INFINITE WORDS AND LENGTH FUNCTIONS

by

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Abstract

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Denis Serbin

Advisor: Alexei Miasnikov

Let $F = F(X)$ be a free group with basis X and $\mathbb{Z}[t]$ be a ring of polynomials in variable t with integer coefficients. The main result of the first part of this thesis is the representation of elements of Lyndon's free $\mathbb{Z}[t]$-group $F^{\mathbb{Z}[t]}$ by infinite words defined as sequences $w : [1, f_w] \to X^{\pm 1}$ over closed intervals $[1, f_w], f_w \geq 0$, in the additive group $\mathbb{Z}[t]^+$, viewed as an ordered abelian group. This representation provides a natural regular free Lyndon length function $w \to f_w$ on $F^{\mathbb{Z}[t]}$ with values in $\mathbb{Z}[t]^+$. The second part of the thesis is concerned with applications of the construction above to finitely generated subgroups of $F^{\mathbb{Z}[t]}$. Finitely generated subgroups of $F^{\mathbb{Z}[t]}$ are associated with combinatorial objects called $(\mathbb{Z}[t], X)$-graphs study of which solves some algorithmic problems for these subgroups such as the membership problem, the conjugacy problem etc.

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Contents

List of Figures

1 Main results

Let G be a group and A be an ordered abelian group. Then a function $l : G \to A$ is called a *(Lyndon) length function* on G if the following conditions hold:

(L1) $\forall\, x \in G : \ l(x) \geq 0$ and $l(1) = 0$;

(L2) $\forall\, x \in G : \ l(x) = l(x^{-1})$;

(L3) $\forall\, x, y, z \in G : \ d(x, y) > d(x, z) \to d(x, z) = d(y, z)$,

where $d(x, y) = \frac{1}{2}(l(x) + l(y) - l(x^{-1}y))$.

A length function $l : G \to A$ is called *free*, if it satisfies

(L4) $\forall\, x \in G : \ x \neq 1 \to l(x^2) > l(x)$

and *regular*, if it satisfies

(L5) $\forall x, y \in G, \exists u, x_1, y_1 \in G$:

$$x = u \circ x_1 \ \& \ y = u \circ y_1 \ \& \ l(u) = d(x, y).$$

Let A be a discretely ordered abelian group and $A^+ = \{a \in A \mid a \geq 0\}$. Let $X = \{x_i \mid i \in I\}$ be a set, $X^{-1} = \{x_i^{-1} \mid i \in I\}$ and $X^{\pm} = X \cup X^{-1}$. An A-word is a function of the type

$$w : [1_A, \alpha] \to X^{\pm},$$

where $\alpha \in A^+$ and 1_A is the minimal positive element of A. The element α is called the length $|w|$ of w.

An A-word w is ***reduced*** if $w(\beta + 1_A) \neq w(\beta)^{-1}$ for each $1_A \leq \beta < |w|$ and $R(A, X)$ denotes the set of all reduced A-words. Let $CDR(A, X)$ denote the set of elements w from $R(A, X)$ which admit cyclic decomposition (precise definitions can be found in Section 8).

It must be noted that the idea of A-words was introduced originally in [7].

The following theorem provides a basis for all further results.

Theorem 1 (**Miasnikov, Remeslennikov, Serbin**) *The function*

$$L : CDR(A, X) \to A$$

defined as $L(w) = |w|$ *satisfies all the axioms of a free Lyndon length function.*

This implies that every group embeddable into $CDR(A, X)$ has a free Lyndon length function with values in A.

Let $F = F(X)$ be a free non-abelian group with basis X. Let $\mathbb{Z}[t]$ be a ring of integer polynomials in variable t and $\mathbb{Z}[t]^+$ be its additive group, viewed as an ordered abelian group. In [31] R.Lyndon defined and studied a free $\mathbb{Z}[t]$-group $F^{\mathbb{Z}[t]}$ which admits exponents in the ring $\mathbb{Z}[t]$.

Using techniques of infinite words we construct an embedding (by no means unique) of $F^{\mathbb{Z}[t]}$ into the set of infinite words.

Theorem 2 (**Miasnikov, Remeslennikov, Serbin**) *There exists an embedding*

$$\phi : F^{\mathbb{Z}[t]} \to CDR(\mathbb{Z}[t]^+, X).$$

Moreover, the free Lyndon length function induced on $F^{\mathbb{Z}[t]}$ *from* $CDR(\mathbb{Z}[t]^+, X)$ *is regular.*

The following result is a corollary of the theorem above and the fact proved in [26] (see Section 5.2), that the coordinate groups of irreducible algebraic sets over F are precisely the finitely generated subgroups of $F^{\mathbb{Z}[t]}$.

Corollary 1 (**Miasnikov, Remeslennikov, Serbin**) *Every coordinate group of an irreducible algebraic set over a free group* $F = F(X)$ *has a free Lyndon length* *function with values in a free abelian group $\mathbb{Z}^n$ of finite rank with the lexicographic order.*

The result above was proved originally in [39], [40] using different approach.

Note, that the regularity of a length function is not inherited by subgroups of the ambient group.

The second part of the thesis investigates properties of finitely generated subgroups of $F^{\mathbb{Z}[t]}$ using special objects called *U-folded $(\mathbb{Z}[t], X)$-graphs.*

Γ is called a *$(\mathbb{Z}[t], X)$-graph*, if it is a combinatorial graph such that every edge in it has a direction and is labeled by a letter from the alphabet $\{X \cup X^{-1}\} \cup \{u^\alpha \mid u \in U, \alpha \in \mathbb{Z}[t]^+\}$, where U is some special subset of $F^{\mathbb{Z}[t]}$.

We define *the language of* Γ *with respect to* its vertex v to be:

$$L(\Gamma, v) = \{\overline{\mu(p)} \mid p \text{ is a reduced path in } \Gamma \text{ from } v \text{ to } v\},$$

where $\overline{\mu(p)}$ is a label of a path p viewed as a reduced $\mathbb{Z}[t]^+$-word.

The following theorem is the most important result in the second part of the thesis.

Theorem 3 *(***Miasnikov, Remeslennikov, Serbin***) Let H be a finitely generated subgroup of $F^{\mathbb{Z}[t]}$. Then there exists a U-folded $(\mathbb{Z}[t], X)$-graph Γ and a vertex v of Γ such that $L(\Gamma, v) = H$.*

Using the apparatus of $(\mathbb{Z}[t], X)$-graphs it is possible to solve many algorithmic problems in $F^{\mathbb{Z}[t]}$.

Theorem 4 *(***Miasnikov, Remeslennikov, Serbin***) The group $F^{\mathbb{Z}[t]}$ has a solvable generalized word problem. That is, there exists an algorithm which, given finitely many standard decompositions of elements $g, h_1, \ldots, h_k$ from $F^{\mathbb{Z}[t]}$, decides whether or not g belongs to the subgroup $H = \langle h_1, \ldots, h_n \rangle$ of $F^{\mathbb{Z}[t]}$.*

Theorem 5 (**Miasnikov, Remeslennikov, Serbin**) *There exists an algorithm which, given finitely many standard decompositions of elements* $h_1, \ldots, h_k, g_1, \ldots, g_m$ *from* $F^{\mathbb{Z}[t]}$, *finds generators of intersection* $H \cap K$, *where* $H = \langle h_1, \ldots, h_k \rangle$, $K = \langle g_1, \ldots, g_m \rangle$.

Theorem 6 (**Miasnikov, Remeslennikov, Serbin**) *Any finitely generated subgroup of* $F^{\mathbb{Z}[t]}$ *has a solvable conjugacy problem. That is, there exists an algorithm which, given standard decompositions of elements* $g, f \in H = \langle h_1, \ldots, h_k \rangle$, *decides whether or not* g *is conjugate to* f *in* H, *and if yes, generates an element* $c \in H$ *such that* $c^{-1}gc = f$.

This thesis is based on the preprints [41] and [42]. The results presented in the first part of the thesis, in particular Theorem 1, Theorem 2 and Corollary 1, were announced in several conference talks.

2 Ordered abelian groups

In this section some well-known results on ordered abelian groups are collected. For proofs and details we refer to the books [16] and [29].

Definition 1 *A set A equipped with addition $+$ and a partial order $\leq$ is called a partially ordered abelian group if:*

(1) $\langle A, +\rangle$ *is an abelian group;*

(2) $\langle A, \leq\rangle$ *is a partially ordered set;*

(3) $\forall a, b, c \in A: \ a \leq b \ \rightarrow \ a + c \leq b + c.$

If the partial ordering is a linear (total) ordering then A is called *an ordered abelian group*

If A is an ordered abelian group then the set of all non-negative elements

$$A^+ = \{a \in A \mid a \geq 0\}$$

forms a semigroup, such that $A^+ \cap -A^+ = 0$ and $A^+ \cup -A^+ = A$. Conversely, if P is a subsemigroup of A such that $P \cup -P = A$ and $P \cap -P = 0$ then the relation

$$a \geq b \Leftrightarrow a - b \in P$$

turns A into an ordered abelian group. We call P the *positive cone* of the ordered abelian group A.

An abelian group A is called *orderable* if there exists a linear order $\leq$ on A, satisfying the condition (3) above. In general, the ordering on A is not unique.

Observe, that every ordered abelian group is torsion-free, since if $0 < a \in A$ then $0 < na$ for any positive integer n. It is easy to see that the reverse is also true, that is, a torsion-free abelian group is orderable. Indeed, by the compactness theorem for first-order logic, a group is orderable if and only if every finitely generated subgroup of it is orderable. Hence it suffices to show that finite direct sums of copies of the infinite cyclic group $\mathbb{Z}$ are orderable. This is easy, one of the possible orderings is the lexicographical order described below.

Let A and B be ordered abelian groups. Then the direct sum $A \oplus B$ is orderable with respect to the *right lexicographic order*, defined as follows:

$$(a_1, b_1) < (a_2, b_2) \Leftrightarrow b_1 < b_2 \text{ or } b_1 = b_2 \text{ and } a_1 < a_2.$$

Similarly, one can define the right lexicographic order on finite direct sums of ordered abelian groups or even on infinite direct sums if the set of indices is linearly ordered. Indeed, let I be a linearly ordered set of indices and $A_i, i \in I$, be a set of ordered abelian groups. Then the *right lexicographic order* on the direct sum $A_I = \oplus_{i \in I} A_i$ is defined by the following condition: an element $a = (a_i)_{i \in I} \in A_I$ is positive if and only if its greatest non-zero component is positive. It follows that the right lexicographic order on A_I extends the order on each group A_i, viewed as a subgroup under the canonical embedding.

For example, let $\mathbb{Z}[t]^+$ be the additive group of the polynomial ring $\mathbb{Z}[t]$. Recall, that as a group $\mathbb{Z}[t]^+$ is the infinite direct sum

$$\mathbb{Z}[t]^+ = \oplus_{n=0}^{\infty} \langle t^n \rangle$$

of copies of $\mathbb{Z}$. Hence, $\mathbb{Z}[t]^+$ has the right lexicographic order induced by this direct decomposition.

Observe, that the construction above allows one to introduce a right lexicographic order on any torsion-free abelian group A. Indeed, there exists an embedding (by no means unique) of A into a divisible abelian group $A_{\mathbb{Q}} = A \otimes_{\mathbb{Z}} \mathbb{Q}$, where $\mathbb{Q}$ is the additive group of rational numbers. Clearly, $A_{\mathbb{Q}}$ is a direct sum of copies of $\mathbb{Q}$, $A_{\mathbb{Q}} = \oplus_{i \in I} \mathbb{Q}$. Since the set of indices I can be linearly ordered (assuming the axiom of choice) the group $A_{\mathbb{Q}}$ is orderable, as well as its subgroup A. The induced order on A is also called lexicographic.

If A is already ordered then the right lexicographic order on $A_{\mathbb{Q}}$, in general, does not extend the order on A. Now we introduce an order on $A_{\mathbb{Q}}$ that extends the existing order on A. Notice, that elements in $A_{\mathbb{Q}}$ can be described as fractions $\frac{a}{m}$, where $a \in A$ and $m \in \mathbb{Z}, m > 0$. Then the relation

$$\frac{a}{k} \leq \frac{b}{m} \Leftrightarrow ma \leq kb$$

gives rise to an order on $A_{\mathbb{Q}}$ which extends the order on A under the embedding $a \to \frac{a}{1}$. We will refer to this order as to a *fraction* order.

Definition 2 *An ordered abelian group A is called archimedean if, given $a, b \in A$ with $b \neq 0$, there exists $n \in \mathbb{Z}$ such that $a < nb$.*

Theorem 7 *[9] An ordered abelian group A is archimedean if and only if there exists an embedding of ordered abelian groups $A \to \mathbb{R}$.*

Obviously, if the set of indices I has at least two elements then the direct sum A_I with the right lexicographic order is non-archimedean. On the other hand, if A is archimedean ordered abelian group then the fraction order on its $\mathbb{Q}$-completion $A_{\mathbb{Q}}$ is also archimedean.

There is one simple fact we can note about the subgroups of $\mathbb{R}$.

Lemma 1 *[9] Let A be a subgroup of the additive group of $\mathbb{R}$. Then either A is cyclic, or it is dense in $\mathbb{R}$.*

For elements a, b of an ordered group A the *closed segment* $[a, b]$ is defined by

$$[a, b] = \{c \in A \mid a \leq c \leq b\}.$$

A subset $C \subset A$ is called *convex*, if for every $a, b \in C$ the set C contains $[a, b]$. In particular, a subgroup B of A is convex if $[0, b] \subset B$ for every positive $b \in B$. In this event, the quotient A/B is an ordered abelian group with respect to the order induced from A.

Lemma 2 *The set of convex subgroups of an ordered abelian group is linearly ordered by inclusion.*

Proof. Suppose A_1 and A_2 are convex subgroups, and it is possible to find $a \in A_1 \setminus A_2$ and $b \in A_2 \setminus A_1$. Replacing a, b by $-a, -b$ as necessary, we can assume that $a > 0, b > 0$. Also, either $a \leq b$ or $b \leq a$. If $0 < a \leq b$, then $a \in A_2$ since A_2 is convex - contradiction. Similarly, if $0 <\leq a$, we obtain the contradiction $b \in A_2$.

$\square$

Thus, we call linearly ordered set of all convex subgroups of A *the complete chain of convex subgroups* in A. In particular, if A is finitely generated then its complete chain of convex subgroups is finite. The following result shows that this chain completely determines the order on A, as well as the structure of A.

Theorem 8 *Let A be a finitely generated ordered abelian group and $0 = A_0 < A_1 < \ldots < A_n = A$ be the complete chain of convex subgroups of A. Then A_i/A_{i-1} is archimedean (with respect to the induced order) and A is isomorphic (as an ordered group) to the direct sum*

$$A_1 \dotplus A_2/A_1 \dotplus \ldots \dotplus A_n/A_{n-1}$$

with the right lexicographic order.

Proof. Observe first, that the canonical epimorphism $A_i \to A_i/A_{i-1}$ is an order epimorphism, so if A_i/A_{i-1} is non-archimedean then the chain is not complete.

Secondly, every convex subgroup is pure, hence

$$A_i \simeq A_{i-1} \dotplus (A_i/A_{i-1}). \tag{1}$$

To finish the proof by induction on n one has to show that A_n is order isomorphic to $A_{n-1} \dotplus (A/A_{n-1})$ with the right lexicographic order. To see this let C be a direct complement of A_{n-1} in A. Then $A = A_{n-1} \dotplus C$ and the C is order isomorphic to A/A_{n-1} (via $\phi : A \to A/A_{n-1}$). Now, $a_1+c_1 \leq a_2+c_2$ in A (where $a_i \in A_{n-1}, c_i \in C$) if and only if $c_1 < c_2$ (applying ϕ) or else $c_1 = c_2$ and then $a_1 \leq a_2$, as required.

□

An ordered abelian group A is called *discretely ordered* if A^+ has a minimal non-trivial element (we denote it by 1_A). In this event, for any $a \in A$ the following hold:

1) $a + 1_A = min\{b \mid b > a\}$,

2) $a - 1_A = max\{b \mid b < a\}$.

3 Free constructions and normal forms

Free constructions of groups are very important tools for building new groups and they lie at the basis of Combinatorial Group Theory. Below the basic definitions and properties of different free constructions are given. For more details see [35], [33]

3.1 Free products

Let A and B be groups with presentations $A = \langle a_1, \dots \mid r_1, \dots \rangle$ and $B = \langle b_1, \dots \mid s_1, \dots \rangle$ respectively, where the sets of generators $\{a_1, \dots\}$ and $\{b_1, \dots\}$ are disjoint. The free product, $A * B$, of the groups A and B is the group with the presentation

$$A * B = \langle a_1, \dots, b_1, \dots \mid r_1, \dots, s_1, \dots \rangle.$$

The groups A and B are called factors of $A * B$.

Lemma 3 *[33] The free product $A * B$ is uniquely determined by the groups A and B. Also, $A * B$ is generated by subgroups $\overline{A}$ and $\overline{B}$ which are isomorphic to A and B respectively, and such that $\overline{A} \cap \overline{B} = 1$.*

For the proof of the lemma above we refer to [33]. We only note that in this proof subgroups $\overline{A}$ and $\overline{B}$ are generated in $A * B$ by $a_1, \dots$ and $b_1, \dots$ respectively and this provides natural isomorphisms $A \simeq \overline{A}$ and $B \simeq \overline{B}$. So, we can identify A with $\overline{A}$, B with $\overline{B}$, and view A and B as subgroups of $A * B$.

Definition 3 *A* reduced sequence *(or* normal form*) is a sequence $g_1, \dots, g_n$, $n \geq 0$, of elements of $A * B$ such that each $g_i \neq 1$, each g_i is in one of the factors, A or B, and successive g_i, g_{i+1} are not in the same factor. (We allow $n = 0$ for the empty sequence).*

Theorem 9 *[33] (The Normal Form Theorem for Free Products)*

Let G be a free product of groups A and B. Then the following equivalent statements hold in G.

(1) If $w = g_1 \cdots g_n, n > 0$, where $g_1, \ldots, g_n$ is a reduced sequence, then $w \neq 1$ in G.

(2) Each element w of G can be uniquely expressed as a product $w = g_1 \cdots g_n$, where $g_1, \ldots, g_n$ is a reduced sequence.

The Normal Form Theorem allows one to define a length for elements of free products. If an element $w = g_1 \cdots g_n$ of $G = A * B$ is such that $g_1, \ldots, g_n$ is a reduced sequence, then the *length* of w, denoted $|w|$, is defined to be n.

3.2 Free Products with Amalgamation

Let $G = \langle x_1, \ldots \mid r_1, \ldots \rangle$ and $H = \langle y_1, \ldots \mid s_1, \ldots \rangle$ be groups. Let $A \leq G$ and $B \leq H$ be such that there exists an isomorphism $\phi : A \to B$. Then the *free product of G and H, amalgamating the subgroups A and B by the isomorphism ϕ* is the group

$$G *_{A=B} H = \langle x_1, \ldots, y_1, \ldots \mid r_1, \ldots, s_1, \ldots, a = \phi(a), a \in A \rangle.$$

That is, one can view $G *_{A=B} H$ as the quotient of the free product of G and H by the normal subgroup generated by $\{a\, \phi(a)^{-1} \mid a \in A\}$ and further we will be using the following notation

$$G *_{A=B} H = \langle G * H \mid a = \phi(a), a \in A \rangle.$$

The free product with amalgamation depends on G, H, A, B and the isomorphism ϕ. The groups G and H are called the *factors* of the free product with amalgamation, while A and $B = \phi(A)$ are called the *amalgamated subgroups*.

Definition 4 *A sequence $c_1, \ldots, c_n$, $n \geq 0$, of elements of $G * H$ is called reduced if*

(1) each c_i is in one of the factors G or H;

(2) successive c_i, c_{i+1} come from different factors;

(3) if $n > 1$ then no c_i is in A or B;

(4) if $n = 1$ then $c_1 \neq 1$.

It is clear that every element of $G *_{A=B} H$ can be presented as the product of the elements in a reduced sequence, but products which come from different reduced sequences can represent the same element.

One can introduce unique *normal forms* in the following way. Choose a set of representatives of the right cosets of A in G, and a set of representatives of the right cosets of B in H. We assume 1 to be the representative of A in G and B in H. Further we assume chosen representatives to be fixed.

Definition 5 *A normal form is a sequence $c_0, c_1, \ldots, c_n$, where $n \geq 0$, such that*

(1) $c_0 \in A$;

(2) each $c_i, i > 0$ is a non-trivial representative either of a coset of A in G or of a coset of B in H;

(3) successive $c_i, c_{i+1}, i > 0$ come from different factors.

Observe that normal forms depend on the choice of representatives.

Theorem 10 *[33], [35] (Normal Form Theorem for Free Products with Amalgamation)*

(I) If $c_1, \ldots, c_n, n \geq 1$ is a reduced sequence then the product $c_1 \cdots c_n \neq 1$ in $G *_{A=B} H$. In particular, G and H are embedded in $G *_{A=B} H$ by the maps $g \to g, g \in G$ and $h \to h, h \in H$.

(II) Every element w of $G *_{A=B} H$ has a unique representation as $w = c_0 \cdots c_n$, $n \geq 0$, where $c_0, c_1, \ldots, c_n$ is a normal form.

The length for elements in $G *_{A=B} H$ is defined as follows. By the Normal Form Theorem for Free Products with Amalgamation any $w \in G *_{A=B} H$ can be presented as a unique product $w = c_0 \cdots c_n$, $n \geq 0$, where $c_0, c_1, \ldots, c_n$ is a normal form. Then, the *length* of w, denoted $|w|$, is defined to be n.

3.3 HNN-extensions

We start with HNN-extensions which were introduced by G.Higman, G.H.Neumann, and H.Neumann in 1949 (see [13]).

Let G be a group, and let A and B be subgroups of G isomorphic under $\phi : A \to B$. The *HNN extension of G relative to A, B and ϕ* is the group

$$G^* = \langle G, t \mid t^{-1}at = \phi(a), a \in A \rangle.$$

The group G is called the *base* of G^*, t is called the *stable letter*, and A and B are called the *associated subgroups*.

Definition 6 *A sequence $g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n$, $(n \geq 0)$, where $g_i \in G, \epsilon_i \in \{1, -1\}$, is said to be* reduced *if there is no consecutive subsequence t^{-1}, g_i, t with $g_i \in A$ or t, g_j, t^{-1} with $g_j \in B$.*

The following important result was proved by J.L.Britton in 1963.

Lemma 4 *[33] (Britton's Lemma)*

If the sequence $g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n$, where $g_i \in G, \epsilon_i \in \{1, -1\}$, is reduced and $n \geq 1$, then $g_0 t^{\epsilon_1} g_1 \cdots t^{\epsilon_n} g_n \neq 1$ in G^*.

We call the product of the elements in a reduced sequence a *reduced word*.

Observe that reduced words obtained from different reduced sequences may represent the same element in G^*. However, one can introduce unique *normal forms* in the following way.

Choose a set of representatives of the right cosets of A in G, and a set of representatives of the right cosets of B in G. We assume 1 to be the representative of both A and B. Further we assume chosen representatives to be fixed.

Definition 7 *A normal form is a sequence $g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n$, where $n \geq 0$, $g_i \in G$, $\epsilon_i \in \{1, -1\}$, such that*

(i) g_0 is an arbitrary element of G,

(ii) if $\epsilon_i = -1$, then g_i is a representative of a coset of A in G,

(iii) if $\epsilon_i = 1$, then g_i is a representative of a coset of B in G, and

(iv) there is no consecutive subsequence $t^{\epsilon}, 1, t^{-\epsilon}$, where $\epsilon \in \{1, -1\}$.

The Normal Form Theorem for HNN-extensions follows from Britton's Lemma and original result of Higman, Neumann and Neumann proved in [13].

Theorem 11 *[33] (The Normal Form Theorem for HNN-extensions)*

Let $G^* = \langle G, t \mid t^{-1}at = \phi(a), a \in A\rangle$ be an HNN-extension. Then

(I) The group G is embedded in G^* by the map $g \to g$. If $g_0t^{\epsilon_1}g_1 \cdots t^{\epsilon_n}g_n = 1$ in G^*, where $n \geq 1$, $g_i \in G$, $\epsilon_i \in \{1, -1\}$, then $g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n$ is not reduced.

(II) Every element w of G^* has a unique representation as $w = g_0t^{\epsilon_1}g_1 \cdots t^{\epsilon_n}g_n$, where $n \geq 0$, $g_i \in G$, $\epsilon_i \in \{1, -1\}$ and $g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n$ is a normal form.

We assign a length to each element z of G^* as follows. Let w be any reduced word which represents z in G^*. If $w = g_0t^{\epsilon_1}g_1 \cdots t^{\epsilon_n}g_n, n \geq 0, g_i \in G, \epsilon_i \in \{1, -1\}$, the *length* of z, written $|z|$, is the number n of occurrences of $t^{\pm 1}$ in w. The correctness of this definition follows from the lemma below.

Lemma 5 *[33] Let $u = g_0 t^{\epsilon_1} g_1 \cdots t^{\epsilon_n} g_n$ and $v = h_0 t^{\delta_1} h_1 \cdots t^{\delta_m} h_m$ be reduced words, where $g_i, h_i \in G$, $\epsilon_i, \delta_i \in \{1, -1\}$, $n, m \geq 0$ and suppose $u = v$ in G^*. Then $m = n$ and $\epsilon_i = \delta_i$, $i = 1, \ldots, n$.*

3.4 Centralizer extensions

The following particular case of free product with amalgamation plays a very important role in all further considerations.

Let G be a group. The centralizer of an element $v \in G$ in G is denoted by $C_G(v)$.

If $G = \langle X \mid R \rangle$ is a presentation of G, Y is a set of words in X and t is a new letter (not in X), then by $\langle G, t \mid [Y, t] = 1 \rangle$ we denote the group with the presentation $\langle X, t \mid R, [y, t]_{(y \in Y)} \rangle$.

Definition 8 *The group $G(v, t) = \langle G, t \mid [C_G(v), t] = 1 \rangle$ is called the direct of rank 1 extension of the centralizer of the element v.*

It is easy to see that $G(v, t)$ can be obtained from G by an HNN-extension with respect to the identity isomorphism $C_G(v) \to C_G(v)$:

$$G(v, t) = \langle G, t \mid t^{-1} a t = a, a \in C_G(v) \rangle.$$

or as a free product with amalgamation

$$\langle G * (C_G(v) \times \langle t \rangle) \mid C_G(v) = \phi(C_G(v)) \rangle$$

with amalgamation by the canonical monomorphism $\phi : C_G(v) \to C_G(v) \times \langle t \rangle$.

Definition 9 *Let G, H be groups, $C_G(v) = C$ the centralizer of an element v from G, then the group*

$$G(v, H) = \langle G, H \mid [C, H] = 1 \rangle$$

is obtained from G by direct extension of the centralizer of v by the group H.

The group $G(v, H)$ has the following decomposition as a free product with amalgamation:

$$\langle G * (C_G(v) \times H) \mid C_G(v) = \phi(C_G(v)) \rangle.$$

Further we refer to $G(v, H)$ as an extension of centralizer of v in G.

Observe that due to the definition above the centralizer of v in $G(v, H)$ becomes $C_G(v) \times H$.

The construction of centralizer extension was introduced in [38] in the course of investigating exponential groups. It turns out that this construction is very important in algebraic geometry over groups.

4 Actions on Λ-trees and length functions

Here we present some important results from the theory of Λ-trees and length functions.

The theory of Λ-trees has its origins in two papers, by I. Chiswell [9] in 1976 and by J.Tits [51] in 1977. Chiswell's paper contains a construction of an $\mathbb{R}$-tree starting from a Lyndon length function on a group, an idea considered earlier by Lyndon [32].

The theory was further developed in an important paper by Alperin and Bass [1] where authors state a fundamental problem in the theory of Λ-trees. Find the group theoretic information carried by a Λ-tree action, analogous to that presented in [50] for the case $\Lambda = \mathbb{Z}$, due to Bass and Serre. This problem is still far from solved and in general is intractable. However, some aspects of the theory of group actions on ordinary trees can be generalized to group actions (isometries) on Λ-trees. One way to proceed is to put restrictions on the action, another one is to put restrictions on Λ.

4.1 Λ-trees and actions

Definition 10 *Let X be a non-empty set, Λ an ordered abelian group. A Λ-metric on X is a mapping $p : X \times X \longrightarrow \Lambda$ such that for all $x, y, z \in X$:*

(1) $p(x, y) \geq 0$.

(2) $p(x, y) = 0$ if and only if $x = y$.

(3) $p(x, y) = p(y, x)$.

(4) $p(x, y) \leq p(x, z) + p(y, z)$.

So a Λ-metric space is a pair (X, p), where X is a non-empty set and p is a Λ-metric on X. If (X, p) and (X', p') are Λ-metric spaces, an *isometry* from (X, p) to (X', p') is a mapping $f : X \to X'$ such that $p(x, y) = p'(f(x), f(y))$ for all $x, y \in X$.

A *segment* in a Λ-metric space is the image of an isometry $\alpha : [a, b]_\Lambda \to X$ for some $a, b \in \Lambda$ and $[a, b]_\Lambda$ is a segment in Λ. The endpoints of the segment are $\alpha(a), \alpha(b)$.

We call a Λ-metric space (X, p) *geodesic* if for all $x, y \in X$, there is a segment in X with endpoints x, y and (X, p) is *geodesically linear* if for all $x, y \in X$, there is a unique segment in X whose set of endpoints is $\{x, y\}$.

It is not hard to see, for example, that (Λ, p) is a geodesically linear Λ-metric space, where $p(a, b) = |a - b|$, and the segment with endpoints a, b is $[a, b]_\Lambda$.

Let (X, p) be a Λ-metric space. Choose a point $v \in X$, and for $x, y \in X$, define

$$(x \cdot y)_v = \frac{1}{2}(p(x, v) + p(y, v) - p(x, y)).$$

Observe, that in general $(x \cdot y)_v \in \frac{1}{2}\Lambda$.

The following simple result follows immediately

Lemma 6 *[10] If (X, p) is a Λ-metric space then the following are equivalent:*

(i) for some $v \in X$ and all $x, y \in X$, $(x \cdot y)_v \in \Lambda$.

(ii) for all $v, x, y \in X$, $(x \cdot y)_v \in \Lambda$.

We now give Gromov's definition of a hyperbolic space (see [17]), which is applicable to any metric space.

Definition 11 *Let $\delta \in \Lambda$ with $\delta \geq 0$. Then (X, p) is δ-hyperbolic with respect to v if, for all $x, y, z \in X$,*

$$(x \cdot y)_v \geq min\{(x \cdot z)_v, (z \cdot y)_v\} - \delta.$$

Lemma 7 *[10] If (X, p) is δ-hyperbolic with respect to v, and t is any other point of X, then (X, p) is 2δ-hyperbolic with respect to t.*

Definition 12 *A Λ-tree is a Λ-metric space (X, p) such that:*

(1) (X, p) is geodesic.

(2) if two segments of (X, p) intersect in a single point, which is an endpoint of both, then their union is a segment.

(3) the intersection of two segments with a common endpoint is also a segment.

Example 1 *Λ together with the usual metric $p(a, b) = |a - b|$ is a Λ-tree. Moreover, any convex set of Λ is a Λ-tree.*

Example 2 *A $\mathbb{Z}$-metric space (X, p) is a $\mathbb{Z}$-tree if and only if there is a simplicial tree Γ such that $X = V(\Gamma)$ and p is the path metric of Γ.*

Observe that in general a Λ-tree can not be viewed as a simplicial tree with the path metric like in Example 2. Even in the case when $\Lambda = \mathbb{R}$ one can have some tricky Λ-trees, which are not trees in the usual sense.

Example 3 *Here is an example of an $\mathbb{R}$-tree. Let $Y = \mathbb{R}^2$ be the plane, but with metric p defined by*

$$p((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1| + |y_2| + |x_1 - x_2| & \text{if } x_1 \neq x_2 \\ |y_1 - y_2| & \text{if } x_1 = x_2 \end{cases}$$

That is, to measure the distance between two points not on the same vertical line, we take their projections onto the horizontal axis, and add their distances to these projections and the distance between the projections (distance in the usual Euclidean sense).

Lemma 8 *[10] Let (X, p) be Λ-tree. Then (X, p) is 0-hyperbolic, and for all $x, y, v \in X$ we have $(x \cdot y)_v \in \Lambda$.*

For all the details, results and discussions about Λ-metric spaces, trees and hyperbolic metric spaces we refer to [17], [10].

Eventually, we say that group G acts on a Λ-tree X if any element $g \in G$ defines an isometry $g : X \to X$.

Note, that every group has a trivial action on a Λ-tree, that is, all its elements act as identity.

Definition 13 *Let a group G act as isometries on a Λ-tree X. $g \in G$ is called* elliptic *if it has a fixed point. $g \in G$ is called an* inversion *if it does not have a fixed point, but g^2 does. If g is not elliptic and not an inversion then it is called* hyperbolic.

A group G acts *freely* and *without inversions* on a Λ-tree X if for all $1 \neq g \in G$, g acts as a hyperbolic isometry. In this case we also say that G is Λ-free.

The case of free actions on Λ-trees is simpler than the general case and many interesting results were obtained in this direction.

The next two theorems are general results in the theory of groups acting freely on Λ-trees. The second one is very important because it provides some kind of induction, which can be used when dealing with an arbitrary ordered abelian group Λ.

Theorem 12 *[21] If $\{G_i \mid i \in I\}$ is a collection of Λ-free groups then the free product $*_{i \in I} G_i$ is Λ-free.*

Theorem 13 *[2] Let a group G act freely and without inversions on a Λ-tree, where $\Lambda = \mathbb{Z} \times \Lambda_0$. Then there is a graph of groups (Γ, Y^*) such that:*

1) $G = \pi_1(\Gamma, Y^*)$;

2) for every vertex $x^* \in Y^*$, a vertex group Γ_{x^*} acts freely and without inversions on a Λ_0-tree;

3) for every edge $e \in Y^*$ with an endpoint x^* an edge group Γ_e is either maximal abelian subgroup in Γ_{x^*} or is trivial and Γ_{x^*} is not abelian;

4) if $e_1, e_2, e_3 \in Y^*$ are edges with an endpoint x^* then $\Gamma_{e_1}, \Gamma_{e_2}, \Gamma_{e_3}$ are not all conjugate in Γ_{x^*}.

Conversely, from the existence of a graph (Γ, Y^*) satisfying conditions 1)–4) it follows that G acts freely and without inversions on a $\mathbb{Z} \times \Lambda_0$-tree in the following cases: Y^* is a tree, $\Lambda_0 \subset Q$ and either $\Lambda_0 = Q$ or Y^* is finite.

Another way of developing the theory of actions on Λ-trees is to put restrictions on Λ.

In the case when $\Lambda = \mathbb{Z}$ the following result is a consequence of the Bass-Serre Theory.

Theorem 14 *[50] A group G acts freely and without inversions on a $\mathbb{Z}$-tree if and only if G is a free group.*

In fact, it is easy to see, without recourse to the general theory, that a free group acts freely on its Cayley graph with respect to a basis, which is a simplicial tree, giving a free action without inversion on the corresponding $\mathbb{Z}$-tree.

The next results deal with free actions on $\mathbb{R}$-trees.

At first, a classical result due to Harrison [21].

Theorem 15 *[21] Let G be a group acting freely and without inversions on an $\mathbb{R}$-tree X, and suppose $g, h \in G \setminus \{1\}$. Then $\langle g, h \rangle$ is either free of rank two or abelian.*

A joint effort of several researchers culminated in a description of finitely generated groups with real-valued free length function ([36], [43], [18], [19], [8]) which is now known as Rips' Theorem.

Theorem 16 *[18] (Rips' Theorem)*

Let G be a finitely generated group acting freely and without inversions on an $\mathbb{R}$-tree. Then G can be written as a free product $G = G_1 * \cdots * G_n$ for some *integer $n \geq 1$, where each G_i is either a finitely generated free abelian group or the fundamental group of a closed surface.*

4.2 Lyndon length functions

Let G be a group and A be an ordered abelian group. Then a function $l : G \to A$ is called a *(Lyndon) length function* on G if the following conditions hold:

(L1) $\forall\, x \in G : \ l(x) \geq 0$ and $l(1) = 0$;

(L2) $\forall\, x \in G : \ l(x) = l(x^{-1})$;

(L3) $\forall\, x, y, z \in G : \ d(x, y) > d(x, z) \to d(x, z) = d(y, z)$,

where $d(x, y) = \frac{1}{2}(l(x) + l(y) - l(x^{-1}y))$.

To define $d(x, y)$ precisely we assume that A is embedded in its divisible completion $A \otimes_{\mathbb{Z}} \mathbb{Q}$ with the fraction order (which extends the order on A). On the other hand, one can formulate the axiom (L3) using the function $d'(x, y) = 2d(x, y)$ instead of $d(x, y)$.

It is not difficult to derive the following two properties $(L4)$ and $(L5)$ from the axioms above:

(L4) $\forall\, x, y \in G : \ l(xy) \leq l(x) + l(y)$;

(L5) $\forall\, x, y \in G : \ 0 \leq d(x, y) \leq min\{l(x), l(y)\}$.

A length function $l : G \to A$ is called *free*, if it satisfies

(L6) $\forall\, x \in G : \ x \neq 1 \to l(x^2) > l(x)$

Let G be a group acting as isometries on a Λ-tree (X, p) and let $x \in X$. Then defining $l_x(g) = p(x, gx)$ gives a Lyndon length function on G.

Of a special importance to us is the following result due to Chiswell [9], where it is proved that a free action of G on a Λ-tree corresponds to the existence of a free length function on G which takes values in Λ.

Theorem 17 *[9] Let G be a group and $l : G \to \Lambda$ a Lyndon length function satisfying the following condition:*

$$\forall g, h \in G : \ d(g, h) \in \Lambda.$$

Then there are a Λ-tree (X, p), an action of G on X and a point $x \in X$ such that $l = l_x$.

Lyndon himself proved that groups with free length functions with values in $\mathbb{Z}$ are just subgroups of free groups with the induced length functions [32] (compare with the corresponding consequence of a Bass-Serre Theory).

In Section 9 we prove the following result.

Theorem 18 *Let A be a discretely ordered abelian group and X be a set. Then the function $L : CDR(A, X) \to A$ defined as $L(w) = |w|$ satisfies all the axioms (L1) - (L6), whenever corresponding products of elements in these axioms are defined.*

This implies that every group embeddable into $CDR(A, X)$ has a free length function with values in A. Moreover, in the special case when $A = \mathbb{Z}[t]^+$, some subgroups of $CDR(A, X)$, in particular, the free $\mathbb{Z}[t]$-group $F^{\mathbb{Z}[t]}$, have free length functions which are easily computable and satisfy the following extra axiom (L7).

For elements $x_1, \ldots, x_n \in G$ we write $x = x_1 \circ \cdots \circ x_n$ if $x = x_1 \cdots x_n$ and $l(x) = l(x_1) + \cdots + l(x_n)$.

The length function $l : G \to A$ is called *regular* if it satisfies the following *regularity* axiom:

(L7) $\forall x, y \in G, \exists u, x_1, y_1 \in G$:

$$x = u \circ x_1 \ \& \ y = u \circ y_1 \ \& \ l(u) = d(x, y).$$

The element u in (L7) is called the *common initial segment* of x and y.

5 Elements of Algebraic Geometry over Groups

The classical algebraic geometry, having started as the theory of curves, now is closely linked to the ideal theory of finitely generated polynomial algebras over fields. In [5] foundations of algebraic geometry over groups were introduced and developed then in a series of papers (see for example [38],[15],[26],[27],[28]). In this section we present some basic notions of this theory and show how thesis results are related to it.

5.1 G-groups and basic definitions

Let G be a fixed group. A group H is called a *G-group* if it contains G as a distinguished subgroup. The class of G-groups forms a category. Morphisms of this category are homomorphisms of G-groups which are identical on G, that is, if H_1, H_2 are G-groups then $\phi : H_1 \to H_2$ is a *G-homomorphism* if $\phi(g) = g$ for all $g \in G$. By $Hom_G(H, K)$ we denote the set of all G-homomorphisms from H into K.

The free object in the category of G-groups is the free product $G * F(X)$, where $F(X)$ is a free group on some alphabet X. This group is called a free G-group with basis X, and we denote it by $G[X]$.

A G-group H is termed finitely generated G-group if there exists a finite subset $B \subset H$ such that the set $G \cup B$ generates H. We refer to [5] for general discussion of G-groups.

Let G be generated by a finite set B and let $X = \{x_1, \ldots, x_n\}$. If $S \subset G[X]$ then $S = 1$ is called *a system* of equations over G. As an element of the free product the left side of every equation in $S = 1$ can be written as a product of some elements from $X \cup X^{-1}$, which are called *variables* and some elements from B, which are called *constants*.

The set $G^n = \{(g_1, \ldots, g_n) \mid g_i \in G\}$ is called *affine n-space over G*.

A *solution* of the system $S(X) = 1$ over a group G is an element $(g_1, \ldots, g_n) \in G^n$ such that after replacement of each x_i by g_i the left hand side of every equation in $S = 1$ turns into the trivial element in G. Equivalently, a solution of the system $S = 1$ over G can be described as a G-homomorphism $\phi : G[X] \to G$ such that $\phi(S) = 1$. Denote by G_S the quotient group $G[X]/ncl(S)$ of $G[X]$ by the normal closure $ncl(S)$ of S in $G[X]$. Then every solution of $S(X) = 1$ in G gives rise to a G-homomorphism $G_S \to G$. By $V_G(S)$ we denote the set of all solutions in G of the system $S = 1$, it is called *the algebraic set defined by* S. $V_G(S)$ uniquely corresponds to the normal subgroup

$$R(S) = \{T(x) \in G[X] \mid \forall g \in G^n \ (S(g) = 1 \to T(g) = 1)\}$$

of the group $G[X]$. Notice that if $V_G(S) = \emptyset$ then $R(S) = G[X]$. The subgroup $R(S)$ contains S, and is called the *radical of* S. The quotient group

$$G_{R(S)} = G[X]/R(S)$$

is the *coordinate group* of the algebraic set $V_G(S)$

5.2 Discrimination and big powers

Let H and K be G-groups. We say that a family of G-homomorphisms $\mathcal{F} \subset Hom_G(H, K)$ *separates (discriminates)* H into K if for every non-trivial element $h \in H$ (every finite set of non-trivial elements $H_0 \subset H$) there exists $\phi \in \mathcal{F}$ such that $h^\phi \neq 1$ ($h^\phi \neq 1$ for every $h \in H_0$). In this case we say that H is *G-separated* (*G-discriminated*) by K. Usually we simply say H is *separated* (*discriminated*) by K, without mentioning G or also H is *residually (fully residually)* K. In the event when K is a free group we say that H is *residually (fully residually) free*.

The following method of discrimination, which is called *big powers method* was introduced in [38]. For all the details about BP-groups we refer to [38] and [24].

Let G be a group. We say that a tuple $u = (u_1, \dots, u_k) \in G^k$ has *commutation* if $[u_i, u_{i+1}] = 1$ for some $i \in [1, k-1]$. Otherwise we call u *commutation-free*.

Definition 14 *A group G satisfies the* big powers condition *(BP) (or called a BP-group), if for any commutation-free tuple $u = (u_1, \dots, u_k)$ of elements from G there exists an integer $n(u)$ (a boundary of separation for u) such that*

$$u_1^{\alpha_1} \cdots u_k^{\alpha_k} \neq 1$$

for any integers $\alpha_1, \dots, \alpha_k \geq n(u)$.

The class of BP-groups is quite extensive. Obviously, a subgroup of a BP-group is a BP-group; a group discriminated by a BP-group is a BP-group ([38]); every torsion-free hyperbolic group is a BP-group ([24]). It follows that every freely discriminated group is a BP-group.

Another important class of groups is defined as follows. We call a group G a *CSA-group* if every maximal abelian subgroup M of G is *malnormal*, that is, $M^g \cap M = 1$ for any $g \in M$. The class os CSA-groups includes all abelian groups, all torsion-free hyperbolic groups ([38]), all groups acting freely on Λ-trees ([2]), and many one-relator groups ([15]).

Let G be a non-abelian CSA-group and $u \in G$ not a proper power. Recall the definition of a centralizer extension of the centralizer $C_G(u)$ by a letter t from Section 3:

$$G(u, t) = \langle G, t \mid g^t = g, g \in C_G(v) \rangle.$$

It is not hard to see that for any integer k the map $t \to u^k$ extends to a G-homomorphism $\phi_k : G(u, t) \to G$.

The result below is the essence of the big powers method of discrimination.

Theorem 19 *[38] Let G be a non-abelian CSA- BP-group and let $G(u,t)$ be a centralizer extension of the centralizer of not a proper power u by t. Then the family of G-homomorphisms $\mathcal{F} = \{\phi_k \mid k \text{ is an integer}\}$ discriminates $G(u,t)$ into G.*

If G is a non-abelian CSA- BP-group and X is a finite set, then the group $G[X]$ is G-embeddable into $G(u,t)$ for any proper power $u \in G$. It follows from the theorem above that $G[X]$ is G-discriminated by G.

Let G be a non-abelian CSA- BP-group and

$$G = G_0 < G_1 < \cdots < G_n$$

be a chain of extensions of centralizers $G_{i+1} = G_i(u_i, t_i)$. Then every n-tuple of integers $p = (p_1, \dots, p_n)$ gives rise to a G-homomorphism $\psi_p : G_n \to G$ which is the composition of homomorphisms $\phi_{p_i} : G_i \to G$ described above. A set P of n-tuples of integers is called *unbounded* if for every integer d there exists a tuple $p = (p_1, \dots, p_n) \in P$ with $p_i \geq d$ for each i. It follows from the theorem above that for every unbounded set of tuples P the set of G-homomorphisms $\Psi_P = \{\psi_p \mid p \in P\}$ G-discriminates G_n into G. Similar results hold for infinite chains of extensions of centralizers (see [38] and [4]).

