Derived noncommutative deformation theory

by

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Abstract

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We define derived deformation theory with parameters over an operad O, and prove that the ∞-category of such theories is equivalent to the ∞-category of $O^!$-algebras.

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Contents

Chapter 1

Introduction

1.1 The story so far

Deformation theory is a structural approach to the study of parametrized families of mathematical objects. A *deformation* of an object X is a collection of objects of the same type as X fibred over a parameter space $\mathcal{B}$, with X as the fibre over the basepoint in $\mathcal{B}$. Because families over $\mathcal{B}$ can be pulled back to families over $\mathcal{B}'$ along morphisms $\mathcal{B}' \to \mathcal{B}$, one obtains a functor $\textbf{Parameter Spaces}^{op} \to \textbf{Sets}$ by considering all possible deformations of X up to equivalence:

$$\mathcal{B} \mapsto \{\text{deformations of } X \text{ over } \mathcal{B}\}/\sim .$$

By passing to rings of functions on the parameter spaces, we obtain a functor from local Artin commutative algebras to sets. Deformation theory provides a framework for analyzing the situation by describing such functors with well-known algebraic structure:

Folk Theorem 1.1.1. *In characteristic zero, deformations of anything are controlled by a differential graded (dg) Lie algebra.*

Grothendieck [Gro59] was the first person to prove theorems of this sort. Schlessinger [Sch68] developed criteria that singled out a class of functors that captured deformation problems without obstructions. These smooth functors fit into the folk theorem as those controlled by abelian Lie algebras. Obstructions in singular deformation problems were observed to organize themselves into nontrivial dg Lie algebras, and examples such as deformations of rings and algebras [Ger64], complex manifolds [KD57], and representations of fundamental groups [GM88] were discovered and studied case by case. It was in the presence of examples that the folk theorem was understood and began to circulate [SS79, Del86]. In 2002, Manetti, using new ideas of Kontsevich—in particular a hidden smoothness philosophy [Kon95] inspired by theoretical physics—promoted the folk theorem to a theorem by adapting Schlessinger's criteria so that it applied, via hidden smoothness, to all deformation problems:

Theorem 1.1.2 (Manetti)**.** *If a deformation functor F can be extended to parameter spaces whose rings of functions are local Artin dg commutative algebras, then F is representable by a dg Lie algebra in the homotopy category. In other words, there is a dg Lie algebra L so that F is naturally isomorphic to the Maurer-Cartan* *functor for L modulo gauge-equivalence. [Man02]*

We set out to develop deformation theory over noncommutative base spaces by proving the analogue of Theorem 1.1.2 in the noncommutative setting, for spaces whose rings of functions are local Artin dg not-necessarily-commutative algebras. To state the analogue correctly, one must make contact with theories of algebraic structures—this is essential in order to discover what kind of algebraic structures govern the noncommutative deformation theory. We use the theory of operads [May72, LV12]. An operad is an object organizing many-to-one operations; there are operads $\mathrm{Com}, \mathrm{As},$ and Lie whose representations are dg commutative associative algebras, dg (not-necessarily-commutative) associative algebras, and dg Lie algebras, respectively. The theory of operads supports a duality phenomenon: to any operad O there is a dual operad $O^!$, and it is this duality that determines what type of algebraic structure governs a type of deformation theory. In the aforementioned examples, Com and Lie are dual, and As is self-dual.

Higher categories

In order to retain maximal information in any given setting, one should not mod out by equivalence, but rather encode the equivalences themselves into the structure being studied. What is obtained, however, requires new language for their definitions and manipulation. Higher category theory provides a toolkit for succinctly describing these structures, and provides a language sufficiently parallel to classical category theory so that classical intuition can be upgraded to higher categorical intuition. Moreover, proof techniques in some fields have long since outpaced the sophistication of mathematical language, forcing statements of theorems to be either incomplete or unwieldy. Higher category theory provides language for researchers to communicate their discoveries more completely without interrupting the narrative arc of their communication.

In deformation theory, we implement these ideas as follows: the set of deformations is a discrete topological space—for every equivalence of deformations, a 1-cell is attached with those deformations as its endpoints. Similarly, for every homotopy between equivalences of deformations, a 2-cell is attached with those equivalences as its boundary. Homotopies of homotopies are encoded by 3-cell attachments, and continuing in this way a cell complex of deformations is constructed. This changes the target of our deformation functors from **Sets** to **Spaces**. Note that **Spaces** is the category of purely topological spaces, not the structured, algebro-geometric spaces we began with as parameter spaces. This work is the main result of this thesis:

Theorem 1.1.3 (H.)**.** *Let O be an operad. The ∞-category of functors from local Artin dg O algebras to spaces satisfying Schlessinger's criteria is equivalent to the ∞-category of $O^!$ algebras.*

It is convenient to introduce some notation. We denote the ∞-category of dg O algebras by $\mathscr{A}_O$, and the ∞-category of functors from local Artin dg O algebras to spaces satisfying Schlessinger's criteria by $\mathscr{M}_O$. By treating the operad O as a variable, a number of classical and modern results are captured as corollaries of the theorem. Setting O to be Com, we obtain $\mathscr{M}_{\mathrm{Com}} \simeq \mathscr{A}_{\mathrm{Lie}}$, choosing O to be As, we obtain $\mathscr{M}_{As} \simeq \mathscr{A}_{\mathrm{As}}$. Applying π_0 on each side of the preceding equivalences recovers Manetti's Theorem 1.1.2, and the analogous structure theorem for noncommutative deformation theory, respectively. When O is E_n—an operad related to embeddings of n-dimensional discs into each other—we recover a recent theorem of Lurie [Lur11] which can be used to prove Deligne's Conjecture [Del93, GV95, Tho10, Lur12]. When O is BD—the Beilinson-Drinfeld operad of Costello [CG11]—Theorem 1.1.3 enables Costello and Gwilliam to bypass obstructions to constructing perturbative quantum field theories [Cos12]. There is also evidence [GTZ12] that Theorem 1.1.3 provides an alternative proof of the existence of the algebra of string topology operations of a manifold.

Another advantage of Theorem 1.1.3 relates to the choice of base field. Deformation theory implicitly requires choosing a base field $\mathbb{K}$ to work over—the base point in every parameter space $\mathcal{B}$ is an algebro-geometric $\mathbb{K}$-point. Theorem 1.1.2 requires that the base field $\mathbb{K}$ be a field of characteristic zero, which excludes interesting examples of deformation theory in positive characteristic—such as deformations of Galois representations [Maz89] and more recently the work of [Păs13]. The operad Com is sensitive to the characteristic of $\mathbb{K}$, but for the operads As or E_n, Theorem 1.1.3 is expected to be true for any field $\mathbb{K}$ regardless of characteristic. This creates the opportunity for positive characteristic deformation functors to be enhanced by being extended to a larger class of algebras, and simultaneously guarantees that any such extension be represented by a single algebra.

1.2 An outline of the proof

The equivalence of 1.1.3 is expected to rely on the duality between O algebras and $O^!$ algebras, and our proof makes use of this dependence explicitly. We exhibit a model for the duality functor

$$\mathbb{D} : \mathrm{Alg}_O^{\mathrm{op}} \to \mathrm{Alg}_{O^!}$$

and restrict it to the subcategory of local Artin O algebras, which we denote by $\mathrm{Art}_O^{\mathrm{op}} \subset \mathrm{Alg}_O^{\mathrm{op}}$. Because the algebras in Art_O are sufficiently finite, the duality functor provides a (homotopical) embedding

$$\mathbb{D} : \mathrm{Art}_O^{\mathrm{op}} \to \mathrm{Alg}_{O^!}.$$

We prove

1. Every object in $\mathrm{Alg}_{O^!}$ can be (functorially) constructed by "homotopical gluing procedures," where the fundamental building blocks being glued to-

gether are the (images under $\mathbb{D}$ of the) objects of Art_O, and the fundamental "transition maps" between the local pieces are the morphisms of Art_O.

2. Every "abstract homotopical gluing of objects of Art_O of a certain shape" can be (functorially) realized as an object of $\mathrm{Alg}_{O^!}$.

How these results are used to prove Theorem 1.1.3 is now explained.

The category of abstract homotopical gluings (with no conditions on the shape) in an ∞-category C is equivalent to the ∞-category of ∞-functors from C^{op} to spaces. Denote the ∞-category corresponding to Alg_O by $\mathscr{A}_O$, and the ∞-category corresponding to Art_O by $\mathscr{A}_O^{\mathrm{sm}}$, and the ∞-category of ∞-functors from $\mathscr{A}_O^{\mathrm{sm}}$ to spaces by $\mathscr{P}\left((\mathscr{A}_O^{\mathrm{sm}})^{\mathrm{op}}\right)$. The ∞-category $\mathscr{P}\left((\mathscr{A}_O^{\mathrm{sm}})^{\mathrm{op}}\right)$ is modeled by the category of simplicial functors $\mathrm{SSet}^{\mathrm{Art}_O}$ with the projective model structure [Lur09, 4.2.4.4].

The duality functor gives us a way to "test" objects of $\mathrm{Alg}_{O^!}$ by the category Art_O: for a given object $L \in \mathrm{Alg}_{O^!}$, the restriction of its Yoneda functor to the image of $\mathbb{D}$ can be considered, inducing a functor from $\mathrm{Alg}_{O^!} \xrightarrow{\mathrm{Sing}} \mathrm{SSet}^{\mathrm{Art}_O}$. Let $\mathrm{Sing}(L)$ be the functor

$$R \mapsto \mathrm{Alg}_{O^!}\left(\mathbb{D}R, L\right).$$

Because $\mathbb{D}$ is (homotopically) an embedding, the diagram

$$\begin{tikzcd}
\mathrm{Art}_O^{\mathrm{op}} \arrow[r, "y"] \arrow[dr, "\mathbb{D}"'] & \mathrm{SSet}^{\mathrm{Art}_O} \\
& \mathrm{Alg}_{O^!} \arrow[u, dotted, "\mathrm{Sing}^{\mathbb{D}}"']
\end{tikzcd}$$

commutes up to natural weak equivalence. Although it is not relevant to the proof, we point out that these test spaces admit a description in terms of solutions to the Maurer-Cartan equation in L with coefficients in R

$$\mathrm{Sing}(L)(R) \simeq \mathrm{MC}_L(R)$$

whenever the latter is well-defined (see [DW12] for a general characterization of operads with such descriptions).

Next, we identify conditions for functors $F : \mathrm{Art}_O \to \mathrm{SSet}$ to behave in a sufficiently geometric way, and we refer to these properties as *derived Schlessinger-Manetti criteria*. We define the category $\mathscr{M}_O$ to be the full sub-∞-category of $\mathscr{P}\left((\mathscr{A}_O^{\mathrm{sm}})^{\mathrm{op}}\right)$ spanned by the functors satisfying the derived Schlessinger-Manetti criteria, along with an additional constraint on . A priori, we have no corresponding model for this category of functors inside the model category $\mathrm{SSet}^{\mathrm{Art}_O}$; one consequence of this work is a model-theoretic description of $\mathscr{M}_O$ via $\mathrm{SSet}^{\mathrm{Art}_O}$.

Not having a model for $\mathscr{M}_O$ is not an obstruction, as it is a full subcategory of an ∞-category with a nice model. By verifying that the functors $\mathrm{Sing}(L)$ satisfy derived Schlessinger-Manetti criteria for every object L, we obtain a factorization of the ∞-functor corresponding to Sing through the subcategory $\mathscr{M}_O$:

$$\begin{tikzcd}
\mathscr{A}_{O^!} \arrow[r, "\mathscr{S}"] \arrow[dr, dotted] & \mathscr{P}\left((\mathscr{A}_O^{\mathrm{sm}})^{\mathrm{op}}\right) \\
 & \mathscr{M}_O \arrow[u, hook]
\end{tikzcd}$$

This establishes the ∞-functor $\mathscr{S} : \mathscr{A}_{O^!} \to \mathscr{M}_O$. It remains to prove that it is an equivalence. There are two steps:

1. Sing : $\mathrm{Alg}_{O^!} \to \mathrm{SSet}^{\mathrm{Art}_O}$ is (homotopically) fully faithful, ie, that for each pair of objects $L_1, L_2 \in \mathrm{Alg}_{O^!}$, the function on mapping spaces

$$\mathrm{Sing} : \mathrm{Alg}_{O^!}(L_1, L_2) \to \mathrm{SSet}^{\mathrm{Art}_O}(\mathrm{Sing}(L_1), \mathrm{Sing}(L_2))$$

   is a weak equivalence.

2. For each functor $F \in \mathscr{M}_O$, there exists an object $L \in \mathrm{Alg}_{O^!}$ such that $\mathrm{Sing}(L) \simeq F$.

To prove 1, we observe that Sing has a left adjoint, Re : $\mathrm{SSet}^{\mathrm{Art}_O} \to \mathrm{Alg}_{O^!}$. The left adjoint is given by decomposing a functor into an abstract homotopical gluing of objects in Art_O, and using $\mathbb{D}$ to realize this abstract homotopical gluing concretely in $\mathrm{Alg}_{O^!}$ [Dug01], where homotopy colimits exists. The adjunction between Re and Sing implies that for every pair $L_1, L_2 \in \mathrm{Alg}_{O^!}$ we have the commuting diagram

$$\begin{tikzcd}
\mathrm{Alg}_{O^!}(L_1, L_2) \arrow[r, "\mathrm{Sing}"] \arrow[dr, "(\varepsilon_{L_1})^*"'] & \mathrm{SSet}^{\mathrm{Art}_O}(\mathrm{Sing}(L_1), \mathrm{Sing}(L_2)) \arrow[d, "\cong"] \\
& \mathrm{Alg}_{O^!}(\mathrm{ReSing}L_1, L_2)
\end{tikzcd} \tag{1.1}$$

where ε_L : ReSing$L \to L$ is the counit of the adjunction. This diagram encodes the universality of the counit, and it demonstrates that Sing is homotopically fully faithful if and only if the counit of the adjunction is a weak equivalence for every $L \in$ Alg$_{O^!}$. We verify that it is.

