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THE FRATTINI SUBGROUP

by

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INTRODUCTION

The Frattini subgroup of a group G, denoted by $\phi(G)$, is the intersection of the maximal subgroups of G; if the group has no maximal subgroups, its Frattini subgroup is defined to be the group itself. In the notation of Gaschutz, G will be called ϕ free if $\phi(G) = 1$.

Interest in the Frattini subgroup arose out of the theory of finite p groups. Burnside first discovered that studying a finite p group modulo its Frattini subgroup greatly facilitated the investigation of its automorphisms. The important fact that $\phi(G)$ is the union of the non-generators of G (that is, those elements which can be omitted from any set of generators for G) was proven for finite groups by Frattini and later for groups in general by B.H. Neumann. Now in connection with his work on the automorphisms of finite p groups, Burnside was interested in the possible sets of generators of the group in question, and he noted that if a set generates $G/\phi(G)$ it must generate G; hence it was enough to consider generators of $G/\phi(G)$. But as the Frattini subgroup of a finite p group is its derived group times the group consisting of the p^{th} powers, $G/\phi(G)$ may be regarded as a vector space over a field of characteristic p and finding a set of generators for a vector space is in general a more tractable matter.

The Frattini subgroup of a finite group is nilpotent, and K.A. Hirsch observed that this is also true of certain other types of groups [7]. This led Noburo Itô to inquire whether this might not hold generally. In answer to his question, Graham Higman and B.H.Neumann constructed some groups with no maximal subgroups which thus formed their own Frattini subgroups but which were in no sense nilpotent. Itô also raised the question of whether a free product of groups necessarily has maximal subgroups, and Higman and Neumann answered this affirmatively; in fact the Frattini subgroup of the free product of nontrivial groups is trivial. The proof of this theorem is elegantly simple; Higman and Neumann produced, for any element g in a free product G, a proper subgroup T of G which together with g generated G. Their argument did not, however, hold for free products with amalgamations; in fact they noted that the Frattini subgroup of such a group can coincide with the amalgamated subgroup. They raised the question, can the Frattini subgroup of a group of this kind be larger than the amalgamated subgroup; indeed does a free product with amalgamations necessarily have maximal subgroups?

In Chapter I some of these questions are answered, and the argument of Higman and Neumann is extended to certain types of generalized free products. If G is the free product of groups G_λ amalgamating the subgroup H, it is shown that, provided the normal closure of H in each of the G_λ is properly contained in G_λ, G does have maximal subgroups; in fact its Frattini subgroup is contained in the normal closure of H in G. Unfortunately, however, the question of whether $\phi(G)$ can actually be larger than H still remains unanswered. In addition, it is proved that if G contains an element which conjugates every nontrivial element of the amalgamated subgroup H outside of H, then G is ϕ free. This remark, together with the fact that the free product of finitely many infinite cycles with amalgamated subgroup is ϕ free will easily yield the result that the free product of finitely many free groups amalgamating a cycle is ϕ free. It will also be shown that the Frattini subgroup of the free product of finitely many finitely generated abelian groups with torsion free amalgamation is trivial, and that this need not be the case if the torsion freeness condition is removed. And consistent with the nilpotency of the Frattini subgroup of a finite group, it is shown that the Frattini subgroup of a residually finite group is residually nilpotent.

Chapter II contains an investigation of the Frattini subgroups of restricted wreath products of groups with known Frattini subgroup. It is shown that the wreath product of two finite groups of coprime order is ϕ free if and only if the bottom group is ϕ free, and the Frattini subgroup of $A \text{ wr } B$ is evaluated when A and B are finitely generated abelian groups. Free groups and free abelian groups are ϕ free; in this chapter the results on wreath products are used to show that all free soluble groups have trivial Frattini subgroup.

I am deeply indebted to my teacher and adviser, Professor Gilbert Baumslag, whose untiring enthusiasm, patience and moral support have carried me through these last few years, and to Professor L.G. Kovacs, who gave so unstintingly of his time and insights into wreath products. I would also like to thank Dr. John Cossey, whose comments and suggestions were of such great help in determing the subgroup structure of certain groups, and Professor Leo Zippin, for his understanding and encouragement when it was most needed.

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Section 1.

We first present some preliminary notation, definitions and conventions.

If G is a group and H a subgroup of G we write $H \leq G$; if H is a normal subgroup of G we write $H \trianglelefteq G$; the symbols $<$ and $\triangleleft$ will denote proper inclusion. The symbol $G \setminus H$ will mean the set of elements belonging to G but not to H , and 1 will denote the identity element of G , as well as the trivial group. The subgroup generated by a subset S of G will be denoted by $\langle S \rangle$. A left (right) transversal T of the subgroup H of G is a set of elements $\{t_i\}$ of G such that G is the disjoint union of the cosets t_iH (Ht_i) . If g and h are elements of G we denote the element $g^{-1}hg$ by h^g and call it the conjugate of h by g . We let $h^G = \{h^g / g \in G\}$ and $H^G = \langle h^G / h \in H \rangle$. H^G is called the normal closure of H in G , usually denoted by $\mathrm{nm}_G H$, and is the smallest normal subgroup of G containing H . It is clear that every element in $\mathrm{nm}_G H$ can be written in the form

$$\prod_{i=1}^{n(i)} h_i^{g_i} \quad \text{where } g_i \in G \text{ and } h_i \in H .$$

H is abnormal in G if $h^g \notin H$ for all $1 \neq h \in H$ and all $g \in G \setminus H$.

The normalizer of a subgroup H of G is the largest subgroup of G containing H as a normal subgroup. The centralizer of the subset S of G is the set $\zeta_G S$ of elements of G which commute with every element in S . The center ζG of G is the centralizer of G in itself. The derived group G' of G is the group generated by the commutators $[x,y] = x^{-1}y^{-1}xy$ where $x, y \in G$, and if H and K are subgroups of G, we define $[H,K]$ to be the subgroup generated by the commutators $[h,k]$ where $h \in H$ and $k \in K$. A normal series of a group G is a finite sequence $G_0, G_1, \ldots, G_n$ of subgroups of G such that $1 = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$. An invariant series of G is a normal series such that $G_i \lhd G$ for $0 \leq i \leq n$. Let $G_0, G_1, \ldots, G_n$ be a normal series of G; the groups G_{i+1}/G_i $(0 \leq i \leq n)$ are called the factors of the series. A group G is called soluble if it has a normal series with abelian factors; G is called nilpotent if it has an invariant series such that G_{i+1}/G_i is in the center of G/G_i. We define recoursively two sets of subgroups of G: (i) $\gamma_1(G) = G$, $\gamma_{i+1}(G) = [G, \gamma_i(G)]$; (ii) $\delta_1(G) = G$, $\delta_{i+1}(G) = [\delta_i(G), \delta_i(G)]$. G is soluble (nilpotent) if and only if there exists $c \geq 0$ such that $\delta_{c+1}(G) = 1$ $(\gamma_{c+1}(G) = 1)$. The least integer c for which this holds is called the solubility depth (nilpotency class) of G. Subgroups and factor groups of soluble (nilpotent) groups are soluble (nilpotent). G is abelian if its solubility depth ≤ 1; metabelian if its solubility depth is ≤ 2; abelian groups are nilpotent and nilpotent groups are soluble.

If G is given in terms of a set of generators $\{x_i\}_{i \in I}$ (where I is some indexing set) and defining relations $R_j(x_i) = 1$, we call this a presentation for G and write $G = \langle x_i ; R_j(x_i) = 1 \rangle$. G is finitely generated if I is finite.

If $\underline{\underline{X}}$ is any class of groups, a group G is said to be residually a member of $\underline{\underline{X}}$ (we write $G \in \underline{\underline{RX}}$) if for each $g \neq 1$ in G there exists $N_g \lhd G$ such that $g \notin N_g$ and $G/N_g \in \underline{\underline{X}}$. Thus $G \in \underline{\underline{RX}}$ if and only if the intersection of all $N \lhd G$ for which $G/N \in \underline{\underline{X}}$ is trivial. Let Ω be a set of operators on the group G; that is to say, each element $\omega \in \Omega$ induces an endomorphism of G. The subgroup H of G is said to be Ω admissible or Ω invariant if $H_\omega \leq H$ for all $\omega \in \Omega$. H is characteristic in G if H is $\mathcal{A}(G)$ admissible, where $\mathcal{A}(G)$ is the group of automorphisms of G. The Frattini subgroup $\phi(G)$ is characteristic in G; this follows easily from the fact that the automorphic image of a maximal subgroup of G is maximal in G, hence an automorphism α of G just permutes its maximal subgroups, leaving their intersection invariant. A group without proper normal (characteristic) subgroups is called simple (characteristically simple).

We recall from the introduction that an element g of a group G is a nongenerator if for every subset S of G, $\langle S,g\rangle = G$ implies $\langle S\rangle = G$. We have in addition the stronger notion of an omissible subset W of G: W is omissible if for every subset S of G, $\langle S,W\rangle = G$ implies $\langle S\rangle = G$. Clearly each element of an omissible set is a nongenerator, but an infinite set of nongenerators need not be omissible.

The fact that $\phi(G)$ is the union of the nongenerators of G plays such a fundamental role in what is to follow that we justify it here. If an element $g \notin \phi(G)$ then there exists a maximal subgroup M of G such that $g \notin M$, and although $M \neq G$, $\langle M,g\rangle = G$. Thus if g is a nongenerator it lies in $\phi(G)$. Conversely, suppose there is a subset S of G with the property that $\langle S,g\rangle = G$ but $\langle S\rangle \neq G$. If $\underline{\underline{L}}$ is the set of subgroups containing S but not g, it is easy to see that Zorn's Lemma will yield a maximal element A. A is a maximal subgroup of G because every larger subgroup must contain g and by assumption must then coincide with G . Hence $g \notin A \geq \phi(G)$ which shows that every element of $\phi(G)$ is a nongenerator.

As was mentioned earlier the Frattini subgroup of a finite group is nilpotent; moreover if G is nilpotent then $\phi(G) \geq G'$, and the Frattini subgroup of a finite p group for the prime p consists of its derived group and the subgroup of the p^{th} powers of its elements [3]. If N is a normal subgroup of G and τ is the canonical homomorphism of G onto G/N , then there is a one to one correspondence between maximal subgroups of G containing N and maximal subgroups of G/N . Thus if $\phi(G/N) = 1$, that is if those maximal subgroups of G containing N intersect in N , we have $\phi(G) \leq N$. Hence if G is residually ϕ free, that is if those normal subgroups of G with ϕ free factor group intersect trivially, then G is itself ϕ free. Similarly, $\frac{\phi(G) \cdot N}{N} \leq \phi(G/N)$ and if $N \leq \phi(G)$, then $\frac{\phi(G)}{N} = \phi(G/N)$.

A group G is called an extension of a group A by a group B if $A \trianglelefteq G$ and $G/A \cong B$. If in addition, B can actually be injected into G , it is easily checked that $A \cap B = 1$, $G = AB$ and that the map $\rho : B \to \mathcal{A}(A)$ defined by $b\rho(a) = a^b$ is a homomorphism. In this case G is said to split over A and is called a splitting extension of A by B . In the particular case when ρ is the trivial homomorphism sending each element of B to the identity automorphism of A , G is the direct product of its subgroups A and B . If G splits over A , A is said to be complemented by B in G ; if G is abelian A is called a direct summand.

The basis theorem for finitely generated abelian groups states that every such group can be represented as the direct product of a finite number of infinite cyclic groups and/or cyclic groups of prime power order, and that this representation is unique up to isomorphism. This is equivalent to saying that there exists at least one generating set $\{a_1,\dots,a_n\}$ (called a basis) for A such that each element $a \in A$ can be represented in the form $a_1^{\epsilon_1} \dots a_n^{\epsilon_n}$ ($\epsilon_i \in Z$ $(1 \le i \le n)$ where this representation is unique up to choice of basis. Moreover if $H \le A$, then one can choose a basis $a_1,\dots,a_n$ for A such that for a suitable choice of integers $\epsilon_1, \epsilon_2, \dots, \epsilon_n$, the elements $a_1^{\epsilon_1}, a_2^{\epsilon_2}, \dots, a_n^{\epsilon_n}$ form a basis for H. Since $\phi(C_\infty) = 1$, $\phi(C(p^n)) = C(p^{n-1})$ and $\phi(A_1 \times \dots \times A_n) = \phi(A_1) \times \dots \times \phi(A_n)$, the basis theorem enables one to determine immediately the Frattini subgroup of all finitely generated abelian groups.

If an abelian group has no elements of finite (infinite) order it is called torsion (torsion-free). In any abelian group A the elements of finite order form a subgroup called the torsion component of A, usually denoted by A_T, and the factor group A/A_T is torsion free. The torsion free component of A, similarly defined, is denoted by A_F. If A is finitely generated, then clearly $A = A_T \times A_F$.

