expert-like thinking and teaching of the topic. The study, in its microanalysis of the topic, adds to a more complete understanding for teaching the mathematical concepts involved in dividing with fractions.

Pilot study

A pilot study with five teachers used division with fraction interview tasks. These problems were developed by Ball (1988) to examine teachers' knowledge of and teaching for dividing with fractions. Ball's interview tasks were specific problems from the curriculum that are challenging and are often taught procedurally (i.e., subtraction with regrouping, multiplying large numbers, division with fractions).

Teachers in the pilot study were presented 5 tasks. Their think alouds were about ways for:

1) Solving and representing number sentences, $1\frac{3}{4} \div \frac{1}{2}$ and $\frac{1}{2} \div 1\frac{3}{4}$.

2) Solving and translating word problems, Wally has $6\frac{2}{3}$ yards of material.

It takes $\frac{5}{6}$ of a yard to make his scarves. How many scarves can he make? And Wally realizes that $\frac{5}{6}$ of a yard of material is left over. He has

promised to give 6 people equal shares and one person $\frac{2}{3}$ of a share of

the material. How much material would each person receive?

3) Creating a concept map of the concepts they believed important for students to understand fractions.

From on the interview data, three of the five teachers were not able to construct any representations, multiple translations, or make connections in the interview tasks. Their results were similar to those of novice teachers from two earlier studies (Ball, 1988; Ma 1999). Two of the five teacher's explanations, however, stood out. The first teacher constructed multiple representations, translations, and connections across the interview tasks, similar to Chinese expert teachers in the study by Ma (1999).

The second teacher of interest combined instances of procedural and conceptual understandings in constructing translations, representations, and connections. This teacher's thinking was suggestive of being *in transition*. From these two interviews, the dissertation goal became examining a larger sample of

these two types of teachers: 1) teachers *in transition* and 2) teachers with *deep knowledge* for understanding and teaching the topic. The purpose of the dissertation would be to confirm and perhaps expand a provisional set of criteria defining teachers with deep knowledge. Further, contrasting larger groups of both DK and IT teachers would help to define the differences in teachers' content knowledge (i.e., SMK and PCK) for division with fractions.

Dissertation study

Fourteen individual interviews with two groups of middle school mathematics teachers were analyzed. The interview tasks were: 1) pre interview questions; 2) tasks for dividing with fractions in number sentence and word problem forms; 3) a concept map activity; and 4) post interview questions on significant professional development (PD) experiences helpful in developing teachers' knowledge and practice.

Mathematics education specialists referred individual teachers for interviews after these recruitment specialists were shown the interview tasks (Appendixes D-H) and the *provisional* criteria designation categories (see Table 2). These specialists were not expected to identify which of their references would be an *in transition* or a *deep knowledge teacher*.

Teachers' explanations/ performances on interview tasks and their alignment with the *final* (i.e., expanded) teacher category criteria (see Table 4) were used to create the two groups. The eight teachers of the in transition (IT) group are teachers transitioning from procedural or algorithmic points of view to

connected, integrated, and reform minded ways of teaching and thinking about fractions. They are building a knowledge base that resembles a deep knowledge teacher's understanding of the topic. The six teachers in the second group are the deep knowledge (DK) teachers. They consistently constructed multiple representations, translations, and made connections between fractions and connected mathematical ideas. All of the participating teachers were currently or had been involved with curricula and/ or workshop style PD projects for at least two years. Throughout the text, the first group will be referred to as in transition (IT) teachers and the second group will be referred to as deep knowledge (DK) teachers.

One critique of research on comparing subject matter (SMK) and pedagogical content (PCK) knowledge of teachers has been the conceptualization of teacher knowledge as opposites: as declarative versus procedural knowledge, conceptual versus procedural understanding, or expert versus novice (Wineburg, 1997). The proposed study does not dichotomize teacher knowledge into extremes, but describes qualitative patterns of teacher understanding both across and within the cognitive interview tasks.

CHAPTER 2

Review of the Literature

Rational numbers, particularly in fraction forms, are considered to be the most complex and least understood topics in the middle school mathematics curriculum. Post, Harel, Behr and Lesh (1991) stress the following reasons why understanding rational numbers is important: 1) from a practical perspective: to effectively deal with “real world” situations; 2) from a psychological perspective: to develop and expand the mental structures necessary for continued intellectual development; and 3) from a mathematical perspective: to provide the foundation for elementary algebraic operations. The topic continues as a frequent research focus for examining mathematics teachers’ knowledge and understanding (Armstrong & Bezuk, 1995; Ball, 1993; Graeber & Tirosh, 1991; Lamon, 1999; Leinhardt & Greeno, 1986; Ma, 1999).

The research areas of interest are detailed in this literature review section. The first part provides background, describing the historical changes in educational research from content-free studies to content issues as primary to understanding teaching and learning. Knowledge of fractions, structural complexity of the topic, and models for dividing with fractions are the second section of the review. Structural research perspectives on fractions are those that describe the “personalities” of rational numbers from mathematical points of view (Behr et. al., 1992; Kieren, 1992; Streefland, 1993). This section also includes research on the types of disconnections in understanding division with fractions

and challenges in dividing by the larger fraction. Students' interpretations of fractions and research on teachers' approaches for dividing with fractions are the third and fourth sections of the literature review. The fifth section reviews research on the characteristics and assessment of expertise in teaching of mathematics. The sixth section describes research on issues of teacher change and teacher development. The proposed study is then described, its purpose, hypotheses, and research questions.

Background: Subject Matter as Focus

Research examining teaching within specific school subject areas is relatively new to the research on teaching. This call to "psychologize" subject matter was initiated through the research and writings of Shulman and others (Ball, 1988, 1991; Leinhardt, 1993, 2001; Leinhardt & Greeno, 1986; Shulman, 1986; Wilson, Shuman, & Richert (1987); Wineburg, 1990) and was summarized by Shulman and Quinlan (1996) in a review, *The Comparative Psychology of School Subjects*. In the review, innovative educational research studies from different school subjects are described, with each research program placing the content as the focal point from which to examine teaching.

Shulman and Quinlan (1996) pointed out that teaching had traditionally been studied and operationally defined as collections of outcome variables. This line of research focused on statistical relationships between teacher knowledge and student test scores and did not examine the qualities of teacher knowledge as relevant features. In these studies, student achievement was hypothesized as

a direct link to teacher understanding and knowledge. Teacher knowledge, as an independent variable, was often operationally defined as the college grade point average of the teacher, the number of courses, or the types of degrees the teacher completed. The dependent variable was student scores on standardized tests. The better the student performed on a standardized test, it was assumed, the more knowledgeable teacher she or he had. This type of research was criticized because it was conducted without regard to the content of instruction, the subject matter in teaching (Ball, 1993; DeCorte, Greer, & Vershaffel, 1996; Koehler & Grouws, 1992).

Wilson (1989) and Leinhardt (2001) argued that these operational definitions of teacher knowledge were actually the result of methodological inconsistencies since the studies had not directly examined teachers' content knowledge (i.e., SMK and PCK). Wilson (1989), Leinhardt (1993; 2001), and others (Shulman, 1986) emphasized that it was not the achievement scores of students that were indicators of teacher knowledge but what teachers knew about what they taught and how they taught what they knew. Teachers' representations, and translations for the subject matter (i.e., their SMK and PCK) were the important variables and were missing. Subsequent research then developed more accurate and detailed measures of teachers' skill in representing subject matter to reveal teacher understanding of the content. These measures were better related to how well students understood and used the content in different subjects (Ball, 1988; Kennedy et. al., 1993; Wilson, 1989; Wineburg,

1990) to assess teachers' subject matter (SMK) and pedagogical content (PCK) knowledge. These researchers suggested a focused analysis of teaching as an intersection of subject matter and pedagogy. As Wineburg and Wilson (1991) pointed out, "Any act of teaching presents both the content of the domain and a model of the domain" (p. 315), stressing the connections between SMK and PCK forms of content knowledge.

Knowledge of Fractions and Structural Complexity of Topic

In this section, the complexity of the fraction topic is discussed along with 1) division with fractions models proposed by mathematics education researchers; 2) types of disconnections students and teachers can make in understanding the meanings of dividing with fractions; and 3) challenges in understanding representations for dividing a smaller fraction by a larger fraction.

Various representations of fractions

Fractions are difficult to teach and difficult to learn because, as numbers, they stand for various representations at the same time (Leinhardt & Smith, 1985). On national assessments for students' performance on mathematics, students often solve fraction problems using algorithms. Mistakes on solutions from the incorrect use of algorithms are thought to demonstrate a reliance on procedural "rules" (i.e., using an algorithm such as invert and multiply) that are incorrectly memorized (National Research Council, 2001). These behaviors reflect the, "detachment of algorithmic behavior from meaningful models of fractions" (Kieren, 1992, p. 326). To address these concerns, research attention

has been given to the general construct of fractions, including semantic analyses of the construct and has emphasized division of fraction knowledge.

The construct complexity of fractions has been well documented (Behr et. al., 1992, 1993; Fosnot & Dolk, 2002; Kieren, 1988, 1992; Streefland, 1993). This complexity requires the learner and teacher to understand conceptually the many different meanings of a fraction: 1) part to whole; 2) quotient; 3) ratio; 4) operator; 5) absolute/relative; and 6) measurement/ linear quantity.

There is agreement by mathematics education researchers (Behr et. al., 1992; Kieren, 1992; Lamon, 1999) that the fraction concept stands for at least six different ideas at the same time. Table 1 summarizes these different meanings. The summary is from reviews of rational numbers from Behr et. al (1992), from the research of Kieren (1992), and from English and Halford (1999).

Using an example of $\frac{3}{4}$, the following representations for the number are possible:

Table 1

Different Meanings of Fractions

Type	Meaning	Example
Part to whole	An object is divided into a specific number of parts in a set model or area/region model.	$\frac{3}{4}$, 3 out of 4 equal-sized shares.
Quotient	Decimal model of .75 is a result	$\frac{3}{4}$, 3 divided by 4.
Ratio	Whole to whole comparison, Parts of a set	$\frac{3}{4}$, 3 whole units compared to 4 whole units.
Operator	Multiplication/Division	$\frac{3}{4}$, three quarters of a number, object, or set where the numerator extends and the denominator shrinks the amount.
Absolute and relative quantities	Two fractions can be relative quantities to each other or absolute quantities by themselves.	$\frac{3}{4}$ and $\frac{9}{12}$ or $\frac{1}{2}, \frac{2}{4}, \frac{4}{8}$ are relative to each other and absolute as well.
Measure model	Any fraction is a distance of measure between other quantities.	$\frac{3}{4}$ is $\frac{1}{4}$ of a distance from 1.

The table categorizes the different meanings a single rational number can represent *prior* to addition, subtraction, multiplication, and division.

Knowledge about the structure of a fraction as communicated in instruction is important. Teachers may understand one meaning better than another-- possibly leading to overgeneralization and subsequently, student difficulty in connecting a familiar fraction idea (i.e., part to whole) with more challenging ones (i.e., operator, absolute/relative). Challenging characteristics also include recognizing fractions as composite units (or units of units) (Steffe,

1988), as special cases of whole numbers (Moss & Case, 1999), and as equivalences or magnifications (Niemi, 1996) since different fractions can be associated as the same fractional quantities, quotients, and operators (such as $\frac{1}{2}, \frac{2}{4}, \frac{4}{8}$).

Such layers of meaning are may cause to disconnect students to whole number relationships to fractions (Armstrong & Bezuk, 1995) and over use whole number models. This consequence may be a matter of not having instructional opportunities to understand and discover connections and differences between rational and whole numbers. Lamon (1999) explains:

As one moves from whole number to fraction, the variety and complexity of the situation that gives meaning to the symbols increase dramatically. Understanding of rational numbers involves the coordination of many different but interconnected ideas and interpretations. There are many different meanings that end up looking alike when they are written in fraction symbols. (p. 30-31).

Research models that have organized these topic features include Kieren's (1988, 1992) theoretical model of rational numbers. Kieren combines the multiple references of fractions into a hierarchical, organizing scheme, "a scheme for personal knowledge of rational number" (Kieren, 1992, p. 347). The goal of the model is to emphasize mathematical features of the construct to assist research on rational number thinking and organization of curriculum development.

An ideal rational number knowledge structure, according to this model, organizes knowledge and representations of rational numbers as having connected and corresponding levels. These levels are structured from the simplest to the most complex. The first level contains facts and surface details of the fraction, such as being able to identify numerator or denominator. The second level includes partitioning, forming units, and developing the idea of equivalency with fractions. The third level includes the constructs of rational numbers as measures, quotients, ratios and operators. The fourth level includes knowledge of different forms of fraction models such as geometric and linear types in rational number thinking. The fifth and sixth levels include operational connections of context such as the role of addition, multiplication, division and the overall organizing force of thinking about fractions, which Kieren refers to as the quotitive field and others have called the multiplicative field (Behr et. al., 1992; Steffe, 1988).

Kieren suggests the model be seen as a whole and that the lower level ideas are also complex. The concepts at the lower levels are about fractional situations, the middle level has constructs about quantitative and relational properties, and the highest level represents a deductive knowledge of rational numbers. A student's and a teacher's proficient knowledge of rational numbers, according to this model, would have connections among and across levels. Students would build on their informal knowledge and images on their knowledge before using the language of notations and algorithms. This is considered

building a “number sense” prior to working with algorithms. Algorithms can be dangerous because as procedural shortcuts they replace the various math strategies students can develop for fraction problems (Fosnot & Dolk, 2002).

Behr et. al. (1992, 1993) extended Kieren’s (1988) model and offer a semantic analysis of division with rational numbers as part of developing understanding of a “multiplicative conceptual field.” Problems involving multiplication or division are part of this field, requiring the learner to transform problem amounts. Their analysis has examined the part to whole and operator features of rational numbers, focusing on discrete and continuous qualities of these numbers. These researchers define the operator construct by focusing on the role of the numerator and the denominator of the fraction, since both parts of the number impact the result. For example, the numerator can extend the amount while the denominator causes a contraction. The numerator and denominator, according to the researchers, can behave as: 1) duplicator and partition reducer; 2) stretcher and divisor; and 3) multiplier and divisor (Behr et. al., 1992). The researchers stress that these interpretations encourage tactile, visual instructional experiences with manipulative aids, real-world situations, pictures, and written symbols. Instruction that encourages reasoning about a fraction before procedure (i.e., being able to interpret the numerator and denominator specific to the problem) is stressed.

Fosnot & Dolk (2002) describe the learning paths of students’ constructions for understanding fractions, decimals, and percents as

“landscapes of learning” with focus on the importance of landmark strategies, big ideas, and the development of models. An example of a landmark strategy is the construction and support of student’s knowledge in using important fractions such as $\frac{1}{2}$ and $\frac{1}{4}$. The “big ideas” in the landscape of learning fractions include 1) relating fractions to both division and multiplication and 2) multiplication and division of rational numbers as the act of “relations on relations” (p. 137). Models of fractions in Fosnot and Dolk’s (2002) model include understanding ideas of fair sharing, partitive division, and fractions as: 1) part/ whole relations; 2) decimals; 3) percents; 4) proportions; and 5) operators.

Division models

The different types of division models add another layer of complexity for understanding fractions and division. Division models have been defined (English & Halford, 1994; Fosnot & Dolk, 2002; Ma, 1999) as product and factor, partitive (fair sharing/grouping) models, and measurement (quotitive) models. Every type of division situation with whole or rational numbers can be represented as any of these models. Examples of each for $\frac{3}{4} \div \frac{1}{2}$ include: 1)

Product and factor model: To find a factor that when the factor is multiplied by $\frac{1}{2}$, the answer is $\frac{3}{4}$. A story representation example: If the area of the box is $\frac{3}{4}$ square feet, its width is $\frac{1}{2}$ foot, what is its length?; 2) *Partitive model:* If $\frac{3}{4}$ is distributed into $\frac{1}{2}$ groups, the whole group is of size $1\frac{1}{2}$. A story representation

is: If it takes $\frac{3}{4}$ of an hour to clean my desk this week and that is $\frac{1}{2}$ as long as it took last week, how long did it take to clean my desk last week? ; 3) *Quotitive*

(measurement) model: To find how many $\frac{1}{2}$ s there are in $\frac{3}{4}$ s. For example, how many $\frac{1}{2}$ inches are there in $\frac{3}{4}$ s of an inch?

The easiest meaning of division to demonstrate is considered the measurement or quotitive model. The partitive and product factor models model have not been given as much attention. In an article (Ott, Snook, & Gibson, 1991) on partitive division in the *Arithmetic Teacher*, the authors found no related articles explaining the partitive meaning of division and no textbooks that gave equal attention to all three models: measurement/ quotitive, partitive, and product/factor models.

Each model (i.e., representation) type: product/ factor, partitive, and quotitive (i.e., measurement) can be created for a division problem with whole numbers or rational numbers. The frequency of these types of models between the deep knowledge and in transition groups on division tasks was analyzed.

Disconnections in understanding division with fractions

The complexity of fractions and division involves considering a number of mathematical connections at the same time (Behr et. al., 1992; Kieren, 1992). This section examines several types of misunderstandings of dividing with fractions and their possible causes.

Misunderstanding the connections between fractions and corresponding operations can result from instruction. For example, teaching procedural rules for different translation (i.e., solution) situations such as changing a percent to a decimal or changing a fraction to a percent can lead to misunderstanding the connections among these forms of rational numbers (National Research Council, 2001; Moss & Case, 1999). The “rules” are taught as procedures with steps (i.e., such as: to change a percent to a decimal, drop the percent sign and move the decimal point over two places to the left or divide by a fraction by multiplying by its reciprocal). These procedures are algorithmic shortcuts, not connected to the relationships between percents and decimals or between division of whole numbers to division with fractions. These unconnected shortcuts lead to confusion since the “rules” are disconnected from the conceptual relationships between the numbers and operations.

The operation of division, like multiplication, adds complexity for working with composite units (i.e., more than one fraction at a time). When students and teachers have limited understanding of these operations it can be reflected in their belief that “multiplication always makes bigger” and “division always makes smaller” (Tirosh, 2000; Tirosh & Graeber, 1990). This is an overgeneralization from dividing with whole numbers where the divisor number can be smaller than the dividend amount (i.e., 12 divided by 3: small answer so division makes smaller and 12 multiplied by 3: large answer so multiplication makes bigger). Students and teachers who approach division in this limited way do not have an

understanding of the shared models (i.e., product/factor, partitive, quotitive) between whole number division and division with fractions. An example of a shared model approach between whole numbers and division can be seen in dividing 12 by 3, we can use a quotitive approach: How many groups of 3 are in 12? The same model can be used when we divide the fractions: $\frac{3}{4}$ divided by $\frac{1}{2}$. How many groups of $\frac{1}{2}$ are there in $\frac{3}{4}$?

The National Research Council's, *Adding it Up*, (2001) suggests that instruction focused solely on “fair sharing” models by teachers and students can also lead to disconnections among the various concepts of fraction and division of fractions, resulting in a rule based, algorithmic focus in instruction. Fair sharing models are those in which an amount can be shared equally. For example: If 4 students are sharing 8 pizzas, how many pizzas does each student receive? Fair sharing models are considered “entry points” for understanding rational numbers. This model does not naturally lend itself to understanding sharing situations when the share is not equal (i.e., when the solution includes a remainder). The subject (SMK) and pedagogical (PCK) specific knowledge base of the teacher may be more responsible for improving understanding above and beyond informal concepts of partitioning units. Ma's (1999) expert teachers, for example, were able to construct various appropriate real-world contexts (e.g., work, acreage, filling up containers) for representing division of fractions problems using quotitive, product and factor and partitive contexts. This may disprove English and Halford (1994) claim, “Division with fractions is difficult to

model and has few real world applications (p.215).” This was not the case for Ma’s 8 expert teachers who created a variety of realistic representations.

Dividing by the larger fraction

Research examining teacher knowledge of division with fractions has predominantly focused on problem situations where the divisor fraction is smaller than the dividend fraction. (Ball, 1988; Kennedy et. al., 1993; Ma, 1999; Schifter, 1997). The inverse of this situation has not been examined closely. Such problems are challenging and more complex to understand when the divisor is in mixed number form (i.e., $\frac{1}{2}$ divided by $1\frac{3}{4}$). The research literature on dividing with fractions has not discussed teacher knowledge in generating models (i.e., representations), translations (i.e., solutions), and connections when the divisor is larger than the dividend. Why is it easy to create story situations in which $\frac{1}{2}$ can be divided by 2 but not $\frac{1}{2}$ divided by $1\frac{3}{4}$? In the interview tasks, teachers are challenged to create representations for problems with a larger fractional divisor than the dividend fraction.

Interpretations

In this section, research, examining the development of students’ interpretations of fractions, is discussed. These studies each propose different “best” practice approaches for students to understand fractions.

Background

Researchers have examined the ways that “natural” or intuitive strategies can be both helpful and distracting for building conceptual models (Niemi, 1996, 1997; Mack, 2001; Moss & Case, 1999; Tirosh, 2000). Researchers have also studied “quantitative” approaches to learning fractions. These approaches are not necessarily mutually exclusive and yet are quite different in terms of where the starting points for instruction can be and where instruction can lead.

The various conceptualizations for learning fractions are developed in instructional contexts. There is not consensus, from the research, on how to initiate this learning in classrooms or how the various interpretations can bridge into instruction of fractions and operations.

The informal models for fractions that students have assumed to develop initially are equal sharing/grouping models that are taught in use with whole numbers rather than with fractions (Mack, 1990, 2001). These studies stress the part/whole meaning of fractions (Kieren, 1992; Mack, 2001; Steffe, 1988). From this knowledge, researchers suggest building subsequent instruction. Other researchers (Moss & Case, 1999; Niemi, 1996) encourage a “quantitative” modeling approach. The quantitative modeling view encourages instruction that focuses on fractions as numbers on a number line between infinite amounts of quantities (National Research Council, 2001). This approach emphasizes the measurement/ linear and absolute/relative meaning of fractions. Research

examples from each of the perspectives, equal sharing/grouping versus measurement/ linear interpretations, are described in the following sections.

Intuitive interpretations

Researchers interested in the development of fraction number concepts in children advocate instruction to explore the intuitive knowledge understandings children bring to classrooms. Students' initial understandings (Kieren, 1992; Mack, 1990, 1995, 2001; Streefland, 1993) are hypothesized as "equal" sharing/grouping concepts. This initial way of knowing is used to encourage understanding of dividing amounts into even parts. A teacher can then use this base to expand a student's knowledge of fractions and operations involving fractions.

Mack's (2001) research emphasizes the intuitive models for fractions that students bring to instruction and how to use those intuitive models to help students understand more complex ideas for multiplying with fractions. One study examined how students could build on their informal knowledge of sharing/grouping or splitting units for multiplying fractions. The goal of the research instruction was to help students understand multiplication with fraction situations that involved taking a part of a part of a whole (i.e. $\frac{2}{3}$ s (one part) of $\frac{3}{4}$ s (another part) of a pizza (whole)). In this experiment, Mack's focus is on the operator and part to whole category of the concept (see Table 1). The sample of fifth grade students, three boys and three girls, were from low to middle SES families. Their mathematics teachers identified each as having average

mathematical skills based on their performance on standardized achievement tests. From an initial screening, all six students showed some knowledge of fraction concepts and could identify fractions such as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{6}$ when given materials or pictures. However, they could not compare fractions to each other. Also none of the students could multiply fractions when the problems were represented symbolically ($\frac{3}{4} \times \frac{2}{3}$) and did not understand that different fractions could represent the same amount (i.e., $\frac{2}{4} = \frac{1}{2}$). None had received instruction on multiplying fractions in their fifth grade classrooms.

Each student received individual instruction with the researcher in thirty-minute sessions over three months (i.e., 18 sessions). Case studies were developed from students' responses. Individual students first solved equal sharing/grouping situations (i.e., share a cookie into thirds), explained their understanding of what the fraction meant, and then solved problems that involved taking a part of a fractional part (i.e., multiplying two fractions together). As the multiplication problems became more difficult, the researcher encouraged students to build on their informal knowledge of sharing/grouping and splitting units. All of the problems were presented orally and the students were asked to think aloud as they came up with solutions. The researcher tailored each session for the individual student based on the previous session's progress.

The results indicated that all six students used their informal knowledge to solve problems that involved “friendly” fractions such as finding $\frac{1}{4}$ of one half and one half of one half. However, they could not solve other, more complicated problems that involved taking part of a fractional quantity and could not connect their solutions to multiplication. Students did not recognize that they were multiplying the parts in the problem. Instead they saw the problems as division. The act of taking apart (i.e., taking $\frac{3}{4}$ of $\frac{1}{2}$ or $\frac{1}{2}$ of $\frac{3}{4}$ in $\frac{1}{2} \times \frac{3}{4}$) when using part/whole relationships may have led students to the conclusion that they were breaking up parts as they had learned in whole number division. Mack proposed that using students’ informal knowledge should be “modified in a way that reflects the critical concepts underlying the multiplication of fractions” (Mack, 2001, p. 293). Mack did not specify, however, how this modification could be carried out instructionally.

Multiple interpretations

Moss and Case (1999) devised an instructional sequence to introduce fourth graders to rational numbers. The researchers developed a curriculum that was a comprehensive approach to developing understanding of the rational number system. The relationships among percents, decimals, and fractions were made explicit in a series of instructional sequences. The researchers hypothesized that the historically poor performance of U.S. students on rational numbers assessments had been the result of weak conceptual understanding of

different rational number models (i.e., fractions, decimals, percents). They tested their curriculum in a design comparing sixteen fourth grade students who received the experimental curriculum with thirteen matched comparison group students who received a traditional curriculum.

The experimental group received twenty 40-minute instructional sessions over a five-month period. The comparison group received twenty-five 40-minute lessons over a slightly shorter period of time. Two different teachers taught the control and the experimental groups. Both teachers used the same text when the topics were not rational numbers and only worked with rational numbers during allotted times. Both teachers used manipulatives, group work, and discussion as their instructional styles. The differences between the two groups were the curriculum they used for rational numbers and the difference in length of the sessions for experimental and comparison classes.

The instructional exercises for the experimental curriculum were given in sequential order. In the first exercises, students assigned a value from 1% to 100% on water levels to indicate fullness of a beaker of water. In a series of what the researchers call “sequence of visual motor splits” (p. 126), students in successive lessons, were asked to halve the initial water levels (100% to 50% to 25%, etc.). These lessons focused on encouraging understanding of percentage values. Then students were then introduced to two place decimals. This was done through measurement line exercises (e.g., 5.25 is a distance 25% of the way between 5 and 6). The final exercises focused on interchangeable use of

fractions, decimals, and percents. The comparison group curriculum did not make sequential connections between percents, decimals, and fractions and the fraction, decimal, and percent topics were taught separately from each other.

A rational number test was given to both groups of students before and after instruction. All students had been initially tested on 41 pretest items. After the end of the curriculum, each student was interviewed individually and given 45 posttest items. The items included nearly equal numbers of percent, decimal, and fraction problems. Question types included the categories: Nonstandard computation, compare and order, misleading appearance, word problems, interchangeability of models, and standard computation. The experimental group performed significantly better on posttest items than the comparison group. They were able to translate among fractions, decimals, and percents on word problems.

Moss and Case (1999) suggest that introduction to rational numbers does not have to use one method only; it can be a combination of rational number models. Their program provides important alternatives for conceptualizing rational numbers, moving away from pie chart or pizza demonstrations by using more variety in encouraging initial understandings of students. The curriculum attempted to introduce students to the complexities of the rational number system rather than what they refer to as a “mismodel” of rational numbers (i.e., as a special class of whole numbers). These researchers do not stress the connections between rational and whole numbers but rather the varieties of

features that make fractions different from whole numbers. This research is supported by Behr et. al.'s (1992) recommendation for perceptual variability in rational number instruction. Moss and Case do not explicitly discuss teacher knowledge of these connections, but it is apparent that teacher knowledge needs to be as connected as the curriculum developed by the researchers.