For 2, we prove that a natural transformation $F \to G$ in $\mathscr{M}_O$ is an equivalence as soon as it is an equivalence on the associated *tangent spaces* $TF \to TG$. We compute directly that the unit maps of the adjunction, $F \to$ SingReF induce equivalences on the associated tangent spaces for every $F \in \mathscr{M}_O$.

As a consequence, we obtain a model for the ∞-category $\mathscr{M}_O$: it is the the Bousfield localization of SSet$^{\mathrm{Art}_O}$ by the maps $\{F \to \mathrm{SingRe}F\}$, and the adjunction

$$\mathrm{Re} : \mathrm{SSet}^{\mathrm{Art}_O} \longleftrightarrow \mathrm{Alg}_{O^!} : \mathrm{Sing}$$

descends to a Quillen equivalence of model categories between Alg$_{O^!}$ and the localization of SSet$^{\mathrm{Art}_O}$.

1.3 Technical considerations

In order to keep the narrative of 1.2 concise, some technical details were hidden or obfuscated. These points are clarified here.

1.3.1 Coalgebras

The first technical point involves the construction of the adjoint to Sing. In order to construct the adjunction Re $\dashv$ Sing, we pass to the model category of coalgebras with the Lefèvre-Hasegawa-Vallette model structure [LH03, Val12]. In [Val12], Vallette defines a model structure on the category of coalgebras over the Koszul dual cooperad to O, $O^{¡}$ and proves that the bar and cobar constructions provide an equivalence between this model category of coalgebras and the model category of algebras over O. We endow the model category of coalgebras over $O^{¡}$ with an SSet-enrichment which behaves nicely with respect to homotopy colimits—it is *tensored* over SSet (see equation 2.2). The tensoring of $\mathrm{coAlg}_{O^{¡}}$ guarantees that we can construct the adjunction Re $\dashv$ Sing, and then we can pre/post-compose the adjunction Re $\dashv$ Sing by the adjunction $\Omega \dashv B$ to obtain the adjunction $\Omega \circ \mathrm{Re} \dashv \mathrm{Sing} \circ B$ between $\mathrm{SSet}^{\mathrm{Art}_O}$ and $\mathrm{Alg}_{O^!}$.

1.3.2 Derived adjunction

The diagram in 1.1 is not quite right. The problem is that the space

$$\mathrm{SSet}^{\mathrm{Art}_O}\left(\mathrm{Sing}(L_1), \mathrm{Sing}(L_2)\right)$$

may not be a model for the mapping space in the ∞-category $\mathscr{P}\left((\mathscr{A}_O^{\mathrm{sm}})^{\mathrm{op}}\right)$. Specifically, the simplicial model structure on $\mathrm{SSet}^{\mathrm{Art}_O}$ is only required to provide the correct mapping spaces when the source is cofibrant and the target is fibrant. We think of this failure as the price we pay for the strictness of the model category. Much in the same way that one can rectify homotopy algebra structures by taking larger, but homotopy equivalent, chain complexes, on which the algebra structure on the non-closed elements does not appear to be homotopy invariant. The non-fibrant/cofibrant objects play a similar role in buffering the model category with objects on which the principal structure—the mapping spaces—does not appear to be homotopy invariant.

Since every algebra is fibrant, and right Quillen functors preserve fibrant objects, $\mathrm{Sing}(L_2)$ is fibrant. We have no similar guarantee that $\mathrm{Sing}(L_1)$ is cofibrant, and so in order to obtain the correct mapping space, we must consider a cofibrant replacement of $\mathrm{Sing}(L_1)$: $|Q\mathrm{Sing}(L_1)| \to \mathrm{Sing}(L_1)$. The correct diagram is:

$$\begin{tikzcd}
\mathrm{Alg}_{O^!}(L_1, L_2) \arrow[r, "\mathrm{Sing}"] \arrow[dr, "(\varepsilon_{L_1} Q)^*"'] & \mathrm{SSet}^{\mathrm{Art}_O}(|Q\mathrm{Sing}(L_1)|, \mathrm{Sing}(L_2)) \arrow[d, "\cong"] \\
 & \mathrm{Alg}_{O^!}(\mathrm{Re}|Q\mathrm{Sing} L_1|, L_2).
\end{tikzcd}$$

It is the replacement $|Q(-)|$ that introduces the homotopical decompositions of the functor F, as we describe in Remark 8.

We compute that this map is an equivalence by reducing to a homotopy colimit over a Reedy diagram, and applying the theory of homotopy colimits over Reedy diagrams in model categories [Hir02, Chapter 15]. In particular we use the fact that over a Reedy diagram, the homotopy colimit can be computed as the total derived functor of the colimit functor. This implies that homotopy colimits of Reedy cofibrant diagrams coincide with the classical colimit of the diagram.

1.3.3 Dugger's Universal Homotopy Theories

The goal of [Dug01] is to create methods of comparing general model categories with simplicial model categories. As such, Dugger only considers universal homotopy theories on Set-categories C. We extend his constructions to the situation where $\underline{C}$ is SSet-enriched, where the universal homotopy theory is given by the SSet-enriched category of SSet-enriched functors $\mathrm{SSet}^{\underline{C}^{\mathrm{op}}}$.

Chapter 2

Background

2.1 Conventions

We follow cohomological convention, so that the degree of the differentials in our chain complexes is $+1$.

Throughout this document, we work over a field k of characteristic zero.

We work with finitely generated operads O satisfying $O(0) = 0, O(1) = k$, and finitely cogenerated cooperads C satisfying $C(0) = 0$, $C(1) = k$.

2.2 Model category of algebras

Definition 2.2.1. *The symbol Alg_O denotes the category of O algebras in the symmetric monoidal category of $\mathbb{Z}$-graded cochain complexes.*

Definition 2.2.2. *Let S be a class of morphisms in a category M. A map $A \xrightarrow{i} B$ is in LLP(S), or* has the left lifting property with respect to S *if for every solid diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "s"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

with $s \in S$, the dotted arrow exists.

Theorem 2.2.3 ([Hin97]). *The category Alg$_O$ has the structure of a Quillen model category, with*

$$W = \{\textit{quasi-isomorphisms}\}$$

$$F = \{\textit{surjections}\}$$

$$C = \textit{LLP}\,(W \cap F)$$

Corollary 2.2.4. *The free-forgetful adjunction between Alg$_O \xrightarrow{U}$ Ch$_k$ and Ch$_k \xrightarrow{O(-)}$ Alg$_O$ is a Quillen-pair.*

Proof. The forgetful functor U is the right adjoint and preserves weak equivalences and fibrations by their definition in Alg$_O$. □

Definition 2.2.5. *A map of dg O-algebras $A \xrightarrow{i} B$ is* quasi-free *if there is a set I and a (non-dg) algebra isomorphism $B \cong A * O(g_\alpha)_{\alpha \in I}$ so that the composite map $A \xrightarrow{i} B \cong A * O(g_\alpha)_{\alpha \in I}$ is the natural inclusion.*

Definition 2.2.6. *A quasi-free map i is called* triangular *if there is a set I and isomorphism $B \cong A * O(g_\alpha)_{\alpha \in I}$ which exhibits i as quasi-free, and an order relation $<$ on the set I which satisfies the following properties:*

1. *all descending chains in $<$ are finite*

2. *when the differential in B is transported to $A * O(g_\alpha)_{\alpha \in I}$ via the isomorphism, it has the property that $d(g_\beta)$ is contained in the subalgebra $A * O(g_\alpha)_{\alpha \in I \textit{ and } \alpha < \beta} \subset A * O(g_\alpha)_{\alpha \in I}$*

Proposition 2.2.7. *The class of cofibrations in Alg$_O$ can be identified with the closure of the quasi-free triangular maps under retracts.*

Proof. One can verify directly that quasi-free triangular maps have the left lifting property with respect to acyclic fibrations. To see that every cofibration is a retract of a quasi-free triangular map, we observe that every map $A \xrightarrow{i} B$ in Alg_O can be factored as $A \xrightarrow{\tilde{i}} \tilde{B} \xrightarrow{p} B$ where i is quasi-free triangular, and p is a surjective quasi-isomorphism. In particular, any cofibration $A \xrightarrow{i} B$ has such a factorization $A \xrightarrow{\tilde{i}} \tilde{B} \xrightarrow{p} B$. This gives the solid diagram

$$\begin{tikzcd}
A \arrow[r, "\tilde{i}"] \arrow[d, "i"'] & \tilde{B} \arrow[d, "p"] \\
B \arrow[r, "\mathrm{id}_B"'] \arrow[ur, dotted, "f"] & B
\end{tikzcd}$$

where we have assumed that i has the left lifting property with respect to p, so the dotted arrow f exists. This map realizes the cofibration as a retract of $\tilde{i}$:

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}_A"] \arrow[d, "i"] & A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\tilde{i}"] & A \arrow[d, "i"] \\
B \arrow[r, "f"] & \tilde{B} \arrow[r, "p"] & B.
\end{tikzcd}$$

□

Theorem 2.2.8 ([Hin97]). *The category Alg_O has an SSet-enrichment given by*

$$Alg_O(A,B)_\bullet \equiv Alg_O(A, B\otimes k[\Delta^\bullet])$$

where $k[\Delta^\bullet]$ denotes the functor

$$\begin{aligned}
&\Delta \to Alg_{Com} \\
&n \mapsto k[\Delta^n] \\
&\quad = k[t_0,\dots,t_n,dt_0,\dots,dt_n]\Big/\left(1-\sum_{i=0}^n t_i, \sum_{i=0}^n dt_i\right).
\end{aligned}$$

The composition is given by the multiplication $k[\Delta^n]\otimes k[\Delta^n]\to k[\Delta^n]$: for $f : A \to B\otimes k[\Delta^n]$ and $g : B \to C\otimes k[\Delta^n]$, the composition $g\circ f : A\to C\otimes k[\Delta^n]$ is given by

$$A \xrightarrow{f} B\otimes k[\Delta^n] \xrightarrow{g\otimes \mathrm{id}_{k[\Delta^n]}} C\otimes k[\Delta^n]\otimes k[\Delta^n] \xrightarrow{\mathrm{id}_C\otimes \mu_{\Delta^n}} C\otimes k[\Delta^n]$$

This SSet-structure has the property that for a cofibration $A\to B$ and a fibration $X\to Y$ the map

$$Alg_O(B,X)_\bullet \xrightarrow{(i^*,p_*)} Alg_O(A,X)_\bullet \underset{Alg_O(A,Y)_\bullet}{\times} Alg_O(B,Y)_\bullet \tag{2.1}$$

is a fibration in SSet. Moreover, if either i *or* p *is a weak equivalence, then* (i^*, p_*) *is also a weak equivalence.*

Property 2.1 implies that the mapping spaces between a cofibrant and a fibrant object are homotopy invariant.

Proposition 2.2.9. *Let* $A \xrightarrow{f} B$ *be a weak equivalence between cofibrant algebras, and* $X \xrightarrow{g} Y$ *be a weak equivalence between fibrant algebras. Then the maps*

$$Alg_O(B, X)_\bullet \xrightarrow{f^*} Alg_O(A, X)$$

$$Alg_O(A, X)_\bullet \xrightarrow{g_*} Alg_O(B, X)$$

are weak equivalences of Kan complexes (fibrant simplicial sets).

Proof. The simplicial set of maps between a cofibrant and a fibrant object is a Kan complex by considering the square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, hook] & X \arrow[d] \\ A \arrow[r] & 1. \end{tikzcd}$$

Ken Brown's lemma [Hov99] implies that maps into a fibrant object preserves weak equivalences between cofibrant objects and the dual statement. □

The following theorem relates the homotopy invariant mapping spaces to the natural mapping spaces which arise from the weak equivalences alone.

Theorem 2.2.10. *Let $R_\bullet : M \to M^{\Delta^{op}}$ be a simplicial framing of the model category M. Then for every cofibrant object $A \in M$ and every fibrant object $Y \in M$, the map of simplicial sets*

$$M(A, R_\bullet Y) \to L_H(M, W)(A, Y)$$

is a weak equivalence of fibrant simplicial sets. [Dug06, DK80]

Corollary 2.2.11. *For every cofibrant algebra A and fibrant algebra X, the map*

$$\mathit{Alg}_O(A, X)_\bullet \to L\left(\mathit{Alg}_O, \{\textit{quasi-isomorphisms}\}\right)(A, X)$$

is a weak equivalence of fibrant simplicial sets.

Proof. The space $\mathrm{Alg}_O(A, X)_\bullet$ is precisely $\mathrm{Alg}_O(A, R_\bullet X)$ for the simplicial framing $R_\bullet$ given by tensoring with the simplicial algebra of polynomial differential forms on the n simplex. □

2.3 Model categories of coalgebras

Recall that we intend to use the model category of algebras as a preliminary receptacle for the realization functor from $\mathrm{SSet}^{\mathrm{Art}_O}$. We describe the model category structure now.

