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Geometric Realization of Homotopy Systems

by

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Chapter 1. Introduction

In this paper, we begin an investigation into the geometric realization of homotopy systems. A homotopy system is a crossed complex of free type, originally defined by J.H.C. Whitehead in **[CH II]**, and patterned after a sequence of relative homotopy groups associated with the skeletal filtration of a CW complex. More specifically, if $(X,*)$ is a pointed connected CW complex with skeleta $\{X^n\}$, then the corresponding homotopy system for X is the crossed complex

$$\rho(X,*) = \cdots \to \pi_k(X^k, X^{k-1}) \to \pi_{k-1}(X^{k-1}, X^{k-2}) \to \cdots \to \pi_2(X^2, X^1) \to \pi_1(X^1)$$

in which all homotopy groups are taken with the basepoint $*$, which will usually be suppressed from the notation. Each group in the sequence has operators from $\pi_1(X)$ for $n \geq 3$ and from $\pi_1(X^1)$ for $n = 1, 2$ and is free in an appropriate sense.

$\rho(X)$ is closely related to the usual chains on the universal cover of X—the two agree in dimensions greater than two. In dimensions 1 and 2, $\rho(X)$ has the non-abelian structure of a free crossed module.

There is a category of homotopy systems with properties analogous to those of the category of free chain complexes, and ρ is a functor from the category of pointed CW complexes and skeleta-preserving maps to the category of homotopy systems. Each category has a definition of homotopy, and ρ is also functor on the respective homotopy categories.

The geometrical realization problem asks whether a given homotopy system can be obtained as $\rho(X)$ for some CW complex X. The low-dimensional non-abelian information in the homotopy system is definitive enough to guarantee that any homotopy system can be partially realized through dimension four, with the first obstruction occurring in dimension five **[CH II]**.

1.1. The Main Results.

The attempt to realize the relative homotopy groups in a homotopy system by attaching cells leads naturally to the consideration of filtrations other than the CW filtration. Given a homotopy system C, we shall build a cell complex KC, unique up to homotopy, that has a filtration $\{J^nKC\}$. The filtered space KC_{J} has a homotopy system

$$\rho(KC_{\mathrm{J}}) = \cdots \to \pi_k(J^kKC, J^{k-1}KC) \to \pi_{k-1}(J^{k-1}KC, J^{k-2}KC) \to \\ \cdots \to \pi_2(J^2KC, J^1KC) \to \pi_1(KC^1)$$

that realizes C. KC is constructed by attaching cells and so has a CW filtration, but the CW filtration on KC is not necessarily the same as the J-filtration mentioned above that accomplishes the geometric realization. In general, $J^{n+1}KC$ is obtained from J^nKC by attaching cells in dimensions $n+1$ and $n+2$, and roughly speaking, the inability to dispense with the $n+2$ cells constitutes an obstruction to geometric realizability.

The rough observation is made precise in the following fashion: Given C, construct the CW complex KC with its CW- and J-filtrations, and identify C with $\rho(KC_{\mathrm{J}})$. The identity map of KC induces a map $\alpha\colon \rho(KC_{\mathrm{CW}}) \to C$, and the chain homotopy class of α contains a chain epimorphism $\bar{\alpha}$. Our main result is that the existence of a chain splitting of this epimorphism is a necessary and sufficient condition for geometric realization, in other words, C can be geometrically realized if and only if it is a retract of $\rho(KC_{\mathrm{CW}})$. This observation is the starting point for an obstruction theory for geometric realization.

The filtration $\{J^nKC\}$ mentioned above is an example of what we call a *J-filtration*, generalizing J.H.C. Whitehead's definition of a J-complex in **[CH II]**. The defining property of a J-filtration on a space Y is that the maps $\pi_nJ^nY \to \pi_n(J^nY, J^{n-1}Y)$ for $n \geq 2$ are monomorphisms. The critical fact about J-filtered spaces is that if Y_{J} is a J-filtered space and X_{CW} is a CW complex, then there is a natural bijection of filtered homotopy classes of filtered maps

$$[\,X_{\mathrm{CW}},\, Y_{\mathrm{J}}\,] \leftrightarrow [\,\rho(X_{\mathrm{CW}}),\, \rho(Y_{\mathrm{J}})\,].$$

When $Y_{\mathrm{J}} = KC_{\mathrm{J}}$, then there is an isomorphism $\rho(KC_{\mathrm{J}}) \simeq C$ that is natural for maps of C, and we then obtain a natural bijection

$$[\,X_{\mathrm{CW}},\, KC_{\mathrm{J}}\,] \leftrightarrow [\,\rho(X_{\mathrm{CW}}),\, C\,].$$

When C can be geometrically realized, this bijection provides us with a homotopy splitting for the map α mentioned above.

The J-filtration on KC has enough connectivity to allow us to approximate any continuous function $X \to KC$ by a filtration preserving map $X_{\mathrm{CW}} \to KC_{\mathrm{J}}$. A bijection

$$[X, KC] \leftrightarrow [X_{\mathrm{CW}}, KC_{\mathrm{J}}]$$

results, where the left-hand side denotes homotopy classes of maps. Composing this with the previous bijection, we obtain a natural bijection

$$[X, KC] \leftrightarrow [\rho(X_{\mathrm{CW}}), KC_{\mathrm{J}}]$$

for any CW complex X and any homotopy system C. This suggests that we view KC as a "generalized Eilenberg-MacLane space", and there are other reasons for advancing this analogy: First, if C has homology in just one dimension n, then KC is a $K(H_nC, n)$. Second, the universal covering space $\widetilde{KC}$ of KC has the homotopy type of the product

$$\widetilde{KC} = \prod_{n=2}^{\infty} K(H_nC, n).$$

A particular case of this occurs when C has homology in dimensions 1 and $n > 1$. In this case, $\widetilde{KC} = K(H_nC, n)$ with the action of $H_1(C)$ prescribed by the structure of C. Thus KC has the homotopy type of the classifying space for cohomology with local coefficients $H_1(C)$.