Measurement interpretations

Niemi (1996, 1997) created mathematical assessments to capture the deeper ideas within the fraction domain. These types of assessments were aligned to the recommendations by reform groups of mathematics education including NCTM and the National Research Council. For Niemi, previous research on explaining fractional knowledge development in students had narrowly focused on part/whole relationships, interfering with recognizing fractions in various meanings and not only as parts of wholes.

Niemi (1997) stressed a model of the rational number concept as an expression of a relationship between two measured quantities with the following principles: 1) all intervals in the measured quantity must be equal; 2) any quantity can be measured by some smaller quantity or partitioned into smaller quantities; 3) a fraction is a number representing a relation between two numbers; 4) between any two numbers, there is an infinite amount of fractions; 5) for any fraction, there are an infinite number of equivalent fractions; 6) any quantities of the same type can be compared by measurement; 7) two quantities can be

compared more easily if they are in terms of the same measurement unit and; and 8) fractions can represent relations between quantities (Niemi, 1997, p.352).

There were three hypotheses to test the assessments: 1) those students who understood the deep meaning of a fraction would be able to see the multiple ways to represent the same fraction (i.e., equivalent fractions); 2) representational knowledge would be measured by, “the ability to identify correct representations of a given fraction” (Niemi, 1997, p. 352); and 3) those students who could identify more correct models and avoid misleading representations were hypothesized to be able to develop and use models on explanation tasks.

Five hundred and forty fifth grade students in 22 classrooms participated. Students spent two class sessions completing assessment measures on representational knowledge, problem solving, justification, and explanations. Fifth graders were chosen because the researcher hypothesized that if the construct of fraction was not developed by this time, then subsequent understanding would become procedural and algorithmic. It was assumed that a conceptual understanding of the topic in elementary school would be the base for understanding in later grades.

Students were tested on three measures of fraction knowledge. The first, a fluency task, asked students to circle fractions that represented the same amount. Students chose among correct equivalent representations as well as distracters. The second, a problem-solving measure asked students to solve six symbolically presented fraction problems, to compare fractions of a distance, to

evaluate the truth of addition problems and equivalency statements, and to find a fraction between two other fractions. The third, an explanation measure, asked students to write and draw pictures of their understanding of fractions. Niemi's use of the three measures was created to follow NCTM's recommendations regarding multiple sources of evidence, including measures of implicit (i.e., representational knowledge) and explicit (i.e., justification, explanation) understanding.

Results confirmed the initial hypothesis that students with higher levels of representational knowledge produced more explanations and justifications and were better at problem solving. From scores on the measures, students in the top 25% were compared with the bottom 25%. Levels of representational knowledge development predicted performance on how well the two groups could problem solve, justify, and explain tasks involving fractions. Even though Niemi did not specifically comment on the role of teacher knowledge, it follows that when instruction is connected to the various representations of fractions, then teacher understanding is connected as well.

Approaches assessing student interpretations of fractions from these studies indicate that students who are able to think about fractions in multiple ways not only are more successful in problem solving but also are able to justify and explain their answers. In the next section, research examining the types of models of thought and action that teachers have about fractions is presented.

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teach and learn mathematics topics. For her dissertation (1999), she wanted to compare the responses of the US teachers with Chinese teachers on the same set of TELT questions for elementary mathematics. From TELT transcripts, Ma chose to analyze the responses of twenty-three in service American elementary school teachers: 1) 11 were participating in a national PD program and, on average, had 11 years of teaching experience, 2) 12 were first year teachers who were about to complete their Master degrees. Ma interviewed 72 Chinese teachers using the same TELT elementary mathematics interview protocol. The Chinese mathematics teachers were from five elementary schools in urban and rural settings. Forty of the Chinese teachers had less than five years of teaching experience, twenty-four had more than five years, and eight Chinese teachers had an average of 18 years of teaching experience. These eight also were the ones with the most developed knowledge across all interview tasks and were identified as teachers a profound understanding of fundamental mathematics (PUFM).

The interview protocol Ma used was drawn from the elementary mathematics part of the TELT study and a dissertation (Ball, 1988). Both were developed to evaluate prospective or beginning teacher knowledge (Ball, 1988; Kennedy et. al., 1993). Each interview question presented the teachers with a hypothetical task in which they were asked to: 1) teach a topic; 2) respond to a student's mistake; and 3) generate a model (i.e., a representation) of the topic. The interview questions focused on difficult content in the elementary

mathematics curriculum (e.g., subtraction with regrouping, multidigit multiplication, division with fractions, the relationship between area and perimeter). The interview format was a unique opportunity to go beyond a questionnaire measure in order to find out what and how teachers thought about the given mathematical topics.

In Ma's study, less than half (43%) of the 23 of the American teachers computed the division by fraction problem correctly (i.e., solved for the right answer of $1\frac{3}{4}$ divided by $\frac{1}{2}$). The remaining American teachers had partial ideas about how to use the algorithm or gave an incomplete algorithm and an incomplete answer. On the representation portion of the problem, none generated representative models for the problem. Nearly half of these teachers had been procedurally correct in being able to use the algorithm of invert and multiply. Yet, they were unable to explain why the algorithm worked or how the symbolic problem ($1\frac{3}{4}$ divided by $\frac{1}{2}$) could be presented as a model in a context. Their incorrect representations were categorized according as: 1) confusing division by $\frac{1}{2}$ with division by 2; 2) multiplying by $\frac{1}{2}$; 3) no representation; and 4) inappropriate representations (i.e., If each child gets a half a Twinkie and I have 1 and $\frac{3}{4}$ of a Twinkie, how many kids will get a piece of the Twinkie?) (Ma, 1999, p.67).

Ma concluded that the American teachers had an inadequate understanding of dividing by fractions, which affected their ability to create

representations and appropriate models. The eight Chinese expert teachers were able to generate several models and representations for the problem. As discussed earlier, fractions, particularly division with fractions, are excellent content areas to examine the depth of teacher knowledge and teaching of the topic.

Examining students' mistakes

Researchers have categorized the types of mistakes that students make on division of fraction problems and assessed teacher awareness of mistakes. In this research, a teacher's capacity to diagnose students' initial errors was thought to reveal teachers' knowledge. Tirosh (2000) examined prospective elementary teachers' knowledge of the types of mistakes that students make when dividing with fractions. The mistakes were categorized as algorithmic, intuitive, and formal. In subsequent research, Tirosh (2000) designed a methods course for increasing teacher awareness of the types of students' mistakes on fractions, thereby refining and extending teacher knowledge of the topic.

Tirosh's argument was that the mistakes made by students in understanding fractional numbers were products of missing connections in teachers' understandings of their algorithmic, formal, and intuitive mistakes. Algorithmic mistakes were defined as procedural "bugs," as in forgetting which number to invert in following the division of fractions algorithm. Intuitive based mistakes were defined as overgeneralization errors from using a sharing/grouping approach to division. The third category of mistakes, mistakes

based on formal knowledge, was a combination of limited understanding of the fraction concept and an inadequate knowledge of the operations. An example of a mistake based on formal knowledge would be one in which students believe that division is commutative so that 1 divided by $\frac{1}{2}$ should be $\frac{1}{2}$ since $\frac{1}{2}$ divided by 1 is $\frac{1}{2}$.

Thirty prospective elementary teachers were given a content specific questionnaire as a pretest to assess their subject matter and pedagogical content knowledge of fractions. The researcher also interviewed each prospective teacher. Each of the 30 participants was enrolled in a mathematics methods course designed by the researcher to expose and familiarize teachers with students' ways of thinking on rational numbers. Each activity revolved around questioning participants on, "Why is this so?" so that teachers would question procedures and be able to think about their own understanding of mathematical structures. The goal of the intervention was to change prospective teachers' awareness of the common mistakes that students make with division of fractions and the reasons to which the teachers attributed these difficulties.

Initially, the majority of participants said that one source for students' incorrect response was in not memorizing the steps in the procedure or confusing some steps. Another source of error described by the prospective teachers was that students found the idea of a larger answer when dividing by a fraction unreasonable.

On the pretest items, 26 out of the 30 participants said that an incorrect application of the invert and multiply algorithm and not remembering the steps caused difficulties for students. They did not attribute mistakes to missing or incomplete knowledge to possible models of division and fractions that students may have had. Tirosh then designed activities for the methods course that focused on challenging teachers' conceptions of the sources of students' mistakes. Class sessions were structured around group discussions of topics such as "Can fractions be divided in an easier way?"

At the end of the course, teachers' knowledge of students' knowledge about division of fractions improved and included more connected/ conceptual interpretations than only noticing algorithmically based mistakes. They became aware of the restrictions of sharing/grouping models for division with fractions and were able to construct different models to interpret division of fraction problems.

Focusing on student's mistakes as a way of developing teacher knowledge may have resulted in improved teacher knowledge. However, this focus may still be limiting since the possible responses that students can give are more varied than the three organizing categories of algorithmic, intuitive, and formal knowledge. Students, for example, may still not understand other models such as product/ factor or partition models. Further, being aware of student mistakes may not be the same as being aware of students' knowledge. The links

between reflecting on and changing one's own teaching practice as a result of access to student mistakes is not explicitly discussed in Tirosh's research.

Teacher change and development

The work of Schifter and her colleagues (1995,1997), through the Teaching to the Big Ideas (TBI) project, focused on teachers' examining their own understandings of mathematics topics such as dividing with fractions. Teachers were assisted in workshops to connect the mathematics they teach to the "big ideas," the conceptual links that connected topics in the curriculum. In workshop settings, the teachers experienced the math as students would.

The workshop activities focused on division of fractions revealed important ideas about how teachers initially thought about the topic and how their involvement in the project developed deeper understanding. The project staff developed exercises emphasizing ways to represent dividing with fractions in diagrams and real world examples. These activities were opportunities for teachers to make connections between related mathematical ideas. The difficult idea of keeping track of the reference unit in a fraction problem was an important discovery for the participating teachers. A problem illustrating the complexity in understanding the reference unit was: Wanda really likes cake. She has decided that a serving can be $\frac{3}{5}$ of a cake. If she orders 4 cakes, how many servings can she make? (Schifter & Bastable, 1995; Schifter, 1997). Teachers had difficulty not in computing an answer, but what the answer meant. In the fractional part of the answer, $6\frac{2}{3}$, what did that $\frac{2}{3}$ refer to? The answer was not just how many pieces

of cake there were. Each “whole” unit of cake was $\frac{3}{5}$ of a cake. The remainder part of the answer, $\frac{2}{3}$, referred to two thirds of $\frac{3}{5}$ s (one unit) of cake. Behr et. al. (1992), in their review of rational numbers, have also discussed the difficulty that teachers and students have in recognizing what the means in division problems.

Schifter and her colleagues documented how teacher understanding of the unit of reference concept developed. After the institute, classroom observations found participating teachers were less reliant on textbooks and more aware of assessing other ideas their students knew about fractions and recognizing what the students needed to learn. Teachers’ thinking changed and became more linked to the big idea connections between operations and representation.

The project demonstrated that the initial knowledge that teachers held regarding dividing with fractions was not different from the knowledge their students had. As their mathematical understanding strengthened, choices of problems, and pedagogical approaches changed. Teachers developed a deeper understanding of the content they taught.

Perspectives on Expertise in Mathematics Teaching

Understanding expert thinking has been an important research focus for educational researchers (Leinhardt, 1983, 1993; Leinhardt, Putnam, Stein, & Baxter, 1991; Leinhardt & Smith, 1985; Niemi, 1997). Expert/ novice comparisons are a widely used method to describe accomplished thinking in subject matter areas (Leinhardt, 1983, 2001; Turner-Bisset, 1999; Wineburg, 1990, 1997). This methodology is also used and reviewed to describe qualities in competent

teaching. These qualities are then incorporated into applications towards teacher learning and development (Munby et. al., 2001; Richardson, 1997; Richardson & Placier, 2001; Schifter & Bastable, 1995). The use of this method has shifted research from defining teacher success indirectly through the achievement outcomes of their students. The comparison or expertise method describes qualities of competent teaching, not a prescribed step-by-step approach to turn “novices” into “experts”. In the following section, research on expertise in subject matter domains is described with particular emphasis on mathematics. An historical background to expert/novice studies and general characteristics of expert thinking are described. Research on assessing expertise and instances from subject specific research in examining expertise are provided.

A classic expert/novice study

The initial research on expert/ novice comparisons (summarized in Bransford et. al., 1999) was by DeGroot whose focus was on the differences between expert and novice chess players on their ability to recognize and implement patterns on a chessboard. He examined how chess masters were consistently able to “out think” opponents.

DeGroot’s initial hypothesis was that master chess players had a depth and breadth of chess knowledge that the novices did not have. DeGroot asked chess participants to think aloud as they considered specific chess moves. Chess experts did not use more strategies but the quality of their chess searches was different. They were more likely to recognize meaningful chess patterns

leading to better chess moves than the novices. Their knowledge of the domain was hypothesized to be highly organized, more aware of meaningful patterns of information. The knowledge of the novice chess players was not organized in the same way, and they were not able to make expert chess moves. Subsequently, researchers attempted to confirm and expand DeGroot's findings among other groups, and focus on differences in the content areas such as mathematics English, and history (Ball, 1988; Leinhardt, 1993, 2001; Ma, 1999; Wilson, 1989; Wineburg, 1990).

Expert mathematics teachers show accomplished skill in recognizing patterns, representations (i.e., models), translations (i.e., solutions), and connections of mathematical concepts (Leinhardt et. al., 1991; Ma, 1999). This quality of expert thinking in the teaching of mathematics is the subject specific version of what the chess masters accomplished, leading to current notions of expertise in teaching as a profound or connected understanding of their content knowledge: subject matter (SMK) and pedagogical strategies for teaching (PCK) the subject matter (Ma, 1999; Goldsmith & Schifter, 1997).

General characteristics of expert thinking

While early work on expertise focused on general areas such as chess and solving puzzles, this research did contribute to our understanding certain characteristics of expert thinking. According to a recent review (Bransford et. al., 1999), experts "do" the following:

1. They notice meaningful, not surface patterns of information.

2. They have an organized content knowledge that is reflected by their “deep” understanding of subject matter.
3. Their knowledge is connected to a set of larger organizing principles of the content (i.e., the big ideas).

Bransford et. al. (1999) caution, however, that expertise in a discipline does may not guarantee expertise in teaching.

Subject specific examples in studying expert thinking and teaching

Current research in expert thinking focuses the subject matter as central (Leinhardt, 1993, 2001; Shulman, 1986; Shulman & Quinlan, 1996; Wilson, 1989; Wineburg, 1990). The body of knowledge gathered from this work has identified specific characteristics that distinguish expert thinking within a subject area and has described the connections and organization which experts show around the major concepts within the subject area. Leinhardt and Smith’s (1985) and Leinhardt’s (1993) examinations of expert teachers in mathematics, Ma’s (1999) analysis of select Chinese elementary teachers understanding of the arithmetic curriculum, and Wineburg’s (1990) discussion of historians’ understandings of historical moments illustrate the contextualization of subject matter in research on expert thinking.

Leinhardt and Smith (1985) described the organization of content knowledge in terms of subject matter (SMK) and pedagogy (PCK) for a group of “expert” mathematics teachers from their lessons. Eight fourth grade mathematics teachers were selected from an ongoing three-year study on

teaching and expertise. Four of the selected teachers were experts and four were novices. Each teacher was 1) observed for three months out of the school year and 2) their in classroom teaching was videotaped for 10 hours. They were also interviewed on their fraction knowledge in how they planned their lessons (i.e., in this case, reducing fractions). The analyses of the videotapes are discussed here since the mode of analysis is useful to the dissertation study.

The teachers videotaped had all used the same texts, used similar examples, and taught reducing fractions lesson in nearly the same order. The researchers were looking for differences in how the teacher communicated the content and how the classroom performances reflected differences in the organization of teacher's knowledge. Teacher knowledge was constructed with the researcher building a semantic net representation of the teacher's representation of the material. The semantic net is a "node link structure containing two types of information: concepts and relationships among concepts" (p.251). The *concepts* the teachers presented in the lesson are the *nodes* and the *relationships* are the *links* connecting the nodes. The researchers expected that the more complex the domain was, the more nodes (i.e., concepts) and links (i.e., relationships) the teacher's semantic net would have.

Comparing the nets of the participating teachers, researchers observed: 1) variability between teachers on the number of fraction concepts; 2) experts had more categories than novices ; and 3) differences among the teachers initially defined as "experts" in their content knowledge. The researchers

organized these differences as “components of competency” (p.269) in which an expert would be more likely to display 1) multiple representations; 2) multiple connections to related mathematical ideas for the topic (i.e., in this case of reducing fractions a related mathematical idea was the identity function); and 3) multiple links across concepts. Through the semantic net technique, the researchers illustrated the organization and content of a teacher’s lesson and were able to describe instances of expert thinking on a topic that another method may not have “caught.”

In a later study, Leinhardt (1993) developed a model of expertise through detailed observation, interviews, and collaborations with expert teachers. Using a descriptive approach, Leinhardt captured the components of the act of teaching as the routines, agendas, scripts, explanations, and representations or the “strands of teaching.” In one of her analyses, she examined the instructional explanations of two teachers, one of them, M. Lampert.

In her analysis of Lampert, Leinhardt focused on a single episode of teaching, within a specific area of subject matter (i.e., multiplication with fractions). The lesson was the unit of analysis, within which she analyzed the strand structures of routine, agenda, script, explanation, and representation. Classroom observations, field notes, materials used by the teacher were analyzed. Blocks of lessons were observed and videotaped. The teacher watched the videotapes and commented on instructional moments. Data on

students was also gathered in pre and post unit interviews, as well as in class work and homework.

Leinhardt observed a number of qualities in expert routines. Expert agendas contained principles of content and pedagogy and revealed a teacher's knowledge system. Representations and explanations had coherence, relating to the content and the substantive and syntactic features of subject matter knowledge (SMK). The experts were able to "demonstrate, justify, and support problem solving and reasoning in the process of developing understanding" (p. 52).

Wineburg (1990) also discovered similar qualities of expertise within the subject matter of history. He described the differences in the organization of knowledge between eight historians with a variety of specializations and eight high achieving high school seniors (as measured by performance from a pretest measure) from an AP history class. Each group was given a set of tasks, one a questionnaire of history facts and another a talk aloud in which each group made sense of an historical event, an important battle during the Revolutionary War.

The study, in two parts, included two measures for each group, analyzing historical documents and a fact questionnaire. In the first phase, the historians read historical documents related to a Revolutionary War battle (Battle of Lexington), thinking aloud as they read. They evaluated a series of paintings about the same battle and also answered a fact questionnaire. In the next phase, high school students followed the same tasks.

On the fact questionnaire, the seniors performed as well as the historical experts. The multiple choice test, a traditional way of assessing historical knowledge did not relate to the depth and breadth of subject matter knowledge as demonstrated in the think aloud measure.

On the think aloud, each group was asked how they made sense of the same set of historical documents (i.e., assessing deeper historical understanding). The results showed that the historical experts were far more elaborate in their understandings. They could develop alternative (i.e., multiple models) explanations based on how they used evidence materials from other sources. Regardless of their historical specialties, they effectively recognized patterns from the historical documents, an important feature of having expertise. The historians also used corroborating evidence whereas the students did not.

Each participant's think aloud was transcribed and formed the data for analysis. The protocol analysis was conducted inductively. Initially, Wineburg had developed a coding system for the possible different types of models of historical texts that subjects could produce (e.g., representation of text, representation of event, representation of subtext or purpose.). Protocols were reviewed and compared on the differences and similarities from the preliminary created categories with those that emerged from the protocols. The goal of the analysis was to, "describe the cognitive activities of historians and high school students as they evaluate historical evidence and to investigate the frequency of these various cognitive processes" (p.51). Wineburg developed a cognitive model for

the types of “cognitive representations in the minds of historians” (p. v). The model developed from verbal protocols between experts and novices in the use of historical materials was in terms of three distinct types of representations: 1) of text; 2) of event; and 3) of subtext.

Other researchers have examined their own teaching practice as a means of describing connections of subject matter and pedagogical content knowledge. Lampert’s (1990) analysis of her teaching shows expertise in subject matter as put into action through pedagogy. Expert teachers such as Lampert are aware of the parts of the discipline that are difficult or easy for students and they recognize the conceptual complexities of the domain.

Lampert analyzed her own multiplication with fractions lessons from a public school elementary class of 28 fourth graders. The lessons she created demonstrated a deep knowledge of the interrelationships between mathematical principles. An example of this knowledge was in Lampert’s use of the development of students’ arguments as evidence of their understanding. In addition, her description of the instruction of the class make it clear that she has a deep knowledge of multiplicative structures, using several representations (models) and problem situations related to multidigit multiplication. Moreover, her lesson planning included understanding of student’s development of mathematical ideas. Lampert’s research examined the qualities of reasoning and justification of arguments in students in her own classrooms. Her knowledge was

interrelated and connected to the conceptual issues of the curriculum in order for her to organize, observe, and support mathematical reasoning in the classroom.

Ma's (1999) 8 Chinese teachers demonstrated what Ma called a "profound understanding of mathematics." These characteristics have four dominant features. These qualities were:

1. Connectedness- Expert teachers make explicit connections among mathematical concepts and procedures on the arithmetic problem solving protocols.
2. Multiple perspectives- Expert teachers represented various approaches to solutions.
3. Basic ideas- Expert teachers were aware and used the relationships of basic concepts and principles in mathematics to connect to the problems in the protocol.
4. Longitudinal coherence- Expert teachers had knowledge of the curriculum that their students had experienced in earlier grades as well as curriculum to come. This vision and understanding of the elementary mathematics curriculum guided them in instruction and pedagogical choices. (p. 122)

On the arithmetic task of interest, Ma asked teachers to come up with a computational procedure to solve a the problem in number sentence form as well as construct a representation that would capture the meaning of the problem.

The problem was $1\frac{3}{4}$ divided by $\frac{1}{2}$. The expert Chinese teachers could make

sense of the “flip and multiply” algorithm and were able to come up with other algorithms that could also solve the problem. They proposed several alternatives to the algorithm, including converting the fractions to decimals, dividing without taking the reciprocal of the divisor (common denominator method), and using the distributive property.

They also created several different models of division by fractions such as the partitive, quotitive, and a factors and product model (i.e., using multiplication to derive an answer). They discussed the connections between multiplication and division and developed many topics representing different models of division by fractions. They discussed the relationship of division to the other operations and emphasized the relationship of division of fractions with multiplying fractions and whole numbers.

The eight expert Chinese teachers had less formal education than their American counterparts. However, the mathematics departments of the schools where they taught were organized around formal and informal staff development and support. They also had taught a range of grade levels, familiarizing themselves with the curriculum through the different grades. Their explanations were connected to the bigger conceptual ideas in the curriculum.

As the research discussed in this section demonstrates, an essential quality of expertise is recognizing patterns within the domain. In order to develop understanding of the problem, experts think in terms of essential concepts, the big ideas of the domain and the connections among them. They can offer

multiple approaches or methods in reaching a solution. The research programs discussed in this section address an issue raised by Munby et. al. (2001) in their review of teacher knowledge: “Better conceptions are needed of what experts need to know, not simply what distinguishes them from novices” (p. 880).

Assessing mathematics teachers understanding of mathematics

Expert mathematical knowledge for teaching is not compartmentalized (Ball & Bass, 2000; Lamon, 1999; Mack, 2001; Behr et al., 1993). Ball (1988) has identified teachers’ understanding of mathematics as “interweaving ideas of and about the subject”, proposing three criteria to assess teachers’ substantive and syntactic knowledge: correctness, meaning, and connectedness. Ma’s (1999) criteria for profound understanding was depth, breadth, thoroughness, and longitudinal coherence between mathematical ideas. Leinhardt and Smith’s (1985) categories for expertise were 1) multiple representations; 2) multiple connections within a topic; and 3) multiple connections across related topics.

The present study follows in the tradition of research designed to assess teacher understanding and extends the division with fraction interview task used by Ball (1988), Ma (1999), and the TELT study package created by Kennedy et. al (1993) by focusing on division of fractions at the micro level in number sentence and word problem forms.

Teacher Change in Mathematics Knowledge and Practice

Background

In a reformed vision of teaching (NCTM, 1991; 2000; National Research Council, 2001) the development of deep understanding of mathematical ideas within the curriculum is the key to change. The research programs discussed in this section developed in opposition to the types of teaching that have been labeled “traditional mathematics instruction” (i.e., instruction as sets of clear instructions, correct information about procedures, and automaticity of procedures through practice). The teachers and textbooks, in traditional situations, are the mathematical authorities and instruction is a process of transmission, from authority of text to teacher to student. Moving away from traditional ways of teaching, several models of teacher change have developed in research on teachers transitioning to reformed practice.

Educational researchers have examined issues of teacher change (Fennema & Franke, 1992; Franke, et al, 2001; Munby et al., 2001; Nelson, 1995; Richardson, 1997; Richardson & Placier, 2001; Sherin, 2002; Stein & Brown, 1997; Schifter & Simon, 1992). In a recent review on the topic, Richardson and Placier (2001), categorized two perspectives on change in the research literature: approaches that examine small group/individual issues in changing practice and those approaches examining change from organizational and structural points of view. Three studies reviewed in this section describe changes for individual elementary school teachers (Franke et al., 1997;

Goldsmith & Schifter, 1997; Sherin 2002) and changes in school mathematics programs in middle schools (Stein & Brown, 1997).

The teacher change programs described here attempt to systematically research and conduct teacher education in innovative ways in order to examine teacher understanding of the subject matter. In the following studies, the research goals were: 1) to assist teachers in creating opportunities to learn from instruction and to understand how and why mathematical ideas work (Goldsmith & Schifter, 1997); 2) to assist teachers in recognizing the development of student thinking about mathematics, the various strategies students use, and to use problem solving strategies in teaching the subject (Carpenter et al, 1989; Peterson et al., 1989; Franke et al., 2001); 3) to observe how teacher change of content knowledge occurs in the implementation of reform material in classrooms (Sherin, 2002); and 4) to foster teacher development through the development of a community of practice approach to middle school mathematics (Stein & Brown, 1997).

These approaches emerged from constructivist and social constructivist frameworks (Richardson, 1997; Richardson & Placier, 2001), perspectives in which the learners, individual teachers or collective mathematics departments, are participants in the knowledge building process. Each research program examines the growth of teacher knowledge and links the experiences they go through as research project participants. The respective research programs are uniquely different in how participating teachers receive information. Also, of

particular interest of this study, the programs provide teacher descriptions and reflections on the changes in their own constructions and representations of mathematical understanding and problem solving.

Models for understanding change in teaching

Schifter and Simon's (1992) and Goldsmith and Schifter's (1997) model of becoming a constructivist teacher relates the complexity of incorporating change into teaching practice. The data used to develop this model of effective teaching was collected over several years of observations and activities, included field notes, and audio and video taped responses of participating teachers. Data was gathered in Summer Math workshops and in observations of participating teachers' classrooms as well as other teacher development projects taking place through the Educational Development Center (Nelson, 1995; Schifter, 1995, 1997). The researchers organized the data into a developmental model of four stages that teachers go through to become constructivist. A constructivist teacher teaches at the highest level of the model. The levels from least constructivist to most constructivist teaching are:

1. Ad hoc accumulation of facts, definitions, and computational routines.
2. Student centered activity with little or no attempts at systematic inquiry into issues of mathematical structure and validity.
3. Student centered activity directed towards systematic inquiry into issues of mathematical structure and validity.