Definition 2.3.1. *The symbol coAlg_C denotes the category of C coalgebras in the category of $\mathbb{Z}$-graded cochain complexes.*

Definition 2.3.2. *A map* $X \xrightarrow{p} Y$ *is in RLP*(S) *or* has the right lifting property with respect to S *if for every solid diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "s"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

with $s \in S$*, the dotted arrow exists.*

In the presence of Koszul duality between C and O, we have bar and cobar constructions between Alg_O and coAlg_C, denoted Ω_α and B_α (see Appendix A). There is a model structure on coAlg_C so that the cobar construction preserves weak equivalences by definition.

Theorem 2.3.3 ([Val12])**.** *The category coAlg*$_C$ *has the structure of a Quillen model category, with*

$$\begin{aligned} W &= \{\textit{preimage under } \Omega_\alpha \textit{ of quasi-isomorphisms in } Alg_O\} \\ C &= \{\textit{inclusions}\} \\ F &= \textit{RLP}\,(W \cap C)\,. \end{aligned}$$

We call this model structure the Lefèvre-Hasegawa-Vallette model structure on coAlg$_C$*.*

Proposition 2.3.4 ([Val12])**.** *The weak equivalences in the Lefèvre-Hasegawa-Vallette model structure are contained in the quasi-isomorphisms.*

Corollary 2.3.5. *The cofree-forgetful adjunction between coAlg*$_O \overset{U}{\to}$ *Ch*$_k$ *and Ch*$_k \xrightarrow{C(-)}$ *Alg*$_O$ *is a Quillen-pair.*

Proof. The forgetful functor is the left adjoint, and it preserves cofibrations. Since weak equivalences are contained in the quasi-isomorphisms, the forgetful functor preserves weak equivalences. □

Theorem 2.3.6 ([Val12])**.** *For C and O Koszul dual, the Quillen adjunction Ω_α : coAlg*$_C \leftrightarrow$ *Alg*$_O$: B_α *is a Quillen equivalence.*

Remark 1. *There is a more natural candidate for the model structure on coAlg*$_C$*:*

$$W = \{\textit{quasi-isomorphisms}\}$$

$$C = \{\textit{inclusions}\}$$

$$F = \textit{RLP}\,(W \cap C)\,.$$

It is a theorem of Hess and Shipley [HS12] that this forms a model category structure. This model structure is not useful in our context because we are not in fact interested in coalgebras or their homotopy theory—we are interested in C-coalgebra models for O-algebras. Another, less conceptual problem with the Hess-Shipley model structure is that the weak equivalences in the Hess-Shipley model category structure are not well behaved with respect to the cobar construction, in the sense that there are quasi-isomorphisms not preserved by the cobar construction. [LV12, Proposition 2.4.3]

Let $C_*(\Delta^\bullet)$ be a cosimplicial resolution of the ground field k as a dg cocommutative coassociative coalgebra in the Lefèvre-Hasegawa-Vallette model structure.

Proposition 2.3.7. *For a coalgebra $X \in \mathit{coAlg}_{O^\vee}$, the tensor product $X \otimes C_*(\Delta^\bullet)$ is a cosimplicial resolution of the coalgebra X.*

Proof. We need to prove that $X \otimes C_*(\Delta^\bullet)$ is a Reedy cofibrant diagram, and that the map $X \otimes C_*(\Delta^\bullet) \to X$ is a weak equivalence. Over a field k, everything is flat, so the second condition follows from the fact that $C_*(\Delta^\bullet) \to k$ is a weak equivalence. To see that $X \otimes C_*(\Delta^\bullet)$ is Reedy cofibrant, we observe that colimits in coAlg_C can be computed in Ch_k, and in Ch_k the tensor product commutes with colimits, so the latching maps of $X \otimes C_*(\Delta^\bullet)$ are given by the tensor product of X with the latching maps of $C_*(\Delta^\bullet)$, and the tensor product of an inclusion by the identity is an inclusion. □

Corollary 2.3.8. *The category coAlg_C is an SSet-category with the mapping spaces given by*

$$\mathit{coAlg}_C(A, B)_\bullet \equiv \mathit{coAlg}_C(A \otimes C_*(\Delta^\bullet), B). \tag{2.2}$$

Moreover, the mapping spaces have the property that if $A \to B$ is a cofibration and $X \to Y$ a fibration, the map

$$\mathit{coAlg}_C(B, X)_\bullet \xrightarrow{(i^*,p_*)} \mathit{coAlg}_C(A, X)_\bullet \underset{\mathit{coAlg}_C(A,Y)_\bullet}{\times} \mathit{coAlg}_C(B, Y)_\bullet \tag{2.3}$$

is a fibration in SSet. If either i or p is a weak equivalence, then (i^*, p_*) is also a weak equivalence.

Proof. The coalgebra structure on $C_*(\Delta^\bullet)$ gives a composition structure on the mapping spaces $\mathrm{coAlg}_{O^\vee}(B \otimes C_*(\Delta^\bullet), D)$ so that $\mathrm{coAlg}_{O^\vee}$ is an SSet-category. For coalgebra maps $f : A \otimes C_*(\Delta^n) \to B$ and $g : B \otimes C_*(\Delta^n) \to C$, the composition $g \circ f : A \to C \otimes k[\Delta^n]$ is given by

$$A \otimes C_*(\Delta^n) \xrightarrow{\mathrm{id}_A \otimes \Delta_{C_*(\Delta^n)}} A \otimes C_*(\Delta^n) \otimes C_*(\Delta^n) \xrightarrow{f \otimes \mathrm{id}_{C_*(\Delta^n)}} B \otimes k[\Delta^n] \xrightarrow{g} C.$$

The fibration property follows from [Hir02, Proposition 16.4.6]. □

Remark 2. *We point out that constructing a cosimplicial resolution of k as a coalgebra seems to be more complicated than its simplicial analogue as an algebra, but our evidence for this fact is that we could not succeed at the former whereas the latter is well-known, and that the Lawrence-Sullivan model for the interval involves the Bernoulli numbers. [LS06]*

Corollary 2.3.9. *Maps to a fibrant coalgebra preserve weak equivalences between cofibrant coalgebras, and maps from a cofibrant coalgebra preserve weak equivalences between fibrant coalgebras.*

Proof. The proof is the same as 2.2.9. □

Theorem 2.3.10. *Let $Q_\bullet : M \to M^\Delta$ be a cosimplicial framing of the model category M. Then for every cofibrant object $A \in M$ and every fibrant object $Y \in M$, the map of simplicial sets*

$$M(Q_\bullet A, Y) \to L_H(M, W)(A, Y)$$

is a weak equivalence of fibrant simplicial sets. [Dug06, DK80]

Corollary 2.3.11. *For every cofibrant coalgebra A and fibrant coalgebra X, the map*

$$\mathit{coAlg}_C(A, X)_\bullet \to L\left(\mathit{coAlg}_C, \{\mathit{weak\text{-}equivalences}\}\right)(A, X)$$

is a weak equivalence of fibrant simplicial sets.

Proof. The space $\mathrm{coAlg}_C(A, X)_{bullet}$ is precisely $\mathrm{coAlg}_O(Q_\bullet A, X)$ for the cosimplicial framing $Q_\bullet$ given by tensoring with the cosimplicial coalgebra of chains on the n-simplex. $\square$

Chapter 3

Local Artin Algebras

There is a step in 1.2 where the fact that a natural transformation of derived deformation problem is an equivalence as soon as it is an equivalence on the tangent spaces of the deformation problems. The key ingredient we use to prove this result is 3.1.4. This guarantees that we can perform a kind of induction on algebras—known as Artinian induction—with the zero-dimensional algebra as the base case. We describe this lemma for local Artin O algebras.

3.1 Local Artin algebras over O

In this section, O is an operad satisfying the conditions of A.0.19.

Definition 3.1.1. *A differential graded commutative k-algebra R is a* local Artin algebra *if and only if it is finite dimensional and nilpotent.*

Remark 3. *The words "local" and "Artin" have their own meaning in the context* *of commutative algebra. One can prove that the combination of these two words is equivalent to the definition in 3.1.1.*

Definition 3.1.2. *A differential graded O algebra over k is a* local Artin algebra *if and only if it is finite dimensional and nilpotent.*

Definition 3.1.3. *We denote by Art_O the full subcategory of Alg_O spanned by the local Artin O algebras.*

3.1.1 Extensions

An *extension* is an exact sequence of algebras

$$0 \to I \to A \to B \to 0 \tag{3.1}$$

where the ideal I annihilates itself, in other words, any operation $\mu \in O(n)$ is 0 in A as soon as two elements of I are input. The extension in 3.1 is *small* when the ideal I carries the trivial module structure, ie, when an operation $\mu \in O(n)$ is 0 in A as soon as an element of the ideal I is input. A small extension is *infinitesimally small* when it is small and I is 1-dimensional. The extension in 3.1 is *acyclic* when the map $A \to B$ is a weak equivalence, or equivalently, when $H_n(I) = 0$ for every $n \in \mathbb{Z}$.

Lemma 3.1.4 (Small Extension)**.** *If $A \to B$ is a surjective map of local Artin O algebras, then it is the composite of a finite sequence of small extensions.*

Proof. It is sufficient to prove that if f is not already a small extension then we can factor the map f as a small extension with nontrivial kernel e followed by a surjective map g:

$$\begin{tikzcd} & C \arrow[d, two heads, "g"] \\ A \arrow[ur, "e"] \arrow[r, two heads, "f"] & B \end{tikzcd}$$

In order to find this extension, we take a nonzero element of the kernel of f, x_0. If x_0 annihilates A, then we can quotient by x_0 to obtain our factorization of f into a small extension followed by the quotient of the remainder of the kernel. If x_0 does not annihilate A, then there exists a nonzero element $x_1 \equiv \mu(a_1, a_2, \ldots, x_0, \ldots, a_n) \in \ker(f)$, and we can ask the same question about x_1: does it annihilate A? If so, we have what we want, and if not, this means there exists a nonzero element $x_2 \equiv \mu(b_1, b_2, \ldots, x_1, \ldots, b_m) \in \ker(f)$. Continuing in this way, we have a sequence of elements $x_0, x_1, x_2, \ldots$ that are obtained by successive applications of operations in Com, and as we have assumed A is nilpotent, we conclude that eventually this sequence must be identically 0, and the last nonzero element in the sequence x_N is in the kernel of f and annihilates A.

We point out that with a little more care for the differential in A, one can decompose f as a sequence of infinitesimally small extensions. More explicitly, if $d_A x_N = 0$, then the differential ideal generated by x_N is one dimensional, and the small extension obtained is infinitesimally small. If not, then $d_A x_N \neq 0$ is an element of the kernel which annihilates all of A and the differential ideal it generates is one dimensional, so the extension obtained by the quotient is infinitesimally small. ☐

Corollary 3.1.5. *Every local Artin algebra is obtained by a finite sequence of infinitesimally small extensions of the* 0 *algebra.*

Proof. The augmentation of O guarantees that there is a 0 algebra, and again, the map to the 0 algebra is surjective. ☐

The following corollary describes the induction we're after, often called *Artinian induction*.

Corollary 3.1.6. *Let* $P(A)$ *denote a property of the local Artin* O *algebra* A*, and suppose that*

1. *the statement* $P(0)$ *is true,*
2. *whenever the statement* $P(B)$ *is true and* $A \xrightarrow{\alpha} B$ *is a small extension then* $P(A)$ *is true,*

then $P(A)$ *is true for every local Artin algebra* A*.*

Proof. Follows from induction on the dimension of the algebras. ☐

3.2 Art$_O$ and its homotopy theory

In order to study derived deformation functors, we need to equip the category Art$_O$ with the structure of a higher category: we want to define mapping spaces Art$_O(A, B)$. There is a naïve choice, namely the mapping spaces $\mathrm{Alg}_O(A, B)_\bullet$ defined in 2.2.8, but we know that these spaces are only homotopy invariant when A is cofibrant. The characterization of cofibrations in 2.2.7 implies that there are no cofibrant local Artin O algebras besides 0 when O does not have finite *total* dimension.

We first describe the general process of obtaining mapping spaces on subcategories without good resolutions.

Definition 3.2.1. *Let C_0 be a collection of objects of M, and let W be a class of weak equivalences of M. We denote by sub(C, M, W) the full SSet-subcategory of the localization of M by W, ie SSet-category defined by*

$$sub(C, M, W)(c_0, c_1) = L(M, W)(c_0, c_1).$$

Proposition 3.2.2. *Suppose C is a full subcategory of a model category M which is closed under weak equivalences, ie, $c \in C, Y \overset{\sim}{\leftarrow} c \overset{\sim}{\rightarrow} X \Rightarrow X, Y \in C$. Suppose further that M is equipped with a (tensored and/or cotensored) SSet-structure satisfying 2.1. Then the category C with weak equivalences $W \cup C$ is* *a model for sub(C, M, W), and both are modeled by the restriction of the SSet-structure of M to C.*

Proof. Because C is closed under weak equivalences in M, we know that fibrant and cofibrant replacements exist in C, and that

$$\begin{aligned} C(QA, RX)_\bullet &= M(QA, RX)_\bullet \\ &\xrightarrow{\sim} \mathrm{L}(M, W)(QA, RX) \\ &= \mathrm{sub}(C, M, W)(QA, RX). \end{aligned}$$

The space $C(QA, RX)_\bullet$ models $\mathrm{L}(C, C \cap W)$ by the arguments of [DK80, Dug06] applied to the C via the model category M. □

Remark 4. *Our definition of local Artin algebras as strictly nilpotent and finite dimensional means that Art$_O$ is not closed under weak equivalences in Alg$_O$ and so we cannot use 3.2.2 to obtain alternate models of Art$_O$. This will be a problem in 4.2.7, where we need the fact that $($Art$_O, \{$quasi-isomorphisms$\})$ is a model for sub$($Art$_O$, Alg$_O, \{$quasi-isomorphisms$\})$. We are working on a definition of Art$_O$ (following an idea of [Lur11]) that*

1. *is closed under weak equivalences*

2. *satisfies a (homotopical) version of the small extension lemma 3.1.4*

3. *still yields under the computational techniques of 8.*

3.2.1 A model for the duality functor

Fix a Koszul pair $C \overset{\alpha}{\to} O$ as in Appendix A.