The exponent of a group is the smallest integer n which is divisible by the order of each element of the group. (A group with elements of infinite order will have exponent 0). If an abelian group A has exponent p for a prime p it is called an elementary abelian p group and may be considered as a vector space over the field of integers modulo p. Then since every subgroup of A may be treated as a subspace of A and as such is complemented in A, it is clear that every subgroup of an elementary abelian p group is a direct summand.*

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*Moreover if an abelian group A is the direct product of elementary abelian p groups for various primes p, then every subgroup of A is a direct summand, since each subgroup is the direct product of its intersections with the p^{th} components of A.

The treatment of an elementary abelian p group A as a vector space over a field of characteristic p, particularly in the context of a finite group G that splits over A, and in which A has order prime to its index,gives rise to some representation theory and to some powerful results due to Maschke and Schur and Zassenhaus, which will be used in Chapter II. A representation of a group G is a homomorphism ρ of G into the algebra of linear transformations of a finite dimensional vector space V over a field P. The representation ρ of G is termed completely reducible if V can be decomposed into minimal G invariant subspaces (where G is thought of as a group of operators on V). Maschke's theorem states that every representation of a finite group G in a field P whose characteristic does not divide the order of G is completely reducible.

The Schur-Zassenhaus theorem states that if A is a normal subgroup of G with order prime to its index, then G splits over A, and any two complements for A are conjugate by an element in A. (It is required that either A or G/A be soluble, but the fact that groups of odd order are soluble, due to Feit and Thompson, assures this.)

Finally, the abelian group A is called divisible if the equation $x^n = a$ has a solution in A for every positive integer n and every a in A, and a divisible abelian group is a direct sum of groups each isomorphic to the additive group of rationals or to $Z_p(\infty)$ for various primes p, where $Z_p(\infty)$ is the subgroup of the additive group of rationals modulo the integers consisting of those elements with denominator a power of p.

We now discuss some important methods of constructing new groups from given groups. A group G is said to be the free product of its subgroups G_λ (where λ ranges over some index set Λ) if the subgroups G_λ generate G, that is, every element g of G is a product of a finite number of the elements of the G_λ

(1) $$g = g_1 g_2 \ldots g_n \qquad g_i \in G_{\lambda_i} \qquad i = 1,2,\ldots,n$$

and every element g of G, $g \neq 1$, has a unique representation in the form (1) subject to the condition that all elements g_i are nontrivial and that in (1) no two adjacent elements are in the same subgroup G_λ -- although the product (1) may in general contain several factors from one and the same subgroup. The free product is denoted by the symbol

(2) $$G = \prod_{\lambda \in \Lambda}^{*} G_\lambda$$

and if $\Lambda = \{1\ 2,\ldots,k\}$ is finite, by the symbol

$$G = G_1 * G_2 * \ldots * G_k .$$

The subgroups G_λ are called the free factors in the decomposition (2) of G; the expression (1) is called the normal form of the element g in the decomposition (2) and the number n the length of g in this decomposition; we write $n = \ell(g)$.

(3) We have the following characterization for a free product of groups: if a group G is generated by subgroups G_λ (where λ ranges over an index set Λ) then G is the free product of these subgroups if and only if for every group H and every set of homomorphic mappings φ_λ of each G_λ into H there exists a homomorphic mapping φ of G into H that coincides with φ_λ on G_λ.

A group which can be presented without defining relations is called a free group; such a group is a free product of its cyclic subgroups

$<x_\lambda>$ where $\{x_\lambda\}_{\lambda \in \Lambda}$ is a set of free generators for the group. In this special case the characterization (3) is called the universal mapping property for free groups. Subgroups of free groups are free and free groups are Hopfian, that is they have no automorphisms with nontrivial kernel.

For an arbitrary collection of groups G_λ a word in the G_λ is an ordered system of elements $w = g_1 g_2 \dots g_n$ where the length $n \geq 1$, where each g_i and g_{i+1} belong to different groups G_λ. We define a product of two words w and w' by writing down the words in juxtaposition and performing the necessary cancellations and contractions to obtain a word in nontrivial elements in which adjacent terms come from different groups. (See Kurosch, [10], vol. II, p.12). With this multiplication and with the empty word (in the case $n = 0$) acting as the identity element, the set of all words in the G_λ form a group isomorphic to the free product of the G_λ and every word of length > 0 is nontrivial.

We now turn to an even more general construction than that of free products. Let G_λ be groups where λ ranges over a set of indices Λ, and let a proper subgroup H_λ be chosen in every G_λ such that each H_λ is isomorphic to a fixed group H. By i_λ we denote a specific isomorphic mapping of H_λ onto H; then $i_{\lambda\mu} = i_\lambda i_\mu^{-1}$ is an isomorphic mapping of H_λ onto H_μ. The free product of the groups G_λ with the amalgamated subgroup H, denoted by $\overset{*}{\prod}_{\lambda \in \Lambda} (G_\lambda; H)$ and by $(G_1 * G_2 * \dots * G_k; H)$ when Λ is finite, is defined as the factor group G of the free product of the groups G_λ with respect to the normal subgroup generated by all the elements of the form $h_\lambda h_\mu^{-1}$ where $h_\mu = h_\lambda i_{\lambda\mu}$, where h_λ

ranges over the whole subgroup H_λ and where λ and μ are all possible index pairs. In other words, if every group G_λ is given by a system of generators X_λ and a system of defining relations R_λ between these generators, then G has as a system of generators the union of all sets X_λ, as a system of defining relations the union of the sets R_λ and in addition, all relations obtained by identifying those elements of different subgroups H_λ and H_μ which are mapped by the isomorphisms i_λ and i_μ onto the same element of H. If in each group G_λ we select a left transversal T_λ (including one) of the subgroup H_λ then a word in G is an expression

$$t_1 \, t_2 \, \ldots \, t_n \, h \tag{4}$$

where $n \geq 0$, where h is any element of H, possibly 1, where each t_i is a nontrivial coset representative of the left coset H_λ in G_λ for some λ and where adjacent representatives t_i, t_{i+1} $(1 \leq i < n)$ belong to distinct groups G_λ. If G is the free product of the groups G_λ with amalgamated subgroup H, then every element of G can be expressed in this form. In particular, if $g \in G \backslash H$ then since $t_n h \in G_\lambda \backslash H_\lambda$ for some $\lambda \in \Lambda$, g can be expressed as a word whose terms belong to $G_\lambda \backslash H_\lambda$ and in which adjacent terms lie in distinct groups G_λ. The expression (4) is called the normal form for g in G and is unique up to choice of transversals T_λ; the number n the (unique) length of g. Elements in H have length 0; all elements of length $n > 0$ are nontrivial.

Conversely, if a group G is generated by subgroups $G_\lambda (\lambda \in \Lambda)$ such that any pair of distinct subgroups intersect in the same subgroup H and if, for any choice of left transversals T_λ (including one) of H in G_λ, each element of G has a unique representation in the form (4), then G is the free product of its subgroups G_λ amalgamating H. From this it is not difficult to deduce that if $L_\lambda \leq G_\lambda$ and $L_\lambda \cap H = L_\mu \cap H$ for all $\lambda, \mu \in \Lambda$, then the groups $L_\lambda (\lambda \in \Lambda)$ generate their free product amalgamating their common intersection with H.

The following analogue of the characterization (3) for free products states that the free product with amalgamations is in some sense the freest thing of its kind. Let G be the free product of the groups G_λ amalgamating a common subgroup H and let P be a group which contains to each G_λ a homomorphic copy $\bar{G}_\lambda = G_\lambda \varphi_\lambda$ in such a way that two homomorphisms $\varphi_\lambda, \varphi_\mu$ are compatible with $i_{\lambda\mu}$, that is, if $h \in H_\lambda$ then $h\varphi_\lambda = h\, i_{\lambda\mu}\, \varphi_\mu$. Then all of the φ_λ can be extended to a homomorphic mapping of G into P which agrees with φ_λ on every G_λ. [12]

In the following discussion of free products with amalgamations let it be understood that unless otherwise stated all of the two or more constituent groups (factors) are nontrivial and all of the amalgamations are proper subgroups of each of the factors. For the sake of convenience we will often de-emphasize the amalgamating isomorphisms $i_{\lambda\mu}$ and denote all of the amalgamated subgroups H_λ by H.

Section 2

We first show that a free product with amalgamations necessarily has maximal subgroups, provided the normal closure of the amalgamated subgroup in each of the factors is properly contained in that factor.

Proposition 1. Let G be the free product of the groups $\{G_\lambda\}_{\lambda \in \Lambda}$ amalgamating the subgroups H_λ of G_λ via the isomorphisms $i_{\lambda\mu}$, and suppose $nm_{G_\lambda} H_\lambda < G_\lambda \ \forall \lambda \in \Lambda$. Then $\phi(G) \leq nm_G H$, where H is the amalgamated subgroup of G .

Proof: If we denote $\prod^{*}_{\lambda\in\Lambda} G_\lambda/\mathrm{nm}_{G_\lambda}H_\lambda$ by G^{*} , then by Higman and Neuman's result for the Frattini subgroup of free products of groups it is enough to prove that $G/\mathrm{nm}_G H \cong G^{*}$, for as was noted in the preliminaries, this will imply the proposition. In order to justify this isomorphism, consider the epimorphisms $\varphi_\lambda : G_\lambda \longrightarrow G_\lambda/\mathrm{nm}_{G_\lambda}H_\lambda$. Since $h \in H_\lambda$ implies $h\varphi_\lambda = 1 = hi_{\lambda\mu}\varphi_\mu$, the maps φ_λ may be extended to an epimorphism $\varphi : G \longrightarrow G^{*}$. It will suffice to show that the kernel of φ is the normal closure of H in G . If $g \in \mathrm{nm}_G H$, $g = \prod_{i=1}^{n} h_i^{g_i}$ where $g_i \in G$, $h_i \in H$ $(1 \le i \le n)$; hence $g\varphi = 1$ and $g \in \ker\varphi$. On the other hand, if an element g in the kernel of φ is put in normal form $g = g_1 g_2 \dots g_n h$ where $g_i \in G_{\lambda_i}$, $h \in H$, $\lambda_i \neq \lambda_{i+1}$, we have $1 = (g_1 g_2 \dots g_n h)\varphi = g_1\varphi\, g_2\varphi \dots g_n\varphi$ where $g_i\varphi \in G_{\lambda_i}/\mathrm{nm}_{G_{\lambda_i}}H_{\lambda_i}$ and $\lambda_i \neq \lambda_{i+1}$, which implies, by virtue of the fact that G^{*} is a free product, that $g_i\varphi = 1$ for $1 \le i \le n$, i.e., that $g_i \in \mathrm{nm}_{G_{\lambda_i}}H_{\lambda_i}$. Hence $g \in \mathrm{nm}_G H$.

The proposition yields two corollaries which will be of future use.

Corollary 1. If $G, \Lambda, G_\lambda, H_\lambda$ and H are defined as in proposition 1 and $H_\lambda \triangleleft G_\lambda$ for all $\lambda \in \Lambda$, then $\Phi(G) \le H$.

Corollary 2. Let $G, \Lambda, G_\lambda, H_\lambda$ and H be defined as in proposition 1. Then $\mathrm{nm}_G H \cap G_\lambda = \mathrm{nm}_{G_\lambda}H_\lambda$ for all $\lambda \in \Lambda$.

Proof: Clearly $\mathrm{nm}_{G_\lambda}H_\lambda \le \mathrm{nm}_G H \cap G_\lambda$ for all $\lambda \in \Lambda$. But if $g \in \mathrm{nm}_G H \cap G_\lambda$, by proposition 1 $g \in \ker\varphi \cap G_\lambda = \mathrm{nm}_{G_\lambda}H_\lambda$ since $\varphi = \varphi_\lambda$ on G_λ .

In their short proof that the free product G of nontrivial groups is ϕ free, Higman and Neuman found for any element g of G a proper subgroup T_g of G which together with g generated G. The following proposition is a generalization of their argument.

Proposition 2. If in a free product G with amalgamated subgroup H there exists an element which conjugates each nontrivial element of H outside of H then G is ϕ free.

Proof: Let $G = \prod_{\lambda\in\Lambda}^{*}(G_\lambda;H_\lambda)$ and let μ be an element of the nonempty indexing set Λ. If $A = G_\mu$, $H_A = H_\mu$ and $B = \prod_{\substack{\lambda\in\Lambda\\ \lambda\neq\mu}}^{*}(G_\lambda;H_\lambda)$ with H_B the amalgamated subgroup of B, then $G = (A * B;\ H_A = H_B)$; hence it is enough to prove the proposition for the free product with amalgamations of two groups A and B. So let $G = (A * B;\ H_A \overset{i}{\cong} H_B)$ where H_A and H_B are isomorphic (via i) subgroups of A and B respectively, let $H = H_A = H_B$, and let $c \in G$ be such that $h^c \notin H$ for all nontrivial h in H. Note then that $h^{c^{-1}} \notin H$ for all nontrivial h in H. (For if $h^{c^{-1}} = h'$ for nontrivial elements $h, h' \in H$, then $(h')^c = (h^{c^{-1}})^c = h$, contrary to assumption.) We use the normality of $\phi(G)$ to conclude that since no nontrivial subgroup of H is normal in G, it is enough to exclude elements of length ≥ 1 from $\phi(G)$, and secondly, in order to exclude an arbitrary element g in G of length at least one, it is enough to exclude any conjugate of g by an element of G (or, in fact, any power of g or g^{-1}). We do this by showing that the element in question is not a nongenerator, that is by producing a proper subgroup T of G which together with our element generates G.