1.2. Organization.

Chapter Two contains the necessary algebraic preliminaries on crossed modules, crossed complexes, and homotopy systems. Chapter Three contains the necessary topological preliminaries. In it we analyze completely the way in which filtered homotopies of filtered spaces induce chain homotopies of the crossed complexes associated with those filtered spaces. The literature contains of assertions of these facts, but proofs are either lacking or are widely scattered among the works of different authors, with different assumptions and notations. When some of the details are worked out, they are verified for spaces with CW filtrations, and I am not aware of a complete analysis for an arbitrary filtered space. Consequently, we have devoted the chapter to a complete derivation of all of the required facts about induced chain homotopies. In Chapter Four, we define the concept of a J-filtration and work out the basic properties of J-filtered spaces referred to in the previous section. In Chapter Five we use the information in a homotopy system C to construct a J- filtered space KC_{J} with $\rho(KC_{\mathrm{J}}) \cong C$. This construction contains the basic insight of the paper, namely that it is easy to realize C by inductively attaching cells if one doesn't require the filtration to be a CW filtration. Using the results of Chapter Four, we verify various properties of KC, principally its invariance up to homotopy. In Chapter Six we calculate the homotopy type of the universal covering space of KC, and in Chapter Seven we prove that C is geometrically realizable if and only if it is a retract of $\rho(KC_{\mathrm{CW}})$.

Some of these ideas have seen extensive development in other contexts and for other purposes. The next three sections review these approaches.

1.3. The Work of Brown and Higgins.

Although Brown and Higgins do not discuss the problem of geometric realizability of CW complexes, they do develop the machinery described above (in a more general setting) as part of their work on generalized van Kampen theorems. In particular, the natural bijection that we denote as

$$[X, KC] \leftrightarrow [\rho X, C]$$

appears in their work as

$$[X, BC] \leftrightarrow [\pi X, C]. \tag{1.1}$$

Brown and Higgins do not relate BC to the geometric realizability of C. Their construction of BC is more general than ours in two respects: First, they avoid the problems and restrictions of basepoints by using crossed complexes over groupoids rather than over groups. (This extra level of generality may not be of much help for geometric realization problems, although it is essential for van Kampen theorems.) Secondly, they do not restrict their attention to homotopy systems (in which the groups involved are all free in some sense) and obtain the bijection for any crossed complex C. While our construction is *ad hoc*, building the space KC by attaching cells according to the information in C, Brown and Higgins' result depends on four categories, $\mathcal{FTOP}$, $\mathcal{CUBS}$, $\mathcal{CRS}$, and ω-$\mathcal{GPD}$, and the relations between them defined by eight functors.

For the remainder of this section we use the notation and conventions of Brown and Higgins, writing BC instead of our KC and πX instead of our ρX. Brown and Higgins use ρ to denote another functor. In order to lessen the confusion, we denote their functor rho by the symbol ϱ and keep ρ for our functor, which they denote by π.

The categories involved in Brown and Higgins' generalization of our result are

(1) $\mathcal{FTOP}$ is the category of *filtered weakly compactly generated topological spaces.* (The category of weakly compactly generated spaces is denoted here by $\mathcal{WTOP}$.) There is an internal product

$$(X \otimes Y)_n = \bigcup_{p+q=n} X_p \times X_q$$

(where "$\times$" is the product in $\mathcal{WTOP}$), an internal hom

$$\mathrm{FTOP}(X,Y)_n = \{\, f \in \mathcal{WTOP} \mid f(X_p) \subset Y_{p+n} \,\},$$

and an exponential law giving $\mathcal{FTOP}$ the structure of a symmetric monoidal closed category.

(2) $\mathcal{CUBS}$ is the category of cubical sets. A *cubical set* is a family of sets $\{ K_n \}$ together with *face maps* $\delta_i^\alpha \colon K_n \to K_{n-1}$, $i = 1, \ldots, n$, $\alpha = 0, 1$ and *degeneracy maps* $\epsilon_i \colon K_{n-1} \to K_n$, $i = 1, \ldots, n$ satisfying axioms on how these compose. (The degeneracies model singular cubes that result from crossing a lower dimensional cube with I.) In [**B-H 3**], a product structure on $\mathcal{CUBS}$ (mimicking the topological product of cubes) is described, making $\mathcal{CUBS}$ monoidal closed.

(3) ω-$\mathcal{GPD}$ is the category of ω-groupoids. An *ω-groupoid*, defined in [**B-H 1**], is a cubical set with two additional types of structures: There are *connections* $\Gamma_i \colon K_n \to$

K_{n-1} which model singular cubes with adjacent degenerate faces, and *compositions* consisting of n partial compositions $+_j$ and n unary operations $-_j$ for each set K_n. These model the homotopy compositions definable along the axes of the cubes. An intricate collection of axioms (involving 31 cases) governs the interactions between the various degeneracies, face maps, and compositions. The product structure for $\mathcal{CUBS}$ mentioned above extends to ω- groupoids and so ω-$\mathcal{GPD}$ is also monoidal closed.

(4) $\mathcal{CRS}$ is the category of crossed complexes over groupoids, described in [**B-H 2**] and alluded to above. Let X_{F} be a filtered space and let $C_1 = \pi_1(X, F^0X)$ be the fundamental groupoid of X relative to the set of basepoints F^0X. For $n \geq 2$, let

$$C_n = \{\, \pi_n(F^nX, F^{n-1}X, x) \mid x \in F^0X \,\}.$$

The collection of these $\{ C_n \}$ together with the various boundary maps and actions of C_1 is called the *fundamental crossed complex* of the filtered space X_{F} and is denoted by πX_{F}. The objects of $\mathcal{CRS}$ are algebraic models for πX_{F}; each is called a *crossed complex over a groupoid.* Here are some details:

A groupoid is, of course, a small category in which every morphism is invertible. If G is a groupoid, write $G_1 = \mathrm{Mor}(G)$, $G_0 = \mathrm{Ob}(G)$. There are the source and target maps $\delta^0, \delta^1 \colon G_1 \to G_0$ and the "inclusion" map $\iota \colon G_0 \to G_1$ satisfying $\delta^0\iota = 1 = \delta^1\iota$.

If G is a groupoid with the structure defined above, a *G-module* C is a family of groups $\{ C_n(v) \mid v \in G_0 \}$ with a right action of G_1, $C(v) \times G_1(v, w) \to C(w)$ $(v, w \in G_0)$. These actions give C the structure of a groupoid over (i.e. with object set) G_0. It is conventional to write the $C(v)$ additively, whether or not they are abelian, and write composition in G_1 multiplicatively.