4. Systematic inquiry organized around investigation of “big” mathematical ideas. (Schifter, 1995; Schifter & Fosnot, 1993, p. 18)

The model addresses what a reconstructed view of teaching practice looks like. Case study analysis led the researchers to create, “a developmental logic to the process of teacher change” (Schifter, 1995, p.18). The model can 1) assist researchers and staff developers in recognizing possible content-related issues in the process of teacher transformation and 2) can serve as a framework for interpreting teacher change. The researchers stress that their model is not a traditional stage model with an invariant set of steps; rather they view the model as a “qualitative organization of understanding”, which can vary from teacher to teacher.

Franke et. al. (1997) proposed another developmental model for discussing teacher change. Their model summarizes patterns of changes in beliefs, classroom practice, and knowledge based on observing and analyzing participating teacher involvement with CGI (Cognitively Guided Instruction) project. The initial CGI project, over the course of four years, was a collection of workshops designed to assist teachers in understanding their student’s thinking. Classroom observations and after class discussions with project staff helped teachers notice how they thought about their students’ thinking about mathematics problems. These discussions were supported by opportunities to plan lessons with teachers and project staff and then pilot these lessons with students.

Data collected from teacher observations and discussions from workshops led to the documentation of levels of change in beliefs and in practice. Change was examined in 21 teachers over a four-year period. The levels evolved through a constant recoding and revising of categories. The researchers looked for emerging patterns rather than counting instances of specific types of teacher behaviors or teacher statements. Teacher knowledge was not assessed explicitly but connected to assessing beliefs and practice.

The researchers presented a four level model, with the highest, the fourth level having two sublevels. Level 1 teachers' beliefs and practices were that students can only learn mathematics if they are taught it. For them, student knowledge was not incorporated into their teaching practice and students were not encouraged to devise their own strategies for solving problems. A level 2 teacher recognized that students do bring their own knowledge of mathematics into mathematics classrooms. However, these teachers were not consistent in their teaching practice for understanding and incorporating student's ideas. Level 3 teachers believed that students could solve problems without a teacher giving them the strategy first. They paid attention to their students' thinking and looked for ways to incorporate these ideas into their teaching practice. A level 3 teacher, however, did not recognize the "implications" of students' statements. Level 4A teachers stressed student thinking as the organizing influence for their instructional decisions. Yet these teachers were not consistent in their attempts at basing their practice on the knowledge of student's thinking. Level 4B teachers

were consistent about basing their practice on student's knowledge. They believed that understanding individual student's thinking was the most important influence for instructional decisions. CGI researchers concluded that a change in beliefs was essential in order to support a change in classroom practice.

While the studies on teacher change previously described focus on macro issues regarding teacher knowledge and practice, the work of Leinhardt (1993, 2001) and more recently, Sherin (2002) focus in on content particulars to explain processes of teacher change. Sherin (2002) observed the process of change in content knowledge for two teachers who agreed to use a new linear functions curriculum in their classes. Sherin expected to observe changes in what she termed "content knowledge complexes" defined as "composite structures within teachers' knowledge with tight interconnections among the components" (p.125). These cognitive structures contained content knowledge forms: SMK and PCK.

The new curriculum for teaching linear function emphasized connections among representations, building conceptual understanding of slope and intercept, and real world contexts for investigating linear functions. The new curriculum was very different from the previous texts used by the teachers that stressed procedures for solving linear equations.

Two experienced teachers volunteered to participate in the 6-week unit. Each teacher was interviewed three times before the start of the project in order to understand their current knowledge of linear equations and their ideas for teaching the topic. The linear functions unit was in 5 classes and each class was

videotaped for each day of the unit. There were two cameras in each class, one on the teacher and the other on the students. The researcher created written observations from the videotapes focusing on places where the teachers' SMK and or PCK were evident. Observations investigated the following research questions: How did they discuss the important concepts of the topic? How did they present the materials? How did they respond to student questions and responses? The teachers also watched excerpts of videotapes of their classrooms together. They discussed the mathematics in the lessons. These interactions were also videotaped.

From the data sources, the researcher created three categories of interactions between the teachers' content knowledge and how they used that knowledge in implementing this new curriculum. These categories, *transform*, *adapt*, and *negotiate*, "represent the different ways that teachers apply their content knowledge as they attempt to use different materials" (p.129). In *transform* moments, teachers changed the lesson from its original purpose and did not change their content knowledge. The teachers continued to implement familiar routines for the content of the lesson. In *adapt* moments; teachers changed their content knowledge, not because of the curriculum but because of students' comments and ideas. In *negotiate* moments, teachers developed new content knowledge and made changes in the lesson by using new strategies for instruction. Teachers, in negotiate moments, were in the process of negotiating

how they viewed the lesson, the students, and their knowledge about linear functions.

Each of the 17 observed lessons was coded for evidences of transformation, adaptation, and negotiation. Transformation occurred in 53% of the lessons, adaptations in 82% of the lessons, and negotiation instances in 29% of the lessons. Teachers changing the ways in which they taught and understood the content were expected to be those who were rethinking their SMK and PCK in relation to the domain, the lesson, and to students' understandings.

These examples of teacher change have important implications for teaching practice. Two studies examined overall change in practice and beliefs and one (Sherin, 2002) was a microanalysis of teacher change. The models contribute to an understanding of the changes in SMK and PCK.

Teacher as student

In an earlier section, the work of Schifter was discussed in relation to research approaches for examining teachers' understandings of dividing with fractions. In this section, the Summer Math Course developed by Schifter & Simon (1992) and expanded in further research by Schifter & Fosnot (1993) and Schifter (1995, 1997) is discussed. Again, findings from these models are included because teachers in the study have attended similar programs. In the Summer Math courses, project staff designed activities to stimulate thinking about the "big ideas" in the mathematics curriculum and to help teachers to develop possible alternative methods for instruction on a topic. Activity topics

included whole number, integer, fraction, decimal, exponents, area, and perimeter operations. Researchers were particularly interested in those teachers who were transitioning from one level of knowledge to the next as their learning was supported through the participation in the project.

The idea of the project was to enable mathematics teachers to learn mathematics while they taught it. The developmental model came out of the many case studies of teachers participating in the project. One case, Linda (Schifter & Fosnot, 1993) illustrated the impact of participation in PD projects. Linda was a middle school mathematics teacher who had taught the procedures of mathematics but did not understand the connections between procedures and representations. Through the workshop activities, discussions, and readings, Linda brought what she learned to her classroom and began the process of “exploring mathematical relationships, conjecturing, generalizing... As she evaluated different ways of making meaning, she was able to pose to her students deep questions about the nature of mathematics” (p.81). For the teachers involved in the project, transition and change occurred through understanding the mathematics they taught. This learning helped them rethink and reconstruct the types of pedagogical choices they made, particularly representations or models.

Researchers were also able to evaluate teacher development by observing how teachers changed in their thinking about their own and their students' representations and strategies on alternative methods to solutions. Of

the many examples of teacher development, three qualitative changes occurred. First, learning to use diagrams in activities increased awareness of making mathematical sense of the mathematics problem. Second, cases of other teachers were presented. Third, considering student confusion as helpful information allowed teachers to rethink their own understandings about mathematics.

The Summer Math project supported teachers whose knowledge of mathematics was not initially deep. Through the in-service activities, these teachers were able to develop better understanding of the meanings behind mathematical procedures.

Using research to effect teacher change

Teacher change programs such as Cognitively Guided Instruction (CGI) incorporated knowledge about student's thinking about mathematics as a means for initiating change in teacher thinking and understanding. The professional development programs of CGI have been documented extensively (Carpenter et al., 1996; Fennema et al., 1993; Fennema et al., 1996; Fennema & Franke, 1992; Franke et al., 2001). Using research-based knowledge of student development of mathematical understanding and strategies, teachers in the CGI programs reflected on their own practice and integrated their own knowledge with research knowledge in workshop sessions. The CGI project worked with elementary school teachers, and introduced teachers to research findings about students' mathematical thinking in order to facilitate teacher development of pedagogical

strategies for understanding operations such as addition and subtraction. An experimental study from CGI (Carpenter et al, 1989) and follow up study (Franke et al, 2001) with these teachers four years after the end of the project are discussed in this section, including their implications for teacher development. A detailed analysis of the experimental study can be found in Peterson, Fennema, & Carpenter (1989), presenting significant correlations between teacher knowledge of student strategies and student achievement.

Cognitively Guided Instruction's research focus was to bridge the gaps between teacher classroom practice and research on 1) children's development of strategy use; and 2) the types and developmental difficulty of mathematics word problems. The research question of interest in the study by Carpenter et al (1989) was: Does providing explicit knowledge from subject specific research on student's thinking influence teaching practice and student achievement?

Researchers presented teachers possible different addition and subtraction models/strategies their students may use depending on the type of word problem (Carpenter et al, 1989). By introducing the developmental research on both the structure of mathematical word problems and solution strategies, teachers had a framework on which to base both assessment of students and instructional decisions. The continual focus on assessment of student's knowledge as the primary means to inform pedagogical decisions is a characteristic distinguishing CGI from other teacher development programs.

Forty first grade teachers were randomly assigned to either a control or experimental group. The experimental group participated in a month long workshop to study videotapes presenting the development of student problem solving strategy thinking about addition and subtraction concepts in the context of solving word problems. The videotapes and workshop discussions were ways to assist teachers in developing a conceptual framework for problem types and strategies used by students. Instruction would then be based on individual student's development of mathematical knowledge. The twenty other first grade teachers were assigned to a control group. They participated in workshops focused on problem solving only. After the workshops, all forty teachers were observed during mathematics instruction.

At the end of the year, teachers were asked how individual students would solve specific problems. Teachers' predictions were matched with students' performance. The prediction variable was a measure of teachers' knowledge of student performance. Students in the forty classrooms had completed a pretest achievement measure in the beginning of the year and the same measure as a posttest at the end of the year. Students were also interviewed as they solved various addition and subtraction problems. Videotapes of students' problem solving strategies were documented and instructional materials were reviewed. Detailed classroom observation procedures were created in which a sampling of behavior and activities of mathematics content, setting, and strategy use were coded.

Results indicated that teacher participation in the CGI group did make a difference. Significant differences between the two groups of teachers were found on: time engaged in addition and subtraction activities, teacher knowledge of student strategies for number facts and problem solving, teacher beliefs about mathematics, and student achievement (controlling for prior mathematics achievement).

In a follow up study with 22 teachers involved in the CGI experimental program from 1991 to 1993, Franke et. al (2001) examined the characteristics of the teachers post professional development. Each teacher was interviewed and 20 of the teachers' classrooms were observed. For classroom observations, interactions between student and teacher were coded according to: 1) the types of problems the teacher posed; 2) the strategies students used to solve the problems; 3) teacher facilitation of student's sharing strategies; and 4) teacher's input when student's strategies for solution were incomplete or incorrect. The interview questions focused on teacher's use of student thinking and their own self-perceptions regarding change in their knowledge and practice. The teachers first discussed the lesson the researcher had observed and why she had chosen the problems she chose and what was important to point out about the strategies the students used to solve the problems. The interview also included questions on how and why the teacher had changed since the end of their participation in the 1993 CGI program and the type of current support for their mathematics teaching in their current schools.

The researchers identified instances of “self sustaining generative change” in teachers who reported teaching in schools where the majority of teachers also participated in PD programs. Support between teachers included planning together, talking to their colleagues, and moral support. As important, teachers who focused on understanding student’s mathematical thinking saw the process as an opportunity to become learners themselves. The researchers propose that it is teacher engagement with student thinking and ongoing collegial support that is most related to developing deep knowledge in teachers.

Teacher change through participation in communities of practice

Stein and Brown’s (1997) research approach to assessing teacher change examined not individual teachers but mathematics departments in schools. The research interest was to examine the impact of a community of practice approach to teaching mathematics or a “transformation of participation in the practices of a community” (p. 155). This research was carried out in a large-scale reform project, Quantitative Understanding: Amplifying Student Achievement and Reasoning (QUASAR) (Silver & Stein, 1996).

QUASAR was a teaching reform project based in middle schools with economically disadvantaged students. At each QUASAR school, the mathematics faculty as well as administrators and local mathematics educators worked together. Teacher change was defined as a collaborative process with teachers, principals, district supervisors, and local mathematics educators working towards the goal of reforming mathematics teaching practice. The

researchers conceptualized teacher development as a sociocultural phenomenon, focusing on the changes that occur as a result of participation in forms of group practice, not at the level of the individual teacher or the individual student. Participation in practice included: 1) the group's selection of curriculum materials; 2) agreement on ways for assessing students; 3) planning lessons; and 4) working with other teachers and administrators. The researchers observed and documented changes in patterns of participation by teachers in QUASAR sites, particularly those who were new teachers to the schools.

Two of the five QUASAR sites are described in detail, Riverside Middle School and Norton Middle School (Stein & Brown, 1997). Mathematics teachers at Riverside worked inside and outside their classrooms to improve their mathematics program. They shared a set of beliefs encouraging the importance of discussion in their mathematics classroom and the importance of student understanding. New mathematics teachers coming into the department were included as participating members of the mathematical community of the school. Riverside mathematics teachers regularly worked with each other and the new teachers said that these interactions were very important for their development. There was also collaborative work on the products created by Riverside teachers. Products included systems for portfolios, communication with parents, and presentations of their mathematics program at local and national meetings. Such factors created a community of practice at Riverside that the researchers felt

placed teachers in the school's mathematics department in alignment with NCTM's (1991) recommendations for reform in teaching practice.

In contrast with the Riverside school, the Norton Middle School teachers did not develop reform goals, as had the Riverside teachers. The researchers identified possible reasons for this difference. One important factor was the impact of the "chains of assistance" (p.173) at Riverside, support given to teachers by their principals and district supervisors. Principals and supervisors provided opportunities for teachers to interact with each other and support such activities consistently and frequently. Norton school did not have such assistance in place to support teachers.

Norton's participation in the QUASAR project did not lead to the formation of a reform community of mathematics teachers because principals and supervisors for the school did not support the change due to pressures from the district. Initially, Norton's district supervisors and principal agreed to the project goal of reforming the school's mathematics program. Yet, the district supervisors continued to hold principals and teachers accountable for student performance on standardized tests. As a result, the principal and teachers at Norton found that the two goals, reforming their instructional program and complying with district requirements on assessments, were incompatible. For example, the school had agreed to set up a common planning time where teachers would work together in developing lessons and sharing/grouping ideas related to practice. The teachers, however, did not meet regularly. When planning time did occur, teachers did not

share and develop lesson plans for the mathematics curriculum. In the third year of the project, however, Norton teachers began to make progress in assisting each other during the planning times. One reason to account for this change was the district's adoption of a curriculum complementing NCTM's goals rather than the previous assessment focused curriculum. Norton's barriers to organizational change as created by the district's requirements had discouraged a community of practice approach to reforming instruction in the school. The roles of participation did not improve for the Norton teachers as it had for the Riverside teachers.

The three projects discussed in this section are quite different from each other in terms of focus for change. A difference between CGI and Summer Math, for example, is the role of the teacher in each. In the Summer Math workshops, the teacher was a learner and confronted his or her own knowledge and its limitations in activity sessions and reflections. These served as beginnings to recognize larger mathematical ideas in the curriculum. In the CGI project, teachers had access to models of student developmental progression of solutions for addition and subtraction problems and how understanding the categories of problem type and problem solving strategies could be used to develop instruction and curriculum. Here, the teachers' initial knowledge of subtraction and addition was not assessed directly. Instead, teachers' understandings of subject matter developed as a result of understanding students' solution strategies of different types of mathematical problems. The third project included in the section, QUASAR, examined teacher change from a

participation in a community of practice approach and identified change as an organizational shift of “assistance and collaboration” rather than an individual teacher shift.

An important similarity between these three major teacher development projects are the experiences of teachers, in groups, discussing mathematics and strategies for learning mathematics as key to sustaining change beyond the course of the PD program. In the present study, teachers, in the follow up interview questions, are asked about their own formal and informal experiences of PD as well as their own “communities of practice”. The follow up questions are designed to capture teachers’ awareness of connections between their own practice and knowledge development to their schools, colleagues, and PD experiences.

Study: Purpose

The purpose of this study is to describe the mathematical content knowledge: subject matter (SMK) and pedagogical content (PCK) knowledge of fractions and division with fractions for two selected groups of middle school mathematics teachers, eight teachers in transition (IT) and six teachers with deep knowledge (DK). All the mathematics teachers have been involved with professional development (PD) programs for at least two years. Teacher deep knowledge of the topic area is defined as well developed, organized, and connected as evident through multiple representations (i.e., models) and multiple translations (i.e., solutions) for the concept and connections between the

concept, representations, translations and related “big ideas” (Fosnot & Dolk, 2002) in the mathematics curriculum.

Each teacher was interviewed individually on the problem tasks, concept map, and follow up questions. All teachers had filled out a pre interview set of questions detailing their educational background, years of teaching, types of mathematics courses taught, PD experiences, and had read and signed the consent form (see Appendix A) prior to the interview appointment.

During the interview, teachers were presented five tasks and follow up questions in a standardized format. The tasks are contextualized as teaching situations. The first two elicit representations (i.e., models). These first two tasks of Tasks 1 and 2 are in number sentence forms. Teachers were asked first to solve the number sentence and then represent it in a story. The standardized prompt for the *first* task was:

Division by fractions is often confusing for students. People have different approaches to solving problems involving division with fractions.

1) How would you solve a problem like this one?

$$1\frac{3}{4} \div \frac{1}{2}$$

Imagine that you are teaching division with fractions. To make this meaningful for students, many teachers try to relate mathematics to other things. Sometimes they try to come up with real world situations or story problems to show the application of some particular piece of content.

2) What would you say would be a good story or model for:

$$1\frac{3}{4} \div \frac{1}{2}$$

3) If teacher gives a story, ask: Why do you think this is a good example?

4) If teacher cannot come up with a story, ask, why do you think it is difficult to come up with a story for the problem situation?

The *second* task included the same questions as task 1 but the number sentence was changed to: $\frac{1}{2} \div 1\frac{3}{4}$.

The third and fourth interview tasks elicited translations (i.e., solutions).

They were presented in word problem forms. The *third* task's prompt was:

Students must often solve word problems involving division with fractions. People have different approaches to solving such word problems.

1) How would you solve this one:

Wally has $6\frac{2}{3}$ yards of material. It takes $\frac{5}{6}$ of a yard to make his scarves. How many scarves can he make?

2) Would a word problem like this help students learn about division with fractions?

3) This is only one way/ one method. What other ways can you think of to solve this problem?

The fourth interview task, also in word problem form, was created to capture a context that would fit task 3's inverse, $\frac{5}{6} \div 6\frac{2}{3}$.

The first four interview tasks are introduced to teachers as problems contextualized in instruction.

The *fifth* interview task was a talk aloud about the concepts teachers would stress as essential for students to understand fractions (Appendix H). For the *sixth* and final part of the interview, each teacher was asked to discuss their experiences in developing their math knowledge and practice. The questions were:

- 1) Please tell me about a specific career experience that you see as critical in your dedication and motivation to keep developing your mathematical knowledge and teaching practice?
- 2) With whom do you discuss a) your teaching practice and b) mathematics topics
- 3) If a teacher gives a name, X, ask, "Is X in your school?" If not how do you know X?
 - a) In talking to X about your teaching practice, can you give me an example of a recent conversation?
 - b) In talking with X about mathematical topics, can you give me an example of a recent conversation?

The set of questions chosen and created for the interview tasks were designed to closely examine teacher subject matter (SMK) and pedagogical content knowledge (PCK) for dividing with fractions as well as finding out about teacher involvement and perceptions about professional development (PD) experiences.

Guiding hypotheses and research questions

A set of hypotheses and research questions guide this study:

Hypothesis 1: There will be differences between DK (deep knowledge) and IT (in transition) teachers on their SMK and PCK for dividing with fractions. SMK and PCK were analyzed as types and frequencies of representations and translations on division tasks (i.e., tasks 1 through 3).

Q1. What types and how many *representations* (i.e., models) of division with fractions do deep knowledge (DK) and in transition (IT) teachers present in tasks 1 and 2?

Q2. What types and how many *translations* (i.e., solution methods) do DK and IT teachers present in task 3?

Hypothesis 2: There will be differences between DK and IT mathematics teachers on their connected knowledge of the fraction concept. SMK and PCK were analyzed as types and frequencies of concepts and relationships (i.e.,links) on the concept map activity.

Q3. What types of fraction concepts and how many levels of connections do teachers have on the concept map activity?

Hypothesis 3: IT teachers will attribute failure to create representations in tasks 1 and 2 to a lack of their mathematical knowledge.

Q4. What types of reasons do IT teachers give when they fail to represent the problems in tasks 1 and 2 (see Appendixes D-F) in representational or story word forms?

Hypothesis 4: There will be fewer representations on interview task 2 on IT and DK teachers' representations and translations.

Q5. Do types and numbers of representations and translations differ when DK and IT teachers solve problem 2 (i.e., when the divisor fraction is larger than the dividend), when compared to performance on division with fraction problems such as tasks 1?

Hypothesis 5: IT teachers will exhibit variable patterns in the use of their knowledge across tasks, whereas DK teachers will be consistent.

Q6: Do DK and IT teachers' response patterns across tasks differ?

Study methods, procedures, results, and conclusions are discussed in the following sections.

CHAPTER 3

Methodology

This section includes details on study participants, materials, procedures, and techniques and categories used for data coding and analysis.

Participants: Recruitment and Informed Consent

The DK and IT teachers from the pilot study had been active in formal (i.e., curriculum and institute style workshops) and informal (i.e., in school discussions with colleagues) PD activities. In order to find the specific teachers for the dissertation sample, it was expected they would also have had similar professional development (PD) experiences.

In order to recruit teachers for the study, referral specialists were contacted. The specialists were mathematics education faculty (N=1) and staff developers (N=3) with links to New York City schools and in-service middle school teachers. These teacher education and professional development specialists were shown all of the interview tasks (Appendix D-H) as well as the types of expected models, translations, and knowledge connections expected for the tasks in the preliminary “teacher criteria designation” table (Table 2).

Table 2 served to identify the knowledge base for the specialists to use in recommending teachers with whom they had worked. Table 2 was created from the responses of an expert teacher who participated in the pilot study and from related research examining teachers’ models and representations of division and

fractions. This knowledge criteria table served as an initial guide to identify possible participants as “expert” teachers.

Table 2

Preliminary Guide for Deep Knowledge Teacher Designation: SMK and PCK

Tasks 1 & 2:	Tasks 3 & 4:	Concept map:	Task 2 & 4:
Types of pedagogical models and representations	Types of translations or solution methods	Connections among mathematical ideas related to the concept of fraction	Impact of reversal problem
1) Quotitive models	1) Estimation	1) Part to whole, ratio, quotient, operator, measure, absolute/relative	1) On representations
2) Partitive models	2) Partitive		2) On translations
3) Product/factor models	3) Quotitive		3) Recognizing the reference unit
4) Variety of topics for pedagogical models (i.e., time, work, cooking, construction forms)	4) Reference unit	2) Types of models (i.e., linear, measurement, set, part/whole, comparisons)	
5) Multiple models	5) Common denominator	3) Connection between whole number division and division with fractions	
		4) Connection of multiplication to division	
		5) Connections to rational number ideas (percent, decimal forms)	

The recruitment specialists referred thirty teachers. Letters were sent to the entire group and fourteen middle school teachers agreed to participate in the study. At the time of the interview, all participants were middle school mathematics teachers and current or former participants in PD programs such as Math in The City, Summer Math at Mt. Holyoke, or MS². Twelve taught in public schools and 2 taught in the same private school.

The participating teachers were sent 1) the cover letter/ consent letter (Appendix A) explaining the study and their rights as participants ; and 2) the pre interview questions on their years teaching, educational background, and professional development experiences (Appendix B). Interview appointment dates were set. Responses to pre interview questions and the consent forms were completed prior to the interview date. The interviews took place after school at locations convenient for the teacher.

Materials

Materials included pre interview questions, dividing with fractions tasks (i.e., tasks in number sentence and word problem forms), concept map activity, and follow up questions on professional development (PD) experiences. The materials were chosen (i.e., Task/ Task1) and created (i.e., Tasks/ Tasks 2-5) because of their relevance to teaching the topic, their representativeness of the types of dividing with fractions situations teachers face, and they were structured to provide a view into teachers' knowledge (Kennedy et al, 1993). The

professional development questions were adapted from a 1991 teacher survey (McLaughlin & Talbert, 2001).

Pre-interview

The first material source, pre interview questions (Appendix B), provided information on individual teachers' past experiences in teaching mathematics as well as their educational and PD backgrounds. Questions included: 1) years teaching math, 2) highest degree obtained, 3) undergraduate degree major, 4) the number and type of mathematics courses they currently teach and have taught, and 5) description of their experiences in sustained PD experiences.

Dividing with fractions: Tasks

The contexts in each of the four division tasks were hypothetical. Previous research using similar tasks and their corresponding questions show that problem solving tasks can reveal what a teacher knows about the concept (e.g., multiple representations, translations, and connections between concept to related mathematical ideas) when asked to talk aloud their understanding as contextualized in a teaching situation.

The format of the first two division questions (i.e., questions asked within tasks for prompting the participant) were originally created for the elementary mathematics portion of the Teacher Education and Learning to Teach (TELT) Study (Kennedy et al., 1993) and Ball's (1988) dissertation. The third interview task was similar to those used by Schifter (1995, 1997) in the Summer Math project. The interview task questions for Task 3 were designed around what the

TELT creators called “the common tasks of teaching” (Kennedy, et. al, 1993, p. 17) and were similar to the ones asked in the first two. The teachers, when presented with the interview tasks, were asked what they would do or say if the same situation came up in their own teaching.

Prompts for division tasks

Task 1 and 2's questions are contextualized to prompt teachers to discuss how they would teach the following hypothetical situation to their class. The prompts for the problem situations in Task 1 were:

Division by fractions is often confusing for students. People have different approaches to solving problems involving division with fractions.

a) How would you solve a problem like this one?

$$1\frac{3}{4} \div \frac{1}{2}$$

b) Imagine that you are teaching division with fractions. To make this meaningful for students, many teachers try to relate mathematics to other things. Sometimes they try to come up with real world situations or story problems to show the application of some particular piece of content.

What would you say would be a good story or model for:

$$1\frac{3}{4} \div \frac{1}{2}$$

c1) If teacher gives a story, ask: Why do you think this is a good example?

c2) If teacher cannot come up with a story, ask, why do you think it is difficult to come up with a story for the problem situation?

These questions, used in both task 1 and 2, reveal teachers' understandings of the subject matter, how they would represent that understanding in a story form, and if the teacher could not create a story, why this was difficult to do. Task 2 was created as an extension of task 1. The

interview questions for it were the same but the number sentence was the inverse of Task 1, $1\frac{3}{4} \div \frac{1}{2}$. The purpose of task 2 $\frac{1}{2} \div 1\frac{3}{4}$ was to examine what would happen to teacher knowledge when the fractional divisor was larger than the dividend.

For these the first two tasks, teachers were asked how they would solve the given problems and then how they could apply the problem to a “real world” situation. These questions were expected to reveal how teachers themselves understand the subject matter and how they would communicate and use that knowledge in their teaching.

In the third task, the problem was given in word problem form and teachers solved and talked aloud different solution methods (i.e., translations). Task 3 questions were contextualized in the different methods teachers could generate for solutions and the teaching value of using word problems in instruction. The prompt for this task was:

Students must often solve word problems involving division with fractions. People have different approaches to solving such word problems. a) How would you solve this one:

Wally has $6\frac{2}{3}$ yards of material. It takes $\frac{5}{6}$ of a yard to make his scarves. How many scarves can he make?

b) Would a word problem like this help students learn about division with fractions?

c) This is only one way/ one method. What other ways can you think of to solve this problem?

The prompt questions in task 3 were created to extend knowledge of the multiple translations teachers would offer in solving the given word problem. Task

4 (i.e., inverse of Task 3 in word problem form) is not included here. It was excluded from the analysis due to its incorrect construction (see Discussion section). A copy of it is included in Appendix G.