We consider four cases, corresponding to the four possibilities for a normal form for c.

<u>Case I</u>: $c = \alpha_1\beta_1\ldots\alpha_m\beta_m$ where α_i, $\beta_i \in A\backslash H$, $B\backslash H$ respectively $(1\le i\le m)$ and $m \ge 1$. If g is an arbitrary element of G of length at least one, conjugation of g by a suitable element of G will yield one of the two normal forms (i) $\bar\beta_1\bar\alpha_1\ldots\bar\beta_n\bar\alpha_n\bar\beta_{n+1}$ or (ii) $\bar\alpha_1\bar\beta_1\ldots\bar\alpha_n\bar\beta_n$, where $n \ge 1$; hence we may assume that g is in one of these two forms.

(i) If $g = \bar\beta_1\bar\alpha_1\ldots\bar\beta_n\bar\alpha_n\bar\beta_{n+1}$ we exclude $z = g^{c\alpha_1 c^{-1}}$ from $\phi(G)$. If $T < \langle z^{-1}Az , B\rangle$ it is clear that $\langle T,z\rangle = G$. If $z \in T$, z would equal a product of terms coming alternately out of $z^{-1}Az$ and B and multiplication of both sides by z^{-1} would yield one of the following equations, where we may assume the a_i and b_j are nontrivial:

a) $1 = a_1^z b_1 a_2^z b_2\ldots a_k^z b_k\; z^{-1}$ $(k \ge 1)$

b) $1 = b_1(a_1^z b_2\ldots a_k^z b_{k+1}\; z^{-1})$ $(k \ge 0)$

c) $1 = (a_1^z b_1 a_2^z b_2\ldots a_k^z b_k\; z^{-1})a_{k+1}$ $(k \ge 0)$

d) $1 = b_1(a_1^z b_2\ldots a_k^z b_{k+1}\; z^{-1})a_{k+1}$ $(k \ge 0)$.

Equation (a) becomes

$$1 = c\alpha_1^{-1}c^{-1}g^{-1}\underbrace{c\alpha_1c^{-1}a_1c\alpha_1^{-1}c^{-1}}gc\alpha_1\underbrace{c^{-1}b_1c}\,\alpha_1^{-1}c^{-1}g^{-1}\underbrace{c\alpha_1c^{-1}a_2c\alpha_1^{-1}c^{-1}}gc\alpha_1c^{-1}b_2\ldots$$
$$\ldots c\alpha_1^{-1}c^{-1}g^{-1}\underbrace{c\alpha_1c^{-1}a_k\;c\alpha_1^{-1}c^{-1}}\;gc\alpha_1\underbrace{c^{-1}b_kc}\,\alpha_1^{-1}c^{-1}g^{-1}c\alpha_1c^{-1}\;. \qquad (1)$$

But if $a_1 \in H$ then a_1^c is a word beginning and ending with a term from $B\backslash H$ which cannot cancel with α_1 or α_1^{-1} . If $a_1 \notin H$ and $a_1^{c\alpha_1^{-1}} \in H$, then the first bracketed expression begins and ends with a term from $A\backslash H$ which will not cancel with $\bar\beta_1$ or $\bar\beta_1^{-1}$. Similarly, if $b_1 \in H$ then b_1^c begins and ends with a term from $B\backslash H$, while if $b_1 \in B\backslash H$ no cancellation can occur at all. Continuing this process of combining terms we see that at each a_i and b_j after combining at most seven terms we are left with an expression with factors coming alternately out of $A \backslash H$ and $B \backslash H$, and that the terms consumed at an a_i or b_j never overlap. Hence the expression in (1) cannot collapse to 1 . Moreover, with closer scrutiny we observe that even if the right hand side of equation (1) were preceded with a nontrivial element $b \in B$ and/or followed by a nontrivial element $a \in A$, the cancellation stops at c and c^{-1} ; hence equations (b) through (d) cannot occur. From this it follows that $z \notin T$, i.e., that z and hence $g \notin \phi(g)$.

(ii) If $g = \bar{\alpha}_1\bar{\beta}_1 \ldots \bar{\alpha}_n\bar{\beta}_n$ ($\alpha_i \in A \backslash H$, $\beta_j \in B \backslash H$), choose an integer k such that $kn > m$. Then $\ell(g^k) = 2kn > 2m = \ell(c)$, and

$$cg^kc^{-1} = \overbrace{\alpha_1\beta_1 \ldots \alpha_m\beta_m}^{c} \; \overbrace{\bar{\alpha}_1\bar{\beta}_1 \ldots \bar{\alpha}_n\bar{\beta}_n \ldots \bar{\alpha}_1\bar{\beta}_1 \ldots \bar{\alpha}_n\bar{\beta}_n}^{g^k} \, \overbrace{\beta_m^{-1}\alpha_m^{-1} \ldots \beta_1^{-1}\alpha_1^{-1}}^{c^{-1}} \; . \qquad (2)$$

Cancel and combine as much as possible at the only spot where cancellation can occur -- between $\bar{\beta}_n$ and β_m^{-1} . Note that c remains untouched. In fact at worst one is left with c followed by two terms coming out of $A \backslash H$ and $B \backslash H$, say $cg^kc^{-1} = c\alpha\beta$. If after cancellation and contraction cg^kc^{-1} ends in a $\beta \in B \backslash H$, i.e., $cg^kc^{-1} = c\alpha \ldots \beta$, square it to obtain $c\alpha \ldots \beta c\alpha \ldots \beta = z$. We exclude z from $\phi(G)$; for this purpose let $T = < z^{-1}B\, z, A >$. Again, under the assumption that $z \in T$ we obtain one of the following equations:

a′) $1 = b_1^z a_1 b_2^z a_2 \ldots b_k^z a_k z^{-1}$ $\quad$ $(k \geq 1)$

b′) $1 = a_1(b_1^z a_2 \ldots b_k^z a_{k+1} z^{-1})$ $\quad$ $(k \geq 0)$

c′) $1 = (b_1^z a_1 b_2^z a_2 \ldots b_k^z a_k z^{-1}) b_{k+1}$ $\quad$ $(k \geq 0)$

d′) $1 = a_1(b_1^z a_2 \ldots b_k^z a_{k+1} z^{-1}) b_{k+1}$ $\quad$ $(k \geq 0)$

As in the preceding case we examine only equation (a′); that equations (b′) through (d′) cannot occur will follow as before from the limited contractions possible in this equation.

Equation (a′) becomes

$$\begin{aligned}1 = {} & \beta^{-1}\ldots\alpha^{-1}c^{-1}\beta^{-1}\ldots\alpha^{-1}\underbracket{c^{-1}b_1c}\,\alpha\ldots\beta \qquad (2') \\ & \cdot\underbracket{c\alpha\ldots\beta a_1\beta^{-1}\ldots\alpha^{-1}c^{-1}}\,\beta^{-1}\ldots\alpha^{-1}\underbracket{c^{-1}b_2c}\,\alpha\ldots\beta c\alpha\ldots\beta a_2\,\ldots \\ & \ldots\beta^{-1}\ldots\alpha^{-1}c^{-1}\beta^{-1}\ldots\alpha^{-1}\underbracket{c^{-1}b_nc}\,\alpha\ldots\beta\underbracket{c\alpha\ldots\beta a_n\beta^{-1}\ldots\alpha^{-1}c^{-1}}\beta^{-1}\ldots\alpha^{-1}c^{-1}\,.\end{aligned}$$

As the bracketed expressions involving b_i begin and end with terms coming from $B\backslash H$ and those involving a_i begin and end with terms coming from $A\backslash H$, the entire expression in (2′) cannot collapse to 1; in fact equations (b′) through (d′) are also impossible, implying as before that $z \notin T$ and hence $g \notin \phi(G)$.

If after cancellation in (2) $cg^kc^{-1} = c\ldots\alpha$ where $\alpha \in A\backslash H$, let $z = c^{-1}\beta_1cg^kc^{-1}\beta_1^{-1}c = c^{-1}\beta_1c\ldots\alpha\beta_1^{-1}c$, and let $T = < z^{-1}Az,B >$. $z \in T$ only if an equality of type (a),(b),(c) or (d) holds; restricting our attention as before to equation (a) we have:

$$\begin{aligned}1 = {} & c^{-1}\beta_1\alpha^{-1}\ldots c^{-1}\beta_1^{-1}\underbracket{ca_1c^{-1}}\beta_1\underbracket{c\ldots\alpha\beta_1^{-1}cb_1c^{-1}\beta_1\alpha^{-1}\ldots c^{-1}} \qquad (3) \\ & \cdot\beta_1^{-1}\underbracket{ca_2c^{-1}}\beta_1c\ldots\alpha\beta_1^{-1}cb_2\,\ldots \\ & \ldots c^{-1}\beta_1\alpha^{-1}\ldots c^{-1}\beta_1^{-1}\underbracket{ca_nc^{-1}}\beta_1\underbracket{c\ldots\alpha\beta_1^{-1}cb_nc^{-1}\beta_1\alpha^{-1}\ldots c^{-1}}\beta_1^{-1}c\,.\end{aligned}$$

Consider the bracketed expressions involving the a_i. If $a_i \notin H$ no cancellation can occur at all; if $a_i \in H$ then a_i^c begins and ends with a term from $A\backslash H$, hence cannot cancel with β_1 or β_1^{-1}. Similarly, in those bracketed expressions involving b_i, if $b_i \in H$ cancellation stops at $b_i^{c^{-1}}$. If $b_i \notin H$ and $b_i^c \in H$, cancel and combine as much as possible, noting that at worst the bracketed expressions will be of the form chc^{-1} (where $h \in H$), which begins and ends with terms from $A \setminus H$; hence cannot combine with β_1 or β_1^{-1}. As before we must conclude that $z \notin T$ and $g \notin \phi(G)$. Thus if c is in form I, $\phi(G) = 1$.

Case II: $c = \beta_1\alpha_1 \ldots \beta_m\alpha_m$ ($\alpha_i, \beta_i \in A\setminus H, B\setminus H$ respectively for $1 \le i \le m$; $m \ge 1$). Since by conjugating g (where g is an arbitrary element of G subject to $\ell(g) \ge 1$) by a suitable element of G we may also assume that either (i′) $g = \bar{\alpha}_1\bar{\beta}_1 \ldots \bar{\alpha}_n\bar{\beta}_n\bar{\alpha}_{n+1}$ ($n \ge 0$) or (ii′) $g = \bar{\beta}_1\bar{\alpha}_1 \ldots \bar{\beta}_n\bar{\alpha}_n$ ($n \ge 1$), Case II follows from I by interchanging A and B.

Case III: $c = \alpha_1\beta_1 \ldots \alpha_m\beta_m\alpha_{m+1}$ ($\alpha_i, \beta_j \in A\setminus H, B\setminus H$ respectively, $1 \le i \le m+1$, $1 \le j \le m$, $m \ge 0$).

Again, given an element g in G of length at least one we conjugate it judiciously to obtain (i) or (ii). If g can be put in form (i), that is, $\bar{\beta}_1\bar{\alpha}_1 \ldots \bar{\beta}_n\bar{\alpha}_n\bar{\beta}_{n+1}$ ($\bar{\beta}_i \in B\setminus H$ $1 \le i \le n+1$, $\bar{\alpha}_j \in A\setminus H$ $1 \le j \le n$, $n \ge 0$) we exclude $z = g^{c^{-1}\beta_1 c}$ by choosing $T = \langle z^{-1}Az, B \rangle$ and observing that while $\langle T, z \rangle = G$, if $z \in T$ one of the equations (a) through (d) must hold, or, again confining ourselves to (a) and expanding,

$$\begin{aligned} 1 = {} & c^{-1}\beta_1^{-1}cg^{-1}\underbrace{c^{-1}\beta_1 c a_1 c^{-1}\beta_1^{-1}c}\, gc^{-1}\beta_1 c\underbrace{b_1 c^{-1}} \\ & \cdot \beta_1^{-1}cg^{-1}\underbrace{c^{-1}\beta_1 c a_2 c^{-1}\beta_1^{-1}c}\, gc^{-1}\beta_1 c b_2 \ldots \\ & \ldots c^{-1}\beta_1^{-1}cg^{-1}\underbrace{c^{-1}\beta_1 c\, a_n\, c^{-1}\beta_1^{-1}c}\, gc^{-1}\beta_1 c\underbrace{b_n c}{}^{-1}\beta_1^{-1}cg^{-1}c^{-1}\beta_1 c . \end{aligned} \tag{4}$$

Note that after all possible cancellations are performed the bracketed expressions begin and end with terms from $A\setminus H$ and we are left with a nontrivial expression. We complete the argument as before.