A *crossed G-module* C is a G-module C together with a morphism of groupoids over G_0 $\partial \colon C \to G$ such that

(1) $\partial C(v) = G_1(v, v)$ for all $v \in G_0$.

(2) $\partial(c^g) = g^{-1}(\partial c)g$ for all $c \in C(v)$, $g \in G_1(v, w)$.

(3) $c_1^{\partial c_2} = -c_1 + c_2 + c_1$ for all $c_1, c_2 \in C(v)$.

An *abelian G-module* is a G-module C with all groups $C(v)$ abelian. Finally, a *crossed complex over G* is a sequence

$$\begin{tikzcd} \cdots \arrow[r, "\partial_{n+1}"] & C_n \arrow[r, "\partial_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] & \cdots \arrow[r, "\partial_3"] & C_2 \arrow[r, "\partial_2"] & C_1 \arrow[r, shift left, "\delta^0"] \arrow[r, shift right, "\delta^1"'] & C_0 \end{tikzcd}$$

such that

(1)

$$\begin{tikzcd} C_1 \arrow[r, shift left, "\delta^0"] \arrow[r, shift right, "\delta^1"'] & C_0. \end{tikzcd}$$

is a groupoid G.

(2) $C_2 \to C_1$ is a crossed G-module.

(3) C_n is an abelian G-module for $n \geq 3$.

(4) ∂_n is a morphism of groupoids over C_0 (and so is a morphism of G-modules.)

(5) $\partial_{n-1}\partial_n$ is trivial for $n \geq 3$.

(6) ∂C_2 acts trivially on C_n for $n \geq 3$.

If C_0 is a single point, these definitions reduce to the definition of a crossed complex used in this paper (see Chapter 2), except that our actions are left actions rather than right actions.

If G and H are groupoids, M is a G-module and N is an H-module, then a *module morphism* $f\colon M \to N$ consists of a pair $(\{f(v)\}, \phi)$ where $\phi\colon G \to H$ is a morphism of groupoids and $\{f(v)\}$ is a family of operator homomorphisms $f(v)\colon M(v) \to N(v)$, $(v \in G_0)$, i.e. $f(w)(m^g) = (f(v)m)^{\phi g}$ for all $v, w \in G_0$, $m \in M(v)$, $g \in G_1(v,w)$. A *morphism of crossed complexes* $f\colon C \to D$ consists of a family of module morphisms $f_n\colon C_n \to D_n$ that commute with the boundary maps and preserve the conditions on the actions. We thus obtain the category $\mathcal{CRS}$.

$\mathcal{CRS}$ has an internal hom and product that yield a symmetric monoidal closed structure. The definitions are complicated—the relations for the product, for example, involve fifteen cases. The definitions and properties are obtained in [**B-H 3**] by first defining an internal hom and product for cubes in ω-$\mathcal{GPD}$ and then transporting these definitions to $\mathcal{CRS}$ by the equivalence of categories γ described below.

The functors involved in establishing the bijection (1.1) are displayed in the following diagram:

$$\begin{tikzcd}
\mathcal{FTOP} \arrow[d, shift right, "\pi"'] \arrow[dr, "\varrho" near end] & \mathcal{CUBS} \arrow[l, "|\ldots|"'] \\
\mathcal{CRS} \arrow[u, shift right, "B"'] \arrow[ur, "N" near end] \arrow[r, shift left, "\lambda"] & \omega\text{-}\mathcal{GPD} \arrow[u, "U"] \arrow[l, shift left, "\gamma"]
\end{tikzcd}$$

π is the fundamental crossed complex functor defined above, and U is the functor that forgets the connection and composition structures of ω-$\mathcal{GPD}$. The remaining six functors can be described as follows:

(1) $|\ldots|\colon \mathcal{CUBS} \to \mathcal{FTOP}$ is the cubical geometrical realization functor.

(2) $\varrho\colon \mathcal{FTOP} \to \omega$-$\mathcal{GPD}$ is the *fundamental ω-groupoid* functor defined in [**B-H 2**] (and revised in [**B-H 4**]). For a filtered space X_{F}, $\varrho_n(X_{\mathrm{F}})$ is the quotient $\mathcal{FTOP}(I^n, X)$ by the relation of filter homotopy rel vertices.

(3) $\lambda\colon \mathcal{CRS} \to \omega$-$\mathcal{GPD}$ is defined in [**B-H 1**] by $(\lambda C)_n = \mathcal{CRS}(\pi I^n, C)$.

(4) $\gamma\colon \omega$-$\mathcal{GPD} \to \mathcal{CRS}$ is defined in [**B-H 1**] so that $(\gamma G)_0 = G_0$, $(\gamma G)_1 = G_1$, and $(\gamma G)_n(v)$ is the set of all n-cubes x, all of whose faces except $\delta_0^1 x$ are concentrated at $v \in G_0$.

(5) $N\colon \mathcal{CRS} \to \mathcal{CUBS}$ is $U\lambda$. For any crossed complex C, it can be shown that NC is a Kan complex.

(6) $B\colon \mathcal{CRS} \to \mathcal{FTOP}$ is $|N\ldots|$.

The internal products introduced in these categories allow the definition of homotopies (and n-fold homotopies) in each category. It is then possible to show that both π and B are homotopy functors.

The following pair of results underlie the proof of the natural bijection (1.1):

(1) λ and γ are an inverse pair of equivalences of categories. [**B-H 1**] These equivalences are a critical ingredient in many of the proofs. Features of $\mathcal{CRS}$ are transported to ω-$\mathcal{GPD}$, a proof is carried out in ω-$\mathcal{GPD}$, and the result is transported back to $\mathcal{CRS}$.

(2) For a pair of filtered spaces X_{F} and Y_{G}, there is in $\mathcal{CRS}$ a natural Eilenberg-Zilber morphism $\pi X_{\mathrm{F}} \otimes \pi Y_{\mathrm{G}} \to \pi(X_{\mathrm{F}} \otimes Y_{\mathrm{G}})$ which is an isomorphism when F and G are CW filtrations of X and Y respectively. (This is a good example of a result proved in ω-$\mathcal{GPD}$ and transported to $\mathcal{CRS}$.)