Table 3 summarizes the three problem solving tasks (i.e., division with fraction tasks) and offers the levels of analyses for which task was examined:

Table 3

Problem Types in Tasks and Level of Analysis

Task	Problem type	Problem prompt	Level of analysis
1	$1\frac{3}{4} \div \frac{1}{2}$	Teacher: a) solves b) creates stories/ representations for problem.	a) Solution(s) b) Multiple models c) Explanations d) Representations
2	$\frac{1}{2} \div 1\frac{3}{4}$	Teacher: a) solves b) creates stories/ representations for problem.	a) Solution(s) b) Multiple models c) Explanations d) Representations
3	Word problem for $6\frac{2}{3} \div \frac{5}{6}$	Teacher: a) translates b) uses multiple methods.	a) Translations of story problem into solutions b) Multiple methods

In terms of solutions for the problem solving tasks (i.e., division tasks), there are multiple ways of translating and representing each type of problem regardless of presented form. Examining teachers' understandings and use of multiple methods for translations (task 3) and multiple models for representations (tasks 1 and 2) are similar to previous researchers' analyses of the links between multiple representations, translations, and connections with a deep understanding of the subject matter and ways to teach it (Ball & Bass, 2001).

Concept map

For the concept map, teachers talked out loud about the essential ideas they would stress for understanding fractions. The rationale for using a “concept map” was to access teacher knowledge of the topic using other approaches and to bridge specific instructional contexts (i.e., tasks 1-3) with a traditional assessment tool, the concept map, to examine teacher knowledge (Osmundson, Chung, Herl, & Klein, 1999). The concept map prompt was (Appendix H): Please draw out your “concept map” of your knowledge of the topic fraction. A “concept map” can look like a road map or like a web of connections. Include in your map the connected ideas of the different parts of the fraction topic that you believe are important for students to know. Additional questions asked: 1) What do you think are the connections among fractions and other related mathematical topics? Specifically, division of fractions? ; and 2) how do you think about fractions in relation to: multiplication, decimals, ratios, whole numbers, and proportions

Information from this activity would add to understanding the structure of teachers’ subject matter and pedagogical content knowledge for the fraction topic.

The process used to create the concept maps was adapted from Leinhardt and Smith (1985). They analyzed videotaped observations of mathematics classrooms and transcripts to create “semantic nets” to capture the thinking of the teachers as they taught division with fractions. The concepts teachers presented in the instruction were displayed as “nodes” and relationships as links

between the nodes. For this study, teacher think alouds for the concept map prompt were transcribed and maps were created for each teacher using a similar technique.

Follow up interview

Follow up interview questions (Appendix I) were asked after the “concept map” activity (Appendix H). Follow up questions asked teachers to: 1) relate a specific career experience critical to their dedication and motivation to continue to develop their knowledge and practice; and 2) identify specific opportunities they used to discuss teaching practice and math knowledge, as for example, recent conversations at school with colleagues.

The follow up questions were adapted from McLaughlin and Talbert’s (2001) 1991 Teacher Questionnaire. The teacher questionnaire is a survey instrument asking teachers about their work within the contexts of institutional environments, social systems, and administrative contexts. It was expected that teacher responses to the follow up questions would suggest the impact of workshops, reform style curricula, and structures within schools (i.e., planning lessons, department meetings) on their development of subject matter knowledge and teaching practice.

Procedures

Institutional Review Board (IRB) applications from NYC Board of Education and CUNY’s Graduate Center were completed and permission to conduct research was granted.

Teachers who agreed to participate were sent consent letters, cover letters (Appendix A), and the pre interview questions (Appendix B). Interview dates and locations were set. Teachers returned signed consent forms and completed pre interview questionnaires before the start of the interview. Interviews were conducted individually, lasted approximately forty-five to ninety minutes, and were audio taped. To ensure consistency and comparability of the interviews, the protocol was followed verbatim for each interview. The standardized interview protocol including consent forms, pre interview questions, interviewer and interviewee versions of each task, and the follow up questions are located in Appendixes A-I.

Each teacher was given a page with the task problem typed on the page (i.e., the interviewee version) and each interview task was presented one at a time by the interviewer. Each task had its own specific standardized set of questions to be asked orally (as reviewed in the materials section). After the problem solving tasks (i.e., division and concept map tasks), each teacher was asked the same post interview questions.

Think aloud procedure

A “think aloud” procedure was the primary method of this study. The think aloud is an established method for examining features of expert/novice thinking, teaching, and learning (Ericsson & Simon, 1993). Teachers verbalized their knowledge for solving division with fraction problems, knowledge of the fraction

topic, and answered follow up questions regarding their professional development experiences.

At the start of the interview, teachers were read a set of standardized instructions for the division tasks and for the concept map activity to encourage them to talk aloud their thoughts on how they would think about and approach each task. These instructions were:

Thank you very much for participating in today's interview. I am trying to learn about how you think about teaching fractions and division with fractions. I will present you with four different tasks. Each task has a series of questions that I would like for you to answer. I will ask you the questions orally. For each task, I'm just as interested in how you think about things that puzzle you as I am in learning exactly what you'd do. For each one, I'd like to know what you think you would do or say and why that's what you'd do.

The tasks were presented in order, one at a time. For the division tasks, teachers were asked to imagine themselves in instructional situations in which they would be faced with each task. They "thought out loud" as they generated representations (Tasks 1 and 2) or translated solution methods (Tasks 3 and 4). In the think alouds for the concept map activity, teachers were asked to state the ideas in their talk alouds about the concepts they believed were important for students to know fractions.

I am also interested in how you would "map out" your knowledge of fractions. I would like for you to talk aloud a map of this knowledge.

It can be a web of words or a flow chart of words that captures the different parts of this topic that you believe are important for a student to know about fractions.

You may want to list all the ideas that you think are important for a student to know in order to understand fractions. Talk aloud how those words related to each other as ideas and the links between the ideas.

For the follow up questions, teachers discussed the influences in participating in professional development inside and outside of their schools and current conversations they were having which influenced their knowledge and practice. Teachers' responses to the set of interview tasks were audiotaped and transcribed for analysis.

Data Analysis

Data sources for analysis included: 1) the written responses to pre interview questions; 2) transcripts of responses to division of fraction tasks; 3) transcripts for the concept maps; and 4) transcripts of oral responses from the post interview questions.

Hypotheses 1 through 5 and research questions 1 through 6 were analyzed using the categories from the *final* deep knowledge teacher designation criteria table (Table 4). This final table includes more categories than the *provisional* table (see Table 2), which was used to identify teachers to interview.

Incorporating emerging categories into analysis is a dominant feature of a "constant comparative" approach to data analysis. The constant comparative approach to data analysis allows flexibility in combining existing (i.e., provisional

categories) with emerging new categories from the transcripts. Glaser and Strauss (1967) in first presenting this approach to analysis methods, stressed the approach as an opportunity to code and analyze at the same time.

The initial provisional categories were used in examining and coding the teacher transcripts for each of the division and concept map protocols. This analysis was looking for differences and similarities between the provisional criteria and teacher responses. The provisional model was then adjusted to include the new, emerging categories for teachers' content knowledge: subject matter (SMK) and pedagogical content knowledge (PCK) features for the tasks. Table 4 presents the *final* category designation model used to identify deep knowledge (DK) and in transition (IT) teachers on the division and concept map tasks. The new categories used in transcript analysis are identified by italics in Table 4. SMK and PCK are not separated since they both contribute to teachers' content knowledge. The *final guide* in Table 4 was used in the transcript analyses for the fourteen teachers.

Table 4

Final Guide for Deep Knowledge Teacher Designation: SMK and PCK

Tasks1 & 2:	Task 3:	Concept map:	Task 2:
Types of pedagogical models and representations	Types of translations or solution methods	Connections among mathematical ideas related to the concept of fraction	Impact of reversal problem
1) Quotitive models	1) Quotitive	1) Part to whole, ratio, quotient, operator, measure, absolute/relative	1) On representations
2) Partitive models	2) Partitive	2) Types of models (i.e., linear, measurement, set, part/whole, comparisons)	2) On translations
3) Product/factor models	3) Reference unit	3) Connection between whole number division and division with fractions	3) Recognizing reference unit
4) Variety of topics for pedagogical models (i.e., time, work, cooking, construction forms)	4) Common denominator	4) Connection of multiplication to division	
5) Multiple models	5) <i>Graphic translation</i>	5) Connections to rational number ideas (percent, decimal forms)	
6) <i>Stress context</i>	6) <i>Connect to students' strategies</i>	6) <i>Fraction concept as set of processes versus as set of objects</i>	
7) <i>Stress reference unit</i>	7) <i>Algorithm (includes invert and multiply, repeated subtraction, repeated addition)</i>		

Each of the division with fraction tasks and concept map responses was examined separately, one at a time, using qualitative analysis (i.e., case samples) and quantitative analysis (i.e., frequencies, t tests, means, standard deviations) methods. The number of models/ representations, translations, solutions, and knowledge connections were counted and the two groups – in

transition (IT) and deep knowledge (DK) teachers compared on these categories for each task. To test hypotheses related to the division with fraction tasks 1, 2, and 3 (i.e., Hypotheses 1, 3, and 4), frequencies of the types of representations, translations, and connection categories between the two teacher groups were counted. Means, standard deviations and t test values were calculated to compare the groups for Hypotheses 1 through 4. Sample variances between groups were used to analyze Hypothesis 5.

Division tasks

Hypotheses 1, 3, and 4 examine differences on the three division tasks, tasks 1 through 3. Task 4 was dropped from the analysis because its inaccurate construction did not lead teachers to create various representations and translations.

Hypothesis 1: There will be differences between DK (deep knowledge) and IT (in transition) teachers on their SMK and PCK for dividing with fractions. SMK and PCK are defined as types and frequencies of representations and translations on tasks 1 through 3.

The *first* hypothesis tests possible differences in the number and types of representations (i.e., models) and translations (i.e., solutions) between the DK and IT groups for the first three tasks.

Hypothesis 3: IT teachers will attribute failure to create representations in tasks 1 and 2 to a lack of their mathematical knowledge.

The *third* hypothesis tests the types of reasons that IT teachers may give for not representing the number sentences: $1\frac{3}{4} \div \frac{1}{2}$ on Task 1 and $\frac{1}{2} \div 1\frac{3}{4}$ on Task 2.

Hypothesis 4: There will be a fewer representations on interview task 2 on IT and DK teachers' representations and translations.

The *fourth* hypothesis suggests that there may be an overall decrease in the number of representations for both groups when the divisor fraction is larger than the dividend as in $\frac{1}{2} \div 1\frac{3}{4}$, on Task 2.

In the three division tasks, each teacher's transcript for the task was analyzed for: 1) types of specific models (i.e., quotitive, partitive, product / factor); 2) types of translation or solution methods; 3) topic variety in story representation; 4) stressing reference unit (i.e., what does answer represent); 5) stressing context; 6) graphic translations; 7) different algorithm approaches; and 8) connection to student strategies. These are the categories represented in Table 4.

Concept map concepts

Hypothesis 2: There will be differences between DK and IT mathematics teachers on their connected knowledge of the fraction concept.

The *second* hypothesis tests for differences between IT and DK teachers in the number and types of concepts and links (i.e., relationships) between concepts on the constructed concept maps, Task 5.

The analyses of the concept map task included counting the number of links (i.e., relationships) and number of concepts teachers gave in their think alouds on the fraction concept. The process for coding and analysis was adapted from a study by Leinhardt and Smith (1985). The researchers had created semantic nets from their observations and transcripts of lesson videotapes on the teaching of division with fractions. Concept maps for the present study were from the audio and written transcripts from teachers on Task 5, counting concepts and links between the concepts. The maps were constructed using a program called Cmap Tools (<http://cmap.ihmc.us/>). It was assumed that a well-developed knowledge structure would be one with many links among concepts.

Each teacher's transcript was reviewed multiple times in order to create a concept map. Each concept map idea (i.e., concept) was counted as well as the number of levels (i.e., connections), and the number of connections *between* ideas.

The analysis for the concept maps included frequency of: 1) examples of the mathematical concept of fraction as a series of processes versus objects (i.e., does the teacher give examples of fraction models and representations or does the teacher present fractions as discrete objects, such as in naming parts or operations); 2) links between concepts; and 3) concepts (e.g., fractions as comparisons, as proportions). Tapes were reviewed several times to capture any missed details.

Variability in responses

Hypothesis 5: IT teachers will exhibit variable patterns in the use of their knowledge across tasks, whereas DK teachers will be consistent.

The *fifth* hypothesis assesses whether DK teachers are more similar to each other in terms of consistencies in translations and representations, for the division tasks. The sample variances for the two groups were compared.

Professional development questions

Responses to the follow up questions were also audio taped and transcribed. These questions requested examples of activities relating to teachers' professional development. Of particular interest were teacher discussions of influential: 1) PD workshops; 2) in school conversations with peers (i.e., conversations about subject matter and practice); and 3) reform type curricula.

Teacher Category Designation

As the *final* teacher category designation table was developed (see Table 4), teachers were coded as deep knowledge (DK) and in transition (IT) teachers. This process involved several stages. After reviewing each transcript, participants were identified as DK or IT in terms of interview fit with the *final* designation criteria from Table 4. The process of confirmation took three cycles of close review, and required confirming which criteria were met and not met.

Six teachers qualified as deep knowledge (DK) and eight as in transition (IT) teachers. The six DK teachers' responses included and exceeded the features initially predicted they would contain from the preliminary deep

knowledge categories (Table 2) and related research on teaching and mathematics knowledge.

Regardless of the task, the IT teachers were not novices. They were able to generate models/ representations and solutions for division tasks. Novice teachers in other studies did not generate any representations and gave “thin” or limited responses to similar tasks (Tirosh, 2000; Ball, 1988; Ma, 1999). This was not the case for the IT teachers. However, the frequency and quality of representations and translations of the IT teachers were not as many or as thorough as those of the deep knowledge (DK) teachers. DK teachers were expected to consistently have more representations, translations, and connections in their content knowledge (i.e., SMK and PCK) on fractions and division with fractions than the IT teachers. The group comparisons, reported in the Results, Chapter 4, are based on these expectations.

CHAPTER 4

Results

This chapter is organized with results from analyses of the: 1) demographics from pre-interview questions; 2) research questions and hypotheses; and 3) follow up questions for each teacher group.

Demographic characteristics

Participating teachers (N=14, 8 female, 6 male) had a variety of academic experiences in their undergraduate and graduate years including civil engineering, applied linguistics, liberal arts, and mathematics education. They were current public middle school math teachers, teaching 6th through 8th grade arithmetic and middle school pre algebra in New York City schools. Two of the 14 were math coaches at the time of the interview. Ten of the fourteen (71%) teachers had master's degrees in education; four (28%) had masters in mathematics education. The teachers' undergraduate degrees were in: communications, government, linguistics, psychology, biology, international relations, civil engineering, sociology, computer science, electrical engineering, science, finance, and business.

All of them had experiences in two forms of professional development (PD): curricula training with reform texts and/or participation in institutes. Sixty percent (N=8) of the teachers were using reform curriculum in their classes at the time of the interview. Examples of these reform texts are the *Connected Mathematics Project (CMP)*, *TERC: Investigations in Data, Number, and Space*,

and *Math in Context* (MiC). Each program conducted extensive out of school professional development activities to assist teachers in using the texts.

Other PD programs did not revolve around set curricula but were designed as institutes for teachers to attend during or after the school year. Fifty percent (N=7) of the teachers had attended PD institutes/ workshops, such as MS², Math in the City, or Summer Math. These were conducted as group experiences in which teachers become students and were challenged to examine their thinking about mathematical content concepts while becoming aware of students' strategies for problem solving for specific mathematics topics. Two of the teachers were using reform curricula and had attended workshop style teacher projects. The majority of the teachers interviewed worked at schools where there were informal (60%) (i.e., conversations with other teachers) and formal (71%) (i.e., weekly math department meetings) and continuing PD experiences as well.

Table 5 summarizes the participating teachers' 1) years in mathematics teaching and 2) educational backgrounds. Six DK teachers (4 male, 2 female) and eight IT teachers (2 male, 6 female) were identified from the interviews. Means and standard deviations for years of teaching are given as well as the percent of master degrees in education for each group.

Table 5

Educational Background and Average Years Teaching Mathematics

Teacher Group	Years teaching		Percent of Master Degrees in Education
	Mean	SD	
Deep Knowledge (DK)	5.67	1.45	50%
In Transition (IT)	15.00	4.0	87%

DK= 6 teachers, IT= 8 teachers

Teaching experience

The average (mean) years teaching for the 14 teachers was 11 years (SD= 9.9) with a minimum of 2 and a maximum of 33 years. For the 8 IT teachers, the mean years for mathematics teaching were 15 years (SD=4.0) versus 5.7 (SD=1.45) years for the 6 DK teachers. Half of the DK teachers had master degrees in education as compared to 87% of the IT teachers. The IT teachers Masters were in elementary education (N=4), mathematics education (N=2), reading (N=1) and ESL (N=1). DK teachers with masters were in elementary education (N=1) and in mathematics education (N=2). All had been involved with either institute or curricula forms of professional development.

Performance on Division and Concept map Tasks

In this section, results for each hypothesis connected to the division tasks and their corresponding research questions are presented. As stated earlier, task 4 was dropped from analysis because the word problem, as created, did not make it possible for teachers to create representations and translations. This task is in Appendix G.

The research questions are used to assess the frequency of 1) multiple representations; 2) multiple solutions; 3) connections; 4) the challenge to represent on Task 2; and 5) patterns of response for the two groups on division tasks 1, 2 and 3. Content knowledge is defined here as: subject matter (SMK) and pedagogical content (PCK) knowledge. The wording of the directions for the task tasks resulted in teacher responses that integrated SMK and PCK. That is, their responses were focused on how the subject matter was presented and made comprehensive to students. While subject matter knowledge can be identified, it was not separate from pedagogical thinking about it. This content knowledge, SMK and PCK, was operationally defined in the frequencies of representations, translations, and connections of division of fractions by teachers on the problem solving tasks and concept maps. The *second* hypothesis tests for differences between IT and DK teachers in the number and types of concepts and links (i.e., relationships or connections) between concepts on the constructed concept maps, Task 5.

Hypothesis 1: Solutions and Representations for Tasks 1 and 2

Hypothesis 1: There will be differences between DK (deep knowledge) and IT (in transition) teachers on their content knowledge for dividing with fractions. The content knowledge variables, SMK and PCK, for Hypothesis 1 are defined as types and frequencies of representations and translations on tasks 1 through 3 as specified in the teacher category table (see Table 4).

Research question 1 focused on the frequency and type of representations (i.e., models) and translations (i.e., solutions) teachers constructed for division task 1: $1\frac{3}{4} \div \frac{1}{2}$ and division on task 2: $\frac{1}{2} \div 1\frac{3}{4}$.

Question 1: What types and how many representations of division with fractions do deep knowledge (DK) and in transition (IT) teachers present in tasks 1 and 2?

The number and types of models (i.e., representations) and translations (i.e., solutions) for each teacher in the first two tasks were counted. Different types of representations and translations were identified, counted and averaged for each group. The results are presented in Tables 6 (Task 1), 7 (Task 2), and 8 (Task 1 and 2 combined). In each table, the means, standard deviations, t test results, degrees of freedom, and p values for each category of representation or translation for Task1, 2, and a combination of the two are given.

Table 6

Solutions and Representations for Task 1: $1\frac{3}{4} \div \frac{1}{2}$: DK and IT Teachers' Means, Standard Deviations, t Values

Solutions and Representations	Deep Knowledge		In Transition		t	df	p
	Mean	SD	Mean	SD			
Quotitive models	2.33	(1.21)	1.13	(.99)	2.06	12	.06
Partitive models	1.17	(.75)	.00		4.45	12	.00**
Product / Factor models	.50	(.84)	.00		1.71	12	.11
Multiple models (combination)	3.83	(2.31)	1.25	(.89)	2.91	12	.01*
Algorithm [^]	.67	(.52)	.88	(.35)	.90	12	.39
Context	1.00	(.00)	.50	(.53)	2.27	12	.04*
Unit	.67	(.52)	.00		3.70	12	.00**

DK teachers= 6, IT teachers = 8** = Significance at $p < .01$, * = Significance at $p < .05$, ^ = Algorithm category includes use of invert/multiply, common denominator, distribution, and conversion approaches to solving the first part of the two tasks.

All of the teachers solved the number sentence problem correctly for the answer, $3\frac{1}{2}$. When asked to create a story that would fit the problem situation: $1\frac{3}{4} \div \frac{1}{2}$, DK teachers were different only in the number of partitive stories they created. Groups were similar in the mean number of quotitive and product factor models they created and neither group was more likely than the other to use algorithm methods as their primary source for solution. DK teachers created

more contextualized stories (i.e., used multiple models) than IT teachers. They were also more likely to discuss the importance of considering what the answer, $3\frac{1}{2}$, would mean in reference to whatever was the whole unit in the story they created. They also stressed how the context they created in the stories would be important for how the student would then think about solving the problem.

Table 7 includes results from task 2. In task 2, the order of the first problem was reversed to $\frac{1}{2} \div 1\frac{3}{4}$. This problem was included to examine what would happen to the types and frequencies of stories (i.e., representations) and contexts teachers would give in the first problem if the numbers were now reversed, complicating the initial task. The divisor as the larger number becomes much more challenging to represent because it is 1) a larger number than the dividend and 2) a mixed number.

Table 7

Solutions and Representations for Task 2: $\frac{1}{2} \div 1\frac{3}{4}$: DK and IT Teachers' Means, Standard Deviations, t Values

Solutions and Representations	Deep Knowledge		In Transition		t	df	p
	Mean	SD	Mean	SD			
Quotitive models	1.00	(.00)	.13	(.35)	6.00	12	.00**
Partitive models	1.17	(.75)	.00		4.45	12	.00**
Product / Factor models	.17	(.41)	.25	(.46)	.35	12	.73
Multiple models (combination)	2.17	(.41)	.13	(.35)	10.01	12	.00**
Algorithm [^]	1.00	(.00)	.88	(.35)	.86	12	.41
Context	1.00	(.00)	.13	(.35)	6.00	12	.00**
Unit	1.00	(.00)	.13	(.35)	3.70	12	.00**

DK teachers= 6, IT teachers= 8** = Significance at $p < .01$, * = Significance at $p < .05$, ^ = Algorithm category includes use of invert/multiply, common denominator, distribution, and conversion approaches to solving the first part of the two tasks.

On this division task, both DK and IT teachers created far fewer stories than in the first task. DK teachers did construct more stories than IT teachers, specifically more quotitive and product/ factor types for this challenging problem. They also, as was the case in the first task, stressed the contexts of the problem situations and what the answer $\frac{2}{7}$ would now refer to, $\frac{2}{7}$ s of $1\frac{3}{4}$ (i.e., the reference unit) more than the IT teachers did for this task.

Table 8 presents the means, standard deviations, t test values for the combined data of tasks 1 and 2. Across the first two division tasks, DK teachers created more models, particularly quotitive and partitive ones, and stressed context and unit issues in creating stories and considering the meaning of answers as compared to the IT teachers.

Table 8

Solutions and Representations for Tasks 1 and 2: DK and IT Teachers' Means, Standard Deviations, t Values

Solutions and Models	Deep Knowledge		In Transition		t	df	p
	Mean	SD	Mean	SD			
Quotitive models	3.33	(1.21)	1.25	(.89)	3.73	12	.01**
Partitive models	2.33	(1.50)	.00		4.45	12	.00**
Product / Factor models	.67	(1.03)	.25	(.46)	1.02	12	.33
Multiple models (combination)	6.00	(2.60)	1.37	(.92)	4.70	12	.01**
Algorithm [^]	1.67	(.52)	1.75	(.46)	.312	12	.76
Context	2.00	(.00)	.63	(.52)	6.44	12	.00**
Unit	1.67	(.52)	.13	(.36)	6.65	12	.00**

N= 6 DK teachers, 8 IT teachers

** = Significance at $p < .01$, * =Significance at $p < .05$, ^ = Algorithm category includes use of invert/multiply, common denominator, distribution, and conversion approaches to solving the first part of the two tasks.

Cases: Story samples from task 1

When asked for a “real world example”, teachers produced the following types of models or representations: quotitive (measurement/ grouping), partitive,

and product and factor models. Examples of the different types of interpretations for 2 divided by $\frac{1}{2}$ can be:

- 1) Quotitive (Measurement/grouping) model: Find how many $\frac{1}{2}$ s there are in 2.
- 2) Partitive: Find a number such that $\frac{1}{2}$ of it is 2.
- 3) Product and Factor: Find a factor that multiplied by $\frac{1}{2}$ will be 2.

Each of these interpretations yield the same result, in this case, 4, but they are very different in the types of story contexts that would be created to capture the quotitive, partitive, and product/factor idea.

Each DK teacher presented at least two different stories for the number sentence, $1\frac{3}{4} \div \frac{1}{2}$. One DK teacher, T6, created two stories for the first problem.

The first was partitive representation and the second, a quotitive one:

1) I think you could also think of it as now it takes 1 and 3 quarters to do this job. That's half of what it used to take. How long did the job used to take?

2) You have a half bag of potatoes and in total you have 1 and 3 fourths of these things, how many $\frac{1}{2}$ bags can you fit these $1\frac{3}{4}$ lbs. into? (T6)

In the first example, the teacher constructs a partitive context. In this situation, the goal is to find a number such that one half of $3\frac{1}{2}$ is the dividend,

$1\frac{3}{4}$. The second example is a grouping example. The goal is to find the number

of bags of potatoes that can be made from a total of 1 and $\frac{3}{4}$ lbs of potatoes.

Here, the whole is $\frac{1}{2}$ and the answer related to the number of $\frac{1}{2}$ s (wholes) in the

amount $1\frac{3}{4}$. In each case, the answer is $3\frac{1}{2}$, but each story is different in its

meaning because of the representational contexts are different in each.

Many of the IT teachers were also able to create stories. Teacher 14 explains on the solution part of division task 1 how she would solve it. First she gives the invert and multiply algorithm and then says:

Another way to think about it is like thinking about what would happen when we have 1 divided by 2. How many times or how many $\frac{1}{2}$ are in $1\frac{3}{4}$ s? I guess I could find how many halves are

there (starts counting). So this is one $\frac{1}{2}$, two $\frac{1}{2}$, three $\frac{1}{2}$ s and then a

half of the $\frac{1}{2}$ is left. So this would be 3 whole $\frac{1}{2}$ s and then $\frac{1}{2}$ of a

half. That makes sense. I went to whole number division first cause

I wanted to get at the idea that what division is about. (T14)

Interestingly, this same teacher when asked for a representation/ story on the second part of the problem said:

I never really thought about it. Let's think when would you want to find out how many halves you have, um, I am not sure. Like $\frac{1}{2}$ dollars, but that doesn't work, I am stuck. I don't know. In real life, we don't really work with halves, you know. We divide by two but we don't really divide by halves. It depends on what we are dividing by (T14).

T14 does not connect her own conceptual understanding of thinking about the unit of halves and counting how many halves are in $1\frac{3}{4}$ s (i.e., in the solution part of her answer) as an opportunity to construct a story. She has knowledge for the model but doesn't realize she has and so does not transfer her knowledge to a pedagogical story situation.

Other IT teachers were able to create stories, particularly for task 1:

$1\frac{3}{4}$ divided by $\frac{1}{2}$.

1) I can understand this division with fraction problem as you have this much of something and you want to know how many half cups you can get out of it. (T9)

2) You have one and three quarter liters of lemonade, how many half liters can you give away? Now that works since you can have $\frac{1}{2}$ of $\frac{1}{2}$. (T11)

In each, the teacher creates context using a quotitive model. One half represents the whole unit of reference in each of these situations.