5.3 Lyndon's free $\mathbb{Z}[t]$-group $F^{\mathbb{Z}[t]}$

Here we introduce the central object of investigation in this thesis. For all the details we refer to [31], [37], [38] and [6]

Let A be an associative unitary ring. A group G is termed an A-*group* if it comes equipped with a function (*exponentiation*) $G \times A \to G$:

$$(g, \alpha) \to g^\alpha$$

satisfying the following conditions for arbitrary $g, h \in G$ and $\alpha, \beta \in A$:

(E1) $g^1 = g$, $g^{\alpha+\beta} = g^\alpha g^\beta$, $g^{\alpha\beta} = (g^\alpha)^\beta$;

(E2) $g^{-1}h^\alpha g = (g^{-1}hg)^\alpha$;

(E3) if g and h commute, then $(gh)^\alpha = g^\alpha h^\alpha$.

A homomorphism $\phi : G \to H$ between two A-groups is termed A-homomorphism if $\phi(g^\alpha) = \phi(g)^\alpha$ for every $g \in G$ and $\alpha \in A$. It is not hard to prove (see, [37] or [38]) that for every group G there exists an A-group H (which is unique up to an A-automorphism) and a homomorphism $\mu : G \longrightarrow H$ such that for every A-group K and every A-homomorphism $\theta : G \longrightarrow K$, there exists a unique A-homomorphism $\phi : H \longrightarrow K$ such that $\phi\mu = \theta$. We denote H by G^A and call it *A-completion* of G. In particular, there exists a $\mathbb{Z}[t]$-completion $F^{\mathbb{Z}[t]}$ of a free group F. It was introduced by R. Lyndon in [31] who used different methods, and it is called now Lyndon's free $\mathbb{Z}[t]$-group.

In [38] an effective construction of $F^{\mathbb{Z}[t]}$ in terms of extensions of centralizers was given. For a group G choose a set $S = \{C_i \mid i \in I\}$ of representatives of conjugacy classes of proper cyclic centralizers in G, that is, every proper cyclic centralizer in G is conjugate to one from S, and no two centralizers from S are conjugate. Then the HNN-extension

$$H = \langle\, G, s_{i,j} \quad (i \in I, j \in \mathbb{N}) \;\mid\; [s_{i,j}, u_i] = [s_{i,j}, s_{i,k}] = 1, (u_i \in C_i, i \in I, j, k \in \mathbb{N}) \,\rangle,$$

where $\mathbb{N}$ stands for the set of positive natural numbers, is termed an *extension of all cyclic centralizers* in G. Let F be a free non-abelian group. We obtain the group $F^{\mathbb{Z}[t]}$ as union of the following infinite chain of groups:

$$F = G_0 < G_1 < \ldots < G_n \ldots < \ldots.$$

where G_{i+1} is obtained from G_i by extension of all cyclic centralizers in G_i.

5.4 Freely discriminated groups and $F^{\mathbb{Z}[t]}$

Now, we are finally ready to introduce the connection between algebraic geometry over groups and the results of the thesis. This connection lies in the following theorems.

Theorem 20 *[5],[25] Let F be a free non-abelian group. Then a finitely generated F-group G is the coordinate group of a non-empty irreducible algebraic set over F if and only if G is freely F-discriminated by F.*

Theorem 21 *[26] Let F be a free non-abelian group. Then a finitely generated F-group G is the coordinate group of a non-empty irreducible algebraic set over F if and only if G is F-embeddable into Lyndon's free $\mathbb{Z}[t]$-group $F^{\mathbb{Z}[t]}$.*

The last theorem implies that finitely generated freely discriminated groups are finitely presented.

Now it is clear why it is important to study finitely generated subgroups of $F^{\mathbb{Z}[t]}$ - this study gives answers to many questions in algebraic geometry over free groups. One way of investigating properties of finitely generated subgroups of $F^{\mathbb{Z}[t]}$ is the use of classical methods of combinatorial group theory: free constructions, Bass-Serre theory, JSJ-decompositions, etc. In the present thesis we suggest another way. We represent elements of the group $F^{\mathbb{Z}[t]}$ by infinite words in the alphabet $X^{\pm 1}$. These words are functions of the type

$$w : [1, f_w] \to X^{\pm 1},$$

where $f_w \geq 0$, $f_w \in \mathbb{Z}[t]^+$ and $[1, f_w] = \{g \in \mathbb{Z}[t]^+ \mid 1 \leq g \leq f_w\}$ is a closed interval in $\mathbb{Z}[t]^+$, the additive group of $\mathbb{Z}[t]$, viewed as an abelian ordered group with respect to the right lexicographical order $\leq$. The function $L : w \longrightarrow f_w$ gives rise to a regular free Lyndon length function on $F^{\mathbb{Z}[t]}$ with values in $\mathbb{Z}[t]^+$.

Once the representation of elements of $F^{\mathbb{Z}[t]}$ by infinite words is established, a host of problems about $F^{\mathbb{Z}[t]}$ can be solved pretty much in the same way as for the standard free group F, replacing the standard length function by the one constructed above. In particular, we associate finitely generated subgroups of $F^{\mathbb{Z}[t]}$ with special combinatorial objects called $(\mathbb{Z}[t], X)$-graphs, and then using combinatorial techniques we investigate properties of $(\mathbb{Z}[t], X)$-graphs which reveal underlying properties of finitely generated subgroups of $F^{\mathbb{Z}[t]}$.

The results presented in this thesis have many interesting applications. In particular, one of the applications stems from the regularity of the length function L. The regularity condition is crucial for Nielsen cancellation method, which is the base for Makanin technique for solving equations over F (see [34]). It turns out, that if G is a coordinate group of an irreducible algebraic set over F with a computable regular free Lyndon length function $l : G \longrightarrow \mathbb{Z}^n$ then a Makanin's type argument can be applied for solving equations over G (see [28]). This plays an important part in proving decidability of the elementary theories of free groups.

Observe, that one can derive from the description of $F^{\mathbb{Z}[t]}$ in [38] and the results of H.Bass ([2]) that $F^{\mathbb{Z}[t]}$ acts freely on a Λ-tree, where Λ is a free abelian group of countable rank with the lexicographic order. This implies, that $F^{\mathbb{Z}[t]}$, indeed, has a free Lyndon length function with values in $\mathbb{Z}[t]^+$, but the method does not provide any information whether this length is regular, or computable.

Our method for constructing free length functions on groups is quite general and can be applied to a wide class of groups. It is based on the study of infinite A-words in an alphabet $X^{\pm}$, where A is a discretely ordered abelian group. An A-word is a function $w : [1, \alpha] \longrightarrow X^{\pm}$, where 1 is the minimal positive element of A and $[1, \alpha]$ is a closed segment in A. One can consider a set $R(A, X)$ of all reduced A-words which comes equipped with the natural partial multiplication and length function $w \longrightarrow \alpha_w$. For a given group G every embedding of G into $R(A, X)$ provides immediately a natural free length function on G inherited from $R(A, X)$. It turned out (quite unexpectedly) that Stalling's pregroups supply the most adequate technique to study $R(A, X)$.

6 Stallings' pregroups and their universal groups

For detailed discussions and results on prees and pregroups we refer to [3], [22], [30], [44], [45] and [46].

Let P be a *pree*, that is, let P be a non-empty set with a partial operation $m : D \to P$, where $D \subseteq P \times P$. (We say pq is *defined* if $(p, q) \in D$ and we will usually denote $m(p, q)$ by pq). The term "pree" was introduced by Lipschutz in [30].

Baer [3] defined a pregroup P as a pree satisfying certain axioms and Stallings, who invented the name *pregroup*, proved additional properties of pregroups in [45] and [46]. These results were extended by Rimlinger in [44].

Further we use the definition of a pregroup P and its universal group $U(P)$ given by Stallings.

A *pregroup* P is a set P, with a distinguished element ϵ, equipped with a partial multiplication, that is a function $m : D \to P$, $m : (x, y) \to xy$, where $D \subset P \times P$, and an inversion, that is a function $P \to P$, $x \to x^{-1}$, satisfying the following axioms:

(P1) for all $u \in P$ $u\epsilon$ and ϵu are defined and $u\epsilon = \epsilon u = u$;

(P2) for all $u \in P$, $u^{-1}u$ and uu^{-1} are defined and $u^{-1}u = uu^{-1} = \epsilon$;

(P3) for all $u, v \in P$, *if* uv is defined, then so is $v^{-1}u^{-1}$, and $(uv)^{-1} = v^{-1}u^{-1}$;

(P4) for all $u, v, w \in P$, if uv and vw are defined, then $(uv)w$ is defined if and only if $u(vw)$ is defined, in which case

$$(uv)w = u(vw);$$

(P5) for all $u, v, w, z \in P$ if uv, vw, and wz are all defined then either uvw or vwz is defined.

It was noticed (see [20]) that (P3) follows from (P1), (P2), and (P4), hence can be omitted.

Stallings defined the *universal group* $U(P)$ of a pregroup (P, D) to be the free group on the set P modulo the relations $\{xy = z \mid x, y, z \in P \text{ and } z = m(x, y)\}$.

$U(P)$ can be characterized by a universal property. At first, a map $\phi : P \to Q$ of pregroups is a *morphism* if for any $x, y \in P$ whenever xy is defined in P, $\phi(x)\phi(y)$ is defined in Q and equal to $\phi(xy)$.

Now the group $U(P)$ can be characterized by the following universal property: there is a morphism of pregroups $\lambda : P \to U(P)$, such that, for any morphism $\phi : P \to G$ of P into a group G, there is a unique group homomorphism $\psi : U(P) \to G$ for which $\psi\lambda = \phi$. This shows that $U(P)$ is a group with a generating set P and a set of relations $xy = z$, where $x, y \in P$, xy is defined in P, and equal to z.

There exists an explicit construction of $U(P)$ due to Stallings [46]. Namely, a finite sequence $(u_1, \ldots, u_n)$ of elements from P is called a *reduced* P-sequence if for any $1 \leq i \leq n-1$ the product $u_i u_{i+1}$ is not defined in P. The group $U(P)$ can be described as the set of equivalence classes on the set of all reduced P-sequences modulo equivalence relation $\sim$, where $(u_1, \ldots, u_n) \sim (v_1, \ldots, v_m)$ if and only if $m = n$ and there exist elements $a_1, \ldots, a_{n-1} \in P$ such that $v_i = a_{i-1}^{-1} u_i a_i$ for $1 \leq i \leq n$ (here $a_0 = a_n = 1$). The multiplication on $U(P)$ is given by concatenation of representatives and reduction of the resulting sequence.

If G is isomorphic to $U(P)$, we say that G has a *pregroup structure* (P, D). Stallings showed in [45] that free products with amalgamation and HNN-extensions have pregroup structures. Later, Rimlinger in [44] proved more general results - if G is a fundamental group of a graph of groups satisfying some particular conditions then there exists a pregroup structure for G.

Observe that a pregroup P provides a very economic way to describe normal forms of elements of $U(P)$.

Pregroups are used in this thesis as a convenient language to describe various presentations of groups by infinite words. In Section 8.2 we prove that the set $R(A, X)$ with the partial multiplication $*$ and the inversion $^{-1}$ satisfies the axioms (P1) - (P4). In general, $R(A, X)$ does not satisfy (P5), but some subsets of it, which play an important role in our constructions, do.

7 X-graphs

Finally, to complete preliminary material we present basic notions from the theory of Stallings foldings of X-graphs introduced in [23]. This material will be used in the second part of the thesis.

The theory of X-graphs is based on the fact that a free group F can be identified with the fundamental group of a topological graph (which we may think of as a 1-complex). Then any subgroup of F corresponds to a covering map from another graph to the original graph. The topological viewpoint was studied in detail by J.Stallings in [47]. In this work J.Stallings introduced an extremely useful notion of a *folding* of graphs. The ideas of Stallings have found many interesting applications, see, for example, [12], [17], [48], [49].

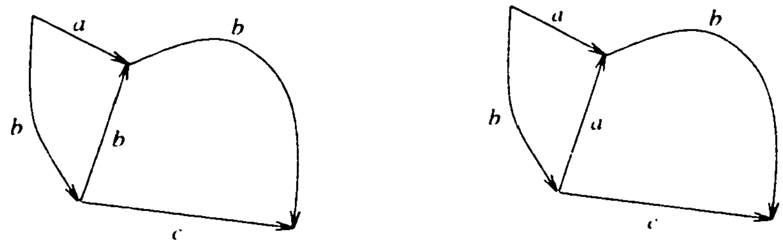
Definition 15 *(X-graph) Let $X = \{x_1, \ldots, x_N\}$ be a finite alphabet. By an X-labeled directed graph Γ (or X-graph) we mean the following:*

Γ is a combinatorial graph where every edge e has an arrow (direction) and is labeled by a letter from X, denoted $\mu(e)$.

For each edge e of Γ we denote the origin of e by $o(e)$ and the terminus of e by $t(e)$.

We can make Γ into an oriented graph labeled by the alphabet $X \cup X^{-1}$. Namely, for each edge e of Γ we introduce a formal inverse e^{-1} of e with label $\mu(e)^{-1}$ and the endpoints defined as $o(e^{-1}) = t(e)$, $t(e^{-1}) = o(e)$. The arrow on e^{-1} points from the terminus of e to the origin of e. For the new edges e^{-1} we set $(e^{-1})^{-1} = e$.

The usual notion of a *path* in Γ can be introduced. Namely, a *path* p in Γ is a sequence of edges $p = e_1, \ldots, e_k$ where each e_i is an edge of Γ and the origin of each e_i (for $i > 1$) is the terminus of e_{i-1}. Such a path p has a naturally defined label $\mu(p) = \mu(e_1)\ldots\mu(e_k)$. Thus $\mu(p)$ is a word in the alphabet $X \cup X^{-1}$.



(a) (b)

Figure 1: Folded and non-folded graphs. The labeling alphabet is $X = \{a, b, c\}$.

Definition 16 *(Folded graphs) Let Γ be an X-graph. We say that Γ is* folded *if for each vertex v of Γ and each letter $a \in X$ there is at most one edge in Γ with origin v and label a and there is at most one edge with terminus v and label a.*

The graph shown in Figure 1(a) is folded and the graph shown in Figure 1(b) is not folded. Note that Γ is folded if and only if for each vertex v of Γ and each $x \in X \cup X^{-1}$ there is at most one edge in Γ with origin v and label x.

One can introduce the following operation on X-graphs - suppose Γ is an X-graph and e_1, e_2 are edges of Γ with common origin and the same label $x \in X \cup X^{-1}$, then *folding* Γ at e_1, e_2 means identifying e_1 and e_2 into a single new edge labeled x. The resulting graph carries a natural structure of an X-graph (see Figure 2). For a more precise definition we refer to [23].

The following theorem follows directly from the results presented in [23] and is one of the most important in the theory of X-graphs.

Theorem 22 *[23] Let Γ be any finite X-graph. Then there exists a folded X-graph Δ which can be obtained from Γ by finitely many foldings.*

Definition 17 *(Language recognized by an X-graph) Let Γ be an X-graph and let v be a vertex of Γ. We define the language of Γ with respect to v to be:*

$$L(\Gamma, v) = \{\mu(p) \mid p \text{ is a reduced path in } \Gamma \text{ from } v \text{ to } v\}$$

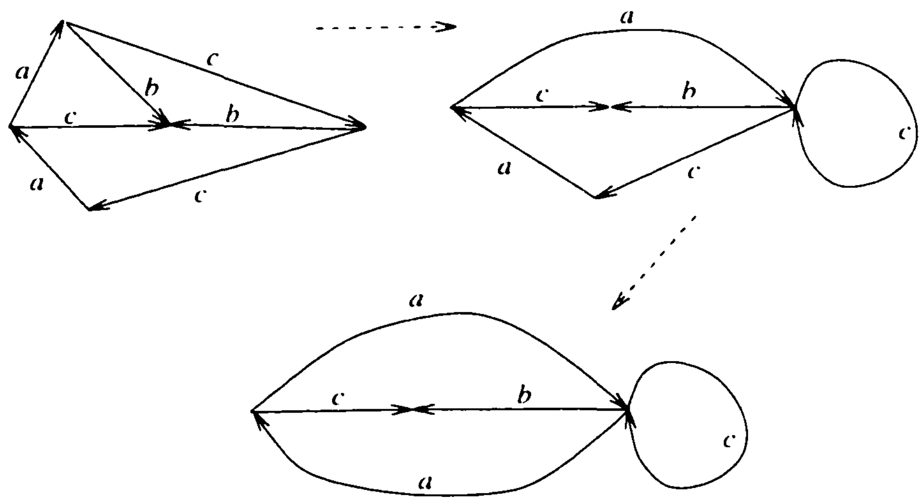


Figure 2: Folding of labeled graphs

The next result connects languages of X-graphs with finitely generated subgroups of $F(X)$.

Proposition 1 *[23] Let Γ be an X-graph and let v be a base-vertex of Γ. Then the set*

$$\overline{L} = \{\overline{w} \mid w \in L(\Gamma, v)\}$$

is a subgroup of $F(X)$.

The converse is also true.

Theorem 23 *[23] Let H be a finitely generated subgroup of $F(X)$. Then there exists a finite X-graph Γ and a vertex v of Γ such that:*

1. *the graph Γ is folded and connected;*
2. *the language of Γ is equal to H, that is $L(\Gamma, v) = H$.*

Moreover, the graph Γ above is unique up to *morphisms* of X-graphs, hence we denote it $\Gamma(H)$.

The correspondence between finitely generated subgroups of $F(X)$ and X-graphs gives a new method of investigating properties of subgroups of free groups. In particular, many algorithmic problems for them can be solved more easily than by using classical methods of combinatorial group theory.

In the second part of the thesis we combine the techniques of infinite words introduced in the first part (Sections 8–13) and methods of X-graphs to solve algorithmic problems for $F^{\mathbb{Z}[t]}$ and its finitely generated subgroups. For this purpose we adjust the definition of X-graph and introduce new combinatorial objects denoted $(\mathbb{Z}[t], X)$-graphs edges of which can be labeled not only by letters from $X^{\pm 1}$, but also by infinite words of some particular type.

8 A-words

8.1 Definitions

Let A be a discretely ordered abelian group and let $A^+ = \{a \in A \mid a \geq 0\}$. By 1_A we denote the minimal positive element of A. Recall that if $a, b \in A$ then the closed segment $[a, b]$ is defined as

$$[a, b] = \{x \in A \mid a \leq x \leq b\}.$$

Every closed segment $[a, b]$ is a convex subset of A. For a function $f : B \to C$ by $dom(f)$ we denote the domain B of f.

Let $X = \{x_i \mid i \in I\}$ be a set. Put $X^{-1} = \{x_i^{-1} \mid i \in I\}$ and $X^{\pm} = X \cup X^{-1}$. As usual we define an involution ${}^{-1}$ on $X^{\pm}$ by $(x)^{-1} = x^{-1}$ and $(x^{-1})^{-1} = x$.

Definition 18 *An A-word is a function of the type*

$$w : [1_A, \alpha] \to X^{\pm},$$

where $\alpha \in A^+$. The element α is called the length $|w|$ of w.

By $W(A, X)$ we denote the set of all A-words. Observe, that $|w| = 0$ if and only if the domain of w is empty ($[1_A, 0] = \emptyset$), i.e. the function w is empty. We denote this function by ε. Also, we say that an element $w \in W(A, X)$ has a *finite length* if $|w| \in \mathbb{Z}$.

Concatenation uv of two words $u, v \in W(A, X)$ is an A-word of length $|u| + |v|$ and such that:

$$(uv)(a) = \begin{cases} u(a) & \text{if } 1_A \leq a \leq |u| \\ v(a - |u|) & \text{if } |u| < a \leq |u| + |v| \end{cases}$$

In particular, $\varepsilon u = u\varepsilon = u$ for any $u \in W(A, X)$. For any A-word w we define an *inverse* w^{-1} as an A-word of the length $|w|$ and such that

$$w^{-1}(\beta) = w(|w| + 1_A - \beta)^{-1} \quad (\beta \in [1_A, |w|]).$$

An A-word w is *reduced* if $w(\beta + 1_A) \neq w(\beta)^{-1}$ for each $1_A \leq \beta < |w|$. We denote by $R(A, X)$ the set of all reduced A-words. Clearly, $\varepsilon \in R(A, X)$.

Of course, concatenation uv of two reduced words u, v may not be reduced. We write $u \circ v$ instead of uv in the case when uv is reduced, i.e. $u(|u|) \neq v(1_A)^{-1}$. Obviously, $\circ$ satisfies the following cancellation conditions:

$$x \circ y = x \circ z \Longrightarrow y = z, \quad y \circ x = z \circ x \Longrightarrow y = z.$$

8.2 Multiplication

In this subsection we introduce a (partial) multiplication on $R(A, X)$ and show that it satisfies the axioms (P1) - (P4) of pregroups.

For $u \in W(A, X)$ and $\beta \in dom(u)$ by $u_\beta = u \mid_\beta$ we denote the restriction of u on $[1_A, \beta]$. If u is reduced and $\beta \in dom(u)$ then

$$u = u_\beta \circ \tilde{u}_\beta,$$

for some uniquely defined $\tilde{u}_\beta$ (namely, $|\tilde{u}_\beta| = |u| - \beta$ and $\tilde{u}_\beta(\gamma) = u(\beta + \gamma)$ for each $\gamma \in [1_A, |u| - \beta]$).

An element $c \in R(A, X)$ is called the *(longest) common initial segment* of A-words u and v if

$$u = c \circ \tilde{u}, \quad v = c \circ \tilde{v}$$

for some (uniquely defined) A-words $\tilde{u}, \tilde{v}$ such that $\tilde{u}(1_A) \neq \tilde{v}(1_A)$. We denote such

e as $c(u, v)$. Notice that, there are words u, v for which the common initial segment $c(u, v)$ does not exist. It is not hard to see that $c(u, v)$ exists if and only if the following element from A is defined:

$$\delta(u, v) = \begin{cases} 0 & \text{if } u(1_A) \neq v(1_A) \\ \max\{\beta \mid u_\beta = v_\beta\} & \text{if such } \beta \text{ exists} \\ undefined & \text{otherwise} \end{cases}$$

In this case

$$c(u, v) = u \mid_{\delta(u,v)} = v \mid_{\delta(u,v)} .$$

Clearly, if the length of u is finite then $\delta(u, v)$ and $\delta(v, u)$ are defined for every $v \in R(A, X)$.

Definition 19 *A pair of reduced A-words (u, v) is called regular if $c(u^{-1}, v)$ is defined. In this event*

$$u^{-1} = c(u^{-1}, v) \circ \tilde{u}, \quad v = c(u^{-1}, v) \circ \tilde{v},$$

for some uniquely defined $\tilde{u}$ and $\tilde{v}$. Put

$$u * v = \tilde{u}^{-1} \circ \tilde{v}.$$

The product $*$ is a partial binary operation on $R(X, A)$.

Example 4 *Let $A = \mathbb{Z}^2$ with the right lexicographic order (in this case $1_A = (1, 0)$). Put*

$$w(\beta) = \begin{cases} x & \text{if } \beta = (s, 0) \text{ and } s \geq 1 \\ x^{-1} & \text{if } \beta = (s, 1) \text{ and } s \leq 0 \end{cases}$$

Then

$$w : [1_A, (0, 1)] \rightarrow X^{\pm}$$

is a reduced A-word. Clearly, $w^{-1} = w$ *so* $w * w = \varepsilon$*. In particular,* $R(A, X)$ *has 2-torsion with respect to* $*$.

One has to be careful with "infinite cancellations". We are going to use sometimes term "cancellation" in the following precise meaning. If $w = uzz^{-1}v$, where w, u, z, v are A-words, then we say that w has cancellation (or reduction) zz^{-1} and uv is obtained from w by cancelling (or reducing) zz^{-1}.

The main result of this section is the following theorem.

Theorem 24 *Let* A *be a discretely ordered abelian group and* X *be a set. Then the set of reduced A-words* $R(A, X)$ *with the partial binary operation* $*$ *satisfies the axioms (P1) - (P4) of a pregroup.*

Proof. The axioms (P1), (P2), and (P3) follow immediately from definitions.

Let u, v, w be reduced A-words such that the products $u * v, v * w$ are defined. Suppose that one of the products $(u * v) * w, u * (v * w)$, say $(u * v) * w$, is defined (the other case is similar). We need to show that the product $u * (v * w)$ is also defined and the equality

$$(u*v)*w = u*(v*w). \tag{2}$$

holds. To this end we consider several cases.

C1). Let $uv = u \circ v$. Since $v * w$ is defined we have

$$v = v_1 \circ v_2, \quad w = v_2^{-1} \circ w_2, \quad v_1 * w_2 = v_1 \circ w_2 \tag{3}$$

for some (perhaps, trivial) $v_1, v_2, w_2 \in R(A, X)$.

a) Let $v_1 \neq \varepsilon$. In this case

$$(u * v) * w = (u \circ v_1 \circ v_2) * (v_2^{-1} \circ w_2) = u \circ v_1 \circ w_2.$$

On the other hand,

$$u * (v * w) = u * \big((v_1 \circ v_2) * (v_2^{-1} \circ w_2)\big) = u * (v_1 \circ w_2) = u \circ v_1 \circ w_2,$$

so it is defined and (2) holds.

b) Let $v_1 = \varepsilon$. Then $w = v^{-1} \circ w_2$ and in this case

$$(u * v) * w = (u \circ v) * (v^{-1} \circ w_2) = u * w_2,$$

which is defined. On the other hand,

$$u * (v * w) = u * (v * (v^{-1} \circ w_2)) = u * w_2,$$

which is also defined and equal to $(u * v) * w$.

C2). Let $u = v^{-1}$. In this case

$$(u * v) * w = (v^{-1} * v) * w = w.$$

On the other hand, in notations from (3), we have

$$u * (v * w) = v^{-1} * (v * w) = (v_2^{-1} \circ v_1^{-1}) * \big((v_1 \circ v_2) * (v_2^{-1} \circ w_2)\big) =$$

$$= (v_2^{-1} \circ v_1^{-1}) * (v_1 \circ w_2) = v_2^{-1} \circ w_2 = w,$$

hence it is defined and (2) holds.

Now we are ready to prove the general case.

C3) Let $u = u_1 \circ u_2^{-1}$, $v = u_2 \circ v_2$ for some (perhaps trivial) elements $u_1, u_2, v_2 \in R(A, X)$. Then $u * v = u_1 \circ v_2$ and the triple u_1, v_2, w satisfies all conditions of C1) (observe, that $v_2 * w$ is defined since $v * w$ is defined). Hence,

$$(u * v) * w = (u_1 * v_2) * w = u_1 * (v_2 * w).$$

On the other hand (using C1) and C2)),

$$u * (v * w) = (u_1 \circ u_2^{-1}) * ((u_2 \circ v_2) * w) \stackrel{C1}{=} (u_1 \circ u_2^{-1}) * (u_2 * (v_2 * w)) \stackrel{C1}{=}$$

$$u_1 * \left(u_2^{-1} * (u_2 * (v_2 * w))\right) \stackrel{C2}{=} u_1 * (v_2 * w) = (u * v) * w,$$

as desired. This proves the theorem.

□

A subset $G \leq R(A, X)$ is called a *subgroup* of $R(A, X)$ if G is a group with respect to $*$. We say that a subset $Y \subset R(A, X)$ *generates a subgroup* $\langle Y \rangle$ in $R(A, X)$ if for any finite sequence of elements $y_1, \ldots, y_n \in Y^{\pm 1}$ the product $y_1 * \ldots * y_n$ is defined.

Example 5 *Let A be a direct sum of copies of $\mathbb{Z}$ with the right lexicographic order. Then the set of all elements of finite length in $R(A, X)$ forms a subgroup which is isomorphic to a free group with basis X.*

8.3 Standard Exponentiation, roots, and conjugation

In this section we study properties of the "standard exponentiation" (by integers) in $R(A, X)$, roots of elements, and conjugation.

Observe, that there are elements $w \in R(A, X)$ for which even the square $w * w$ is not defined. We have to exclude such elements from our considerations related to exponentiation. Put

$$E_n R(A, X) = \{w \in R(A, X) \mid w^k \text{ is defined for every } k \leq n\}.$$

$$E_\infty R(A, X) = \bigcap_n E_n R(A, X).$$

Then the set $E_\infty R(A, X)$ is closed under the "standard" exponentiation by element from $\mathbb{Z}$. Notice, that $E_\infty R(A, X)$ is precisely the set of elements from $R(A, X)$ for which the notion of order is defined. The following definitions provide some tools to classify orders of elements from $E_\infty R(A, X)$. An element $v \in R(A, X)$ is termed *cyclically reduced* if $v(1_A)^{-1} \neq v(|v|)$. We say that an element $v \in R(A, X)$ admits a *cyclic decomposition* if $v = c^{-1} \circ u \circ c$, where $c, u \in R(A, X)$ and u is cyclically reduced. Observe that a cyclic decomposition is unique (whenever exists). Denote by $CR(A, X)$ the set of all cyclically reduced words in $R(A, X)$ and by $CDR(A, X)$ the set of all words from $R(A, X)$ which admit a cyclic decomposition. Obviously, $CDR(A, X) \subset E_\infty R(A, X)$. Not all elements in $E_\infty R(A, X)$ admit cyclic decomposition, for instance, the element w of order 2 from Example 4 in Section 8.2 does not. We will show below that such elements are the only elements in $E_\infty R(A, X)$ which do not admit cyclic decomposition. Put

$$T_2 R(A, X) = \{w \in R(A, X) \mid w * w = \varepsilon\}.$$

Clearly, $T_2 R(A, X) \subset E_\infty R(A, X)$.

Lemma 9 *Let A be a discretely ordered abelian group and X be a set. Then:*

1) $E_2 R(A, X) = CDR(A, X) \cup T_2 R(A, X)$;

2) $E_\infty R(A, X) = E_2 R(A, X)$;

3) every element from $CDR(A, X)$ has infinite order.

Proof. Let $v \in E_2 R(A, X)$, $v \neq \varepsilon$. Then $v = v_1 \circ c = c^{-1} \circ v_2$ for some $v_1, v_2, c \in R(A, X)$ such that $v_1 * v_2 = v_1 \circ v_2$.

If $|v_2| \geq |c|$ then $v_2 = v_3 \circ c$, so $v = c^{-1} \circ v_3 \circ c$ and $v_3 * v_3 = v_3 \circ v_3$. In this case v_3 is cyclically reduced and $v \in CDR(A, X)$.

If $|v_2| < |c|$ then $c = c_1 \circ v_2$, therefore

$$v = v_1 \circ c_1 \circ v_2 = v_2^{-1} \circ c_1^{-1} \circ v_2$$

which implies $c_1 = c_1^{-1}$ and hence $v * v = \varepsilon$, i.e., $v \in TR(A, X)$. Now 1) follows.

To see 2) observe that $E_\infty R(A, X) \subset E_2R(A, X)$ and, as we have mentioned above, $CDR(A, X) \cup T_2R(A, X) \subset E_\infty R(A, X)$. Now 2) follows from 1).

If $v = c^{-1} \circ u \circ c$ and $\varepsilon \neq u \in CR(A, X)$, then $v^k = c^{-1} \circ u^k \circ c$ and $|u^k| = k|u| > 0$. It follows that $|v^k| \geq |u^k| > 0$, hence $v^k \neq \varepsilon$. This proves 3), and the lemma.

□

Since the set $T_2R(A, X)$ is not very interesting from the exponentiation viewpoint, in what follows we bound our considerations to the set $CDR(A, X)$.

Let $v \in CDR(A, X)$ we say that $u \in CDR(A, X)$ is a *k-root* of v if $v = u^k$.

Lemma 10 *Let A be a discretely ordered abelian group, X be a set and let $v \in CDR(A, X)$. Then*

1) If for a given k, v has a k-root, then this k-root is unique.

2) If $A = \mathbb{Z}[t]^+$ then there are only finitely many natural numbers k such that v has a k-root.

Proof. Let $v \in CDR(A, X)$ and $v = c^{-1} \circ w \circ c$ be its cyclic decomposition. It is easy to see that if v has a k-root u then $u = c^{-1} \circ u_1 \circ c$, where u_1 is a k-root of w. Converse is also true, that is there exists a k-root of w then after conjugating it by c we get a k-root for v. Thus, without loss of generality we can assume v to be cyclically reduced.

1) Let k be a fixed natural number. Since any root u of v is an element of $CDR(A, X)$, which is the restriction of v on the segment $[1_A, |v|/k]$, we have the uniqueness of roots automatically.

2) The necessary condition for the existence of a k-root for v is the divisibility of $|v|$ by k in $\mathbb{Z}[t]^+$.

Recall, that as a group $\mathbb{Z}[t]^+$ is the infinite direct sum

$$\mathbb{Z}[t]^+ = \oplus_{i=0}^{\infty}\langle t^i\rangle$$

of copies of $\mathbb{Z}$. Hence, there exists a natural number n such that $|v| \in \oplus_{i=0}^{n}\langle t^i\rangle$.

Now, $|v| = (h_0, \ldots, h_n)$, $h_i \in \mathbb{Z}$ and $|v|$ can be divided by some natural k if and only if all h_i can be divided by this k. Since any integer can be divided only by finitely many natural numbers, then the set $D_i = \{m \in \mathbb{N} \mid m \text{ divides } h_i\}$ is finite for $i \in [0, n]$. Then let $D = \{m \in \mathbb{N} \mid m \text{ divides } |v|\}$. This set is also finite because $D = \bigcap_{i=0}^{n} D_i$.

□

Let $u, v \in CDR(A, X)$ we say that u is a *conjugate* of v if there exists $c \in R(A, X)$ such that the products $c^{-1} * v$, $v * c$, and $(c^{-1} * v) * c$ are defined and $u = c^{-1} * v * c$. We say that u is a *cyclic permutation* of v if $v = v_1 \circ v_2$ and $u = v_2 \circ v_1$ for some elements $v_1, v_2 \in R(A, X)$ (observe that there can be infinitely many different cyclic permutations of a given v).

Lemma 11 *The following hold:*

1) *the set $CDR(A, X)$ is closed under conjugation by elements from $R(A, X)$ (i.e., any conjugate of an element from $CDR(A, X)$ belongs to $CDR(A, X)$;*

2) *if u and v are conjugate and cyclically reduced then $|u| = |v|$ and if $g^{-1}*u*g = v$ for some g such that $|g| \leq |u|$ then u is a cyclic permutation of v.*

Proof. 1) Let $v = c^{-1} \circ u \circ c \in CDR(A, X)$ and $d \in R(A, X)$. We assume that $d^{-1} * v$ and $v * d$ are both defined. Then $c * d$ and so $d^{-1} * c^{-1}$ are also defined and

$$d^{-1} * (c^{-1} \circ u \circ c) * d = (c * d)^{-1} * u * (c * d).$$

In other words we can assume from the beginning that v is cyclically reduced. So, assume $c = \varepsilon$ and $v = u$. Since v is cyclically reduced then either $d^{-1} * v = d^{-1} \circ v$ or $v * d = v \circ d$. Assume the latter.

a) v does not cancel completely in $d^{-1} * (v \circ d)$. Then $v = v_1 \circ v_2, d = v_1 \circ d_1$ and $d^{-1} * (v \circ d) = d_1^{-1} \circ v_2 \circ v_1 \circ d_1$, where $v_2 \circ v_1$ is cyclically reduced as a cyclic permutation of v.

b) v cancels completely in $d^{-1} * (v \circ d)$. Then d and $v \circ d$ have common initial segment w so that $d = w \circ d_2, v \circ d = w \circ d_1$ and $w = v \circ d_3 \neq \varepsilon$. Thus we have $d = d_3 \circ d_1, d = v \circ d_3 \circ d_2$. It follows that $|d_1| > |d_2|$ and moreover d_2 is a terminal segment of d_1. Hence, $d_1 = d_4 \circ d_2$ and we have

$$d^{-1} * (v \circ d) = d_2^{-1} \circ d_1 = d_2^{-1} \circ d_4 \circ d_2.$$

Since d_4 is cyclically reduced we obtained a cyclic decomposition of $d^{-1} * (v \circ d)$.

So, in both cases we showed that $d^{-1} * (v \circ d) \in CDR(A, X)$.

2) Suppose $g^{-1} * u * g = v$ for some g. Then either $g^{-1} * u = g^{-1} \circ u$ or $u * g = u \circ g$ because u is cyclically reduced. Assume the latter. Moreover, g^{-1} cancels completely in $g^{-1} * (u \circ g)$ because v is cyclically reduced, so we have $|v| = |g^{-1} * (u \circ g)| = |u| + |g| - |g^{-1}| = |u|$.

If $|g| \leq |u|$ then $u = g \circ u_1$, but $v = u_1 \circ g$, so u is a cyclic permutation of v.

□

9 A free Lyndon length function on $CDR(A, X)$

The main result of this section is the following theorem.

Theorem 25 *Let A be a discretely ordered abelian group and X be a set. Then the function $L : CDR(A, X) \to A$ defined as $L(w) = |w|$ satisfies all the axioms (L1) - (L6) of a Lyndon length function whenever corresponding products of elements in these axioms are defined.*

Proof. Axioms (L1) and (L2) follow immediately from the definition of the length $|w|$ of an element from $CDR(A, X)$. To prove (L3) recall the definition of the function $\delta(u, v)$ from Subsection 8.2

$$\delta(u, v) = \begin{cases} 0 & \text{if } u(1_A) \neq v(1_A) \\ max\{\beta \mid u_\beta = v_\beta\} & \text{if such } \beta \text{ exists} \\ undefined & \text{otherwise} \end{cases}$$

By definition $\delta(u, v)$ measures the length of the longest common initial segment of u and v. It was shown that the product $u^{-1} * v$ is defined if and only if $\delta(u, v)$ is defined, in which case

$$\delta(u, v) = \frac{1}{2}(|u| + |v| - |u^{-1} * v|) = d(u, v).$$

Now the axiom (L3) easily holds whenever the products $u^{-1} * v, u^{-1} * w, v^{-1} * w$ are defined.

(L4) and (L5) follow from (L1) - (L3). The axiom (L6) follows from the existence of the cyclic decomposition. This proves the theorem.

□

Corollary 2 *Let A be a discretely ordered abelian group and X be a set. Then any subgroup G of $CDR(A, X)$ has a free Lyndon length function with values in A - the* *restriction* $L|_G$ *on* G *of the standard length function* L *on* $CDR(A, X)$.

The following straightforward result shows that one can extend length functions to unions of chains of groups.

Lemma 12 *Let*

$$(G_0, l_0) < (G_1, l_1) < \ldots < (G_n, l_n) \ldots$$

be a chain of groups with (regular) length functions $l_i : G_i \longrightarrow A$ *such that* $l_{i+1}|_{G_i} = l_i$ *for all* i. *Then* $G_\infty = \cup_{i=0}^{\infty} G_i$ *is a group with a regular length function* $l : G_\infty \longrightarrow A$, *where* $l = \cup_{i=0}^{\infty} l_i$.

10 Lyndon's Exponentiation

In this section we describe a very natural and concrete realization of Lyndon's axiomatic approach to exponentiation by polynomials with integer coefficients. Let $\mathbb{Z}[t]^+$ be the additive group of the polynomial ring $\mathbb{Z}[t]$ viewed as a discrete abelian group with respect to the lexicographic order described in Section 2. Below we introduce a $\mathbb{Z}[t]$-exponentiation on $CDR(\mathbb{Z}[t]^+, X)$. Also, from now on, we denote $1_{\mathbb{Z}[t]^+}$ by 1.

The axiom E2) of exponentiation ($(g^{-1}hg)^\alpha = g^{-1}h^\alpha g$) allows one to define exponentiation via representatives of conjugacy classes (extending the exponentiation onto the whole conjugacy class by conjugation). To realize this idea we need the following definition.

Definition 20 *Let $S \subseteq CDR(\mathbb{Z}[t]^+, X)$. A subset $R_S \subseteq CR(\mathbb{Z}[t]^+, X)$ is called a set of representatives of S up to conjugacy (RC-set) if R_S satisfies the following conditions:*

1) R_S does not contain proper powers;

2) for any $u, v \in R_S$ $u \neq v^{-1}$;

3) for each $u \in S$ there exist $v \in R_S, k \in \mathbb{Z}, c \in R(\mathbb{Z}[t]^+, X)$, and a cyclic permutation $\pi(v)$ of v such that

$$u = c^{-1} \circ \pi(v)^k \circ c$$

moreover, such $v, c, k, \pi(v)$ are unique.

It is easy to see that for any $S \subseteq CDR(\mathbb{Z}[t]^+, X)$ there exists a set $R_S \subset CR(\mathbb{Z}[t]^+, X)$ which satisfies conditions 1) 3) above. We show how one can construct such a set in several steps.

Step 1. Let $w \in S$ and $w = c^{-1} \circ \overline{w} \circ c$ be its cyclic decomposition. Put

$$R_1 = \{\overline{w} \mid w \in S\}.$$

Step 2. For any $w \in R_1$, there exists a subset $S_w \subset R_1$ which contains all cyclic permutations of w and all cyclic permutations of w^{-1} which are in R_1. S_w is not empty because it contains w. Thus, choose any element w' in S_w and eliminate from R_1 all elements in $S_w \setminus \{w'\}$. The resulting set is denoted by R_2.

Step 3. If $w \in R_2$ has a maximal k-root v, that is $w = v^k$ then we replace w in R_2 by v. By Lemma 10 such v exists and is unique. After all possible replacements we obtain R_3.

Clearly, R_3 satisfies, by the construction, conditions 1)–3) from the definition of RC-set, so we can set $R_S = R_3$.

Observe that for any given $S \subseteq CDR(\mathbb{Z}[t]^+, X)$ an RC-set is not unique, but all RC-sets for S have the same cardinality and can be obtained one from another by taking inverses and cyclic permutations of elements.

Now let $S \subseteq CDR(\mathbb{Z}[t]^+, X)$ and R_S be a fixed set of representatives for S. Suppose that for every element $u \in R_S$ and every $f(t) \in \mathbb{Z}[t]$ an exponent $u^f \in CDR(\mathbb{Z}[t]^+, X)$ is defined. Then one can extend this exponentiation uniquely to the set S as follows.

Let $u \in S, f(t) \in \mathbb{Z}[t]$. Then by 3) $u = c^{-1} \circ \pi(v)^k \circ c$ for unique $v \in R_S, k \in \mathbb{Z}, c \in R(\mathbb{Z}[t]^+, X)$, and a unique cyclic permutation $\pi(v) = v_1 \circ v_2$ of $v = v_2 \circ v_1$. Put

$$u^{f(t)} = c^{-1} \circ (v_2^{-1} * (v^{f(t)})^k * v_2) \circ c \tag{4}$$

(the product above is defined in $R(\mathbb{Z}[t]^+, X)$ because of the definition of exponents and the fact that $|v_2| \leq |v|$).

Now let $S = CDR(\mathbb{Z}[t]^+, X)$ and $R = R_S$ be a fixed set of representatives for S. It follows from the argument above that to define a $\mathbb{Z}[t]$-exponentiation on $CDR(\mathbb{Z}[t]^+, X)$ it suffices to define $\mathbb{Z}[t]$-exponentiation on R_S. Thus we have a $\mathbb{Z}[t]$-exponentiation function

$$exp_R : CDR(\mathbb{Z}[t]^+, X) \times \mathbb{Z}[t] \to CDR(\mathbb{Z}[t]^+, X)$$

which formally depends on the choice of R, but it will be shown later that in fact this exponentiation function on $CDR(\mathbb{Z}[t]^+, X)$ does not depend on the choice of RC-set.

Recall that as a group $\mathbb{Z}[t]^+$ is a countable direct sum

$$\mathbb{Z}[t]^+ = \oplus_{i=0}^{\infty} \langle t^i \rangle$$

of copies of the infinite cyclic group $\mathbb{Z}$ with the right lexicographic order. It is easy to see that $R(\mathbb{Z}[t]^+, X)$ is the union of the following chain

$$R(\mathbb{Z}, X) \leq R(\mathbb{Z}^2, X) \leq \ldots R(\mathbb{Z}^n, X) \leq \ldots$$

where $\mathbb{Z}^n = \oplus_{i=0}^{n} \langle t^i \rangle$. For an element $w \in R(\mathbb{Z}[t]^+, X)$ the length $|w|$ is a polynomial $g(t) \in \mathbb{Z}[t]$:

$$|w| = g(t) = a_0 + \cdots + a_n t^n,$$

where $a_n > 0$. In this event we say that w has *height* n and write $h(w) = n$. Clearly,

$$h(w) = n \Leftrightarrow w \in R(\mathbb{Z}^n, X) - R(\mathbb{Z}^{n-1}, X).$$

Now we define exponents $v^{f(t)}$ for a given element $v \in R_S$ and a polynomial $f(t) \in \mathbb{Z}[t]$. In fact, because of the way in which we define exponents on R_S we need to define them for any element of $CR(\mathbb{Z}[t]^+, X)$ (after the first step below it can happen that $v^t \notin R_S$).

1) Let $v \in CR(\mathbb{Z}[t]^+, X)$ be not a proper power and $|v| = g(t) = a_0 + \cdots + a_n t^n, a_n > 0$. We define v^t as an element of $CR(\mathbb{Z}[t]^+, X)$ of length $|v^t| = g(t)t$, so, v^t is a function with the domain $[1, g(t)t]$ and $g(t)t = a_0 t + a_1 t^2 + \cdots + a_{n-1} t^n + a_n t^{n+1}, a_n > 0$.

a) If $a_n = 1$ then we set

$$v^t(\beta) = \begin{cases} v(\alpha), & \text{if } \beta = mg(t) + \alpha, m \geq 0, 1 \leq \alpha \leq g(t); \\ v(\alpha), & \text{if } \beta = g(t)t - mg(t) + \alpha, m > 0, 1 \leq \alpha \leq g(t). \end{cases}$$

Observe that the formula above defines $v^t(\beta)$ for any β which belongs either to some initial subsegment of $[1, g(t)t]$ of the form $[1, mg(t)]$ where $m \geq 0$ or to some terminal subsegment of $[1, g(t)t]$ of the form $[g(t)t - mg(t), g(t)t]$ where $m > 0$.