To eliminate g qua form (ii), i.e., $\bar{\alpha}_1\bar{\beta}_1 \ldots \bar{\alpha}_n\bar{\beta}_n$ ($\bar{\alpha}_i, \bar{\beta}_i \in A\setminus H, B\setminus H$ respectively, $n \ge 1$) from $\phi(G)$ we again choose k such that $kn > m$, let

$$z = cg^{-k}c^{-1} = \overbrace{\alpha_1\beta_1\ldots\alpha_m\beta_m\alpha_{m+1}}^{c}\overbrace{(\bar{\beta}_n^{-1}\ldots\bar{\alpha}_1^{-1}\ldots\bar{\beta}_n^{-1}\ldots}^{g^{-k}}\underbrace{\bar{\alpha}_1^{-1})\alpha_{m+1}^{-1}}_{}\overbrace{\beta_m^{-1}\alpha_m^{-1}\ldots\beta_1^{-1}\alpha_1^{-1}}^{c^{-1}} \quad (5)$$

and apply the argument used in I (ii), since the only essential difference between (5) and (2) is that in (5) at worst, $2m+1$ terms of g^{-k} are consumed after cancellation; c however still remains intact.

Case IV: $c = \beta_1\alpha_1\ldots\beta_m\alpha_m\beta_{m+1}$ $(m\geq 0)$. In this case we conjugate our arbitrary g of nonzero length into forms (i′) or (ii′) by a suitable element of G and, interchanging A and B, apply Case III.

Since we have exhausted all the possibilities for c, the proposition follows.

We state again that a subgroup H of a group G is called abnormal if $h^g \notin H$ for all $h \neq 1$ in H and all $g \in G\backslash H$.

Corollary 3. Let $\{G_\lambda\}_{\lambda\in\Lambda}$ be a family of groups, $H_\lambda < G_\lambda$ such that for all $\lambda,\mu \in \Lambda$ $H_\lambda \cong H_\mu$, and suppose that for some $\gamma \in \Lambda$, H_γ is abnormal in G_γ. Then the free product G of the groups G_λ amalgamating the subgroups H_λ is ϕ free.

Corollary 4. Let $\{G_\lambda\}_{\lambda\in\Lambda}$, $\{H_\lambda\}_{\lambda\in\Lambda}$, G be as in Corollary 3, with each H_λ a cycle of order p for some prime p and suppose for some $\gamma \in \Lambda$, H_γ is not normal in G_γ. Then $\phi(G) = 1$.

Proof: By assumption there exists $g \in G_\gamma$ such that $h^g \notin H_\gamma$ for some $\neq 1$ in H_γ which implies that $(h^\lambda)^g \notin H_\gamma$ $(1\leq\lambda<p)$, that is the hypothesis of proposition 2 is satisfied.

Proposition 3. Let $X_1, X_2, \ldots, X_n$ be infinite cyclic groups, with $X_i = < x_i >$ and let $H_i < X_i$, say $H_i = < x_i^{m_i} >$ $(m_i>1, 1\leq i\leq n)$.

If G is the free product of the groups X_i amalgamating the subgroups H_i, then $\phi(G) = 1$.

Proof: If we denote the amalgamated subgroup of G by H, then by Corollary 1, $\phi(G) \le H$. So let $h = x_i^{\lambda m_i}$ $(1\le i\le n,\ \lambda>0)$ be an arbitrary element of H. Choose an integer $\rho > 1$ such that $(\rho,\lambda m_i) = 1$ for $i = 1,\dots,n$, and let $T = \langle x_i^{\rho};\ 1\le i\le n \rangle$. Then $\langle T,h \rangle = G$. But $T \neq G$. To see this, let $Y_i = \langle x_i^{\rho} \rangle$ for all i and note that for $1\le j,k\le n$ $Y_j \cap H = \langle x_j^{\rho m_j} \rangle = \langle x_k^{\rho m_k} \rangle = Y_k \cap H$; hence $T = \langle Y_i,\ 1\le i\le n \rangle = \prod_{i=1}^{n}{}^{*}(Y_i;Y_i \cap H)$. Suppose now, that $x_1 \in T$. Since $x_1 \notin Y_1$, the normal form for x_1 in T must have length ≥ 2, say $x_1 = x_{i_1}^{r_1\rho} x_{i_2}^{r_2\rho} \dots x_{i_\ell}^{r_\ell\rho}$ where $x_{i_k}^{r_k\rho} \in Y_{i_k} \setminus H_{i_k}$ $(1\le k\le \ell, \ell\ge 2)$ and $i_k \neq i_{k+1}$ $(1\le k<\ell)$, i.e., $1 = x_1^{-1}x_{i_1}^{r_1\rho} x_{i_2}^{r_2\rho} \dots x_{i_\ell}^{r_\ell\rho}$. If $x_1^{-1}x_{i_1}^{r_1\rho} \notin H$ we have in G a product of terms lying outside of H such that no two successive terms belong to the same factor equal to one, which is impossible. If $i_1 = 1$ and $x_1^{r_1\rho-1} \in H$ then $x_1^{r_1\rho-1} x_{i_2}^{r_2\rho} \in Y_{i_2} \setminus H$, again giving the same contradiction. Thus $x_1 \notin T$ and $h \notin \phi(G)$. Since h was arbitrary, $\phi(G) = 1$.

With the aid of propositions 2 and 3 and the following lemma, we will be in a position to deduce the main result of this section.

Lemma 1: Let g and h be nontrivial elements of a free group F. If $g^{-1}h^{\lambda}g = h^{\rho}$, then $\lambda = \rho$ and $[g,h] = 1$.

Proof: Consider $K = \langle g,h \rangle$. K is free, generated either by one or two free generators. If K is free on one generator, no proof is required, so suppose K is freely generated by x and y . Let φ be the epimorphism of K mapping x to g and y to h . Since K is free it is Hopfian; thus if any word $w(g,h)$ in g and h is trivial, then $w(x,y) = 1$, that is $w(g,h)$ collapses trivially. Hence g and h freely generate K and the lemma follows.

Corollary 5. Let F be a free group on a set of free generators $X = \{x_1, x_2, \ldots\}$ where X may have any cardinality greater than one, and let $H = < h >$ be a cyclic subgroup of F . Then there exists $c \in F$ such that $(h^n)^c \not\in H$ for all nonzero integers n .

Proof: By lemma 1 it is enough to produce an element c such that $[h,c] \neq 1$. Express h uniquely as a word $w(x_i)$ in the given generators for F and let S be the subset of X consisting of those x_i that appear in $w(x_i)$. If there exists an $x_k \in X \backslash S$, then $[h,x_k] \neq 1$. If $S = X$ then $w(x_i)$ involves at least two free generators, either one of which will not commute with h .

Collecting our results we now have the following

Theorem 1. The free product G of a finite number of free groups amalgamating a proper cyclic subgroup is Φ free.

For by proposition 3 we may assume one of the free factors is noncyclic, which implies, by Corollary 5, that the hypothesis of proposition 2 is satisfied.

In some sense this result is as it should be, for we are dealing with a free product of free groups with a "slim" (in this case cyclic) amalgamation. It is interesting to note, however, that the theorem is false if we relax the finiteness condition on the number of factors. To see this, let

$G = < a_1, a_2, a_3, \ldots, a_i, \ldots \; ; \; a_1^2 = a_2^3 = a_3^5 = \ldots = a_i^{p_i} = \ldots >$ where p_i denotes the i^{th} positive prime. G is the free product of a countable number of infinite cyclic groups with amalgamated subgroup H ; let $H = < h >$. By Corollary 1, $\phi(G) \leq H$; in order to show that $\phi(G)$ actually coincides with H , it is enough to show that $h \in \phi(G)$. Suppose not and let M be a maximal subgroup of G excluding h . Then, since $G = < M, h >$ and $H = \zeta(G)$, we have for elements m_i in M , $a_i = m_i h^{\alpha_i} = m_i a_i^{p_i \alpha_i}$ $(\alpha_i \in Z,\ i=1,2,\ldots)$ which implies that $m_i = a_i^{1-p_i\alpha_i}$ and thus $m_i^{p_i} = a_i^{p_i(1-p_i\alpha_i)} = h^{(1-p_i\alpha_i)} \in M \cap H$. Let $p_{i_1}, \ldots, p_{i_n}$ be the prime divisors of $1-2\alpha_1$. Then the integers $1-2\alpha_1$, $1-p_{i_1}\alpha_{i_1}, \ldots, 1-p_{i_n}\alpha_{i_n}$ are relatively prime; hence there exist integers $\beta_1, \beta_{i_1}, \ldots, \beta_{i_n}$ such that $\beta_1(1-2\alpha_1) + \sum_{k=1}^{n} \beta_{i_k}(1-p_{i_k}\alpha_{i_k}) = 1$ and therefore

$$m_1^{2\beta_1} \cdot \prod_{k=1}^{n} m_{i_k}^{p_{i_k}\beta_{i_k}} = h^{(1-2\alpha_1)\beta_1 + \sum_{k=1}^{n}(1-p_{i_k}\alpha_{i_k})\beta_k} = h \; ,$$

which puts h in M , a contradiction which establishes that $h \in \phi(G)$ and hence $H = \phi(G)$.

Section 3

In this section we utilize proposition 3 to draw some conclusions about the Frattini subgroups of free products with amalgamations when the factors are finitely generated abelian groups. We need the following lemma.

Lemma 2: Let $A_1, A_2, \dots, A_n$ be nontrivial finitely generated abelian groups, and for $1 \leq i \leq n$, let H_i be proper subgroups of A_i and $\varkappa_i : H_i \longrightarrow H$ isomorphisms mapping H_i to a fixed group H. Let $G = \prod_{i=1}^{n}{}^{*} (A_i ; H_i)$ with the amalgamated subgroup denoted by H, and let h be any nontrivial torsion free element of H. Then there exists a homomorphism φ mapping G onto the free product of n free abelian groups $B_i (1 \leq i \leq n)$ amalgamating a cyclic subgroup C such that $1 \neq h\varphi \in C$.

Proof: We choose bases $a_{i1}, a_{i2}, \dots, a_{is}$ for the groups $H_i (1 \leq i \leq n)$ by first selecting a basis $a_{11}, a_{12}, \dots, a_{1s}$ for H_1 and letting $a_{ij} = a_{1j} \varkappa_1 \varkappa_i^{-1}$ $(1 < i \leq n,\ 1 \leq j \leq s)$. Then by virtue of the amalgamating isomorphisms, $h = \prod_{\ell=1}^{s} a_{1\ell}^{e_\ell} = \dots = \prod_{\ell=1}^{s} a_{n\ell}^{e_\ell}$ $(e_\ell \in Z)$. Since h is not one and of infinite order we may assume, by rearrangement of basis elements if necessary, that $e_1 \neq 0$ and that the a_{i1} are of infinite order. Let $N_i = < a_{i2}, \dots, a_{is} >$. A_i/N_i, as finitely generated groups, can be written in the form $F_i/N_i \times T_i/N_i$, where the F_i/N_i are nontrivial free abelian groups and the T_i/N_i are torsion groups;

moreover the images of the a_{i1} lie nontrivially in F_i/N_i. Let $\varphi_i\colon A_i \longrightarrow A_i/T_i$, let $\bar{G} = \prod_{i=1}^{n}{}^{*}(A_i/T_i\ ;\ < a_{i1}T_i >)$ with the cyclic amalgamated subgroup denoted by C. The A_i/T_i, since isomorphic to F_i/N_i, are free abelian. It is easily checked that the homomorphisms φ_i are compatible with the amalgamating isomorphisms $\varkappa_i\varkappa_j^{-1}$ $(1 \le i \le n$, $1 \le j \le n)$; hence the φ_i may be extended to a homomorphism $\varphi\colon G \longrightarrow \bar{G}$. φ is an epimorphism because each factor of $\bar{G}$ is the image of some φ_i, and a generating set for $\bar{G}$ is contained in the union of given generating sets for each of the factors. Moreover, since $h\varphi$ $(= a_{11}^{e_1}T_1 = \ldots = a_{n1}^{e_1}T_n)$ is a nontrivial element of C, φ is the desired homomorphism.

Theorem 2. Let G be the free product of a finite number of finitely generated abelian groups amalgamating a subgroup H. Then $\phi(G)$ is contained in the torsion subgroup of H.