The proof of (1.1) has two basic ingredients: One ingredient is the exploitation of the properties of the geometric realization functor. This makes it possible to regard any CW complex X as the geometric realization of its cubical singular complex SX. Using the definition of B given above, one gets

$$\mathrm{WTOP}(X, BC) = \mathrm{WTOP}(|SX|, |NC|).$$

Using the fact that NC is a Kan complex and the adjointness relations afforded by the monoidal closed structures in $\mathcal{WTOP}$ and $\mathcal{CUBS}$, one finds a natural weak homotopy equivalence

$$\mathrm{WTOP}(|SX|, |NC|) \simeq \mathrm{CUBS}(SX, NC).$$

Here WTOP and CUBS are the internal homs in $\mathcal{WTOP}$ and $\mathcal{CUBS}$.

The second ingredient is the exploitation of the equivalence of the categories $\mathcal{CRS}$ and ω-$\mathcal{GPD}$, the monoidal closed structures in those categories, and the Eilenberg-Zilber isomorphism described above. These make it possible to obtain a natural isomorphism of cubical sets

$$\mathrm{CUBS}(SX, NC) \cong N(\mathrm{CRS}(\pi|SX|, C).$$

String the preceding equivalences together to get a weak homotopy equivalence

$$\mathrm{WTOP}(X, BC) \simeq N(\mathrm{CRS}(\pi|SX|, C).$$

(1.1) is the bijection of path components induced by this last equivalence.

1.4. The Work of H.J. Baues.

In **[Bau]** , H.J. Baues describes a construction he calls a *tower of categories* that generalizes and unifies Whitehead's approach and encodes the geometric realization problem for homotopy systems. Since his main target is homotopy classification problems, Baues is more interested in counting actual realizations than in identifying obstructions to realizations. Nonetheless, Whitehead's approach to calculating obstructions to geometric realizability can be described in the language of the tower of categories, and the elaborate machinery of the tower may provide tools for analyzing and calculating these obstructions.

One of the thrusts of Brown and Higgins' work is to transcend the restrictions of the category of pointed spaces. One of the thrusts of Baues' work is to develop relative versions of Whitehead's ideas. For simplicity, our description below ignores the additional generality inherent in the relative nature of all of Baues' constructs. A second thrust of Baues' work that we do not address here is an axiomatization of homotopy theory that makes the results and proofs simultaneously available in several categories.

We first must deal with a conflict of terminology. Baues chooses to use the term "homotopy system" for an object that is related to but different from the free crossed complexes that Whitehead called homotopy systems. In this paper, we use the word homotopy system in Whitehead's sense, so we feel obliged for the sake of clarity to rename Baues' homotopy systems.

Baues defines categories $\mathbf{H}_n$ of *homotopy systems of order n.* We are going to rename $\mathbf{H}_n$ the category of *partial realizations through dimension n.* A partial realization through dimension n is a triple (C, f_n, X^{n-1}) in which C is essentially a homotopy system, X^{n-1} is a CW complex, $C_k \cong \pi_k(X^k, X^{k-1})$ for $1 < k < n-1$ and $C_1 \cong \pi_1 X^1$, $f_n\colon C_n \to$

$\pi_{n-1}(X^{n-1})$, and there is a commutative diagram

$$\begin{tikzcd}
 & & & \pi_{n-1}X^{n-1} \arrow[r] & \pi_{n-1}(X^{n-1},X^{n-2}) \arrow[r] & \cdots \arrow[r] & \pi_2(X^2,X^1) \arrow[r] & \pi_1X^1 \\
\cdots \arrow[r] & C_{n+1} \arrow[r] & C_n \arrow[ur, "f_n"] \arrow[rr, "d_n"'] & & C_{n-1} \arrow[u, "\cong"] \arrow[r] & \cdots \arrow[r] & C_2 \arrow[u, "\cong"] \arrow[r] & C_1 \arrow[u, "\cong"]
\end{tikzcd}$$

(We say "essentially" above because Baues uses for C a chain complex that models the chains on the universal covering space of X^n. This complex agrees with the free crossed complex we have called C in dimensions above 2 and has related abelian groups in dimensions 1 and 2.) There are appropriate concepts of morphisms and chain homotopy for these partial realizations, and so there is a category $\mathbf{H}_n$ and the associated homotopy category $\mathbf{H}_n/\simeq$.

From the point of view of the ultimate geometrical realization of the homotopy system C, there may be partial realizations through dimension n that are "dead ends", meaning that a partial realization through dimension n cannot be extended to the next dimension. A necessary and sufficient condition for a partial realization (C, f_n, X^{n-1}) to be extended to dimension $n+1$ is the *cocycle condition* on the map f_n given by $f_n d_{n+1} = 0$, where $d_{n+1}\colon C_{n+1} \to C_n$ is the boundary map. The full subcategory $\mathbf{H}_n^c$ of $\mathbf{H}_n$ consists of all partial realizations through dimension n that satisfy the cocycle condition—roughly speaking this is all homotopy systems with partial realizations through dimension n that can be extended to the next dimension.

Each partial realization (C, f_{n+1}, X^n) satisfying the cocycle condition can obviously be viewed as a partial realization (C, f_n, X^{n-1}) satisfying the cocycle condition by forgetting $f\colon C_{n+1} \to \pi_n X^n$ and X^n. This shift of viewpoint is formalized by the forgetful functor $\lambda\colon \mathbf{H}_{n+1}^c/\simeq \to \mathbf{H}_n^c/\simeq$. The chain of these categories of partial realizations in consecutive dimensions, linked by the forgetful functors λ, forms the superstructure of the tower of categories, and Baues' primary goal is to find algebraic categories equivalent to the categories in the tower. The tower of categories has far more mandated structure than we have described here. An intricate set of axioms and definitions describes an obstruction theory for each stage of the tower.

In the tower of categories, the geometrical realization problem has the following description: Let B be an object in $\mathbf{H}_n^c/\simeq$, and suppose B has a representative (C, f_n, X_{n-1}). Since f_n satisfies the cocycle condition, there is an object E in $\mathbf{H}_{n+1}$ with $\lambda E = B$. If $E \notin \mathbf{H}_{n+1}^c$, then E is a "dead end." If B has no preimages under λ in $\mathbf{H}_{n+1}^c$, then there is no geometric realization of C that agrees with the homotopy system for X up through dimension n. What Baues shows is that *if* an E exists in $\mathbf{H}_{n+1}^c$ with $\lambda E = B$, then there is a cohomology group that acts transitively and effectively on the set of all such $E \in \mathbf{H}_{n+1}^c$, thereby reducing the enumeration of realizations to the calculation an approriate sequence of cohomology groups. Baues does not discuss the question of obstructions to realization, i.e. conditions under which all preimages E under λ have "attaching maps" that fail to satisfy the cocycle condition.