Definition 3.2.3. *For any algebra $A \in Alg_O$, the coalgebra $B_\alpha A$ is weight graded by the arity of the cooperad C. We define $\mathbb{D}A$ to be the weight-graded-wise linear dual of $B_\alpha A$.*

We observe that $(B_\alpha A)^\vee$ is an $O^!$-algebra, so that $\mathbb{D}$ is a functor from $\mathrm{Alg}_O{}^{\mathrm{op}}$ to $\mathrm{Alg}_{O^!}$.

Definition 3.2.4. *Let C and D be categories, each equipped with subcategories of weak equivalences. A functor $C \to D$ is* homotopical *if it carries weak equivalences in C to weak equivalences in D.*

Proposition 3.2.5. *Homotopical functors $(C, W) \to (D, W)$ induce SSet-functors on the simplicial localizations $L(C, W) \to L(D, W)$.*

Proof. Hammocks of weak equivalences in C are sent to hammocks in D under a homotopical functor. Functoriality implies that the face and degeneracy maps are preserved. □

Proposition 3.2.6. *The functor $\mathbb{D} : Alg_O{}^{op} \to Alg_{O^!}$ is homotopical.*

Proof. By [Val12] B_α takes quasi-isomorphisms to weak equivalences, and since weak equivalences are always quasi-isomorphisms of coalgebras, they are preserved by the linear dual functor. $\square$

Corollary 3.2.7. *When restricted to the subcategory Art_O, the duality functor is naturally isomorphic to the composite*

$$Art_O{}^{op} \xrightarrow{(-)^\vee} coAlg_{O^\vee} \xrightarrow{\Omega_{\bar{\alpha}}} Alg_{C^\vee}$$

Proof. There is a natural linear map $A^\vee \to (B_\alpha A)^\vee$ which extends to a dg-algebra isomorphism $\Omega_{\bar{\alpha}} A^\vee \to (B_\alpha A)^\vee$. $\square$

Proposition 3.2.8. *The functor $\mathbb{D} : Art_O{}^{op} \to Alg_{O^!}$ has the following properties:*

1. *For every small extension $A \to C$, $\mathbb{D}C \to \mathbb{D}A$ is a cofibration.*
2. $\mathbb{D}(0) = 0$.
3. $\mathbb{D}(A \times_C B) = \mathbb{D}A *_{\mathbb{D}C} \mathbb{D}B$.
4. *The induced functor $\mathbb{D} : sub(Art_O, Alg_O, qi) \to L(Alg_{O^!}, qi)$ is homotopically fully faithful.*

Proof. The image of objects under $\Omega_{\bar{\alpha}}$ are cofibrant, the image of small extensions under $(-)^\vee$ are injections of coalgebras, and the image of injections under $\Omega_{\bar{\alpha}}$ are cofibrations. The fact that 0 is carried to 0 follows from direct observation, and pullbacks go to pushouts because $(-)^\vee$ carries pullbacks in algebras to pushouts of coalgebras, and $\Omega_{\bar{\alpha}}$ preserves all colimits as it is a left adjoint.

To see that $\mathbb{D}$ is homotopically fully faithful, recall that $B_\alpha : \mathrm{sub}(\mathrm{Art}_O, \mathrm{Alg}_O, \mathrm{qi}) = \mathrm{L}(\mathrm{Alg}_O, \mathrm{qi}) \xrightarrow{\sim} \mathrm{L}(\mathrm{coAlg}_C, W)$ because it is part of a Quillen equivalence. To see that the linear dual functor is homotopically fully-faithful on the image of Art_O under B_α, we use the SSet-enriched mapping space $\mathrm{coAlg}_C(B_\alpha A, B_\alpha C)_\bullet$ and observe

$$\begin{aligned}\mathrm{coAlg}_C(B_\alpha A \otimes C_*(\Delta^\bullet), B_\alpha C) &\xrightarrow{(-)^\vee} \mathrm{Alg}_{O^!}(\mathbb{D}C, (B_\alpha A \otimes C_*(\Delta^\bullet))^\vee) \\ &\xleftarrow{\cong} \mathrm{Alg}_{O^!}(\mathbb{D}C, (B_\alpha A)^\vee \otimes (C_*(\Delta^\bullet))^\vee) \\ &\xrightarrow{\sim} \mathrm{L}(\mathrm{Alg}_{O^!}, \mathrm{qi})(\mathbb{D}C, \mathbb{D}A \otimes (C_*(\Delta^\bullet))^\vee).\end{aligned}$$

The last equivalence follows from the fact that $\mathbb{D}C$ is cofibrant and the fact that $- \otimes (C_*(\Delta^\bullet))^\vee$ is a simplicial framing on the model category $\mathrm{Alg}_{O^!}$, which means the simplicial set of maps from $\mathbb{D}C$ to $\mathbb{D}A \otimes (C_*(\Delta^\bullet))^\vee$ models the mapping space in the simplicial localization. [Hir02] □

Chapter 4

Universal Homotopy Theories

Before tackling universal homotopy theories, we first collect results about the analogous ideas in classical category theory. We extend these results to the ∞-categorical context by extending to the SSet-enriched context, where we leverage a well behaved SSet-enrichment and its relation to the model structure to realize the analogous ∞-categorical constructions. The big picture and the technical details are contained in more detail in [Dug01].

4.1 Categories of gluings

In this section we describe in what way the category of functors $\mathrm{Fun}(C^{\mathrm{op}}, \text{Set})$ is the category of abstract gluings of objects in the category C.

4.1.1 Singular functors

Given any functor $C \xrightarrow{D} M$, there is an induced functor

$$M \xrightarrow{\text{Sing}_D} \text{Set}^{C^{op}}$$

$$L \mapsto M\left(D(-), L\right).$$

and a natural transformation $\eta_D : y(-) \to \text{Sing}(D(-))$ making the diagram of categories

$$\begin{tikzcd}
C \arrow[r, "y"] \arrow[dr, "D"'] & \text{Set}^{C^{\text{op}}} \\
& M \arrow[u, "\text{Sing}_D"']
\end{tikzcd}$$

commute, where $y : C \to \text{Set}^{C^{\text{op}}}$ is the Yoneda embedding: $y(A) = C(-, A)$. The natural transformation η_D is given by

$$\eta_D(-) : y(A)(-) \cong C(-, A) \xrightarrow{D} M(D-, DA) \cong \text{Sing}_D(DA)(-) \qquad (4.1)$$

with naturality of η_D given by functoriality of D. We regard the functor Sing_D as giving the category M "as seen by the category C through D." This perspective is inspired by the "functor of points" perspective, provided by the Yoneda Lemma: an object L is uniquely determined by the functor $M(-, L) : M^{op} \to \text{Set}$, and a map $f : L \to K$ is uniquely determined by the natural transformation $M(-, L) \to M(-, K)$. The functor of points construction is realized as Sing of the functor id_M.

If $C \xrightarrow{D} M$ is fully faithful, then each of the components of η is an isomorphism (4.1), so η is a natural equivalence. In such a case, we can consider $L \in M$ where $L \ncong K$ for any K in C, and we can ask: how much does $\mathrm{Sing}(L)$ remember about L? In good cases, the surprising answer is: everything.

4.1.2 Realization Functors

We'll set up some categorical machinery for obtaining an adjoint to Sing, and then use the adjointness to analyze the properties of Sing.

Given any functor $F \in \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})$, we have an indexing category, $y \downarrow F$, the objects of which are representable functors with a map to F, the morphisms of which are maps of representable functors making the diagrams commute. This indexing category comes with a natural map to the category $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})$ by

$$(y(X) \to F) \mapsto y(X).$$

This gives a diagram in the category $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})$, and the diagram comes with a natural map to the functor F.

Proposition 4.1.1. *For any functor $F \in \mathrm{Fun}(C^{op}, Set)$, the map*

$$\left(\varinjlim_{y(X) \to F \in y \downarrow F} y(X) \right) \to F$$

induced by the map from the diagram to F is an isomorphism. Moreover, the assignment that takes F to the diagram in $\mathrm{Fun}(C^{op}, Set)$ is functorial in F.

Proof. The fact that the map from the colimit to F is an isomorphism is equivalent to the statement that the natural map from the diagram to F realizes F as the colimit of the diagram. This follows from a diagram chase and applications of the Yoneda Lemma. The diagram is functorial because maps $F \xrightarrow{\theta} G$ allow one to push-forward maps $y(X) \xrightarrow{\alpha} F$ to $y(X) \xrightarrow{\theta\alpha} G$. □

Because every functor can be decomposed into a "natural" diagram of representable functors of which it is the colimit, if we have a functor from $C \xrightarrow{D} M$, we can attempt to extend it to a functor $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set}) \to M$ by first decomposing the functor F into the canonical diagram of representable functors, and then applying the functor D to each piece. The only thing required is that M be equipped with a way to put the diagram back together.

Proposition 4.1.2. *Let $C \xrightarrow{D} M$ be a functor, and $Sing_D$ be as in 4.1.1. We obtain the solid part of the diagram*

$$\begin{tikzcd}
C \arrow[r, "y"] \arrow[dr, "D"'] & \mathrm{Fun}(C^{op}, Set) \arrow[d, dotted, shift left, "Re_D"] \\
 & M. \arrow[u, shift left, "Sing_D"]
\end{tikzcd}$$

Suppose further that small colimits exist in M; then the dotted arrow exists, makes the triangle commute, and is the unique colimit preserving functor to do so. Moreover, it is left adjoint to $Sing_D$.

Proof. Any two functors that make the triangle commute and commute with colimits must agree by 4.1.1. If it exists, Re_D must be defined as described: we break F into its diagram of representables, apply D to each piece of the diagram, and use the colimit in M to reassemble the pieces:

$$\mathrm{Re}_D(F) \equiv \varinjlim_{y(X)\to F \in y\downarrow F} DX.$$

Because the diagram $y \downarrow F$ is functorial in F, this construction is functorial as well. To see that $\mathrm{Re}_D \circ y \cong D$, observe that $y \downarrow y(Y)$ has an initial object, namely $y(Y) \xrightarrow{\mathrm{id}_Y} y(Y)$, so the colimit

$$\varinjlim_{y(X)\to F \in y\downarrow y(Y)} DX = \mathrm{Re}_D y(Y)$$

is given by the diagram evaluated at the initial object, namely DY.

Finally, to show that Re_D does in fact commute with all colimits, it is sufficient to show that it is the left adjoint in an adjunction. To see that it is left adjoint to

Sing_D, we use the string of identities:

$$\begin{aligned}
M(\mathrm{Re}_D F, L) &= M(\varinjlim_{y(X)\to F\in y\downarrow F} DX, L) && (4.2)\\
&\cong \varprojlim_{y(X)\to F\in y\downarrow F} M\left(DX, L\right) && (4.3)\\
&\cong \varprojlim_{y(X)\to F\in y\downarrow F} \mathrm{Sing}_D(L)(X) && (4.4)\\
&\cong \varprojlim_{y(X)\to F\in y\downarrow F} \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})(y(X), \mathrm{Sing}_D(L)) && (4.5)\\
&\cong \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})(\varinjlim_{y(X)\to F\in y\downarrow F} y(X), \mathrm{Sing}_D L) && (4.6)\\
&\cong \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})(F, \mathrm{Sing}_D(L)). && (4.7)
\end{aligned}$$

where 4.2 is the definition of Re_D, 4.3 follows from the universal property of the colimit, 4.4 is the definition of Sing_D, 4.5 follows from the Yoneda lemma, 4.6 follows from the universal property of the colimit, and 4.7 follows from 4.1.1. □

We ignore issues of size to offer the following interpretation of 4.1.2.

Proposition 4.1.3. *Consider the category of categories, Cat, and the category of categories with all small colimits, coComplete (where functors between such categories are colimit preserving functors). The forgetful functor coComplete* $\xrightarrow{U}$ *Cat has a left adjoint and it is given by* $C \mapsto \mathrm{Fun}(C^{op}, \mathit{Set})$. *The unit of the adjunction is the Yoneda embedding,* $C \hookrightarrow \mathrm{Fun}(C^{op}, \mathit{Set})$.

Proof. This is a restatement of 4.1.2. □

This is the sense in which $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})$ is describing "abstract gluings of objects in C"—it is describing free gluings, or colimits, in C.

4.1.3 Reformulation in terms of functor tensor products

The previous arguments do not generalize so nicely to the enriched case, so we reformulate the discussion in terms of functor tensor product which do generalize nicely. Our reference for all the material in this section, and for enriched category theory related to homotopy theory is [Rie12].

Proposition 4.1.4. *If M has coproducts, then M is* tensored *over Set, with the tensoring given by*

$$M \times \mathit{Set} \dot{\to} M$$

$$L, S \mapsto \coprod_{s \in S} L.$$

Proof. The universal property of the coproduct implies that

$$M(\coprod_{s \in S} L, L') = \mathrm{Set}(S, M(L, L'))$$

as required. ☐

Definition 4.1.5. *Given a functor $F : C^{op} \to$ Set and a functor $D : C \to M$, the* functor tensor product of F and γ over C *is the coend*

$$\int^{c \in C} Fc \cdot Dc.$$

We denote the functor tensor product by $F \otimes_C D$.