Proof: Let A_i, H_i, $\varkappa_i$ be as in lemma 2 and let H_T and H_F denote the torsion and torsion free components, respectively, of H. Since by Corollary 1 $\phi(G) \le H$, we choose an arbitrary nontrivial element h of H_F and exclude it from $\phi(G)$. By lemma 2 there exists a homomorphism φ mapping G onto $\bar{G} = \prod_{i=1}^{n}{}^{*}(B_i;C)$ where each B_i is free abelian and C is infinite cyclic, such that h is mapped nontrivially into C. Now we are free to choose a basis $b_{11}, b_{12}, \ldots, b_{1r_1}$ for B_1 such that for some integers $e_{11}, e_{12}, \ldots, e_{1r_1}$ the elements $b_{11}^{e_{11}}, b_{12}^{e_{12}}, \ldots, b_{1r_1}^{e_{1r_1}}$ form a basis for the subgroup C . Since C is cyclic, only one b_{1i} $(1 \le i \le r_1)$ can occur nontrivially as a basis element of C , so we may assume $C = < b_{11}^{e_{11}} >$. In the same way we choose bases $b_{i1}, \ldots, b_{ir_i}$ for the groups B_i such that $C = < b_{i1}^{e_{i1}} >, (1 < i \le n)$. Let $M_i = < b_{i2}, \ldots, b_{ir_i} >$, let $\sigma_i : B_i \longrightarrow B_i/M_i$ and let $\bar{\bar{G}} = (B_1/M_1 * \ldots * B_n/M_n ; b_{11}^{e_{11}} M_1 = \ldots = b_{n1}^{e_{n1}} M_n)$. If $c \in C$, $c\sigma_i = b_{i1}^{e_{i1}} M_i = b_{j1}^{e_{j1}} M_j = c\sigma_j$; that is, the homomorphisms $\sigma_i (1 \le i \le n)$ agree with the amalgamating isomorphisms of $\bar{G}$. Thus we extend the σ_i to a homomorphism $\sigma : \bar{G} \longrightarrow \bar{\bar{G}}$ with the properties that $h\varphi\sigma \neq 1$ and σ is onto $\bar{\bar{G}}$. But now we are in good shape, for by proposition 2 $\bar{\bar{G}}$, as the generalized free product of a finite number of infinite cycles, has trivial Frattini subgroup . So for any nontrivial h in H_F we have constructed a homomorphism $\varphi\sigma$ of G onto a Φ free group with the property that $h \notin \ker(\varphi\sigma)$. Since $\Phi(G) \le \ker(\varphi\sigma)$ and h was arbitrary, the theorem follows.

Corollary 6. The free product of a finite number of finitely generated abelian groups amalgamating a torsion free subgroup has trivial Frattini subgroup.

That Theorem 2 is the best possible result with respect to the

ϕ freeness of such a group is illustrated by the following generalization of some results of Gaschutz [3]:

Remark 1: Let N be a finitely generated normal subgroup of a group G, K a subgroup of G with $N \leq \phi(K)$. Then $N \leq \phi(G)$.

Before proving the remark we note that in particular, when G is the free product of a family of groups $\{G_\lambda\}_{\lambda \in \Lambda}$ with amalgamated subgroup H, and N is a finitely generated subgroup of H such that $N \lhd G$ and $N \leq \phi(G_\lambda)$ for some $\lambda \in \Lambda$, then $N \leq \phi(G)$. And in the case when the groups G_λ are finitely generated and abelian, if H intersects the Frattini subgroup of any one of the factors non-trivially, this intersection, which is of course a torsion group, is contained in $\phi(G)$.

To prove the remark we need the following simple lemma.

Lemma 3: A finitely generated normal subgroup N of G is contained in $\phi(G)$ if and only if $G \neq NL$ for some proper subgroup L of G.

Proof: Let $N = < n_1, n_2, \ldots, n_k >$ be contained in $\phi(G)$ and suppose $G = N \cdot L$ with $L \leq G$, that is $G = < n_1, n_2, \ldots, n_k, L >$. Then since each n_i $(1 \leq i \leq k)$ is a nongenerator, $G = L$. On the other hand, if $N \nleq \phi(G)$ then for some i $(1 \leq i \leq k)$ $n_i \notin \phi(G)$, from which it follows that there exists a maximal subgroup L of G which does not contain n_i. Hence $G = < n_i, L > = NL$, which shows that if G does not have this property, N must be contained in $\phi(G)$.

Proof of Remark 1: If $N \nleq \phi(G)$ then by lemma 3 $G = NL$ with $L < G$. Since $N \nleq L$, and since by assumption $N \leq K$, we have $N \nleq L \cap K < K$. But then $K = N(L \cap K)$ which by the lemma contradicts our assumption that $N \leq \phi(K)$.

Gaschutz has shown that for any group G , $G' \cap \zeta(G) \leq \phi(G)$. It is interesting to compare this with the following:

Remark 2: Let G be the free product with amalgamations of groups G_λ , $\lambda \in \Lambda$ and let $L = \bigcap_{\lambda \in \Lambda} \phi(G_\lambda) \cap \zeta(G)$. Then $L \leq \phi(G)$.

Proof: Let $\ell \in L$ and suppose that $G = < T, \ell >$ with $T < G$. For each $\lambda \in \Lambda$ and each $g \in G_\lambda$ we have $g = t\ell^n$ $(t \in T, n \in Z)$ which implies that $t \in G_\lambda$. Hence if $T_\lambda = T \cap G_\lambda$ then $G_\lambda \leq < T_\lambda, \ell >$, from which it follows that $G_\lambda \leq T_\lambda$. Since λ was arbitrary we have $T = G$ and $\ell \in \phi(G)$.

CHAPTER II.

Section 1.

We recall that a group G is residually $\underline{\underline{X}}$ (where $\underline{\underline{X}}$ is any class of groups) if for any nontrivial g in G there exists $N_g \lhd G$ such that $g \notin N_g$ and $G/N_g \in \underline{\underline{X}}$. It has already been mentioned that the Frattini subgroup of a finite group is nilpotent. We make use of this fact to prove the following

Remark 3: If a group G is residually finite, then $\phi(G)$ is residually nilpotent.

Proof: If g is a nontrivial element of $\phi(G)$ we seek a normal subgroup M of $\phi(G)$ such that $g \notin M$ and $\phi(G)/M$ is nilpotent. By assumption there exists $N \lhd G$ such that $g \notin N$ and G/N is finite; hence $\phi(G/N)$ is nilpotent. Let $M = \phi(G) \cap N$. Then $g \notin M$ and $\phi(G)/M \cong \frac{\phi(G) \cdot N}{N} \leq \phi(G/N)$; thus $\phi(G)/M$ is nilpotent.

It was shown in Chapter I that if N is a finitely generated normal subgroup of a group G and $N \leq \phi(K)$ for some subgroup K of G, then $N \leq \phi(G)$. In this chapter we will need a similar proposition, with different demands placed on the normal subgroup N. We first prove a lemma.

Lemma 4: Let N be a normal subgroup of a group G. If each left (right) transversal of N in G generates G then $N \leq \phi(G)$. Conversely, if N is nilpotent and the normal closure in G of finitely many elements, then $N \leq \phi(G)$ implies that each left (right) transversal of N in G generates G .

Proof: Let $H \leq G$ such that $HN = G$ and suppose that each left transversal of N in G generates G . If gN is a coset of N in G , then for each $n \in N$, $gn = h\hat{n}$ where $h \in H$ and $\hat{n} \in N$. Hence H contains $gn\hat{n}^{-1}$, an element of gN , that is, H contains a left transversal for N in G . Thus $H = G$ and $N \leq \phi(G)$; in fact N is omissible as a subgroup of G .

To prove the converse statement, we first note that if N is nilpotent, say of class c , then N' is omissible as a subgroup of N . We prove this by induction on c ; it is clearly true if $c = 1$, so assume $c > 1$ and that nilpotent groups of class $< c$ have omissible derived group. Let $R \leq N$ be such that $RN' = N$. Then since $(N/\gamma_c(N))' = N'/\gamma_c(N)$, we have inductively that

$$\frac{R\cdot\gamma_c(N)}{\gamma_c(N)} = \frac{N}{\gamma_c(N)} \quad . \tag{1}$$

Let $[n_1,n_2]$ $(n_1 \in N,\ n_2 \in \gamma_{c-1}(N))$ be an arbitrary generator of $\gamma_c(N)$. By (1) $n_1 = r_1n_1'$ and $n_2 = r_2n_2'$ where r_1, $r_2 \in R$ and n_1', $n_2' \in \gamma_c(N)$. Since $\gamma_c(N) \leq \zeta(N)$ we have $[n_1,n_2] = [r_1,r_2] \in R$, that is $\gamma_c(N) \leq R$, which implies that $R = N$, as was required.

Now suppose that $N \leq \phi(G)$, where $N = nm_G(x_1,\ldots,x_k)$ is nilpotent and let $\mathcal{T}$ be a left transversal of N in G , with $<\mathcal{T}> = T$. Then since $TN = G$ and since for all $g \in G$ $x_i^g = x_i[x_i,g] = x_i[x_i,tn]$ $(1 \leq i \leq k,\ t \in T,\ n \in N)$, $N = < x_1,\ldots,x_k,[x_i,tn]/t \in t,\ n \in N,\ i=1,\ldots,k >$. But $[x_i,tn] = x^{-1}n^{-1}x^tn = [x,t]\cdot[x^t,n]$ where $[x^t,n] \in N'$, and by the omissibility of N' we have

$N = < x_1,\ldots,x_k,[x_i,t]/t \in T,\ i = 1,\ldots,k >$ and hence

$G = TN = < x_1,\ldots,x_k,\mathcal{J} >$. But since $N \leq \Phi(G)$ the finitely many x_i are nongenerators; hence $\mathcal{J}$ generates G .

The proposition which we desire and which will prove quite useful in the future is an easy consequence of this lemma.

Proposition 4. Let G be any group, $H \leq G$, N nilpotent and the normal closure in G of finitely many elements, and suppose that $N \leq \Phi(H)$. Then $N \leq \Phi(G)$.

Proof: Let $\mathcal{J}_G$ be a left transversal of N in G . By lemma 4 it is enough to show that $\mathcal{J}_G$ generates G . Let $\mathcal{J}_H = \mathcal{J}_G \cap H$. Note that the complex tN (where $t \in \mathcal{J}_G$) is contained in H if and only if $t \in \mathcal{J}_H$. It follows from this that $\bigcup_{t\in\mathcal{J}_H} tN = H$; hence $\mathcal{J}_H$ contains a left transversal of N in H , whence by lemma 4, $\mathcal{J}_H$ generates H . If $g \in G\backslash H$, then there exists $t \in \mathcal{J}_G\backslash\mathcal{J}_H$ such that $g \in tN \subset tH$, that is, $\mathcal{J}_G\backslash\mathcal{J}_H$ contains a left transversal of H in G . Thus $\mathcal{J}_G$ generates G .

It will be convenient in the future to refer to the following restatement of proposition 4:

Corollary 7. Let H be a nilpotent subgroup of a group G , and suppose that $\Phi(H)$ is the normal closure in G of finitely many elements. Then $\Phi(H) \leq \Phi(G)$.

Section 2

Before proceeding further, we stop to define the wreath product of groups, and to introduce some associated notation. For groups A and B the cartesian power A^B of A is the set of all functions from B to A with multiplication defined componentwise (not to be confused with the normal closure of A in B). In symbols, $A^B = \{f/f: B \longrightarrow A\}$ and if $g, f \in A^B$ then fg is the element of A^B for which, for all $b \in B$, $(fg)b = f(b)g(b)$. The direct power $A^{(B)}$ of A is that subgroup of A^B consisting of those functions f such that $f(b) = 1$ for all but finitely many b in B. If we define the support $\sigma(f)$ of a function $f \in A^B$ by $\sigma f = \{b \in B / f(b) \neq 1\}$ then $A^{(B)}$ consists of those functions in A^B whose supports are finite. The restricted wreath product of A by B is denoted by $W = A \text{ wr } B$ and is defined as a splitting extension of $A^{(B)}$ by B using the following action: for $f \in A^{(B)}$ and b in B we define $f^b \in A^{(B)}$ by $f^b(\beta) = f(b^{-1}\beta)$ for all $\beta \in B$. (The standard or unrestricted wreath product is defined as above using A^B in place of $A^{(B)}$. If B is finite, these concepts obviously coincide).

We refer to B as the "top" group and A as the "bottom" group of W. If we denote by 1 the function in $A^{(B)}$ for which $1(b) = 1 \in A$ for every b in B, then the set of all elements $(b,1)$, $b \in B$ forms a subgroup of W isomorphic to B which we identify with B. Similarly, the set of all elements $(1f)$ with $f \in A^{(B)}$ forms a subgroup of W which we identify with $A^{(B)}$ and call the base group of W, denoted by K. With these conventions, K is a normal subgroup of W and is complemented by B. Elements of W can be factorized in a unique way as products bf with $b \in B$ and $f \in A^{(B)} = K$.

The next proposition will give us some understanding of the structure of normal subgroups and factor groups of the restricted wreath product of abelian groups.

Proposition 5. Let A and B be groups, with $A_* \lhd A$, $B_* \lhd B$, A/A_* abelian, and $\mathcal{J}$ a transversal of B_* in B . Let $W = A \mathrm{wr} B$ with base group K , and let $N = nm_W(A_*,B_*)$. Then $W/N \cong A/A_* \mathrm{wr} B/B_*$ and $N = \{bf \,/\, b \in B_* \text{ and } \prod_{b \in tB_*} f(b) \in A_* \text{ for all } t \in \mathcal{J}\}$.