1.5. The Work of Justin Smith.

In the papers [**JS1**], [**JS2**], and [**JS3**], Justin Smith considers the problem of geometric realizability in the following essentially equivalent form: Let C be an ordinary abelian chain complex of $\mathbf{Z}G$ modules satisfying conditions necessary for it to be the G-equivariant normalized chains on the universal cover of a space—namely $H_0C \cong \mathbf{Z}$ and $H_1C = 0$. Is there a space X with universal cover $\widetilde{X}$ such that $\pi_1 X \cong G$ and C is chain homotopic to $C_*(\widetilde{X})$, the normalized chains on $\widetilde{X}$? Smith's approach is to attempt to construct X by building its Postnikov tower

$$X_0 = K(G,1) \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_n \leftarrow \cdots$$

along with, at the n^{th} stage, a chain homotopy $\Lambda_n \colon C^n \to C_*(\widetilde{X_n})$. Here C^n is the n-skeleton of C, $C_*(\widetilde{X_n})$ is computed as a twisted tensor product, and having succeeded in the previous dimensions, there is an obstruction to obtaining Λ_{n+1}. This obstruction uses the homological k-invariants of C and is an element of $H^{n+1}(C; H_n(\widetilde{X_n}))$. The calculations are very technical and use the detailed results of a number of papers of other authors. They do, however, ultimately yield specific realizability and non-realizability results for certain equivariant Moore spaces.

Chapter 2. Algebraic Preliminaries

2.1. Crossed Modules.

In the definition below, we have the groups C and G acting on themselves on the left by conjugation, and a left action of G on C. These actions will all be denoted by upper left exponents. We shall adopt the convention of writing G multiplicatively and C additively. The notation is not intended to convey any information about whether or not either of these groups is abelian.

DEFINITION. A *crossed module* (C, ∂, G) consists of

(1) Groups C and G, acting on themselves on the left by conjugation.

(2) A left action of G on C.

(3) A homomorphism of G-groups $\partial \colon C \to G$ satisfying

$$ {}^{\partial(c)}d = {}^{c}d \quad (= c + d - c) $$

for all $g \in G$, $c \in C$.

We shall sometimes supress ∂ and refer to C as a crossed G-module, and at other times suppress C and G, referring to the triple (C, ∂, G) as ∂.

The basic motivating example of a crossed module is $\partial \colon \pi_2(X^2, X^1) \to \pi_1(X^1)$, where X is a CW-complex with skeleta $\{X^n\}$. This example has the additional property of freeness defined below.

These definitions have the following immediate consequences:

PROPOSITION 2.1.

(1) $\partial C \triangleleft G$

(2) $ker\, \partial < \mathcal{Z}(C)$ *where* $\mathcal{Z}(C)$ *is the center of* C*.*

(3) $\mathcal{Z}(C)$ *and* C^{ab} *inherit a* $(G/\partial C)$*-module structure from the action of* G *on* C*, and* $ker\, \partial$ *is a submodule of* $\mathcal{Z}(C)$ *with respect to this structure.*

DEFINITION. Let (C, ∂, G) and (D, δ, H) be crossed modules, and suppose the actions of G on C and H on D are given by $\phi \colon G \times C \to C$, $\psi \colon H \times D \to D$ respectively. A *morphism of crossed modules* is a pair of group homomorphisms $r \colon C \to D$, $s \colon G \to H$ that fit into the commutative diagram

$$\begin{tikzcd}
G \times C \arrow[r, "\phi"] \arrow[d, "s \times r"'] & C \arrow[r, "\partial"] \arrow[d, "r"'] & G \arrow[d, "s"'] \\
H \times D \arrow[r, "\psi"] & D \arrow[r, "\delta"] & H
\end{tikzcd}$$

In other words, $\delta r = s\partial$ and $r({}^{g}c) = {}^{s(g)}r(c)$ for all $g \in G$, $c \in C$.

Let $\mathcal{XMOD}$ denote the category of crossed modules and let $\mathcal{GP}2$ denote the category of group homomorphisms and commutative squares. There is a forgetful functor $U \colon \mathcal{XMOD} \to \mathcal{GP}2$ that ignores the crossed module structure. U has a left adjoint F that produces the *free crossed module* $F(\lambda)$ on a group homomorphism $\lambda \colon A \to G$.

A special case of this, the only case that will concern us, occurs when A is a free group on a set S. A set map $m \colon S \to G$ (We shall suppress forgetful functors in the notation here) determines a group homomorphism $\mu \colon A \to G$, and in this case we denote the corresponding free crossed module on μ by (C_m, ∂_m, G). There is an injective map $i \colon S \to C_m$ which allows us to think of S as a subset of C_m, and we refer to C_m as a free crossed G-module with *basis* S. Let $\delta \colon D \to H$ be a crossed module and $\gamma \colon G \to H$ a group homomorphism. If there is a set map b making the following diagram, without β, commute in the category of sets

$$\begin{tikzcd}
& C_m \arrow[r, "\partial_m"] \arrow[dd, dashed, "\beta"'] & G \arrow[dd, "\gamma"] \\
S \arrow[ur, "i"] \arrow[dr, "b"'] & & \\
& D \arrow[r, "\delta"'] & H
\end{tikzcd}$$

then b extends to a unique crossed module morphism $(\beta, \gamma) \colon (C_m, \partial_m, G) \to (D, \delta, H)$ such that $\beta i = b$ and $\delta\beta = \gamma\partial_m$. This case is the one originally studied by J.H.C. Whitehead in **[CH II]**. The motivating example is Whitehead's theorem, alluded to above, that

$\pi_2(X^2, X^1) \to \pi_1(X^1)$ is a free crossed module with basis the homotopy classes of attaching maps of the two-cells of X.