The majority of the DK teachers stated that the story they would create for tasks 1 and 2 had a lot to do with what their answer would mean (i.e., context).

An illustration of this point:

Well there are a few actually. It depends on how I want kids to think about it to accurately solve the problem. Each way I think is going to yield different strategies in the ways that kids solve the problem.

(T5)

The DK teachers and some of the IT teachers recognized that the answer, $3\frac{1}{2}$, drove the context in which the problem would have to be set up. A DK

teacher, T1's response illustrates this point:

You have 1 and three quarters of pizza pies and you want to give everyone a half a slice, how many people can you serve? No, that's not right.

And corrects with this example:

Well if you had one hour and three quarters, um and you had half hour appointments, how many half hour appointments can you have? (T1)

In the second example, T1 uses a quotitive model. The answer $3\frac{1}{2}$ refers to the number of half hour appointments that could be made within the span of 1

and three quarters of an hour. She realizes the first example is incorrect because the answer, $3\frac{1}{2}$ would not fit a sharing story between people. She is aware of more appropriate contexts, such as work, time, building, or set examples.

Cases: Story samples from task 2

On task 2, both DK and IT teachers were challenged to come up with a real world story for the problem with a fractional divisor that was larger than the dividend, $\frac{1}{2} \div 1\frac{3}{4}$. DK teachers created the following stories:

1) If 1 and 3/4s pound of flour is a dollar, how much is a half-pound of flour?

2) Or I make 1.75 an hour, how long does it take me to make 50 cents? (T5)

3) Here's $\frac{1}{2}$ an hour and that's 1 and three fourths of what it used to take. So what did it used to take? (T6)

4) What piece of $1\frac{3}{4}$ s is $\frac{1}{2}$? This is a part to whole relationship.

Two sevenths of $1\frac{3}{4}$ is $\frac{1}{2}$. (T7)

In these stories, the DK teachers attempt to solve the challenge of having a fractional divisor that is larger than the dividend. The types of stories they create resemble partitive models, in which the goal is to find the whole that 1 and three fourths is part of. In such cases, none of the teachers used a quotitive model.

Hypothesis 1: Task 3 Solutions

The second question connected to Hypothesis 1 was to examine the number and types of translations teachers would generate when the division was in a word problem form, not a number sentence form.

Question 2: What types and how many translations (i.e., multiple solutions) do DK and IT teachers present in task 3? In task 3, the word problem was: Wally has $6\frac{2}{3}$ yards of material. It takes $\frac{5}{6}$ of a yard to make his scarves. How many scarves can he make? The word problem contextualized the division story and did not require teachers to create one of their own.

Table 9 presents the results for the third task and presents the mean number, standard deviation, t test results, degrees of freedom, and p values for each category of translation for task.

Table 9

Task 3: From Representations to Translations: DK and IT Teachers' Means, Standard Deviations, t Values

Translations	Deep Knowledge		In Transition		t	df	p
	Mean	SD	Mean	SD			
Invert and multiply	.50	(.55)	.25	(.46)	.90	9.79	.389
Unit of reference	.83	(.41)	.25	(.46)	2.45	11.59	.029*
Connect to student strategies	.50	(.55)	.13	(.35)	1.56	12	.145
Quotitive	1.00	(.00)	1.00	(.00)			
Partitive	.17		.00		1.17	12	.264
Graphic translation	.83	(.41)	.87	(.35)	.20	9.9	.846

Note: N= 6 DK teachers, 8 IT teachers, *= Significance level $p < .05$

The word problem in task 3 was already in representational form and there were no differences between the two groups on performance in creating translations. In their responses all correctly solved the problem, teachers discussed the importance of drawing out the problem situation (e.g., bars for partitioning) and were equally as likely to use the traditional invert and multiply algorithm. DK teachers were different only in terms of being more likely to use a common unit translation to solve the word problem (i.e., emphasizing the unit, $1/6$ in solving the story).

Cases: Translation samples from task 3

The story problem was given in a quotitive/ measurement representation (i.e., how many scarves of $\frac{5}{6}$ yard length can be made from fabric of $6\frac{2}{3}$ yard length?)

Examples of DK and IT teachers' responses are:

- 1) There's the create the improper fraction and invert and multiply. There's the quotitive situation, I could create a common denominator and just focus on the numerator. Let kids approach it in different ways so that they can approach it in ways to make sense to them. Try to make connections to kids strategies, why they work and make sense. How many $\frac{5}{6}$ s are in $6\frac{2}{3}$ s? It is division but why is it division? (DK 5)

2) This is a word problem. I would want to think of it in $\frac{1}{6}$ ths. I would

want to look at it visually. Would want my class to do this visually and chart our work. Pictures can then support our work. I could start

chopping $\frac{5}{6}$ ths, this is one $\frac{5}{6}$ ths, this is another, and I would keep

going until I used up the fabric. (IT 9)

3) So, in my mind, if you put a word next to your answer like 5

scarves and you have a piece left over, well what are you trying to

find, is it a fraction of a scarf or is it a fraction of a yard. The answer

relates to the number of scarves, the piece left over is a part of the

scarf (DK 3).

These teachers emphasized a focus on: 1) students' strategies for solution; 2) graphic representation for the story; and 3) understanding the meaning of the answer.

Hypothesis 2: Concept map connections and links

Hypothesis 2: There will be differences between DK and IT mathematics teachers on their connected knowledge of the fraction concept.

The *second* hypothesis tests for differences between IT and DK teachers in the number and types of concepts and links (i.e., relationships or connections) between concepts on the constructed concept maps, Task 5.

Hypothesis 2 focused the analysis of the concept map to the 1) number of concepts and 2) number of links between concepts (i.e., connections) as

indicator of a deep understanding of the topic. The analyses that were made for the final guide (Table 4) resulted in two other concepts: process type and object type. Thus, in addition to assessing the frequency of concepts as well as links between concepts, the maps were analyzed on the number of statements made by the teachers that focused on processes for connecting fraction representations. *Process type concepts* include discussion of: 1) fraction relationships; 2) types of models; 3) unit of reference as implied by problem construction; 4) comparison types of relationships; 5) context; and 6) connection to landmark fractions.

Maps were also analyzed on the number of object type characteristics of the fraction topic. *Object characteristics* included discussion of: 1) naming denominator and numerator; 2) naming operations--add, multiply, divide, and subtract; 3) naming percents and decimals; and 4) naming fractions as "pieces." In summary, the concept maps were analyzed for number and types of concepts, links (i.e., connections), process type, and object type.

The connections (i.e., number of links) between concepts on teachers' concept maps were not significantly different for the two groups. DK teachers, on average, had 17.33 (SD=5.78) relationships (i.e., links) on their concept maps. For IT teachers there were 14.37 relationship connections (SD=3.85) with no difference between groups $t=1.152$, $df=12$, $p>.27$. Chi square analysis for possible differences between groups on the representation of the fraction concept as a *process* showed no significant differences between the two groups

in terms of presenting more process (i.e., relationships, model types, etc.) type concepts, $\chi^2 = 2.73$, $df = 1$, $p < .098$. The chi square to test whether either group was more likely to present object type of concepts (i.e., naming the parts of fraction, naming operations, etc.) did show differences between groups, $\chi^2 = 6.52$, $df = 1$, $p < .011$. In transition (IT) teachers were more likely than deep knowledge (DK) teachers to identify more object type concepts when asked what ideas students should know in order to understand fractions.

Cases: Constructed maps of DK and IT teachers

Concept maps for all teachers are given in Appendix J. Four specific cases are presented here. These cases were chosen for their contrasting features. The first is a DK teacher's comprehensive view of the fraction concept with the emphasis on fractions as a web of connected models and representations. The second is a concept map of an IT teacher who has acquired the terminology of the subject matter knowledge. However, this teacher's performance on the division tasks indicates the qualities of "in transition." That is, the teacher on the division tasks 1 and 2 did not have the types and numbers of representations and translations the DK teachers had.

The third map is that of an IT teacher who stressed object (i.e. object features) of fractions rather than the process types for fractions (see p. 112). The fourth map is from the field study and is that of one of the novice teachers interviewed. This map is included in order to contrast the likely differences between a novice teacher, in transition teachers and deep knowledge teachers.

DK teacher, T5

The map below was constructed from the transcript of T5, a DK teacher. Throughout his transcript, the “big idea” from his map was to view fractions as links of various models, contexts and comparisons. This teacher discussed all the major types of models: area, quotitive, grouping, partitive, linear, and equivalency. He also identified the importance of strategies for comparing fractions such as landmark fractions (e.g., comparison to “friendly” fractions such as $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{3}$), and the missing piece strategy (e.g., in comparing two fractions, consider how far away each is from 1, the larger the missing piece from one, the smaller the original fraction for comparison).

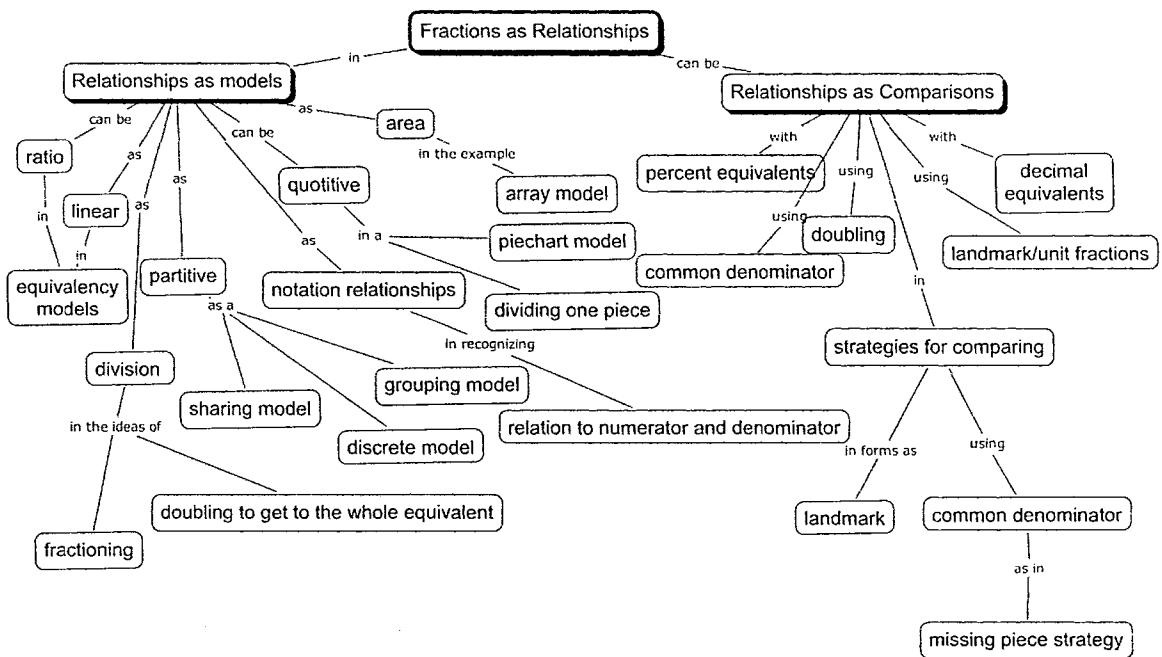


Figure 1. Concept map of DK Teacher, T5

An IT teacher, T9

The second concept map was constructed from the transcript of an IT teacher. As with the DK teacher, there are a number of concepts and numerous

links between concepts. On the division tasks tasks, 1-3, this teacher did not create many representations and translations but her concept map contains as many links as the DK teachers.

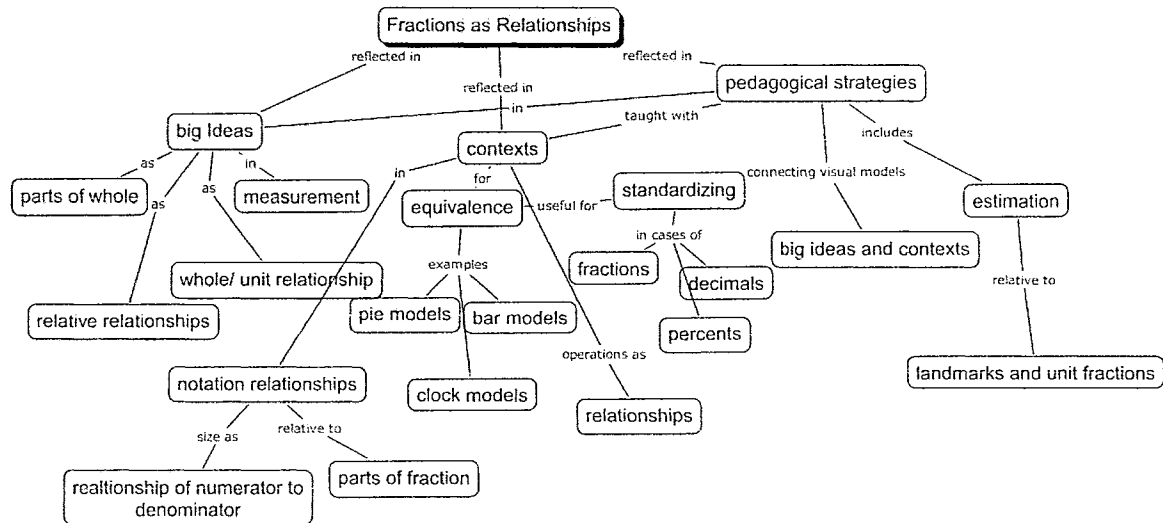


Figure 2. Concept map of IT Teacher, T9.

This teacher’s performance on the division tasks was more like that of an IT teacher 9 (i.e., fewer representations and translations). The concept map, however, is thorough and contains more examples of connected representations and models than are usual for the IT teachers in the study.

IT teacher, T14

The concept map below is that of an IT teacher who stressed forms of fractions (i.e., decimals, percents) more than models and representations of fractions.

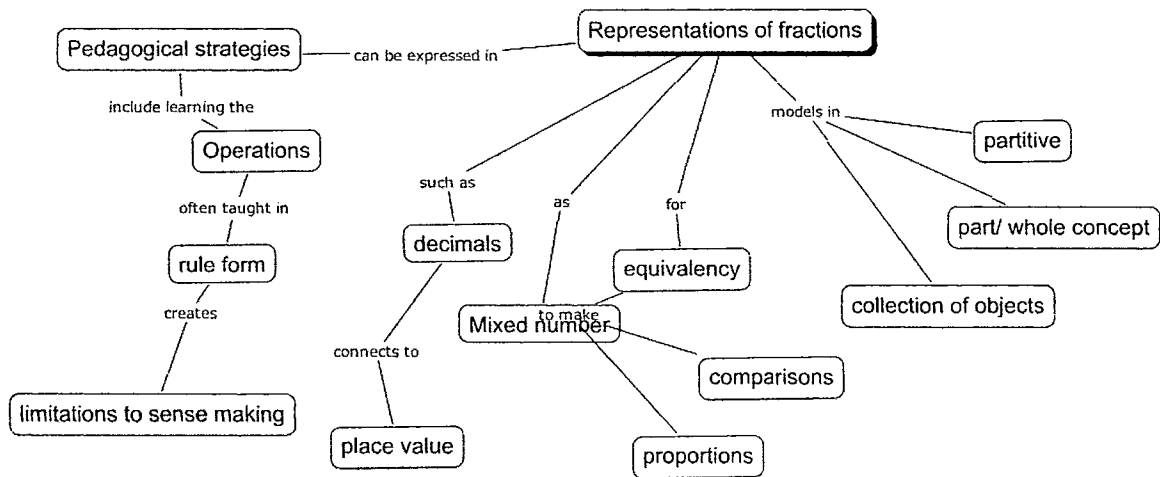


Figure 3. Concept map of IT Teacher, T14

The teacher discusses the different types of representations from fractions as well as naming operations and fraction types (i.e., decimals, mixed number, and place value).

Novice teacher

The fourth concept map is that of a novice teacher concept map from the field study. The map has very few concepts and fewer connections between them. Here, it is clear that the teacher views the fraction concept as a set of discrete objects. This map was included to place in transition knowledge as separate from novice knowledge of the fraction. It clearly illustrates the differences in knowledge between a teacher in transition to deep knowledge and a novice teacher with far less detailed knowledge about the concept.

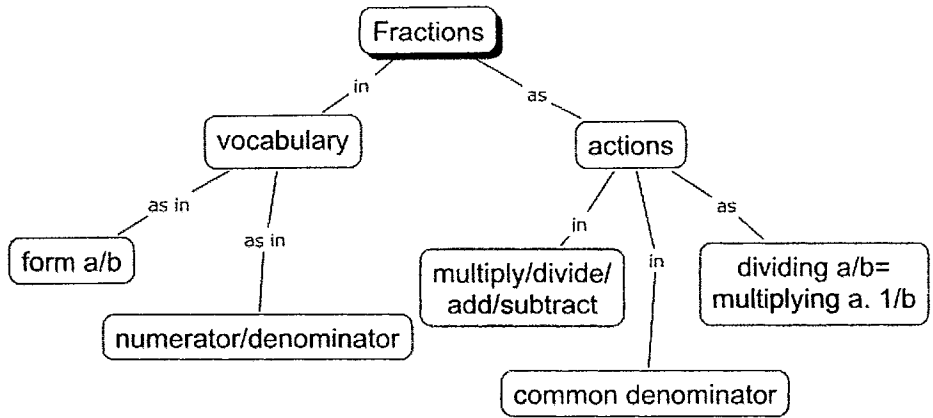


Figure 4.Concept map of Novice Teacher

In summary, IT and DK teachers’ concept maps were similar in the number of concepts and in the number of links (i.e., connections) between concepts. IT teachers were more likely to present object type forms of fractions than DK teachers who stressed the process forms of fractions (i.e., relationships and connected models) for students to understand fractions. The remaining 6 concept maps (in Appendix J) of IT teachers show examples of their knowledge development of fractions.

Teachers discussed the importance of students having a variety of models for understanding fractions.

I think the most important thing is that students have a bunch of different models of what a fraction is before they do any operations with them. To understand fractions in a variety of contexts like parts of a whole, splitting the whole, or ascertain subsets of a set, understanding distance models, number line models, and partitive models (DK teacher 6).

DK teacher 13 emphasized the importance of students developing connections between pictures, words, numbers, and manipulatives in their experiences with fractions.

What's important for students is to help them develop links between the pictures, the numbers, the words, and the manipulatives. Students will link the picture to the words to the manipulatives, to the number sentences and different number sentences to each other (DK teacher 13).

In their think alouds, the DK teachers related features for understanding fractions as relationships and the importance of contextualizing fractions as points of comparison. They also stressed strategies for understanding equivalency and having different instructional experiences (i.e., visual, numerical) of fraction problems.

Hypothesis 3: Types of reasons given for representation challenges

Hypothesis 3: IT teachers will attribute failure to create representations in tasks 1 and 2 to a lack of their mathematical knowledge. *Research Question 4:* What types of reasons do IT teachers give when they fail to represent the problems in tasks 1 and 2 in representational or story word forms?

The reasons for not being able to create a story for these two division tasks were expected to include: 1) teacher acknowledges limitations in content knowledge; 2) teacher believes that any story for the number sentence would be

unrealistic; 3) teacher cites lack of models in current curricula with similar situations.

Five of the 8 IT teachers created stories for task 1, $1\frac{3}{4} \div \frac{1}{2}$, and one also created a story for task 2, $\frac{1}{2} \div 1\frac{3}{4}$. For task 1, those who did not create stories believed that no real situations or support curricula existed to create an application for $1\frac{3}{4} \div \frac{1}{2}$. For example:

Reason 2 (above): In real life we don't work with halves, we divide by two but we don't divide by halves (T14).

Reason 3 (above): We have kids say that when we divide by $\frac{1}{2}$, we are not dividing by 2 but we are multiplying by 2. Dividing by $\frac{1}{2}$ is not used very often. That's why I am having a hard time coming up with a story (T6).

On task 2, 7 of the 8 IT teachers did not create representations. Of those, 45% of their reasons were on lack of such problems in the curricula they work with and 55% said they couldn't envision a story situation for such a problem.

Examples of each type of reason are:

Reason 2) This one's very hard to come up with as a story. It doesn't exist in the world unless it's like $\frac{1}{2}$ divided by 2. You will find these as computational problems not as story problems (T7).

Reason 3) I have never thought about how to put a problem like this into words, because usually the bigger fraction comes first. When it comes to story models in the 8th grade curriculum, I haven't seen these types of story problems. It is difficult to put this into a word problem format (T11).

One of the IT teachers who did not create a story for the division tasks acknowledged that representations may exist but that her current content knowledge may be the reason for not being able to construct a story.

I am sure a real life story does exist. I know there are real life concrete examples for this problem. I just don't know any. One of the things that professional development programs are getting me to think about are real life examples (T2).

This teacher was aware that story situations existed and was developing the knowledge to do so from her PD experiences.

Hypothesis 4 and Question 5: Dividing by the larger fraction

Hypothesis 4: There will be less representations for division task 2: $\frac{1}{2} \div 1\frac{3}{4}$

on IT and DK teachers' representations and translations. To answer question 5, the average (i.e., mean) number and types of representations were compared for the groups on task 2 and also compared to representations for task 1. This was an opportunity to examine the difficulty of challenging the IT and DK teachers to create a problem, that of the larger fraction in the divisor role.

It is assumed that the switching of numbers, making the divisor the larger fraction than the dividend, would increase the complexity and be difficult for both groups of teachers. It was hypothesized that deep knowledge teachers might be more likely to represent and translate when this situation occurred but they would be challenged as well. Within each group of teachers, types and frequencies of representations (models and translations) are expected to be fewer for task 2 than for task 1.

The average (i.e., mean) number and types of representations within the groups for task 2 were compared to those for task 1. Table 10 presents the mean differences for IT and DK groups on division task 1 performance versus division task 2. Performance is defined as the variable: multiple models. This variable is the sum of the models each group presented per task. Means and standard deviations for each group are given.

Table 10

Means and Standard Deviations for DK and IT Teachers on Multiple Models in Division Task 1 and Task 2

Teacher	Multiple Models, Task 1		Multiple Models, Task 2	
	Mean	SD	Mean	SD
Deep Knowledge	3.83	(2.32)	2.17	(.41)
In Transition	1.21	(.89)	.13	(.35)

Each group's average number of models decreased from task 1 to task 2, when the challenge of creating the representation increased. There were no differences between groups on the decrease in models in task 1, $\chi^2 = 6.51$, $df = 6$,

$p < .34$. Groups were different in their constructions of models for task 2, $\chi^2 = 14.0$
 $df = 3$, $p < .00$.

Hypothesis 5: Variability in responses

Hypothesis 5: IT teachers will exhibit variable patterns in the use of their knowledge across tasks, whereas DK teachers will be consistent. *Question 6:* Do DK and IT teachers' response patterns across tasks differ? This hypothesis examined whether DK teachers are more similar to each other in terms of consistencies in translations and representations for the division tasks. The sample variances for the two groups were compared. Each teacher's representations and translations were counted for each group. These were summed and the sample variance was calculated. The sample variance for the DK group was $s^2 = 10.7$, $SD = 3.27$ versus the IT group's sample variance was $s^2 = .70$, $SD = .83$. The IT teachers on average had 2.1 representations and translations (out of 9 possible) across tasks 1, 2, and 3. The DK teachers had a range of 5 to 9 representations and translations (out of 9 possible). They gave on average, 4.5 representations and translations across tasks 1, 2, and 3. The IT teachers were more similar to each other than the DK teachers. DK teachers had a wider range in the numbers of representations and translations they created than the IT teachers, contrary to the initial statement of Hypothesis 5. In response to Question 6, the analyses indicate that there is a difference in response patterns across the three tasks. These analyses are considered further in the discussion chapter.

Professional Development Questions

At the end of the interview, teachers were asked: to discuss 1) a particular moment or teaching experience that had an impact on the development of their mathematics knowledge and teaching; 2) with whom they currently have math conversations; and 3) how such discussions have affected their knowledge and practice. These follow up questions were created in order to elicit and document teachers' perceptions and experiences in learning the mathematical content, perhaps in new ways and in experiences with others. Teachers' mathematics knowledge may have been shaped by different combinations of sustained professional development experiences and involvement in training experiences with math curricula.

Teachers' transcripts were reviewed for examples and citations related to 1) PD workshops; 2) in school conversations with peers (i.e., conversations about practice); and 3) reform style curricula. These examples were assessed in relation to their development of subject matter and pedagogical content knowledge. Nine of the teachers (out of the 14) stressed the importance of the PD workshops; 86% (N=11) discussed the value of working together and communicating in content related conversations in their schools in the continual search for "better ways to teach"; and 78% emphasized the importance of using reform style curricula in their classrooms, with the resulting focus away from "traditional" texts to lessons focused on investigations.

Percentage of responses for the DK and IT groups for the three categories: workshops, conversations about practice, and reform curricula are presented in Table 11.

Table 11

Percentage of responses for DK and IT teachers on Professional Development Influences

Teacher	Workshops	Conversations about practice	Reform curricula
Deep Knowledge	66.7%	100%	100%
In Transition	62.5%	75%	62.5%

DK teachers=6, IT teachers= 8

Chi square to test differences between groups on discussing workshops was not significant $\chi^2 = .03$ $df = 1$, $p < .87$. Chi square testing differences between groups on discussing conversations about practice was not significant, $\chi^2 = 1.75$ $df = 1$, $p < .19$ and neither was chi square to test discussions about the impact of reform curricula, $\chi^2 = 2.86$ $df = 1$, $p < .09$. There were no differences between groups on the frequency of DK and IT teachers' verbalizations on professional development influences that contributed to changing their knowledge and teaching.

For the teachers, conversations with their peers at school were very important. Through the conversations, they were able to "do" math with their colleagues, examine students' work, share and argue about the mathematics they were teaching. For others, their experiences in institute style workshops were key in helping to shape their content knowledge and practice, allowing them

to experience the mathematics they taught from the larger, big idea approach. IT and DK teachers discussed their in-school communities for teaching math and emphasized the experiences of workshops and reform curricula as factors contributing to their knowledge and practice.

I am currently teaching this investigation seminar, I am hunting for new material all the time. I have become a math person through experiences where I keep wanting to know more about mathematics. I have really open, thoughtful colleagues. We talk about teaching math a lot. (T1, DK)

I was never interested in math until I became a math teacher. Having the opportunity to work with professional development instructors was fantastic. I like the standard that they set. I saw it as a vision of how I wanted to teach math. (T14, IT)

CHAPTER 5

Discussion

This chapter contains: 1) a summary of study results; 2) a proposed content knowledge model of subject matter (SMK) and pedagogical content (PCK) knowledge features for dividing with fractions; 3) study strengths; 4) study limitations; 5) conceptual understandings for dividing with fractions; 6) ideas for future directions; and 7) conclusions.

In this study, teaching practice and content knowledge were described in middle school teachers' own words as they related their understanding of how they would solve, represent, and translate the difficult to teach and difficult to learn topic, division with fractions. Participating teachers provided examples and insights on deep knowledge (DK) and in transition (IT) teachers' content and pedagogical knowledge of the topic.

Years of Teaching and Performance

The participating middle school mathematics teachers had a variety of undergraduate and graduate experiences and were either currently or had been involved intensively with professional development experiences. These were institute style workshops or trainings in using reform curricula and texts for middle school mathematics. IT teachers were more likely than the DK teachers to hold master's degrees in education, but this may have to do with the difference in years of teaching for the two groups. DK teachers, on average, had been teaching about 6 years ($SD=1.45$) whereas the IT teachers had, on average, 15

years ($SD=4.0$) of teaching experience. The IT teachers, with more years experience, were more likely to have completed their master degrees.

In educational research, expertise in teaching has often been defined as years of teaching. In this study, teachers with fewer years of teaching were more expert in their performances on division of fraction tasks and more expert in the quality and frequency of representations, translations, and connections even though they had considerably less years of teaching experience.