Proof: Let $M = \{bf \,/\, b \in B_* \text{ and } \prod_{b \in tB_*} f(b) \in A_* \text{ for all } t \in \mathcal{J}\}$ and let $\varphi: W \longrightarrow A/A_* \mathrm{wr} B/B_*$ be defined as follows: if $(b,f) \in W$, $(b,f)\varphi = (bB_*,g)$ where $g(tB_*) = \prod_{b \in tB_*} f(b)A_*$ for all $t \in \mathcal{J}$. Clearly, M is the kernel of the map φ . We show (i) that φ is a homomorphism onto $A/A_* \mathrm{wr} B/B_*$ and (ii) that $M = N$, and these two facts will establish both assertions of the proposition.

(i) Since A/A_* is abelian φ is well defined; we check that φ is a homomorphism. If (b_1,f_1) and $b_2,f_2)$ are any two elements of W , $[(b_1,f_1)(b_2,f_2)]\varphi = (b_1b_2,f_1^{b_2}f_2)\varphi = (b_1b_2B_*,g)$ where $g(tB_*) = \prod_{b \in tB_*} f_1^{b_2}f_2(b)A_*$ for all $t \in \mathcal{J}$. On the other hand $(b_1,f_1)\varphi \cdot (b_2,f_2)\varphi = (b_1B_*,g_1)(b_2B_*,g_2) = (b_1b_2B_*,g_1^{b_2B_*}g_2)$ where $g_i(tB_*) = \prod_{b \in tB_*} f_i(b)A_*$ for all $t \in \mathcal{J}$, $i = 1,2$. But if $t \in \mathcal{J}$ then

$g_1^{b_2B_*}g_2(tB_*) = g_1^{b_2B_*}(tB_*)g_2(tB_*) = g_1(tb_2B_*)g_2(tB_*) =$

$$= \prod_{b\in tb_2B_*} f_1(b)A_* \cdot \prod_{b\in tB_*} f_2(b)A_* = \prod_{b\in tB_*} f_1(bb_2)f_2(b)A_* = \prod_{b\in tB_*} f_1^{b_2}f_2(b)A_* = g(tB_*).$$

Since this is true for any t in $\mathcal{J}$, $g_1^{b_2B_*}g_2 = g$ and φ is a homomorphism. To check that φ is onto, let $(tB_*,g) \in A/A_*\mathrm{wr}B/B_*$. Let $f \in K$ be defined by $f(t) = g(tB_*)$ for all $t \in \mathcal{J}$, $f(b) = 1$ for all $b \in B\setminus\mathcal{J}$. Then $(t,f)\varphi = (tB_*,g)$.

(ii) Identify with A_* the subgroup of W consisting of elements of the form $(1,f)$ where $f(1) = a \in A_*, f(b) = 1$ for all $b \neq 1$ in B. Then M, the kernel of φ, is clearly a normal subgroup of W containing A_* and B_*. Hence $N \leq M$. On the other hand, if $(b,f) \in M$, to verify that $M \leq N$ we need only show that $f \in N$. If $\sigma f = (b_1,\dots,b_n)$ let $\sigma_t = \{b\in\sigma f \,/\, b\in tB_*\}$ for all $t \in \mathcal{J}$. Define $g_t\colon B \longrightarrow A$ as follows: if $b\in\sigma_t$ $g_t(b) = f(b)$; if $b\notin\sigma_t g_t(b)=1$. Since $f = \prod_{t\in\mathcal{J}} g_t$ it is enough to show each $g_t \in N$. So let $\sigma_t = \{b_{t_1},\dots,b_{t_k}\}$ and let $g_t(b_{t_j}) = a_j \in A$. Let

$$h_t(b) = \begin{cases} \prod_{j=1}^{k} a_j & \text{if } b=b_{t_1} \\ 1 & \text{if } b\neq b_{t_1} \end{cases} \text{; then } g_t = h_tg_t' \text{ where } g_t'(b) = \begin{cases} \left(\prod_{j=2}^{k} a_j\right)^{-1} & \text{if } b=b_{t_1} \\ g_t(b) & \text{if } b\neq b_{t_1} \end{cases}$$

Since $\prod_{j=1}^{k} a_j \in A_*$, $h_t \in N$; hence it suffices to show $g_t' \in N$. Define h_i by $h_i(b_{t_i}) = a_{i+1}a_{i+2}\dots a_k$, $h_i(b) = 1$ for all $b\neq b_{t_1}$ in B $(1\leq i\leq k)$.

Then

$$g_t' = [h_1,b_{t_1}^{-1}b_{t_2}][h_2,b_{t_2}^{-1}b_{t_3}]\dots[h_{k-2},b_{t_{k-2}}^{-1}b_{t_{k-1}}][h_{k-1},b_{t_{k-1}}^{-1}b_{t_k}].$$

Since $b_{t_i}^{-1}b_{t_{i+1}} \in B_*, (1 \leq i < k)$, each of these commutators lies in N ; hence $g_t' \in N$, and $M \leq N$. Thus $M = N$ and proposition 5 is proved.

Section 3

We now examine the Frattini subgroup of the wreath product of finite groups of coprime order. Throughout the rest of this chapter, W will denote the restricted wreath product of A by B with base group K .

Remark 4: Let G be an extension of a finite abelian group A by a finite group of coprime order, and suppose A is its own centralizer in G . Then $\phi(G) = 1$ if and only if $\phi(A) = 1$.

Proof: One part of the proof of this remark is an immediate consequence of Corollary 7, which states that $\phi(A) \leq \phi(G)$, so suppose that $\phi(A) = 1$ and let M be a minimal normal subgroup of G . Then since $M \cap A$ is normal in G , we must have either $M \leq A$ or $M \cap A = 1$. In the latter case M would centralize A (since $[M,A]$ is normal in G and is contained in M) contrary to assumption. Hence M , as a subgroup of the direct product A of elementary abelian groups, is a direct summand of A . Under these conditions L.G. Kovac's adaptation of Mashke's theorem [9] can be applied to deduce that M has a G admissible direct complement N in A .

Note that by the Schur-Zassenhaus theorem G splits over A ; let B be a complement for A in G . We show that NB is maximal in G . If $NB < L \leq G = MNB$, then $L = NB(L \cap M)$. $L \cap M$ is normalized by NB and of course is normal in the abelian group M , hence $L \cap M$

is a nontrivial normal subgroup of G contained in M which implies that $L \cap M = M$ and $L = G$. Hence NB is maximal in G and $M \cap NB = 1$, from which it follows, since $\phi(G)$ must contain some minimal normal subgroup of G, that $\phi(G) = 1$.

Corollary 8. Let A and B be finite groups of coprime order with A abelian. Then $\phi(W) = 1$ if and only if $\phi(A) = 1$.

Proof: Since $\phi(K) = 1$ if and only if $\phi(A) = 1$ and K is its own centralizer in W, remark 4 may be applied with $W = G$ and $K = A$.

In order to determine $\phi(W)$ when A is nonabelian, we first prove a lemma.

Lemma 5: Let A and B be finite groups of coprime order and let H be a normal subgroup of W which is not contained in K. Then H is not nilpotent.

Proof: Since $H \cap K \lhd H$ and $H/H \cap K$ is isomorphic to a subgroup of B, $H \cap K$ has order prime to its index, whence by the Schur-Zassenhaus, Feit-Thompson results, H splits over $H \cap K$. If C is a complement for $H \cap K$ in H, we have $KC = KH$. If $C^* = B \cap KH$ then $KC^* = K(B \cap KH) \leq KH$, but on the other hand, if $kh \in KH$, then since $kh = \bar{k}b$ $(\bar{k} \in K, b \in B)$ we have $b = \bar{k}^{-1}kh$, that is $b \in B \cap KH$ and $KH \leq K(B \cap KH)$. Hence $KC = KH = KC^*$. Since K has order prime to its index in KC, again by Schur-Zassenhaus, the two complements C and C^* for K in KC are conjugate in $H \cap K$. Thus we have shown that C is conjugate to a subgroup of B. Note that $H \cap K \neq 1$ for if so, the normal subgroup H would be contained in a conjugate of B and hence in B which is impossible. We now produce two elements of coprime order in H which do not commute, and since in a nilpotent group this situation cannot occur, the lemma will follow. Let c be a nontrivial element of C. (By assumption $C \neq 1$). Since $c = b^f$ where $b \in B \cap H$, $f \in H \cap K$, and since b and c have the same order, we may assume $c \in B$. Let $g \in K$ be defined by choosing $g(c)$ to be a nontrivial element a of A and $g(b) = 1$ for all $b \neq c$ in B. Since A and B have coprime order we are assured that not both a and c have order 2; assume for explicitness, that $a^2 \neq 1$. Note that $h = [g,c] \in H \cap K$, hence has order prime to c; moreover

$$h(c) = g^{-1}(c)\, g^{c}(c) = a^{-1} \quad \text{while} \quad h^{c}(c) = (g^{-1})^{c}(c)\, g^{c^2}(c) = \begin{cases} 1 & \text{if } c^2 \neq 1 \\ a & \text{if } c^2 = 1 \end{cases}$$

In any case $h \neq h^c$; hence H is not nilpotent.

Now let G be any finite group. The Fitting subgroup $\Psi(G)$ is defined to be the product of the normal nilpotent subgroups of G. Fitting has shown that $\Psi(G)$ is itself normal in G and nilpotent. The socle of G, denoted by $S(G)$, is the product of the minimal normal subgroups G_μ of G. It is easily verified that $S(G)$ is the direct product of the G_μ and that any G normal subgroup of $S(G)$ has a G normal complement in $S(G)$. Let $S(G)_\alpha$ be the maximal soluble G normal subgroup of $S(G)$. Then for each minimal normal subgroup G_μ of G either $G_\mu \cap S(G)_\alpha = 1$ or $G_\mu \leq S(G)_\alpha$; hence $S(G)_\alpha$ is the direct product of the minimal normal subgroups contained in it and the latter, as subgroups of a soluble group, are soluble. Moreover they are all direct products of prime cycles, for if not they would contain characteristic subgroups which would thus be normal in G, contradicting their minimality. Hence $S(G)_\alpha$ is the direct product of cycles of order p and is called the abelian component of the socle of G. We note two results of Gaschutz [3] for finite groups: (i) $\phi(G) = 1$ implies $\Psi(G) = S(G)_\alpha$ and (ii) $\phi(G) = 1$ if and only if G splits over $S(G)_\alpha$. With these results and with lemma 5 we are now in a position to prove the analogue of Corollary 8.

Remark 5: Let A and B be finite groups of coprime order, with A nonabelian. Let $W = AwrB$ with base group K. Then $\phi(W) = 1$ if and only if $\phi(A) = 1$.

Proof: As in the proof of remark 4, Corollary 7 yields immediately that $\phi(W) = 1$ only if $\phi(A) = 1$. Conversely, if $\phi(A) = 1$ then by (i) $\Psi(A) = S(A)_\alpha = C(p_1)\times\ldots\times C(p_n)$ for primes $p_1,\ldots,p_n$. Since $\phi(K) = 1$, by (ii) K splits over $\Psi(K)$, but $\Psi(K) = \Psi(A)^{(B)}$. (One part of this equality, namely that $\Psi(A)^{(B)} \leq \Psi(K)$ is trivial; on the other hand, if K^* is a nilpotent normal subgroup of K, then for each $b \in B$ the projection $\pi_b\colon K \longrightarrow A$ defined by $f\pi_b = f(b)$ maps K^* onto a nilpotent normal subgroup of A which is thus contained in $\Psi(A)$ and hence $K^* \leq \Psi(A)^{(B)}$. Since true for all such subgroups K^*, $\Psi(K) \leq \Psi(A)^{(B)}$ and equality holds). Hence $\Psi(K)$, as a nilpotent normal subgroup of W, is contained in $\Psi(W)$. By lemma 5, however, every normal nilpotent subgroup H of W is contained in $\Psi(K)$, so $\Psi(W) = \Psi(K) = \Psi(A)^{(B)}$, and $W = K\cdot B = \Psi(W)\cdot K_1\cdot B$, where $\Psi(W) \cap K_1 = 1$, implies that $\Psi(W) \cap K_1\cdot B = 1$. In other words, W splits over $\Psi(W)$, so by virtue of (ii) it is enough to show that $\Psi(W) = S(W)_\alpha$. Clearly for any group W, $S(W)_\alpha \leq \Psi(W)$. On the other hand,

$$\Psi(W) = \Psi(A)^{(B)} = (C(p_1)\times\ldots\times C(p_n))^{(B)} = C(p_1)^{(B)}\times\ldots\times C(p_n)^{(B)}$$

that is $\Psi(W)$ is a product of minimal normal subgroups of W and is abelian, which implies the other inclusion and thus equality.

Collecting the results of Section 3 we have shown

Proposition 6. The wreath product of finite groups of coprime order is ϕ free if and only if the bottom group is ϕ free.

Section 4

If S is a finite set of elements of a group G, let us say that a normal subgroup H of G separates S if all the elements of S belong to distinct cosets of H in G. And a family $\{B_\lambda\}_{\lambda\in\Lambda}$ of groups is closed with respect to finite intersection if the intersection of any finite collection of members of the family is a member of the family. After these introductory comments we prove a useful fact.