Both the special case we have described and the generalization of it due to Brown admit descriptions in terms of generators and relations. See [**CH II**] and [ref. Brown]

2.2. Crossed Complexes and Homotopy Systems.

DEFINITION. A *crossed complex* C is a sequence of groups and homomorphisms

$$C\colon \cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \to C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1$$

such that:

(1) $\partial_2\colon C_2 \to C_1$ is a crossed module.

(2) C_n for $n \geq 3$ is a P-module where $P = \operatorname{coker} \partial_2$.

(3) ∂_n for $n \geq 4$ is a P-module map.

(4) ∂_3 is a homomorphism of the C_1-operator groups C_3 and C_2, where C_3 is viewed as a C_1-module via the quotient map $C_1 \to P$.

(5) $\partial^2 = 0$.

Note also that, because of items (2) and (3) in the definition of a crossed module, $\ker \partial_2$ and consequently $\operatorname{im} \partial_3$ are also P-modules. We shall sometimes refer to C as a *crossed complex over P*.

DEFINITION. If C_1 is a free group on a basis S_1, $\partial_2\colon C_2 \to C_1$ is a free crossed module on a basis S_2, and if C_n for $n \geq 3$ is a free P-module on a basis S_n, we call C a *homotopy system*, following J.H.C. Whitehead. Although we do not always mention them explicitly, we think of a choice of bases $\{S_1, S_2, \ldots, S_n, \ldots\}$ as part of the given data of the homotopy system.

DEFINITION. A *morphism of crossed complexes* is a sequence of group homomorphisms $\phi_n\colon C_n \to C'_n$ for $n = 1, 2, 3, \ldots$ such that:

(1) $\phi_n \partial_{n+1} = \partial'_{n+1} \phi_{n+1}$ for $n = 1, 2, 3, \ldots$

(2) (ϕ_2, ϕ_1) is a crossed module morphism.

(3) ϕ_n for $n \geq 3$ is a P-module homomorphism.

We shall often risk confusion with the abelian case by calling ϕ a *chain map.*

DEFINITION. Let C and C' be crossed complexes over P and P' respectively, and suppose that $\phi^0, \phi^1 \colon C \to C'$ are chain maps. A *chain homotopy* $\Lambda \colon \phi^0 \simeq \phi^1$ is a collection of maps $\Lambda_n \colon C_n \to C'_{n+1}$ for $n = 1, 2, 3, \ldots$, where

(1)

$$\Lambda_n \text{ is } \begin{cases} \text{a crossed homomorphism associated with } \phi_1^0 & \text{for } n = 1 \\ \text{a } C_1\text{-homomorphism} & \text{for } n = 2 \\ \text{a } P\text{-homomorphism} & \text{for } n \geq 3 \end{cases}$$

and such that

(2)

$$\partial'_{n+1}\Lambda_n(x) = \begin{cases} \phi_1^1(x)\phi_1^0(x)^{-1}, & \text{for } n = 1 \\ -\phi_n^0(x) - \Lambda_{n-1}\partial_n(x) + \phi_n^1(x), & \text{for } n \geq 2 \end{cases}$$

The statement that Λ_1 is a *crossed homomorphism* associated with ϕ_1^0 means that

$$\Lambda_1(xy) = \Lambda_1(x) + {}^{\phi_1^0(x)}\Lambda_1(y) \quad \text{for all } x, y \in C_1$$

Note that in part (1) of the definition, the statement that Λ_2 is a C_1-homomorphism assumes a C_1-action on C'_3 without specifying whether ϕ_1^0 or ϕ_1^1 is to be used to induce this action. The lack of specificity is justified by the fact that the two actions are the same by virtue of the $n = 1$ case of part (2) of the definition.

With these definitions, the relation of chain homotopy is an equivalence relation on the morphisms $C \to C'$. The definition of chain homotopy used here is the appropriate one to model pointed homotopies of pointed maps of CW complexes. Whitehead's original more complicated definition modeled the features of free homotopies of pointed maps of CW complexes.

Chapter 3. Topological Preliminaries

3.1. Filtered Spaces.

Definition. A *filtered space* is a strictly pointed CW complex X and a filtration

$$\{ F^n X \mid n = 0, 1, 2, \ldots \}$$

with $F^0 X = *$, $F^1 X = X^1$, and $F^2 X = X^2$. (*Strictly pointed* means that $X^0 = *$.) The filtration will be called *free* if $\pi_n(F^n X, F^{n-1} X)$ is free for $n \geq 3$. The filtration F will be called *well-connected* if $\pi_n(F^r X, F^{r-1} X) = 0$ for all $r < n$. The filtration F will be called a *J-filtration* if $\pi_n F^n X \to \pi_n(F^n X, F^{n-1} X)$ is a monomorphism for $n \geq 3$. (It is automatically a monomorphism for $n = 2$ by definition of F^2 and F^1.) The letter J, when used as a filtration symbol, will always denote a J- filtration.

When we regard X as a space with the filtration F, we shall write X_{F}. The space X with its CW filtration will be denoted by X_{CW}. If X_{F}, X_{G} are filtered spaces, the notation $f\colon X_{\mathrm{F}} \to X_{\mathrm{G}}$ will always mean that f is filtration-preserving. If we wish to denote a continuous function from X to Y that is not necessarily filtration-preserving, we omit the filtration subscripts on the domain and codomain and write $f\colon X \to Y$.

Because our definitions require a filtration F to begin as the CW filtration in dimensions 0,1, and 2, $\pi_1 F^1 X$ is a free group and $\pi_2(F^2 X, F^1 X) \to \pi_1 F^1 X$ is a free crossed module with basis indexed by the 2-cells of X

Definition. If X_{F} is a filtered space, let $\rho(X_{\mathrm{F}})$ be the crossed complex

$$\rho(X_{\mathrm{F}}) = \cdots \to \pi_n(F^n X, F^{n-1} X) \to \cdots \to \pi_2(F^2 X, F^1 X) \to \pi_1 F^1 X.$$

ρ is a functor from the category of filtered spaces and filtration-preserving maps to the category of crossed complexes. If F is a free filtration, then $\rho(X_{\mathrm{F}})$ is a homotopy system.