Summary of Representations, Translations, and Connections

A summary of the types of representations, translations, and connections deep knowledge and in transition teachers verbalized on think aloud tasks are provided along with a proposed model for considering teacher knowledge of and teaching for division with fractions (Table 4).

Across the division tasks, tasks 1 through 3, teachers' responses were examined for occurrences of specific types of representations and translations such as partitive, quotitive, and product factor models and methods. Teachers' descriptions of context and unit of reference were also analyzed. On the concept map tasks, the constructed maps of the teachers were analyzed for the frequency of fraction concepts and links between concepts the teachers verbalized and whether these concepts related more to the processes of understanding fractions (i.e., models) or to object like features of the concept (i.e., naming operations, and parts of a fraction).

On division task 1, DK teachers created more partitive contexts and, on average, presented 3.83 models versus 1.25 models of IT teachers in the number sentence as a story form. The IT teachers did create stories to match the meaning of the number sentence. DK teachers also constructed a greater variety of stories for division task 1 and were more likely to include all three types of possible models such as partitive, quotitive, and product/factor. Trend in the direction for significance with quotitive models was also observed.

On division task 2, both groups of teachers average number of representations decreased from division task 1. As stated earlier, division task 2 was difficult to conceptualize as representations since the divisor, a mixed number, was larger than the dividend. On this task, DK teachers created 2.17 stories ($SD=.41$) versus, on average, less than one story for each IT teacher (i.e., $M=.17$, $SD=.35$). Task 2 was an “inversion” of division task 1 and was created to assess a further layer of DK teacher knowledge, particularly with challenging division with fraction problems. DK teachers continued to represent more types and varieties of models.

Throughout both division tasks (i.e., tasks 1 and 2), DK teachers also pointed out the contexts for the stories they created. IT teachers were not as likely to stress context or unit in the first two division number sentence problems, Tasks 1 and 2.

There were fewer differences on representations and translations between the two groups on Task 3. Since the problem was given in word problem form, it

was not surprising that both groups of teachers performed equally well in making translations (i.e., solutions). The problem was contextualized into a story format and both groups of teachers were comfortable in creating methods for solution. Interestingly, all of the teachers drew the division situation (i.e., made a visual model) in the story, but none did so when the division with fraction problem was in number sentence forms as in tasks 1 and 2.

This stress on creating a visual model in order to solve the problem may come from the influence of constructivist principles in learning and teaching mathematics. That is, through the process of working out visual models, students can discover the answer $3\frac{1}{2}$ by first organizing the calculation visually.

Concept maps were constructed from teachers' verbalizations. Analyzing the concept map included counting: 1) the number of concepts; 2) the number of links (i.e., relationships) between concepts; 3) number of process types of fraction relationships; and 4) number of object types of fraction versus the object features of the topic. Both groups of teachers were very knowledgeable about fraction concepts and the connections between concepts. They emphasized the importance of students understanding representations of fractions. The only difference between the two groups was that IT teachers were more likely to stress the object components of fractions more than DK teachers. This may also be a sign of the development of the knowledge base for the teachers. That is, as teachers develop their SMK and PCK for the topic, they will be less likely to rely on the surface features of the concept of fraction.

Proposed SMK and PCK Features for Dividing with Fractions

Subject matter and pedagogical knowledge features for dividing with fractions are studied together. The teachers' responses revealed what they knew about the topic (i.e., SMK) and how they would teach what they knew (i.e., PCK). These two features of content knowledge are not separated since the prompts in the interview tasks were contextualized in possible teaching situations. From the division and concept map tasks, a model of deep knowledge understanding for dividing with fractions is proposed as including the following subject matter and pedagogical content knowledge features: 1) multiple representations; 2) varied representations; 3) multiple translations; 4) emphasis on context; 5) emphasis on reference unit; and 6) fractions as sets of connected representations. The six DK teachers in the study sample exhibited these features and the 8 IT teachers, from their transcripts, are moving towards this knowledge base.

Teacher Reasons for not Creating Representations

In transition teachers were expected to attribute failure for creating models in tasks 1 and 2 to their mathematical knowledge. This expectation was not supported. One IT teacher did mention her awareness of models existing that she was beginning to understand, but the majority stressed that stories for such problems did not have real applications or were not in the present curriculum. Interestingly, in task 3, they were given a problem in which the division was in story form and all teachers were able to translate the solution correctly. When asked to construct a story for the number sentence in tasks 1 and 2, the IT

teachers who were not able to do so, were more likely to state the “un realness” of the situation. This outcome was surprising. Since the IT teachers had extensive involvement with professional development situations, it was expected that they would connect their developing state of content knowledge as the reason why they were not able to create stories to represent the number sentence situation.

Connections to Conceptual Meaning of Division

An example of the levels of knowledge DK teachers were building on is the case of T5. In his response to how he would solve task 1, he connects the conceptual idea of division to the calculation. His response to the “how would you solve part” of problem 1 is:

Well it depends, really. If it's there like this and not given a context, I'm not given a chance to think about it in any kind of way.

You know so, there's always a few ways, I can invert and multiply or

I can think of how many one-halves are in 1 and three quarters. So I can think about it like that.

Or, I can think of what is $\frac{1}{2}$ of 3 and $\frac{1}{2}$. (T5)

He stresses the importance of context as the main issue in creating a calculation and then offers algorithm, quotitive, and product and factors approaches. He blends representation and calculation. This shows a well-connected understanding of the meaning of division to the problem situation, regardless of the types of numbers involved. Each teacher group was able to use

the algorithm correctly, but both groups emphasized the importance of understanding the concept of division in the problem above procedural accuracy in “doing” the algorithm correctly.

Study Strengths

The strengths of the study include: 1) sampling focus; 2) microanalysis of topic; 3) use of interview protocol; and 4) “think aloud” analysis.

The purposive sampling technique was important for locating specific groups of teachers, those whose thinking aligns with Ma’s (1999) experts and those who are changing their thinking and action similarly to the scores of elementary and middle school teachers involved in curriculum and teacher change efforts at institutes, workshops, and within their schools (Ball & Bass, 2000; NCTM, 2000)

Another strength of the study is its microanalysis of one idea, division, with one type of number, fraction divided by fraction. Within fractions, division is implied within each rational number given its form, and when fractions are being divided by fractions, the model of division is amplified. The specific examination of the topic provides rich samples of what teacher understanding for the subject may be.

Research on knowledge of fractions and teaching fractions has often been examined separately. Researchers have examined student methods for understanding fractions (Mack, 1993; 2001). Others have described the mathematical structural components of fractions (Behr et al, 1992) and described

pedagogical issues (Lampert, 1990; Schifter, 1997) in teaching fractions. The study protocol brings together these different threads of research and examines the two components, mathematical structure and mathematics pedagogy for dividing with fractions, at the same time. The interview questions used in the study suggests the importance of using assessments that allow teachers to solve open ended problems in a variety of ways within possible teaching situations (Shulman, 1989).

Many studies have used a think aloud procedure, “a powerful tool for understanding the nature of thought” (Wineburg, 1990, p. 75). Think alouds are opportunities to capture the process of thought and give researchers a view into the complex parts of thinking about a topic. Using a similar method was a strength of this study, because of the rich amount of data available from talk aloud verbalizations (Ericsson & Simon, 1993).

Given that “rational numbers provide the first number system in which all the operations of arithmetic are possible” (National Research Council, 2001, p. 415), investigating how middle school teachers understand them is important. Researchers agree successful problem solving with rational numbers involves understanding thoroughly their different representations, translations, and connections. Teachers must have a deep and developed understanding of this type of number. Professional development opportunities, formal and informal, are places where teachers can re-learn and re-discover the various identities of rational numbers. Through workshops, conversations about practice, and use of

reform curricula, teachers develop a “repertoire of pedagogical techniques and the ability to use those techniques to accomplish lesson goals” (National Research Council, 2001, p.429)

Study Limitations

There were several limitations to this study. The first, the sample size of 14, is too small to consider the possible meaning of these “findings” beyond the scope of the teachers interviewed. Even though the sample in this study is small and chosen purposefully, there is no assumption that the sample or “results” are representative of all American or New York middle school math teachers. That was never the purpose of this study. It was to describe and understand, for these teachers, their knowledge about a focused, circumscribed topic, which math researchers and curriculum developers have all stated is one of the most difficult topics in mathematics to learn conceptually. In retrospect, another major limitation was the faulty construction of task 4. In this division task, a word problem was created to represent the situation, $\frac{5}{6}$ divided by $6\frac{2}{3}$ s, as a parallel to task 2 in relation to task 1.

The word problem in Task 4, as written, confused teachers. Word problems where the divisor was larger than the dividend are not frequently used in curricula. Because the problem was difficult to begin with, none of the teachers connected the numbers to task 3 and said that the answer must be the inverse of the task 3. In reviewing other research, I later found an article (Ott, Snook, & Gibson, 1991) discussing partitive division. If the problem had been written as a

partitive question it might have been more appropriate for the revised Task 4 to be: If $\frac{5}{6}$ of a yard of material has to be broken up to $6\frac{2}{3}$ sets, how much material would go in each set? This structure of the problem might have assisted in teachers' solutions.

The coding of the transcripts had only one rater and this may have been a potential area of bias. For future research assessing the knowledge categories, multiple raters may be used.

Another limitation is that the study focused on teachers' knowledge and did not examine teacher beliefs regarding their students or the content. The questions in the interview prompts were focused on how teachers thought about division with fractions and how they would teach the topic rather than asking them to state conceptions about their teaching or how students learn.

Future Directions

The study's focus was to expand and confirm the research on analyzing mathematics teacher knowledge for dividing with fractions. Future directions for related research may include using contextualized problem solving tasks to evaluate mathematics teachers' knowledge for other important topics in the middle school and high school curriculum, to continue to build further content knowledge models for those topics. More micro analytical studies, particularly with teachers transitioning in their knowledge bases may inform these future knowledge models. Specific problem solving tasks assessing the details for the types of models, translations, and connections offered in the teacher category

table (see table 4) may also be useful. Systematic investigations on the influence of problem situations (i.e., numbers sentence versus word problem forms) on representations, translations, and connections would also enhance content knowledge models.

Incorporating think aloud training for participating teachers before the interview would be helpful to help teachers make their thinking transparent.

Specific training in think aloud application may also give teachers the opportunity to construct their own concept maps and those results would be interesting to compare to the researcher developed maps from interview transcripts.

Developing a longitudinal design to follow the IT teachers in the study would also benefit our understanding of how to improve and expand the types of professional development experiences teachers join.

Classroom observations of the teachers interviewed would add another layer of understanding of thought in action for dividing with fractions. Content knowledge models can then inform further collaborations in studying how teachers think about what they teach and how they teach what they know.

Conclusions

The goal of the study, in its inception, had been to examine how cognitive representations (i.e., models, translations, and connections) of understanding subject matter are manifested in teachers' explanations in division with fraction

problem solving situations for two important groups of teachers, experts and those in transition to expertise via professional development opportunities.

In a recent interview for the Eisenhower National Clearinghouse for Mathematics and Science Education (Herrera, 2002), Liping Ma, a senior scholar at the Carnegie Endowment for the Advancement of Teaching, discussed her concern about the current state of American teachers' substantive knowledge on the difficult to teach mathematic topics their grade work covered. "Teachers themselves should ask "Why?" all the time. When they teach multiplication to their students, they can think about why it works. Then they will learn elementary math in a deeper way through their own teaching of it" (p. 5). Teachers in this study, DK and IT, ask themselves these questions as evident through their use and constructions of multiple models, translations, and connections.

In Ma's 1999 book, *Knowing and Teaching Elementary Mathematics*, a sentence in the conclusion was noted: "It does not seem an accident that not one of a group of above average US teachers displayed a profound understanding of elementary mathematics. While we want to work on improving students' mathematics education, we also need to improve their teachers' knowledge of school mathematics" (p.144).

This study does not disagree with Ma's claim of the need to improve teacher's knowledge of the content they teach. But it does suggest that within our schools, there are instances of teachers whose knowledge and pedagogy are

responses to the post interview questions (Appendix I). They were positively affected by the implementations of reform-based curricula and workshops in their schools, out-of-school institutes, and workshops they attended to build their knowledge. One IT teacher's involvement in projects echoes many of the other teachers' experiences. In discussing a set of institutes and workshops she had been a part of for the past three years she stated:

It was the first time I had been introduced to a way of learning math that was very different from my own experiences in school. The program was aiming for depth in understanding and I enjoyed "re learning" middle school math as I prepared my lessons. I saw the need to teach for understanding when I saw the kinds of errors my students made. I learned to grapple with problems in workshops and spent time in groups analyzing and extending them. We were engaged in exploring and making connections. In these seminars, I experienced the learning of math in fresh and exciting ways and realized that this is how I want my students to learn math (IT14)

This teacher recognized the value and importance of rediscovering the complexities of understanding middle school mathematics and through her own grappling with subject matter in an effort to understand the content, she articulated the ways in which she wants her students to learn, thorough connections, extensions, and deep understanding.

DK teachers also were influenced by their participation in reform based curriculum workshops. DK teacher 3 focused on bringing inquiry based learning into his classroom from the ideas shared and problems tackled in the summer institute he attended:

At the workshops, we would spend time working on really rich problems and then talk about the different math topics contained within that topic. So it helped shift from “what is the problem” to what is the math topic and one problem could be used 25 different ways and your job was to find the ideas you wanted out of it and focus your class. Another important experience was observing over time a teacher who has taught CMP math for 6 years and it was a wow experience, this is what it’s supposed to look like, having kids know what the big math ideas were and keeping track of them. (DK, T3)

The teachers interviewed have and are developing deep understandings, making meaning through a mathematical framework. In their PD experiences all of the teachers were able to re learn mathematics by modeling situations and constructing solutions. Through these opportunities their visions of mathematics pedagogy shifts. The DK teachers I interviewed were once in transition. They have had more experiences and are committed to push their understanding, content knowledge, and teaching practice and have been supported by their schools and colleagues in the process.

The transformation in teaching from in transition to deep knowledge requires the ongoing development of a knowledge base of developed subject matter and pedagogical content features of the topics, fractions and division with fractions, as evidenced by: 1) multiple representations; 2) varied representations; 3) multiple translations; 4) emphasis on context; 5) emphasis on reference unit; and 6) recognizing fractions as sets of representations. These features complemented and extended what is currently discussed as features of accomplished thinking in the domain (Ball & Bass, 2000; Leinhardt, 2001). The knowledge base for this difficult topic, division with fractions, requires the teacher to be well versed in translating, representing, and connecting models, context, and related processes for dividing with fractions. The development of this knowledge base most likely requires ongoing, consistent, and systematic opportunities for teachers to be participants of formal and informal professional development experiences, conversations about practice, and instruction and training with mathematics reform curricula.

The teachers, in their think alouds, showed their mathematical thinking as layered sets of cognitions directly related to their subject matter knowledge and pedagogical content knowledge for the topic. The knowledge features were not sets of facts or procedures but woven relationships between subject matter and pedagogical features. Teachers' knowledge of mathematics may be changing, as teachers' understandings of the mathematics they teach improves. Teachers are the primary communicators of content and, as such, teacher knowledge

developments hold promise for how students can then think, reason, and understand the mathematics of the middle school curriculum.

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Appendix A

Cover Letter to Participating Teachers

Date : _____

Dear participating teacher,

Thank you very much for agreeing to participate in this study. I am interested in examining teacher thinking on an important topic in arithmetic, fractions. Your participation in the interview will help me to understand how teachers think about this topic.

Prior to our interview, please read and sign the enclosed consent form. In addition to the consent form, please answer the pre interview questions. The first set of questions focus on your teaching experience in mathematics (i.e., math courses you have taught, grade levels, and content of those courses). The second part of the pre interview questions asks you about your experiences in professional development programs aimed at implementing reform-based curricula in classrooms. Please feel free to write as much as you would like. You may include your response on the back of the pre interview questions form.

Your responses to the pre interview questions and the interview questions will be kept confidential. I will never use your name in any written or oral reports of the study. I will assign a numeric code to each participating teacher as a way of protecting identities. Once again, I greatly appreciate your involvement in this project. Your ideas will help me to understand teacher knowledge.

Sincerely,

Yasemin Jones

718.260.5680

Appendix B

Pre Interview Questions

Please answer the following questions and read and sign the consent form prior to our interview.

number _____

Code

Name _____

Date _____

Years teaching math _____

Highest degree obtained and specialty _____

Undergraduate degree major _____

Indicate the number of mathematics courses currently teaching, the corresponding grade level, and the name of the classes

Math course name	Grade level	Course content
------------------	-------------	----------------

Please identify other types of math courses you have taught

Math course name	Grade level	Course content
------------------	-------------	----------------

Have you ever or are you currently participating in a sustained set of experiences to deepen your knowledge of mathematics and teaching practice? For example, you may have participated in professional development projects such as training to implement CMP , MiC, Everyday Math, or other curriculum based programs in your classrooms. **If yes, list the projects (include length of involvement) and please turn this page over and briefly describe your experience in these projects.**

Appendix C

Standardized Protocol for Interviews

All interviews begin with these instructions.

*Thank you very much for participating in today's interview. I am trying to learn about how you think about teaching fractions and division with fractions. I will present you with five different tasks. Each task has a series of questions that I would like for you to answer. I will ask you the questions orally. For each task, I'm just as interested in how you think about things that puzzle you as I am in learning exactly what you'd do. For each one, I'd like to know what you think you would do or say and **why** that's what you'd do.*

Each teacher was given a page with the task problem typed on the page. Each problem was presented one at a time.

Each task had its own specific standardized set of questions to be asked orally.

After the problem solving tasks, each teacher was asked the same post interview questions.

At the end of the interview, the teacher was thanked and asked if he/she would like an abstract of the completed study.

Appendix D

Task 1: From Number Sentence to Representation**Interviewer version**

Division by fractions is often confusing for students. People have different approaches to solving problems involving division with fractions. **a)** How would you solve a problem like this one?

$$1\frac{3}{4} \div \frac{1}{2}$$

b) Imagine that you are teaching division with fractions. To make this meaningful for students, many teachers try to relate mathematics to other things. Sometimes they try to come up with real world situations or story problems to show the application of some particular piece of content.

What would you say would be a good story or model for :

$$1\frac{3}{4} \div \frac{1}{2}$$

c1) If teacher gives a story, ask: Why do you think this is a good example ?

c2) If teacher cannot come up with a story, ask, why do you think it is difficult to come up with a story for the problem situation?

Appendix D

Task 1

Task 1: From Number Sentence to Representation

Interviewee version

a) How would you solve a problem like this one?

$$1\frac{3}{4} \div \frac{1}{2}$$

b) What would you say would be a good story or model for :

$$1\frac{3}{4} \div \frac{1}{2}$$

Appendix E

Task 2: From Number Sentence to Representation with Reversal**Interviewer version**

a) How would you solve a problem like this one?

$$\frac{1}{2} \div 1\frac{3}{4}$$

b) Imagine that you are teaching division with fractions. To make this meaningful for students, many teachers try to relate mathematics to other things. Sometimes they try to come up with real world situations or story problems to show the application of some particular piece of content. What would you say would be a good story or model for :

$$\frac{1}{2} \div 1\frac{3}{4}$$

c1) If teacher gives a story, ask: What kind of student(s) did you have in mind when you came up with this example? Why do you think this was a good example for them?

c2) If teacher cannot come up with a story, ask, why do you think it is difficult to come up with a story for the problem situation?

Appendix E

Task 2: From Number Sentence to Representation with Reversal**Interviewee version**

a) How would you solve a problem like this one?

$$\frac{1}{2} \div 1\frac{3}{4}$$

b) What would you say would be a good story or model for :

$$\frac{1}{2} \div 1\frac{3}{4}$$

Appendix F

Task 3: From Representation to Translation**Interviewer version**

Students must often solve word problems involving division with fractions. People have different approaches to solving such word problems. a) How would you solve this one :

Wally has $6\frac{2}{3}$ yards of material. It takes $\frac{5}{6}$ of a yard to make his scarves. How many scarves can he make?

- a) Would a word problem like this help students learn about division with fractions?
- b) This is only one way/ one method. What other ways can you think of to solve this problem?

Appendix F

Task 3: From Representation to Translation**Interviewee version**

How would you solve this one:

Wally has $6\frac{2}{3}$ yards of material. It takes $\frac{5}{6}$ of a yard to make his scarves. How many scarves can he make?

Appendix G

Task 4: From Representation to Translation with Reversal**Interviewer Version**

a) How would you solve this one:

Wally realizes that $\frac{5}{6}$ of a yard of material is left over. He has promised to give 6 people equal shares and one person $\frac{2}{3}$ of a share of the material. How much material would each person receive?

b) Would a word problem like this help students learn about division with fractions?

c) This is only one way/ one method. What other ways are there to solve this problem?

Appendix G

Task 4: From Representation to Translation with Reversal**Interviewee Version**

How would you solve this one:

Wally realizes that $\frac{5}{6}$ of a yard of material is left over. He has promised to give 6 people equal shares and one person $\frac{2}{3}$ of a share of the material. How much material would each person receive?

Appendix H

Task 5: Concept map**Interviewer version**

I am also interested in how you would “map out” your knowledge of fractions. I would like for you to talk aloud a map of this knowledge. It can be a web of words or a flow chart of words that captures the different parts of this topic that you believe are important for a student to know about fractions.

You may want to list all the ideas that you think are important for a student to know in order to understand fractions. Talk aloud how those words related to each other as ideas and the links between the ideas.

What do you think are the connections among fractions and other related mathematical topics? Specifically, division of fractions?

How do you think about fractions in relation to: multiplication, decimals, ratios, whole numbers, and proportions

What do you think are the connections among fractions and other related mathematical topics? Specifically, division of fractions?

How do you think about fractions in relation to: multiplication, decimals, ratios, whole numbers, and proportions

Appendix H

Task 5: Concept map

Interviewee version

Please talk aloud your “concept map” of your knowledge of the topic fraction. A “concept map” can look like a road map or like a web of connections. Include in your map the connected ideas of the different parts of the fraction topic that you believe are important for students to know.

Appendix I

Post Interview Questions

Thank you very much for all your help. These last questions will help me to understand about your continuing development as a mathematics teacher.

- 1) Please tell me about a specific career experience that you see as critical in your dedication and motivation to keep developing your
 - a. mathematical knowledge
 - b. and teaching practice?

- 2) With whom do you discuss a) your teaching practice and b) mathematical topics

If a teacher gives a name, X, ask, "Is X in your school?" If not how do you know X?

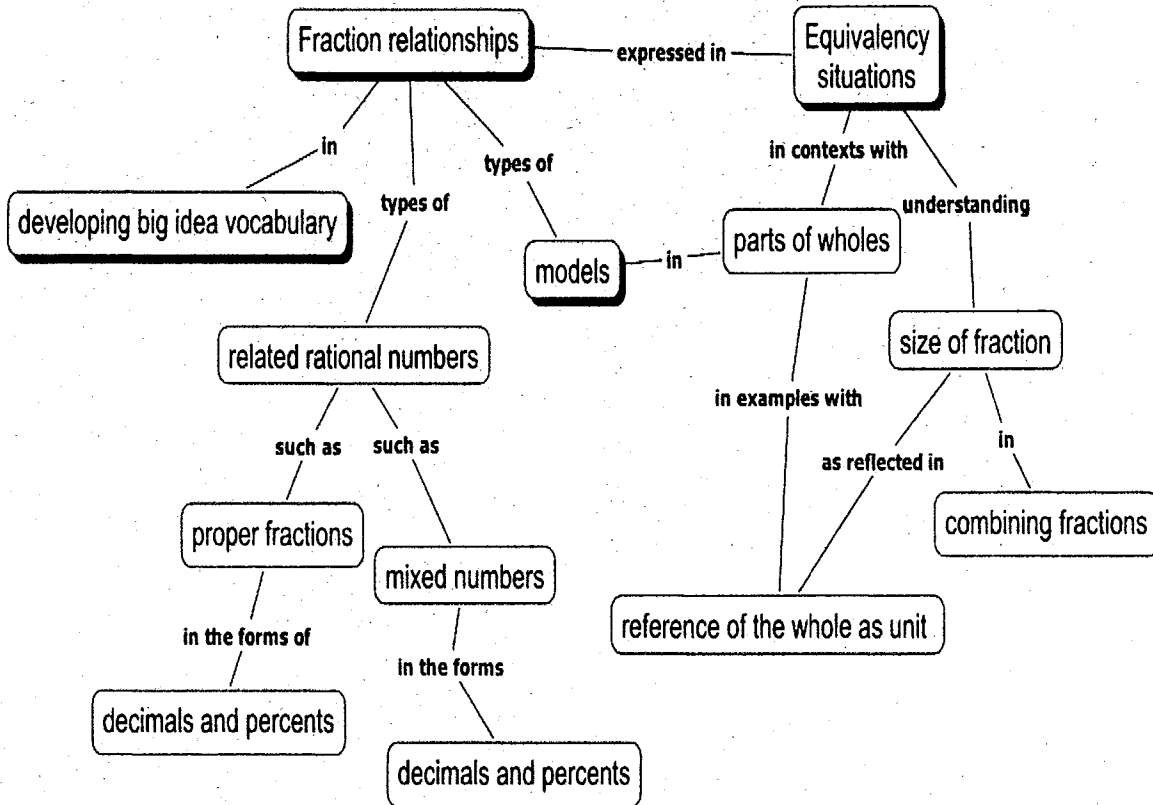
- a) In talking to X about your teaching practice, can you give me an example of a recent conversation?
- b) In talking with X about mathematical topics, can you give me an example of a recent conversation?

Thank you very much for this opportunity to interview you today.