Remark 6: Suppose B is a group with a family $\{B_\lambda\}_{\lambda\in\Lambda}$ of subgroups closed with respect to finite intersection, and such that $\bigcap_{\lambda\in\Lambda} B_\lambda = 1$.

Then if A is an abelian group and $W = A\text{wr}B$ with base group K, $\bigcap_{\lambda\in\Lambda} \text{nm}_W B_\lambda = 1$.

Proof: Let $H = \bigcap_{\lambda\in\Lambda} \text{nm}_W B_\lambda$. If $M_\lambda = [K,B_\lambda]$ then since $\text{nm}_W B_\lambda = M_\lambda \cdot B_\lambda$ [13], $H = \bigcap_{\lambda\in\Lambda}(M_\lambda \cdot B_\lambda)$. We show that $H = \bigcap_{\lambda\in\Lambda} M_\lambda$. Clearly H contains $\bigcap_{\lambda\in\Lambda} M_\lambda$, but if $fb \in H$ (where $f \in K$, $b \in B$) then for each $\lambda \in \Lambda$, $fb = f_\lambda b_\lambda$ ($f_\lambda \in M_\lambda, b_\lambda \in B_\lambda$), that is, $f_\lambda^{-1}f = b_\lambda b^{-1}$, which implies that $f = f_\lambda$ and $b = b_\lambda$. Since this is true for all $\lambda \in \Lambda$, $fb \in \bigcap_{\lambda\in\Lambda} M_\lambda \cdot \bigcap_{\lambda\in\Lambda} B_\lambda = \bigcap_{\lambda\in\Lambda} M_\lambda$. It remains to show $\bigcap_{\lambda\in\Lambda} M_\lambda = 1$, so suppose $f \in \bigcap_{\lambda\in\Lambda} M_\lambda$ and let $\sigma f = \{b_1, \ldots, b_n\}$. Choose $B_{ij} \in \{B_\lambda\}$ such that $b_i b_j^{-1} \notin B_{ij}$ $(1 \le i \le n, 1 \le j \le n, i \ne j)$. Then $B_\mu = \bigcap_{\substack{i,j=1 \\ i\ne j}}^{n} B_{ij} \in \{B_\lambda\}_{\lambda\in\Lambda}$ and B_μ separates σf . Since $f \in \mathrm{nm}_W B_\mu$, $f = 1$ by proposition 5.

It is well known that if A is abelian, $\Phi(A) = \bigcap_p A^p$ where A^p is the group consisting of the p^{th} powers of elements of A and the intersection runs over all primes p . To see this, note that since A/A^p is an elementary abelian p group $\Phi(A) \le A^p$ for all primes p . On the other hand, let $a \in \bigcap_p A^p$ (i.e., a is a p^{th} power for all primes p) and let M be a maximal subgroup of A . Then $A/M = C(q)$ which implies $M \ge A^q \ni a$. Since this is true for all maximal subgroups of A , $a \in \Phi(G)$.

Hence if A is any Φ free abelian group then for some (not necessarily unique) set π of primes A is a residually simple π group, that is there exists a family $\{A_\alpha\}_{\alpha\in\mathcal{A}}$ of subgroups of A such that $\bigcap_{\alpha\in\mathcal{A}} A_\alpha = 1$ and $A/A_\alpha = C(p)$ where $p \in \pi$.

Proposition 7. Let A be a Φ free abelian group and let π be chosen so that A is residually a simple π group. If for all $p \in \pi$

B is residually a finite p' group, then $\phi(W) = 1$.

Proof: Associate with each A_α, $\alpha \in \mathcal{A}$ (where $A/A_\alpha = C(p)$ for some $p \in \pi$) the family $\{B_{\alpha\lambda}\}_{\lambda \in \Lambda_\alpha}$ of subgroups of B such that $B/B_{\alpha\lambda}$ is a finite p' group and $\bigcap_{\lambda \in \Lambda_\alpha} B_{\alpha\lambda} = 1$. Let $W_{\alpha\lambda} = nm_W(A_\alpha, B_{\alpha\lambda})$ for all $\alpha \in \mathcal{A}, \lambda \in \Lambda_\alpha$. Then by propositions 5 and 6, $\phi(W/W_{\alpha\lambda}) \cong \phi(A/A_\alpha wr B/B_{\alpha\lambda}) = 1$; hence $\phi(W) \leq \bigcap_{\substack{\alpha \in \mathcal{A} \\ \lambda \in \Lambda_\alpha}} W_{\alpha\lambda}$; denote this intersection by W^* .

(Observe that we may assume $A \neq C(p)$, for in this case $\pi = \{p\}$, $\mathcal{A} = \{\alpha\}$ where $A_\alpha = 1$ and $W_{\alpha\lambda} = nm_W\{B_{\alpha\lambda}\}$. Since the family $\{B_{\alpha\lambda}\}_{\lambda \in \Lambda}$ is closed with respect to finite intersection, by remark 6 $W^* = 1$ and hence $\phi(W) = 1$). If $bf \in W^*$, by proposition 5, $b = 1$. Suppose $f \neq 1$ and choose $\alpha \in \mathcal{A}$ such that $f(b) \notin A_\alpha$ for at least one $b \in \sigma f$. Now choose λ such that $B_{\alpha\lambda}$ separates σf . Again by proposition 5 $f \notin W_{\alpha\lambda}$ and this contradiction implies $f = 1$ and hence $\phi(W) = 1$.

If G is any abelian group, let G_p be the subgroup of G consisting of elements of p power order.

Corollary 9. Let A be a ϕ free abelian group and suppose that $A_p = 1$. Then if B is residually a finite p group, $\phi(W) = 1$.

Proof: By proposition 7 it suffices to show that A is residually a finite p' cycle, that is $\bigcap_{q \neq p} A^q = 1$, where q runs through the primes with the exception of p . If $a \in \bigcap_{q \neq p} A^q$ then $a^p \in \bigcap_p A^p = \Phi(A) = 1$, which implies by assumption that $a = 1$.

Lemma 6: Let D be an abelian normal subgroup of a group G and suppose G splits over D . If M is a maximal subgroup of G , either M contains D or $M \cap D$ is a maximal G admissible subgroup of D . Conversely, every maximal G admissible subgroup of D occurs as $M \cap D$, where M is a maximal subgroup of G .

Proof: Let C be a complement for D in G and let M be maximal in G . If $D \not\leq M$ then $MD = G$; hence $M \cap D \lhd G$. If $M \cap D < N \leq D$ with $N \lhd G$ then $M < NC$. To verify this we show $M \leq NC$; the fact that $M \cap D < N$ will then imply that $M < NC$. Let $m = dc (d \in D, c \in C)$ be an arbitrary element of M . Since $N \not\leq M$ we have $NM = G$, hence $d = nm' \, (n \in N, m' \in M)$ and thus $m' = n^{-1}d \in M \cap D < N$, from which it follows that $d \in N$ and $m \in NC$. Therefore $M < NC$ whence $NC = G$ and $N = D$. Now suppose H is a maximal G admissible subgroup of D . If $d \in D \backslash H$, then $nm_G(d,H) = D$ which implies, since D is abelian, that $< d, HC > = G$, from which it follows that HC is maximal in G .

The characterization of the Frattini subgroup of an abelian group A as the intersection of its subgroups A^p as p runs through all the primes yields immediately that an abelian divisible group has no maximal subgroups. On the other hand, if A has no maximal subgroups then $A = \bigcap_p A^p$; hence if $a \in A$, n is any integer and $n = p_1 p_2 \cdots p_k$ is the unique factorization of n into primes, then there exist a_i $(1 \geq i \geq k)$ in A such that $a_i^{p_i} = a_{i+1}$ $(1 \leq i < k)$, $a_k^{p_k} = a$. Thus $a_1^n = a_1^{p_1 \cdots p_k} = a$ and A is divisible. Combining these results we have

Remark 7: An abelian group is divisible if and only if it has no maximal subgroups, and this leads to the following

Remark 8: Let A and B be groups without maximal subgroups and suppose A is an abelian torsion group. Then $W = A \mathrm{wr} B$ has no maximal subgroups.

Proof: We note first that by the above comments A is divisible; hence A and thus K is a direct sum of groups each isomorphic to $Z(p^\infty)$ for various primes p. We assume W contains a maximal subgroup M and look for a contradiction. If $K \leq M$ then $M = K \cdot M \cap B$, hence there exists a proper subgroup B_* of B such that $M \cap B < B_*$ and $M < KB_*$. So $K \nleq M$ and by lemma 6 $K \cap M$ is a maximal W normal subgroup of K, i.e., $K/K \cap M$ is a characteristically simple torsion group, which implies, since the subgroup of elements of order p for a simple prime p form a characteristic subgroup of an abelian group, that $K/K \cap M$ is a group of exponent p for various primes p. But this clearly contradicts the fact that $K/K \cap M$, as the homomorph of a divisible group, is divisible.

Section 5

We now confine our attention to restricted wreath products of finitely generated abelian groups and evaluate the Frattini subgroup of such a group. As before, we let $W = A \mathrm{wr} B$, where in this case A and B are both finitely generated and abelian, where K is the base group of W , π_A and π_B the sets of primes occurring in the cyclic decomposition of A and B respectively, $\pi = \pi_A \cap \pi_B$ and for each $p \in \pi$, W_p the Sylow p subgroup of W . For a fixed prime p let P,Q be the Sylow p subgroups of A and B respectively, with P_1,Q_1 complements for P in A and Q in B . We first prove a simple lemma.

Lemma 8: Let p be a fixed prime in π , and let P,Q and W_p be as defined above. Then W_p' , the derived group of $P^{(B)} \cdot Q$, is the normal closure in W of a finite number of elements.

Proof: $N = \mathrm{nm}_W[P,Q]$ is the normal closure in W of a finite group and clearly $N \leq W_p'$. For $b \in B$ $[P^b,Q] = [P,Q]^b \leq N$; thus N contains $[P^{(B)},Q] = W_p'$ and so $N = W_p'$.

Proposition 8. Let A and B be finitely generated groups and suppose that $\phi(A) = 1$.

(i) If $\pi = \emptyset$, W is ϕ free.

(ii) If $\pi = \pi_A = \{p\}$ then $\phi(W) = W_p' \cdot \phi(Q)$ where Q is the Sylow p subgroup of B .

(iii) If $\pi \neq \emptyset$ and $|\pi_A| > 1$ then $\phi(W) = \prod_{p \in \pi} W_p'$.

Proof: (i) follows immediately from proposition 7 since B is residually finite p' for all $p \in \pi_A$.

(ii) If Q_1 is a complement for Q in B , then $W_p = KQ = A^{Q_1}\mathrm{wr}Q$ is an extension of an abelian p group of exponent p by a finite p group and hence is nilpotent. [1]. Therefore the Frattini subgroup of W_p contains W_p' and the subgroup of W_p generated by the p^{th} powers of its elements, and so $\phi(W_p)$ surely contains $W_p' \cdot \phi(Q)$. But $\phi(W_p/W_p' \cdot \phi(Q)) \cong \phi(A \times Q/\phi(Q)) = 1$ and so $\phi(W_p) = W_p' \cdot \phi(Q)$. Since $W_p \lhd W$, $\phi(W_p)$ is a nilpotent normal subgroup of W ; moreover by lemma 8 it is the normal closure in W of finitely many elements; thus by proposition 4, $W_p' \cdot \phi(Q) \leq \phi(W)$. On the other hand, $\phi(W/W_p' \cdot \phi(Q)) = \phi(Q/\phi(Q) \times (P\mathrm{wr}Q_1)) = 1$ by proposition 7, which implies that $\phi(W) = W_p' \cdot \phi(Q)$.

In case (iii) we first note that $\phi(W)$ is contained in the torsion component of K . To see this we denote by A_π the product of the Sylow p subgroups of A for all $p \in \pi$ and $A_{\pi'}$ its complement in A and consider two possibilities: (a) $A_\pi < A$ and (b) $A_\pi = A$. If (a) holds then $\phi(W/A_\pi^{(B)}) \cong \phi(A_{\pi'}\mathrm{wr}B) = 1$ by (i); hence $\phi(W) \leq A_\pi^{(B)}$. If (b) is the case, where say, $A = A_{p_1} \times A_{p_2} \times \ldots \times A_{p_n}$ and $B = B_{p_1} \times B_{p_2} \times \ldots \times B_{p_n} \times B_{\pi'}$, let

$$R = [A_{p_1}, B/(B_{p_2} \times B_{\pi'})] \cdot [A_{p_2}, B/B_{p_3}] \ldots [A_{p_{n-1}}, B/B_{p_n}] \cdot [A_{p_n}, B/B_{p_1}] .$$

It is easy to check that R is normal in W and is contained in the torsion part of K. Moreover

$$\phi(W/R) = \phi[(A_{p_1}\,\mathrm{wr}(B_{p_2} \times B_{\pi'}))\times(A_{p_2}\,\mathrm{wr}B_{p_3})\times\ldots\times(A_{p_{n-1}}\,\mathrm{wr}B_{p_n})\times(A_{p_n}\,\mathrm{wr}B_{p_1})] = 1$$

by (i), which by the now familiar argument implies our assertion.