Now, any $\beta \in [1, g(t)t]$ is a polynomial $\beta = r(t) = r_1(t) + b_p t^p \in \mathbb{Z}[t]$, where $b_p > 0, deg(r_1) < deg(r)$, and either $p < n+1$ or $p = n+1, b_p = 1, r_1(t) < 0$. In the former case there exists $m \geq 0$ such that $mg(t) > r(t)$, so that $[1, \beta]$ is an initial subsegment of $[1, mg(t)]$ and $\beta \in [1, mg(t)]$. In the latter case there exists $m > 0$ such that $g(t)t - mg(t) < r(t)$, so that $[\beta, g(t)t]$ is a terminal subsegment of $[g(t)t - mg(t), g(t)t]$ and $\beta \in [g(t)t - mg(t), g(t)t]$.

b) If $a_n > 1$ then we present $[1, g(t)t]$ as the union of disjoint closed segments

$$\left(\bigcup_{k=0}^{a_n - 2} [kt^{n+1} + 1, (k+1)t^{n+1}] \right) \bigcup [(a_n - 1)t^{n+1} + 1, g(t)t]$$

and define v^t on these segments as follows.

For any $k \in [0, a_n - 2]$ and $\beta \in [kt^{n+1} + 1, (k+1)t^{n+1}]$ we set

$$v^t(\beta) = \begin{cases} v(\alpha), & \text{if } \beta = kt^{n+1} + mg(t) + \alpha, m \geq 0, 1 \leq \alpha \leq g(t); \\ v(\alpha), & \text{if } \beta = (k+1)t^{n+1} - mg(t) + \alpha, m > 0, 1 \leq \alpha \leq g(t). \end{cases}$$

and for $\beta \in [(a_n - 1)t^{n+1} + 1, g(t)t]$ we set

$$v^t(\beta) = \begin{cases} v(\alpha), & \text{if } \beta = (a_n - 1)t^{n+1} + mg(t) + \alpha, m \geq 0, 1 \leq \alpha \leq g(t); \\ v(\alpha), & \text{if } \beta = g(t)t - mg(t) + \alpha, m > 0, 1 \leq \alpha \leq g(t). \end{cases}$$

For any $k \in [0, a_n - 2]$, the first formula above defines $v^t(\beta)$ for any β which belongs to some initial subsegment of $[kt^{n+1} + 1, (k+1)t^{n+1}]$ of the form $[kt^{n+1}, kt^{n+1} + mg(t)]$ where $m \geq 0$ or to some terminal subsegment of $[kt^{n+1} + 1, (k+1)t^{n+1}]$ of the form $[(k+1)t^{n+1} - mg(t), (k+1)t^{n+1}]$ where $m > 0$. The second formula given above defines $v^t(\beta)$ for any β which belongs to any initial subsegment of $[(a_n - 1)t^{n+1}, g(t)t]$ of the form $[(a_n - 1)t^{n+1}, (a_n - 1)t^{n+1} + mg(t)]$ where $m \geq 0$ or to any terminal subsegment of $[(a_n - 1)t^{n+1}, g(t)t]$ of the form $[g(t)t - mg(t), g(t)t]$ where $m > 0$.

In the same way as in a) one can show that these formulas define $v^t(\beta)$ for any $\beta \in [1, g(t)t]$.

Now, if $v \in CR(\mathbb{Z}[t]^+, X)$ is such that $v = u^k$ for some $u \in CR(\mathbb{Z}[t]^+, X)$ then we set $v^t = (u^t)^k$.

Clearly, $|v^t| = g(t)t = |v|t$, v^t starts with v and ends with v. In particular, $v^t \in CR(\mathbb{Z}[t]^+, X)$. It follows that $v^t * v = v^t \circ v = v \circ v^t = v * v^t$, hence $[v^t, v] = \varepsilon$.

2) Now for $v \in CR(\mathbb{Z}[t]^+, X)$ we define exponents v^{t^k} by induction. Since $v^t \in$

$CR(\mathbb{Z}[t]^+, X)$ one can repeat the construction from 1) and define

$$v^{t^{k+1}} = (v^{t^k})^t.$$

3) Now we define $v^{f(t)}$, where $f(t) \in \mathbb{Z}[t]$, by linearity, i.e., if $f(t) = m_0 + m_1 t + \ldots + m_k t^k$ then

$$v^{f(t)} = v^{m_0} * (v^t)^{m_1} * \ldots * (v^{t^k})^{m_k}.$$

Observe that the product above is defined because of the definition in 1) and 2). We have $v^{f(t)} \in CR(\mathbb{Z}[t]^+, X)$, $|v^{f(t)}| = g(t)|f(t)| = |v||f(t)|$ and $[v^{f(t)}, v] = \varepsilon$.

Thus, following the steps 1)–3) above, for any given $v \in CR(\mathbb{Z}[t]^+, X)$ and $f(t) \in \mathbb{Z}[t]$ one can define $v^{f(t)}$ which is again an element of $CR(\mathbb{Z}[t]^+, X)$. Also, it follows from the definition above that if $w = v^k \in CR(\mathbb{Z}[t]^+, X)$ and $f(t) \in \mathbb{Z}[t]$ then $w^{f(t)} = (w^{f(t)})^k$.

Then, suppose we have $S = CR(\mathbb{Z}[t]^+, X)$, $f(t) \in \mathbb{Z}$, and $u \in S$ is a cyclic permutation of $v \in R_S$, where R_S is any RC-set for S. Then, at first, we can define $u^{f(t)}$ according to steps 1)–3) above, but on the other side we can define $u^{f(t)}$ like in (4), via its representative v, that is

$$u^{f(t)} = v_1^{-1} * v^{f(t)} * v_1,$$

where $v = v_1 \circ v_2$, $u = v_2 \circ v_1$. For now it is not clear why definition via representatives gives the same result as the definition in steps 1)–3). The following lemma clarifies this matter.

Lemma 13 *Let $u, v \in CR(\mathbb{Z}[t]^+, X)$ be such that $v = v_1 \circ v_2$, $u = v_2 \circ v_1$ and $f(t) \in \mathbb{Z}[t]$. Then $(v_1^{-1} * v * v_1)^{f(t)} = v_1^{-1} * v^{f(t)} * v_1$.*

Proof. The statement is obviously true for any constant $f(t) \in \mathbb{Z}$. We prove the statement for $f(t) = t$. We want to show

$$(v_2 \circ v_1)^t = v_1^{-1} * (v_1 \circ v_2)^t * v_1.$$

We have $|v_2 \circ v_1| = |v_1 \circ v_2|$ and $(v_2 \circ v_1)^t(\beta) = (v_1 \circ v_2)^t(\beta + |v_1|), \beta \in [1, |v|t - |v_1|]$. When we conjugate $(v_1 \circ v_2)^t$ by v_1 we cancel the initial segment of $(v_1 \circ v_2)^t$ of length $|v_1|$ and add a terminal segment of length $|v_1|$, so we have $(v_2 \circ v_1)^t(\beta) = (v^{-1} * (v_1 \circ v_2)^t * v_1)(\beta), \beta \in [1, |v|t]$, and $(v_2 \circ v_1)^t = v_1^{-1} * (v_1 \circ v_2)^t * v_1$.

Since v^t and u^t are cyclic permutations of each other and belong to $CR(\mathbb{Z}[t]^+, X)$ we can apply induction and get the statement for any $f(t) \in \mathbb{Z}[t]$.

□

So, the lemma above shows that the exponentiation function defined in (4) coincides on $CR(\mathbb{Z}[t]^+, X)$ with the exponentiation finction defined according to steps 1) 3) above.

Lemma 14 *Let R_1 and R_2 be two RC-sets for $CDR(\mathbb{Z}[t]^+, X)$. Let*

$$exp_{R_i} : CDR(\mathbb{Z}[t]^+, X) \times \mathbb{Z}[t] \to CDR(\mathbb{Z}[t]^+, X)$$

be the $\mathbb{Z}[t]$-exponentiation function defined in (4). Then

$$exp_{R_1} = exp_{R_2}$$

on $CDR(\mathbb{Z}[t]^+, X)$.

Proof. Let $u \in R_1$. Then there exists an element $v \in R_2$ such that $v = v_1 \circ v_2, u = v_2 \circ v_1$. If $f(t) \in \mathbb{Z}[t]$ then by Lemma 13 we have

$$u^{f(t)} = (v_1^{-1} * v * v_1)^{f(t)} = v_1^{-1} * v^{f(t)} * v_1.$$

Now, let $w \in CDR(\mathbb{Z}[t]^+, X)$ be such that

$$w = c^{-1} \circ \pi(v)^k \circ c$$

for some cyclic permutation $\pi(v)$ of v. Without loss of generality we can assume $\pi(v) = u$. So we have

$$exp_{R_1}(w, f(t)) = w^{f(t)} = c^{-1} \circ (u^{f(t)})^k \circ c$$

$$exp_{R_2}(w, f(t)) = w^{f(t)} = c^{-1} \circ (v_1^{-1} * (v^{f(t)})^k * v_1) \circ c$$

and from the argument above it follows that $exp_{R_1}(w, f(t)) = exp_{R_2}(w, f(t))$.

□

The lemmas above make it possible to define $v^{f(t)}$ for any $v \in CDR(\mathbb{Z}[t]^+, X)$ with a cyclic decomposition $v = c^{-1} \circ u \circ c$ and $f(t) \in \mathbb{Z}[t]$ as:

$$v^{f(t)} = c^{-1} \circ u^{f(t)} \circ c. \qquad (5)$$

This definition is equivalent to the one given in (4) and it does not depend on any RC-set at all.

Thus we have defined $\mathbb{Z}[t]$-exponentiation function

$$exp : CDR(\mathbb{Z}[t]^+, X) \times \mathbb{Z}[t] \to CDR(\mathbb{Z}[t]^+, X)$$

on the whole set $CDR(\mathbb{Z}[t]^+, X)$.

It can be seen easily from the steps 1)–3) above that the $\mathbb{Z}[t]$-exponentiation is not unique and its definition can be different. The reason why we have chosen precisely this one is that the following property holds for it - if $v \in CR(\mathbb{Z}[t]^+, X)$ and $f(t) \in \mathbb{Z}[t]$ then $|v^{f(t)}| = |v||f(t)|$.

Lemma 15 *Let $u, v \in CR(\mathbb{Z}[t]^+, X)$ and $f(t), g(t) \in \mathbb{Z}[t]$ be such that $u^{f(t)} = v^{g(t)}$. Then $[u, v]$ is defined and is equal to ε.*

Proof. Since $[u, u^{f(t)}] = \varepsilon$ and $[v, v^{g(t)}] = \varepsilon$ then $[u, v^{g(t)}] = \varepsilon$ and $[v, u^{f(t)}] = \varepsilon$. From the latter equalities we will derive the required statement.

Observe that if $|u| = |v|$ then it follows automatically that $u = v^{\pm 1}$. Indeed, by the definition of exponents $u^{f(t)}$ and $v^{g(t)}$ have correspondingly $u^{\pm 1}$ and $v^{\pm 1}$ as initial segments. Since $u^{f(t)} = v^{g(t)}$ then initial segments of length $|u|$ in both coincide.

We can assume $|u| < |v|$ and consider $[u, v^{g(t)}] = \varepsilon$ (if $|u| > |v|$ then we consider $[v, u^{f(t)}] = \varepsilon$ and apply the same arguments). Also, $g(t) > 1$, otherwise we have nothing to prove.

Thus we have $u * v^{g(t)} = v^{g(t)} * u$. Since u and v are cyclically reduced and equal $\mathbb{Z}[t]^+$-words have equal initial and terminal segments of the same length then $[u, v]$ is defined and we have two cases.

a) $u * v = u \circ v$.

Thus, automatically we have $v * u = v \circ u$

$u \circ v^{g(t)}$ and $v^{g(t)} \circ u$ have the same initial segment of length $2|v|$. So $v = u \circ v_1 = v_1 \circ v_2$ and $|u| = |v_2|$. Comparing terminal segments of $u \circ v^{g(t)}$ and $v^{g(t)} \circ u$ of length $|u|$ we have $u = v_2$ and from $u \circ v_1 = v_1 \circ u$ it follows that $[u, v] = \varepsilon$.

b) There is a cancellation in $u * v$.

Then, from $u^{f(t)} = v^{g(t)}$ it follows that $v^{-1} = v_1^{-1} \circ u$ and so $v = u^{-1} \circ v_1$. Using the same arguments as in a) we obtain $v = u^{-1} \circ v_1 = v_1 \circ v_2$, $|u| = |v_2|$ and $u^{-1} = v_2$. It follows immediately that $[u, v] = \varepsilon$.

□

Lemma 16 *Let $u, v \in CDR(\mathbb{Z}[t]^+, X)$ be such that $h(u) = h(v)$ and $[u, v] = \varepsilon$. Then $[u^{f(t)}, v] = \varepsilon$ for any $f(t) \in \mathbb{Z}[t]$ provided $[u^{f(t)}, v]$ is defined.*

Proof. We can assume that either u or v is cyclically reduced. This is always possible because both elements belong to $CDR(\mathbb{Z}[t]^+, X)$. Suppose we have $v^{-1} * u * v = u$, where u is cyclically reduced.

a) $|u| < |v|$

Since u is cyclically reduced either $v^{-1} * u = v^{-1} \circ u$ or $u * v = u \circ v$. Assume the former. Then v has to cancel completely in $v^{-1} * u * v$ because this product is equal to u which is cyclically reduced. So v has the form $v = u^k \circ w$, where $k < 0$ is the smallest possible and w does not have u as an initial segment. We have then

$$v^{-1} * u * v = w^{-1} * u * w = w^{-1} * (u \circ w) = u.$$

and w^{-1} cancels completely. In this case the only possibility is that $|w| < |u|$ (otherwise we have a contradiction with the choice of k) and $[u, w] = \varepsilon$. So now we reduced everything to the case b) because clearly $[u^{f(t)}, u^k] = \varepsilon$ for any $f(t) \in \mathbb{Z}[t]$.

b) $|u| > |v|$

We have $v^{-1} * u * v = u$, u is cyclically reduced, moreover, u is a cyclic permutation of itself that is $v^{-1} * u * v = u$. Finally, since $[u^{f(t)}, v]$ is defined then

$$v^{-1} * u^{f(t)} * v = u^{f(t)}$$

follows from Lemma 13.

□

We summarize the properties of the exponentiation exp in the following theorem.

Theorem 26 *The $\mathbb{Z}[t]$-exponentiation function*

$$exp : (u, f(t)) \rightarrow u^{f(t)}$$

defined in (5) satisfies the following axioms (here $u, v \in CDR(\mathbb{Z}[t]^+, X)$, $f, g \in \mathbb{Z}[t]$):

E1) $u^1 = u, \quad u^{fg} = (u^f)^g, \quad u^{f+g} = u^f * u^g.$

*E2)** $(v^{-1} * u * v)^f = v^{-1} * u^f * v$

provided $[u, v] = \varepsilon$ *and* $h(u) = h(v)$, *or* $u = v \circ w$, *or* $u = w^\alpha, v = w^\beta$ *for some* $w \in CDR(\mathbb{Z}[t]^+, X)$ *and* $\alpha, \beta \in \mathbb{Z}[t]$;

*E3)** *if* $[u, v] = \varepsilon$ *and* $u = w^\alpha, v = w^\beta$ *for some* $w \in CDR(\mathbb{Z}[t]^+, X)$, $\alpha, \beta \in \mathbb{Z}[t]$ *then*

$$(u * v)^f = u^f * v^f$$

Proof. Let $u \in CDR(\mathbb{Z}[t]^+, X)$ and $\alpha, \beta \in \mathbb{Z}[t]$.

The equalities $u^1 = u$ and $(u^f)^g = u^{fg}$ follow directly from the definition of exponentiation. To show E1) we need to prove only that $u^{f+g} = u^f * u^g$. Let

$$u = c^{-1} \circ u_1^k \circ c$$

be a cyclic decomposition of u. Then

$$u^f = c^{-1} \circ (u_1^f)^k \circ c, \quad u^g = c^{-1} \circ (u_1^g)^k \circ c.$$

Now

$$u^{f+g} = c^{-1} \circ (u_1^{f+g})^k \circ c = (c^{-1} \circ (u_1^f)^k \circ c) * (c^{-1} \circ (u_1^g)^k \circ c),$$

as required.

E2)* If $u = w^\alpha, v = w^\beta$ for some $w \in CDR(\mathbb{Z}[t]^+, X)$ and $\alpha, \beta \in \mathbb{Z}[t]$, then result follows from the definition of exponentiation. If $[u, v] = \varepsilon$ and $h(u) = h(v)$ then result follows from Lemma 16. If $u = v \circ w$ then result follows from Lemma 13.

E3)* We have $(u*v)^f = (w^{\alpha+\beta})^f = w^{(\alpha+\beta)f} = w^{\alpha f} * w^{\beta f} = (w^\alpha)^f * (w^\beta)^f = u^f * v^f$.

□

11 Extensions of centralizers

Let A be a discretely ordered abelian group. Recall that by $CDR(A, X)$ we denote the subset of all A-words w from $R(A, X)$ which admit cyclic decomposition $w = c^{-1} \circ v \circ c$.

Recall that a subset $G \leq R(A, X)$ is called a *subgroup* if G is closed under multiplication $*$ and inversion $^{-1}$. In this section we will prove that for any subgroup $G \leq CDR(A, X)$ the extension G^* of all cyclic centralizers of G by free abelian group of infinite rank has a natural embedding into $CDR(A, X)$. This is the main technical result, it provides the induction argument for an embedding of $F^{\mathbb{Z}[t]}$ into $CDR(\mathbb{Z}[t]^+, X)$.

11.1 Extending pregroups by non-standard powers

At first we give several definitions which are important for our further considerations.

Definition 21 *Let G be a group with a length function $l : G \to \Lambda$. We say that G satisfies the* stabilizing condition (S) *if for any elements $v, w, g \in G$ such that:*

1) u, v are not proper powers,

2) $C_G(u) \simeq C_G(v) = \mathbb{Z}$,

3) either v is not conjugate to $w^{\pm 1}$ or $v = w^{\pm 1}$ but $[g, v] \neq 1$

there exists a natural number $r = r(v, w, g)$ such that for all $n, m \geq t$

$$v^n g w^m = v^{n-r} \circ_{\delta_1} v^r g w^r \circ_{\delta_2} w^{m-r},$$

where $2\delta_1 \leq l(v), 2\delta_2 \leq l(w)$ and $xy = x \circ_\alpha y$ means $l(xy) \geq l(x) + l(y) - \alpha$.

Let G be a group. We say that a tuple $u = (u_1, \ldots, u_k) \in G^k$ *has commutation* if $[u_i, u_{i+1}] = 1$ for some $i \in [1, k-1]$. Otherwise we call u *commutation-free*.

Definition 22 *A group G satisfies the* big powers condition *(BP) (or called a BP-group), if for any commutation-free tuple $u = (u_1, \ldots, u_k)$ of elements from G there exists an integer $n(u)$ (a boundary of separation for u) such that*

$$u_1^{\alpha_1} \cdots u_k^{\alpha_k} \neq 1$$

for any integers $\alpha_1, \ldots, \alpha_k \geq n(u)$.

The notion of big powers condition was introduced in [38] and provides a method of discrimination for a wide class of groups. We refer to [38] and [24] for details about BP-groups.

In fact, we are interested in the following weak form of the big powers condition.

Definition 23 *A group G satisfies the* weak big powers condition *(WBP) (or called a WBP-group), if for any commutation-free tuple $u = (u_1, \ldots, u_k)$ of elements from G such that*

1) u_i is not a proper power,

2) $C_G(u_i) \simeq \mathbb{Z}$,

3) u_i is not conjugate to $u_{i+1}^{\pm 1}$ for any $i \in [1, k-1]$

there exists an integer $n(u)$ (a boundary of separation for u) such that

$$u_1^{\alpha_1} \cdots u_k^{\alpha_k} \neq 1$$

for any integers $\alpha_1, \ldots, \alpha_k \geq n(u)$.

Lemma 17 *Let $G \leq CDR(\mathbb{Z}[t]^+, X)$ satisfy the stabilizing condition (S). Then G is a WBP-group.*

Proof. We want to prove the stronger statement, from which the WBP-condition follows easily:

Claim. If $u = (u_1, \ldots, u_k) \in G^k$ is a commutation-free tuple such that u_i is not conjugate to $u_{i+1}^{\pm 1}$ for any $i \in [1, k-1]$ and centralizers of all u_i are cyclic, then there exist integers $n(u), \beta$ such that for any integers $\alpha_1, \ldots, \alpha_k \geq n(u)$

$$u_1^{\alpha_1} * \cdots * u_k^{\alpha_k} = (u_1^{\alpha_1} * \cdots * u_{k-1}^{\alpha_{k-1}} * u_k^{\beta}) \circ_\delta u_k^{\alpha_k - \beta} \neq \varepsilon$$

where $\delta \in \mathbb{Z}[t]$ does not depend on any α_i.

We use induction on the length of $u = (u_1, \ldots, u_k)$.

a) $k = 2$

From the stabilizing condition we can find an integer r such that for all $n, m > r$

$$u_1^n * u_2^m = u_1^{n-r} \circ_{\delta_1} (u_1^r * u_2^r) \circ_{\delta_2} u_2^{m-r},$$

where $\delta_1, \delta_2 \in \mathbb{Z}[t]$ and $2\delta_1 \leq |u_1|, 2\delta_2 \leq |u_2|$. Thus

$$|u_1^n * u_2^m| \geq |u_1^{n-r}| + |u_1^r * u_2^r| + |u_2^{m-r}| - \delta_1 - \delta_2.$$

By Corollary 2 the length function on G induced from $CDR(\mathbb{Z}[t]^+, X)$ is free, so we can choose an integer s such that for all $n, m > s$ we get $|u_1^{n-r}| + |u_2^{m-r}| > \delta_1 + \delta_2$. Now, if we take $M = \max\{s, r\}$ then

$$u_1^n * u_2^m = (u_1^n * u_2^M) \circ_{\delta_2} u_2^{m-M} \neq \varepsilon$$

for any integers $m, n \geq M$. So in this case $n(u) = M, \beta = M, \delta = \delta_2$.

b) Suppose we proved the Claim for all tuples of the length $k = p - 1$. Consider $u = (u_1, \ldots, u_p)$. By induction there exist integers N, β_1 such that

$$u_1^{\alpha_1} * \cdots * u_{p-1}^{\alpha_{p-1}} = (u_1^{\alpha_1} * \cdots * u_{p-1}^{\beta_1}) \circ_{\delta_1} u_{p-1}^{\alpha_{p-1} - \beta_1} \neq \varepsilon$$

for any integers $\alpha_1, \ldots, \alpha_{p-1} \geq N$. Then

$$u_1^{\alpha_1} * \cdots * u_{p-1}^{\alpha_{p-1}} * u_p^{\alpha_p} = ((u_1^{\alpha_1} * \cdots * u_{p-1}^{\beta_1}) \circ_{\delta_1} u_{p-1}^{\alpha_{p-1}-\beta_1}) * u_p^{\alpha_p} = (w \circ_{\delta_1} u_{p-1}^{\alpha_{p-1}-\beta_1}) * u_p^{\alpha_p},$$

where we denote $u_1^{\alpha_1} * \cdots * u_{p-1}^{\beta_1}$ by w. By the stabilizing condition there exists an integer r such that for all $\alpha_{p-1} - \beta_1, \alpha_p \geq r$

$$u_1^{\alpha_1} * \cdots * u_p^{\alpha_p} = (w \circ_{\delta_1} u_{p-1}^{\alpha_{p-1}-\beta_1-r}) \circ_{\delta_2} (u_{p-1}^r * u_p^r) \circ_{\delta_3} u_p^{\alpha_p - r},$$

where $\delta_2, \delta_3 \in \mathbb{Z}[t]$ and $2\delta_2 \leq |u_{p-1}|, 2\delta_3 \leq |u_p|$. Again using length argument we can choose $M > \max\{r, N\}$ to be such that for all $\alpha_{p-1} - \beta_1, \alpha_p \geq M$ we have

$$|u_{p-1}^{\alpha_{p-1}-\beta_1-r}| + |u_p^{\alpha_p - r}| > \delta_2 + \delta_3.$$

Finally, take $n(u) = M + \beta_1, \beta = M$ and

$$u_1^{\alpha_1} * \cdots * u_p^{\alpha_p} = (u_1^{\alpha_1} * \cdots * u_{p-1}^{\alpha_{p-1}} * u_p^{\beta}) \circ_{\delta_3} u_p^{\alpha_p - \beta} \neq \varepsilon$$

for all $\alpha_1, \ldots, \alpha_p \geq n(u)$.

□

The next result is analogous to one for BP-groups proved in [24].

Lemma 18 *Let G be a WBP-group, $u = (u_1, \ldots, u_k) \in G^k$ be a commutation-free tuple such that u_i is not conjugate to $u_{i+1}^{\pm 1}$ for any $i \in [1, k-1]$ and centralizers of all u_i are cyclic. Let $h \in G$. Then there exists a constant $n(u/h)$ such that*

$$u_1^{\alpha_1} \cdots u_k^{\alpha_k} \neq h$$

for every $\alpha_1, \ldots, \alpha_k > n(u/h)$.

Proof. Suppose that for any integer n there exist integers $n_1, \ldots, n_k > n$ such that

$$u_1^{n_1} \cdots u_k^{n_k} = h.$$

Observe that the tuple

$$w = (u_1, \ldots, u_k, u_{k-1}^{-1}, \ldots, u_1^{-1})$$

is commutation-free, all elements in w have cyclic centralizers and any two successive elements are not conjugate. Let $n(w)$ be a boundary of separation for w.

Then, by our assumption there are $\alpha_1, \ldots, \alpha_k > n(w)$ such that

$$u_1^{\alpha_1} \cdots u_k^{\alpha_k} = h.$$

Similarly, there are some $\beta_1, \ldots, \beta_k > n(w) + \alpha_k$ such that

$$u_1^{\beta_1} \cdots u_k^{\beta_k} = h.$$

So

$$u_1^{\beta_1} \cdots u_k^{\beta_k - \alpha_k} (u_{k-1}^{-1})^{\alpha_{k-1}} \cdots (u_1^{-1})^{\alpha_1} = 1$$

and all the exponents above are greater than $n(w)$ - a contradiction.

□

Now we prove the main technical result of the first part of the thesis.

Proposition 2 *Let $G \leq CDR(\mathbb{Z}[t]^+, X)$ satisfy the stabilizing condition (S), B be a subgroup of $\mathbb{Z}[t]^+$ containing $\mathbb{Z}$ and $C \subset G$ contain all elements $v \in G$ such that the centralizer $C_G(v)$ is generated by v in G. Let G satisfy the exponentiation axiom (E2) with respect to B, that is $(v^{-1} * u * v)^\alpha = v^{-1} * u^\alpha * v$ for all $u, v \in G$ and*

$\alpha \in B$. Let R_C be the set of representatives for C and

$$P = P(G, C, B) = \{g * u^\alpha * h \mid g, h \in G, u \in R_C, \alpha \in B\}.$$

$$D_P = \{(p, q) \in P \times P \mid p * q \in P\} \subset P \times P.$$

Then

1) $P \subset CDR(\mathbb{Z}[t]^+, X)$;

2) (P, D_P) forms a pregroup.

Remark 1 *Here, in the proof, we call exponents from $B - \mathbb{Z}$ infinite and exponents from $\mathbb{Z}$ finite.*

Proof. 1) Take any $g * u^\alpha * h \in P$. If α is finite then $u^\alpha \in G$ and $g * u^\alpha * h$ is defined.

Suppose α is infinite. By the stabilizing condition there exists an integer r such that for all $n \geq 2r$

$$g * u^n * h = (g * u^r) \circ_\delta u^{n-2r} \circ_\delta (u^r * h),$$

where $\delta \in \mathbb{Z}[t]^+, 2\delta \leq |u|$. Let us fix any $n > r$. Observe that in fact we have

$$g * u^n * h = (g * u^r) \circ u^{n-2r} \circ (u^r * h)$$

because $u \in R_C \subset CR(\mathbb{Z}[t]^+, X)$. Moreover $g * u^n * h \in G$, thus $g * u^n * h \in CDR(\mathbb{Z}[t]^+, X)$ and $c^{-1} \circ (u_1 \circ u^k \circ u_2) \circ c$ is its cyclic decomposition, where $u = u_3 \circ u_1 = u_2 \circ u_4, k \leq n - 2r$. Therefore,

$$c^{-1} \circ (u_1 \circ u^\beta \circ u_2) \circ c,$$

where $\beta = \alpha + k - n$, is a cyclic decomposition of $g * u^{\alpha} * h$ and $P \subset CDR(\mathbb{Z}[t]^{+}, X)$.

2) Observe that if $x, y \in P$ and xy is defined in P then $x * y$ is defined in $R(\mathbb{Z}[t]^{+}, X)$. Now it follows that axioms (P1) - (P4) hold in P since they hold in $R(\mathbb{Z}[t]^{+}, X)$ (Theorem 24). To complete the proof it suffices to check that the axiom (P5) holds in (P, D_P):

(P5) for every $u, v, w, z, \in P$ if $u * v, v * w$, and $w * z$ are all defined then either $u * v * w$ or $v * w * z$ is defined in P.

Observe that if $c^{\alpha} = c_1^{\beta}$ for some $\alpha, \beta \in B$ and $c_1 \in R_{C}$, then it follows from Lemma 15 that $[c, c_1] = \varepsilon$. Since c has cyclic centralizer in G, it follows that $c_1 = c^k$ which is possible only if $c_1 = c^{\pm 1}$. Also, if $\alpha \in B$ is infinite then $c^{\alpha} \notin G$ for any $c \in R_{C}$. Otherwise $[c^{\alpha}, c] = \varepsilon$ and since $C_G(c)$ is cyclic generated by c then $\alpha \in \mathbb{Z}$ - contradiction.

It is easy to see that any element $p \in P$ has many representations as $g * c^{\alpha} * h$, where $g, h \in G, c \in R_{C}, \alpha \in B$. Let us fix the representation for every $p \in P, p = g_p * c_p^{\alpha_p} * h_p$, where $g_p, h_p \in G, c_p \in R_{C}, \alpha_p \in B$.

Claim. Let $p = g_p * c_p^{\alpha_p} * h_p, q = g_q * c_q^{\alpha_q} * h_q \in P$, where α_p, α_q are infinite and $p * q \in P$. Then $c_p = c_q^{\pm 1}$ and $h_p * g_q$ belongs to the cyclic subgroup generated by c_p.

Since $p * q \in P$ then there exists $x \in P$ such that $p * q = x$. Therefore,

$$(g_p * c_p^{\alpha_p} * h_p) * (g_q * c_q^{\alpha_q} * h_q) = g_x * c_x^{\alpha_x} * h_x$$

or in other words

$$(g_p * c_p^{\alpha_p} * h_p) * (g_q * c_q^{\alpha_q} * h_q) * (h_x^{-1} * c_x^{-\alpha_x} * g_x^{-1}) = \varepsilon.$$

Thus we have

$$(g_p * c_p^{\alpha_p} * g_p^{-1}) * (g_p * h_p) * (g_q * c_q^{\alpha_q} * h_q) * (h_r^{-1} * c_r^{-\alpha_r} * g_r^{-1}) = \varepsilon.$$

$$(c_p^{\alpha_p})^{g_p^{-1}} * (g_p * h_p * g_q * c_q^{\alpha_q} * g_q^{-1} * h_p^{-1} * g_p^{-1}) * (g_p * h_p * g_q * h_q) * (h_r^{-1} * c_r^{-\alpha_r} * g_r^{-1}) = \varepsilon.$$

$$(c_p^{\alpha_p})^{g_p^{-1}} * (c_q^{\alpha_q})^{g_q^{-1} \bullet h_p^{-1} \bullet g_p^{-1}} * (g_p * h_p * g_q * h_q) * (h_r^{-1} * c_r^{-\alpha_r} * g_r^{-1}) = \varepsilon.$$

and finally

$$(c_p^{\alpha_p})^{g_p^{-1}} * (c_q^{\alpha_q})^{g_q^{-1} \bullet h_p^{-1} \bullet g_p^{-1}} * (c_r^{-\alpha_r})^{h_r \bullet h_q^{-1} \bullet g_q^{-1} \bullet h_p^{-1} \bullet g_p^{-1}} * (g_p * h_p * g_q * h_q * h_r^{-1} * g_r^{-1}) = \varepsilon.$$

Since G satisfies the exponentiation axiom (E2) we can rewrite the left-hand side of the expression above as

$$(c_p^{g_p^{-1}})^{\alpha_p} * (c_q^{g_q^{-1} \bullet h_p^{-1} \bullet g_p^{-1}})^{\alpha_q} * ((c_r^{-1})^{h_r \bullet h_q^{-1} \bullet g_q^{-1} \bullet h_p^{-1} \bullet g_p^{-1}})^{\alpha_r} * (g_p * h_p * g_q * h_q * h_r^{-1} * g_r^{-1}) = \varepsilon.$$

By the assumption, G satisfies the stabilizing condition, so by Lemma 17, G is a WBP-group. Hence, by Lemma 18, there exists a natural number $r > 0$ such that

$$(c_p^{g_p^{-1}})^{k} * (c_q^{g_q^{-1} \bullet h_p^{-1} \bullet g_p^{-1}})^{m} * ((c_r^{-1})^{h_r \bullet h_q^{-1} \bullet g_q^{-1} \bullet h_p^{-1} \bullet g_p^{-1}})^{n} * (g_p * h_p * g_q * h_q * h_r^{-1} * g_r^{-1}) \neq \varepsilon$$

for all natural $k, m, n \geq r$ unless one of the commutators $[c_p^{g_p^{-1}}, c_q^{g_q^{-1} \bullet h_p^{-1} \bullet g_p^{-1}}]$, $[c_q^{g_q^{-1} \bullet h_p^{-1} \bullet g_p^{-1}}, (c_r^{-1})^{h_r \bullet h_q^{-1} \bullet g_q^{-1} \bullet h_p^{-1} \bullet g_p^{-1}}]$ is trivial or either $c_p^{g_p^{-1}}$ is conjugate to $c_q^{g_q^{-1} \bullet h_p^{-1} \bullet g_p^{-1}}$ or $c_q^{g_q^{-1} \bullet h_p^{-1} \bullet g_p^{-1}}$ is conjugate to $(c_r^{-1})^{h_r \bullet h_q^{-1} \bullet g_q^{-1} \bullet h_p^{-1} \bullet g_p^{-1}}$.

But then it also holds for any $\alpha_p, \alpha_q, \alpha_r > r$. Indeed, we can obtain

$$(c_p^{g_p^{-1}})^{\alpha_p} * (c_q^{g_q^{-1} \bullet h_p^{-1} \bullet g_p^{-1}})^{\alpha_q} * ((c_r^{-1})^{h_r \bullet h_q^{-1} \bullet g_q^{-1} \bullet h_p^{-1} \bullet g_p^{-1}})^{\alpha_r} * (g_p * h_p * g_q * h_q * h_r^{-1} * g_r^{-1})$$

from

$$(c_p^{g_p^{-1}})^{2k} * (c_q^{g_q^{-1} * h_p^{-1} * g_p^{-1}})^{2m} * ((c_r^{-1})^{h_r * h_q^{-1} * g_q^{-1} * h_p^{-1} * g_p^{-1}})^{2n} * (g_p * h_p * g_q * h_q * h_r^{-1} * g_r^{-1})$$

by inserting $c_p^{\alpha_p - 2k}, c_q^{\alpha_q - 2m}$ and $(c_r^{-1})^{\alpha_r - 2n}$ without cancellation, using the fact that c_p, c_q and c_r are cyclically reduced (here we assume implicitly all $\alpha_p, \alpha_q, \alpha_r$ to be positive, but this does not restrict the argument because one can always consider c_y^{-1} instead of c_y, where $y \in \{p, q, r\}$).

a) Suppose $c_p^{g_p^{-1}}$ is conjugate to $c_q^{g_q^{-1} * h_p^{-1} * g_p^{-1}}$, that is, $f^{-1} * c_p^{g_p^{-1}} * f = c_q^{g_q^{-1} * h_p^{-1} * g_p^{-1}}$ for some $f \in G$. Then, since $c_p, c_q \in R_C$, it can happen only if $c_p = c_q^{\pm 1}$ and $g_q^{-1} * h_p^{-1} * g_p^{-1} * f * g_p = c_p^k, k \in \mathbb{Z}$.

If $h_p * g_q \in \langle c_p \rangle$, then we proved the claim.

If $h_p * g_q \notin \langle c_p \rangle$, then by the stabilizing condition there exists an integer N such that for all $k_1, k_2 \geq N$ we have

$$g_p * c_p^{\alpha_p - k_1} \circ (c_p^{k_1} * h_p * g_q * c_p^{k_2}) \circ c_p^{\alpha_q - k_2} * h_q.$$

It means that in order to maintain

$$(g_p * c_p^{\alpha_p} * h_p) * (g_q * c_q^{\alpha_q} * h_q) * (h_r^{-1} * c_r^{-\alpha_r} * g_r^{-1}) = \varepsilon$$

the following conditions have to be satisfied: $c_r = c_p^{\pm 1}, h_q * h_r^{-1} = c_p^{k_3}, k_3 \in \mathbb{Z}$ and $\alpha_q - \alpha_r$ is infinite. So we have

$$(g_p * c_p^{\alpha_p} * h_p) * (g_q * c_p^{\alpha_q - \alpha_r + k_3} * g_r^{-1}) = \varepsilon.$$

But by our assumption $h_p * g_q \notin \langle c_p \rangle$ and by the stabilizing condition the product above can not be trivial - contradiction.

b) If $c_q^{g_q^{-1} * h_p^{-1} * g_p^{-1}}$ is conjugate to $(c_r^{-1})^{h_r * h_q^{-1} * g_q^{-1} * h_p^{-1} * g_p^{-1}}$ then in the same way as above we get $c_q = c_r^{\pm 1}$. Considering

$$(g_p * c_p^{\alpha_p} * h_p) * (g_q * c_q^{\alpha_q} * h_q) * (h_r^{-1} * c_r^{-\alpha_r} * g_r^{-1}) = \varepsilon$$

we can conclude from the stabilizing condition that either $h_q * h_r^{-1} \notin \langle c_q \rangle$ and automatically we have $c_p = c_q^{\pm 1}$, $h_p * g_q \in \langle c_p \rangle$, or $h_q * h_r^{-1} = c_q^k$ and we have

$$(g_p * c_p^{\alpha_p} * h_p) * (g_q * c_p^{\alpha_q - \alpha_r - k} * g_r^{-1}) = \varepsilon.$$

so again this can happen only if $c_p = c_q^{\pm 1}$, $h_p * g_q \in \langle c_p \rangle$.

c) Now suppose either $[c_q^{g_q^{-1} * h_p^{-1} * g_p^{-1}}, (c_r^{-1})^{h_r * h_q^{-1} * g_q^{-1} * h_p^{-1} * g_p^{-1}}] = \varepsilon$ or $[c_p^{g_p^{-1}}, c_q^{g_q^{-1} * h_p^{-1} * g_p^{-1}}] = \varepsilon$. In the former case we obtain

$$[c_q, (c_r^{-1})^{h_r * h_q^{-1}}] = \varepsilon$$

and it follows that $c_q^{\pm 1} = (c_r^{-1})^{h_r * h_q^{-1}}$, so $c_q = c_r^{\pm 1}$ because $c_q, c_r \in R_C$. Moreover, $[h_r * h_q^{-1}, c_q] = \varepsilon$, so we have $h_r * h_q^{-1} = c_q^m$. Therefore

$$(g_p * c_p^{\alpha_p} * h_p) * (g_q * c_q^{\alpha_q - \alpha_r - m} * g_r^{-1}) = \varepsilon$$

and using the same argument as above we get $c_p = c_q^{\pm 1}$ and $h_p * g_q$ belongs to the cyclic subgroup generated by c_p.

The same observations work in the latter case, that is when $[c_p^{g_p^{-1}}, c_q^{g_q^{-1} * h_p^{-1} * g_p^{-1}}] = \varepsilon$. Here we get $c_p = c_q^{\pm 1}$ and $h_p * g_q = c_p^k$ immediately.

Now we are in the position to complete the proof of 2). We have the following cases:

1) All elements in the set $E = \{\alpha_u, \alpha_v, \alpha_w, \alpha_z\}$ are infinite.

Then by the claim $c_u = c_v = c_w = c_z$ and $h_u g_v, h_v g_w, h_w h_z$ belong to the cyclic subgroup generated by c_u. In this case $uvw \in P$.

2) Precisely one element in E is finite.

Then in the sequence u, v, w, z we have a subsequence a, b, c in which either a, b or b, c satisfy the condition in 1). In this case obviously $abc \in P$.

3) Precisely two elements in E are finite.

Then in the sequence u, v, w, z we have either a subsequence a, b, c in which a, b or b, c satisfy the condition in 1) or a subsequence a, b, c in which exactly b satisfies the condition in 1). In both cases $abc \in P$. This proves the proposition.

□

11.2 Non-standard extension of centralizers

We continue to use notations from the previous section.

Let $G \leq CDR(\mathbb{Z}[t]^+, X)$ be a group such that for any $g \in G$ its cyclically reduced form also belongs to G. Let K be a subset of G containing all elements $w \in G$ such that $C_G(w) \simeq \mathbb{Z}$ and let C be some fixed RC-set of K. Observe that $C \subseteq G$.

Let's fix G and C for the rest of this subsection.

By Proposition 2 if G satisfies the stabilizing condition (S) and the axiom of exponentiation (E2) then the subset

$$P = \{g * u^\alpha * h \mid g, h \in G, u \in C, \alpha \in \mathbb{Z}[t]^+\} \subset CDR(\mathbb{Z}[t]^+, X)$$

forms a pregroup with respect to the partial multiplication $*$ restricted to

$$D_P = \{(p, q) \in P \times P \mid p * q \in P\} \subset P \times P.$$

Theorem 27 *Let G satisfy the stabilizing condition (S) with respect to the length function induced from $CDR(\mathbb{Z}[t]^+, X)$ and axiom (E2). Then the set P generates a* *subgroup* $\langle P\rangle$ *in* $CDR(\mathbb{Z}[t]^+, X)$ *which is isomorphic to* $U(P)$.

Remark 2 *Recall that* $U(P)$ *is a universal group of a pregroup* P. *In the proof we refer to any tuple* $y = (y_1, \dots, y_n) \in P^n$ *as a* P*-sequence and we call* y *reduced if* $y_i y_{i+1} \notin P$ *for* $i \in [1, n-1]$.

Proof. Since all elements of P can be viewed as infinite words, that is, there exists an embedding $\iota : P \to CDR(\mathbb{Z}[t]^+, X)$, we can define a map $\phi : U(P) \to CDR(\mathbb{Z}[t]^+, X)$ as a unique extension of ι (by the categorical property of a universal group of a pregroup). In other words, if we have $y \in U(P)$ which can be viewed as a P-sequence $y = (y_1, \dots, y_n) \in P^n$, then

$$y^{\phi} = y_1 * \dots * y_n,$$

where we identify y_i with its image y_i^{ι} in $CDR(\mathbb{Z}[t]^+, X)$.

Claim 1. Let $y = (y_1, \dots, y_n)$ be an arbitrary P-sequence. Then the product

$$y^{\phi} = y_1 * \dots * y_n$$

is defined in $CDR(\mathbb{Z}[t]^+, X)$.

We use induction on the length of a y. Observe that we can assume y to be reduced, because if $y_j y_{j+1} \in P$, then their product is defined in $R(\mathbb{Z}[t], X)$ and we can consider a new P-sequence $y' = (y_1, \dots, y_j y_{j+1}, \dots, y_n)$. Therefore, we have all α_i infinite (not in $\mathbb{Z}$).

a) $k = 2$

We have $y_1 * y_2 = (g_1 * c_1^{\alpha_1} * h_1) * (g_2 * c_2^{\alpha_2} * h_2)$. Since all $c_i \in CR(\mathbb{Z}[t]^+, X)$, using the stabilizing condition we can find a natural number r such that for all $n_1, m_1, m_2, n_2 > r$

$$y_1 * y_2 = (g_1 * c_1^{m_1}) \circ c_1^{\alpha_1 - m_1 - n_1} \circ (c_1^{n_1} * h_1 * g_2 * c_2^{m_2}) \circ c_2^{\alpha_2 - m_2 - n_2} \circ (c_2^{n_2} * h_2).$$

Indeed, either $c_1 \neq c_2^{\pm 1}$ or $c_1 = c_2^{\pm 1}, [h_1 g_2, c_1] \neq \varepsilon$. We also use the fact that G is a group and finite powers of c_1, c_2 are elements from G, so the products $g_1 * c_1^{m_1}, c_1^{n_1} * h_1 * g_2 * c_2^{m_2}, c_2^{n_2} * h_2$ are defined.

b) Suppose we proved the claim for all R_c-sequences of the length $n = p - 1$. Consider $y = (y_1, \ldots, y_p)$. By induction $y_1 * \ldots * y_{p-1}$ is defined. Moreover, there exists a natural number r_1 such that

$$y_1 * \ldots * y_{p-1} = (g_1 * c_1^{m_1}) \circ c_1^{\alpha_1 - m_1 - n_1} \circ (c_1^{n_1} * h_1 * g_2 * c_2^{m_2}) \circ \ldots$$

$$\ldots \circ (c_{p-2}^{n_{p-2}} * h_{p-2} * g_{p-1} * c_{p-1}^{m_{p-1}}) \circ c_{p-1}^{\alpha_{p-1} - m_{p-1} - n_{p-1}} \circ (c_{p-1}^{n_{p-1}} * h_{p-1})$$

for all $m_i, n_i > r_1, i \in [1, p-1]$.