Remark 5. *We recall that the coend of a functor* $H : C^{op} \times C \to M$ *is an object* $\int^{c\in C} H(c,c)$ *in* M *with maps* $H(c,c) \xrightarrow{u_c} \int^{c\in C} H(c,c)$ *so that for every map* $c \xrightarrow{f} c' \in C$ *the diagram*

$$\begin{tikzcd}
H(c',c) \arrow[r, "H(f,1)"] \arrow[d, "H(1,f)"'] & H(c,c) \arrow[d, "u_c"] \\
H(c',c') \arrow[r, "u_{c'}"'] & \int^{c\in C} H(c,c)
\end{tikzcd}$$

commutes, and $\int^{c\in C} H(c,c)$ *is initial in the category of objects of* M *equipped with maps from* $H(c,c)$ *which make the diagrams commute.*

If M *has colimits, the coend can be described by the "point-wise" formula*

$$\textit{coeq}\left(\coprod_{c',c} C(c,c') \cdot H(c',c) \rightrightarrows \coprod_{c} H(c,c)\right) \tag{4.8}$$

where one of the arrows maps factor $H(c',c)$ *labeled by* $f \in C(c,c')$ *to* $H(c',c')$ *by pushing forward by* f *in the second coordinate of* $H(c',c)$*, and the other arrow pulls back along the first coordinate of* $H(c',c)$.

Proposition 4.1.6. *For a representable functor,* $y(X) : C^{op} \to$ *Set and a functor* $D : C \to M$*, the functor tensor product* $y(X) \otimes_C D$ *is naturally isomorphic to* $D(X)$.

Proof. This is the co-Yoneda lemma. $\square$

Proposition 4.1.7. *Given a functor $D : C \to M$ where M has colimits, the functor $- \otimes_C D : Set^{C^{op}} \to M$ preserves colimits.*

Proof. This follows from the point-wise formula of the functor tensor product 4.8 and the fact that colimits commute with colimits. $\square$

Corollary 4.1.8. *The functor tensor product with D, $- \otimes_C F$, and the realization with respect to D, Re_D, coincide.*

Proof. By 4.1.6 and 4.1.7, the functor $- \otimes_C D$ satisfies the properties that, according to 4.1.2, uniquely define Re_D. $\square$

4.2 Categories of homotopical gluings

In the previous section, we described how functors C^{op} to Set were the free colimit complete category on C. In this section, we walk through a similar argument to show that $\mathrm{SSet}^{C^{\mathrm{op}}}$ is the free homotopy-colimit complete category on the SSet-category C. In [Dug01], Dugger proves that the ∞-category of homotopical gluings of objects in a category C is modeled by the model category of functors $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{SSet})$. In this section we extend his constructions to the case that C has an SSet-enrichment, in which case the ∞-category of homotopical gluings is modeled by the model category of SSet-functors $\mathrm{SSet}^{C^{\mathrm{op}}}$.

4.2.1 Tensored SSet-categories

Definition 4.2.1. *An SSet-category M is* tensored *if for every object $A \in \operatorname{ob} M$ and every simplicial set K, the functor*

$$B \mapsto M(A, B)^K$$

is corepresentable. In that case, call the corepresenting object for the functor $A \otimes K$. The assignment

$$\operatorname{ob} M \times \operatorname{ob} \textit{SSet} \to \operatorname{ob} M$$

$$(A, K) \mapsto A \otimes K$$

naturally extends to a functor by identification with the functorial assignment that defines it,

$$M^{op} \times \textit{SSet}^{op} \to \left(\textit{SSet}^{M}\right)^{op}$$

$$(A, K) \mapsto M(A, -)^K$$

Remark 6. *Because the functor which defines the tensoring of M preserves colimits in the SSet variable, and every simplicial set is the canonical colimit of n-simplices 4.1.1, it is sufficient to verify that the functors $M(A, -)^{\Delta^n}$ are representable for every $n \geq 0$, or in other words, to provide the objects $A \otimes \Delta^n$ for every $n \geq 0$.*

Example 4.2.2. *The category SSet with the standard mapping spaces is tensored over itself by the functor* $K, \Delta^n \mapsto K \times \Delta^n$.

Example 4.2.3. *The category* $coAlg_C$ *with the mapping spaces defined in 2.3.8 is tensored over itself by the functor* $B, \Delta^n \mapsto B \otimes C_*(\Delta^n)$.

4.2.2 Simplicial realization functors

Proposition 4.2.4. *Let* C, M *be SSet categories, and* $\mathbb{D} : C \to M$ *an SSet-functor. There is an induced SSet-functor* $Sing_{\mathbb{D}} : M \to SSet^{C^{op}}$ *given by*

$$L \mapsto M(\mathbb{D}-, L).$$

Proof. Composites of SSet-functors are SSet-functors. □

Proposition 4.2.5. *Let* $C \xrightarrow{\mathbb{D}} M$ *be an SSet-functor, and* $Sing_{\mathbb{D}}$ *be as in 4.2.4. We obtain the solid part of the diagram*

$$\begin{tikzcd}
C \arrow[r, "y"] \arrow[dr, "\mathbb{D}"'] & \mathrm{Fun}(C^{op}, Set) \arrow[d, dotted, shift left, "Re_{\mathbb{D}}"] \\
& M. \arrow[u, shift left, "Sing_{\mathbb{D}}"]
\end{tikzcd}$$

Suppose further that colimits exist in the underlying category of M *and that* M *is tensored over SSet. Then the dotted arrow exists makes the triangle commute. Moreover, it is left adjoint to* $Sing_{\mathbb{D}}$.

Proof. We define the functor $\mathrm{Re}_{\mathbb{D}} F$ by the *enriched* functor tensor product [Rie12, Chapter 7] $F \otimes_{\underline{C}} \mathbb{D}$. Since M is tensored over SSet, the enriched The proof of adjointness with $\mathrm{Sing}_{\mathbb{D}}$ follows from the point-wise formula for enriched coends [Rie12]:

$$\begin{aligned} M(\mathrm{Re}_{\mathbb{D}} F, L) &= M(F \otimes_{\underline{C}} \mathbb{D}, L) \\ &= \mathrm{SSet}^{\underline{C}^{\mathrm{op}}}(F, \underline{M}(\mathbb{D}, L)) \\ &= \mathrm{SSet}^{\underline{C}^{\mathrm{op}}}(F, \mathrm{Sing}_{\mathbb{D}}(L)). \end{aligned}$$

□

Remark 7. *We point out that we have not proved that the functors are in SSet-adjunction with each other, just in classical adjunction.*

Corollary 4.2.6. *If M is a model category with an SSet-enrichment which is cotensored, so that the enrichment satisfies 2.3, $C \xrightarrow{\mathbb{D}} M$ is an SSet-functor with values in the cofibrant objects of M, then the adjunction $Re_{\mathbb{D}} \dashv Sing_{\mathbb{D}}$ is a Quillen pair.*

Proof. The condition 2.3 applies to $\mathrm{Sing}_{\mathbb{D}}(L)$ because $\mathbb{D}$ takes values in cofibrant objects, and implies that the right adjoint, $\mathrm{Sing}_{\mathbb{D}}$, preserves fibrations and weak equivalences between fibrations. □

Remark 8. *The derived functor of a left Quillen functor Re is computed by applying Re to a cofibrant replacement functor Q. In our case, the category $SSet^{C^{op}}$ has a cofibrant replacement functor given by a taking a canonical cosimplicial resolution by representables, analogous to 4.1.2, see [Dug01, Proposition 2.9]. Taking the geometric realization of this cosimplicial resolution gives a cofibrant replacement of F; we denote the cosimplicial resolution by $\mathcal{Q}$, and the geometric realization of $\mathcal{Q}F$ by $|\mathcal{Q}F|$. Because coends commute with*

4.2.3 A simplicial model for the duality functor

Proposition 4.2.7. *We can endow the category $Art_O{}^{op}$ with the SSet-structure of $coAlg_{O^\vee}$ by*

$$\underline{Art_O}^{op}(B,C) = coAlg_{O^\vee}(B^\vee, C^\vee).$$

Remark 9. *This construction does not endow Art_O with an SSet-structure which is equivalent to $sub(Art_O, Alg_O, qi)$. However, we expect that when we adjust the definition of Art_O to be homotopy invariant, these constructions will coincide.*

Corollary 4.2.8. *The definition implies that $(-)^\vee : \underline{Art_O}^{op} \to coAlg_{O^\vee}$ is an SSet-functor.*

Corollary 4.2.9. *The functor $(-)^\vee$ induces a Quillen adjunction*

$$\begin{tikzcd} SSet^{\underline{Art_O}} \arrow[r, shift left, "Re_{(-)^\vee}"] & coAlg_{O^\vee} \arrow[l, shift left, "Sing_{(-)^\vee}"] \end{tikzcd}$$

Definition 4.2.10. *We define the main Quillen adjunction that we'll use as the composite of the Quillen adjunction above with the bar cobar adjunction:*

$$\mathit{Re}_{\mathbb{D}} \equiv \Omega_{\bar{\alpha}} \circ \mathit{Re}_{(-)^{\vee}}$$

$$\mathit{Sing}_{\mathbb{D}} \equiv \mathit{Sing}_{(-)^{\vee}} \circ B_{\bar{\alpha}}.$$

Chapter 5

Deformation Functors

5.1 Derived deformation functors

Definition 5.1.1. *An SSet-functor* $F \in \mathit{SSet}^{\underline{Art_O}}$ *is a* derived O deformation functor *iff*

1. *F is homotopical.*
2. *$F(0)$ is weakly contractible.*
3. *For diagrams*

$$\begin{tikzcd} & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"'] & C \end{tikzcd}$$

   the natural map $F(A \times_C B) \to FA \times^h_{FC} FB$ is a weak equivalence when either α or β is small.

4. *There is a cofibrant replacement of F in $\mathit{SSet}^{\underline{Art_O}}$, $\mathbf{Q}F$, and a fibrant replace-*

ment of Re$_{(-)^\vee}(\mathbf{Q}F)$ *in coAlg*$_{O^\vee}$*, so that the derived unit of the adjunction*

$$\mathbf{Q}F(k^i) \xrightarrow{Sing(\mathbf{R})\circ\eta_{\mathbf{Q}F}} Sing\mathbf{R}Re\mathbf{Q}F(k^i)$$

is a π_0*-equivalence of spaces for every* $i \in \mathbb{Z}$*.*

Remark 10. *The last condition is not truly an analogue of Schlessinger's criteria, and we hope to be able to prove in future work that it follows from the other axioms.*

Definition 5.1.2. *The* $(\infty, 1)$-category of derived deformation functors *is the full sub-quasicategory of* $\mathscr{P}\left((\mathscr{A}_O^{sm})^{op}\right)$ *spanned by the derived deformation functors, and we denote it by* $\mathscr{M}_O$*.*

5.1.1 First consequences

Some Schlessinger-type theorems in the derived setting are stated.

Proposition 5.1.3. *For any derived deformation functor* F*, and any diagram*

$$\begin{tikzcd} & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"'] & C \end{tikzcd}$$

with either α *or* β *surjective, the natural map* $F(A \times_C B) \to FA \times^h_{FC} FB$ *is a weak-equivalence equivalence.*

Proof. By Lemma 3.1.4, any surjective map of Artin algebras is a composition of a finite sequence of small extensions. The result follows from the fact that pullbacks and homotopy pullbacks can be computed iteratively. □

For any Artin algebras A and B, the maps to $C = 0$ are surjective, and hence

Proposition 5.1.4. *Derived deformation functors are weakly product-preserving. That is, for any deformation functor F, we have that the map*

$$F(A \times B) \to FA \times FB$$

is a weak equivalence.

Proof. By assumption $F(A \times B) \xrightarrow{\sim} FA \times^{\mathrm{h}} FB$ and since for any simplicial sets X, Y, the natural map $X \times Y \xrightarrow{\sim} X \times^{\mathrm{h}} Y$, the result follows from the 2-out-of-3 property of weak equivalences. □

Proposition 5.1.5. *A derived O-deformation functor carries small extensions to fiber sequences.*

Proof. If the extension $0 \to I \xrightarrow{i} A \xrightarrow{\alpha} B \to 0$ is small, then the pullback diagram

$$\begin{tikzcd}
I \arrow[r, "i"] \arrow[d] & A \arrow[d, "\alpha"] \\
0 \arrow[r] & B
\end{tikzcd}$$

is preserved by derived O-deformation functors, but the statement that FI is the homotopy pullback of the map $FA \to FB$ along the map $F(0) \to FB$, is precisely the statement that FI is the homotopy fiber of the map $FA \to FB$, since $F(0)$ is contractible. $\square$

5.1.2 Tangent complexes

The following fact about the category of finite dimensional vector spaces motivates the ideas in this section.