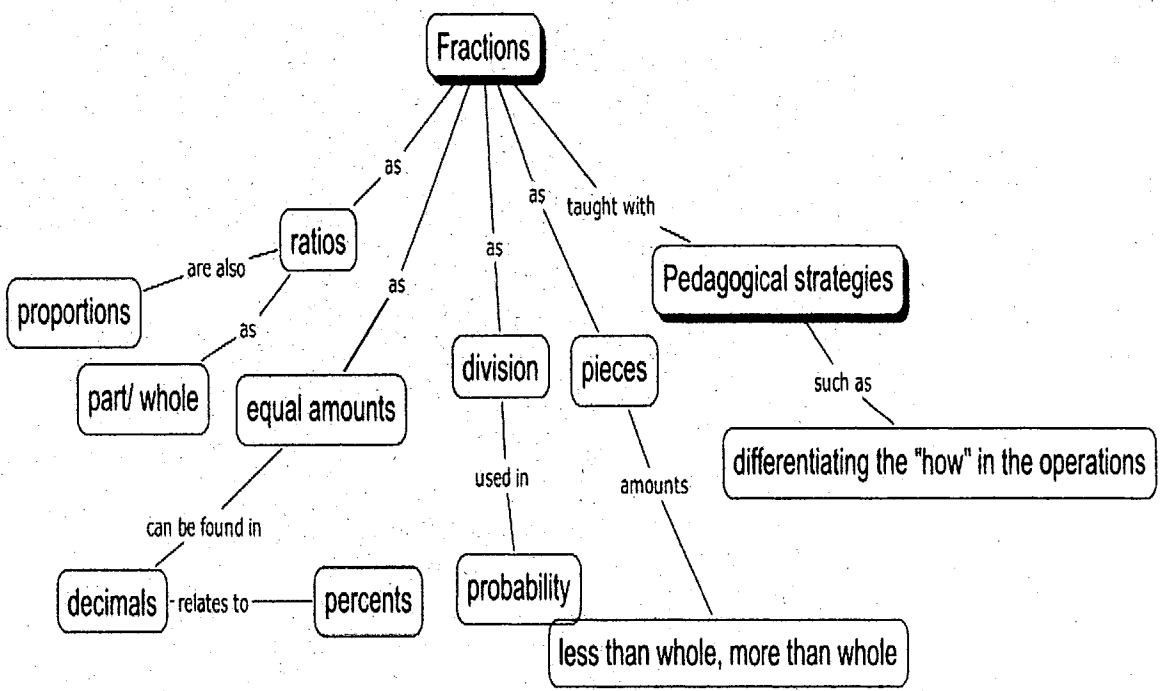
Appendix J

Concept maps from Deep Knowledge and In Transition Teachers

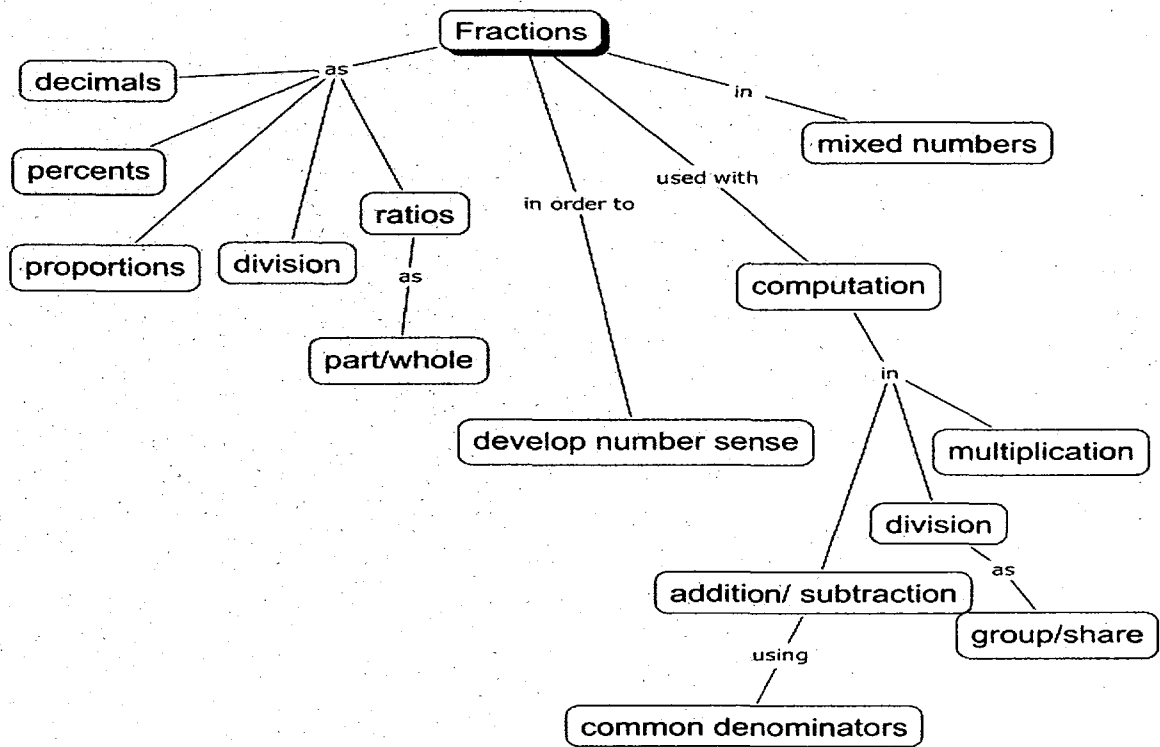
T1: DK



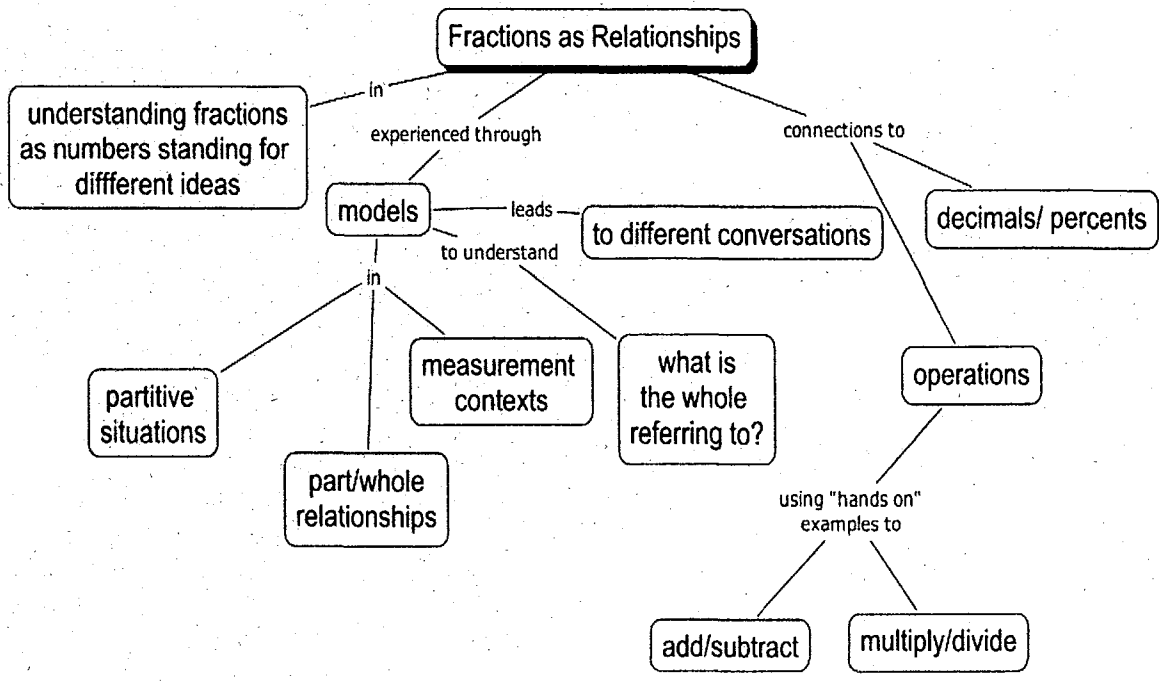
T2: IT Teacher



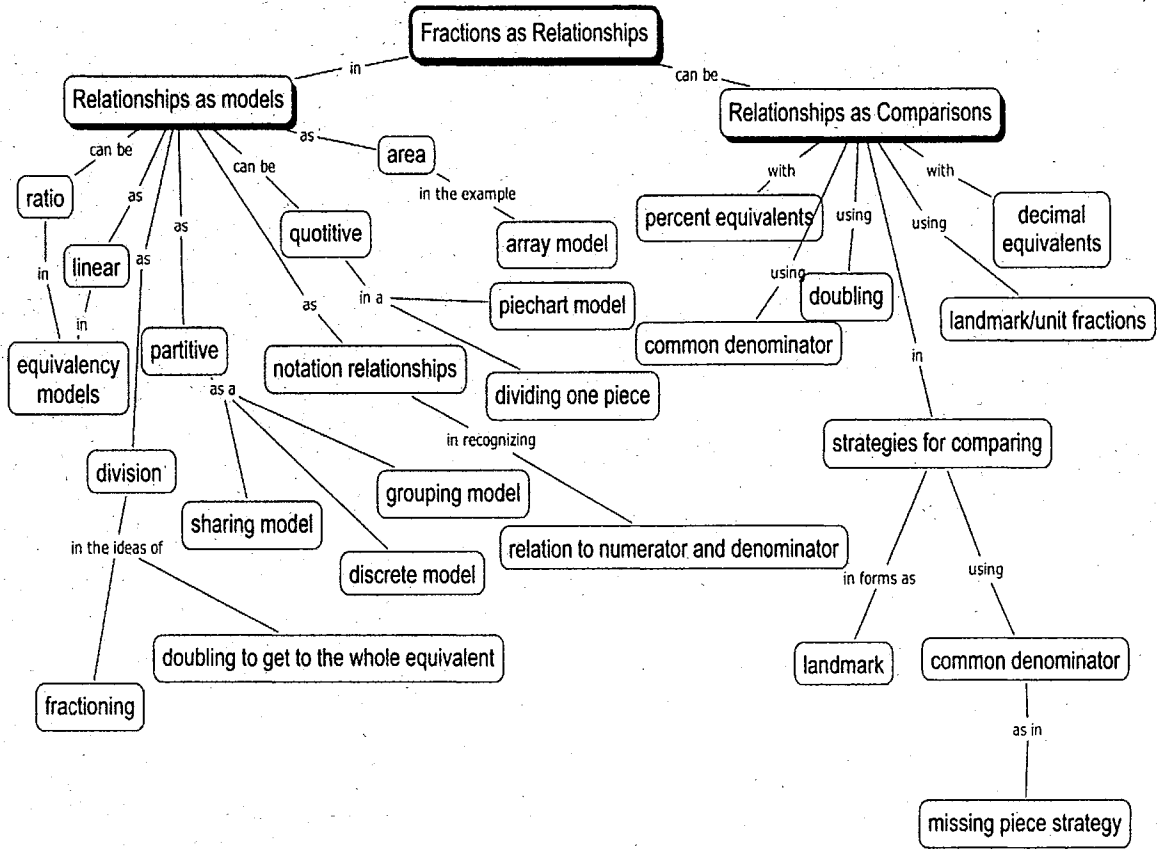
T3: IT teacher



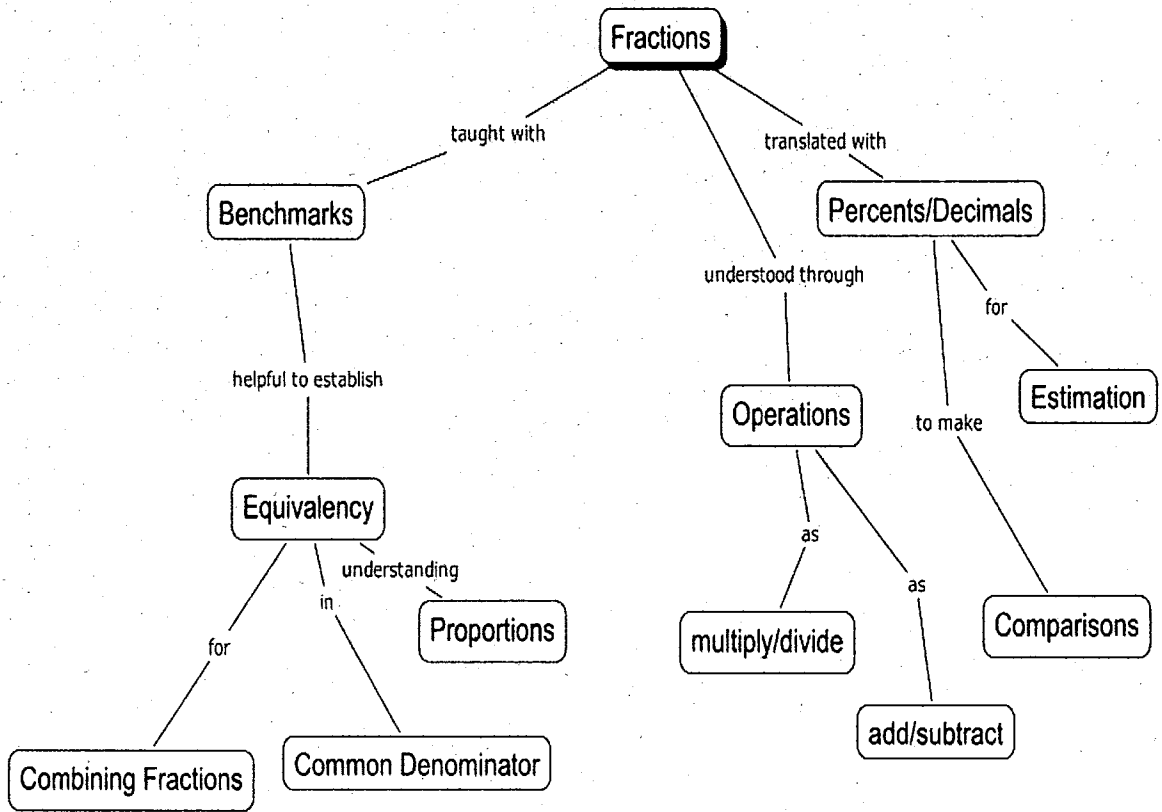
T4: IT Teacher



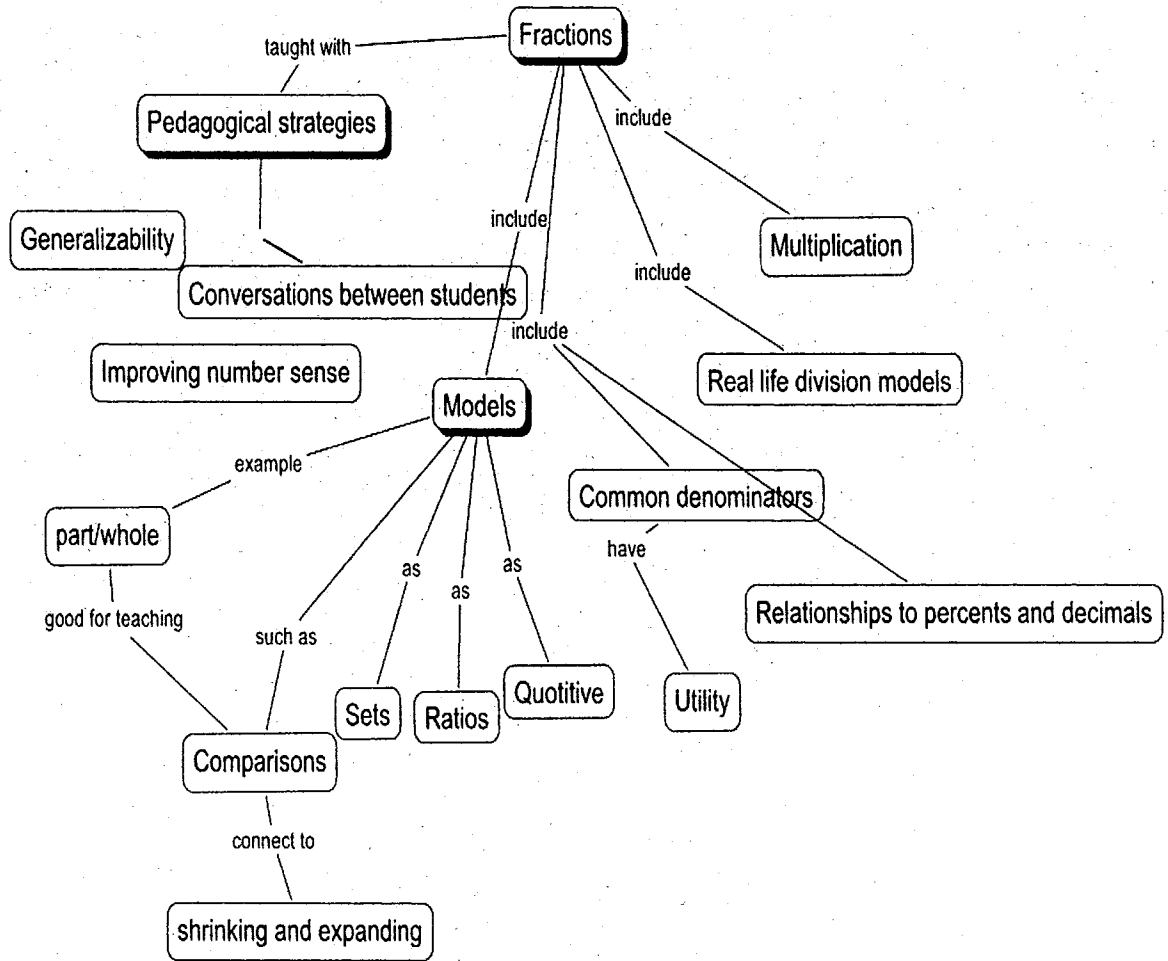
T5: DK Teacher



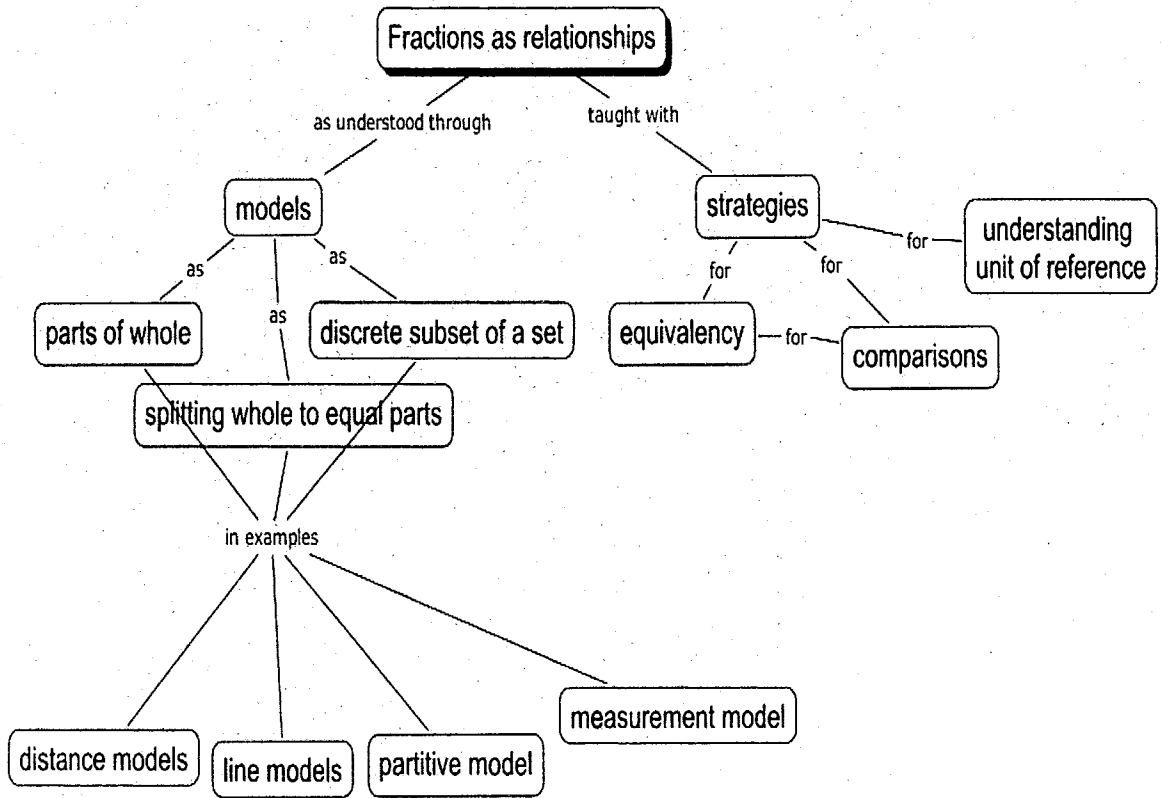
T6: IT Teacher



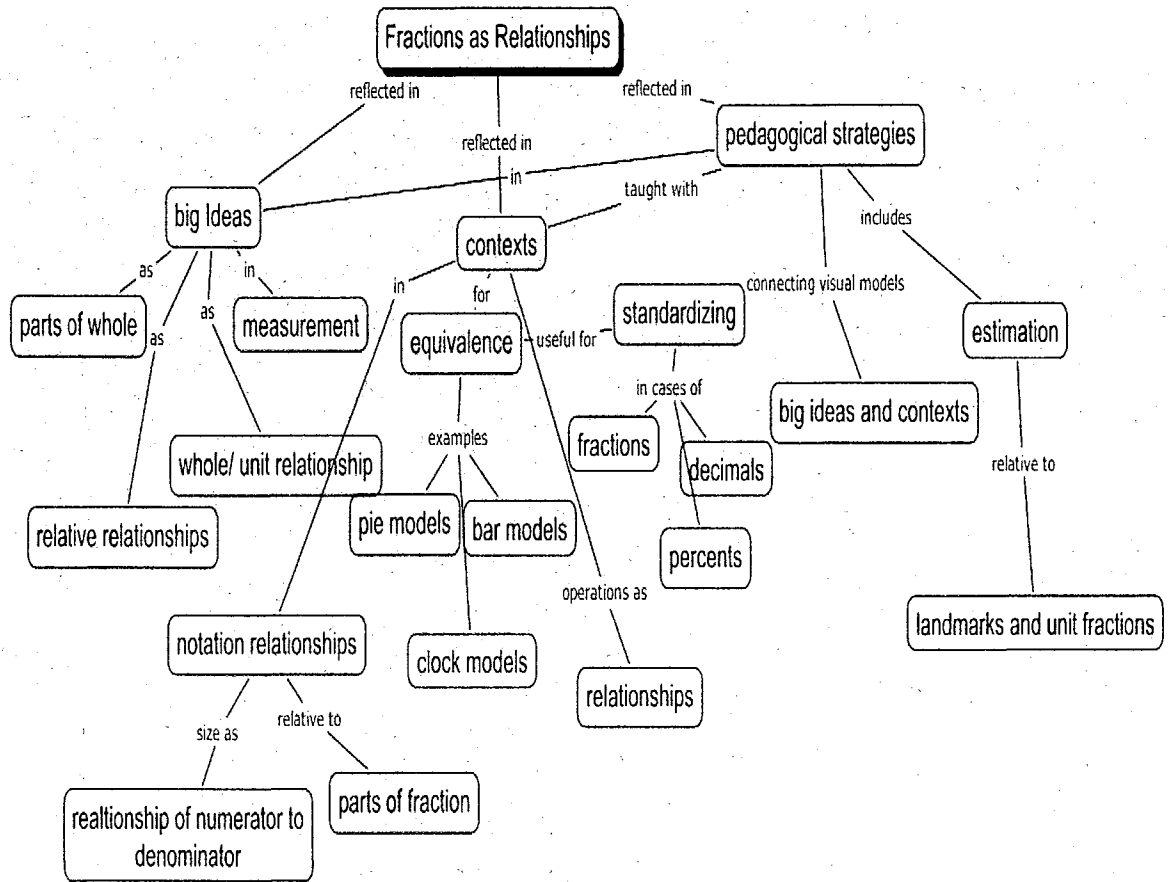
T7: IT Teacher



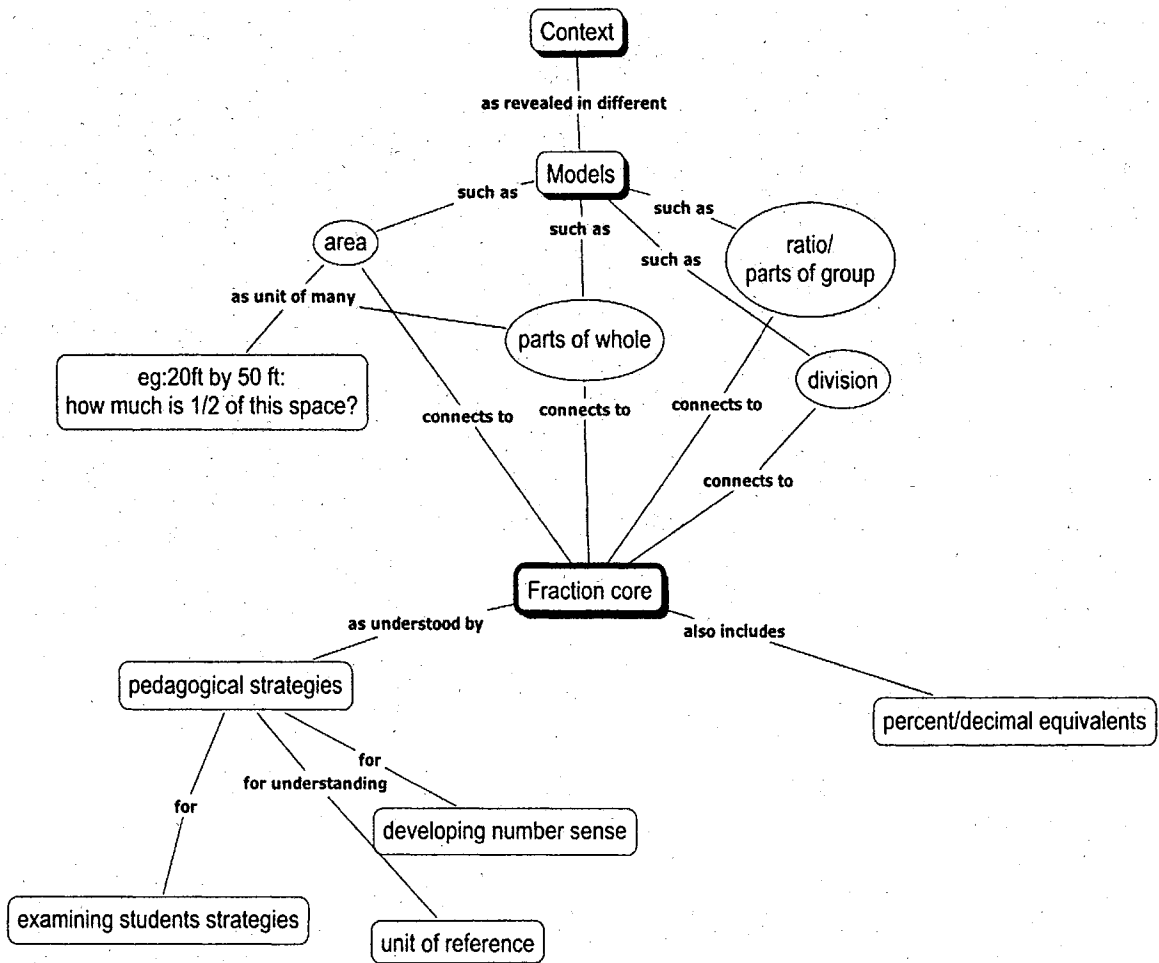
T8: DK teacher



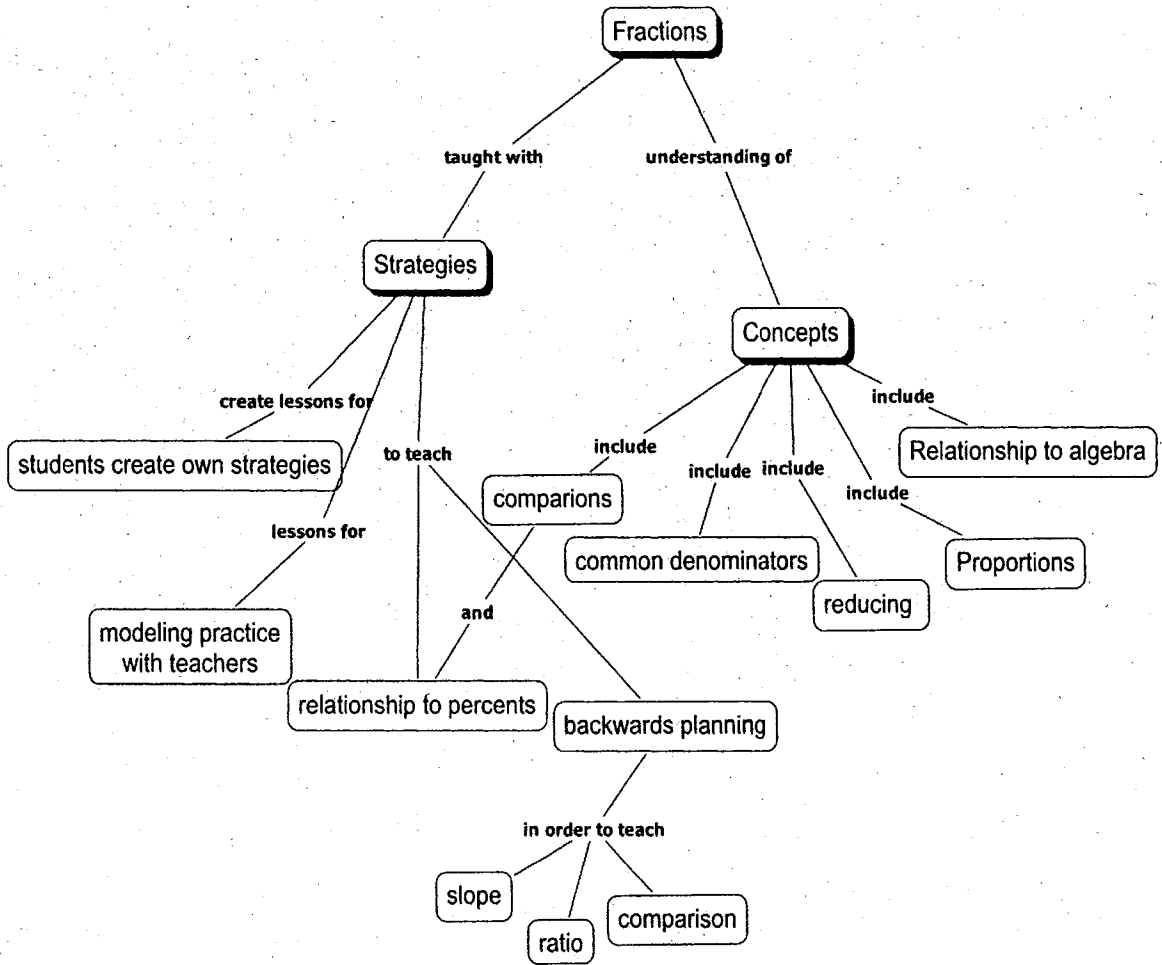
T9: IT Teacher



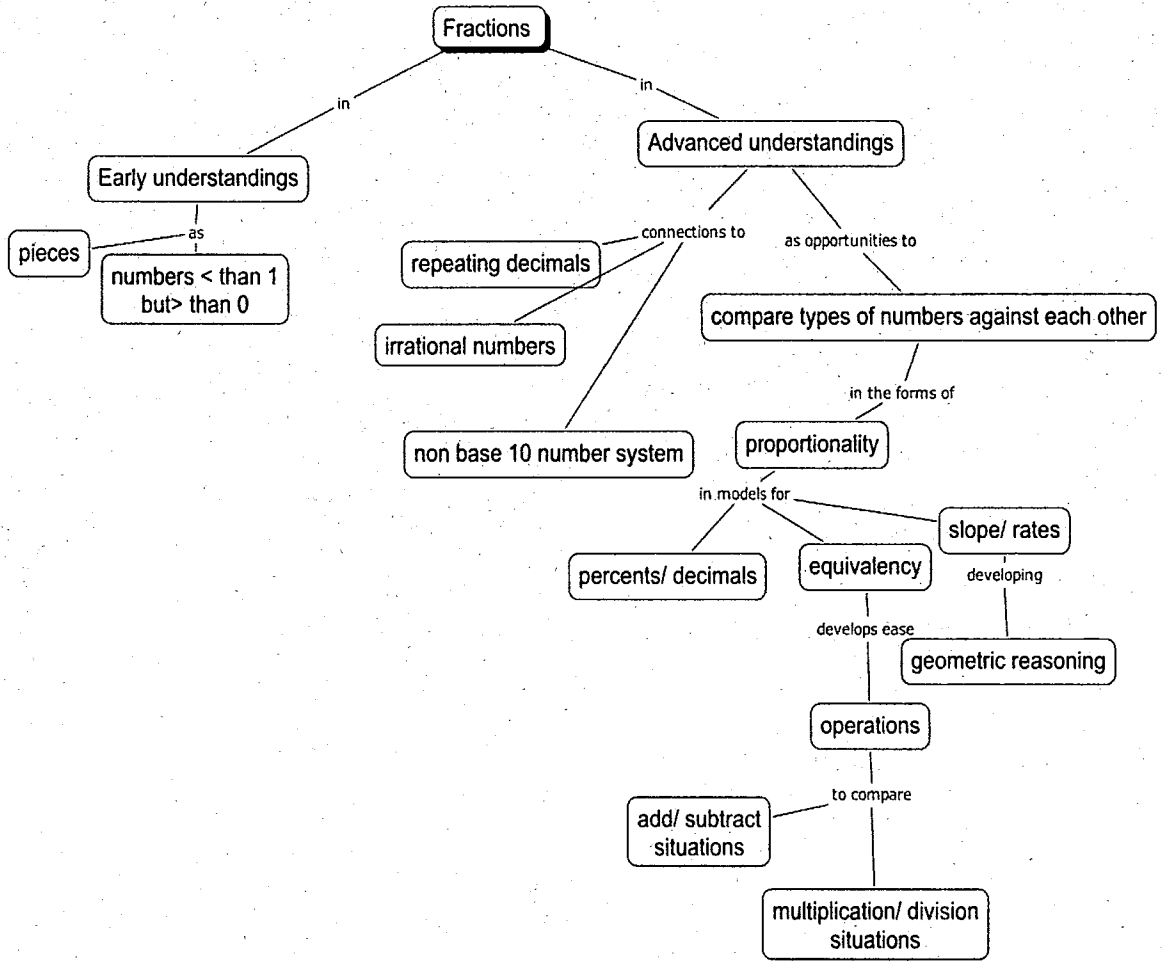
T10: DK teacher



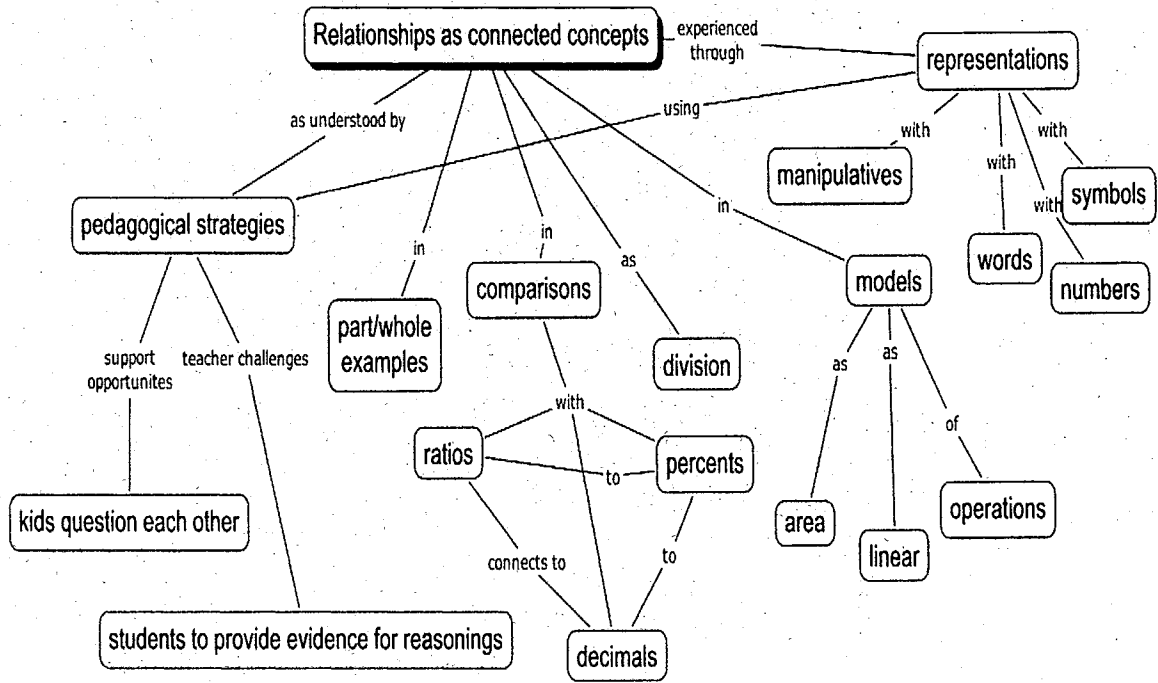
T11: IT Teacher



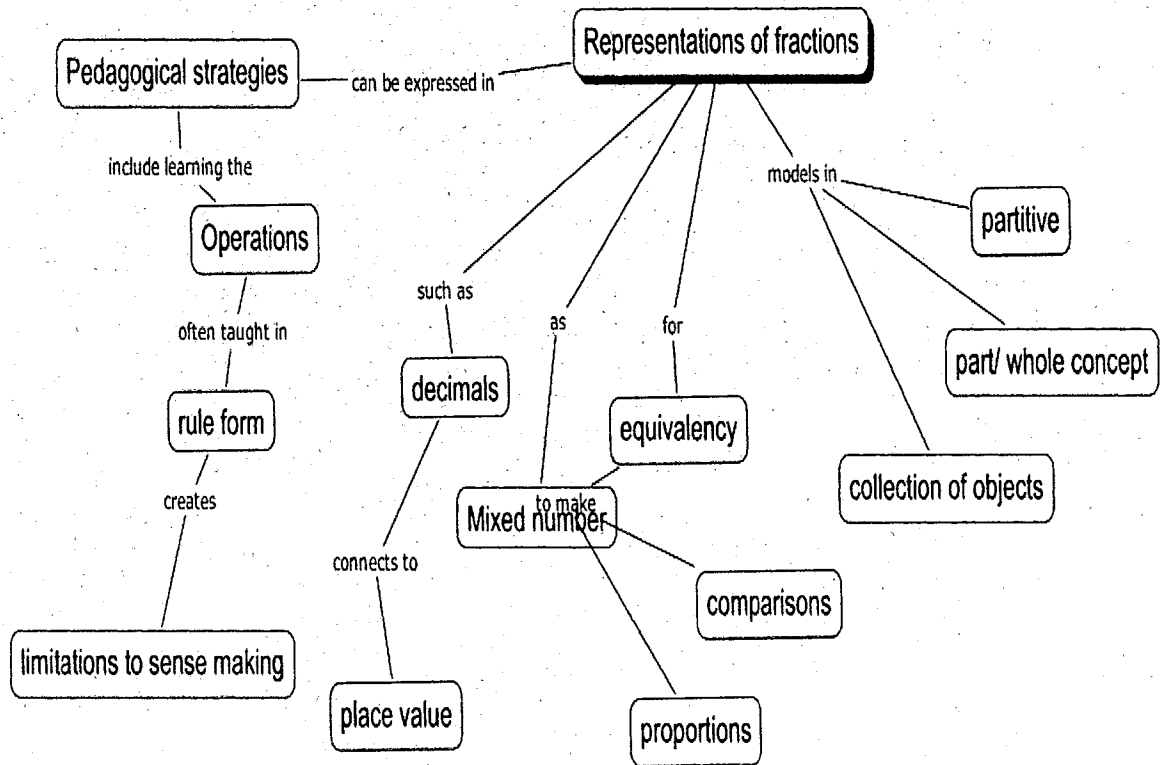
T12: DK Teacher



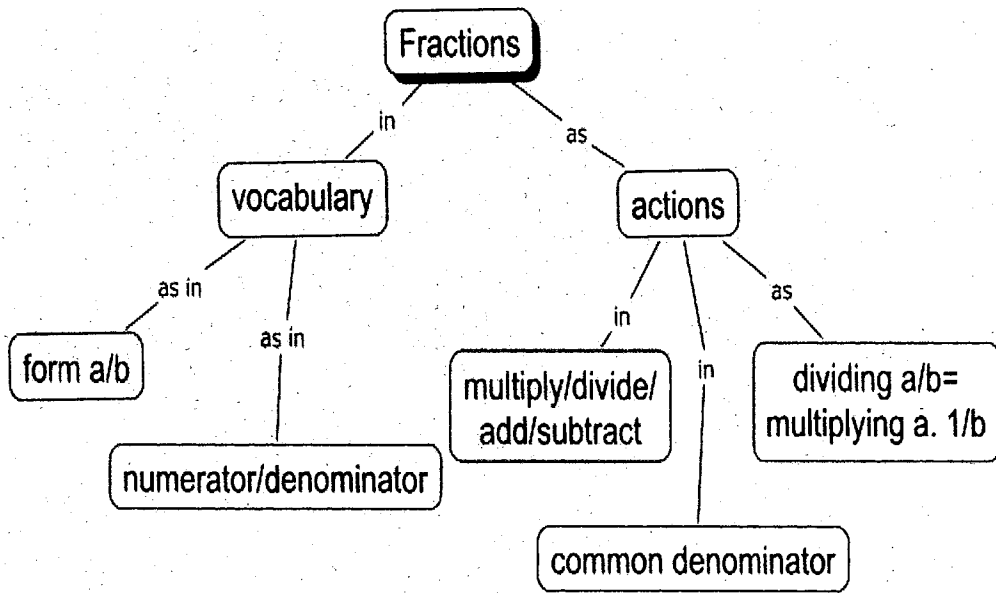
T13: DK Teacher



T14: IT Teacher



T15: Novice teacher- Concept map from pilot study



References

- Armstrong, B.E., & Bezuk, N.S. (1995). Multiplication and division of fractions: The search for meaning. In J.T. Sowder & B.P. Schapelle (Eds.), *Providing a foundation for teaching mathematics in the middle grades* (p. 85-119). Albany, NY: SUNY Press.
- Ball, D. L. (1988). *Knowledge and reasoning in mathematical pedagogy: Examining what prospective teachers bring to teacher education*. Unpublished doctoral dissertation, East Lansing, Michigan State University.
- Ball, D. L. (1991). Research on teaching mathematics: Making subject matter knowledge part of the equation. In J.E. Brophy (Ed.), *Advances in research on teaching: Teachers' subject matter knowledge and classroom instruction* (Vol.2, pp.1-48). Greenwich, CT: JAI Press, Inc.
- Ball, D. L. (1993). Halves, pieces, and twos: Constructing and using representational contexts in the teaching of fractions. In T. Carpenter, E. Fennema, & T. Romberg (Eds.), *Rational numbers: An integration of research* (pp.157-195). Hillsdale, NJ: Lawrence Erlbaum.
- Ball, D., & Bass, H. (2000). Making believe: The collective construction of public mathematical knowledge in the elementary classroom. In M. Early (Ed.), *Constructivism in education: Opinions and second opinions on controversial issues, Part I, Ninety Ninth Yearbook of the National Society for the Study of Education* (pp. 193- 224). Chicago: University of Chicago Press.
- Ball, D., & Lampert, M. (1999). Multiples of evidence, time, and perspective: Revising the study of teaching and learning. In E.C. Lagemann & L.S. Shulman (Eds.), *Issues in Educational Research* (pp. 371- 398). San Francisco: Jossey-Bass.
- Ball, D. & Mosenthal, J. (1990). *The construction of new forms of teaching: subject matter knowledge in in-service education*. (Report No. 90-8) East Lansing, MI: National Center for Research on Teacher Education. (ERIC Document Reproduction Services No. ED 323208).
- Behr, M., Havel, G., Lesh, R., & Post, T. (1992). Rational number, ratio, and proportion. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp.296-333) . New York: Macmillan.

- Behr, M.J., Havel, G., Post, T., & Lesh, R. (1993). Rational numbers: Toward a semantic analysis –Emphasis on the operator construct. In T.P. Carpenter, E. Fennema, & T.A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 13-47). Hillsdale, NJ: Lawrence Erlbaum.
- Bransford, J., Brown, A L., Cocking, R. R. (Eds.), (1999). *How people learn: Brain, mind, and experience*. Washington, DC: National Academy Press.
- Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal*, 97(1), 3-20.
- Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C.P., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal*, 26, 499-531.
- Chi, M.T.H., & Ceci, S. J. (1987). Content knowledge: Its role, representation, and reconstructing in memory development. *Advances in child development and behavior*, 20, 91-142.
- DeCorte, E., Greer, B., & Verschaffel, L. (1996). Mathematics teaching and learning. In D. C. Berliner & R. C. Calfee (Eds.), *Handbook of educational psychology* (pp. 491-550). New York: Macmillan.
- English, L. D. & Halford, G. S. (1999). *Mathematics Education: Models and Processes*. Mahwah, NJ: Lawrence Erlbaum.
- Ericsson, K.A. & Simon, H.A. (1993). *Protocol analysis: Verbal reports as data*. Cambridge, MA: MIT press.
- Even, R., & Tirosh, D. (1995). Subject matter knowledge and knowledge about students as sources of teacher presentations of the subject matter. *Educational Studies in Mathematics*, 29, 1-20.
- Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-164). New York: Macmillan.
- Fosnot, C. T. & Dolk, M. (2002). *Young mathematicians at work: Constructing fractions, decimals, and percents*. Heinemann: Portsmouth, New Hampshire.
- Franke, M., Fennema, E., & Carpenter, T. P. (1997). Changing teachers: Interactions between beliefs and classroom practice. In E. Fennema & B.

S. Nelson (Eds.), *Mathematics teachers in transition* (pp. 155-191). Mahwah, NJ: Lawrence Erlbaum.

- Franke, M. L., Carpenter, T. P., Levi, L., & Fennema, E. (2001). Capturing teachers' generative change: A follow-up study of professional development in mathematics. *American Educational Research Journal*, 38, 653- 689.
- Glaser, B.G., Strauss, A.L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Chicago: Aldine Publishing.
- Goldin, G., & Kaput, J. (1996). A joint perspective on the idea of representation in learning and doing mathematics. In L. Steffe, P. Nesher, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 397-430). Hillsdale, NJ: Erlbaum.
- Goldsmith, L.T., & Schifter, D. (1997). Understanding teachers in transition: Characteristics of a model for the development of mathematics teaching. In E. Fennema & B.S. Nelson (Eds.), *Mathematics teachers in transition* (pp.19-54). Mahwah, NJ: Lawrence Erlbaum.
- Graeber, A., & Tirosh, D. (1991). The effect of problem type and common misconceptions on preservice elementary teachers' thinking about division. *School Science and Mathematics*, 91, 157-164.
- Grossman (1990). *The making of a teacher: Teacher knowledge and teacher education*. New York: Teachers College Press.
- Herrera, T. (2002) An Interview with Liping Ma. *ENC Focus: Increasing your mathematics and science content knowledge*, 9(3), 5-10.
- Hiebert, J., Gallimore, R., Stigler, J. W. (2002). A knowledge base for the teaching profession: What it looks like and how can we get one? *Educational Researcher*, 31(5), 3-15.
- Kennedy, M. M., Ball, D., & McDiarmid, G. W. (1993). *A study package for examining and tracking changes in teachers' knowledge* (Technical Series 93-1). East Lansing, MI: The National Center for Research on Teacher Education.
- Kieren, T. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 53-92). Reston, VA: National Council of Teachers of Mathematics.

- Kieren, T.E. (1992). Rational and fractional numbers as mathematical and personal knowledge: Implications for curriculum and instruction. In G. Leinhardt & R. T. Putnam (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 323-371). Hillsdale, NJ: Lawrence Erlbaum.
- Koehler, M., S., & Grouws, D. A. (1992). Mathematics teaching practices and their effects. In D. A. Grouws (Ed.), *Handbook of research in mathematics teaching and learning* (pp. 115- 130). New York: Macmillan.
- Lamon, S. J. (1993). Ratio and proportion: Connecting content and children's thinking. *Journal for Research in Mathematics Education*, 24, 41-61.
- Lamon, S. J.(1999). *Teaching fractions and ratios for understanding*. Mahwah, NJ: Lawrence Erlbaum.
- Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, 27 (1), 29-63.
- Lampert, M. (1999). Knowing teaching from the inside out: Implications of inquiry in practice for teacher education. In G. Griffin & M. Early (Eds.), *The education of teachers* (pp. 167-184). Chicago, IL: National Society for the Study of Education.
- Leinhardt, G. (1983). Novice and expert knowledge of individual student's achievement. *Educational Psychologist*, 18(3),165-179.
- Leinhardt, G. (1993). On teaching. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 4, pp. 1-54). Hillsdale, NJ: Lawrence Erlbaum.
- Leinhardt, G. (2001). Instructional explanations: A commonplace for teaching and location for contrast. In V. Richardson (Ed.), *Handbook of Research on Teaching* (see Richardson & Placier 4th edition reference pp. 333- 357). Washington, DC: AERA.
- Leinhardt, G., & Greeno, J. G. (1986). The cognitive skill of mathematics teaching. *Journal of Educational Psychology*, 78, 75-95.