3.2. Chain Homotopy.

In this section, we study the manner in which a filtered homotopy of filtered spaces $H\colon I \underline{\times} X_{\mathrm{F}} \to Y_{\mathrm{G}}$ gives rise to a chain homotopy $\Lambda\colon \rho(X_{\mathrm{F}}) \to \rho(Y_{\mathrm{G}})$ of the associated crossed complexes. After defining the appropriate product filtration on $I\underline{\times}X$, we adopt the perspective that a chain homotopy Λ can be realized as the composition $H_* \circ \Psi$, where $H_*\colon \rho(I\underline{\times}X_{\mathrm{F}}) \to \rho(Y_{\mathrm{G}})$ is the morphism of crossed complexes induced by H and $\Psi\colon \rho(X_{\mathrm{F}}) \to \rho(I\underline{\times}X_{\mathrm{F}})$ is a certain natural map of graded groups of degree 1. We use this perspective to define what we mean by the geometric realization of a chain homotopy. The definition also allows us to obtain the properties of Λ from an analysis of the properties of Ψ. The results of this analysis and the verification that Λ has the properties required of a chain homotopy are presented in Theorems 3.1 and 3.2.

The basic intuition about Ψ is that Ψ_n should take a homotopy class $[f]$, where

$$f\colon (D^n, S^{n-1}) \to (F^nX, F^{n-1}X),$$

to the class $[I \times f]$, where, essentially,

$$I \times f\colon (D^{n+1}, S^n) \to (\partial I\underline{\times}F^nX \cup I\underline{\times}F^{n-1}X)$$

is obtained by extending f over $I \times D^n$.

The details of the proof of Theorem 3.1 are relegated to section 3. There we narrow the focus further by studying maps $\Phi_n\colon \pi_n(X,A) \to \pi_{n+1}(I\underline{\times}X, \partial I\underline{\times}X \cup I\underline{\times}A)$. These maps are then used to define Ψ. This amounts to concentrating for as long as possible on pairs $(F^nX, F^{n-1}X)$ before considering triples $(F^{n+1}X, F^nX, F^{n-1}X)$.

In the actual proofs, we use cubes rather than spheres, for the convenience of identifying $I \times I^{n-1}$ with I^n, and because we can obtain the explicit combinatorial boundary formulas (as given in Lemma 3.4) that ultimately give us the boundary formulas for Λ.

DEFINITION. Let X_{F} be a filtered space with basepoint $*$. Let I denote the unit interval, ∂I its boundary $\{0\} \cup \{1\}$, and define

$$I\underline{\times}X = \frac{I \times X}{I \times *}.$$

We will take $\{I \times *\}$ as the basepoint of $I\underline{\times}X$ and denote this basepoint by $\underline{*}$.

DEFINITION. A filtration P on $I\underline{\times}X$ is defined by

$$\begin{aligned} P^0(I\underline{\times}X) &= \underline{*} \\ P^n(I\underline{\times}X) &= \frac{\partial I \times F^nX \cup I \times F^{n-1}X}{I \times *} \\ &= \partial I\underline{\times}F^nX \cup I\underline{\times}F^{n-1}X, \quad \text{for } n \geq 1. \end{aligned}$$

When X is a filtered space with filtration F, it is understood that $I\underline{\times}X$ has the P-filtration, and we will use $I\underline{\times}X_{\mathrm{F}}$ to denote $I\underline{\times}X$ with the P-filtration.

THEOREM 3.1. *For each filtered space X_{F} and each positive integer n, there are maps*

$$\Psi_n(X_{\mathrm{F}})\colon \rho_n(X_{\mathrm{F}}) \to \rho_{n+1}(I\underline{\times}X_{\mathrm{F}})$$

satisfying

(1) *Naturality: If $g\colon X_{\mathrm{F}} \to W_L$ is a map of filtered spaces, then*

$$\begin{tikzcd} \rho_n(X_{\mathrm{F}}) \arrow[r, "\Psi_n(X_{\mathrm{F}})"] \arrow[d, "(g_n)_*"'] & \rho_{n+1}(I\underline{\times}X_{\mathrm{F}}) \arrow[d, "(I\underline{\times}g)_{n*}"] \\ \rho_n(W_L) \arrow[r, "\Psi_n(W_L)"'] & \rho_{n+1}(I\underline{\times}W_L) \end{tikzcd}$$

commutes for all $n \geq 1$.

(2) *$\Psi_1(X_{\mathrm{F}})$ is a crossed homomorphism and $\Psi_n(X_{\mathrm{F}})$ is a π_1X-homomorphism for $n \geq 2$.*

(3) *For $n = 1$, let $\alpha \in \rho_1(X_{\mathrm{F}})$ be represented by $f\colon (I, \partial I) \to (X^1, *)$. For $\partial_2^{\times} : \rho_2(I\underline{\times}X_{\mathrm{F}}) \to \rho_1(I\underline{\times}X_{\mathrm{F}})$, we have*

$$\partial_2^{\times}\Psi_1(X_{\mathrm{F}})(\alpha) = [I \times f|1 \times I][I \times f|0 \times I]^{-1}.$$

For $n \geq 2$, let $\alpha \in \rho_n(X_{\mathrm{F}})$ be represented by $f\colon (I^n, \partial I^n, J^{n-1}) \to (F^nX, F^{n-1}X, *)$. (See section 3 for the definition of J^{n-1}.) Then for the maps in the commutative diagram

$$\begin{tikzcd} \rho_n(X_{\mathrm{F}}) \arrow[r, "\Psi_n(X_{\mathrm{F}})"] \arrow[d, "\partial_n"'] & \rho_{n+1}(I\underline{\times}X_{\mathrm{F}}) \arrow[d, "\partial_{n+1}^{\times}"] \\ \rho_{n-1}(X_{\mathrm{F}}) \arrow[r, "\Psi_{n-1}(X_{\mathrm{F}})"'] & \rho_n(I\underline{\times}X_{\mathrm{F}}) \end{tikzcd}$$

we have the formula

$$\partial^{X}_{n+1}\Psi_n(X_{\mathrm{F}})(\alpha) = -[I \times f|0 \times I^n] - \Psi_{n-1}(X_{\mathrm{F}})\partial_n\alpha + [I \times f|1 \times I^n].$$

PROOF: The properties of Ψ require a large number of verifications which are relegated to section 3. ◊

Once we have Ψ with all its properties, the verification that Λ is a chain homotopy is routine:

THEOREM 3.2. *Let $X_{\mathrm{F}}, Y_{\mathrm{G}}$ be filtered spaces, $H\colon I\underline{\times}X_{\mathrm{F}} \to Y_{\mathrm{G}}$ a filtered homotopy, and denote $H|(t \times X_{\mathrm{F}})\colon X_{\mathrm{F}} \to Y_{\mathrm{G}}$ by H_t. Let Λ_n be the composition*

$$\Lambda_n\colon \rho_n(X_{\mathrm{F}}) \xrightarrow{\Psi_n(X_{\mathrm{F}})} \rho_{n+1}(I\underline{\times}X_{\mathrm{F}}) \xrightarrow{H_*} \rho_{n+1}(Y_{\mathrm{G}}).$$

Then Λ_n is a chain homotopy of $(H_0)_*$ with $(H_1)_*$.