By induction the product $y_{p-1} * y_p$ is also defined and there exists a natural number r_2 such that

$$y_{p-1} * y_p = (g_{p-1} * c_{p-1}^{m_{p-1}}) \circ c_{p-1}^{\alpha_{p-1} - m_{p-1} - n_{p-1}} \circ (c_{p-1}^{n_{p-1}} * h_{p-1} * g_p * c_p^{m_p}) \circ c_p^{\alpha_p - m_p - n_p} \circ (c_p^{n_p} * h_p)$$

for all $m_{p-1}, n_{p-1}, m_p, n_p > r_2$. In fact we can choose r_2 to be such that

$$(c_{p-2}^{n_{p-2}} * h_{p-2}) * y_{p-1} * y_p = (c_{p-2}^{n_{p-2}} * h_{p-2} * g_{p-1} * c_{p-1}^{m_{p-1}}) \circ c_{p-1}^{\alpha_{p-1} - m_{p-1} - n_{p-1}} \circ$$

$$(c_{p-1}^{n_{p-1}} * h_{p-1} * g_p * c_p^{m_p}) \circ c_p^{\alpha_p - m_p - n_p} \circ (c_p^{n_p} * h_p).$$

for all $n_{p-2}, m_{p-1}, n_{p-1}, m_p, n_p > r_2$.

Now, if we take $r = \max\{r_1, r_2\}$, then for all $m_i, n_i > r$ we have

$$y_1 * \ldots * y_p = (g_1 * c_1^{m_1}) \circ c_1^{\alpha_1 - m_1 - n_1} \circ (c_1^{n_1} * h_1 * g_2 * c_2^{m_2}) \circ \ldots$$

$$\ldots \circ (c_{p-1}^{n_{p-1}} * h_{p-1} * g_p * c_p^{m_p}) \circ c_p^{\alpha_p - m_p - n_p} \circ (c_p^{n_p} * h_p).$$

which proves **Claim 1**.

Claim 2. The set P generates a subgroup $H = \langle P \rangle$ in $CDR(\mathbb{Z}[t]^+, X)$.

Let $y = (y_1, \ldots, y_n)$ and $z = (z_1, \ldots, z_k)$ be two P-sequences. We have to show that $(yz)^\varphi = y^\varphi * z^\varphi$ that is

$$y_1 * \ldots * y_n * z_1 * \ldots * z_k = (y_1 * \ldots * y_n) * (z_1 * \ldots * z_k).$$

Since by **Claim 1** the left part of the required equality is defined in $R(\mathbb{Z}[t]^+, X)$, then by Theorem 24 the right part is also defined and required equality is true.

Finally for any P-sequence $y = (y_1, \ldots, y_n)$ its inverse is $y^{-1} = (y_n^{-1}, \ldots, y_1^{-1})$ which is a P-sequence and by **Claim 1**, $y_n^{-1} * \ldots * y_1^{-1}$ is defined in $R(\mathbb{Z}[t]^+, X)$. By previous considerations we have $(yy^{-1})^\varphi = \varepsilon = y^\varphi * (y^{-1})^\varphi$ and we can conclude that $H = \langle P \rangle$ in $CDR(\mathbb{Z}[t]^+, X)$.

To show that H is isomorphic to $U(P)$ we follow the construction of the universal group $U(P)$ described above.

Let $y = (y_1, \ldots, y_n)$ be a reduced P-sequence. It satisfies the following condition for any $1 \leq i \leq n-1$:

1) if $n > 1$ then all α_i are infinite, where $y_i = g_i * c_i^{\alpha_i} * h_i$;

2) if $y_i = g_i * c_i^{\alpha_i} * h_i, y_{i+1} = g_{i+1} * c_{i+1}^{\alpha_{i+1}} * h_{i+1}$ then either $c_i \neq c_{i+1}^{\pm 1}$ or $c_i = c_{i+1}^{\pm 1}$ and $[c_i, (h_i * g_{i+1})] \neq \varepsilon$.

The following claim follows directly from definitions.

Claim 3. Let x, y be reduced P-sequences. If $x \sim y$ then $x^\varphi = y^\varphi$.

Combining **Claim 2** and **Claim 3** we see that

$$\varphi : U(P) \rightarrow H$$

is a group homomorphism.

Claim 4. φ is onto.

To prove this one needs to show that for every P-sequence y there exists a reduced P-sequence z such that $y \sim z$. This is obvious, since if $y_i y_{i+1}$ is defined in P then replacing the pair of components y_i, y_{i+1} by one component $y_i y_{i+1}$ we have a new (shorter) sequence with fewer reductions than it in y. And the result follows by induction.

Claim 5. ϕ is one-to-one.

Indeed, we need to show that if y, z are reduced P-sequences such that $y^\phi = z^\phi$ then $y \sim z$. To this end, we prove first that for any P-sequence y there exists a unique P-sequence $u = (u_1, \ldots, u_m)$ such that $y \sim u$ and

$$u^\phi = u_1 \circ \cdots \circ u_m.$$

Let $y = (y_1, \ldots, y_m)$ be any reduced P-sequence. Since y is reduced we have

$$y = (g_1 * c_1^{\alpha_1} * h_1, g_2 * c_2^{\alpha_2} * h_2, \ldots, g_m * c_m^{\alpha_m} * h_m).$$

where $g_i, h_i \in G$ and $\alpha_i \in \mathbb{Z}[t]$ are infinite. Thus,

$$y^\phi = f_0 * c_1^{\alpha_1} * f_1 * c_2^{\alpha_2} * \cdots * g_{m-1} * c_m^{\alpha_m} * f_m,$$

where $f_0 = g_1, f_i = h_i * g_{i+1}, i \in [1, m-1], f_m = h_m$.

Now, using the fact that G satisfies the stabilizing condition we rewrite y^ϕ in the unique way and then find from the obtained representation a new reduced P-sequence u such that $u \sim y$ and $u^\phi = y^\phi$.

In the same way as in **Claim 1**, using the stabilizing condition in G we can rewrite y^ϕ in the following way

$$y^\phi = w_0 \circ c_1^{\gamma_1} \circ w_1 \circ c_2^{\gamma_2} \circ \cdots \circ w_{m-1} \circ c_m^{\gamma_m} \circ w_m.$$

where $w_0 = f_0 * c_1^{p_1}$, $w_m = c_m^{k_m} * f_m$, $w_i = c_i^{k_i} * f_i * c_{i+1}^{p_{i+1}}$, $\gamma_i = \alpha_i - p_i - k_i$, $i \in [1, m]$. Observe that this representation of y^σ is not unique and we proceed as follows.

If w_0 contains an exponent of c_1 as a terminal segment and $w_1 \circ c_2^{\gamma_2}$ contains an exponent of c_1 as an initial segment then we adjoin these exponents to $c_1^{\gamma_1}$. Observe that these exponents are finite because of the stabilizing condition. Thus we obtain

$$y^\sigma = w'_0 \circ c_1^{\beta_1} \circ w''_1 \circ c_2^{\gamma'_2} \circ \cdots \circ w_{m-1} \circ c_m^{\gamma_m} \circ w_m.$$

where w'_0 does not have $c_1^{\pm 1}$ as a terminal segment and $w''_1 \circ c_2^{\gamma'_2}$ does not contain $c_1^{\pm 1}$ as an initial segment. That is, β_1 is uniquely defined.

Then, if w''_1 contains an exponent of c_2 as a terminal segment and $w_2 \circ c_3^{\gamma_3}$ contains an exponent of c_2 as an initial segment then we adjoin these exponents to $c_2^{\gamma'_2}$. Again, by the stabilizing condition these exponents can be only finite. Thus we obtain

$$y^\sigma = w'_0 \circ c_1^{\beta_1} \circ w'_1 \circ c_2^{\beta_2} \circ w''_2 \circ c_3^{\gamma'_3} \circ \cdots \circ w_{m-1} \circ c_m^{\gamma_m} \circ w_m.$$

where w'_1 does not have $c_2^{\pm 1}$ as a terminal segment and $w''_2 \circ c_3^{\gamma'_3}$ does not contain $c_2^{\pm 1}$ as an initial segment. That is, β_2 is also uniquely defined.

We continue in the same way with all the other infinite exponents in y^σ.

Finally we obtain

$$y^\sigma = w'_0 \circ c_1^{\beta_1} \circ w'_1 \circ c_2^{\beta_2} \circ \cdots \circ w'_{m-1} \circ c_m^{\beta_m} \circ w'_m.$$

which is a unique representation of y^σ.

Now, let

$$u = (w'_0 \circ c_1^{\beta_1} \circ w'_1, c_2^{\beta_2} \circ w'_2, \ldots, c_m^{\beta_m} \circ w'_m).$$

Observe that u is reduced and $u \sim y$ by the construction.

To complete the proof of **Claim 5** we take two P-sequences y, z such that $y^{\wp} = z^{\wp}$. By previous considerations there are normal P-sequences u, w such that $y \sim u$, $z \sim w$. Due to uniqueness of normal forms and fact that $u^{\wp} = w^{\wp}$ we have $u = w$. This proves the **Claim 5** and the Theorem.

□

Remark 3 *While proving* **Claim 5** *in the theorem above we showed that for any reduced P-sequence $y = (y_1, \ldots, y_m)$ one can find the unique P-sequence $y' = (y'_1, \ldots, y'_m)$ such that $y \sim y'$ and $(y'_i y'_{i+1})^{\wp} = (y'_i)^{\wp} \circ (y'_{i+1})^{\wp}$, $1 \leq i \leq m-1$. We call y' a* normal sequence *for y.*

Observe that from the proof above and the stabilizing condition it follows that any element $g \in H$ has a unique representation as an infinite word which comes from the corresponding normal sequence for g in $U(P)$

$$g = g_1 \circ c_1^{\alpha_1} \circ g_2 \circ c_2^{\alpha_2} \circ \cdots \circ c_m^{\alpha_m} * g_{m+1},$$

where $c_i, g_i \in G$, $\alpha_i \in \mathbb{Z}[t]^+$ are infinite. We call such unique representation a unique reduced form *of g.*

We will use this fact in Section 14 after we construct an embedding of $F^{\mathbb{Z}[t]}$ into infinite words. Then in Section 15 we summarize different types of representation for elements of $F^{\mathbb{Z}[t]}$ and the ways to obtain them.

Now we describe the algebraic structure of the group $H = \langle P \rangle$.

Definition 24 *Let us enumerate all elements of C so that $C = \{c_i \mid i \in I\}$. Let $S = \{s_{i,j} | i \in I, j \in \mathbb{N}\}$. Then we define*

$$G(C, S) = \langle G, S | [c_i, s_{i,j}] = [s_{i,j}, s_{k,j}] = 1, i \in I, (j, k \in \mathbb{N}) \rangle$$

Theorem 28 $\langle P \rangle \simeq G(C, S)$.

Proof. We define a map $\phi : P \to G(C, S)$, by

$$g_i * c_i^{\alpha} * h_i \xrightarrow{\phi} g_i s_{i,n}^{a_n} s_{i,n-1}^{a_{n-1}} \cdots s_{i,1}^{a_1} c_i^{a_0} h_i,$$

where $\alpha = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$. Since $H = U(P)$, by the universal property ϕ extends to a homomorphism $\psi : H \to G(C, S)$ and this homomorphism is unique.

Claim. ψ is bijective.

Take any reduced P-sequence

$$y = (g_1 * c_1^{\alpha_1} * h_1, g_2 * c_2^{\alpha_2} * h_2, \ldots, g_m * c_m^{\alpha_m} * h_m).$$

Due to the definition of a reduced sequence and the fact that in any $\alpha_i = a_{n_i} t^n + a_{n-1_i} t^{n-1} + \cdots + a_{1_i} t + a_{0_i}$ we have not all a_{k_i} equal to zero, $\psi(y) \neq 1$ by [35]. Also since $G(C, S)$ is generated by $G \cup S$ then $\psi(P)$ generates $G(C, S)$ and ψ is surjective.

□

Observe that any element of $G(C, S)$ has a representation (not unique) as an element of HNN-extension. But also the isomorphism constructed in Theorem 24 makes it possible to introduce unique normal forms in the following way. Let $w \in G(C, S)$. Then $w = y^{\psi}$ for some $y \in U(P)$ and $\psi : U(P) \to G(C, S)$ is the isomorphism from Theorem 24. So we have

$$y = (g_1 * c_1^{\alpha_1} * h_1, g_2 * c_2^{\alpha_2} * h_2, \ldots, g_m * c_m^{\alpha_m} * h_m)$$

and

$$w = g_1 f_1 g_2 f_2 \cdots g_m f_m g_{m+1},$$

where

$$f_i = s_{i,n}^{a_{i,n}} s_{i,n-1}^{a_{i,n-1}} \cdots s_{i,1}^{a_{i,1}} c_i^{a_{i,0}} \text{ and } g_i \in G.$$

We say that w is in the *normal form* if y is a normal P-sequence. Observe that the normal form for any element from $G(C, S)$ can be computed directly (without its preimage in $U(P)$) using the stabilizing condition in the same way as for normal sequences.

12 Regular free length functions on extensions of centralizers

Let G be a subgroup of $CDR(\mathbb{Z}[t]^+, X)$ such that the length function on G induced from $CDR(\mathbb{Z}[t]^+, X)$ is regular and we assume that the stabilizing condition (S) holds in G together with the exponentiation axiom (E2) with respect to $\mathbb{Z}[t]^+$. Let C and P be defined as in the previous section. Then, as we have shown above, the universal group $U(P)$, which is isomorphic to the extension of all cyclic centralizers of G by free abelian groups of infinite rank, is naturally embeddable into $CDR(\mathbb{Z}[t]^+, X)$ by means of the map $\varphi : U(P) \to CDR(\mathbb{Z}[t]^+, X)$, constructed in Theorem 27. Hence, there exists a length function on $U(P)$, which is induced from $CDR(\mathbb{Z}[t]^+, X)$. In this section we show that this length function is regular and computable.

At first we show how one can compute a length of any element in $U(P)$.

Take any reduced P-sequence y' and let y be a normal sequence such that $y' \sim y$. Then we set $|y'| = |y|$ and define $|y|$ as follows. We have

$$y = (g_0 \circ c_1^{\alpha_1} \circ g_1, c_2^{\alpha_2} \circ g_2, \ldots, c_m^{\alpha_m} \circ g_m),$$

where $g_i, c_i \in G$ and $\alpha_i \in \mathbb{Z}[t]^+$ are infinite. We want to express $|y|$ in terms of $|g_i|$, $|c_i|$ and α_i.

Since y is normal, we have

$$y^{\varphi} = g_0 \circ c_1^{\alpha_1} \circ g_1 \circ c_2^{\alpha_2} \circ g_2 \circ \cdots \circ c_m^{\alpha_m} \circ g_m$$

and

$$|y| = \sum_{i=0}^{m} |g_i| + \sum_{i=1}^{m} |c_i^{\alpha_i}|.$$

Finally, we have

$$|y| = \sum_{i=0}^{m} |g_i| + \sum_{i=1}^{m} |c_i||\alpha_i|.$$

That is, the length of y is the length of its unique reduced form in $CDR(\mathbb{Z}[t]^+, X)$.

Now we can compute the length of elements from $G(C, S)$ in the same way using the fact that $G(C, S)$ is isomorphic to $U(P)$. Indeed, we set the length of any $x \in G(C, S)$ to be the length of its preimage $y \in U(P)$ under the isomorphism between $G(C, S)$ and $U(P)$. Since to compute the length of y we have to obtain the normal sequence which corresponds to y, it means that we can compute the length of x via the normal form for x.

Let $x \in G(C, S)$

$$x = z_1 h_1 z_2 h_2 z_3 \ldots h_m z_m h_{m+1},$$

where $h_i = s_{i,n}^{b_{i,n}} s_{i,n-1}^{b_{i,n-1}} \cdots s_{i,1}^{b_{i,1}} c_i^{b_{i,0}}$ and $z_i \in G$. Suppose

$$g_1 f_1 g_2 f_2 g_3 \ldots g_m f_m g_{m+1},$$

where $f_i = s_{i,n}^{a_{i,n}} s_{i,n-1}^{a_{i,n-1}} \cdots s_{i,1}^{a_{i,1}} c_i^{a_{i,0}}$ and $g_i \in G$, is the normal form for x. Then we have the following formula:

$$|x| = \sum_{i=1}^{m+1} |g_i| + \sum_{i=1}^{m} |(a_{i,0}, a_{i,1}, \ldots, a_{i,n})| \, |c_i|.$$

Thus, if we are given any reduced P-sequence or HNN-extension representation of an element from $U(P)$ then we can compute its length effectively.

Now, take two elements $x, y \in U(P)$. Then, we have $x^{\phi} = z' \circ x_1, y^{\phi} = z' \circ y_1$, where $l(z') = d(x, y)$. To prove regularity of length function induced on $U(P)$ from $CDR(\mathbb{Z}[t]^+, X)$ we have to show that there exists $z \in U(P)$ such that $z^{\phi} = z'$.

By Remark 3 we have

$$x^{\circ} = h_0 \circ c_1^{\alpha_1} \circ h_1 \circ c_2^{\alpha_2} \circ h_2 \circ \cdots \circ c_m^{\alpha_m} \circ h_m.$$

where $h_i, c_i \in G$, $\alpha_i \in \mathbb{Z}[t]^+$ and α_i are infinite.

An initial subword z' of x° in general has the following form

$$z' = h_0 \circ \ldots \circ c_k^{\beta_k} \circ c'_k \circ h'_k.$$

where either $h'_k = \varepsilon$, $|\beta_k| < |\alpha_k|$ and c'_k is an initial segment of c_k, or $c'_k = \varepsilon$, $\beta_k = \alpha_k$ and h'_k is an initial segment of h_k. In the former case let $z \in U(P)$ be defined as

$$z = (h_0 \circ c_1^{\alpha_1}, h_1 \circ c_2^{\alpha_2}, \ldots, h_{k-1} \circ c_k^{\beta_k} \circ c'_k)$$

and in the latter case we set

$$z = (h_0 \circ c_1^{\alpha_1}, h_1 \circ c_2^{\alpha_2}, \ldots, h_{k-1} \circ c_k^{\alpha_k} \circ h'_k).$$

Observe that in both, z is indeed a reduced sequence from $U(P)$ because $c'_k, h'_k \in G$, $\beta_k \in \mathbb{Z}[t]^+$. Also, it is easy to see that $z^{\circ} = z'$. Hence, $U(P)$ has a regular length function induced from $CDR(\mathbb{Z}[t]^+, X)$.

13 Embedding $F^{\mathbb{Z}[t]}$ into $CDR(\mathbb{Z}[t]^+, X)$

In this section we realize constructively the formal description of $F^{\mathbb{Z}[t]}$ suggested by Lyndon [31]. In his initial definition Lyndon started with a free group $F(X)$ and went on adding formal exponents from $\mathbb{Z}[t]$, identifying some of these formal expressions to get a group. We show below that precisely this method can be realized using infinite $\mathbb{Z}[t]^+$-words.

We start by recalling the description of $F^{\mathbb{Z}[t]}$ as the union of some chain of extensions of centralizers (see Section 5.3). For a group G choose a set $S = \{C_i \mid i \in I\}$ of representatives of conjugacy classes of cyclic centralizers in G, i.e., every cyclic centralizer in G is conjugate to one from S, and no two centralizers from S are conjugate. Then the HNN-extension

$$H = \langle\, G, t_{i,j} \ \ (i \in I) \mid [t_{i,j}, u_i] = [t_{i,j}, t_{i,k}] = 1 \ (u_i \in C_i, i \in I, j, k \in \mathbb{N})\,\rangle.$$

is termed an *extension of all cyclic centralizers* in G (here u_i is a generator of C_i).

Let F be a free non-abelian group. We obtain the group $F^{\mathbb{Z}[t]}$ as union of the following infinite chain of groups:

$$F = G_1 < G_2 < \ldots < G_n \ldots < \ldots . \qquad (6)$$

where G_{i+1} is obtained from G_i by extension of all cyclic centralizers in G_i.

In Subsection 11.2 we showed that if G is a subgroup of $CDR(\mathbb{Z}[t]^+, X)$ and the stabilizing condition holds in G together with the exponentiation axiom (E2) then a group obtained from G by extension of all cyclic centralizers is also a subgroup of $CDR(\mathbb{Z}[t]^+, X)$. Thus we can use this fact by induction on n in the series above.

To be able to apply the induction we need to show now that the free group F satisfies the stabilizing condition and the exponentiation axiom (E2) with respect to $\mathbb{Z}[t]^+$.

Lemma 19 *A free group F satisfies the stabilizing condition (S).*

Proof. We start with the following known result.

Lemma 20 *[33] Let $v, w \in F$ be cyclically reduced and not proper powers. If v^m and w^n, $m, n > 0$ have common initial segment of length $|v| + |w|$ then $v = w$.*

Now we prove the stabilizing condition (S). Observe that we can assume v and w to be cyclically reduced. Indeed, suppose $v = c^{-1} \circ v_1 \circ c, w = d^{-1} \circ w_1 \circ d$, then

$$v^n g w^m = (c^{-1} \circ v_1 \circ c)^n g (d^{-1} \circ w_1 \circ d)^m = (c^{-1} \circ v_1^n \circ c) g (d^{-1} \circ w_1^m \circ d)$$

so finally we have

$$v^n g w^m = c^{-1}(v_1^n (cg(d^{-1}) w_1^m) d$$

and if (S) holds for the triple (v_1, cgd^{-1}, w_1) then it holds for the triple (v, g, w).

a) Consider the case when v is not conjugate to $w^{\pm 1}$. Assume that (S) condition breaks on the triple (v, g, w). Without loss of generality we can assume that (S) is not met for a triple (v, g, w) such that $gw^n = g \circ w^n, |g| \leq |v|$.

Now if $|g| = |v|$ then for sufficiently large numbers n, m elements v^{-n}, w^m have common initial segment of length greater than $|v| + |w|$. Therefore, by Lemma 20, $v = w$ or $v = w^{-1}$. This is a contradiction with our assumption.

Let $|g| < |v|$. Without loss of generality we can consider the case $m, n > 0$. Then $v = v_1 \circ g^{-1}$ and elements $(v_1^{-1} \circ g)^n$ and w^m have common initial segment of length greater than $|v| + |w|$ for reasonably large n, m. By Lemma 20, $v_1^{-1} \circ g = w$, thus v and w^{-1} are conjugates - a contradiction.

b) Let $1 \neq v = w, [g, v] \neq 1$ and condition (S) breaks on the triple (v, g, v). As before we can assume $gv^m = g \circ v^m, |g| \leq |v|$. Besides, let the numbers m, n be of the same sign (for example $n, m \geq 0$). In this case $v = v_1 \circ g^{-1}$ and for sufficiently large n, m elements $(v_1^{-1} \circ g)^n$ and v^m have common initial segment of length greater than $2|v|$. By Lemma 20, $v_1 \circ g^{-1} = g \circ v_1^{-1}$, thus $v = 1$ - contradiction.

If numbers n, m are of opposite signs we have the equality $v_1 \circ g = g \circ v_1$. Therefore $[g, v_1] = 1$ and the element v is a proper power - contradiction with the choice of v.

□

Now we check the axiom of exponentiation (E2) for any G_i in the series (6).

Lemma 21 *Any G_i in the series (6) satisfies the exponentiation axiom (E2) with respect to $\mathbb{Z}[t]^+$.*

Proof. We show that $(u^{-1} * v * u)^\alpha = u^{-1} * v^\alpha * u$ for any $u, v \in G_i$ and $\alpha \in \mathbb{Z}[t]^+$.

Observe, that for any element $x \in G_i$ its centralizer is either cyclic or is isomorphic to $\mathbb{Z}[t]^+$.

If $[u, v] = \varepsilon$ then u and v belong to the same centralizer, that is $u = w^\delta, v = w^\gamma$ for some $w \in G_{i-1}, \delta, \gamma \in \mathbb{Z}[t]^+$. So by definition of exponentiation we have (E2) for the pair (u, v) automatically.

Suppose $[u, v] \neq \varepsilon$. Because of the definition of exponentiation we can assume v to be cyclically reduced. We can also assume that u does not have an initial segment w_1 commuting with v because in this case by observation above we have

$$(u^{-1} * v * u)^\alpha = (w^{-1} * v * w)^\alpha,$$

$$u^{-1} * v^\alpha * u = w^{-1} * v^\alpha * w,$$

where $u = w_1 \circ w$.

Since v is cyclically reduced we have either $u^{-1} * v = u^{-1} \circ v$ or $v * u = v \circ u$. Suppose the latter. Due to assumptions made above v does not cancel completely in $u^{-1} * (v \circ u)$. So we have $u = v_1 \circ u_2$, $v = v_1 \circ v_2$ and $u^{-1} * (v \circ u) = u_2^{-1} \circ v_2 \circ v_1 \circ u_2$. Notice that $v_2 \circ v_1$ is cyclically reduced because $v = v_1 \circ v_2$ is. Now, by definition of exponentiation we have

$$(u^{-1} * v * u)^\alpha = u_2^{-1} \circ (v_2 \circ v_1)^\alpha \circ u_2,$$

and by Lemma 13 we have

$$u^{-1} * v^\alpha * u = (u_2^{-1} \circ v_1^{-1}) * (v_1 \circ v_2)^\alpha * (v_1 \circ u_2) = u_2^{-1} * (v_2 \circ v_1)^\alpha * u_2.$$

To complete the proof observe that

$$u_2^{-1} * (v_2 \circ v_1)^\alpha * u_2 = u_2^{-1} \circ (v_2 \circ v_1)^\alpha \circ u_2.$$

□

Lemma 21 means automatically that F has axiom (E2) and now we have a base of induction.

Once the base of induction is established, we can apply the results of Subsection 11.2 to carry out the embedding of $F^{\mathbb{Z}[t]}$ into $CDR(\mathbb{Z}[t]^+, X)$. Indeed, we may assume by induction that the group G_i (from the chain (6)) is embedded into $CDR(\mathbb{Z}[t]^+, X)$ and satisfies the stabilizing condition with respect to the length function induced from $CDR(\mathbb{Z}[t]^+, X)$ and also has the exponentiation axiom (E2). Then by Theorems 27 and 28 the group G_{i+1} is embeddable into $CDR(\mathbb{Z}[t]^+, X)$. To complete the induction step we need only to prove that G_{i+1} also satisfies the stabilizing condition and the exponentiation axiom (E2). We have (E2) in G_{i+1} by Lemma 21 so let us prove the stabilizing condition.

Lemma 22 *G_{i+1} satisfies the stabilizing condition (S) with respect to the length function induced from $CDR(\mathbb{Z}[t]^+, X)$.*

Proof. In the proof we need the following result, which is a generalization in some sense of Lemma 20.

Claim. Let $v, w \in G_{i+1}$ be cyclically reduced with cyclic centralizers and not proper powers. If v^m and $w^n, m, n > 0$ have common initial segment of length $|v| + |w|$ then $v = w$.

Without loss of generality we can assume $|w| \geq |v|$. Then we have $w = v^k \circ w_1, |v| > |w_1|, k \geq 1$ and $v = w_1 \circ v_1$. Observe also that since $k \geq 1$ and w starts with the power v^k then v has v_1 as an initial segment. On the other hand v has w_1 as a terminal segment. So we have $v = w_1 \circ v_1 = v_1 \circ w_1$ and it follows that $[w_1, v_1] = \varepsilon$ and we have $[w_1, v] = \varepsilon$ immediately. Then, since $w = v^k \circ w_1$, we have $[v, w] = \varepsilon$, so they belong to the same centralizer and by our assumption v and w are finite powers of the same element. But the only possibility is that $v = w$ - otherwise a contradiction with assumprions of the Claim.

Now we can conduct exactly the same argument as in Lemma 19 using the **Claim** above instead of Lemma 20 wherever necessary.

□

The induction step is complete and an embedding of $F^{\mathbb{Z}[t]}$ into $CDR(\mathbb{Z}[t]^+, X)$ is constructed.

14 Conjugacy and power problems in $F^{\mathbb{Z}[t]}$

In this section we apply techniques of infinite A-words for solving conjugacy and power problems in $F^{\mathbb{Z}[t]}$.

We fix embedding of $F^{\mathbb{Z}[t]}$ into $CDR(\mathbb{Z}[t]^+, X)$ constructed in the previous section. Now we can view elements from $F^{\mathbb{Z}[t]}$ as infinite $\mathbb{Z}[t]^+$-words.

I) Conjugacy problem - given two elements $f, g \in F^{\mathbb{Z}[t]}$, we need to find an algorithm which decides if there exists $x \in F^{\mathbb{Z}[t]}$ such that $x^{-1} f x = g$.

Obviously, it is enough to solve conjugacy problem for the case when f and g are cyclically reduced.

Lemma 23 *Let $f, g \in F^{\mathbb{Z}[t]}$ be cyclically reduced and $w \in F^{\mathbb{Z}[t]}$ such that $w^{-1} * f * w = g$. Then $|f| = |g|$.*

Proof. Since f is cyclically reduced in $w^{-1} * f * w$ we have either $w^{-1} * f = w^{-1} \circ f$ or $f * w = f \circ w$. Assume the first. Also since g is cyclically reduced then w cancels completely in $w^{-1} * f * w$. Then $|g| = |w^{-1} * f * w| = |w| + |f| - |w| = |f|$.

□

Lemma 24 *Let $f, g \in F^{\mathbb{Z}[t]}, f \neq g$ be cyclically reduced and $w \in F^{\mathbb{Z}[t]}$ be such that $w^{-1} * f * w = g$. Then $w = w_1 \circ h$, where $[w_1, f] = \varepsilon, |h| < |f|$ and g is a cyclic permutation of f.*

Proof. We can present w in the following form $w = w_1 \circ h$, where w_1 belongs to the centralizer of f and h does not contain as an initial segment any element commuting with f. Thus we have to show that $|h| < |f|$ and $h^{-1} * f * h = g$, which means automatically, that g is a cyclic permutation of f.

Suppose on the contrary $|h| \geq |f|$. Since f is cyclically reduced we have either $h^{-1} * f = h^{-1} \circ f$ or $f * h = f \circ h$. Assume the former. Then we have

$$w^{-1} * f * w = (h^{-1} \circ f) * h$$

and h has to cancel completely in $(h^{-1} \circ f) * h$. Thus we have that h has f^{-1} as an initial segment - contradiction, which proves the lemma.

□

From previous lemmas it follows that two cyclically reduced elements from $F^{\mathbb{Z}[t]}$ are conjugate if and only if they have the same length and, moreover, they are cyclic permutations of each other. So, to check if two elements from $F^{\mathbb{Z}[t]}$ are conjugate we have to compare their cyclic permutations. But unlike free groups, there are infinitely many cyclic permutations of an element in $F^{\mathbb{Z}[t]}$. Thus we have to reduce the number of checks somehow.

Let $f, g \in F^{\mathbb{Z}[t]}$ be cyclically reduced. Then there exist natural numbers $n_1, n_2 > 0$ such that $f \in G_{n_1+1}$ but $f \notin G_{n_1}$, and $g \in G_{n_2+1}$ but $g \notin G_{n_2}$.

From the proof of Theorem 27 and from Remark 3 it follows that f and g have unique reduced forms

$$f = f_1 \circ v_1^{\alpha_1} \circ f_2 \circ \ldots \circ v_k^{\alpha_k} \circ f_{k+1},$$

where $f_i, v_i \in G_{n_1}, \alpha_i \in \mathbb{Z}[t]^+$ and

$$g = g_1 \circ u_1^{\beta_1} \circ g_2 \circ \ldots \circ u_l^{\beta_l} \circ g_{l+1},$$

where $g_j, u_j \in G_{n_2}, \beta_j \in \mathbb{Z}[t]^+$.

Suppose $n_1 < n_2$. It is easy to see that since $f \in G_{n_2}$, it can not be some cyclic permutation of g because f does not contain infinite exponents of elements from G_{n_2}. Because of the same reason g can not be conjugate to f if $n_1 > n_2$.

Thus, if f is conjugate to g then $n_1 = n_2$.

Lemma 25 *Let $f, g \in G_{n+1}, f, g \notin G_n$ be cyclically reduced. Suppose f and g have unique reduced forms*

$$f = v_1^{\alpha_1} \circ f_1 \circ \ldots \circ v_k^{\alpha_k} \circ f_k,$$

where f_k does not have $v_1^{\pm 1}$ as a terminal segment and

$$g = g_1 \circ u_1^{\beta_1} \circ g_2 \circ \ldots \circ u_l^{\beta_l} \circ g_{l+1}.$$

Then from $f = g$ it follows that $l = k, g_1 = 1, g_i = f_{i-1}, i \in [2, l+1], u_j = v_j, \alpha_j = \beta_j, j \in [1, l]$.

Proof. If $u_1 \neq v_1$ then from the stabilizing condition $g^{-1} * f$ can not be cancelled completely. So $u_1 = v_1$ and $[g_1, u_1] = 1$ which is impossible because otherwise $g_1 = u_1^p$ - contradiction. Thus $g_1 = 1$

Now we can cancel out initial $u_1^{\alpha_1}$ in f and g if $|\alpha_1| < |\beta_1|$ or $u_1^{\beta_1}$ if $|\beta_1| < |\alpha_1|$. If $\alpha_1 \neq \beta_1$ in both cases we get a contradiction with the assumption that f and g are given as unique reduced forms. So $\alpha_1 = \beta_1$ and we can proceed by induction on the number of syllables.

□

Finally, suppose $f, g \in G_{n+1}, f, g \notin G_n$ are cyclically reduced and have unique reduced forms

$$f = v_1^{\alpha_1} \circ f_1 \circ \ldots \circ v_k^{\alpha_k} \circ f_k,$$

where $f_i, v_i \in G_n, \alpha_i \in \mathbb{Z}[t]^+$, f_k does not have $v_1^{\pm 1}$ as a terminal segment, and

$$g = g_1 \circ u_1^{\beta_1} \circ g_2 \circ \ldots \circ u_l^{\beta_l} \circ g_{l+1},$$

where $g_j, u_j \in G_n, \beta_j \in \mathbb{Z}[t]^+$. Without loss of generality we can assume such a form of f - we can obtain it using cyclic permutations.

If f is a conjugate of g then it is a cyclic permutation of g. But by Lemma 25 this cyclic permutation of g has to be of a particular type

$$g(j) = u_j^{\beta_j} \circ g_{j+1} \circ \ldots \circ g_{l+1} \circ g_1 \circ u_1^{\beta_1} \circ \ldots \circ u_{j-1}^{\beta_{j-1}} \circ g_j$$

otherwise it can not be equal to f. There is only a finite number of such permutations and $|g| = |g(j)|$ for any $j \in [1, l]$.

To decide if f is conjugate to g we check if $f = g(j)$ for some $j \in [1, l]$.

II) Power problem - given $g \in F^{\mathbb{Z}[t]}$, we need to find an algorithm which decides if there exists a natural number n such that $g = f^n, f \in F^{\mathbb{Z}[t]}$.

Let $g \in G_{n+1} \setminus G_n$. We can assume g to be cyclically reduced. We have a unique reduced form

$$g = g_1 \circ u_1^{\beta_1} \circ g_2 \circ \ldots \circ u_l^{\beta_l} \circ g_{l+1},$$

where $g_i, u_i \in G_n, \beta_i \in \mathbb{Z}[t]^+$ and we have that $l \geq 1, \beta_1 \neq 0$.

Consider the case $l = 1$. If $g_1 \neq 1$ then g isn't a power unless g_1 is a power of u_1 which is impossible. Now let $g = u_1^{\beta_1}$ - if β_1 is a multiple of some γ then g is a power otherwise it is not. So, everything reduces in this case to computations in free abelian group of finite rank, where we can check easily if an element is a proper power.

Now suppose $l > 1$. We take all divisors of l - the number of syllables in g. They form a finite set D. Consider initial subwords s_i of g defined as follows: $s_i : [1, \alpha_i] \to X^{\pm}$, where $\alpha_i = |g|/d_i, d_i \in D$ such that we can divide $|g|$ coordinatewise by d_i. Since D is finite then the number of s_i defined above is also finite. For each s_i we check if $g = s_i \circ \cdots \circ s_i$ where the product of s_i is taken d_i times. If it is for some i then g is a proper power, otherwise it is not. This follows from the fact that we have a regular length function on $F^{\mathbb{Z}[t]}$.

Thus, the procedure described above can effectively determine if any given element from $F^{\mathbb{Z}[t]}$ is a proper power in $F^{\mathbb{Z}[t]}$.

15 Reduced forms for elements in $F^{\mathbb{Z}[t]}$

In this section we summarize the material of Sections 11 – 14 by recalling the construction of $F^{\mathbb{Z}[t]}$ and ways of presenting its elements as reduced infinite words. The corresponding notation is introduced which will be used in the next sections.

15.1 Lyndon's free $\mathbb{Z}[t]$-group

Recall the construction of $F^{\mathbb{Z}[t]}$. We obtain the group $F^{\mathbb{Z}[t]}$ as union of the following infinite chain of groups:

$$F = G_0 < G_1 < \ldots < G_n \ldots < \ldots . \tag{7}$$

where G_{i+1} is obtained from G_i by extension of all cyclic centralizers in G_i. In G_i, we choose K to be a subset of G_i containing all elements with cyclic centralizers and U_i to be some fixed RC-set of K (see Subsection 11.2). Recall that U_i satisfies the following conditions:

1) U_i does not contain proper powers;

2) for any $u, v \in U_i$, $u \neq v^{-1}$;

3) for any $u \neq v \in U_i$ the elements $u^{\pm 1}, v^{\pm 1}$ are not conjugate;

4) any $u \in U_i$ is cyclically reduced.

5) for each $u \in K$ ($C_{G_i}(u)$ is cyclic) there exist $v \in U_i, k \in \mathbb{Z}, c \in CDR(\mathbb{Z}[t]^+, X)$, and a cyclic permutation $\pi(v)$ of v such that

$$u = c^{-1} \circ \pi(v)^k \circ c,$$

moreover, such $v, c, k, \pi(v)$ are unique.

Eventually, to obtain G_{i+1} from G_i we extend centralizers of all elements from U_i (thus, all cyclic centralizers are extended automatically as well).

Of course, the existence of an embedding of $F^{\mathbb{Z}[t]}$ into the set of infinite words implies automatically the fact that all subgroups of $F^{\mathbb{Z}[t]}$ also belong to $CDR(\mathbb{Z}[t]^+, X)$, that is, their elements can be viewed as infinite words. From now on we assume the embedding $\rho : F^{\mathbb{Z}[t]} \longrightarrow CDR(\mathbb{Z}[t]^+, X)$ to be fixed. Moreover, for simplicity we identify $F^{\mathbb{Z}[t]}$ with its image $\rho(F^{\mathbb{Z}[t]})$.

We introduce an order on each U_i. At first, consider $v \in U_i$. There exists an infinite subset $K_i(v) \subset U_i$ such that for any $u \in K_i(v)$ one has $|u| = |v|$. In fact $K_i(u) = K_i(v)$ if $u, v \in U_i$ and $|u| = |v|$. Thus there exist elements $u_1^i, \ldots, u_n^i, \ldots$ in U_i such that

$$U_i = \bigcup_j K_i(u_j^i).$$

By Zorn's Lemma we can assume $K_i(u_j^i)$, $j \in \mathbb{N}$ to be well-ordered under some order $>_j$. Let $u, v \in U_i$, then

$$u > v \text{ iff } \begin{cases} |u| > |v| \\ \text{or} \\ |u| = |v| \text{ but } u >_j v, \text{ where } u, v \in K_i(u_j^i) \text{ for some } u_j^i \end{cases}$$

Observe that U_i is well-ordered under $>$ and this order induces an order on

$$U = \bigcup_i U_i$$

- if $u, v \in U_i$ then we compare them using the order defined on U_i and if $u \in U_i, v \in U_j$ then $u < v$ if and only if $i < j$.

Further we will need the definition of abelian height for elements from $\mathbb{Z}[t]^+$. Recall that $\mathbb{Z}[t]^+$ is a countable direct sum

$$\mathbb{Z}[t]^+ = \oplus_{i=0}^{\infty}\langle t^i\rangle$$

of copies of the infinite cyclic group $\mathbb{Z}$ with the right lexicographic order. So for every element α from $\mathbb{Z}[t]^+$ there exists a natural number n such that α belongs to

$$\mathbb{Z}^n = \oplus_{i=0}^{n}\langle t^i\rangle.$$

Thus, we say that an *abelian height* $h_{ab}(\alpha) = n$ if $\alpha \in \mathbb{Z}^n \setminus \mathbb{Z}^{n-1}$.

Compare the definition of abelian height with the definition of a *height* given in Section 10. Observe that if $w = u^{\alpha} \in CR(\mathbb{Z}[t]^+, X)$ then $h(w) = h(u)h_{ab}(\alpha)$.

Finally, for the text below we make the following important assumption (see Subsection 15.2 for all the definitions used). Let $v \in U_{i+1}$ have a u-reduced form $h_1 \circ u^{\beta_1} \circ h_2 \circ \cdots \circ u^{\beta_p} \circ h_{p+1}$, where $u \in U_i$. Then we assume this u-form to be cyclically u-reduced. This is possible because elements in any set U_i can be chosen up to cyclic permutations and Lemma 28 below ensures the existence of an appropriate one.

15.2 Reduced forms

As usual, we call elements from $\mathbb{Z}[t]^+ \backslash \mathbb{Z}$, nonstandard (infinite) numbers. Also the following notation will be used - if $\alpha \in \mathbb{Z}[t]^+, \alpha > 0$ is nonstandard then we write $\alpha >> 1$.

Recall that the group $F^{\mathbb{Z}[t]}$ is a union of the infinite chain (7). In fact, to obtain G_{i+1} from G_i we extend centralizers of elements from some special (infinite) set $U_i = \{u_{i_1}, u_{i_2}, \ldots\} \subset G_i$. Let I_i denote the set of indices $i_1, i_2, \ldots$ of elements from U_i. Thus, any $g \in G_{i+1}, g \notin G_i$ has the following representation as a reduced infinite word:

$$g = g_1 \circ u_{n_1}^{\alpha_1} \circ g_2 \circ \cdots \circ u_{n_l}^{\alpha_l} \circ g_{l+1}. \tag{8}$$

where $n_1, n_2, \ldots, n_l \in I_i, g_k \in G_i, [g_k, u_{n_k}] \neq 1, [g_{k+1}, u_{n_k}] \neq 1, |\alpha_k| >> 1$. Observe that this representation is not unique because it is possible that for each $u_{n_k}^{\alpha_k}$, g_k has $u_{n_k}^{p_k}$ as a terminal segment and g_{k+1} has $u_{n_k}^{m_k}$ as an initial segment, so we can adjoin these finite exponents of u_{n_k} from the left and from the right to $u_{n_k}^{\alpha_k}$ to form a new infinite word representation of the same element g:

$$g = h_1 \circ u_{n_1}^{\beta_1} \circ h_2 \circ \cdots \circ u_{n_l}^{\beta_l} \circ h_{l+1}.$$

where $\beta_k = \alpha_k + p_k + m_k, g_1 = h_1 \circ u_{n_1}^{p_1}, g_{l+1} = u_{n_l}^{m_l} \circ h_{l+1}, g_k = u_{n_{k-1}}^{m_{k-1}} \circ h_k \circ u_{n_k}^{p_k}, 1 < k < l+1$.

Remark 4 *It is not hard to see that there are infinitely many representations of g in the form (8) - it is possible not only to perform the transformation described above, that is, to adjoin finite exponents to each $u_{n_k}^{\alpha_k}$ from left and right, but also to perform inverse transformation, that is, to take finite exponents from each $u_{n_k}^{\alpha_k}$ in order to obtain infinitely many tuples of interleaving elements $f_1, \ldots, f_{l+1}$.*

Lemma 26 *Let $g \in G_{i+1} \backslash G_i$ have two representations*

$$g = g_1 \circ u_{s_1}^{\alpha_1} \circ g_2 \circ \cdots \circ u_{s_l}^{\alpha_l} \circ g_{l+1}.$$

where $s_j \in I_i, g_j \in G_i, [g_j, u_{s_j}] \neq 1, [g_{j+1}, u_{s_j}] \neq 1, |\alpha_j| >> 1, j \in [1, l]$ and

$$g = h_1 \circ u_{r_1}^{\beta_1} \circ h_2 \circ \cdots \circ u_{r_l}^{\beta_l} \circ h_{m+1}.$$

where $t_k \in I_i, h_k \in G_i, [h_k, u_{r_k}] \neq 1, [h_{k+1}, u_{r_k}] \neq 1, |\beta_k| >> 1, k \in [1, m]$.

Then

1) $l = m$;

2) $u_{r_j} = u_{s_j}, j \in [1, l]$;

Proof. We have

$$(g_{l+1}^{-1} \circ u_{s_l}^{-\alpha_l} \circ \cdots \circ u_{s_1}^{-\alpha_1} \circ g_1^{-1}) * (h_1 \circ u_{r_1}^{\beta_1} \circ h_2 \circ \cdots \circ u_{r_l}^{\beta_l} \circ h_{m+1}) = \varepsilon.$$

Since $F^{\mathbb{Z}[t]}$ satisfies the stabilizing condition it follows that the equality above can hold only if $u_{r_1} = u_{s_1}$ and $g_1^{-1} * h_1 = u_{s_1}^{k_1}$. Thus, we have

$$(g_{l+1}^{-1} \circ u_{s_l}^{-\alpha_l} \circ \cdots \circ g_2^{-1}) * u_{s_1}^{p_1} * (h_2 \circ \cdots \circ u_{r_l}^{\beta_l} \circ h_{m+1}) = \varepsilon.$$

where p_1 is finite. Again, by the stabilizing condition $u_{r_2} = u_{s_2}$ and $g_2^{-1} * u_{s_1}^{p_1} * h_2 = u_{s_2}^{k_2}$ and we can cancel $(u_{s_2}^{-\alpha_2} \circ g_2^{-1}) * u_{s_1}^{p_1} * (h_2 \circ u_{r_2}^{\beta_2})$ into a finite exponent $u_{s_2}^{p_2}$.