Proposition 5.1.6. *A product preserving functor F from finite dimensional vector spaces to sets is representable by a natural vector space structure on its value on the ground field.*

Proof. Because F preserves products, the structure maps

$$k \times k \xrightarrow{+} k$$
$$k \xrightarrow{\cdot\lambda} k$$

are carried to maps

$$F(k) \times F(k) \xrightarrow{+} F(k)$$
$$F(k) \xrightarrow{\cdot\lambda} F(k)$$

which, by functoriality, satisfy the axioms of a vector space. The natural map $F(V) \times V^\vee \to F(k)$ gives rise, by adjointness, to the map

$$\text{Vect}\left((-)^\vee, F(k)\right) \to F(-).$$

To see this map is an isomorphism, choose a basis for V and use the fact that F preserves products. □

In 5.1.4 we saw that derived deformation functors are weakly product-preserving, and by restricting to the full subcategory of finite dimensional chain complexes inside Art_O, we obtain a weakly product-preserving functor on finite chain complexes. Morally, these should be homotopy-representable by some version of a vector space up to homotopy. In this section, we make this result precise, and provide two models for the representing object as an up-to-homotopy vector space: as a k-module spectrum, and as a cochain complex over k.

In this section, F is a derived O deformation functor.

Proposition 5.1.7. *The sequence of spaces* $\{F(k^0), F(k^{-1}), F(k^{-2}), \ldots\}$ *form an* Ω*-spectrum.*

Proof. Recall that an Ω-spectrum is a sequence of spaces $\{X_0, X_1, X_2, \ldots\}$ with structure maps $X_i \xrightarrow{\sim} \Omega_* X_{i+1}$. As the based loop space of X is obtained as the homotopy pullback of the diagram

$$\begin{tikzcd} & * \arrow[d] \\ * \arrow[r] & X \end{tikzcd}$$

and as the homotopy pullback is a homotopy functor on diagrams of spaces, we can obtain a model for $\Omega_* X$ by the homotopy pullback

$$\begin{tikzcd} & C_1 \arrow[d] \\ C_0 \arrow[r] & X \end{tikzcd}$$

where both C_0, C_1 are contractible. This in turn means we can define an Ω-spectrum by giving a sequence of spaces $X_{i,j}$ and maps $X_{i,j} \to X_{i-1,j}$ and $X_{i,j} \to X_{i,j-1}$ so that $X_{i,i-1} \xrightarrow{\sim} * \xleftarrow{\sim} X_{i-1,i}$ and the squares

$$\begin{tikzcd} X_{i,i} \arrow[r] \arrow[d] & X_{i+1,i} \arrow[d] \\ X_{i,i+1} \arrow[r] & X_{i+1,i+1} \end{tikzcd}$$

commute and are homotopy pullback squares. To this end, we define the functor $\Sigma^\infty(k) : (\mathbb{N} \times \mathbb{N}) \to \mathrm{Art}_O$

$$(n, m) \mapsto \begin{cases} 0 & \text{if } i < j, \\ k^{-i} & \text{if } i = j, \\ \mathrm{coCo}\,(\Sigma^\infty(k)(i-1, j)) & \text{if } i > j. \end{cases}$$

The off-diagonals are either 0 or the cocone on an object and so are contractible. The squares lying on the diagonals

$$\begin{tikzcd} k^{-i} \arrow[r] \arrow[d] & \mathrm{coCo}\, k^{-i+1} \arrow[d] \\ 0 \arrow[r] & k^{-i+1} \end{tikzcd}$$

are pullbacks with the maps $\mathrm{coCo}\,(k^{-i+1}) \to k^{-i+1}$ small. So applying the derived deformation functor F to the diagram $\Sigma^\infty(k)$ yields a spectrum-object in SSet, or an Ω-spectrum, with the spaces given by $F\Sigma^\infty(k)(i,i) = F(k^{-i})$. □

Definition 5.1.8 (Tangent space I)**.** *We call the spectrum* $\Omega_* F\Sigma^\infty(k)$ *the* tangent spectrum to F *and we denote it by* TF.

The following proposition offers a more tractable view of the spectrum $F\Sigma^\infty(k)$:

Proposition 5.1.9. *For any derived deformation functor, the homotopy groups* $\pi_* F\Sigma^\infty(k)$ *are naturally vector spaces over* k.

Proof. The construction above defines an assignment $V \mapsto \pi_* F\Sigma^\infty V$, which is a product preserving functor from the category of vector spaces to graded sets, and is therefore representable by a natural (graded) vector space structure on its value on the ground field. □

Definition 5.1.10 (Tangent space II)**.** *We define* the tangent space of F *to be the graded vector space* TF *where*

$$\begin{aligned} TF^i &= \pi_{-i}\Omega_* F\Sigma^\infty(k) \\ &= \varinjlim_j \pi_{-i+1+j} F\Sigma^\infty(k^{-j}). \end{aligned}$$

We denote is by TF.

Remark 11. *Although we now have one symbol, TF, for for the tangent spectrum of F and the tangent space of F, we find this acceptable for the following reason. The $(\infty, 1)$-category of k-module spectra (sequences of spaces with structure maps along with an action of the field k) is equivalent to the $(\infty, 1)$-category of $\mathbb{Z}$-graded chain complexes over the field k [Rob87], which is turn is equivalent to $\mathbb{Z}$-graded cochain complexes over k by reindexing. Under the composite of these equivalences, the tangent spectrum of F is carried to the tangent space of F.*

Proposition 5.1.11. *The construction T is a homotopy functor from deformation problems to graded vector spaces; given a natural transformation $\theta : F \to G$ we write $d\theta : TF \to TG$ for the induced map.*

Proof. An equivalence of functors induces an equivalence of the diagrams defining the tangent spectra. □

5.2 Singular functors are deformation functors

We prove that the image of the singular functor, the functors $\mathrm{Sing}(L)$, are derived deformation functors. We use the following lemmas.

Proposition 5.2.1. *The 0 algebra is ∞ initial and terminal in Alg_O.*

Proof. We observe that 0 is cofibrant, and for any cofibrant replacement functor Q, the mapping spaces

$$\mathrm{Alg}_O(0,Y)_\bullet \cong_{\mathrm{SSet}} \{0\}$$

$$\mathrm{Alg}_O(QA,0)_\bullet \cong_{\mathrm{SSet}} \{0\}$$

and by 2.2.10 we see that the mapping space from or to the 0 algebra is contractible. □

Proposition 5.2.2. *For any diagram*

$$\begin{tikzcd} & A \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"'] & C \end{tikzcd}$$

in Art_O *we have that* $Alg_{O^!}(\mathbb{D}(A \times_C B), R_\bullet L)$ *is the pullback of*

$$\begin{tikzcd} & Alg_{O^!}(\mathbb{D}(A), R_\bullet L)\ . \arrow[d, "\mathbb{D}(\alpha)^*"] \\ Alg_{O^!}(\mathbb{D}(B), R_\bullet L) \arrow[r, "\mathbb{D}(\beta)^*"'] & Alg_{O^!}(\mathbb{D}(C), R_\bullet L) \end{tikzcd} \tag{5.1}$$

Proof. The functor $\mathbb{D}$ carries $A \times_C B$ to $\mathbb{D}A *_{\mathbb{D}C} \mathbb{D}B$ (3.2.8), and the functor $\mathrm{Alg}_{O^!}(\mathbb{D}(-), R_n L)$ from Art_O to Set takes pushouts of algebras to pullbacks in sets. This means that $\mathrm{Alg}_{O^!}(\mathbb{D}(A \times_C B), R_\bullet L)$ is the level-wise pullback of the simplicial diagram above, and so it is the pullback. □

Theorem 5.2.3. *For any object* $L \in Alg_{O^!}$*, the functor* $Sing(L) = Alg_{O^!}(\mathbb{D}(-), L) : Art_O \to SSet$ *is a derived deformation functor.*

Proof. The functor $\mathrm{Sing}(L)$ is homotopical because $\mathbb{D}$ is homotopical, and so we obtain an SSet-functor. To see that $\mathrm{Sing}(L)(0) \simeq \{0\}$, we recall that by 3.2.8 $\mathbb{D}(0) = 0$, and by 5.2.1, $\mathrm{Alg}_{O^!}(0, L) \simeq \{0\}$.

To see that $\mathrm{Sing}(L)$ takes pullbacks along small extensions to homotopy pullbacks of simplicial sets, first recall that the image of a small extension under $\mathbb{D}$ is a cofibration 1, and by 2.1, we know that the corresponding map in 5.1 is a fibration of simplicial sets. This in turn means that the pullback of the diagram, which by 5.2.2 is the space $\mathrm{Alg}_{O^!}(\mathbb{D}(A \times_C B), R_\bullet L)$, is the homotopy pullback. $\square$

Chapter 6

Representability

6.1 Inverse Functor Theorem

According to 5.1.11, the assignment $F \mapsto TF$ is a homotopy functor. This is precisely the statement that weak equivalences of derived deformation functors give weak equivalences of tangent complexes. In this section, we prove that maps which induce equivalences on the tangent complex are in fact equivalences.

Theorem 6.1.1. *A map of derived O deformation functor $\theta : F \to G$ is an equivalence if and only if $d\theta : TF \to TG$ is an equivalence.*

The first step is to reduce to the case over the trivial operad.

Definition 6.1.2. *We call derived* id*-deformation functors* derived chain deformation functors.

Proposition 6.1.3. *Suppose $\theta : F \to G$ is a natural transformation of derived O*

deformation functors, and suppose $\theta(V) : F(V) \to G(V)$ *is a weak equivalence for any chain complex* $V \in Art_O$*. Then* $\theta : F \to G$ *is an equivalence.*

Proof. We prove this by Artinian induction (3.1.6). The base case is guaranteed by the fact that F and G are deformation functors. Now suppose $\theta : F(B) \to G(B)$ is an equivalence and

$$0 \to I \xrightarrow{i} A \xrightarrow{\alpha} B \to 0$$

is a small extension. Then applying F, G, and θ to the small extension we obtain the diagram

$$\begin{tikzcd}
FI \arrow[r] \arrow[d, "\theta_I"] & FA \arrow[r] \arrow[d, "\theta_A"] & FB \arrow[d, "\theta_B"] \\
GI \arrow[r] & GA \arrow[r] & GB
\end{tikzcd} .$$

By 5.1.5, each of the horizontal rows is a fiber sequence, and by naturality, the map θ is a map of fiber sequences. By inductive hypothesis, the map θ_B is an equivalence, and as we began by assuming that θ was an equivalence on chain complexes, and the extension is small, θ_I is an equivalence. As a map of fiber sequences, θ is an equivalence on the base and the fiber and so it is an equivalence on the total space, and θ_A is an equivalence. Therefore θ must be an equivalence on every local Artin O algebra. □

Because the restriction of a derived O deformation functor to the subcategory of chain complexes is a derived chain deformation functor, and natural transformations θ which restrict to equivalences on this subcategory must have been equivalences from the start (6.1.3), it is sufficient to prove that the tangent space construction detects equivalences on derived chain deformation functors.

6.1.1 Chain deformation functors

In this section we prove that maps of chain deformation functors which induce equivalences on tangent spaces are in fact equivalences.

Proposition 6.1.4. *Let $F \xrightarrow{\theta} G$ be a map of derived chain deformation functors which is an equivalence on each k^i. Then the map $\eta : F \to G$ is an equivalence.*

Proof. To see that each map $\theta(C) : F(C) \to G(C)$ is an equivalence, choose an equivalence $H(C) \hookrightarrow C$, and choose a basis $\{x_i\}$ for $H(C)$. Since $H(C)$ is finite dimensional, and both functors preserve products weakly, we have the diagram

$$\begin{tikzcd}
F(C) \arrow[r, "\theta_C"] & G(C) \\
F(H(C)) \arrow[u, "\simeq"] \arrow[r, "\theta_{H(C)}"] & G(H(C)) \arrow[u, "\simeq"'] \\
F(\oplus_i k^{n_i}) \arrow[u, "\cong"] \arrow[r, "\theta_{(\oplus_i k^{n_i})}"] & G(\oplus_i k^{n_i}) \arrow[u, "\cong"'] \\
\prod_i F(k^{n_i}) \arrow[u, "\simeq"] \arrow[r, "\prod_i \theta_{k^{n_i}}"] & \prod_i G(k^{n_i}). \arrow[u, "\simeq"']
\end{tikzcd}$$

By assumption the map $\prod_i \theta_{k^{n_i}}$ is a product of equivalences of spaces and so is an equivalence, and so $\theta(C)$ is an equivalence. □

Proposition 6.1.5. *A map of derived chain deformation functors $\theta : F \to G$ is an equivalence on each k^i if and only if it induces an equivalence on tangent spaces.*

Proof. The tangent space (either one) is constructed from the spaces $F(k^i)$ for i in a certain range, or from certain homotopy groups of $F(k^i)$ in a certain range. This means that an equivalence on each k^i will give rise to an equivalence of tangent spaces. To see the other direction, first recall that the square

$$\begin{tikzcd}
k^{i-1} \arrow[r] \arrow[d] & \mathrm{coCo}(k^i) \arrow[d] \\
0 \arrow[r] & k^i
\end{tikzcd}$$

is a pullback square and the map $\mathrm{coCo}(k^i) \to k^i$ is small for every $i \in \mathbb{Z}$. For any deformation functor, the square

$$\begin{tikzcd}
F(k^{i-1}) \arrow[r] \arrow[d] & F(\mathrm{coCo}(k^i)) \arrow[d] \\
F(0) \arrow[r] & F(k^i)
\end{tikzcd}$$

is a homotopy pullback square. As $F(\mathrm{coCo}(k^i)) \simeq \{0\} \simeq F(0)$, we can identify $F(k^{i-1})$ with the loop space of $F(k^i)$. This implies that the higher homotopy groups of $F(k^i)$ are identifiable with π_0 of $F(k^{i-N})$ for some $N \in \mathbb{Z}$. These are precisely the components of the tangent space TF, so a natural transformation

$F \to G$ which induces an equivalence $TF \to TG$ must induce an equivalence on each of the homotopy groups of k^i for every $i \in \mathbb{Z}$. $\square$

Remark 12. *The proof of 6.1.5 implies for a map of derived deformation functors $\theta : F \to G$ for which $F(k^i) \to G(k^i)$ is a π_0-equivalence for every $i \in \mathbb{Z}$, then $F(k^i) \to G(k^i)$ is a weak equivalence for every $i \in \mathbb{Z}$.*

Corollary 6.1.6. *If θ is a natural transformation of derived chain deformation functors, and induces an equivalence $d\theta : TF \xrightarrow{\sim} TG$, then θ is an equivalence.*

Proof. This follows from 6.1.5 and 6.1.4. $\square$

6.1.2 Representability over O

Using the results of the previous section, we can prove Theorem 6.1.1 and prove that every derived O deformation functor is representable.