- Leinhardt, G., Putnam, R., Stein, M., & Baxter, J. (1991). Where subject matter knowledge matters. In J. Brophy (Ed.), *Advances in research on teaching* (Vol. 2, pp. 87-113). Greenwich, CT: JAI Press.
- Leinhardt, G., & Smith, D. (1985). Expertise in mathematics instruction: Subject matter knowledge. *Journal of Educational Psychology*, 77, 247-271.

- Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Erlbaum.
- Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education*, 21, 16-32.
- Mack, N. K. (1995). Confounding whole numbers and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education*, 26, 422-441.
- Mack, N.K. (2001). Building on informal knowledge through instruction in a complex content domain: Partitioning units and understanding multiplication of fractions. *Journal for Research in Mathematics Education*, 32, 267-296.
- McLaughlin, M. W., & Talbert, J. E. (2001). *Professional communities and the work of high school teaching*. Chicago, IL: The University of Chicago Press.
- Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education*, 30, 122-147.
- Munby, H., Russell, T., & Martin, A. K. (2001). Teachers' knowledge and how it develops. In V. Richardson (Ed.), *Handbook of research on teaching* (see Richardson & Placier 4th edition reference pp. 877-904). Washington, DC: AERA.
- National Center for Research on Teacher Education (1991). *Findings from the teacher education and learning to teach study: Final Report*. East Lansing, MI: National Center for Research on Teacher Education.
- National Council of Teachers of Mathematics (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.
- National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
- National Research Council (2001). *Adding it up: Helping children learn mathematics*. J.Kilpatrick, J. Swafford, and B. Findell (Eds.). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.

- Nelson, B.S. (1995). Introduction. In B.S. Nelson (Ed.), *Inquiry and development of teaching: Issues in the transformation of mathematics teaching* (pp. 1-8). Newton, MA: Education Development Center, Center for the Development of Teaching.
- Nelson, B. S. (1999). Reconstructing teaching. In M. Z. Solomon (Ed.), *The diagnostic teacher: Constructing new approaches to professional development* (pp. 1-21). New York, NY: Teachers College Press.
- Niemi, D. (1996). Assessing conceptual understanding in mathematics; Representations problem solutions, justifications, and explanations. *Journal of Educational Research*, 89, 350-364.
- Niemi, D. (1997). Cognitive science, expert- novice research, and program assessment. *Theory into Practice*, 36, 239-246.
- Osmundson, E., Chung, G.K.W.K., Herl, H.E., & Klein, D.C.D. (1999). *Knowledge mapping in the classroom: A tool for examining the development of students' conceptual understandings*. (CSE Tech. Report 507). Los Angeles: University of California: Center for Research on Evaluation, Standards, and Student Testing.
- Ott, J.M., Snook, D.L., & Gibson, D.L. (1991). Understanding partitive division with fractions. *Arithmetic Teacher*, 7-11.
- Peterson, P.L., Fennema, E., & Carpenter, T.P. (1989). Teachers' knowledge of students' mathematics problem solving knowledge. On J. Brophy (Ed.), *Advances in research on teaching: Teachers' subject matter knowledge* (Vol II, pp.195-221). Greenwich, CT: JAI Press.
- Post, T.R., Harel, G., Behr, M.J., & Lesh, R. (1991). Intermediate teachers' knowledge of rational number concepts. In E. Fennema, T.P. Carpenter, & S.J. Lamon (Eds.), *Integrating teaching and learning mathematics* (pp.177-198). Albany, NY: State University of New York.
- Richardson, V. (1997). *Constructivist teacher education: Building a world of new understandings*. London: Falmer Press.
- Richardson, V., & Placier, P. (2001). Teacher change. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 905-947). Washington, D.C.: American Educational Research Association.
- Romberg, T. A., & Middleton, J. A. (1994). Conceptions of mathematics and mathematics education held by teachers. In N. L. Webb & T. A. Romberg

(Eds.), *Reforming mathematics education in America's cities: The urban mathematics collaborative project*. New York; Teachers College Press.

Schifter, D. (1995). Teachers' changing conceptions of the nature of the mathematics classroom. In B. S. Nelson (Ed.), *Inquiry and development of teaching: Issues in the transformation of mathematics teaching*. [Center for the Development of Teaching Paper Series]. Newton, MA: Center for the Development of Teaching, Education Development Center, Inc.

Schifter, D. (1997). *Learning mathematics for teaching: Lessons in/from the domain of fractions*. [Center for the Development of Teaching Paper Series]. Newton, MA: Center for the Development of Teaching, Education Development Center, Inc.

Schifter, D. & Bastable, V. (1995, April). *From the teachers' seminar to the classroom: The relationship between doing mathematics and teaching mathematics, an example from fractions*. Paper presented at the Annual Meeting of American Educational Research Association. San Francisco, CA.

Schifter, D. & Fosnot, C.T. (1993). *Reconstructing mathematics education: Stories of teachers meeting the challenge of reform*. New York: Teachers College Press.

Schifter, D., & Simon, M. (1992). Assessing teacher's development of a constructivist view of mathematics learning. *Teaching and Teacher Education*, 8(2), 187-197.

Schneider, Y. (2001). *Knowing how and knowing why: Unpublished field study*, Graduate Center and University Center, City University of New York, New York.

Sherin, M, S. (2001). When teaching becomes learning. *Cognition and Instruction* 20(2), 119-150.

Shulman, L. (1986). Those who understand teach: Knowledge growth in teaching. *Educational Researcher*, 57(1), 1-22.

Shulman, L. (1989). *The paradox of teacher assessment*. New Directions for Teacher Assessment: Proceedings of the 1988 ETS International Conference. Princeton, NJ: Educational Testing Service.

Shulman, L. S., & Quinlan, K. M. (1996). The comparative psychology of school subjects. In D.C. Berliner & R. C. Calfee (Eds.), *Handbook of educational psychology* (pp. 399-422). New York: Macmillan Library Reference USA.

- Silver, E., & Stein, M.K. (1996). The Quasar project: The revolution of the possible in mathematics instructional reform in urban middle schools. *Urban Education, 30*, 476-521.
- Sowder, J.T., Phillipp, R.A., Armstrong, B.E., & Schappelle, B.P. (1998). *Middle-grade teachers' mathematical knowledge and its relationship to instruction: A research monograph*. Albany, NY: SUNY Press.
- Steffe, L. (1988). Children's construction of number sequences and multiplying schemes. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 119-140). Reston, VA: National Council of Teachers of Mathematics.
- Stein, M. K., & Brown, C. A. (1997). Teacher learning in a social context: Integrating collaborative and institutional processes with the study of teacher change. In E. Fennema & B.S. Nelson (Eds.), *Mathematics teachers in transition* (pp. 155-191). Mahwah, NJ: Lawrence Erlbaum.
- Streefland, L. (1993). Fractions: A realistic approach. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp.289-325). Hillsdale, NJ: Lawrence Erlbaum.
- Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31*, 5-25.
- Tirosh, D. & Graeber, A.O. (1990). Inconsistencies in preservice elementary teachers' beliefs about multiplication and division. *Focus on Learning Problems in Mathematics, 12* (3& 4), 65-74.
- Turner-Bisset, K. (1999). The knowledge base of the expert teacher. *British Educational Research Journal, 25*(1), 39- 55.
- Wilson, S. (1989). *Understanding historical understanding: Subject matter knowledge and the teaching of U.S. history*. Unpublished doctoral dissertation, Stanford University, Palo Alto, CA.
- Wilson, S.M., Shulman, L.S., & Richert, A. E. (1987). 150 different ways of knowing: representations of knowledge in teaching. In J. Calderhead (Ed.), *Exploring teachers' thinking* (pp. 104-124). London: Cassell.
- Wineburg, S. (1990). *Historical problem solving: A study of the cognitive processes used in the evaluation of documentary evidence*. Unpublished doctoral dissertation, Stanford University, Palo Alto, CA.

Wineburg, S. (1997). Beyond "breadth and depth": Subject matter knowledge and assessment. *Theory into Practice*, 36, 255-261.

Wineburg, S., & Wilson, S. (1991). Content knowledge in the teaching of history. In J. Brophy (Ed.), *Advances in research on teaching* (Vol.2, pp. 305-347). Greenwich, CT: JAI Press.