The required result follows by induction.

□

It follows from the lemma above that in any representation of g as an infinite word the number of infinite nonstandard exponents $u_{n_k}^{\alpha_k}$ in g (the number of syllables in some sense) is the same, which is equal to l. This observation makes it possible to introduce a natural characterization of any representation of g as an l-tuple $\{|\gamma_1|, |\gamma_2|, \ldots, |\gamma_l|\}$, where $|\gamma_k| >> 1$ and

$$g = f_1 \circ u_{n_1}^{\gamma_1} \circ f_2 \circ \cdots \circ u_{n_l}^{\gamma_l} \circ f_{l+1} \tag{9}$$

We call such a representation of g, U_t*-reduced* if the ordered l-tuple $\{|\gamma_1|, |\gamma_2|, \ldots, |\gamma_l|\}$ is maximal with respect to the right lexicographic order among all possible representations of g.

In the proof of Theorem 27 we introduced unique reduced forms for elements from $F^{\mathbb{Z}[t]}$. Now, we redefine the procedure for obtaining them and show that in fact unique reduced forms defined in the proof of Theorem 27 coincide with U_i-reduced representations (in the future we will refer to a U_i-reduced representation as a *U_i-form*).

Suppose $g \in G_{i+1} \backslash G_i$ and

$$g = g_1 \circ u_{n_1}^{\alpha_1} \circ g_2 \circ \cdots \circ u_{n_l}^{\alpha_l} \circ g_{l+1}.$$

From the stabilizing condition (S) it follows that for g_1 and g_2 there exist natural numbers p_1 and m_1 correspondingly such that $g_1 = h_1 \circ u_{n_1}^{p_1}, g_2 \circ u_{n_2}^{\alpha_2} = u_{n_1}^{m_1} \circ g'_2 \circ u_{n_2}^{\gamma_2}$ and h_1 does not have $u_{n_1}^{\pm 1}$ as a terminal segment, $g'_2 \circ u_{n_2}^{\gamma_2}$ does not have $u_{n_1}^{\pm 1}$ as an initial segment. Now we present g as

$$g = h_1 \circ u_{n_1}^{\beta_1} \circ g'_2 \circ u_{n_2}^{\gamma_2} \circ \cdots \circ u_{n_l}^{\alpha_l} \circ g_{l+1}.$$

where $\beta_k = \alpha_k + p_k + m_k, g_1 = h_1 \circ u_{n_1}^{p_1}, g_2 \circ u_{n_2}^{\alpha_2} = u_{n_1}^{m_1} \circ g'_2 \circ u_{n_2}^{\gamma_2}$. Next we take the subword of g

$$g' = g'_2 \circ u_{n_2}^{\gamma_2} \circ g_3 \circ \cdots \circ u_{n_l}^{\alpha_l} \circ g_{l+1}$$

and perform exactly the same procedure of maximizing γ_2. The whole construction follows by induction on l. After a finite number of steps we get

$$g = h_1 \circ u_{n_1}^{\beta_1} \circ h_2 \circ \cdots \circ u_{n_l}^{\beta_l} \circ h_{l+1}. \tag{10}$$

Lemma 27 *1) (10) is unique;*

2) (10) is a U_i-reduced form for g.

Proof. 1) Follows immediately from the construction.

2) Suppose (10) is not a U_i-reduced form for g, that is there exists an l-tuple

$\{|\gamma_1|, |\gamma_2|, \ldots, |\gamma_l|\}$ such that $\{|\gamma_1|, |\gamma_2|, \ldots, |\gamma_l|\} > \{|\beta_1|, |\beta_2|, \ldots, |\beta_l|\}$ and

$$g = f_1 \circ u_{n_1}^{\gamma_1} \circ f_2 \circ \cdots \circ u_{n_l}^{\gamma_l} \circ f_{l+1}.$$

From the inequality above it follows that there exists $k_0 \in [1, l]$ such that $\gamma_k = \beta_k, 1 \leq k < k_0$ and $\gamma_{k_0} > \beta_{k_0}$.

At first we show that $h_k = f_k, k \in [1, k_0 - 1]$. Assume that $k_0 > 1$.

Observe that $|h_1| \leq |f_1|$ otherwise $h_1 = f_1 \circ u_{n_1}^m, |m| > 0, m \in \mathbb{Z}$ and β_1 is not maximal - contradiction (here we use the stabilizing condition by which $f_1^{-1} * h_1 \in \langle u_{n_1} \rangle$). If $|h_1| < |f_1|$ then since $\beta_1 = \gamma_1$ it follows that $h_2 \circ u_{n_2}^{\beta_2}$ contains u_{n_1} as an initial segment - contradiction with the choice of β_1. Thus $|h_1| = |f_1|$ and $h_1 = f_1$. In the same way, using induction one can prove that $h_k = f_k, k \in [2, k_0 - 1]$.

The simple fact proved above shows that we can assume without loss of generality that $k_0 = 1$, that is, $\gamma_1 > \beta_1 > 0$. Again $|h_1| \leq |f_1|$ otherwise we get a contradiction. As above if $|h_1| < |f_1|$ then since $\beta_1 < \gamma_1$ it follows that $h_2 \circ u_{n_2}^{\beta_2}$ contains at least u_{n_1} as an initial segment - contradiction with the choice of β_1.

Thus we proved that $\{|\beta_1|, |\beta_2|, \ldots, |\beta_l|\}$ is the maximal l-tuple with respect to the right lexicographical order. So the representation (10) is U_i-reduced for g.

□

Let $g \in G_{i+1} \backslash G_i$, so it has a representation as an infinite word

$$g = g_1 \circ u_{n_1}^{\alpha_1} \circ g_2 \circ \cdots \circ u_{n_l}^{\alpha_l} \circ g_{l+1}.$$

where $n_1, n_2, \ldots, n_l \in I_i, g_k \in G_i, [g_k, u_{n_k}] \neq \varepsilon, [g_{k+1}, u_{n_k}] \neq \varepsilon, |\alpha_k| >> 1$. Now we fix some $u_{n_{k_0}}$ from the list $u_{n_1}, u_{n_2}, \ldots, u_{n_l}$ of elements from U_i taken to nonstandard powers in the representation of g (it follows from Lemma 26 that this list does not depend on particular representation of g as a reduced word). Consider now a representation of g in which we "mark" only nonstandard exponents of $u_{n_{k_0}}$, that is

$$g = h_1 \circ u_{n_{k_0}}^{\beta_1} \circ h_2 \circ \cdots \circ u_{n_{k_0}}^{\beta_p} \circ h_{p+1}, \tag{11}$$

where $\beta_j = \alpha_{m_j}, m_j \in [1, l], j \in [1, p], h_1 = g_1 \circ u_{n_1}^{\alpha_1} \circ \cdots \circ g_{m_1}, h_{p+1} = g_{m_p+1} \circ \cdots \circ g_{l+1}, h_k = g_{m_k+1} \circ \cdots \circ g_{m_{k+1}}, 1 < k \leq p$. Observe that all h_k, in general, do not belong to G_i any more. We call (11) a $u_{n_{k_0}}$*-representation* or $u_{n_{k_0}}$*-form* for g.

One can prove a statement analogous to Lemma 26 for $u_{n_{k_0}}$-forms of g and this means that we can associate with any such form a p-tuple $\{|\beta_1|, |\beta_2|, \ldots, |\beta_p|\}$ of elements from $\mathbb{Z}[t]^+$, which are exponents from (11).

We call a $u_{n_{k_0}}$-form of g, $u_{n_{k_0}}$*-reduced* if the ordered p-tuple $\{|\beta_1|, |\beta_2|, \ldots, |\beta_p|\}$ is maximal with respect to the right lexicographic order among all possible $u_{n_{k_0}}$-representations for g.

Suppose

$$h_1 \circ u_{n_{k_0}}^{\beta_1} \circ h_2 \circ \cdots \circ u_{n_{k_0}}^{\beta_p} \circ h_{p+1} \tag{12}$$

is a $u_{n_{k_0}}$-form for g and g is cyclically reduced. Then obviously

$$(h_1 \circ u_{n_{k_0}}^{\beta_1} \circ h_2 \circ \cdots \circ u_{n_{k_0}}^{\beta_p} \circ h_{p+1}) \circ (h_1 \circ u_{n_{k_0}}^{\beta_1} \circ h_2 \circ \cdots \circ u_{n_{k_0}}^{\beta_p} \circ h_{p+1}) \tag{13}$$

is a $u_{n_{k_0}}$-form of g^2. So, we call (12) *cyclically* $u_{n_{k_0}}$*-reduced* if (13) is $u_{n_{k_0}}$-reduced.

Lemma 28 *Let*

$$h_1 \circ u^{\beta_1} \circ h_2 \circ \cdots \circ u^{\beta_p} \circ h_{p+1}$$

be a u-reduced form of $g \in G_{i+1} \backslash G_i$, $u \in U_i$. Then there exists a cyclic permutation of g such that its u-reduced form is cyclically u-reduced.

Proof. The proof is based on the following observation.

Claim. If $|w| < |u|$ then in $u^\alpha \circ w \circ u^\beta, \alpha, \beta > 0, \alpha, \beta \in \mathbb{Z}$ then if $w \circ u^\beta$ has u

as an initial segment then $u^\alpha \circ w$ can not have u as a terminal segment.

Suppose on the contrary that $u = w \circ u_1, u = u_1 \circ u_2$ and $u = u_3 \circ w, u = u_4 \circ u_3$ at the same time. From these equalities we have that $|w| = |u_2|, |u_1| = |u_3|, |u_4| = |w|$. Since $u = w \circ u_1, u = u_4 \circ u_3$ and $|u_4| = |w|$ it follows that $w = u_4, u_1 = u_3$. Thus $w = u_2$ and $u = u_1 \circ u_2 = u_2 \circ u_1$ but this is possible only if $u_1 = a^\delta$ and $u_2 = a^\gamma$ and so $u = a^{\delta+\gamma}$ so the centralizer of u in G_i is not cyclic - contradiction with the choice of U_i.

Observe also that if $\alpha > 0, \beta < 0$ then neither $w \circ u^\beta$ has u as an initial segment nor $u^\alpha \circ w$ has u^{-1} as a terminal segment. Indeed, if for example $w \circ u^\beta$ has u as an initial segment then $u = w \circ u_1, u^{-1} = u_1 \circ u_2$. Thus, $u = w \circ u_1 = u_2^{-1} \circ u_1^{-1}$ and $u_1 = u_1^{-1} = \varepsilon, u = w$ - contradiction.

Now we complete the proof of the lemma. Without loss of generality we can assume $h_1 = 1$ and that h_{p+1} does not have u as a terminal segment (using cyclic permutation we can always obtain these properties). We have two cases

1. $|h_{p+1}| \geq |u|$

Since $u^{\beta_1} \circ h_2 \circ \cdots \circ u^{\beta_p} \circ h_{p+1}$ is u-reduced it follows that $u^{\beta_1} \circ h_2 \circ \cdots \circ u^{\beta_p} \circ h_{p+1} \circ u^{\beta_1} \circ h_2 \circ \cdots \circ u^{\beta_p} \circ h_{p+1}$ is u-reduced because h_{p+1} does not have u as an initial or terminal segment.

2. $|h_{p+1}| < |u|$

a) If $u^{\beta_p} \circ h_{p+1}$ has u as a terminal segment then by the Claim above $h_{p+1} \circ u^{\beta_1}$ does not have u as an initial segment. Since $u^{\beta_1} \circ h_2 \circ \cdots \circ u^{\beta_p} \circ h_{p+1}$ is u-reduced then the ordered $2p$-tuple $\{|\beta_1|, |\beta_2|, \ldots, |\beta_p|, |\beta_1|, |\beta_2|, \ldots, |\beta_p|\}$ is maximal with respect to the right lexicographic order among all possible u-forms of g^2. So, $u^{\beta_1} \circ h_2 \circ \cdots \circ u^{\beta_p} \circ h_{p+1}$ is cyclically u-reduced.

b) If $h_{p+1} \circ u^{\beta_1}$ has u as an initial segment then by the Claim $u^{\beta_p} \circ h_{p+1}$ does not have u as a terminal segment. Thus, take a cyclic permutation $g' = h_2 \circ u^{\beta_2} \cdots \circ u^{\beta_p} \circ h_{p+1} \circ u^{\beta_1}$ of g so that $h_2 \circ u^{\beta_2} \cdots \circ u^{\beta_p - 1} \circ b \circ u^{\beta_1 - 1}$ is a u-reduced form of g', where

$u = h_{p-1} \circ a = a \circ b$. But then $(h_2 \circ u^{\beta_2} \cdots \circ u^{\beta_p + 1} \circ b \circ u^{\beta_1 - 1}) \circ (h_2 \circ u^{\beta_2} \cdots \circ u^{\beta_p + 1} \circ b \circ u^{\beta_1 - 1})$ is u-reduced.

□

15.3 Standard decomposition of elements in $F^{\mathbb{Z}[t]}$

Let $g \in F^{\mathbb{Z}[t]}$. It follows that $g \in G_{n+1}$ for some finite n in the series (7). Then g has a U_n-reduced form

$$g = g_1 \circ u_{m(n)_1}^{\alpha_1} \circ g_2 \circ \cdots \circ u_{m(n)_l}^{\alpha_l} \circ g_{l+1},$$

where $m(n)_1, m(n)_2, \ldots, m(n)_l \in I_n$, $g_k \in G_n$, $[g_k, u_{m(n)_k}] \neq \varepsilon$, $[g_{k+1}, u_{m(n)_k}] \neq \varepsilon$, $|\alpha_k| >> 1$, that is g belongs to the extension of centralizers of finitely many elements $u_{m(n)_1}, \ldots, u_{m(n)_l}$ from U_n. Now, by induction one has U_{n-1}-reduced form for g_i

$$g_i = g_{i,1} \circ u_{m(n-1)_{i,1}}^{\beta_{i,1}} \circ g_{i,2} \circ \cdots \circ u_{m(n-1)_{i,l_i}}^{\beta_{i,l_i}} \circ g_{i,l_i+1},$$

where $m(n-1)_{i,k} \in I_{n-1}$, $k \in [1, l_i]$, $i \in [1, l_i + 1]$, $|\beta_{i,k}| >> 1$, $g_{i,k} \in G_{n-1}$. One can get down to a free group F with such a decomposition of g, where step by step subwords between nonstandard powers of elements from U_k are presented as U_{k-1}-forms, $k \in [1, n]$.

Observe that, since on each step of this decomposition of g, we have infinite exponents of only finitely many elements we can form finitely many ordered sets $U_j(g) = \{u_{m(j)_1}, \ldots, u_{m(j)_{k(j)}}\} \subset U_j$, $j \in [1, n]$ (order on $U_j(g)$ is induced from U_j) which are unique for g.

Thus one can form for g the following series:

$$F < H_{m(0)_1} < H_{m(0)_2} < \ldots < H_{m(0)_{k(0)}} < H_{m(1)_1} < \ldots < H_{m(1)_{k(1)}} < \ldots \quad (14)$$

$$\ldots < H_{m(n-1)_{k(n-1)}} < H_{m(n)_1} < \ldots < H_{m(n)_{k(n)}},$$

where $H_{m(j)_1}, \ldots, H_{m(j)_{k(j)}}$ are subgroups of G_{j+1} which do not belong to G_j and $H_{m(j)_i}$ is obtained from $H_{m(j)_{i-1}}$ by centralizer extension of a single element $u_{m(j)_i}$ from $H_{m(j)_{i-1}} < G_j$. We call series (14) an *extension series* for g. (14) obviously exists and is unique since all $U_j(g), j \in [1, n]$ are fixed and ordered.

Using extension series above we can decompose g in the following way. $g \in H_{m(n)_{k(n)}}$ has a $u_{m(n)_{k(n)}}$-reduced form

$$h_1 \circ u_{m(n)_{k(n)}}^{\beta_1} \circ h_2 \circ \cdots \circ u_{m(n)_{k(n)}}^{\beta_l} \circ h_{l+1},$$

where all h_j are $u_{m(n)_{k(n)-1}}$-reduced forms representing elements from $H_{m(n)_{k(n)-1}}$. This gives one a decomposition of g related to its extension series. We call this decomposition a *standard decomposition* or *standard representation* of g. Observe that for any $g \in F^{\mathbb{Z}[t]}$, its standard decomposition can be viewed as a finite product $b_1 b_2 \cdots b_m$, where $b_i \in B = \{X \cup X^{-1}\} \cup \{u^\alpha \mid u \in U, \alpha \in \mathbb{Z}[t]^+\}$. We denote this product by $\tilde{g}$ so we have

$$\tilde{g} = \widetilde{h_1} u_{m(n)_{k(n)}}^{\beta_1} \widetilde{h_2} \cdots u_{m(n)_{k(n)}}^{\beta_l} \widetilde{h_{l+1}},$$

where $\widetilde{h_i}$ is a finite product in the alphabet B corresponding to h_i, and from now on, by standard decomposition of an element g we understand not the representation of g as a reduced infinite word but the finite product $\tilde{g}$.

It is easy to notice that in general $\widetilde{g_1 \circ g_2} \neq \tilde{g}_1 \tilde{g}_2$.

The standard decomposition for given g allows one to use an induction on the number of members in the series (14).

16 Graphs labeled by infinite $\mathbb{Z}[t]$-words

In this section we give the definitions of $(\mathbb{Z}[t], X)$-graph, its special components and introduce operations on such graphs which make it possible to transform them into some suitable form, which has properties, important for the proof of Proposition 4.

16.1 Labeled graphs

In this subsection we introduce definitions, analogous to those given in [24] and cited in Section 7. Of course, we are not working in a free group any more, so we have to adjust these definitions to our case. We use the notation and definitions given in the previous section.

Definition 25 *By an $(\mathbb{Z}[t], X)$-labeled directed graph ($(\mathbb{Z}[t], X)$-graph) Γ we mean the following:*

1) Γ is a combinatorial graph where every edge has a direction and is labeled either by a letter from X or by an infinite word $u^\alpha \in F^{\mathbb{Z}[t]}$, $u \in U$, $\alpha \in \mathbb{Z}[t]^+$, $\alpha > 0$, denoted $\mu(e)$.

2) for each edge e of Γ we denote the origin of e by $o(e)$ and the terminus of e by $t(e)$.

For each edge e of $(\mathbb{Z}[t], X)$-graph we introduce a formal inverse e^{-1} of e with label $\mu(e)^{-1}$ and the endpoints defined as $o(e^{-1}) = t(e), t(e^{-1}) = o(e)$ that is the direction of e^{-1} is reversed with respect to the direction of e. For the new edges e^{-1} we set $(e^{-1})^{-1} = e$. The new graph, endowed with this additional structure we denote by $\hat{\Gamma}$. In fact in many instances we will abuse notation by disregarding the difference between Γ and $\hat{\Gamma}$.

Now we have a partition $E(\hat{\Gamma}) = E(\Gamma) \cup \overline{E(\Gamma)}$ and we say that edges of Γ are *positively oriented* in $\hat{\Gamma}$ while their formal inverses e^{-1} are *negatively oriented* in $\hat{\Gamma}$.

Definition 26 *A path p in Γ is a sequence of edges $p = e_1 \cdots e_k$, where each e_i is an edge of $\widehat{\Gamma}$ and the origin of each e_i is the terminus of e_{i-1}.*

Observe that $\mu(p) = \mu(e_1) \dots \mu(e_k)$ is a word in the alphabet $\{X \cup X^{-1}\} \cup \{u^\alpha \mid u \in U, \alpha \in \mathbb{Z}[t]^+\}$ and we denote by $\overline{\mu(p)}$ a reduced infinite word $\mu(e_1) * \cdots * \mu(e_k)$.

We will be using two different notions of the length of a path $p = e_1 \cdots e_k$ in Γ:

1) *combinatorial length* $|p|$ set equal to k

and

2) *word length* $wl(p) = \sum_{i=1}^{k} l(\mu(e_i))$.

In fact, from these two definitions above arise two possible meanings of irreducible path: the first one - path is irreducible in the combinatorial sense and another one - path is irreducible if its label viewed as an infinite word in $F^{\mathbb{Z}[t]}$ is reduced. Here are formal definitions.

Definition 27 *(Reduced path). A path $p = e_1 \cdots e_k$ in a $(\mathbb{Z}[t], X)$-graph Γ is called reduced if $e_i \neq e_{i+1}^{-1}$ for all $1 \leq i < k$.*

Definition 28 *(Label reduced path). A path $p = e_1 \cdots e_k$ in a $(\mathbb{Z}[t], X)$-graph Γ is called* label reduced *if*

1) p is reduced;

2) if $e_{k_1} \cdots e_{k_2}$, $k_1 \leq k_2$ is a subpath of p such that $\mu(e_i) = u^{\alpha_i}, u \in U, \alpha_i \in \mathbb{Z}[t]^+$, $i \in [k_1, k_2]$ and $\mu(e_{k_1-1}) \neq u^\beta, \mu(e_{k_2+1}) \neq u^\beta$ for any $\beta \in \mathbb{Z}[t]^+$, provided $k_1 - 1, k_2 + 1 \in [1, k]$, then $\alpha = \alpha_{k_1} + \cdots + \alpha_{k_2} \neq 0$ and $\mu(e_{k_1-1}) * u^\alpha = \mu(e_{k_1-1}) \circ u^\alpha$, $u^\alpha * \mu(e_{k_2+1}) = u^\alpha \circ \mu(e_{k_2+1})$.

16.2 Partial foldings

Here we define partial foldings and partially folded $(\mathbb{Z}[t], X)$-graphs. Observe that the definition of a partial folding below is exactly the same as the corresponding definiiton of a folding in [24].

Definition 29 *Let Γ be a finite $(\mathbb{Z}[t], X)$-graph and let v be a vertex of Γ. The valence of v denoted $val(v)$ is the number of all edges in Γ which have v as origin or terminus.*

Let Γ be a $(\mathbb{Z}[t], X)$-graph. Suppose v_0 is a vertex of Γ and f_1, f_2 are two distinct edges of $\widehat{\Gamma}$ such that $o(f_1) = o(f_2) = v_0, \mu(f_1) = \mu(f_2) = x \in X^{\pm 1}$ or $\mu(f_1) = \mu(f_2) = u^{\alpha}, u \in U, \alpha \in \mathbb{Z}[t]^{+}$. Let h_i be the positive edge of Γ corresponding to f_i (that is $h_i = f_i$ if f_i is positive and $h_i = f_i^{-1}$ if f_i is negative).

Let Δ be a $(\mathbb{Z}[t], X)$-graph with the following sets of vertices and edges.

$$V(\Delta) = (V(\Gamma) - \{t(f_1), t(f_2)\}) \cup \{v\}, E(\Delta) = (E(\Gamma) - \{h_1, h_2\}) \cup \{h\}.$$

The endpoints and arrows for the edges of Δ are defined in the following way. Let $e \in E\Delta, e \neq h$ then

1. we put $o_{\Delta}(e) = o_{\Gamma}(e)$ if $o_{\Gamma}(e) \neq t(f_i)$ and $o_{\Delta}(e) = v$ if $o_{\Gamma}(e) = t(f_i)$ for some i;

2. we put $t_{\Delta}(e) = t_{\Gamma}(e)$ if $t_{\Gamma}(e) \neq t(f_i)$ and $t_{\Delta}(e) = v$ if $t_{\Gamma}(e) = t(f_i)$ for some i.

For the edge h we put $o_{\Delta}(h) = v_0, t_{\Delta}(h) = v$ if $h_1 = f_1, h_2 = f_2$ and $o_{\Delta}(h) = v, t_{\Delta}(h) = v_0$ otherwise.

We define labels on the edges of Δ as follows: $\mu_{\Delta}(e) = \mu_{\Gamma}(e)$ if $e \neq h$ and $\mu_{\Delta}(h) = \mu_{\Gamma}(h_1) = \mu_{\Gamma}(h_2)$.

In other words we obtain Δ by identification of two edges f_1 and f_2 in Γ. In this situation we say that Δ is obtained from Γ by a *partial folding* (or by *partial folding the edges f_1 and f_2*).

There can be introduced a notion of a *morphism* between two $(\mathbb{Z}[t], X)$-graphs. That is, if Γ_1, Γ_2 are $(\mathbb{Z}[t], X)$-graphs then a map $\pi : \Gamma_1 \to \Gamma_2$ is called a *morphism* of $(\mathbb{Z}[t], X)$-graphs, if π sends vertices to vertices, directed edges to directed edges, preserves labels of directed edges, and has the property that $o(\pi(e)) = \pi(o(e)), t(\pi(e)) = \pi(t(e))$ for any edge e of Γ_1.

If ϕ is a partial folding defined above then it is easy to see that ϕ is a morphism between Γ and Δ.

Lemma 29 *Let Γ_1 be a $(\mathbb{Z}[t], X)$-graph obtained by a partial folding from a graph Γ. Let v be a vertex of Γ and v_1 be the corresponding vertex of Γ_1. Then the following hold:*

1) If Γ is connected then Γ_1 is connected.

2) Let p be the path from v to v in Γ with label w. Then the edgewise image of p in Γ_1 is a path from v_1 to v_1 with label w.

3) If Γ is a finite $(\mathbb{Z}[t], X)$-graph, then the number of edges in Γ_1 is one less than the number of edges in Γ, that is, any partial folding decreases the number of edges in Γ.

Proof. Follows directly from the definition of a partial folding.

□

Definition 30 *$(\mathbb{Z}[t], X)$-graph Γ is called partially folded if there exist no two edges e_1 and e_2 in Γ with $\mu(e_1) = \mu(e_2)$ such that $o(e_1) = o(e_2)$ or $t(e_1) = t(e_2)$.*

Obviously, Γ is a partially folded $(\mathbb{Z}[t], X)$-graph if and only if one can not perform any partial folding in Γ. Moreover the following proposition is true.

Proposition 3 *Let Γ be a $(\mathbb{Z}[t], X)$-graph, which has only a finite number of edges. Then there exists a partially folded $(\mathbb{Z}[t], X)$-graph Δ, which can be obtained from Γ by a finite number of partial foldings.*

Proof. Since Γ has a finite number of edges by Lemma 29 any $(\mathbb{Z}[t], X)$-graph Γ_1 obtained from Γ by a partial folding has fewer edges. This provides one with an inductive argument based on the number of edges in Γ.

□

16.3 u-components

In the present subsection we concentrate on some particular subgraphs of $(\mathbb{Z}[t], X)$-graphs which consist of edges labeled by exponents of elements from U and are very important in all further investigations. Let $u \in U$ be fixed.

Definition 31 *Let Γ be a $(\mathbb{Z}[t], X)$-graph. $v_1, v_2 \in V(\Gamma)$ are called u-equivalent and denoted $v_1 \sim_u v_2$ if there exists a path $p = e_1 \cdots e_k$ in Γ such that $o(e_1) = v_1, t(e_k) = v_2$ and $\mu(e_i) = u_i^{\alpha}, \alpha_i \in \mathbb{Z}[t]^+, 1 \leq i \leq k$.*

$\sim_u$ is clearly an equivalence relation on vertices of Γ, so if Γ is finite then all its vertices can be divided into a finite number of pairwise disjoint equivalence classes. Suppose, $v \in V(\Gamma)$ is fixed. One can take the subgraph of Γ spanned by vertices u-equivalent to v and remove from it all edges with any label except for $u^{\alpha}, \alpha \in \mathbb{Z}[t]^+$. The resulting subgraph of Γ we denote by $Comp_u(v)$ and call *u-component of v*. In other words, the u-component of a vertex v is subgraph of Γ all edges of which are labeled by exponents of u.

Definition 32 *Let Γ be a $(\mathbb{Z}[t], X)$-graph and $v \in V(\Gamma), v_0 \in V(Comp_u(v))$. We define a set $H_u(v_0)$ associated with v_0 as*

$$H_u(v_0) = \{\overline{\mu(p)} \mid p \text{ is a reduced path in } Comp_u(v) \text{ from } v_0 \text{ to } v_0\}.$$

Observe that even when p is a reduced path in $Comp_u(v)$ its label $\mu(p) = u^{\alpha}$ can define an empty infinite word, that is $\overline{\mu(p)} = \varepsilon$.

Lemma 30 *Let Γ be a $(\mathbb{Z}[t], X)$-graph and $v \in V(\Gamma), v_0 \in V(Comp_u(v))$. Then*

1. *$H_u(v_0)$ is isomorphic to a subgroup of $\mathbb{Z}[t]^+$ and moreover, if $Comp_u(v)$ is a finite graph then $H_u(v_0)$ is finitely generated;*
2. *if $v_1 \in V(Comp_u(v))$ then $H_u(v_0) \simeq H_u(v_1)$.*

Proof. 1. Observe that if p is a cycle in $Comp_u(v)$ at v_0 then $\overline{\mu(p)} = u^{\alpha}, \alpha \in \mathbb{Z}[t]^+$. The concatenation p_1p_2 of two cycles in $Comp_u(v)$ at v_0 is again a cycle in $Comp_u(v)$ at v_0 which may or may be not reduced. Let p be the reduced cycle obtained from p_1p_2 by making all possible path reductions. Then $\overline{\mu(p)} = \overline{\mu(p_1)} * \overline{\mu(p_2)} = \overline{\mu(p_2)} * \overline{\mu(p_1)} \in H_u(v_0)$ and $H_u(v_0)$ is closed under multiplication which is commutative.

Finally, since the inverse path $(p_1)^{-1}$ of p_1 is reduced and has label $\mu(p_1)^{-1}$ it follows that $H_u(v_0)$ is closed under taking inverses. Also, since ε is a label of an empty path which is reduced then clearly $\varepsilon \in H_u(v_0)$.

Thus $H_u(v_0)$ is an abelian group.

One can construct a map $\theta : H_u(v_0) \to \mathbb{Z}[t]^+$, where $\overline{\mu(p)} = u^{\alpha} \xrightarrow{\theta} \alpha \in \mathbb{Z}[t]^+$. Obviously, θ is an isomorphism of $H_u(v_0)$ and a subgroup of $\mathbb{Z}[t]^+$. Moreover, if $Comp_u(v)$ is a finite graph then there exists a natural number n such that $n \geq h_{ab}(\alpha)$, where $\mu(e) = u^{\alpha}$ and e ranges through all edges of $Comp_u(v)$. Since for any reduced cycle $p = e_1 \cdots e_k$ in $Comp_u(v)$ such that $\mu(e_i) = u^{\alpha_i}, i \in [1, k]$ and $\overline{\mu(p)} = u^{\alpha_1 + \cdots + \alpha_k} = u^{\alpha}$ one has $h_{ab}(\alpha) \leq \max_{i=1}^{k}\{h_{ab}(\alpha_i)\}$ it follows that $H_u(v_0)$ is a subgroup of $\mathbb{Z}^n$ which is a subgroup of $\mathbb{Z}[t]^+$ of finite rank. Thus $H_u(v_0)$ is a free abelian group of rank not greater than n.

2. Since $v_0 \sim_u v_1$ there exists a path $p = f_1 \cdots f_k$ in $Comp_u(v)$ such that $o(p) = v_0, t(p) = v_1$ and $\overline{\mu(p)} = u^{\gamma}, \gamma \in \mathbb{Z}[t]^+$. The existence of p provides one with correspondence between cycles at v_0 and v_1 - if $p_0 = e_1 \cdots e_m$ is a reduced cycle at v_1 then $p_1 = f_1 \cdots f_k e_1 \cdots e_m (f_k)^{-1} \cdots (f_1)^{-1}$ is a cycle at v_0 and one can obtain a reduced cycle p_2 at v_0 by making all possible path reductions in p_1. Observe that $\overline{\mu(p_2)} = \overline{\mu(p_1)} = \overline{\mu(p)} * \overline{\mu(p_0)} * (\overline{\mu(p)})^{-1} = \overline{\mu(p_0)}$. Thus $H_u(v_1) \leq H_u(v_0)$. In the same way for any cycle at v_0 one can construct a corresponding cycle at v_1 with the same label. So $H_u(v_1) \geq H_u(v_0)$ and finally $H_u(v_1) \simeq H_u(v_0)$ as abelian groups.

□

It follows from Lemma 30 that one can associate a free abelian group of finite rank with any finite u-component in a $(\mathbb{Z}[t], X)$-graph Γ.

16.4 u-folded u-components

In Subsection 16.2 we introduced a notion of partial foldings, but it turns out that for our further investigations it is not enough to have only partially folded graphs. In the present subsection we give a definition of U-foldings, which are operations on partially folded $(\mathbb{Z}[t], X)$-graphs. Then we show how using these operations one can transform u-components ($u \in U$) into a particular form which satisfies an important conditions given in Lemmas 34 and 35.

Let Γ be a $(\mathbb{Z}[t], X)$-graph and let $u \in U$ be fixed throughout this subsection. Let $v \in V(\Gamma)$, $v_0 \in V(Comp_u(v))$.

Partial folding defined in the previous subsection is a simple operation on edges which have the same labels, but it does not cover the case when we have two edges $f_1, f_2 \in E(Comp_u(v))$ such that $o(f_1) = o(f_2) = v_0$, $\mu(f_1) = u^\alpha$, $\mu(f_2) = u^\beta$, $u \in U$, $\alpha, \beta \in \mathbb{Z}[t]^+$, $\alpha \neq \beta$. Since $\alpha \neq \beta$ no partial folding can be applied.

Definition 33 *(u-folding). Let $f_1, f_2 \in E(Comp_u(v))$ be such that $o(f_1) = o(f_2) = v_0$ such that $\mu(f_1) = u^\alpha$, $\mu(f_2) = u^\beta$, $\alpha, \beta \in \mathbb{Z}[t]^+$ and suppose $|\alpha| \geq |\beta|$. Without loss of generality we can assume both edges to be positively oriented, that is, $\alpha, \beta > 0$ (otherwise we can consider f_i^{-1}, $i = 1, 2$ instead of f_i).*

Let Γ_1 be a $(\mathbb{Z}[t], X)$-graph defined as follows.

$$V(\Gamma_1) = V(\Gamma) \cup \{v_1\},\ E(\Gamma_1) = (E(\Gamma) - \{f_1\}) \cup \{e_1, e_2\}.$$

We think of Γ_1 as a new $(\mathbb{Z}[t], X)$-graph obtained from Γ by dividing the edge f_1 into two edges e_1 and e_2. The endpoints and arrows for the edges of Γ_1 are defined in the following way. Let $e \in E(\Gamma_1)$, $e \neq e_1, e_2$ then we put $o_{\Gamma_1}(e) = o_\Gamma(e)$ and

$t_{\Gamma_1}(e) = t_\Gamma(e)$

For the edges e_1, e_2 *we put* $o_{\Gamma_1}(e_1) = o_\Gamma(f_1), t_{\Gamma_1}(e_1) = v_1, o_{\Gamma_1}(e_2) = v_1, t_{\Gamma_1}(e_2) = t_\Gamma(f_1)$.

Finally, $\mu_{\Gamma_1}(e) = \mu_\Gamma(e)$ *if* $e \neq e_1, e_2$ *and* $\mu_{\Gamma_1}(e_1) = \mu_\Gamma(f_2), \mu_{\Gamma_1}(e_2) = u^{\alpha-\beta}$.

Thus, in Γ_1 *we have a pair of edges* e_1, f_2 *with origin* v_0 *and the same label* u^β, *so we can apply a partial folding* ψ *to* Γ_1. *After the identification of* e_1 *and* f_2 *the resulting* $(\mathbb{Z}[t], X)$*-graph we denote by* Δ.

In this situation we say that Δ *is obtained from* Γ *by a* u-folding *(or by* u-folding the edges f_1 and f_2*) (see Figure 3).*

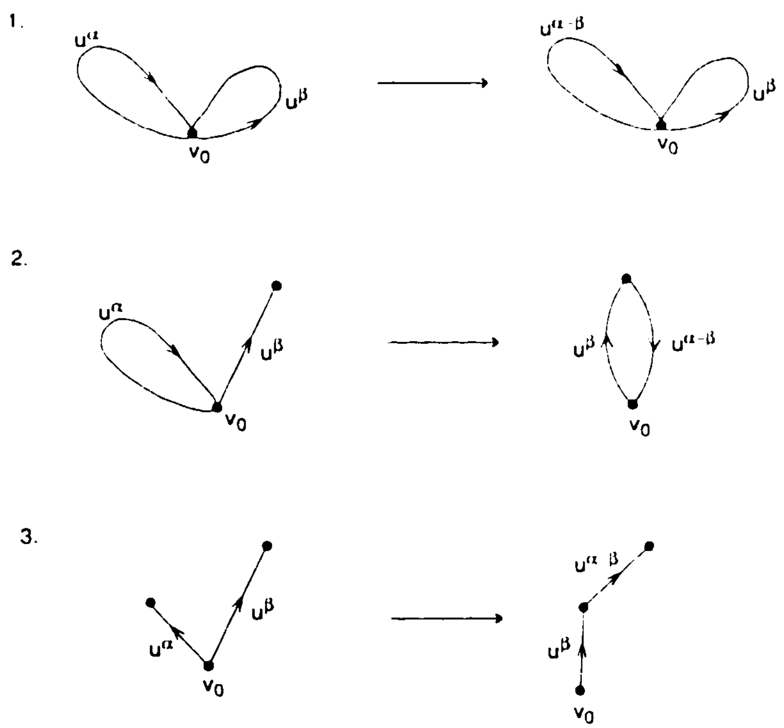


Figure 3: Possible u-foldings.

Observe that if $\alpha = \beta$ then u-folding defined above is just a partial folding.

From now on by *U-foldings* we denote the set of all u-foldings, $u \in U$.

Unlike partial foldings, u-foldings do not define morphisms of $(\mathbb{Z}[t], X)$-graphs because they involve the operation of division of an edge. Hence, we introduce a notion of *generalized morphism* between two $(\mathbb{Z}[t], X)$-graphs. That is, if Γ_1, Γ_2 are $(\mathbb{Z}[t], X)$-graphs then a map $\pi : \Gamma_1 \longrightarrow \Gamma_2$ is called a *generalized morphism* of $(\mathbb{Z}[t], X)$-graphs, if π sends vertices to vertices, directed edges to reduced paths and has the property that $o(\pi(p)) = \pi(o(p)), t(\pi(p)) = \pi(t(p)), \overline{\mu(\pi(p))} = \overline{\mu(p)}$ for any reduced path p of Γ_1.

Let ϕ be a u-folding defined above which is applied to the pair of edges $\{f_1, f_2\}$ in $E(Comp_u(v))$. By definition, ϕ involves as a final stage a partial folding ι. Then we have $\phi(f_1) = \iota(e_1)\iota(e_2), \phi(e) = \iota(e), e \in E(\Gamma), e \neq f_1$ and $\phi(w) = \iota(w), w \in V(\Gamma)$. Observe that it follows from the definition of ϕ that $o(\phi(e)) = o(\iota(e)) = \iota(o(e)) = \phi(o(e)), t(\phi(e)) = t(\iota(e)) = \iota(t(e)) = \phi(t(e)), \mu(\phi(e)) = \mu(e)$ for any $e \in E(\Gamma), e \neq f_1$ and $o(\phi(f_1)) = o(\iota(e_1)) = \iota(o(e_1)) = \phi(o(f_1)), t(\phi(f_1)) = t(\iota(e_2)) = \iota(t(e_2)) = \phi(t(f_1)), \overline{\mu(\phi(f_1))} = \mu(f_1)$. Hence, it follows that $o(\phi(p)) = \phi(o(p)), t(\phi(p)) = \phi(t(p)), \overline{\mu(\phi(p))} = \overline{\mu(p)}$ for any reduced path p in Γ. Observe that $\phi(p)$ may not be reduced.

Thus, we verified that u-folding is a generalized morphism of $(\mathbb{Z}[t], X)$-graphs. The following result is analogous to Lemma 29 about partial foldings.

Lemma 31 *Let Γ_1 be a $(\mathbb{Z}[t], X)$-graph obtained by a u-folding from a $(\mathbb{Z}[t], X)$-graph Γ. Let v be a vertex of Γ and v_1 be the corresponding vertex of Γ_1. Then the following hold:*

1) If Γ is connected then Γ_1 is connected.

2) Let p be the path from v to v in Γ such that $\overline{\mu(p)} = w$. Then the image of p in Γ_1 is a path p_1 from v_1 to v_1 such that $\overline{\mu(p_1)} = w$.

3) If Γ is a finite $(\mathbb{Z}[t], X)$-graph, then $|V(\Gamma_1)| \leq |V(\Gamma)|$.

Proof. 1) and 3) follow directly from the definition and 2) follows from the fact that u-foldings are generalized morphisms.

□

It is easy to see the difference between Lemma 29 and Lemma 31 above. Unlike partial foldings, u-foldings do not preserve labels of paths just because of the division of edges involved, but any path in Γ and its image in Γ_1 have the the same labels viewed as reduced infinite words.

The following important result follows directly from the Lemma 38 which will be proved in Subsection 17.2 in more general context.

Lemma 32 *Let Δ be a $(\mathbb{Z}[t], X)$-graph obtained by a u-folding ϕ from a graph Γ. Let v be a vertex of Γ such that v belongs to some u-component in Γ. Then $\phi(v)$ belongs to a u-component in Δ and $H_u(v) \simeq H_u(\phi(v))$.*

Any finite u-component can be transformed into a single positively oriented path with associated free abelian group of finite rank. The next results show how one can use u-foldings to get such a form of u-components.

At first, recall that in a connected graph a subgraph is said to be *a spanning tree* if this subgraph is a tree and it contains all vertices of the original graph. If a graph T is a tree then for any two vertices v_1, v_2 of T there is a unique reduced path in T from v_1 to v_2.

Let Γ be a $(\mathbb{Z}[t], X)$-graph and $v \in V(\Gamma)$. We call a path $p = e_1 \cdots e_k$ in $Comp_u(v)$ *positively oriented* (*negatively oriented*) if $\alpha_i > 0$ $(\alpha_i < 0), i \in [1, k]$, where $\mu(e_i) = u^{\alpha_i}$.

Lemma 33 *Let T be a finite tree such that all its edges are labeled by $u^{\alpha}, \alpha \in \mathbb{Z}[t]^+$ and let $v_0 \in V(T)$. Then T can be transformed by finitely many u-foldings into a tree T' such that if $v'_0 \in V(T')$ corresponds to v_0 then for any $v \in V(T')$ the unique reduced path p_v from v'_0 to v is either positively oriented or negatively oriented.*

Proof. Since v_0 can be connected by a unique reduced path to any $v \in V(T)$ and T is finite we have a finite number of such paths $p_v, v \in V(T)$. In each path

$p_v = e_1 \cdots e_k$ there can be positive edges, that is, labeled by $u^{\alpha}, \alpha > 0$ and also negative edges. To prove the statement of the lemma we use induction on the number of vertices in T such that p_v is neither positively nor negatively oriented. Let B_T denote the set of such vertices in T.

If $|B_T| = 0$ then lemma is already true for T. Thus we assume the statement to be true for any tree S with $|B_S| = n - 1$.

Let $|B_T| = n$ and take any vertex v from B_T. Then $p_v = e_1 \cdots e_k$ is a unique reduced path such that $o(p) = v_0, t(p) = v, \mu(e_i) = u^{\alpha_i}, i \in [1, k]$. Without loss of generality we can assume $\alpha_1 > 0$. Since $v \in B$ there exists minimal $j \in [2, k]$ such that $\alpha_{j-1} > 0, \alpha_j < 0$. Thus $t(e_j) \in B_T$ and again, without loss of generality we can assume $v = t(e_j)$. We have two cases.

(1) $\alpha_1 + \cdots + \alpha_{j-1} \geq |\alpha_j|$

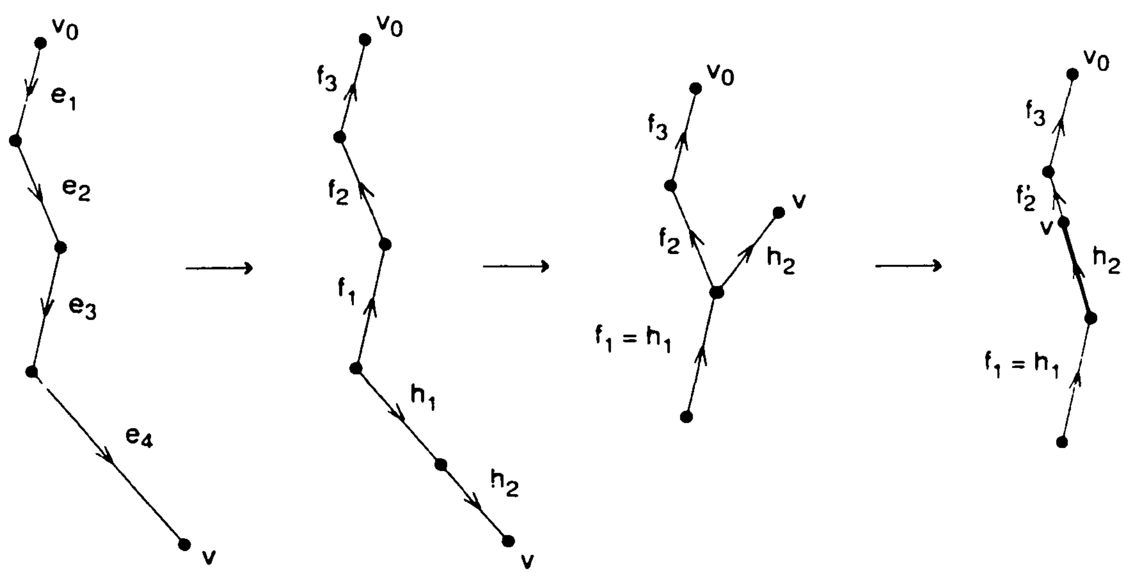


Figure 4: Case (1), $j = 4, i = 2$.