Proof. (Theorem 6.1.1) If $TF \to TG$ is an equivalence, then $F \to G$ is an equivalence on $\mathrm{Ch}_k^{\mathrm{fin}} \subset \mathrm{Art}_O$ by 6.1.6. Then by 6.1.3 $F \to G$ must in fact be an equivalence. $\square$

Every derived O deformation functor is homotopy representable.

Theorem 6.1.7. *If F is a derived O deformation functor, then it is homotopy representable by Sing$\mathbf{R}$Re$\mathbf{Q}F$.*

Proof. By definition of resolution, $\mathbf{Q}F \xrightarrow{\sim} F$. By assumption, we have that $\mathbf{Q}F(k^i) \xrightarrow{\text{Sing}(\mathbf{R})\circ\eta_{\mathbf{Q}F}(k^i)} \text{Sing}\mathbf{R}\text{Re}\mathbf{Q}F(k^i)$ is a π_0-equivalence for each i, so by Remark 12, the natural transformation $\text{Sing}(\mathbf{R}) \circ \eta_{\mathbf{Q}F}$ is an equivalence, and we have

$$F \xleftarrow{\sim} \mathbf{Q}F \xrightarrow{\sim} \text{Sing}\mathbf{R}\text{Re}\mathbf{Q}F.$$

$\square$

Chapter 7

The equivalence of categories $\mathscr{A}_O \simeq \mathscr{M}_O$

Here we collect the results of the previous chapters to conclude that the ∞ category corresponding to $\mathrm{Alg}_{O^!}$ is equivalent to the ∞ category $\mathscr{M}_O$.

Theorem 7.0.8. *The functor Sing* : $Alg_{O^!} \to SSet^{Art_O}$ *is homotopically fully faithful.*

Proof. By the adjunction between Re and Sing, it is sufficient to prove that the counit $\mathrm{ReSing}L \to L$ is a quasi-isomorphism. This is done in 8.1.1. □

Corollary 7.0.9. *The associated* ∞*-functor* $\mathscr{S} : \mathscr{A}_{O^!} \to \mathscr{P}\left((\mathscr{A}_O^{sm})^{op}\right)$ *is* ∞ *fully faithful.*

Proposition 7.0.10. *The induced functor on* ∞*-categories* $\mathscr{S} : \mathscr{A}_{O^!} \to \mathscr{P}\left((\mathscr{A}_O^{sm})^{op}\right)$ *factors through the inclusion* $\mathscr{M}_O \hookrightarrow \mathscr{P}\left((\mathscr{A}_O^{sm})^{op}\right)$.

Proof. By 5.2.3, every object in the image of $\mathscr{S}$ is an object in the subcategory $\mathscr{M}_O$, and so the functor factors through $\mathscr{M}_O$. □

Theorem 7.0.11. *The induced functor on ∞-categories, $\mathscr{S} : \mathscr{A}_{O^!} \to \mathscr{M}_O$ is an equivalence of ∞-categories.*

Proof. The functor $\mathscr{S}$ is fully-faithful by 7.0.9, and is essentially surjective by 6.1.7. □

Chapter 8

Computing the homotopy colimits

Here we pay the computational debts incurred over the last few chapters.

8.1 The derived counit of the adjunction

Proposition 8.1.1. *For every $Alg_{O^!}$ algebra L the derived counit map, $Re_{\mathbb{D}}QSing_{\mathbb{D}}L \to L$ is a quasi-isomorphism.*

We'll break this proposition down into several pieces.

Proposition 8.1.2. *It is sufficient to prove that $Re_{(-)^\vee}QSing_{(-)^\vee}(B_{\bar{\alpha}}L) \to B_{\bar{\alpha}}L$ is an equivalence.*

Proof. Applying $\Omega_{\bar{\alpha}}$ to the map $\mathrm{Re}_{(-)^\vee}Q\mathrm{Sing}_{(-)^\vee}(B_{\bar{\alpha}}L) \to B_{\bar{\alpha}}L$ yields

$$\begin{tikzcd}
\mathrm{Re}_{\mathbb{D}}Q\mathrm{Sing}_{\mathbb{D}}L \arrow[r] \arrow[dr, "\varepsilon_L"'] & \Omega_{\bar{\alpha}}B_{\bar{\alpha}}L \arrow[d] \\
& L
\end{tikzcd}$$

where $\Omega_{\bar{\alpha}} B_{\bar{\alpha}} L \to L$ is the counit of the adjunction $\Omega_{\bar{\alpha}} \dashv B_{\bar{\alpha}}$, which is an equivalence. Since $\Omega_{\bar{\alpha}}$ preserves weak equivaleces between coalgebras, if $\mathrm{Re}_{(-)^\vee} Q\mathrm{Sing}_{(-)^\vee}(B_{\bar{\alpha}} L) \to B_{\bar{\alpha}} L$ were an equivalence, the counit ε_L would be as well. □

Proposition 8.1.3. *The functor* $\mathit{Re}_{(-)^\vee} Q\mathit{Sing}_{(-)^\vee}$ *is a homotopy functor on fibrant coalgebras, and admits a description of the form*

$$Y \mapsto \operatorname*{hocolim}_{\mathit{Art}_{\mathrm{O}}^{op} \times \Delta \downarrow \mathrm{Y}} A^\vee \otimes C_*(\Delta^n)$$

for any fibrant coalgebra Y*.*

Proof. Right Quillen functors preserve weak equivalences between fibrant objects, cofibrant replacements preserve weak equivalences, and left Quillen functors preserve weak equivalences between cofibrant objects. To identify the functor with the homotopy colimit, we choose Q to be the standard resolution of a functor by the geometric realization of the simplicial resolution by representables:

$$F \mapsto \begin{tikzcd} {|Q_\bullet F|} \arrow[d, "\sim"] \\ F \end{tikzcd}$$

where $Q_\bullet F$ is a simplicial object in $\mathrm{SSet}^{\mathrm{Art}_O}$, given by

$$Q_\ell F = \coprod_{\begin{tikzcd}[ampersand replacement=\&] y(A_0)\otimes\Delta^{n_0} \arrow[r] \arrow[d, "f_0\otimes\theta_0"'] \& F \\ \vdots \arrow[d, "f_{\ell-1}\otimes\theta_{\ell-1}"'] \& \\ y(A_\ell)\otimes\Delta^{n_\ell} \arrow[uur] \& \end{tikzcd}} y(A_0)\otimes\Delta^{n_0}$$

with the simplicial boundaries given by contracting morphisms in the indexing, or by applying the pushforward of the indexing map to the factor that it labels (see [Dug01, Rie12] for more detailed descriptions of this construction). For the functor $\mathrm{Sing}_{(-)^\vee}(Y)$, the Yoneda lemma and the tensored SSet structure of $\mathrm{coAlg}_{O^\vee}$ allow us to describe the resolution indexed by maps in the category $\mathrm{coAlg}_{O^\vee}$:

$$Q_\ell \mathrm{Sing}_{(-)^\vee}(Y) = \coprod_{\begin{tikzcd}[ampersand replacement=\&] A_0^\vee\otimes\Delta^{n_0} \arrow[r] \arrow[d, "f_0\otimes\theta_0"'] \& Y \\ \vdots \arrow[d, "f_{\ell-1}\otimes\theta_{\ell-1}"'] \& \\ A_\ell^\vee\otimes\Delta^{n_\ell} \arrow[uur] \& \end{tikzcd}} y(A_0)\otimes\Delta^{n_0}.$$

Then the object $\mathrm{Re}_{(-)^\vee}|Q_\bullet \mathrm{Sing}_{(-)^\vee}(Y)|$ is given by the formula

$$|Q_\bullet \mathrm{Sing}_{(-)^\vee}(Y)| \underset{\underline{\mathrm{Art}_O}^{\mathrm{op}}}{\otimes} (-)^\vee.$$

Because realizations of simplicial objects are coends, and the functor tensor product is a coend, we can commute the tensor into the realization:

$$\mathrm{Re}_{(-)^\vee}|Q_\bullet\mathrm{Sing}_{(-)^\vee}(Y)|= |Q_\bullet\mathrm{Sing}_{(-)^\vee}(Y) \underset{\underline{\mathrm{Art}_O}^{\mathrm{op}}}{\otimes} (-)^\vee|,$$

where $Q_\bullet\mathrm{Sing}_{(-)^\vee}(Y) \underset{\underline{\mathrm{Art}_O}^{\mathrm{op}}}{\otimes} (-)^\vee$ is the level-wise tensor product of the simplicial object $Q_\bullet\mathrm{Sing}_{(-)^\vee}$ with $(-)^\vee$ to obtain a simplicial object in $\mathrm{coAlg}_{O^\vee}$. By 4.1.6, we can identify the ℓ-th level of the resolution with

$$\coprod_{\begin{tikzcd} A_0^\vee \otimes \Delta^{n_0} \arrow[r] \arrow[d, "f_0\otimes\theta_0"'] & Y \\ \vdots \arrow[d, "f_{\ell-1}\otimes\theta_{\ell-1}"'] & \\ A_\ell^\vee \otimes \Delta^{n_\ell} \arrow[uur] & \end{tikzcd}} A_0^\vee \otimes C_*(\Delta^{n_0}). \tag{8.1}$$

Finally, the realization of the simplicial object given levelwise by 8.1 is

$$\underset{\mathrm{Art}_O^{\mathrm{op}}\times\Delta\downarrow \mathrm{Y}}{\mathrm{hocolim}}\ A^\vee \otimes C_*(\Delta^n)$$

as desired. □

Corollary 8.1.4. *The derived counit* $\mathit{Re}_{(-)^\vee}\mathit{QSing}_{(-)^\vee}(B_{\bar\alpha}L) \to B_{\bar\alpha}L$ *is a weak equivalence if and only if the derived counit of a minimal model of* $B_{\bar\alpha}L$ *is a weak equivalence.*

Proof. For any minimal model $X \xrightarrow{I} B_{\bar{\alpha}}L$, we have

$$\begin{tikzcd}
\mathrm{Re}_{(-)^\vee} Q\mathrm{Sing}_{(-)^\vee}(X) \arrow[r] \arrow[d, "\mathrm{Re}_{(-)^\vee} Q\mathrm{Sing}_{(-)^\vee}(I)"'] & X \arrow[d, "I"] \\
\mathrm{Re}_{(-)^\vee} Q\mathrm{Sing}_{(-)^\vee}(B_{\bar{\alpha}}L) \arrow[r] & B_{\bar{\alpha}}L
\end{tikzcd}$$

with both vertical arrows weak equivalences. □

Proposition 8.1.5. *The natural map*

$$\operatorname*{hocolim}_{Art_{\mathrm{O}}^{op} \times \Delta \downarrow \mathrm{X}} A^\vee \to \operatorname*{hocolim}_{Art_{\mathrm{O}}^{op} \times \Delta \downarrow \mathrm{X}} A^\vee \otimes C_*(\Delta^n)$$

is a weak equivalence.

Proof. It is possible to choose $C_*(\Delta^0)$ to be k, and the maps $s^0 : C_*(\Delta^0) \to C_*(\Delta^n)$ to be an inclusion, which is a cofibration. This means the induced maps $A^\vee \cong A^\vee \otimes C_*(\Delta^0) \xrightarrow{s^0} A^\vee \otimes C_*(\Delta^n)$ are inclusions. As the natural map in question comes from a map of diagrams which is an object-wise cofibration and an object-wise weak equivalence of object-wise cofibrant diagrams, the map on the homotopy colimit is a weak equivalence. [Hir02, Theorem 19.4.2] □

Proposition 8.1.6. *The natural map*

$$\operatorname*{hocolim}_{Art_{\mathrm{O}}^{op} \downarrow \mathrm{X}} A^\vee \to \operatorname*{hocolim}_{Art_{\mathrm{O}}^{op} \times \Delta \downarrow \mathrm{X}} A^\vee$$

is an equivalence.

Proof. First, we recall that the forgetful functor $U : \mathrm{coAlg}_{O^\vee} \to \mathrm{Ch}_k$ is a left Quillen functor 2.3.5. This implies that $\operatorname*{hocolim}_{\mathrm{i \in I}} UX(i) \xrightarrow{\sim} U(\operatorname*{hocolim}_{\mathrm{i \in I}} X(i))$ for any indexing category I and diagram X. As weak equivalences of coalgebras are detected by the forgetful functor, it is equivalent to verify that the natural map $\operatorname*{hocolim}_{\mathrm{Art}_{\mathrm{O}}^{\mathrm{op}} \downarrow \mathrm{X}} A^\vee \to \operatorname*{hocolim}_{\mathrm{Art}_{\mathrm{O}}^{\mathrm{op}} \times \Delta \downarrow \mathrm{X}} A^\vee$ is an equivalence.