Thus $\phi(W)$ is an abelian torsion group, in fact a π group. We know from the discussion in case (ii) that for a fixed prime p, $W_p = P^{(B)}\cdot Q = P^{Q_1}\mathrm{wr}Q$ is a nilpotent subgroup of W and that $\phi(W_p) = W'_p\cdot\phi(Q)$. We examine now the subgroup $N = KQ = (P^{(B)}\times P_1^{(B)})Q$, where P_1 is a nontrivial complement for P in A. $\phi(N/P^{(B)}) = \phi(P_1\mathrm{wr}Q) = 1$ by (i), giving $\phi(N) \leq P^{(B)}$. Hence in determining $\phi(N)$ we may disregard those maximal subgroups of N containing $P^{(B)}$ since their intersection meets the remainder of N trivially. If M is a maximal subgroup of N which does not contain $P^{(B)}$, then by the arguments used in lemma 6, $M \cap P^{(B)}$ is a maximal N normal subgroup of $P^{(B)}$, and since the maximal N normal subgroups of $P^{(B)}$ coincide with its maximal W_p normal subgroups, we have $\phi(N) = \phi(W_p) \cap P^{(B)} = W'_p$. Since $N \lhd W$, W'_p is normal in W; in fact it is the normal closure in W of finitely many elements and is nilpotent. By proposition 4 then, $\phi(W) \geq W'_p$, and since this holds for all $p \in \pi$,

$$\phi(W) \geq \prod_{p\in\pi} W'_p \quad . \tag{1}$$

Again with a fixed prime $p \in \pi$ in mind, let $L = P\mathrm{wr}B$. By (ii) $\phi(L) = W_p' \cdot \phi(Q)$. We note that if $\{M_\lambda\}_{\lambda \in \Lambda}$ is a family of subgroups of L, then since $L \cap P_1^{(B)} = 1$ we have

$$\bigcap_{\lambda \in \Lambda} \left(M_\lambda \cdot P_1^{(B)}\right) = \left(\bigcap_{\lambda \in \Lambda} M_\lambda\right) \cdot P_1^{(B)} \quad . \tag{2}$$

Now observe that if $M \underset{\max}{<} L$, then $MP_1^{(B)} \underset{\max}{<} W$; hence

$\phi(W) \le \bigcap_{\substack{M < L \\ \max}} \left(MP_1^{(B)}\right)$ which implies by (2) that

$$\phi(W) \le \phi(L) \cdot P_1^{(B)} = W_p' \cdot \phi(Q) \cdot P_1^{(B)} \quad .$$

But since $\phi(W)$ is a π subgroup of K, the Sylow p subgroup of $\phi(W)$ must be contained in W_p'. This holds for all $p \in \pi$ and hence $\phi(W) \le \prod_{p \in \pi} W_p'$. Combining this with (1) yields the desired result.

To complete our investigation of $\phi(A\mathrm{wr}B)$ where A and B are finitely generated and abelian, we relax the demand that A be ϕ free. But this case is easy to handle; $\phi(W)$ is simply an extension of $\phi([A/\phi(A)]\mathrm{wr}B)$ by $\phi(A)^{(B)}$. For $\phi(K) = \phi(A)^{(B)}$ is clearly nilpotent and the normal closure in W of finitely many elements; hence $\phi(K) \le \phi(W)$ and $\phi(W)/\phi(K) = \phi(W/\phi(K)) = \phi([A/\phi(A)]\mathrm{wr}B)$.

Utilizing the results of propositions 7 and 8 we have the following

<u>Remark 9</u>: Let A and B be finitely generated abelian groups. Then W is ϕ free if and only if $\phi(A) = 1$ and B is residually a finite p' group.

Proof: The sufficiency of these conditions is an immediate consequence of proposition 7; we prove only the necessity. It is equally clear from proposition 4 that the ϕ freeness of A is necessary; so suppose that B is not residually a finite p' group for some $p \in \pi_A$. Since B is finitely generated and abelian, this means that B has a nontrivial Sylow p subgroup Q; hence by proposition 8, $\phi(W) \geq W'_p \neq 1$.

Section 6

A free soluble group of solubility depth d is a group of the form $F/\delta_{d+1}(F)$ for some (absolutely) free group F. G. Baumslag, B.H. Neumann, H. Neumann and P.M. Neumann have shown [2] that given any free soluble group B of solubility depth $d > 1$ and any finite set of nontrivial elements of B, there exists a normal subgroup N of B which excludes those elements and such that $B/N \leq C(\infty)\mathrm{wr}B_1$ where B_1 is a free soluble group of solubility depth $d - 1$. We refer to this fact as (*) and use it together with some of our results on wreath products to show that free soluble groups are ϕ free. If G is any group we denote, following P. Hall, by sG the class of subgroups of G. If G has the property that all of its subgroups are ϕ free, we write $\phi(sG) = 1$. We begin with a lemma which utilizes some concepts in representation theory - in particular the theorem of Maschke.

Lemma 9: If D is a finite group with $\phi(sD) = 1$ and the prime p does not divide the order of D, then $\phi(s(C(p)\mathrm{wr}D)) = 1$.

Proof: Let $W = C(p)\mathrm{wr}D$ with base group K and let $H \leq W$. If $H \leq K$ $\phi(H) = 1$, so by Schur-Zassenhaus and the analysis of lemma 5 we may assume $H = (H \cap K)C$ where H splits over $H \cap K$ and C is conjugate to a subgroup of D. Since $\phi(C) = 1$, $\phi(H)$ is contained in the elementary abelian p group $H \cap K$, hence we may regard $\phi(H)$ as a C submodule of the C module $H \cap K$. Again using Kovac's adaptation of Maschke's theorem, $\phi(H)$ is complemented in $H \cap K$ by a C admissible subgroup M; thus $H = < \phi(H), M, C > = < M, C >$ and hence $H \cap K = M$ and $\phi(H) = 1$.

Proposition 9. Let $W = A \mathrm{wr} B$ with $A = < a >$ an infinite cycle and B free soluble and let K be the base group of W. Let $S = \{w_1, w_2, \ldots, w_n\}$ be a finite set of nontrivial elements of W. Then there exists a normal subgroup N of W such that $w_i \notin N$ for $1 \le i \le n$ and $W/N = C(p) \mathrm{wr} D$ where $p \nmid |D|$ and $\phi(sD) = 1$.

Proof: We proceed by induction on the solubility depth d of B. If $d = 1$, that is $W = A \mathrm{wr} B$ where B is free abelian, we note that given any finite set of nontrivial elements of B they can be mapped nontrivially into a finite elementary abelian p group. Now denote each w_i in S by $b_i f_i (b_i \in B, f_i \in K, 1 \le i \le n)$. Let $\sigma f_i = \{b_{i_1}, \ldots, b_{i_{\rho(i)}}\}$ $1 \le i \le n$, let $R_i = \{b_{i_j} b_{i_k}^{-1} / 1 \le i, j \le \rho(i), \ j \neq k\}$ with $R = \bigcup_{i=1}^{n} R_i$, let $V = \{w_k \in S \ / \ b_k \neq 1\}$ and $U = S \backslash V$. Now choose $B_* \lhd B$ such that the finitely many nontrivial elements of $R \cup (V \cap B)$ are mapped nontrivially in B/B_*, where B/B_* is a finite elementary abelian p group. Then B_* separates $\sigma(f_i)$ for $i = 1, \ldots, n$, that is, each element of $\sigma(f_i)$ represents a distinct coset of B_* in B. If we denote by J the set of subscripts occurring on those w_j in U, then since $f_j \neq 1$ for all $j \in J$ we may choose $b_{j_\ell} \in \sigma(f_j)$ such that $f(b_{j_\ell}) = a^{\mu_j} \neq 1$. If $\mu = \max\limits_{j \in J} \mu_j$, choose the prime $p > \mu$ and such that p does not divide the order of B/B_*. Let $A_* = < a^p >$ and $N = nm_W(A_*, B_*)$. Then by proposition 5 if $\mathcal{J}$ is a transversal of B_* in B, $N = \{bf/b \in B_*$ and $\prod\limits_{b \in tB_*} f(b) \in A_*$ for all $t \in \mathcal{J}\}$.

Hence if $w_i \in V$ our choice of B_* insures that $w_i \notin N$; while if $w_i \in U$ then $f_i \neq 1$ and by our choice of B_*

$f_i \in N \rightleftarrows f_i(b) \in A_*$ for all $b \in \sigma(f_i)$. But we have arranged that $f_i(b_{i_\ell}) = a^{\mu_i} \notin A_*$. Hence each w_i is mapped nontrivially in

$W/N = A/A_* \mathrm{wr} B/B_* = C(p)\mathrm{wr}D$ where p does not divide the order of D and $\phi(sD) = 1$, and our induction is off the ground. So we may assume the lemma holds for the restricted wreath product of an infinite cycle by a free soluble group of depth $d - 1$ and let $W = A\mathrm{wr}B$ where B is free soluble of depth $d > 1$. Let S, U, V, R and J be defined as before. By virtue of (*) there exists a homomorphism φ mapping B onto a subgroup of $C(\infty)\mathrm{wr}B_1$ where B_1 is free soluble of depth $d - 1$ such that φ maps the finite number of elements in $R \cup (V \cap B)$ nontrivially. And by induction there exists a homomorphism ψ mapping $C(\infty)\mathrm{wr}B_1$ onto $C(q)\mathrm{wr}D$ (where the prime q does not divide the order of D and $\phi(sD) = 1$) such that $(b\varphi)\psi \neq 1$ for all $b \in R \cup (V \cap B)$.

Thus $B\varphi\psi \leq C(q)\mathrm{wr}D$ and if we choose B_* to be the kernel of the composite map $\varphi\psi$ then as before, B_* separates $\sigma(f_i)(1 \leq i \leq n)$ and $b_j \notin B_*$ for all $j \in J$. Again for w_j in U choose b_{j_ℓ} such that $1 \neq f(b_{j_\ell}) = a^{\mu_j}$ and with $\mu = \max_J \mu_j$, choose the prime $p > \mu$ and such that p does not divide the order of $C(q)\mathrm{wr}D$. If we let $A_* = < a^p >$ and $N = nm_W(A_*, B_*)$ we see that $S \cap N = \emptyset$ and

$$W/N = A/A_*\mathrm{wr}B/B_* = C(p)\mathrm{wr}(s(C(p)\mathrm{wr}D)) .$$

By lemma 9 $\Phi(s(C(p)\mathrm{wr}D)) = 1$ which completes the proof.

Corollary 10. Free soluble groups have trivial Frattini subgroup.

Proof: Let F be a free soluble group of depth d. Since the assertion is true for free abelian groups we may assume $d > 1$. By (*) $F \in R\ s(C(\infty)\mathrm{wr}B)$ where $\delta(B) = d - 1$ and by proposition 9, $C(\infty)\mathrm{wr}B \in R\ C(p)\mathrm{wr}D$ where p does not divide the order of D and $\Phi(sD) = 1$. Thus $F \in R\ s(C(p)\mathrm{wr}D)$ which implies by lemma 9 that F is residually Φ free and hence F is Φ free.

BIBLIOGRAPHY

[1] Baumslag, G., *Wreath Products and p Groups*, Proc. London Philos. Soc. 55, Part 3, (1959) pp.224-231.

[2] Baumslag, G., Neumann, B.H., Neumann, H., and Neumann, P.M., *On Varieties Generated by a Finitely Generated Group*, Math. Zeitschr. 86, (1964) pp.93-122.

[3] Gaschutz, W., *Über die Φ-Untergruppe endlicher Gruppen*, Math. Zeitschr. 58, (1953) pp.160-170.

[4] Hall, M., *The Theory of Groups*, Macmillan, New York (1959).

[5] Hall, P., *The Frattini Subgroups of Finitely Generated Groups*, Proc. London Math. Soc. (3) 11, (1961) pp.327-352.

[6] Higman, G. and Neumann, B.H., *On Two Questions of Itô*, Journal London Math. Soc., 29, (1954) pp.84-88.

[7] Hirsch, K.A., *On Infinite Soluble Groups* (V), Journal London Math. Soc. 29, (1954) pp.250-251.

[8] Kaplansky, I., *Infinite Abelian Groups*, The University of Michigan Press, Ann Arbor, (1954).

[9] Kovacs, L.G. and Newman, M.F., *Direct Complementation in Groups with Operators*, Archiv der Math., 13, (1962) pp.427-433.

[10] Kurosch, A.G., *The Theory of Groups*, Volume II, Chelsea Publishing Co., New York, (1956).

[11] Magnus, W.,Karrass, A., and Solitar, D., *Combinatorial Group Theory*, Interscience Publishers, New York, (1966).

[12] Neumann, B.H., An Essay on Free Products of Groups with Amalgamations, Royal Soc. of London, Philos. Transactions, 246, (June, 1954) pp.503-544.

[13] Neumann, P.M., On the Structure of Standard Wreath Products of Groups, Math. Zeitschr. 84, (1964) pp.343-373.

AUTOBIOGRAPHICAL STATEMENT

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