There exists maximal $i \in [1, j-1]$ such that $\alpha_i + \cdots + \alpha_{j-1} > |\alpha_j|$ but $\alpha_{i+1} + \cdots + \alpha_{j-1} < |\alpha_j|$. Hence, e_j can be folded step by step with the negatively oriented path $e_{j-1}^{-1} \cdots e_i^{-1} = f_1 \cdots f_{j-i}, e_{j-s}^{-1} = f_s, s \in [1, j-i]$ as follows.

e_j is divided into negative edges $h_1, \dots, h_{j-i}$ by new vertices $v_1, \dots, v_{j-i-1}$ so that $o(h_1) = o(e_j), t(h_s) = o(h_{s+1}) = v_s, s \in [1, j-i-1], t(h_{j-i}) = t(e_j), \mu(h_s) = \mu(f_s) = u^{-\alpha_{j-s}}, s \in [1, j-i-1], \mu(h_{j-i}) = u^{\alpha_j+\alpha_{i+1}+\cdots+\alpha_{j-1}}$. Finally, the sequence of u-foldings identifies each h_s with f_s for all $s \in [1, j-i-1]$ and h_{j-i} becomes an initial part of $f_{j-i} = e_i^{-1}$. We denote this sequence of u-foldings by ϕ and let T' denote the result of applying ϕ to T. Observe that after ϕ is applied $v = t(e_j)$ is identified with a point on edge e_i, so $\phi(v)$ is connected to v_0 by a unique reduced positively oriented path. Obviously T' is also a tree and we have that $\phi(v) \notin B_{T'}$. Observe also that if $w \notin B_T$ then $\phi(w) \notin B_{T'}$. Indeed, if a vertex $w \in T$ is connected to v_0 by a unique path p_w such that $e_j \in p_w$ then p_w also contains $e_1 \cdots e_{j-1}$ and $w \in B_T$. But such vertices are the only ones for which a path leading to v_0 is changed under ϕ. So if $w' \notin B_T$ then $p_{w'}$ is not affected by ϕ and $\phi(p_{w'}) = p_{w'}$. It follows that $|B_{T'}| < |B_T|$.

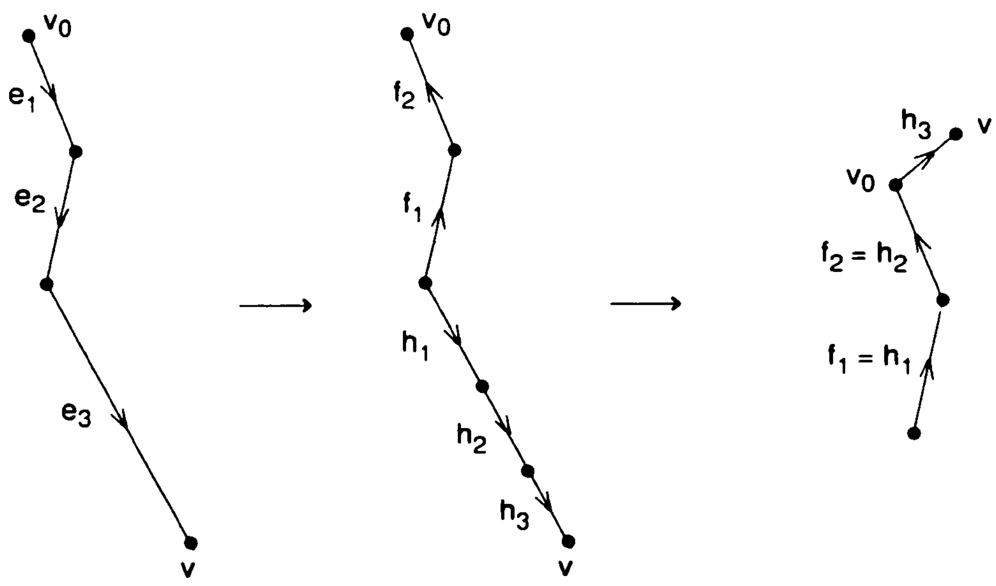


Figure 5: Case (2), $j = 3$.

(2) $\alpha_1 + \cdots + \alpha_{j-1} < |\alpha_j|$

e_j can be folded step by step with the negatively oriented path $e_{j-1}^{-1} \cdots e_1^{-1} =$

$f_1 \cdots f_{j-1}, e_{j-s}^{-1} = f_s, s \in [1, j-1]$ as follows.

e_j is divided into negative edges $h_1, \ldots, h_j$ by new vertices $v_1, \ldots, v_{j-1}$ so that $o(h_1) = o(e_j), t(h_s) = o(h_{s+1}) = v_s, s \in [1, j-1], t(h_j) = t(e_j), \mu(h_s) = \mu(f_s) = u^{-\alpha_{j-s}}, s \in [1, j-1], \mu(h_{j-1}) = u^{\alpha_j + \alpha_1 + \cdots + \alpha_{j-1}}$. Finally, the sequence of u-foldings identifies each h_s with f_s for all $s \in [1, j-1]$. We denote this sequence of u-foldings by ϕ and let T' denote the result of applying ϕ to T. Then T' is also a tree and we have that $\phi(v) = \phi(t(e_j))$ is connected to $\phi(v_0) = v_0$ in T' by a unique reduced path which consists of a single negative edge h_j. So $\phi(v) \notin B_{T'}$ and by the same argument as in (1) we have that $|B_{T'}| < |B_T|$.

Thus, in both cases we obtained a new tree T' for which the statement can be obtained by induction.

□

As a corollary of the Lemma 33 we get the following important result.

Corollary 3 *Let T be a finite tree such that all its edges are labeled by $u^\alpha, \alpha \in \mathbb{Z}[t]^+$ and let $v_0 \in V(T)$. Then T can be transformed into a simple positively oriented path by finitely many u-foldings.*

Proof. By Lemma 33, T can be transformed by finitely many u-foldings into a tree T' with $v'_0 \in V(T')$ corresponding to v_0, such that for any $v \in V(T')$ the unique reduced path p_v from v'_0 to v is either positively oriented or negatively oriented. Now, to complete the proof it is enough to notice that any two finite simple positively oriented paths which have the same origin w can be folded together into one simple positively oriented path with the same origin w. So, after finitely many u-foldings one transforms T' into two paths leading from v'_0 - one positively oriented and another one negatively oriented. Their concatenation is a reduced positively oriented path.

□

Now, we return to u-components in a $(\mathbb{Z}[t], X)$-graph Γ.

Lemma 34 *Let Γ be a $(\mathbb{Z}[t], X)$-graph, $v \in V(\Gamma)$ and $C = Comp_u(v)$ be finite. Then there exist a $(\mathbb{Z}[t], X)$-graph Δ obtained from Γ by finitely many u-foldings such that $v' \in V(\Delta)$ corresponds to v and $C'' = Comp_u(v')$ consists of a simple positively oriented path $P_{C''}$, some vertices of which may be connected by single edges not in $P_{C''}$.*

Proof. Let T be any spanning tree of C. Then, by Corollary 3 there exists a finite sequence $\{\phi_1, \ldots, \phi_n\}$ of u-foldings which transforms T into a simple positively oriented path $P_{C''}$. Observe that some vertices in $P_{C''}$ can be connected by images of edges from $C - T$. Thus, Δ is the image of Γ under $\{\phi_1, \ldots, \phi_n\}$.

□

We call C'' from the lemma above *a reduced u*-component. Since $P_{C''}$ is a simple path there exists a vertex $z_{C''} \in V(P_{C''})$ such that $val_{P_{C''}}(z_{C''}) = 1$ and the only edge in $P_{C''}$ which has $z_{C''}$ as an origin is positive. We call $z_{C''}$ a *base-point* of C''.

Observe that because of arbitrary choice of a spanning tree for C and arbitrary order of performing sequences of u-foldings it follows that reduced u-component C'' corresponding to C is not unique.

Let Γ be any $(\mathbb{Z}[t], X)$-graph, $v \in V(\Gamma)$ and let $C = Comp_u(v)$ be finite reduced u-component. Then there exists a simple positively oriented path P_C in C which originates at a base-point z_C of C. Since C is finite then there are finitely many edges $h_1, \ldots, h_l$ in $C \backslash P_C$. Any h_i connects two vertics in P_C so there exists a unique reduced path q_i in P_C such that $o(h_i) = o(q_i), t(h_i) = t(q_i)$. Moreover, $h_i q_i^{-1}$ is a cycle in C, so $\mu(h_i) * \overline{\mu(q_i)}^{-1} = c \in H_u(z_C)$.

Now, let $p = e_1 \cdots e_n$ be a reduced path in C. Suppose some of its edges $e_{i_1}, \ldots, e_{i_k}, i_j \in [1, n]$ belong to $C \backslash P_C$. Then we construct another path p' in the following way: every edge e_{i_j} is equal to some h_{i_j} in the list of edges of $C \backslash P_C$, so in p we substitute e_{i_j} by the path q_{i_j}. The resulting path p' may be not reduced, so we perform all possible reductions and obtain a reduced path p'' all edges of which belong to $P_{C'}$. We have $o(p'') = o(p), t(p'') = t(p), \overline{\mu(p'')} = \overline{\mu(p)} * h, h \in H_u(z_{C'})$ because of the definition of the path q_i for each $h_i \in C' \backslash P_{C'}$. Finally, observe that p'' is uniquely defined for p because it is a reduced subpath of $P_{C'}$.

Hence, for any reduced path p in C' there exists a unique reduced subpath q of $P_{C'}$ with the same endpoints as p and such that $\overline{\mu(p)} * \overline{\mu(q)}^{-1} \in H_u(z_{C'})$. Further, we will use the notation $q = [p]$.

The converse is also true, that is, if q is a reduced path in $P_{C'}$ and $c \in H_u(z_{C'})$ then since any element from $H_u(z_{C'})$ can be realized as a reduced label of some loop at $t(q)$ in C', there exists a reduced path p in C' such that $o(p) = o(q), t(p) = t(q), \overline{\mu(p)} = \overline{\mu(q)} * c$. Observe that q is not unique with respect to permutation of cycles.

This correspondence above shows that any finite reduced u-component C' in a $(\mathbb{Z}[t], X)$-graph is characterized completely by the pair $(P_{C'}, H_u(z_{C'}))$.

If C' is finite then there exist finitely many subpaths $q_1, \ldots, q_s$ of $P_{C'}$ such that for any path p in C', $[p] = q_i$ for some $i \in [1, s]$. Moreover, let $P_{C'} = f_1 \cdots f_m$, where $o(f_1) = z_{C'}$. Let $v_0 = z_{C'}, v_i = t(f_i), i \in [1, m]$ and let $p_0, p_1, \ldots, p_m$ be reduced subpaths of $P_{C'}$ such that $o(p_i) = z_{C'}, t(p_i) = v_i, i \in [0, m]$. It follows that all p_i are positively oriented. Then for every reduced subpath $p_{i,j}$ of $P_{C'}$ such that $o(p_{i,j}) = v_i, t(p_{i,j}) = v_j$ we have $\overline{\mu(p_{i,j})} = \overline{\mu(p_j)} * \overline{\mu(p_i)}^{-1}$. By Lemma 30, $H_u(z_{C'})$ is finitely generated and is isomorphic to a subgroup of $\mathbb{Z}^r, r \in \mathbb{N}$. So if p is a reduced path in C' such that $o(p) = v_i, t(p) = v_j$ then by definition of $[p]$ we have $\overline{\mu(p)}$ and $\overline{\mu([p])} = \overline{\mu(p_{i,j})} = \overline{\mu(p_j)} * \overline{\mu(p_i)}^{-1}$ are in the same coset in $\mathbb{Z}^r$ by $H_u(z_{C'})$. That is, we can express the label of any reduced path in C' in terms of labels of $p_i, i \in [0, m]$ and elements from $H_u(z_{C'})$. We call a set of paths $p_0, p_1, \ldots, p_m$ a *set of path representatives associated with* C' and denote this set by $Rep(C')$. The following result holds.

Lemma 35 *Let C' be a finite reduced u-component in a $(\mathbb{Z}[t], X)$-graph Γ, $v \in V(C')$*

and let $\alpha \in \mathbb{Z}[t]^{+}$. *If* $\overline{\mu(p_i)} * \overline{\mu(p_j)}^{-1} \notin H_u(z_{C'})$ *for any* $p_i, p_j \in Rep(C'), i \neq j$ *then either there exists a unique reduced path* p *in* $P_{C'}$ *such that* $o(p) = v$ *and* $u^{\alpha} \in \overline{\mu(p)} * H_u(z_{C'})$ *or there exists no path* q *in* C *with this property.*

Proof. Suppose on the contrary that there exist two reduced paths p, q in $P_{C'}$ such that $o(p) = v, o(q) = v$ and $u^{\alpha} = \overline{\mu(p)} * h_1 = \overline{\mu(q)} * h_2, h_1, h_2 \in H_u(z_{C'})$. Then $\overline{\mu(p)} * \overline{\mu(q)}^{-1} = h_1^{-1} * h_2 \in H_u(z_{C'})$.

On the other hand we have that $v = v_i, t(p) = v_j, t(q) = v_k, j \neq k$. So we have $\overline{\mu(p)} = \overline{\mu(p_j)} * \overline{\mu(p_i)}^{-1}, \overline{\mu(q)} = \overline{\mu(p_k)} * \overline{\mu(p_i)}^{-1}$. So $\overline{\mu(p)} * \overline{\mu(q)}^{-1} = (\overline{\mu(p_j)} * \overline{\mu(p_i)}^{-1}) * (\overline{\mu(p_k)} * \overline{\mu(p_i)}^{-1})^{-1} = \overline{\mu(p_j)} * \overline{\mu(p_k)}^{-1} \in H_u(z_{C'})$ - contradiction.

□

If C is reduced and $Rep(C)$ satisfies the condition from Lemma 35 then we call C *a u-folded u-component.*

Let C be a finite reduced u-component in a $(\mathbb{Z}[t], X)$-graph Γ which is not u-folded. That is, there exist two vertices $v_i, v_j, i < j$ in $P_{C'}$ such that $\overline{\mu(p_i)} * \overline{\mu(p_j)}^{-1} \in H_u(z_{C'})$. Consider a graph Δ which is obtained from Γ by identification of vertices v_i, v_j in C into one new vertex v. We call this operation *a collapse of* v_i *and* v_j. The resulting u-component in Δ we denote by C''. In fact, a collapse can be obtained as a finite sequence of u-foldings. Indeed, since $\overline{\mu(p_i)} * \overline{\mu(p_j)}^{-1} = h \in H_u(z_{C'})$ there exists a positive loop at v_i with the label h. This means that if we add a single edge e to C so that $o(e) = t(e) = v_i, \mu(e) = h$ then $H_u(z_C)$ is not changed. Then we can apply a sequence of $j - i$ u-foldings to e and the subpath $q = e_{i+1} \cdots e_j$ of $P_{C'}$ connecting v_i and v_j. After these u-foldings are implemented, v_i is identified with v_j because $\overline{\mu(q)} = h$. Since u-foldings do not change $H_u(z_C)$ in u-components a collapse of v_i and v_j is a valid operation. Observe that $|V(C)| = |V(C'')| + 1$. If C'' is not reduced so using finitely many u-foldings one can reduce it and since u-foldings do not increase the number of vertices for the resulting reduced u-component C''' we have $|V(C)| > |V(C'')| \geq |V(C''')|$. Thus, the following result holds.

Lemma 36 *Let* Γ *be a* $(\mathbb{Z}[t], X)$*-graph which has finitely many u-components all of which are finite and reduced. Then there exists a partially folded* $(\mathbb{Z}[t], X)$*-graph* Δ *which is obtained from* Γ *by finitely many u-foldings such that all its u-components are u-folded.*

17 U-folded $(\mathbb{Z}[t], X)$-graphs and subgroups of $F^{\mathbb{Z}[t]}$

In the previous sections we defined all necessary operations on $(\mathbb{Z}[t], X)$-graphs which are tools for constructing U-folded graphs (see Remark 5). In the Proposition 4 we introduce the procedure which transforms any $(\mathbb{Z}[t], X)$-graph into a U-folded one and also we show the correspondence between finite $(\mathbb{Z}[t], X)$-graphs and finitely generated subgroups of $F^{\mathbb{Z}[t]}$.

17.1 Languages associated with $(\mathbb{Z}[t], X)$-graphs

In this section we adjust the definition of a language recognized by a graph from [24] (Definition 2.7) to $(\mathbb{Z}[t], X)$-graphs.

Definition 34 *Let Γ be a $(\mathbb{Z}[t], X)$-graph and let v be a vertex of Γ. We define the language of Γ with respect to v to be*

$$L(\Gamma, v) = \{\overline{\mu(p)} \mid p \text{ is a reduced path in } \Gamma \text{ from } v \text{ to } v\}.$$

If w belongs to $L(\Gamma, v)$, we will also sometimes say that w is *accepted* by (Γ, v) (or just by Γ if v is fixed).

The following result establishes a connection between $(\mathbb{Z}[t], X)$-graphs and subgroups in $F^{\mathbb{Z}[t]}$.

Lemma 37 *Let Γ be a finite $(\mathbb{Z}[t], X)$-graph and let $v \in V(\Gamma)$. Then $L(\Gamma, v)$ is a subgroup of $F^{\mathbb{Z}[t]}$.*

Proof. Observe, at first, that $L(\Gamma, v)$ is a subset of $F^{\mathbb{Z}[t]}$ by definition.

Let $g_1, g_2 \in L(\Gamma, v)$. Then there are reduced paths p_1 and p_2 from v to v in Γ such that $\overline{\mu(p_i)} = g_i, i = 1, 2$.

The concatenation q of p_1 and p_2 is a path in Γ from v to v such that $\overline{\mu(q)} = \overline{\mu(p_1)} * \overline{\mu(p_2)}$, but q may not be reduced . Let p be the reduced path obtained from

q by making all possible path reductions. This means that the label $\overline{\mu(p)} = \overline{\mu(q)}$ and $o(p) = t(p) = v$. Therefore $\overline{\mu(p_1)} * \overline{\mu(p_2)} = \overline{\mu(p)} \in L(\Gamma, v)$.

Thus $g_1 * g_2 \in L(\Gamma, v)$ and $L(\Gamma, v)$ is closed under multiplication $*$ of infinite words.

It is easy to see that the inverse path $(p_1)^{-1}$ of p_1 is reduced and $\overline{\mu(p_1^{-1})} = \overline{\mu(p_1)^{-1}} = \overline{\mu(p_1)}^{-1}$. This implies that $L(\Gamma, v)$ is closed under taking inverses. Also, obviously $\varepsilon \in L(\Gamma, v)$.

Thus $L(\Gamma, v)$ is a subgroup of $F^{\mathbb{Z}[t]}$.

□

In the previous section we introduced operations on an arbitrary $(\mathbb{Z}[t], X)$-graph Γ (partial foldings and u-foldings, $u \in U$) and now we prove that they do not change the language associated with Γ.

Lemma 38 *Let Γ be a finite $(\mathbb{Z}[t], X)$-graph and let $v \in V(\Gamma)$. Let Δ_1 be a $(\mathbb{Z}[t], X)$-graph obtained from Γ by a single partial folding and let Δ_2 be a $(\mathbb{Z}[t], X)$-graph obtained from Γ by a single u-folding for some $u \in U$, so that $v_1 \in V(\Delta_1)$ and $v_2 \in V(\Delta_2)$ correspond to v. Then*

$$L(\Gamma, v) = L(\Delta_1, v_1) = L(\Delta_2, v_2).$$

Proof. At first we prove $L(\Gamma, v) = L(\Delta_1, v_1)$. If Δ_1 is obtained from Γ by identification of edges $e_1, e_2 \in \widehat{\Gamma}, \mu(e_1) = \mu(e_2) = x \in X^{\pm 1}$ then the result follows from the proof of Lemma 3.4 in [24]. If $e_1, e_2 \in \widehat{\Gamma}$ are labeled by the same exponent of some $u \in U$ then the partial folding which identifies e_1 and e_2 is a u-folding and, thus, in this case, we reduced the proof of $L(\Gamma, v) = L(\Delta_1, v_1)$ to the proof of $L(\Gamma, v) = L(\Delta_2, v_2)$.

So let us prove $L(\Gamma, v) = L(\Delta_2, v_2)$.

Suppose Δ_1 is obtained from Γ by folding two edges e_1, e_2 in $\widehat{\Gamma}$ which have the same initial vertex w and $\mu(e_1) = u^\alpha, \mu(e_2) = u^\beta$. Without loss of generality we can assume $\alpha \geq \beta > 0$. Then e_1 becomes a path $h_1 h_2$ in Δ_2 such that $\mu(h_1) = u^\beta, \mu(h_2) = u^{\alpha-\beta}$ and h_1 is identified with e_2 in Δ_2 into an edge h.

Suppose p is a reduced path in Γ from v to v, so that $\overline{\mu(p)} \in L(\Gamma, v)$. The image of p in Δ_1 by Lemma 31 is a path p' from v_2 to v_2 such that $\overline{\mu(p)} = \overline{\mu(p')}$. However, p' need not be reduced. Namely, p' is reduced if and only if p does not contain any subpaths of the form $e_2^{-1}e_1$ or $e_1^{-1}e_2$. Let p'' be the path obtained from p' by performing all possible path reductions in Δ_2. Then $\overline{\mu(p)} = \overline{\mu(p')} = \overline{\mu(p'')}$ and $\overline{\mu(p'')} \in L(\Delta_2, v_2)$. Thus we have shown that $L(\Gamma, v) \subseteq L(\Gamma', v')$.

Suppose now that p' is an arbitrary reduced path in Δ_2 from v_2 to v_2. We claim that there is a reduced path p in Γ from v to v such that $\overline{\mu(p)} = \overline{\mu(p')}$. We will construct this path explicitly.

(1) $\alpha = \beta$

In this case h_2 is an empty edge and e_1, e_2 are identified with the edge h in Δ_2. The occurrences of $h^{\pm 1}$ (if any) subdivide p' into a concatenation of the form:

$$p' = p_0 f_0 p_1 f_1 \ldots f_k p_{k+1}.$$

where $f_i = h^{\pm 1}$ and the paths p_i do not involve $h^{\pm 1}$.

Suppose that for some i we have $f_i = h$. Since p_i and p_{i+1} do not involve the edge h, they can also be considered as paths in Γ. Moreover, by the definition of u-folding, in the graph Γ the terminal vertex of p_i is joined with the initial vertex of p_{i+1} by either the edge e_1 or e_2. We denote this edge by d_i (so that $d_i \in \{e_1, e_2\}$). Note that now $p_i d_i p_{i+1}$ is a reduced path in Γ with the same label as the path $p_i f_i p_{i+1}$ in Δ_2.

Similarly, if for some i we have $f_i = h^{-1}$, we can find $d_i \in \{e_1^{-1}, e_2^{-1}\}$ such that $p_i d_i p_{i+1}$ is a reduced path in Γ with the same label as the path $p_i f_i p_{i+1}$ in Δ_2.

Then

$$p = p_0 d_0 p_1 \ldots d_k p_{k+1}$$

is a reduced path in Γ from v to v with the same label as p'. Thus $\overline{\mu(p')} \in L(\Gamma, v)$ and therefore $L(\Delta_2, v_2) \subseteq L(\Gamma, v)$.

(2) $\alpha > \beta$

This is a general case when e_1 becomes a path hh_2 and e_2 becomes the edge h in Δ_2. Observe that $\mu(h) = \mu(e_2), \mu(h_2) = \mu(e_2)^{-1} * \mu(e_1)$ and $o(h) = o(e_2) = o(e_1), t(h) = t(e_2), o(h_2) = t(h) = t(e_2), t(h_2) = t(e_1)$ in Δ_2.

Similarly to (1) we subdivide p' by the occurrences of $h^{\pm 1}$ and $h_2^{\pm 1}$ so that

$$p' = p_0 f_0 p_1 f_1 \ldots f_k p_{k+1},$$

where $f_i = h^{\pm 1}$ or $f_i = h_2^{\pm 1}$ and the paths p_i do not involve $h^{\pm 1}, h_2^{\pm 1}$.

Any entry of edge h_2 we substitute by a path $e_2^{-1} e_1$ and every entry of h we substitute by e_2 so that h_2^{-1} is substituted by $e_1^{-1} e_2$ and h^{-1} by e_2^{-1}.

Suppose that for some i we have $f_i = h$. Since p_i and p_{i+1} do not involve the edge h, they can also be considered as paths in Γ. We have $t(p_i) = o(h) = o(e_2), o(p_{i+1}) = t(h) = t(e_2)$, so the substitution $h \to e_2$ is valid. If on the other hand for some j we have $f_j = h_2$ then $t(p_j) = o(h_2) = t(e_2), o(p_{j+1}) = t(h_2) = t(e_1)$ and again the substitution $h_2 \to e_2^{-1} e_1$ is valid. Finally, $\mu(h) = \mu(e_2), \mu(h_2) = \mu(e_2)^{-1} * \mu(e_1)$ implies that after all possible substitutions in p' are made, the resulting path p'' in Γ from v to v and $\overline{\mu(p'')} = \overline{\mu(p')}$, but p'' may not be reduced. Let p be the path obtained from p'' by performing all possible path reductions in Γ. Then $\overline{\mu(p)} = \overline{\mu(p'')} = \overline{\mu(p')}$ and $\overline{\mu(p)} \in L(\Gamma, v)$.

Hence in both cases (1),(2) above $L(\Delta_2, v_2) \subseteq L(\Gamma, v)$ which completes the proof.

□

If $g \in F^{\mathbb{Z}[t]}$, then it follows that $g \in G_{n+1}$ for some finite n in the series (7). Let

$$F < H_{m(0)_1} < H_{m(0)_2} < \ldots < H_{m(0)_{k(0)}} < H_{m(1)_1} < \ldots < H_{m(1)_{k(1)}} < \ldots \quad (15)$$

$$\ldots < H_{m(n-1)_{k(n-1)}} < H_{m(n)_1} < \ldots < H_{m(n)_{k(n)}},$$

be an extension series for g. So $g \in H_{m(n)_{k(n)}}$ and let us denote $u_{m(n)_{k(n)}}$ by w. Then the standard decomposition of g is

$$\tilde{g} = \tilde{h}_1 w^{\beta_1} \tilde{h}_2 \cdots w^{\beta_l} \widetilde{h_{l+1}}.$$

where $h_i \in H_{m(n)_{k(n)}-1}$ and $\tilde{h}_i$ is a standard decomposition for h_i, $i \in [1, l+1]$. As we noted in Subsection 15.3, $\tilde{g}$ can be viewed as a finite word in the alphabet

$$\{X \cup X^{-1}\} \cup \{u^{\alpha} \mid u \in U, \alpha \in \mathbb{Z}[t]^{+}\}.$$

Let $U(g)$ denote a finite subset of U such that if $\tilde{g}$ contains u^{α}, $u \in U$, $\alpha \in \mathbb{Z}[t]^{+}$ then $u \in U(g)$. Observe that $U(g)$ has an order induced from U so we have $U(g) = \{u_1, \ldots, u_m\}$, where $u_i < u_j$ if $i < j$ and $u_m = u_{m(n)_{k(n)}} = w$. So we have

$$\tilde{g} = \tilde{h}_1 u_m^{\beta_1} \tilde{h}_2 \cdots u_m^{\beta_l} \widetilde{h_{l+1}}.$$

Note that if $U(g)$ is empty then $g \in F(X)$.

Definition 35 *Let Γ be a $(\mathbb{Z}[t], X)$-graph and $p = e_1 \cdots e_k$ be a reduced path in Γ, g is as above. We define the property of* the label $\mu(p)$ of p to be equal to $\tilde{g}$ *(that is, $\mu(p) = \tilde{g}$) recursively as follows:*

1) If $|U(g)| = 0$, that is, $\tilde{g} = x_1 \cdots x_r \in F(X)$ then $\mu(p) = \tilde{g}$ means that $k = r$

and $\mu(e_i) = x_i$ *for every* $i \in [1, k]$;

2) Assume that the property $\mu(p) = \tilde{f}$ *is defined for any element* $f \in F^{\mathbb{Z}[t]}$ *if* $|U(f)| < m$;

3) If $|U(g)| = m$ *then* $\mu(p) = \tilde{g}$ *means that* p *can be subdivided into subpaths as*

$$p = p_1 d_1 p_2 \dots d_l p_{l+1},$$

where d_i *is a path in some* u_m*-component of* Γ *and* p_i *is a path in* Γ *which does not contain edges labeled by* $u_m^\alpha, \alpha \in \mathbb{Z}[t]^+$, *so that* $\overline{\mu(d_i)} = u_m^{\beta_i}, i \in [1, l]$ *and* $\overline{\mu(p_i)} = \tilde{h}_i, i \in [1, l+1]$ *are already defined by our assumption because* $|U(h_i)| < m, i \in [1, l+1]$.

It follows immediately that if $\mu(p) = \tilde{g}$ for some $g \in F^{\mathbb{Z}[t]}$ then p is label reduced.

Observe that if Γ is a $(\mathbb{Z}[t], X)$-graph and $v \in V(\Gamma)$ then it follows from Lemma 37 that any element of $L(\Gamma, v)$ is an infinite word in $F^{\mathbb{Z}[t]}$ and thus has a standard decomposition $\tilde{g}$ but it might be impossible to find a label reduced loop p at v in Γ such that $\mu(p) = \tilde{g}$.

The main result of the following subsection is that if Γ is finite and has some particular properties then it is always possible to find such a path p and p is unique in some sense. Moreover, it will be shown how the required properties can be obtained in Γ using partial foldings and u-foldings, $u \in U$.

17.2 U-folded $(\mathbb{Z}[t], X)$-graphs

All $(\mathbb{Z}[t], X)$-graphs which appear in considerations below are supposed to be finite.

Let Γ be a $(\mathbb{Z}[t], X)$-graph. Since Γ is finite, there exist only finitely many edges with labels $u^\alpha, u \in U, \alpha \in \mathbb{Z}[t]^+$. Thus, there exists a natural number $K \geq 0$ such that for an edge e in Γ such that $\mu(e) = u^\alpha$ it follows that $u \in U_j$ and $j \in [0, K]$. For each fixed j and $u \in U_j$ using u-foldings described in the Section 16 one can transform all u-components of Γ into u-folded components. However, because of the nature of the sets U_j, edges of Γ belong to different *levels* (we introduce the precise definition of a level below) and U-foldings do not deal with interactions between these levels. Moreover, u_1-foldings which are applied to some u_1-component can affect u_2-components, $u_1 \in U_i, u_2 \in U_j, i, j \in [0, K], u_2 \neq u_1$. So, one needs some definite procedure which "folds" Γ level by level.

At first we introduce a notion of a *level* in Γ.

Since Γ is finite the set of elements $u \in U$ such that there exists an edge e in Γ labeled by $u^\alpha, \alpha \in \mathbb{Z}[t]^+$ is finite and ordered with the order induced from U. Thus one can associate with Γ an ordered set $U(\Gamma) = \{u_1, \ldots, u_N\}, N > 0, u_i \in U, u_i < u_j$ for $i < j$. Observe that $U(\Gamma)$ can be empty, that is, all edges in Γ are labeled by letters from $X \cup X^{-1}$.

Definition 36 *Let Γ be a $(\mathbb{Z}[t], X)$-graph and let $u_i \in U(\Gamma)$ be fixed. Let $\Gamma(i)$ be a subgraph of Γ which consists only of edges $e \in E(\Gamma)$ such that either $\mu(e) = w \in X^{\pm 1}$ or $\mu(e) = u_j^\alpha, \alpha \in \mathbb{Z}[t]^+, j \leq i$. We call $\Gamma(i)$ an* i-level graph *of Γ (by 0-level graph we understand a subgraph of Γ which consists only of edges with labels from X) and say that Γ has* level n *which we denote by $l(\Gamma)$ if n is the minimal natural number for which $\Gamma = \Gamma(n)$.*

Observe that $\Gamma(n)$ may not be connected for some $i < l(\Gamma)$, but still one can apply to $\Gamma(n)$ partial and u-foldings, $u \in U(\Gamma)$.

Now, the main result of Section 17 about $(\mathbb{Z}[t], X)$-graphs.

Proposition 4 *Let Γ be a finite connected $(\mathbb{Z}[t], X)$-graph. Then there exists a finite, connected $(\mathbb{Z}[t], X)$-graph Δ, which is obtained from Γ by a finite sequence of partial and u-foldings, $u \in U(\Gamma)$, such that for any $u_n \in U(\Gamma)$, Δ satisfies the conditions:*

(i) *Δ is partially folded;*

(ii) *all u_n-components of Δ are u_n-folded and isolated, that is, there exists no reduced path p with $\overline{\mu(p)} = u_n^k, k \in \mathbb{Z}$ in $\Delta(n-1)$ such that p connects two different u_n-components of Δ;*

(iii) *if C is a u_n-component of Δ, $e \in E(P_C)$ and $\mu(e) = u_n^k, k \in \mathbb{Z}$ then there exists a unique label reduced path p in $\Delta(n-1)$ such that $o(p) = o(e), t(p) = t(e), \mu(p) = \widetilde{u_n}^k$;*

(iv) *if C is a u_n-component of Δ and $v \in V(C) \cap V(\Delta(n-1))$ then there exists a unique label reduced path p in $\Delta(n-1)$ such that $o(p) = t(p) = v, \mu(p) = \widetilde{u_n}^k, k \in \mathbb{Z}$ and $H_{u_n}(v) \cap \langle u_n \rangle = \langle u_n^k \rangle$;*

(v) *if C is a u_n-component of Δ and $v_1, v_2 \in V(C)$ are connected by a reduced path p in P_C then either p consists only of edges labeled by finite exponents of u_n or there exists no number $k_p \in \mathbb{Z}$ such that $\overline{\mu(p)} * u^{-k_p} \in H_{u_n}(v_1)$;*

(vi) *for any u_n-component C of Δ and two of its vertices $v_1, v_2, v_1 \neq v_2$ which are joined by some path p in P_C with $o(p) = v_1, t(p) = v_2$ there exists no reduced path r in $\Delta(n-1)$ such that $o(r) = v_1, t(r) = v_2, \overline{\mu(r)} = u_n^k, k \in \mathbb{Z}$ and $\overline{\mu(p)} * \overline{\mu(r)}^{-1} \notin H_{u_n}(v_1)$;*

(vii) *for any u_n-component C of Δ, its vertex v and a reduced path p in $\Delta(n-1)$ such that $o(p) = v, \overline{\mu(p)} = u_n^k, k \in \mathbb{Z}$ it follows that $t(p) \in V(C)$;*

(viii)

(a) *for any u_n-component C of Δ, its vertex v and a label reduced path p in $\Delta(n-1)$ such that $o(p) = v, \overline{\mu(p)} = w, w = u_n^\delta \circ c, \delta \in \{1, -1\}$, there exists a label reduced path $q = q_1 q_2$ in $\Delta(n-1)$ such that $o(q) = v, t(q) = t(q_2) = t(p), \mu(q_1) = \widetilde{u_n}^\delta, t(q_1) \in V(C)$;*

(b) *for any u_n-component C of Δ, its vertex v and a label reduced path p in $\Delta(n)$ such that $p = z_1 z_2, o(p) = v, z_1 \in \Delta(n-1), \overline{\mu(z_1)} = w_1, \overline{\mu(z_2)} = w_2 \circ c = u_n^\gamma \circ c_1, u_n^\delta = w_1 \circ w_2$, there exists a label reduced path $q = q_1 q_2$ in $\Delta(n-1)$ such that $o(q) = v, t(q) = t(q_2) = t(p), \mu(q_1) = \widetilde{u_n}^\delta, t(q_1) \in V(C)$;*

(c) *for any u_n-component C of Δ, its vertex v and a path p in $\Delta(n-1)$ such that $o(p) = v$, $\overline{\mu(p)} = w_1$, $u_n^\delta = w_1 \circ w_2$, $w_2 \neq \varepsilon$, $\delta \in \{1, -1\}$, if there exists an edge e' in C such that $o(e') = v$, $\mu(e') = u^\gamma$, $\gamma\delta > 0$ then there exists a label reduced path p' in $\Delta(n-1)$ such that $o(p') = v$, $\overline{\mu(p')} = u_n^\delta$, $t(p') \in V(C)$ and p is an initial subpath of p';*

(ix) *for any reduced path p in Δ with $\overline{\mu(p)} = w$ there exists a unique label reduced path q such that $o(q) = o(p)$, $t(q) = t(p)$, $\mu(q) = \tilde{w}$;*

(x) *for the standard decomposition $\tilde{g}$ of any $g \in F^{\mathbb{Z}[t]}$ and any $v \in V(\Delta)$ either there exists a unique label reduced path p in Δ starting at v such that $\mu(p) = \tilde{g}$ or for any path q in Δ starting at v it follows that $\overline{\mu(q)} \neq g$.*

Remark 5 *1. We call a $(\mathbb{Z}[t], X)$-graph which has all the properties* **(i)-(x)** *U-*folded.

2. In **(iii)**, **(iv)**, **(ix)** *and* **(x)** *by uniqueness we understand uniqueness with respect to P_C, where C is any u-component, $u \in U(\Delta)$, that is, a path is unique if we disregard edges in $C \backslash P_C$. This is justified in view of Lemma 35.*

Proof. The proof is conducted by induction on the level of Γ. If $l(\Gamma) = 0$, that is all edges in Γ are labeled by letters from $X \cup X^{-1}$, then by Lemma 29 using finitely many partial foldings one can obtain from Γ a partially folded $(\mathbb{Z}[t], X)$-graph Δ which satisfies all the conditions **(i)-(ix)** because it does not contain u-components.

Assume that the statement of the theorem holds for any graph of level $n-1$.

Let $l(\Gamma) = n$. We construct a $(\mathbb{Z}[t], X)$-graph Δ from Γ in several steps.

Step 1. For any edge e in Γ such that $\mu(e) = u_n^\alpha$, $\alpha \in \mathbb{Z}$ we add to Γ a new path from $o(e)$ to $t(e)$ labeled by the standard decomposition of u_n^α. Observe that these new paths belong to $\Gamma(n-1)$.

By induction, using partial foldings and U-foldings one can transform $\Gamma(n-1)$ into some graph $\Delta_0(n-1)$ which is U-folded. So, Γ transforms into a new $(\mathbb{Z}[t], X)$-graph Δ_0.

Let $C_1, \ldots, C_T$ be the list of all u_n-components in Δ_0. We can assume that any C_i either contains at least one edge with the label $u_n^\alpha, |\alpha| >> 1$ or if all its edges are labeled by finite exponents of u_n then the abelian group associated with C_i is not cyclic generated by a finite exponent of u_n. If some u_n-component C_i does not satisfy this condition then we eliminate from C_i every edge $e, \mu(e) = u_n^s, s \in \mathbb{Z}$ and substitute it with a path p_e labeled by the standard decomposition of u_n^s. Since p_e belongs to $\Delta_0(n-1)$ the number of u_n-components reduces.

Consider all vertices in C_1 which also belong to $\Delta_0(n-1)$ - these vertices form a finite set V and let $v \in V$. If $H_{u_n}(v) \cap \langle u_n \rangle = \langle u_n^k \rangle, k > 0$ then we add at v a loop q labeled by the standard decomposition of u_n^k, so $q \in \Delta_0(n-1)$.

Let Θ be a connected component of $\Delta_0(n-1)$ which contains v. Our assumption in the end of Subsection 15.1 means that $\widetilde{u_n^2} = \widetilde{u_n}\,\widetilde{u_n} = \widetilde{u_n}^2$, which implies that a path in Θ starting at v and labeled by $\widetilde{u_n^k}$ contains an initial subpath $\widetilde{u_n}$ and we can find all paths in Θ starting at v and labeled by $\widetilde{u_n^k}$ in the following way. Since $\Delta_0(n-1)$ is U-folded then either there exists a unique path p_1 in Θ such that $o(p_1) = v$ and $\mu(p_1) = \widetilde{u_n}$ (for $\mu(p_1) = \widetilde{u_n^{-1}}$ in the same way) or label of no path starting at v represents u_n as an infinite word. Suppose p_1 exists then again if there exists a continuation p_2 of p_1 such that $o(p_2) = t(p_1), \mu(p_2) = \widetilde{u_n}$ then it is unique and we repeat the process for p_2. Since $V(\Delta_0(n-1))$ is finite then either this process stops in a finite number of steps or there exist minimal natural numbers k_1 and k_2 such that $t(p_{k_1}) = t(p_{k_1+k_2})$ and since $\Delta_0(n-1)$ is U-folded then $p_{k_1+1} = p_{k_1+k_2+1}$, that is after a finite number of steps no new path is added and $P(C_1, v) = \{t(p_i), i \in \mathbb{N}\}$ is finite.

By induction hypothesis $L(\Theta, v)$ is a subgroup of $F^{\mathbb{Z}[t]}$. If k_1, k_2 from above exist then $L(\Theta, v) \cap \langle u_n \rangle = \langle u_n^{k_2} \rangle$ and if $u_n^{k_2} \notin H_{u_n}(v)$ then we add to C_1 an edge e such that $o(e) = t(e) = v, \mu(e) = u_n^{k_2}$, so that, $H_{u_n}(v)$ is changed. Let H be a free abelian group associated with u_n-component $C_1 \cup \{e\}$. If $H_{u_n}(v) \cap \langle u_n \rangle = \langle u_n^{l_1} \rangle, l_1 > 0$ then

$H \cap \langle u_n \rangle = \langle u_n^{l_2} \rangle$ and obviously $l_1 > l_2$. This means that the process of adjoining new edges to C_1 which changes $H_{u_n}(v)$ stops after a finite number of steps.

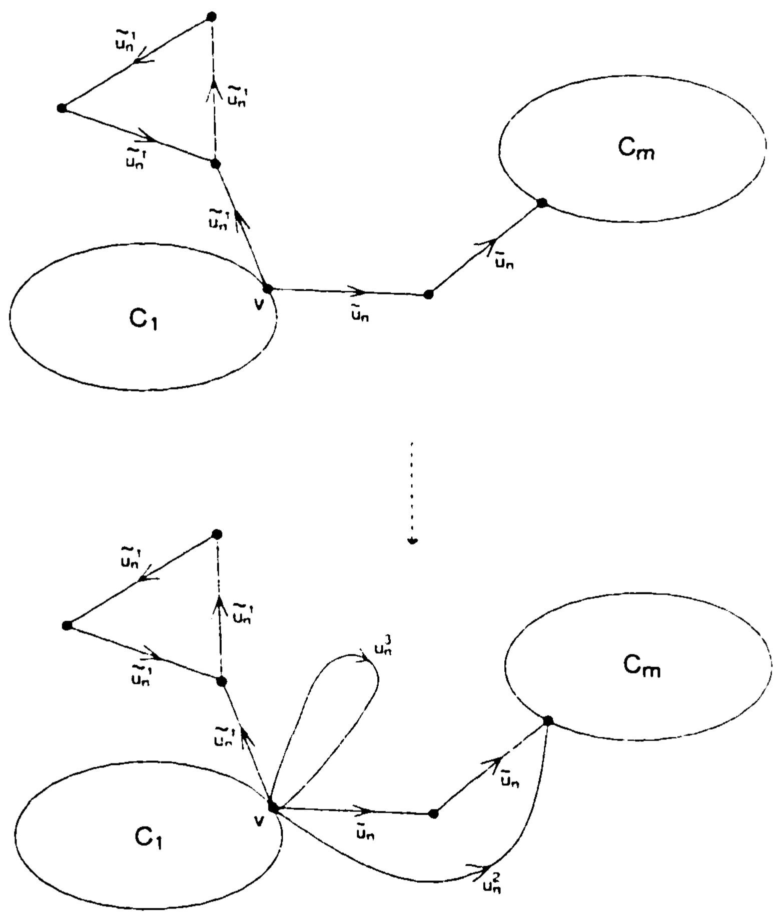


Figure 6: Step 1

If there exists a u_n-component C_m such that $P(C_1, v) \cap V(C_m)$ is not trivial, then it means that there exists a label reduced path p in $\Delta_0(n-1)$ with $\mu(p) = \widetilde{u_n}^r, r \in \mathbb{N}$ connecting C_1 and C_m. So we add an edge e such that $o(e) = v, t(e) = t(p), \mu(e) = u_n^r$ and the number of u_n-components in Δ_0 reduces.

Figure 6 illustrates the process described above.

We perform the procedure described above for all $C_i, i \in [1, T]$ and all $v \in V(C_i)$. Observe that T is finite and $|V(C_i)|, i \in [1, T]$ is also finite thus we stop in a finite number of steps. The resulting graph we denote by Δ'_0. u_n-components in Δ'_0

are not u_n-folded in general, so we apply finitely many u_n-foldings to obtain Δ''_0 from Δ'_0 in which all u_n-components are u_n-folded. Then it might happen that $\Delta''_0(n-1)$ is not U-folded. By induction, $\Delta''_0(n-1)$ can be transformed by finitely many partial and u_m-foldings for $m < n$ into a graph Δ_1 so that $\Delta_1(n-1)$ is U-folded.

Thus one iteration of **Step 1** is complete and we can repeat all the procedures described above for Δ_1. To show that eventually the process converges we introduce the following characteristics of a $(\mathbb{Z}[t], X)$-graph Φ.

Let $M(\Phi) = P_1(\Phi) + P_2(\Phi) + P_3(\Phi)$, where $P_1(\Phi)$ is a number of u_n-components in Φ, $P_2(\Phi)$ is a number of vertices in Φ which belong to u_n-components, $P_3(\Phi) = \sum_{i=1}^{T} l_i$, where $T = P_1(\Phi)$ and $H_{u_n}(v_i) \cap \langle u_n \rangle = \langle u_n^{l_i} \rangle, l_i > 0, v_i \in C_i, C_1, \ldots, C_T$ is a complete list of u_n-components of Φ. Observe that $M(\Phi) \geq 0$ for any $(\mathbb{Z}[t], X)$-graph Φ.