An application of homotopy invariance of cosimplicial framing [Hir02, Theorem 16.7.6] implies we can use the cosimplicial chain complex of (nondegenerate) simplicial chains

$$\Delta^n \mapsto \mathbb{SC}_*(\Delta^n)$$

to compute the $\mathrm{hocolim}$ in Ch_k, rather than the coalgebra model $\Delta^n \mapsto UC_*(\Delta^n)$.

Finally, we use the model of $\mathrm{hocolim}$ given by the weighted SSet-colimit weighted by the nerve of the indexing category [Hir02, Rie12, Definition 19.1.2, Section 7.7] in order to obtain a linear retraction

$$r : \operatorname*{hocolim}_{\mathrm{Art}_{\mathrm{O}}^{\mathrm{op}} \times \Delta \downarrow \mathrm{X}} UA^\vee \to \operatorname*{hocolim}_{\mathrm{Art}_{\mathrm{O}}^{\mathrm{op}} \downarrow \mathrm{X}} UA^\vee$$

which is quasi-inverse to the natural map $i : \operatorname*{hocolim}_{\mathrm{Art}_{\mathrm{O}}^{\mathrm{op}} \downarrow \mathrm{X}} UA^\vee \to \operatorname*{hocolim}_{\mathrm{Art}_{\mathrm{O}}^{\mathrm{op}} \times \Delta \downarrow \mathrm{X}} UA^\vee$. To construct this map, first recall that the weighted SSet-colimit model for $\operatorname*{hocolim}_{\mathrm{Art}_{\mathrm{O}}^{\mathrm{op}} \times \Delta \downarrow \mathrm{X}} UA^\vee$

in Ch_k is a quotient of

$$\coprod_{\begin{tikzcd} A_0^\vee \otimes \Delta^{n_0} \arrow[r] \arrow[d, "f_0\otimes\theta_0"'] & X \\ \vdots \arrow[d, "f_{\ell-1}\otimes\theta_{\ell-1}"'] & \\ A_\ell^\vee \otimes \Delta^{n_\ell} \arrow[uur] & \end{tikzcd}} UA_0^\vee \otimes \mathbb{CS}_*\left(\Delta^{n_\ell}\right),$$

while the weighted SSet-colimit model for $\operatorname*{hocolim}_{\mathrm{Art}_{\mathrm{O}}^{\mathrm{op}}\downarrow \mathrm{X}} UA^\vee$ in Ch_k is a quotient of

$$\coprod_{\begin{tikzcd} A_0^\vee \arrow[r] \arrow[d, "f_0"'] & X \\ \vdots \arrow[d, "f_{\ell-1}"'] & \\ A_\ell^\vee \arrow[uur] & \end{tikzcd}} UA_0^\vee \otimes \mathbb{CS}_*\left(\Delta^{n_\ell}\right).$$

However, one can map each factor labeled by

$$A_0^\vee \otimes \Delta^{n_0} \xrightarrow{f_0\otimes\theta_0} \cdots \xrightarrow{f_{\ell-1}\otimes\theta_{\ell-1}} A_\ell^\vee \otimes \Delta^{n_\ell}$$

to the factor labeled by

$$A_0^\vee \xrightarrow{f_0} \cdots \xrightarrow{f_{\ell-1}} A_\ell^\vee$$

via the identity map of $A_0^\vee \otimes \Delta^\ell$ to obtain the map s. The composite $s \circ i$ is the identity, and the composite $i \circ s$ is homotopic to the identity by the natural prism

$\Delta^\ell \times \Delta^1$ given by

$$\begin{tikzcd}
A_0^\vee \arrow[r, "f_0"] \arrow[d] & \cdots \arrow[r, "f_{\ell-1}"] & A_\ell^\vee \arrow[d] \\
A_0^\vee \otimes \Delta^{n_0} \arrow[r, "f_0\otimes\theta_0"] & \cdots \arrow[r, "f_{\ell-1}\otimes\theta_{\ell-1}"] & A_\ell^\vee \otimes \Delta^{n_\ell}.
\end{tikzcd}$$

□

To compute the homotopy colimit over the category $\mathrm{Art}_O^{\mathrm{op}}\!\downarrow\! X$, we reduce to a subcategory in $\mathrm{Art}_O^{\mathrm{op}}\!\downarrow\! X$ which is homotopy cofinal. We recall the definition and salient properties of homotopy cofinal subcategories.

Definition 8.1.7. *A functor $C \xrightarrow{\varphi} D$ is* homotopy cofinal *if for every object $d_0 \in D$, the category $d_0\!\downarrow\!\varphi$ has contractible nerve.*

Proposition 8.1.8. *If a functor $C \xrightarrow{\varphi} D$ is homotopy cofinal and $D \xrightarrow{X} M$ is object-wise cofibrant, then $\underset{\mathrm{C}}{\mathrm{hocolim}}\, X \circ \varphi \to \underset{\mathrm{D}}{\mathrm{hocolim}}\, X$ is an equivalence. [Hir02, Theorem 19.6.7]*

To identify the homotopy cofinal subcategory, we choose an isomorphism between the minimal cofree coalgebra X with $B_{\bar{\alpha}}L$ for some ∞-algebra structure on the (shifted) graded vector space L of a choice of cogenerators. We fix a homogeneous basis for the cogenerators, $L = \langle v_i \rangle$, and a choice of basis of cooperations $\langle \mu_j \rangle$ for the cooperad $O^\vee$.

Definition 8.1.9. *A subcoalgebra $S \subset B_{\bar{\alpha}}L$ is* standard *if it has a homogeneous basis of monomials $\mu_j\left(v_{i_1}, \ldots, v_{i_k}\right)$ for some cooperations $\mu_i \in O^\vee$. We denote the category of finite dimensional standard subcoalgebras of X, with morphisms given by inclusions, by Stand$\downarrow X$.*

We observe that every finite dimensional subspace of X has a smallest finite dimensional standard subcoalgebra containing it. This fact implies:

Proposition 8.1.10. *The subcategory Stand$\downarrow X$ is cofinal in Art$_O^{op}\downarrow X$.*

Proof. Given a map $A^\vee \xrightarrow{f} X$, we can consider the smallest standard subcoalgebra containing its image, $S_f \subset X$, and we observe that this object is initial in the category $\left(A^\vee \xrightarrow{f} X\right) \downarrow \left(\text{Stand}\downarrow X\right)$, so its nerve is contractible. □

Corollary 8.1.11. *The map*

$$\operatorname*{hocolim}_{\mathrm{S}\in \mathit{Stand}\downarrow \mathrm{X}} S^\vee \to \operatorname*{hocolim}_{\mathrm{A}^\vee\to\mathrm{X}\in \mathit{Art}_{\mathrm{O}}^{op}\downarrow \mathrm{X}} A^\vee$$

is a weak equivalence.

Proof. This follows from 8.1.8. □

The next two propositions are technical, but the important fact is 8.1.15

Proposition 8.1.12. *The category Stand$\downarrow X$ is a Reedy category.*

Proof. Let the degree function $\mathrm{ob}\left(\mathrm{Stand}\downarrow X\right) \to \mathbb{N}$ be the dimension of the finite dimensional standard subcoalgebra, and let all the morphisms be in the increasing subcategory, $\overrightarrow{\mathrm{Stand}\downarrow X}$, and let the decreasing subcategory contain only the identities. Trivially, every non-identity arrow has a unique factorization as a decreasing arrow followed by an increasing arrow, and we are done. □

Remark 13. *See [Hir02, Chapter 15] for an excellent description and detailed definition of Reedy categories.*

Proposition 8.1.13. *The Reedy category $Stand\downarrow X$ has fibrant constants.*

Proof. The matching category of every object is empty. □

Corollary 8.1.14. *The functor* $\operatorname{hocolim}_{Stand\downarrow \mathrm{X}} : coAlg_{O^\vee}^{\left(Stand\downarrow X\right)} \to coAlg_{O^\vee}$ *is the total left derived functor of the left Quillen functor* $\underset{Stand\downarrow X}{colim} : coAlg_{O^\vee}^{\left(Stand\downarrow X\right)} \to coAlg_{O^\vee}$*. [Hir02, Theorem 19.9.1].*

Corollary 8.1.15. *The homotopy colimit of a Reedy-cofibrant diagram over* $Stand\downarrow X$ *coincides with the colimit. In particular, the natural map* $\operatorname{hocolim}_{Stand\downarrow \mathrm{X}} Z(A) \to \underset{Stand\downarrow X}{colim}\ Z(A)$ *for a Reedy-cofibrant diagram Z is a weak equivalence. [Hir02, Theorem 19.9.1]*

Proposition 8.1.16. *Our diagram of interest, namely*

$$Stand\downarrow X \to coAlg_{O^\vee}$$

$$S^\vee \subset X \mapsto S^\vee$$

is Reedy cofibrant.

Corollary 8.1.17. *The map* $\operatorname*{hocolim}_{S^\vee \in \mathit{Stand}\downarrow X} S^\vee \to X$ *is a weak equivalence if and only if the map* $\operatorname*{colim}_{\mathit{Stand}\downarrow X} \to X$ *is a weak equivalence.*

Proof. The diagram

$$\begin{tikzcd}
\operatorname*{hocolim}_{S^\vee \in \mathrm{Stand}\downarrow X} S^\vee \arrow[rr] \arrow[dr] & & \operatorname*{colim}_{S^\vee \in \mathrm{Stand}\downarrow X} S^\vee \arrow[dl] \\
& X &
\end{tikzcd}$$

commutes, and the top arrow is a weak equivalence by 8.1.16 and 8.1.15. □

Proposition 8.1.18. *The map* $\operatorname*{colim}_{S^\vee \in \mathit{Stand}\downarrow X} S^\vee \to X$ *is an isomorphism.*

Proof. There is a natural map $\operatorname*{colim}_{S \in \mathrm{Stand}\downarrow X} S \to X$ which has a linear two-sided inverse. To define the inverse, use the isomorphism to $B_\alpha L$, the basis for L and the basis for $O^\vee$ to obtain a standard basis for X, and send each standard basis element to itself in the finite dimensional standard subcoalgebra it generates in $\operatorname*{colim}_{S \in \mathrm{Stand}\downarrow X} S$. This assignment is well defined, because intersections of standard subcoalgebras are standard, and the intersection will witness and identify any discrepancy in the assignment. □

This completes the proof of 8.1.1.

Appendix A

Koszul dual operads

In this section we will offer an abbreviated treatment of Koszul duality. Moreover, this discussion is concerned only with the homotopical-duality that goes by that name, whereas [LV12] addresses homotopical-duality, algebraic-duality, and the situations in which these two types of duality coincide.

In this section we will assume operads O satisfies the conditions: $O(0) = 0, O(1) = k \cdot 1_O$, and that cooperads are connected weight-graded, in order to make use of the results in [Val12]. We use of the bar and cobar constructions for operads and cooperads [LV12], as well as the bar and cobar constructions for algebras over operads and cooperads.

Definition A.0.19. *An* operad *is an augmented operad in chain complexes satisfying* $O(0) = 0, O(1) = k \cdot 1_O$.

Definition A.0.20. *Given a cooperad* C*, we say a linear map* $C \overset{\alpha}{\to} O$ realizes C

as the dual cooperad to O *if*

$$\Omega C \xrightarrow{\Omega\alpha} P$$

is a quasi-isomorphism of dg-operads. In this situation, we might say C is dual to O via α, though occasionally we may abuse terminology by simply writing that C is dual to O.

Definition A.0.21. *If C is dual to O, the operad $C^\vee$ is called the* Koszul dual operad *to O and is denoted $O^!$.*

Under suitable "homotopy finiteness" conditions on O, the term dual is well-deserved: the double-dual of O will be weakly equivalent to O. We illustrate this fact in the case that O is arity-wise finite dimensional. To do this, we will implicitly use the fact that any two cooperads which are dual to O are weakly equivalent, and so it is sufficient to exhibit O as the double-dual to itself.

Lemma A.0.22. *Suppose O is finite dimensional in every arity, and C is a cooperad dual to O via $C \xrightarrow{\alpha} O$. Then $O^\vee$ is a naturally a cooperad and $O^\vee \xrightarrow{\alpha^\vee} C^\vee$ exhibits $O^\vee$ as dual to $C^\vee$.*

Proof. The dual of a cooperad is naturally an operad, regardless of finiteness: we have a composition map given by

$$C^\vee \circ C^\vee \hookrightarrow (C \circ C)^\vee \xrightarrow{\Delta^\vee} C^\vee.$$

The dual of an operad is not obviously a cooperad:

$$O^\vee \circ O^\vee \hookrightarrow (O \circ O)^\vee \xleftarrow{\gamma^\vee} O^\vee,$$

but when O is finite dimensional in every arity, the inclusion $O^\vee \circ O^\vee \hookrightarrow (O \circ O)^\vee$ is an isomorphism and we can invert it to obtain a cooperad structure on $O^\vee$.

To see that $\alpha^\vee$ realizes $O^\vee$ as dual to $C^\vee$, we make two observations. First, the adjoint map $C \xrightarrow{\mathrm{B}\,\alpha} \mathrm{B}\,O$ is a quasi-isomorphism of dg cooperads [LV12, 6.8.5]. Then, because O is finite dimensional in each arity, the natural map of operads $\Omega(O^\vee) \to (\mathrm{B}\,O)^\vee$ is an isomorphism of operads, and the diagram

$$\begin{tikzcd}
(BO)^\vee \arrow[dr, "(B\alpha)^\vee"] & \\
\Omega(O^\vee) \arrow[u, "\cong"] \arrow[r, "\Omega(\alpha^\vee)"] & C^\vee
\end{tikzcd}$$

commutes. □

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