Remark 6 *Notice that if C is a u_n-component in Δ_0, when we add a cycle labeled by a finite exponent of u_n to C and $H_{u_n}(v_i) \cap \langle u_n \rangle = \varepsilon$ then $P_3(\Gamma)$ can increase on the first iteration of* **Step 1**, *but on all further iterations it can only decrease because once the intersection $H_{u_n}(v_i) \cap \langle u_n \rangle$ becomes nontrivial it can only grow afterwards. So, there exists a natural number $N(\Delta_0)$, which depends only on Δ_0 and does not change with the repetitions of* **Step 1**, *such that starting from N-th iteration $P_3(\Gamma)$ always decreases.*

Thus we have $M(\Delta_0) \geq M(\Delta'_0) \geq M(\Delta''_0) \geq M(\Delta_1)$, where the last two inequalities follow from Lemma 31. $M(\Delta_0) = M(\Delta'_0)$ is possible only if **Step 1** did not change Δ_0, that is, no two different u_n-components were joined by a path labeled by $u_n^k, k \in \mathbb{Z}$ and abelian groups associated with any u_n-component in Δ_0 were not changed. In this case $\Delta_0 = \Delta_1$. Otherwise, if $M(\Delta_0) > M(\Delta'_0)$ then it implies that $M(\Delta_0) > M(\Delta_1)$, so after finitely many iterations of **Step 1** the process stops and we obtain a graph Δ_2 which has properties **(i)-(iv)**. Indeed

(i): Δ_2 is partially folded by construction.

(ii): if there exists a path p in $\Delta_2(n-1)$ with $\overline{\mu(p)} = u_n^k, k \in \mathbb{Z}$ which connects two different u_n-components C_1, C_2 in Δ_2 then, since $\Delta_2(n-1)$ is U-folded by construction, from the property **(ix)** of $\Delta_2(n-1)$ it follows that there exists a unique label reduced path q such that $o(q) = o(p), t(q) = t(p), \mu(q) = \widetilde{u_n}^k$, but then by our construction, C_1 and C_2 are joined by an edge e with $\mu(e) = u_n^k$ - contradiction;

(iii): if C is a u_n-component of Δ_2 then from our construction above it follows that for any edge e in P_C there exists a path p_e in $\Delta_2(n-1)$ such that $o(e) = o(p_e), t(e) = t(p_e), \overline{\mu(p_e)} = \mu(e)$, now the property **(iii)** of Δ_2 follows from the property **(ix)** of $\Delta_2(n-1)$;

(iv): follows from the construction and the property **(ix)** of $\Delta_2(n-1)$ - in the same way as **(iii)**.

Step 2. Let C be a u_n-component in Δ_2. Since C is u_n-folded then by definition there exists a positively-oriented path P_C associated with C such that $V(C) = V(P_C)$. There are only finitely many edges in P_C which are labeled by infinite exponents of u_n and let S_C denote the set of such edges. Let $v_1, v_2 \in V(C)$ be such that there exists a path $p = e_1 \cdots e_k$ in P_C, $o(p) = v_1, t(p) = v_2$ and there exists $j \in [1, k]$ such that $e_j \in S_C$.

Suppose there exists an integer k_p such that $\overline{\mu(p)} * u_n^{-k_p} \in H_{u_n}(v_1)$. If $H_{u_n}(v_1) \cap \langle u_n \rangle$ is trivial then k_p is unique. When $H_{u_n}(v_1) \cap \langle u_n \rangle$ is not trivial there are infinitely many numbers r such that $\overline{\mu(p)} * u^{-r} \in H_{u_n}(v_1)$ all of which belong to the coset $u^{k_p} * (H_{u_n}(v_1) \cap \langle u_n \rangle)$ and this ensures that it does not matter which one of them is chosen. Then we add to C an edge e such that $o(e) = v_1, t(e) = v_2, \mu(e) = u_n^{k_p}$ but eliminate an edge e_j from P_C. Observe that this operation does not change $H_{u_n}(v_1)$.

Since any vertex v in C can be connected to a base-point of z_C by a unique positively oriented label reduced path p_v then any two vertices v_1, v_2 in C can be connected by a path p_{v_1,v_2} which is a terminal subpath of p_{v_1} or p_{v_2} and every edge in P_C belongs to some path p_{w_1,w_2}, $w_i \in V(C)$, $i = 1, 2$. So, to minimize the number of infinite edges in P_C one has to check only finitely many paths.

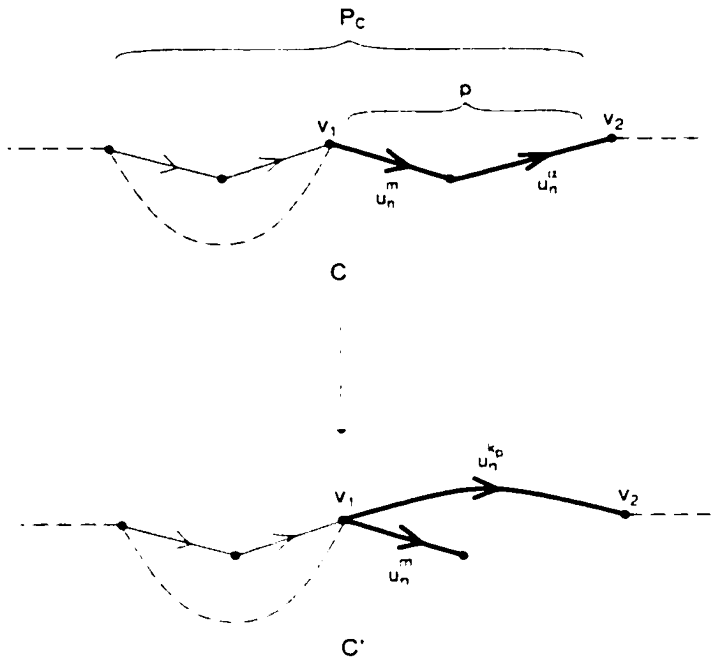


Figure 7: Step 2, $\alpha >> 1$.

If C'' is a u_n-component obtained from C' by all possible substitutions described above then C'' is connected by construction but may be not u_n-folded (see Figure 7).

We perform the described procedure for all u_n-components of Δ_2 and then using u_n-foldings obtain Δ'_2 where all u_n components are u_n-folded, but it might happen that $\Delta'_2(n-1)$ is not U-folded. Thus, by induction we obtain Δ'_2 such that $\Delta''_2(n-1)$ is U-folded. Finally applying **Step 1** finitely many times we obtain Δ_3 which has properties **(i)** - **(iv)** and one iteration of **Step 2** is complete.

Observe that **Step 2** does not increase $M(\Delta_2)$, so $M(\Delta_3) \leq M(\Delta_2)$.

If $P_1(\Phi) = \sum_{i=1}^{P_1(\Phi)} m(C_i)$, where $m(C_i)$ is a number of edges labeled by infinite exponents of u_n in P_{C_i} and C_i is a u_n-component of Φ. Then we have $P_1(\Delta_2) \geq$

$P_4(\Delta'_2) \geq P_4(\Delta''_2) \geq P_4(\Delta_3)$ because U-foldings do not increase the number of infinite edges. If $P_4(\Delta_2) = P_4(\Delta'_2)$ then it means that no u_n-component was changed and the process stops. Otherwise $P_4(\Delta_2) > P_4(\Delta'_2)$ and thus $P_4(\Delta_2) > P_4(\Delta_3)$.

After finitely many iterations of **Step 2** the process stops and we obtain a graph Δ_4 which has properties **(i)-(v)**.

Step 3. Let C be a u_n-component in Δ_3 and $v_1, v_2 \in V(C) \cap V(\Delta_3(n-1)), v_1 \neq v_2$. Suppose there exists a reduced path r in $\Delta_3(n-1)$ for which $o(r) = v_1, t(r) = v_2$ and $\overline{\mu(r)} = u_n^l, l > 0$. Since $\Delta_3(n-1)$ has property **(ix)**, we can assume r to be label reduced and $\mu(r) = \widetilde{u_n}^l$. Then, there exists a unique label reduced path q in $\Delta_3(n-1)$ for which $o(q) = v_1, t(q) = v_2$ and $\mu(q) = \widetilde{u_n}^k, l \geq k > 0$ and such that for any two initial subpaths q_1, q_2 of q such that $\mu(q_1) = \widetilde{u_n}^{k_1}, \mu(q_2) = \widetilde{u_n}^{k_2}, k_1 < k_2 < k$ we have $t(q_1) \neq t(q_2)$.

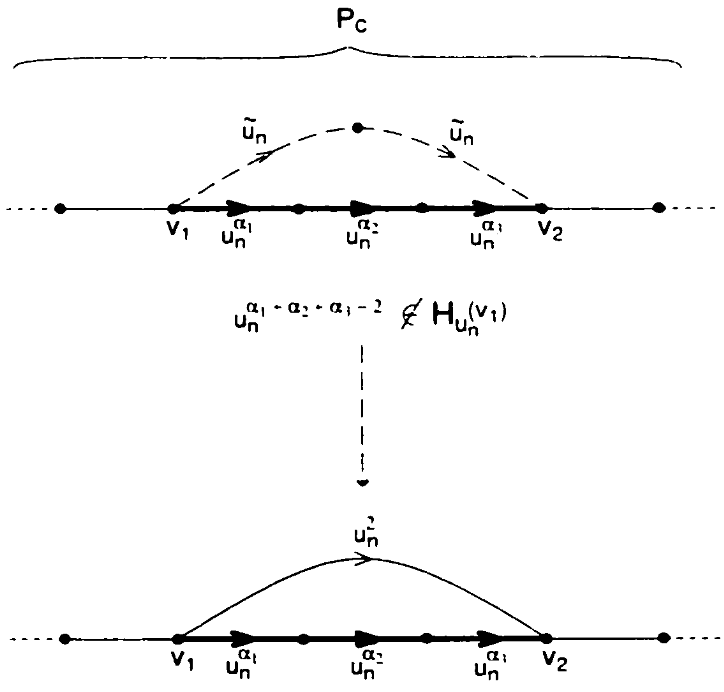


Figure 8: Step 3.

Since $v_1, v_2 \in V(C)$ then there exists a reduced path p in P_C such that $o(p) = v_1, t(p) = v_2, \overline{\mu(p)} = u_n^\alpha, \alpha \in \mathbb{Z}[t]^+$.

1. If p consists only of edges labeled by finite exponents of u_n then from the properties **(iii)**, **(iv)** of Δ_3 and from the fact that $\Delta_3(n-1)$ is U-folded it follows that there exists a unique label reduced path p' in $\Delta_3(n-1)$ such that $o(p') = o(p), t(p') = t(p), \mu(p') = \widetilde{u_n}^{\alpha}$. Hence, $o(q) = o(p'), t(q) = t(p')$. If p is positive then so is p' and we have $\mu(p') = \mu(q)$, otherwise we have a contradiction either with the choice of q or with the fact that $\Delta_3(n-1)$ is U-folded. Thus, by property **(ix)** of $\Delta_3(n-1)$ we have $q = p'$. If p is negative then p' is also negative and we have a positively oriented loop in $\Delta_3(n-1)$ at v_1 which is a concatenation qp'^{-1} so $\overline{\mu(qp'^{-1})} \in H_{u_n}(v_1)$ otherwise a contradiction with **(iv)**.

2. Suppose p contains an edge labeled by infinite exponent of u_n then $\overline{\mu(p)} * \overline{\mu(q)}^{-1} \notin H_{u_n}(v_1)$ - otherwise we have a contradiction with **(v)**. So we add to C an edge e labeled by u_n^k such that $o(e) = v_1, t(e) = v_2$ (see Figure 8). After this operation the new u_n-component C'' may not be u_n-folded, but the number of infinite edges in it can be reduced at least by one - an infinite edge which belongs to p can be deleted now. Thus $P_4(\Delta_3)$ can be reduced by applying **Step 2** while $P_1(\Delta_3), P_2(\Delta_3), P_3(\Delta_3)$ do not increase (see Remark 6).

After a finite number of steps one obtains a graph Δ_4 which satisfies conditions **(i)-(vi)**.

Step 4. Let C be a u_n-component in Δ_4. As in **Step 1** we find all paths labeled by $u_n^m, m \in \mathbb{Z}$ in $\Delta_4(n-1)$ which originate at vertices in C. The endpoints of these paths comprise a finite set B. Now we adjoin to C all vertices from B which do not belong to $V(C)$.

Let $v_1 \in V(C)$ and suppose there exists a reduced path r in $\Delta_4(n-1)$ for which $o(r) = v_1, t(r) = v_2, v_2 \in B, v_2 \notin V(C)$ and $\overline{\mu(r)} = u_n^l, l \in \mathbb{Z}$. Since $\Delta_4(n-1)$ has property **(ix)**, we can assume r to be label reduced and $\mu(r) = \widetilde{u_n}^l$. Without loss of generality we can assume $l > 0$.

Then, there exists a unique label reduced path p in $\Delta_4(n-1)$ for which $o(p) =$

$v_1, t(p) = v_2$ and $\mu(p) = \widetilde{u_n}^k, l \geq k > 0$ and such that for any two initial subpaths p_1, p_2 of p such that $\mu(p_1) = \widetilde{u_n}^{m_1}, \mu(p_2) = \widetilde{u_n}^{m_2}, m_1 < m_2 < k$ we have $t(p_1) \neq t(p_2)$.

We have several cases:

1. $u_n^k \in H_{u_n}(v_1)$

Since $\Delta_4(n-1)$ satisfies the property **(iv)**, that is, there exists a unique label reduced cycle q at v_1 in $\Delta_4(n-1)$ labeled by $H_{u_n}(v_1) \cap \langle u_n \rangle$. Hence $p = q$ because of the choice of p and the fact that $\Delta_4(n-1)$ is U-folded. So $v_1 = v_2$ - contradiction.

2. $u_n^k \notin H_{u_n}(v_1)$

Let v be any vertex in $V(C)$ and let $q_v = e_1 \ldots e_m$ be a label reduced subpath of P_C such that $o(q_v) = v, t(q_v) = v_1, \mu(e_i) = u_n^{\alpha_i}, i \in [1, m]$ (q_v exists since C is u_n-reduced).

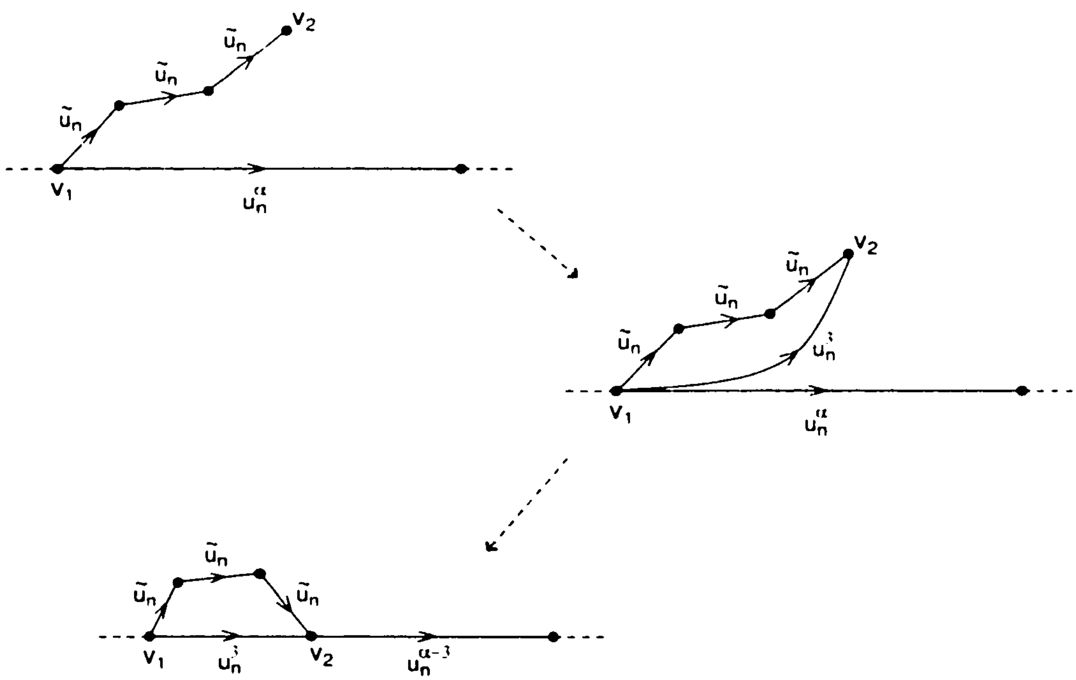


Figure 9: Step 4. 1a).

a) Suppose for any $v \in V(C)$ we have $\overline{\mu(q_v)} * u_n^k \notin H_{u_n}(v_1)$. Then we add to C an edge e such that $o(e) = v_1, t(e) = v_2, \mu(e) = u_n^k$ and then applying a single u_n-folding we fold e with P_C and obtained u_n-component is u_n-folded because of our assumption about k. Observe that the number of infinite edges does not increase and $H_{u_n}(v_1)$ does not change, however the number of vertices in C increases by one (see Figure 9). Also, after this operation $\Delta_4(n-1)$ stays U-folded - v_2 is not identified with any vertex in Δ_4.

b) Suppose there exists $v \in V(C)$ such that $\overline{\mu(q_v)} * u_n^k \in H_{u_n}(v_1)$. Then all edges in q_v are labeled by finite exponents of u_n - otherwise C is not minimal with respect to the number of infinite edges. Since Δ_4 has properties **(iii)**, **(iv)** and $\Delta_4(n-1)$ is U-folded it means that p connects two vertices in C and, hence, $v_2 \in C$ - contradiction.

So, we add to C all vertices from B in finitely many steps and the same can be done for all u_n-components of Δ_4. This procedure leaves $\Delta_4(n-1)$ to be U-folded, but increases $M(\Delta_4)$. After all possible vertices are adjoined to u_n-components, the resulting graph Δ_5 has properties **(i)-(vii)**.

Step 5. Let C be a u_n-component of Δ_5, $v \in V(C)$. Since Δ_5 has all the properties **(i)-(vii)** and $\Delta_5(n-1)$ is U-folded, there exists no reduced path p starting at $v \in V(C)$ such that $\overline{\mu(p)} = u_n^{\pm 1}$ and $t(p) \notin V(C)$.

(a) Let p be a label reduced path in $\Delta_5(n-1)$ such that $o(p) = v, \overline{\mu(p)} = w, w = u_n^\delta \circ c, \delta = 1$ (the case when $\delta = -1$ can be considered in the same way as shown below).

By definition, $\widetilde{u_n}$ is a word in the alphabet $\{X \cup X^{-1}\} \cup \{u^\alpha \mid u \in U(\Delta_5), u < u_n, \alpha \in \mathbb{Z}[t]^+\}$. We have

$$\widetilde{u_n} = \widetilde{h_1} u_{n-1}^{\alpha_1} \widetilde{h_2} \cdots u_{n-1}^{\alpha_m} \widetilde{h_{m+1}}.$$

At first, comparing $\mu(p)$ and $\widetilde{u_n}$, we prove the following claim.

Claim. Either there exists a label reduced path $q = q_1 q_2$ in $\Delta(n-1)$ such that $o(q) = v, t(q) = t(p), \mu(q_1) = \widetilde{u_n}$ or $p = z_1 z_2, \mu(z_1) = \widetilde{h_1} u_{n-1}^{\alpha_1} \widetilde{h_2} \cdots \widetilde{h_m}, \overline{\mu(z_2)} =$

$u_{n-1}^{\alpha_m} \circ h_{m+1} \circ c, u_{n-1} = h_{m+1} \circ u'$.

Let $p = d_1 p_1 d_2 \cdots d_k p_k d_{k+1}$, where p_i is a path in some u_{n-1}-component of $\Delta_5(n-1)$ and d_i is a path in $\Delta_5(n-1)$ which does not contain edges labeled by $u_{n-1}^{\alpha}, \alpha \in \mathbb{Z}[t]^+$. We can assume that there exists no initial subpath d'_i of $d_i p_i \cdots d_k p_k d_{k+1}$ such that $\overline{\mu(d'_i)} = u_{n-1}^{\pm 1}$ and there exists no terminal subpath d''_i of d_i such that $\overline{\mu(d''_i)} = u_{n-1}^{\pm 1}$ - otherwise, since $\Delta_5(n-1)$ is U-folded, by induction we can assume that $t(d'_i)$ and $o(d''_i)$ belong to some u_{n-1}-components and there exist edges e', e'' such that $o(e') = o(d'_i), t(e') = t(d'_i), \mu(e') = u_{n-1}^{\pm 1}, o(e'') = o(d''_i), t(e'') = t(d''_i), \mu(e'') = u_{n-1}^{\pm 1}$, so that we can substitute d'_i, d''_i by e', e'' correspondingly and consider the path p' obtained from p by such substitutions. Let $\overline{\mu(p_i)} = u_{n-1}^{\beta_i}$.

Since $\widetilde{u_n}$ is the standard decomposition of u_n, h_1 does not have u_{n-1} as a terminal segment. Observe that $|\overline{\mu(d_1)}| \geq |h_1|$ because of the choice of h_1.

a) If $|\overline{\mu(d_1)}| = |h_1|$ then automatically $\overline{\mu(d_1)} = h_1$. Now, either $\beta_1 \leq \alpha_1$ or $\beta_1 > \alpha_1, u_{n-1} = h_2 \circ u', u' \neq \varepsilon$ and $\widetilde{u_n} = \tilde{h}_1 u_{n-1}^{\alpha_1} \tilde{h}_2$ - otherwise a contradiction with the choice of h_2.

In the former case, if $\beta_1 = \alpha_1$ then we proceed with d_2, as before with d_1, but if $\beta_1 < \alpha_1$, then $\overline{\mu(d_2)}$ contains u_{n-1} as an initial segment and by induction there exists a path $r = r_1 r_2$ for d_2 such that $o(d_2) = o(r), t(d_2) = t(r), \mu(r_1) = \widetilde{u_{n-1}}$ and $t(r_1) = o(r_2), t(p_1)$ belong to the same u_{n-1}-component D. So, we have either a path $d_1 p_1$ or a path $d_1 p_1 r_1$ labeled by $\tilde{h}_1 u_{n-1}^{\alpha_1}$.

In the latter case, $u_n = h_1 \circ u_{n-1}^{\alpha_1} \circ h_2$ and $p = z_1 z_2$, where $\mu(z_1) = \tilde{h}_1$ and $\overline{\mu(z_2)} = u_{n-1}^{\alpha_1} \circ h_2 \circ c, u_{n-1} = h_2 \circ u'$.

b) If $|\overline{\mu(d_1)}| > |h_1|$ then by the stabilizing condition we should have $\overline{\mu(d_1)} = h_1 \circ u_{n-1}^{k}, k \in \mathbb{Z}$, where the sign of k depends on the sign of α_1 and we can assume $k > 0$. In fact, because of our assumption about d_i we have $k = 1$. By induction there exists a path $r = r_1 r_2$ for d_1 such that $o(d_1) = o(r), t(d_1) = t(r), \mu(r_2) = \widetilde{u_{n-1}}, t(r_1) = o(r_2)$ belongs to some u_{n-1}-component D. Observe that $t(r_1)$ and $t(d_1)$ belong to the same

u_{n-1}-component D.

We proceed with p_1. We have either $\beta_1+1 \leq \alpha_1$ or $\beta_1+1 > \alpha_1$, $u_{n-1} = h_2 \circ u'$, $u' \neq \varepsilon$ and $\widetilde{u_n} = \widetilde{h_1} u_{n-1}^{\alpha_1} \widetilde{h_2}$ - otherwise a contradiction with the choice of h_2.

If $\beta_1+1 = \alpha_1$ then we proceed with d_2, as before with d_1. If $\beta_1+1 < \alpha_1$, then $\overline{\mu(d_2)}$ contains u_{n-1} as an initial segment and by induction there exists a path $r' = r'_1 r'_2$ for d_2 such that $o(d_2) = o(r'), t(d_2) = t(r'), \mu(r'_1) = \widetilde{u_{n-1}}$ and $t(r'_1) = o(r'_2), t(p_1)$ belong to the same u_{n-1}-component D. So, we have either a path $r_1 r_2 p_1$ or a path $r_1 r_2 p_1 r'_1$ labeled by $\widetilde{h_1} u_{n-1}^{\alpha_1}$.

If $\beta_1 + 1 > \alpha_1$, $u_{n-1} = h_2 \circ u'$, $u' \neq \varepsilon$ and $\widetilde{u_n} = \widetilde{h_1} u_{n-1}^{\alpha_1} \widetilde{h_2}$ then $u_n = h_1 \circ u_{n-1}^{\alpha_1} \circ h_2$, $p = z_1 z_2$, where $\mu(z_1) = \widetilde{h_1}$ and $\overline{\mu(z_2)} = u_{n-1}^{\alpha_1} \circ h_2 \circ c$.

We have considered the first step and now the proof of the Claim follows by induction on m.

The Claim above provides us with two possible cases.

1. There exists a label reduced path $q = q_1 q_2$ in $\Delta(n-1)$ such that $o(q) = v, t(q) = t(p), \mu(q_1) = \widetilde{u_n}$.

1.1. There exists an edge e' in P_C such that $o(e') = v, \mu(e') = u_n^{\gamma}, \gamma > 0$.

Observe that if $H_{u_n}(v) \cap \langle u_n \rangle$ is not trivial or $\gamma \in \mathbb{Z}$ then by the properties **(iii)**, **(iv)**, there exists a label reduced path p' labeled by $\widetilde{u_n}$ such that $t(p') \in V(C)$. Since $\Delta_5(n-1)$ is U-folded we have $q_1 = p'$ and $t(q_1) \in V(C)$.

So, assume that $H_{u_n}(v) \cap \langle u_n \rangle$ is trivial and $\gamma >> 1$.

We divide e' by a new vertex v' into two edges e'_1 and e'_2 so that $o(e'_1) = o(e'), t(e'_1) = v', o(e'_2) = v', t(e'_2) = t(e'), \mu(e'_1) = u_n, \mu(e'_2) = u_n^{\gamma-1}$. The resulting u_n-component we denote by C' and now we check that C' is u_n-folded.

Let y be any vertex in $V(C)$ and let $q_y = e_1 \cdots e_m$ be a reduced subpath of P_C such that $o(q_y) = y, t(q_y) = v, \mu(e_i) = u_n^{\gamma_i}, i \in [1, m]$. We have two cases.

a) Suppose for any $y \in V(C)$ we have $\overline{\mu(q_y)} * u_n \notin H_{u_n}(v)$ if q_y is positively oriented and $\overline{\mu(q_y)} * u_n^{-1} \notin H_{u_n}(v)$ if q_y is negatively oriented. Then, it means automatically that C'' is u_n-folded. Finally, we identify v' with $t(q_1)$ and denote the resulting graph by Δ_6 (see Figure 10). Observe that $\Delta_6(n-1)$ is U-folded.

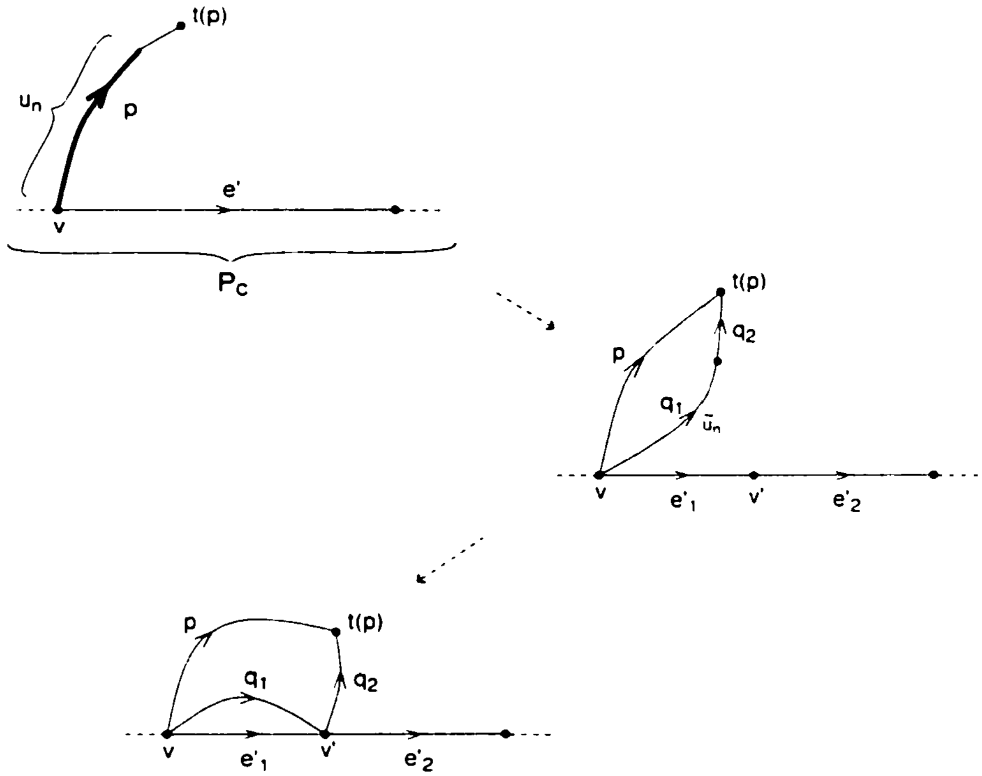


Figure 10: Step 5(a). 1.1

b) Suppose there exists $y \in V(C)$ such that $\overline{\mu(q_y)} * u_n \in H_{u_n}(v)$ and q_y is positively oriented (the case when $\overline{\mu(q_y)} * u_n^{-1} \in H_{u_n}(v)$ and q_y is negatively oriented is considered in the same way). Then all edges in q_y are labeled by finite exponents of u_n - otherwise C is not minimal with respect to the number of infinite edges. But in this case we have a contradiction with the assumption that $H_{u_n}(v) \cap \langle u_n \rangle$ is trivial because $\overline{\mu(q_y)} * u_n \neq \varepsilon$.

1.2. There exists no edge e' in P_C such that $o(e') = v, \mu(e') = u_n^{\gamma}, \gamma > 0$.

We can assume that $H_{u_n}(v) \cap \langle u_n \rangle$ is trivial, otherwise, by the property **(iv)**, there exists a label reduced path p' labeled by $\widetilde{u_n}$ such that $t(p') \in V(C)$. Since $\Delta_5(n-1)$ is U-folded we have $q_1 = p'$ and $t(q_1) \in V(C)$

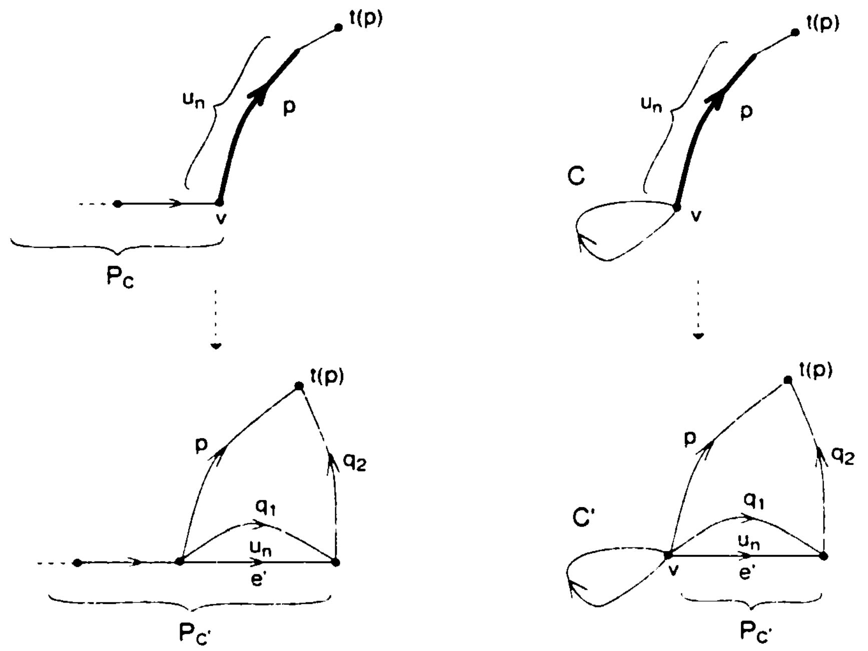


Figure 11: Step 5(a). 1.2: $P_{C'} \neq \emptyset$ (left), $P_{C'} = \emptyset$ (right)

We add an edge e' to $\Delta_5(n-1)$ so that $o(e') = v, t(e') = t(q_1), \mu(e') = u_n$. The resulting graph we denote by Δ_6 (see Figure 11).

Observe that the resulting u_n-component $C'' = C \cup \{e'\}$ is u_n-folded, otherwise, by the same argument as in 1.1.b) we get a contradiction. Also, it is easy to see that $\Delta_6(n-1)$ is U-folded.

2. $p = z_1 z_2$, $\mu(z_1) = \widetilde{h_1} u_{n-1}^{\alpha_1} \widetilde{h_2} \cdots \widetilde{h_m}$, $\overline{\mu(z_2)} = u_{n-1}^{\alpha_m} \circ h_{m+1} \circ c$, $u_{n-1} = h_{m+1} \circ u'$.

Observe that $t(z_1) \in V(D)$ for some u_{n-1}-component D of $\Delta_5(n-1)$.

We add edges e_1, e_2, vertices v_1, v_2 and a path r to $\Delta_5(n-1)$ so that $o(e_1) = t(z_1), o(e_2) = t(e_1) = v_1, t(e_2) = v_2, o(r) = v_1, t(r) = v_2, \mu(e_1) = u_{n-1}^{\alpha_m}, \mu(e_2) = u_{n-1}, \mu(r) = \widetilde{u_{n-1}}$. The resulting graph we denote by Δ'_5.

In fact we added to D a path $e_1 e_2$ and it can happen that $D \cup \{e_1 e_2\}$ is not u_{n-1}-folded. Using u_{n-1}-foldings we make $D \cup \{e_1 e_2\}$ u_{n-1}-folded. The result of these u_{n-1}-foldings we denote by Δ'_6. So, there exists a label reduced path $q = q_1 q_2$ in Δ'_6 such that $o(q) = v, t(q) = t(p), \mu(q_1) = \widetilde{u_n}$ we just have to attach $t(q_1)$ to $V(C)$

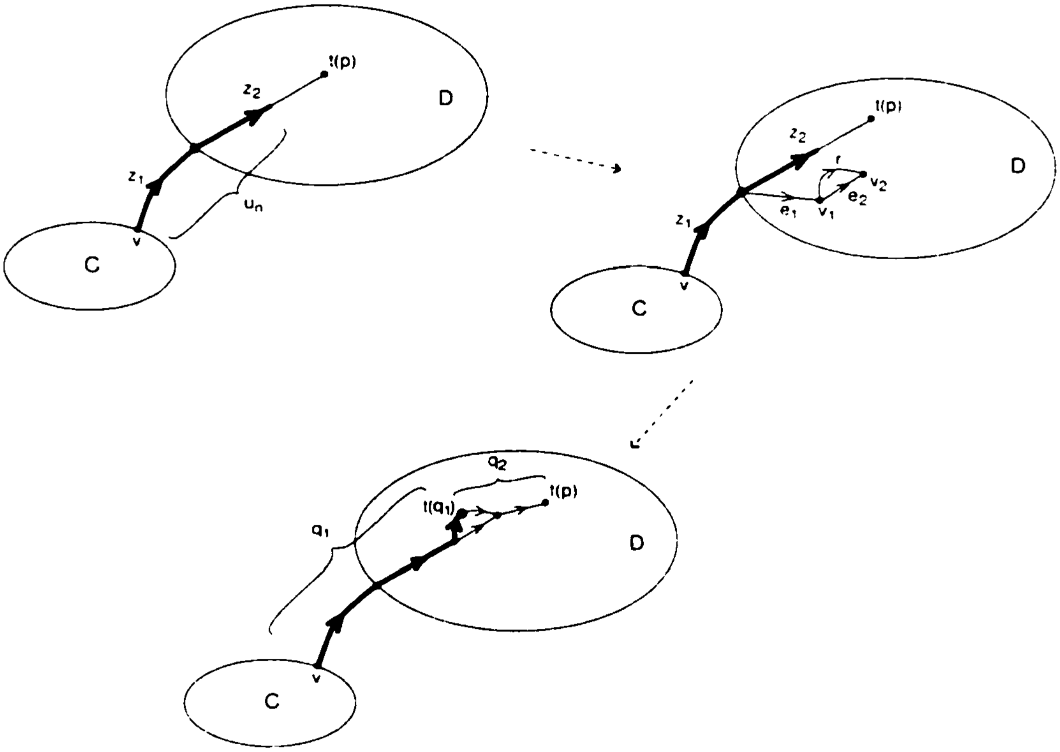


Figure 12: Step 5(a). 2.

(see Figure 12). In fact, we are now in the conditions of case 1, and applying the same argument we obtain a new graph Δ_6 where there exists a label reduced path $q = q_1q_2$ for p in Δ_6 such that $o(q) = v, t(q) = t(p), \mu(q_1) = \widetilde{u_n}$ and $t(q_1) \in V(C)$.

(b) We have $p = z_1z_2, z_1 \in \Delta_5(n-1), \overline{\mu(z_1)} = w_1, u_n = w_1 \circ w_2, \overline{\mu(z_2)} = w_2 \circ c = u_n^\gamma \circ c_1$. That is, $t(z_1) \in V(D)$, where D is a u_n-component of $\Delta_5(n)$.

1. There exists an edge e' in P_D such that $o(e') = t(z_1), \mu(e') = u_n^\gamma, \gamma > 0$.

Without loss of generality we can assume $z_2 = e'$.

Observe that if $H_{u_n}(t(z_1)) \cap \langle u_n \rangle$ is not trivial or $\gamma \in \mathbb{Z}$ then by the properties **(iii)**, **(iv)**, there exists a label reduced path p' labeled by $\widetilde{u_n}$ such that $o(p') = t(z_1), t(p') \in V(D)$. Obviously $\overline{\mu(z_1p')} = u_n \circ c'$ and everything reduces to the case **(a)**. After applying arguments of **(a)** we obtain a new graph Δ_6 in which p satisfies the property **(viii)**.

So, assume that $H_{u_n}(t(z_1)) \cap \langle u_n \rangle$ is trivial and $\gamma >> 1$.

We divide e' by a new vertex v' into two edges e'_1 and e'_2 so that $o(e'_1) = o(e'), t(e'_1) = v', o(e'_2) = v', t(e'_2) = t(e'), \mu(e'_1) = u_n, \mu(e'_2) = u_n^{\gamma-1}$. The resulting graph we denote by Δ'_5, the resulting u_n-component we denote by D' and now we have to check if D' is u_n-folded.

Let y be any vertex in $V(D)$ and let $q_y = e_1 \cdots e_m$ be a reduced subpath of P_D such that $o(q_y) = y, t(q_y) = v, \mu(e_i) = u_n^{\gamma_i}, i \in [1, m]$. We have two cases.

a) Suppose for any $y \in V(D)$ we have $\overline{\mu(q_y)} * u_n \notin H_{u_n}(t(z_1))$ if q_y is positively oriented and $\overline{\mu(q_y)} * u_n^{-1} \notin H_{u_n}(t(z_1))$ if q_y is negatively oriented. Then, it means automatically that D' is u_n-folded. We add a path p' to Δ'_5 so that $o(p') = t(z_1), t(p') = v', \mu(p') = \widetilde{u_n}$ (see Figure 13). We denote the resulting graph by Δ'_6 and $\overline{\mu(z_1 p')} = u_n \circ c'$ so everything reduces to the case **(a)**. After applying arguments of **(a)** we obtain a new graph Δ_6 in which p satisfies the property **(viii)**.

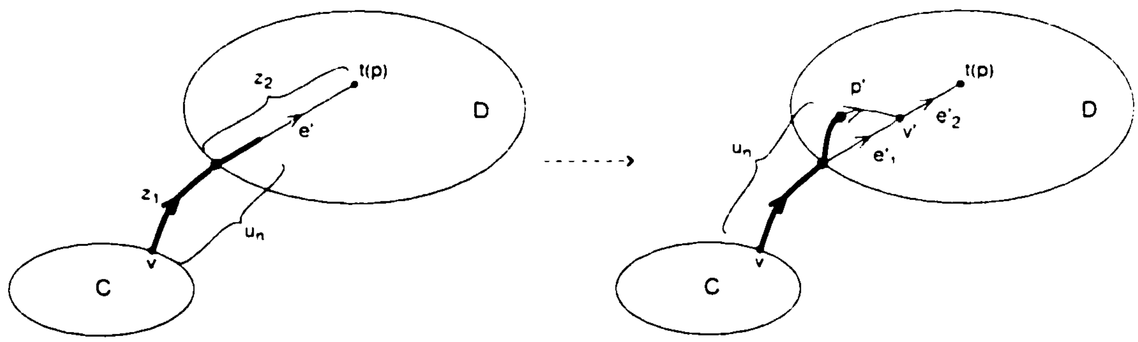


Figure 13: Step 5(b). 1.

b) Suppose there exists $y \in V(D)$ such that $\overline{\mu(q_y)} * u_n \in H_{u_n}(t(z_1))$ and q_y is positively oriented (the case when $\overline{\mu(q_y)} * u_n^{-1} \in H_{u_n}(t(z_1))$ and q_y is negatively oriented is considered in the same way). Then all edges in q_y are labeled by finite exponents of u_n - otherwise D is not minimal with respect to the number of infinite edges. But in this case we have a contradiction with the assumption that $H_{u_n}(t(z_1)) \cap \langle u_n \rangle$ is trivial because $\overline{\mu(q_y)} * u_n \neq \varepsilon$.

2. There exists no edge e' in P_D such that $o(e') = t(z_1), \mu(e') = u_n^{\gamma}, \gamma > 0$.

We can assume that $H_{u_n}(t(z_1)) \cap \langle u_n \rangle$ is trivial, otherwise, by the property **(iv)**, there exists a label reduced path p' labeled by $\widetilde{u_n}$ such that $o(p') = t(z_1), t(p') \in V(D)$. Then, $\overline{\mu(z_1 p')} = u_n \circ c'$ and everything reduces to the case **(a)**.

We add an edge e', a vertex v_1 and a path r to $\Delta_5(n-1)$ so that $o(e') = o(r) = t(z_1), t(e') = t(r) = v_1, \mu(e') = u_n, \mu(r) = \widetilde{u_n}$. The resulting graph we denote by Δ'_5.

In fact we add to D an edge e' and it can happen that $D \cup \{e'\}$ is not u_n-folded. Using u_n-foldings we make $D \cup \{e'\}$ u_n-folded. The result of these u_n-foldings we denote by Δ'_6 (see Figure 14). So, there exists a label reduced path $q = z_1 r$ in $\Delta'_6(n-1)$ such that $o(z_1) = v, t(p') = t(e'), \overline{\mu(z_1 r)} = u_n \circ c'$ and everything reduces to the case **(a)**. After applying arguments of **(a)** we obtain a new graph Δ_6 in which p satisfies the property **(viii)**.

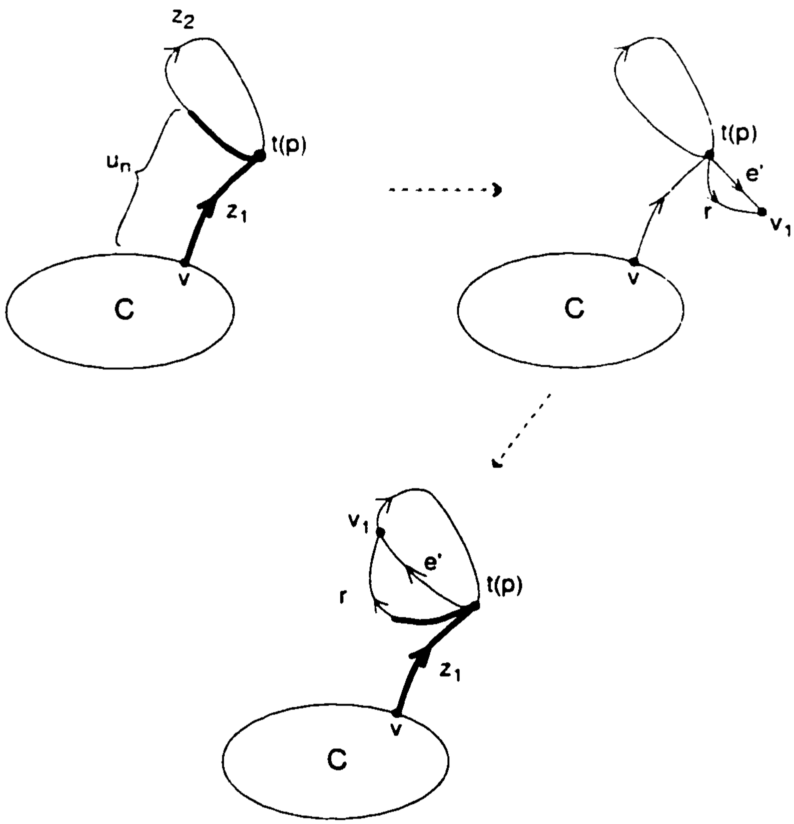


Figure 14: Step 5(b). 2.

(c) We have $o(p) = v, \overline{\mu(p)} = w_1 u_n^{\delta} = w_1 \circ w_2, w_2 \neq \varepsilon, \delta \in \{1, -1\}$. Without loss of generality we can assume $\delta = 1$.

Also, there exists an edge e' in C such that $o(e') = v, \mu(e') = u^{\gamma}, \gamma > 0$.

1. $e' \in P_C$.

Observe that if $H_{u_n}(v) \cap \langle u_n \rangle$ is not trivial or $\gamma \in \mathbb{Z}$ then by the properties **(iii)**, **(iv)**, there exists a label reduced path p' labeled by $\widetilde{u_n}$ such that $t(p') \in V(C)$ and p is an initial subpath of p'.

Further we assume that $H_{u_n}(v) \cap \langle u_n \rangle$ is trivial and $\gamma >> 1$.

We divide e' by a new vertex v' into two edges e'_1 and e'_2 so that $o(e'_1) = o(e'), t(e'_1) = v', o(e'_2) = v', t(e'_2) = t(e'), \mu(e'_1) = u_n, \mu(e'_2) = u_n^{\gamma-1}$. The resulting u_n-component we denote by C'' and now we have to check if C'' is u_n-folded.

Let y be any vertex in $V(C)$ and let $q_y = e_1 \cdots e_m$ be a reduced subpath of P_C such that $o(q_y) = y, t(q_y) = v, \mu(e_i) = u_n^{\gamma_i}, i \in [1, m]$. We have two cases.

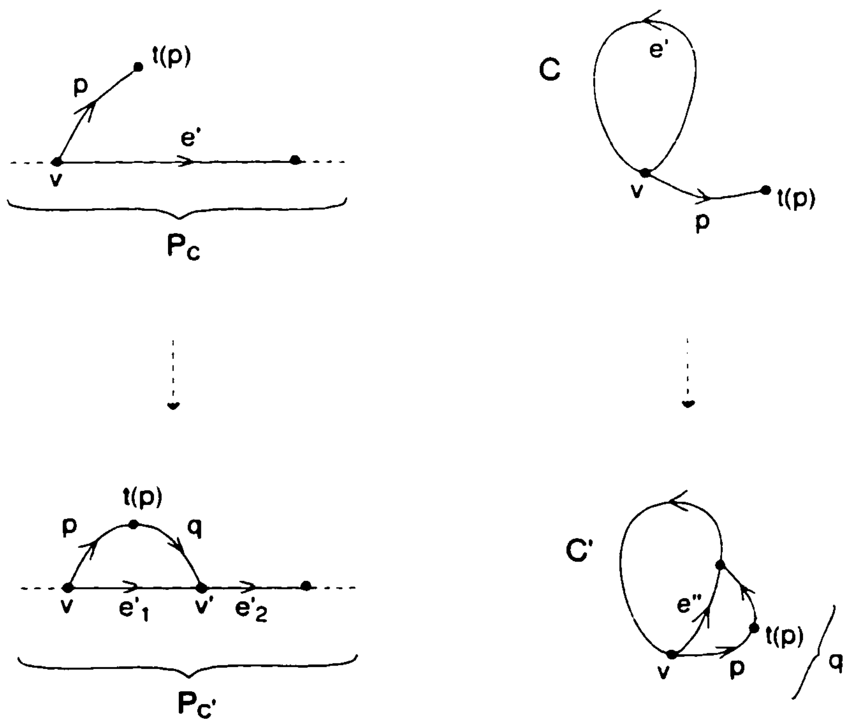


Figure 15: Step 5(c): 1 (left), 2 (right).

a) Suppose for any $y \in V(C)$ we have $\overline{\mu(q_y)} * u_n \notin H_{u_n}(v)$ if q_y is positively oriented and $\overline{\mu(q_y)} * u_n^{-1} \notin H_{u_n}(v)$ if q_y is negatively oriented. Then, it means automatically that C'' is u_n-folded. Finally, take a path q with $\overline{\mu(q)} = w_2$ and we form a new graph $\Delta'_6 = \Delta_5 \cup \{q\}$ so that $o(q) = t(p), t(q) = v'$ (see Figure 15).

$\Delta'_6(n-1)$ may be not U-folded. Using u-foldings for $u < u_n$ we can make $\Delta'_6(n-1)$ U-folded. Observe that these u-foldings do not affect u_n-components, so the resulting graph Δ_6 has all the properties **(i)-(vii)** $\Delta_6(n-1)$ is U-folded. Moreover, there exists a label reduced path p' $\Delta_6(n-1)$, labeled by u_n such that $t(p') \in V(C)$ and p is an initial subpath of p'.

b) Suppose there exists $y \in V(C)$ such that $\overline{\mu(q_y)} * u_n \in H_{u_n}(v)$ and q_y is positively oriented (the case when $\overline{\mu(q_y)} * u_n^{-1} \in H_{u_n}(v)$ and q_y is negatively oriented is considered in the same way). Then all edges in q_y are labeled by finite exponents of u_n - otherwise C is not minimal with respect to the number of infinite edges. But in this case we have a contradiction with the fact that $H_{u_n}(v) \cap \langle u_n \rangle$ is trivial because $\overline{\mu(q_y)} * u_n \neq \varepsilon$.

2. $e' \notin P_C$

$H_{u_n}(v) \cap \langle u_n \rangle$ is trivial, otherwise, by the property **(iv)**, there exists a label reduced path p' labeled by $\widetilde{u_n}$ such that $t(p') \in V(C)$ and p is an initial subpath of p'.

Assume that $H_{u_n}(v) \cap \langle u_n \rangle$ is trivial and $\gamma >> 1$.

We add an edge e'', a vertex v' and a path q to $\Delta_5(n-1)$ so that $o(e'') = v, t(e'') = v', \mu(e'') = u_n, o(q) = v, t(q) = v', \mu(q) = \widetilde{u_n}$ (see Figure 15). The resulting graph we denote by Δ'_6.

Observe that the resulting u_n-component $C'' = C \cup \{e'', v'\}$ is u_n-folded, otherwise, by the same argument as in 1.b) we get a contradiction. But, $\Delta'_6(n-1)$ is not U-folded. We perform necessary u-foldings for $u < u_n$ between q and as a result we obtain a new graph Δ_6 in which there exists a path $q = q_1 q_2$ starting at v such that

$\mu(q) = \widetilde{u_n}.t(q) \in V(C)$ and $p = q_1$ is an initial subpath of q.

We can apply the procedure described above to all cases in Δ_6 when **(viii)** is not satisfied. After all these operations we denote the resulting graph by Δ_7. Observe that Δ_7 has all the properties **(i)**-**(viii)** and $\Delta_7(n-1)$ is U-folded.

Step 6. We verify **(ix)** for $\Delta_7(n)$.

Let p be a path in $\Delta_7(n)$ with $\overline{\mu(p)} = w$. $\mu(p)$ belongs to the alphabet $\{X \cup X^{-1}\} \cup \{u^\alpha \mid u \in U(\Delta_7), u \leq u_n, \alpha \in \mathbb{Z}[t]^+\}$ so let us subdivide p in the following way

$$p = p_1 d_1 p_2 d_2 \cdots p_m d_m p_{m+1},$$

where all edges in p_i do not contain edges labeled by $u_n^\alpha, \alpha \in \mathbb{Z}[t]^+$, so that, we can assume p_i to be a path in $\Delta_7(n-1)$ and d_i contains only edges labeled by $u_n^\alpha, \alpha \in \mathbb{Z}[t]^+$.

Take any p_i. By induction there exists a unique label reduced path q_i such that $o(p_i) = o(q_i), t(p_i) = t(q_i), \mu(q_i) = \tilde{h}_i$, where $\overline{\mu(p_i)} = h_i$. Observe that since $t(p_i)$ belongs to some u_n-component C_i and d_i is a path composed only of edges labeled by exponents of u_n then d_i is a path in C_i such that $\overline{\mu(d_i)} = u_n^{\alpha_i}$. By Lemma 35 there exists a unique path c_i in P_{C_i} such that $o(c_i) = o(d_i)$ and $u_n^{\alpha_i} \in \overline{\mu(c_i)} * H_{u_n}(o(d_i))$. We have that $t(d_i) = t(c_i)$. Indeed, if d_i leads to some other vertex $v \in P_{C_i}$ then there exists a unique positively oriented subpath p_v of P_{C_i} from $o(d_i)$ to v and we have a loop $p_v d_i^{-1}$, so $\overline{\mu(p_v)} * u_n^{-\alpha_i} \in H_{u_n}(o(d_i))$. Thus $u_n^{\alpha_i} = \overline{\mu(p_v)} * h_i, h_i \in H_{u_n}(o(d_i))$ and $\overline{\mu(p_v)} \in \overline{\mu(c_i)} * H_{u_n}(o(d_i))$ which is impossible unless $p_v = c_i$, because C_i is u_n-reduced. So $t(d_i) = t(c_i) = o(p_{i+1})$.

Thus we obtain a reduced path r as the following concatenation

$$q' = q_1 c_1 q_2 c_2 \cdots q_m c_m q_{m+1},$$

where $o(q_i) = o(p_i), t(q_i) = t(p_i), o(c_i) = o(d_i), t(c_i) = t(d_i), \overline{\mu(q_i)} = \overline{\mu(p_i)}, \overline{\mu(c_i)} =$

$\overline{\mu(d_i)} = u_n^{\alpha_i}, i \in [1, m+1]$ and $o(q') = o(p), t(q') = t(p), \overline{\mu(q')} = \overline{\mu(p)}$ and all q_i, c_i are label reduced.

However, q' may be not label reduced, that is, there can be a cancellation in $\overline{\mu(c_i)} * \overline{\mu(q_{i+1})}$. Without loss of generality we can assume $\alpha_i >> 1$ (if not then by **(iii)** we can assume $p_i c_i p_{i+1}$ to be a path in $\Delta_7(n-1)$ and everything follows by induction for $p_i c_i p_{i+1}$).

Suppose $\overline{\mu(c_i)} * \overline{\mu(q_{i+1})} \neq \overline{\mu(c_i)} \circ \overline{\mu(q_{i+1})}$. Then $\overline{\mu(q_{i+1})}$ contains an initial subword $u_n^k \circ g_1, k < 0, u_n^{-1} = g_1 \circ g_2$. Since $\Delta_7(n-1)$ is U-folded, by **(vii)**,**(viii)**, **(ix)** it follows that there exists a path $r_{i+1} = z_{i+1} f_{i+1}$ such that $o(r_{i+1}) = o(q_{i+1}), t(r_{i+1}) = t(q_{i+1}), o(z_{i+1}) = v_1 \in V(C_i), \mu(z_{i+1}) = \widetilde{u_n}^k \tilde{g}_1$ and it can be continued by a path z' to some vertex $v_2 \in V(C_i)$ so that $z_{i+1} z'$ is a label reduced path from v_1 to v_2. Let w_{i+1} be the path in P_{C_i} such that $o(w_{i+1}) = v_1, t(w_{i+1}) = v_2$. Then we have that concatenation $w_{i+1} z'^{-1}$ is a path from $t(c_i) = o(q_{i+1}) = o(z_{i+1})$ to $t(z_{i+1}) = o(f_{i+1})$ and $\overline{\mu(w_{i+1} z'^{-1})} = \overline{\mu(z_{i+1})} = u_n^k \circ g_1$. So we can substitute q_{i+1} by $w_{i+1} z'^{-1} f_{i+1}$ in q. Observe that $\overline{\mu(z'^{-1})} = g_2^{-1}$ and $z'^{-1} f_{i+1}$ is label reduced otherwise $u_n^k \circ g_1$ is not the maximal initial subword of $\overline{\mu(q_{i+1})}$ which cancels in $\overline{\mu(c_i)} * \overline{\mu(q_{i+1})}$. Also, $u_n * g_2^{-1} = u_n \circ g_2^{-1}$.

Thus we have that $c_i q_{i+1}$ becomes $c_i w_{i+1} z'^{-1} f_{i+1}$. We can find a unique label reduced path b_i in C_i which corresponds to $c_i w_{i+1}$ and then $b_i z'^{-1} f_{i+1}$ is label reduced.

If $\overline{\mu(q_i)} * \overline{\mu(c_i)} \neq \overline{\mu(q_i)} \circ \overline{\mu(c_i)}$ then we can substitute q_i by $f_i z''^{-1} w_i$ and using the same argument as above show that we obtain a label reduced path $f_i z''^{-1} b_i$ which corresponds to $q_i c_i$.

After finitely many such substitutions we get a label reduced path q such that $o(q) = o(p), t(q) = t(p), \overline{\mu(q)} = \overline{\mu(p)} = w$. Finally, using the property **(viii)**: **(a)**,**(b)** (similarly to the Claim in **Step 5**) one can construct a unique label reduced path q' such that $o(q') = o(p), t(q') = t(p), \mu(q') = \tilde{w}$.

Step 7. Let $g \in F^{\mathbb{Z}[t]}$ be such that

$$\tilde{g} = \tilde{h}_1 u^{\alpha_1} \tilde{h}_2 \cdots u^{\alpha_m} \widetilde{h_{m+1}}.$$

We check **(x)** for $\Delta_7(n)$.

If $u < u_n$ then **(x)** follows by induction. Suppose $u = u_n$. By induction either there exists a unique label reduced path p_1 in $\Delta_7(n-1)$ for h_1 starting at $v \in \Delta_7(n)$ such that $\mu(p_1) = \tilde{h}_1$ or for any path q_1 in Δ_7 starting at v it follows that $\overline{\mu(q_1)} \neq h_1$. In the latter case there exists no path p for g such that $\overline{\mu(p)} = g$, because if it exists then by **(ix)** there exists a unique path q such that $o(q) = o(p), t(q) = t(p), \mu(q) = \tilde{g}$. But q contains an initial subpath q_1 starting at v with the label $\mu(q_1) = \tilde{h}_1$ - contradiction. In the former case we continue with u^{α_1}. Since $t(p_1)$ belongs to some u-component C_1 of $\Delta_7(n)$ - by Lemma 35 either there exists a unique path c_1 in P_{C_1} such that $o(c_1) = t(p_1)$ and $u^{\alpha_1} \in \overline{\mu(c_1)} * H_u(o(c_1))$ or there exists no continuation of p_1 in C_1 which is labeled by u^{α_1}. Again, if this continuation does not exist then there exists no required path for g, if it does exist then we continue.

Eventually we either construct the required unique path for g or there exists no such path at all.

The induction step is complete and the theorem is proved.

□

It can be seen easily from the Proposition 4 that properties of U-folded graphs are similar to the properties of folded X-graphs introduced in [24]. In [24] X-graphs are used for studying the structure of subgroups in free groups and for solving various algorithmic and combinatorial problems for free groups and their subgroups. In the next subsection we try to do the same for finitely generated subgroups of $F^{\mathbb{Z}[t]}$ using $(\mathbb{Z}[t], X)$-graphs.

17.3 Subgroup graphs and applications of U-folded $(\mathbb{Z}[t], X)$-graphs

In Subsection 17.1 (see Lemma 37) we saw that any finite $(\mathbb{Z}[t], X)$-graph defines a subgroup in $F^{\mathbb{Z}[t]}$. We will see now that the converse statement is also true.

Proposition 5 *Let H be a finitely generated subgroup of $F^{\mathbb{Z}[t]}$. Then there exists a U-folded $(\mathbb{Z}[t], X)$-graph Γ and a vertex v of Γ such that $L(\Gamma, v) = H$.*

Proof. Since H is finitely generated then there are elements $h_1, \ldots, h_k$ in $F^{\mathbb{Z}[t]}$ which generate H. Since $F^{\mathbb{Z}[t]}$ is a union of the following infinite chain of groups:

$$F = G_0 < G_1 < \ldots < G_n \ldots < \ldots,$$

where G_{i+1} is obtained from G_i by extension of all cyclic centralizers in G_i, there exists a minimal natural number n such that $h_i \in G_{n-1}$ for all $i \in [1, k]$.

In Subsection 15.2, it was shown that any element of $F^{\mathbb{Z}[t]}$ has a unique *extension series* and the standard decomposition with respect to these series. If by $U(h_i)$ we denote all elements from U, infinite exponents of which appear in the standard decomposition of some h_i, then $U(h_i)$ is a finite set and so is $\bigcup_{i=1}^{k} U(h_i)$. Thus any $\widetilde{h_i}$ can be viewed as finite word over the alphabet $B = \{X \cup X^{-1}\} \cup \{u^{\alpha} \mid u \in \bigcup_{i=1}^{k} U(h_i), \alpha \in \mathbb{Z}[t]^{+}\}$, of the length n_i.

We define an $(\mathbb{Z}[t], X)$-graph Γ_1 in the following way. Γ_1 is a wedge of k circles wedged at a vertex denoted v_1. The i-th circle is subdivided into n_i edges which are oriented and labeled by B so that the label of the i-th circle (as read from v_1 to v_1) is precisely the word $\widetilde{h_i}$. Note that Γ_1 is connected.

For any cycle p at v_1 in Γ_1, $\overline{\mu(p)} \in L(\Gamma_1, v_1)$ by definition. So $H \subset L(\Gamma_1, v_1)$. The converse is obviously true - if $g \in L(\Gamma_1, v_1)$ then it can be realized as a reduced label of some cycle at v_1 in Γ_1, so it can be obtained as a finite product of basic cycles labeled by $\tilde{h}_i$. Thus we have $H = \langle h_1, \ldots, h_k \rangle \subset L(\Gamma_1, v_1)$.

By Proposition 4, from Γ_1 by finitely many partial and u-foldings, $u \in \bigcup_{i=1}^{k} U(h_i)$ one can obtain a finite U-folded $(\mathbb{Z}[t], X)$-graph Γ which is connected and there exists some vertex v in Γ which corresponds to v_1 in Γ_1. By Lemma 38 we have $L(\Gamma_1, v_1) = L(\Gamma, v)$.

□

Observe that Γ constructed in the proposition above is not unique in general. But all graphs associated with H define the same language which coincides with H.

Proposition 4 only states the existence of $(\mathbb{Z}[t], X)$-graph Γ for H. But, in fact, the following result follows directly from the procedures described in the proofs of Lemma 33, Lemma 34, Lemma 35 and Proposition 4.

Proposition 6 *There is an algorithm which, given finitely many standard decompositions of elements $h_1, \ldots, h_k$ from $F^{\mathbb{Z}[t]}$, constructs a U-folded $(\mathbb{Z}[t], X)$-graph Γ, such that $L(\Gamma, v) = \langle h_1, \ldots, h_k \rangle$.*

The next result is a solution of membership problem for finitely generated subgroups of $F^{\mathbb{Z}[t]}$.

Proposition 7 *Every finitely generated subgroup of $F^{\mathbb{Z}[t]}$ has a solvable membership problem. That is, there exists an algorithm which, given finitely many standard decompositions of elements $g, h_1, \ldots, h_k$ from $F^{\mathbb{Z}[t]}$, decides whether or not g belongs to the subgroup $H = \langle h_1, \ldots, h_n \rangle$ of $F^{\mathbb{Z}[t]}$.*

Proof. We construct a U-folded $(\mathbb{Z}[t], X)$-graph Γ, such that $L(\Gamma, v) = \langle h_1, \ldots, h_k \rangle$ which is a finite algorithmic procedure by Proposition 6.

Let

$$F < H_1 < H_2 < \ldots < H_n,$$

be the extension series for g, where $g \in H_n$ and H_{i+1} is obtained from H_i by a centralizer extension of a single element u_i and let $U(g) = \{u_1, \ldots, u_n\}$ be the subset of U with the induced order. Then we check if $g \in L(\Gamma, v)$ using inductive argument based on $|U(g)|$.

$g \in H = L(\Gamma, v)$ if and only if there exists a reduced cycle p at v in Γ such that $g = \overline{\mu(p)}$. By the property **(ix)** of Γ we can assume that $\mu(p) = \tilde{g}$.

Let

$$\tilde{g} = \tilde{h}_1 u_n^{\alpha_1} \tilde{h}_2 \cdots u_n^{\alpha_m} \widetilde{h_{m+1}}$$

be the standard decomposition of g.

If $|U(g)| = 0$, that is, $\tilde{g}$ is a reduced word in $\{X \cup X^{-1}\}$ then we just try to "read" $\tilde{g}$ in $\Gamma(0)$ starting at the vertex v - this can be done as shown in Proposition 7.2 from [24].

Suppose there is an algorithm which "reads" a standard decomposition of an element $h \in H_{n-1}$, that is $|U(g)| < n$, starting from any point $v' \in V(\Gamma)$ and returns answer "yes" if there exists a path in Γ corresponding to $\tilde{h}$ or "no" if such path does not exist. Then we apply this algorithm to v and $\tilde{h}_1$. If we get "no" as a result it means by the property **(x)** of Γ that there exists no path for h_1 starting at v in Γ and we stop - g does not belong to H. If we get "yes" as a result so we have a path p_1 for $\tilde{h}_1$ and if $t(p_1)$ belongs to some u_n-component C of Γ we try to "read" $u_n^{\alpha_1}$ as follows. A pair $(P_C, H_{u_n}(t(p_1)))$ is associated with C, where P_C is a finite positively oriented path and $H_{u_n}(t(p_1))$ is a finitely generated free abelian group. By Lemma 35 it is enough to check if $u_n^{\alpha_1} \in \overline{\mu(q)} * H_{u_n}(t(p_1))$, where q is a reduced subpath of P_C. Hence, if we can find such q then it is unique and we proceed with h_2. If there exists no such subpath of P_C, we stop - g does not belong to H. In finitely many steps we either find out that $g \notin H$ on some intermediate step or construct a path p from v to some $v_1 \in V(\Gamma)$, which is labeled by $\tilde{g}$. If we managed to find p then we check if $v = v_1$ which holds if and only if $g \in H$.

□

Next application of U-folded $(\mathbb{Z}[t], X)$-graph is the finding of intersection of two finitely generated subgroups in $F^{\mathbb{Z}[t]}$.

Recall the definition of a product-graph from [24].

Definition 37 *Let Θ_1, Θ_2 be graphs labeled by some alphabet A. The* product-graph *$\Theta_1 \times \Theta_2$ is defined as follows. The vertex set of $\Theta_1 \times \Theta_2$ is the set $V(\Theta_1) \times V(\Theta_2)$. For a pair of vertices $(s, t), (s', t') \in V(\Theta_1 \times \Theta_2)$ (so that $s, s' \in V(\Theta_1), t, t' \in V(\Theta_2)$) and a letter $z \in A$ an edge labeled by z with origin (s, t) and terminus (s', t') is introduced, provided there is an edge labeled by z from s to s' in Θ_1 and there is an edge labeled by z from t to t' in Θ_2.*

In [24] this notion was used for constructing a graph corresponding to the intersection of finitely generated subgroups in a free group. In case of U-folded $(\mathbb{Z}[t], X)$-graphs similar notion can be introduced.

At first we prove the following auxiliary result.

Lemma 39 *Let H_1, H_2 be subgroups of $\mathbb{Z}^p$ and let $H = H_1 \cap H_2$. Let $h_1, \ldots, h_{m_1}$ be cosets representatives in $\mathbb{Z}^p$ by H_1 and $g_1, \ldots, g_{m_2}$ be cosets representatives in $\mathbb{Z}^p$ by H_2 (both lists are not necessarily complete). Then there exist coset representatives $p_1, \ldots, p_{m_3}, p_i > 0, m_3 \leq m_1 m_2$ in $\mathbb{Z}^p$ by H such that*

$$\{(h_1 + H_1) \cup \cdots \cup (h_{m_1} + H_1)\} \cap \{(g_1 + H_2) \cup \cdots \cup (g_{m_2} + H_2)\} =$$

$$= \{(p_1 + H) \cup \cdots \cup (p_{m_3} + H)\}.$$

Proof. It is enough to show that the intersection of two cosets $(h_i + H_1) \cap (g_j + H_2)$ is either empty or consists only of one coset $p_k + H$ in $\mathbb{Z}^p$ by H. We have

$$(h_i + H_1) \cap (g_j + H_2) = (h_i + ((r_1 + H) \cup (r_2 + H) \cup \cdots)) \cap (g_j + ((q_1 + H) \cup$$

$$\cup (q_2 + H) \cup \cdots)) = ((a_1 + H) \cup (a_2 + H) \cup \cdots) \cap ((b_1 + H) \cup (b_2 + H) \cup \cdots).$$

where $r_1, \ldots, r_n, \ldots$ are coset representatives in H_1 by H. $q_1, \ldots, q_n, \ldots$ are coset representatives in H_2 by H. a_l is a representative of $h_i + r_l$ in $\mathbb{Z}^p$ by H. b_m is a representative of $g_j + r_m$ in $\mathbb{Z}^p$ by H. Thus we have two sets of cosets in $\mathbb{Z}^p$ by H and their intersection either consists of some number of cosets or is empty. Suppose there are at least two cosets in the intersection, that is say $a_1 = b_1$ and $a_2 = b_2$. Then, there are two elements $t_1, t_2 \in (h_i + H_1) \cap (g_j + H_2)$ such that $t_1 - t_2 \notin H$. It follows, $t_1 - t_2 = a_1 + z_1 - (a_2 + z_2) = h_i + r_1 + z'_1 - (h_i + r_2 + z'_2) = r_1 - r_2 + z'_1 - z'_2 \in H_1$, where $z_1, z_2, z'_1, z'_2 \in H$. On the other hand $t_1 - t_2 = b_1 + z_1 - (b_2 + z_2) = g_j + q_1 + z''_1 - (g_j + q_2 + z''_2) = q_1 - q_2 + z''_1 - z''_2 \in H_2$. So, $t_1 - t_2 \in H$ - contradiction which proves the Lemma.

□

Let Θ_1, Θ_2 be finite U-folded $(\mathbb{Z}[t], X)$-graphs and let $v_i \in V(\Theta_i), i = 1, 2$. We construct a $(\mathbb{Z}[t], X)$-graph Θ_3, such that $L(\Theta_3, v_3) = L(\Theta_1, v_1) \cap L(\Theta_2, v_2)$ as follows. The vertex set of $\Theta_1 \times \Theta_2$ is the set $V(\Theta_1) \times V(\Theta_2)$. For a pair of vertices $(s, t), (s', t') \in V(\Theta_1 \times \Theta_2)$ (so that $s, s' \in V(\Theta_1), t, t' \in V(\Theta_2)$) and a letter $x \in X$, an edge labeled by x with origin (s, t) and terminus (s', t') is introduced, provided there is an edge labeled by x from s to s' in Θ_1 and there is an edge labeled by x from t to t' in Θ_2.

Then let $u \in U(\Theta_1) \cap U(\Theta_2)$. Consider $K_i = Comp_u(w_i), i = 1, 2$ where $w_i \in V(\Theta_i)$. Since $K_i, i = 1, 2$ is finite, by Lemma 30, $H_u(w_i)$ is isomorphic to a subgroup H_i of $\mathbb{Z}^{n_i}$. Thus, we can consider H_1 and H_2 as subgroups of $\mathbb{Z}^p$, where $p = \max\{n_1, n_2\}$. Observe that $H = H_1 \cap H_2$ in $\mathbb{Z}^p$ can be found algorithmically.

From now on we will use additive notation for H_i.

Now, $V(K_1) \times V(K_2) \subset V(\Theta_1 \times \Theta_2)$ and we connect vertices inside $V(K_1) \times V(K_2)$ as follows. Let b_i be a base-point in $K_i, i = 1, 2$. Let $s_1, s_2 \in V(K_1), t_1, t_2 \in V(K_2)$. Since K_i is U-folded, any vertex in K_i is associated with a coset in $\mathbb{Z}^p$ by H_i. Then there exist h_{i_1}, h_{i_2} and g_{j_1}, g_{j_2} which are coset representatives for s_1, s_2 and t_1, t_2

correspondingly. We connect (s_1, t_1) with (s_2, t_2) by a positively oriented edge with label u^{p_k} if and only if $(h_{i_2} - h_{i_1} + H_1) \cap (g_{j_2} - g_{j_1} + H_2) = p_k + H$ for some coset representative p_k.

Observe that $K_1 \times K_2$ can be disconnected and it can consist of several u-components. Also, according to the definition above it contains loops (at least ones which consist of two edges).

Lemma 40 *Let $K_1, K_2, K_1 \times K_2, H_1, H_2, H$ be defined as above and let $K_1 \times K_2$ edge-connected. Then*

(a) if there exists an edge e_1 with label $u^{p_1}, p_1 > 0$ in $K_1 \times K_2$ from (s_1, t_1) to (s_2, t_2) then there exists also an edge e_2 with label $u^{p_2}, p_2 > 0$ from (s_2, t_2) to (s_1, t_1);

(b) if $p = e_1 \dots e_k$ is a loop in $K_1 \times K_2$ such that $\mu(e_i) = u^{f_i}$ then $f_1 + \dots + f_k \in H$;

(c) if $p = e_1 \dots e_k$ is a simple path in $K_1 \times K_2$ such that $\mu(e_i) = u^{f_i}$ then $f_1 + \dots + f_k \notin H$.

Proof. (a) The existence of e_1 means that $(h_{i_2} - h_{i_1} + H_1) \cap (g_{j_2} - g_{j_1} + H_2) = p_k + H$, where h_{i_1}, h_{i_2} are coset representatives in $\mathbb{Z}^p$ by H_1 for s_1, s_2 correspondingly, g_{j_1}, g_{j_2} are coset representatives in $\mathbb{Z}^p$ by H_2 for t_1, t_2 correspondingly and p_k is a coset representative in $\mathbb{Z}^p$ by H. Thus we have $h_{i_2} - h_{i_1} + a_1 = g_{j_2} - g_{j_1} + a_2 = p_k + c$, where $a_i \in H_i, c \in H$. So $h_{i_1} - h_{i_2} + a'_1 = g_{j_1} - g_{j_2} + a'_2 = -p_k + c' = p_l + c'$ for $a'_i \in H_i, c' \in H$ which means that $(h_{i_1} - h_{i_2} + H_1) \cap (g_{j_1} - g_{j_2} + H_2) = p_l + H$ is not empty and there exists an edge from (s_2, t_2) to (s_1, t_1).

(b) Suppose we have a cycle $p = e_1 \dots e_k, \mu(e_i) = u^{f_i}$ in $K_1 \times K_2$. By (a) we can assume all f_i to be positive. We have $o(e_i) = (s_i, t_i), t(e_i) = (s_{i+1}, t_{i+1}), i \in [1, k]$ and $(s_1, t_1) = (s_{k+1}, t_{k+1})$. The rule by which edges in $K_1 \times K_2$ are constructed assumes that there exists g_i, a coset representatives in $\mathbb{Z}^p$ by H_1 corresponding to s_i and h_i, a coset representatives in $\mathbb{Z}^p$ by H_2 corresponding to t_i such that for an edge e_i one has $(h_{i+1} - h_i + H_1) \cap (g_{i+1} - g_i + H_2) = f_i + H, i \in [1, k]$. Thus $h_{i+1} - h_i + a_i =$

$g_{i+1} - g_i + b_i = f_i + c_i$, for some $a_i \in H_1, b_i \in H_2, c_i \in H$. So if we sum all these equalities for $i \in [1, k]$ we obtain $a_1 + \cdots + a_k = b_1 + \cdots + b_k = f_1 + \cdots + f_k + c, c \in H$ and $f_1 + \cdots + f_k \in H$.

(c) Here we use the notation for p introduced in the proof of (b). Thus we have $o(e_i) = (s_i, t_i), t(e_i) = (s_{i+1}, t_{i+1}), i \in [1, k]$, g_i is a coset representatives in $\mathbb{Z}^p$ by H_1 corresponding to s_i and h_i, a coset representatives in $\mathbb{Z}^p$ by H_2 corresponding to t_i for all $i \in [1, k+1]$ so that $(h_{i+1} - h_i + H_1) \cap (g_{i+1} - g_i + H_2) = f_i + H, i \in [1, k]$, that is, $h_{i+1} - h_i + a_i = g_{i+1} - g_i + b_i = f_i + c_i$, for some $a_i \in H_1, b_i \in H_2, c_i \in H$. Again, we take the sum for all $i \in [1, k]$, so $h_{k+1} - h_1 + a = g_{k+1} - g_1 + b = f_1 + \cdots + f_k + c, a \in H_1, b \in H_1, c \in H$. Hence, $(h_{k+1} - h_1 + H_1) \cap (g_{k+1} - g_1 + H_2) = f + H$ is not empty for some coset representative f in $\mathbb{Z}^p$ by H. This means that there exists an edge e labeled by $u^\alpha, \alpha > 0$ in $K_1 \times K_2$ from (s_1, t_1) to (s_{k+1}, t_{k+1}) and $p \notin H$. By (a) there exists also an edge e' from (s_{k+1}, t_{k+1}) to (s_1, t_1) labeled by $u^\beta, \beta > 0$ such that $\alpha + \beta \in H$. Since $q = e_1 \cdots e_k e'$ is a cycle it follows from b) that $f_1 + \cdots + f_k + \beta \in H$ so it follows that $f_1 + \cdots + f_{k+1} \notin H$.

□

Observe that it follows from Lemma 40 that $H_u((b_1, b_2))$ is isomorphic to H.

To complete the construction of all u-components in $K_1 \times K_2$, in any connected u-component of $K_1 \times K_2$ we choose a maximal subtree and then using (a) from Lemma 40 we can change the direction of some edges to make all simple paths starting from the chosen root to leaves of the tree label reduced. Finally, after using u-foldings we obtain a single path from the tree, the origin of this path becomes a base-point of a corresponding u-component and we add at each base-point in $V(K_1 \times K_2)$ cyclic edges labeled by u^{h_i}, where $H = \langle h_1, \ldots, h_k \rangle$.

Observe that labels of all simple paths leading from a base-point (s, t) to any point of a u-component in $K_1 \times K_2$ compose a system of coset representatives by $H_u((s, t))$. Hence all u-component in $K_1 \times K_2$ are u-folded.

Since there are finitely many u-components in Θ_1, Θ_2 for any $u \in U(\Theta_1) \cap U(\Theta_2)$ and $U(\Theta_1) \cap U(\Theta_2)$ is a finite set, this process of construction u-components stops after a finite number of steps. Then we take the connected component of $\Theta_1 \times \Theta_2$ which contains (v_1, v_2) and denote this graph by Θ'_3. Observe that Θ'_3 is partially folded but may be not U-folded. However, all u-components in Θ'_3 are complete and u-folded. By Proposition 4 there exists a U-folded graph finite $(\mathbb{Z}[t], X)$-graph Θ_3 such that $L(\Theta'_3, (v_1, v_2)) = L(\Theta_3, (v_1, v_2))$.

Now, we have to show that $L(\Theta'_3, (v_1, v_2)) = L(\Theta_1, v_1) \cap L(\Theta_2, v_2)$.

Let $g \in L(\Theta_1, v_1) \cap L(\Theta_2, v_2)$ and $\tilde{g} = \tilde{g_1} u^{\alpha_1} \tilde{g_2} \cdots u^{\alpha_k} \widetilde{g_{k+1}}$ be its standard decomposition. Then there exist label reduced cycles z_i in Θ_i at v_i such that $\mu(z_i) = \tilde{g}$. Observe that both z_1 and z_2 belong to the same level in Θ_1 and Θ_2 correspondingly.

If $g \in F(X)$, then by definition $g \in L(\Theta'_3, (v_1, v_2))$. We proceed by induction on the L - the minimal level of Θ_i which z_i belongs to.

Since Θ_1, Θ_2 are U-folded there ia a unique path p_i starting at v_i labeled by $\tilde{g_1}$ in $\Theta_i(L)$ such that $t(p_i) \in V(K_i)$ for some u-component K_i in $\Theta_i, i = 1, 2$. So by induction there exists a label reduced path p_3 in Θ'_3 starting at (v_1, v_2) and ending at $(t(p_1), t(p_2))$ with the label g_1 and $(t(p_1), t(p_2))$ belongs to u-component $K_1 \times K_2$ in Θ'_3.

There is a unique continuation of p_i in K_i because $g \in L(\Theta_1, v_1) \cap L(\Theta_2, v_2)$. This continuation is a path q_i (not unique) with label u^{α_1} and fixed endpoints $o(q_i) = t(p_i), t(q_i) = c_i, i = 1, 2$. Let $H_i \simeq H_u(t(p_i)), H = H_1 \cap H_2$ and b_i be a base-point of $K_i, i = 1, 2$. Then there exist coset representatives β_i, γ_i in $\mathbb{Z}^p$ by H_i which correspond to $o(q_i), t(q_i)$ such that $\alpha_1 \in (\gamma_1 - \beta_1 + H_1) \cap (\gamma_2 - \beta_2 + H_2) \neq \emptyset$. Then by definition of u-components in $K_1 \times K_2$ there exists a coset representative δ_1 in $\mathbb{Z}^p$ by H and a path $q_3, \overline{\mu(q_3)} = u^{\alpha_1}$ from $(t(p_1), t(p_2))$ to $(t(q_1), t(q_2))$ such that $(\gamma_1 - \beta_1 + H_1) \cap (\gamma_2 - \beta_2 + H_2) = \delta_1 + H = \alpha_1 + H$. So the continuation of p_3 in $K_1 \times K_2$ is q_3. q_3 is not unique, but it has fixed end-points.

In the same way one can "read" then $\tilde{g}_2, w^{\alpha_2}$ and so on. Since g has finitely many syllables g_i and w^{α_i} the process of "reading" g in $\Theta_1 \times \Theta_2$ is finite. In Θ_i it ends at v_i, hence in $\Theta_1 \times \Theta_2$ it ends at (v_1, v_2) and this means that $g \in L(\Theta'_3, (v_1, v_2))$. So $L(\Theta_1, v_1) \cap L(\Theta_2, v_2) \subseteq L(\Theta'_3, (v_1, v_2))$.

Now, let $g \in L(\Theta'_3, (v_1, v_2))$. Then there exists a reduced cycle z at (v_1, v_2) in Θ' such that $\overline{\mu(z)} = g$.

If $g \in F(X)$ then by definition of $\Theta_1 \times \Theta_2$ it follows that $g \in L(\Theta_1, v_1) \cap L(\Theta_2, v_2)$ and again as above we use induction on L - the minimal level of Θ'_3 which z belongs to.

We subdivide z as follows

$$z = z_1 d_1 z_2 \cdots d_k z_{k+1},$$

where every z_i does not contain an edge labeled by w^{α} and d_i is a subpath which consists only of edges labeled by w^{α}. Observe that d_i is a reduced path in some w-component C_i of Θ'_3.

By induction there exists a reduced path p_i in Θ_i such that $\overline{\mu(p_i)} = \overline{\mu(z_1)}, o(z_1) = (v_1, v_2), t(z_1) = (t(p_1), t(p_2)), i = 1, 2$.

$t(z_1)$ belongs to w-component C_1 in Θ'_3. Hence $t(z_1) = (s_1, s_2)$ and s_i belongs to a w-component K_i in $\Theta_i, i = 1, 2$. Let $H_i = H_w(s_i), H = H_1 \cap H_2$ and let $d_1 = e_1 \cdots e_m, \mu(e_j) = w^{\beta_j}$.

By definition of $\Theta_1 \times \Theta_2$ we have for each e_j there exist corresponding edges $e_j^1 \in K_1, e_j^2 \in K_2$ such that h_j, h_{j+1} and g_j, g_{j+1} are coset representatives corresponding to $o(e_j^1), t(e_j^1)$ and $o(e_j^2), t(e_j^2)$ and $(h_j - h_{j+1} + H_1) \cap (g_j - g_{j+1} + H_2) = p_j + H$, where $\beta_j \in p_k + H$. So $\beta_j = h_j - h_{j+1} + a_j = g_j - g_{j+1} + b_j, a_j \in H_1, b_j \in H_2$ and $\beta_1 + \cdots + \beta_m = h_1 - h_{m+1} + a = g_1 - g_{m+1} + b, a \in H_1, b \in H_2$. Hence there exists a unique continuation of p_i in $K_i, i = 1, 2$ from $t(p_i)$ to $t(e_m^i)$ (by Lemma 35 and the fact that K_i is w-folded) and we proceed in the same way with z_2.

In a finite number of steps we construct a path r_i in $\Theta_i, i = 1, 2$ such that $o(r_i) = t(r_i) = v_i, \overline{\mu(r_i)} = g$. Thus $g \in L(\Theta_1, v_1) \cap L(\Theta_2, v_2)$ which means that $L(\Theta'_3, (v_1, v_2)) = L(\Theta_1, v_1) \cap L(\Theta_2, v_2)$.

Thus the following result is proved.

Proposition 8 *Let Θ_1, Θ_2 be finite connected $(\mathbb{Z}[t], X)$-graphs which are U-folded and let $v_i \in V(\Theta_i), i = 1, 2$. Then there exists a finite U-folded graph $(\mathbb{Z}[t], X)$-graph Θ_3 such that $L(\Theta_3, v_3) = L(\Theta_1, v_1) \cap L(\Theta_2, v_2)$.*

Observe that construction of $\Theta_1 \times \Theta_2$ is a definite procedure which can be performed in finitely many steps. Hence the following result.

Proposition 9 *There exists an algorithm which, given finitely many standard decompositions of elements $h_1, \ldots, h_k, g_1, \ldots, g_m$ from $F^{\mathbb{Z}[t]}$, finds the generators of the intersection $H \cap K$ which is finitely generated, where $H = \langle h_1, \ldots, h_k \rangle, K = \langle g_1, \ldots, g_m \rangle$.*

Proof. By Proposition 7 there exists an algorithm which constructs U-folded $(\mathbb{Z}[t], X)$-graphs Γ_1 and Γ_2 such that $L(\Gamma_1, v_1) = H$, $L(\Gamma_2, v_2) = K$, $v_i \in V(\Gamma_i)$. Then by Proposition 8 there exists a U-folded $(\mathbb{Z}[t], X)$-graph Γ_3 such that $L(\Theta_3, v_3) = L(\Theta_1, v_1) \cap L(\Theta_2, v_2) = H \cap K$. Γ_3 is finite then we can find all simple loops at v_3 and their reduced labels generate $L(\Theta_3, v_3) = H \cap K$.

□

The following result follows directly from Proposition 9.

Corollary 4 *(Howson Property) The intersection of any two finitely generated subgroups of $F^{\mathbb{Z}[t]}$ is again finitely generated.*

Using the construction of a graph-product described above we are able to solve the conjugacy problem for finitely generated subgroups of $F^{\mathbb{Z}[t]}$.

Proposition 10 *Any finitely generated subgroup of $F^{\mathbb{Z}[t]}$ has a solvable conjugacy problem. That is, there exists an algorithm which, given standard decompositions of elements $g, f \in H = \langle h_1, \ldots, h_k \rangle$, decides whether or not g is conjugate to f in H, and if yes, generates an element $c \in H$ such that $c^{-1}gc = f$.*

Proof. Without loss of generality we can assume f and g to be cyclically reduced elements in $G_{n+1} \cap H$ and applying cyclic permutations we can assume that

$$f = v_1^{\alpha_1} \circ f_1 \circ \ldots \circ v_k^{\alpha_k} \circ f_k$$

and

$$g = u_1^{\beta_1} \circ g_1 \circ \ldots \circ u_l^{\beta_l} \circ g_l$$

are cyclically reduced U_n-forms for f and g.

Recall that we are able to solve the conjugacy problem in $F^{\mathbb{Z}[t]}$ (see Section 14). To determine if f and g are conjugate in $F^{\mathbb{Z}[t]}$ we compare cyclic permutations

$$f(i) = v_i^{\alpha_i} \circ f_i \circ \ldots \circ v_1^{\alpha_1} \circ f_1 \circ \ldots \circ v_{i-1}^{\alpha_{i-1}} \circ f_{i-1}$$

and

$$g(j) = u_j^{\beta_j} \circ g_j \circ \ldots \circ u_1^{\beta_1} \circ g_1 \circ \ldots \circ u_{j-1}^{\beta_{j-1}} \circ g_{j-1}$$

of f and g. If $f(i_0) = g(j_0)$ for some i_0, j_0 then f and g are conjugate in $F^{\mathbb{Z}[t]}$ and, moreover,

$$(u_{j_0}^{\beta_{j_0}} \circ g_{j_0} \circ \ldots \circ u_l^{\beta_l} \circ g_l) * (v_{i_0}^{\alpha_{i_0}} \circ f_{i_0} \circ \ldots \circ v_k^{\alpha_k} \circ f_k)^{-1}$$

is a conjugating element x_{i_0,j_0} such that $x_{i_0,j_0}^{-1} * f * x_{i_0,j_0} = g$. Since there are finitely many such cyclic permutations to check we have a finite procedure of checking.

However, f and g may not be conjugate in H even if they are conjugate in $F^{\mathbb{Z}[t]}$ - x may not belong to H.

Observe that if there exists conjugating element x in $F^{\mathbb{Z}[t]}$ such that $x^{-1} * f * x = g$, then for any element $y = f^m * x$ we have $y^{-1} * f * y = g$.

Now we have that f and g are conjugate in H if and only if there exist $i_0 \in [1, k], j_0 \in [1, l]$ such that $H \cap (x_{i_0, j_0} * \langle f \rangle)$ is not trivial. That is, everything reduces to finding such an intersection.

By Proposition 6, we can find algorithmically a U-folded $(\mathbb{Z}[t], X)$-graph Γ_1 and $v_1 \in V(\Gamma_1)$, such that $L(\Gamma_1, v_1) = H$.

Let us construct a $(\mathbb{Z}[t], X)$-graph Γ_2 which corresponds to $x * \langle f \rangle$ for any given x and f. Let Γ'_2 be composed by a single cycle at v' labeled by standard decomposition of f, and a path, starting at v'_2, ending at v' and labeled by standard decomposition of x. Let

$$K' = \{\overline{\mu(p)} \mid p \text{ is a path in } \Gamma'_2 \text{ from } v'_2 \text{ to } v'\}.$$

Then, clearly, we have $K' = x * \langle f \rangle$.

By Proposition 4, one can obtain algorithmically a U-folded $(\mathbb{Z}[t], X)$-graph Γ_2 and $v, v_2 \in V(\Gamma_2)$ corresponding to v' and v'_2 respectively such that

$$K' = K = \{\overline{\mu(p)} \mid p \text{ is a path in } \Gamma_2 \text{ from } v_2 \text{ to } v\}.$$

Thus we construct a graph product of Γ_1 and Γ_2. The obtained $(\mathbb{Z}[t], X)$-graph Γ'_3 with distinguished vertices (v_1, v_2) and (v_1, v) we transform into a U-folded $(\mathbb{Z}[t], X)$-graph Γ_3 and with abuse of notation we assume $(v_1, v_2), (v_1, v) \in V(\Gamma_3)$.

Finally observe that

$$M = \{\overline{\mu(p)} \mid p \text{ is a path in } \Gamma_3 \text{ from } (v_1, v_2) \text{ to } (v_1, v)\} = H \cap (x * \langle f \rangle).$$

□

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