PROOF: Since $\Lambda_n = H_* \circ \Psi_n$, the homomorphism properties of Ψ_n are transferred to Λ_n and so item (1) in the definition of chain homotopy (see section 2.2) is satisfied.

To obtain the boundary formulas in item (2) of the definition of chain homotopy, begin with the observation that

$$\partial^{Y}_{n+1}\Lambda_n\alpha = \partial^{Y}_{n+1}H_*\Psi_n\alpha = H_*\partial^{X}_{n+1}\Psi_n\alpha.$$

Thus, to finish we have to apply H_* to the boundary formulas in (3) of Theorem 3.1. Using

$$H_*[I \times f|t \times I^n] = (H_t)_*(\alpha) \quad \text{for } t = 1, 2$$

and $H_*\Psi_{n-1} = \Lambda_{n-1}$, we obtain the claimed formulas for Λ_n. ◊

3.3. The Properties of Ψ.

The results of Theorem 3.1 are all consequences of the properties of some maps

$$\Phi_1\colon \pi_1(A, *) \to \pi_2(I\underline{\times}A, \partial I\underline{\times}A)$$

$$\Phi_n\colon \pi_n(X, A) \to \pi_{n+1}(I\underline{\times}X, \partial I\underline{\times}X \cup I\underline{\times}A) \quad n \geq 2$$

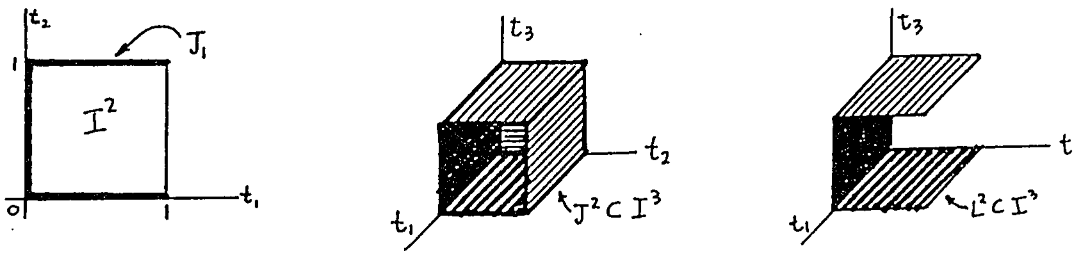

Figure 3.1.

defined for pairs of spaces (X, A). In this section, we define Φ and then prove its relevant properties in Theorem 3.3. We obtain Ψ from Φ in a simple way that allows the demonstrated properties of Φ to become to the asserted properties of Ψ.

We begin with the following list of symbols and conventions. Some of the definitions, for small values of n, are illustrated in Figure 3.1.

$$I^n = \{\, (t_1, t_2, \ldots t_n) \in \mathbf{R}^n \mid 0 \leq t_i \leq 1 \,\} \quad n \geq 2.$$

$$I^0 = \{0\} \cup \{1\}.$$

$$\partial I^n = \{\, (t_1, t_2, \ldots t_n) \in I^n \mid \prod_{i=1}^{n} t_i(1 - t_i) = 0 \,\} \quad n \geq 2.$$

$$\partial I^1 = I^0.$$

$$J^{n-1} = \{\, (t_1, t_2, \ldots t_n) \in I^n \mid t_1 \prod_{i=2}^{n} t_i(1 - t_i) = 0 \,\} \quad n \geq 2.$$

$$J^0 = \{0\} \subset I.$$

$$L^{n-1} = \{\, (t_1, t_2, \ldots t_n) \in I^n \mid t_2 \prod_{i=3}^{n} t_i(1 - t_i) = 0 \,\} \quad n \geq 3.$$

We take the origin $\vec{0} \in \mathbf{R}^n$ as the basepoint of I^n.

We have the relations $J^{n-1} = \partial I^n - (1 \times I^{n-1}) = (I \times \partial I^{n-1}) \cup (0 \times I^{n-1})$ for $n \geq 2$ and $L^{n-1} = I \times J^{n-2} = (I \times \partial I^{n-1}) - I \times 1 \times I^{n-1}$.

For $* \in A$, view $\pi_n(X, A, *)$ as either of the two homotopy sets $[(I^n, \partial I^n, \vec{0}), (X, A, *)]$ or $[(I^n, \partial I^n, J^{n-1}), (X, A, *)]$. The group operation in $\pi_n(X, A, *)$ is defined in the usual way on the last coordinate in the second of the above two homotopy sets. View $\pi_n(X, *)$ as $\pi_n(X, *, *)$, so that if $\alpha \in \pi_n(X, A, *)$ is represented by $f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, *)$, then $\partial\alpha \in \pi_{n-1}(A, *)$ is represented by $f|(1 \times I^{n-1})$.

For $n = 1$, let $\alpha \in \pi_1(A,*)$ be represented by $f\colon (I,\partial I) \to (A,*)$. For $n \geq 2$, let $\alpha \in \pi_n(X,A,*)$ be represented by $f\colon (I^n,\partial I^n, J^{n-1}) \to (X,A,*)$. Make the identification $(I^{n+1},\partial I^{n+1},L^n) \equiv (I \times I^n, \partial I \times I^n \cup I \times \partial I^n, I \times J^{n-1})$. Recall the notation $\underline{*} = \{I \times *\}$.

Definition. For $n = 1$, let $\Phi_1(A)(\alpha) \in \pi_2(I\underline{\times}A, \partial I\underline{\times}A, \underline{*})$ be the element represented by

$$I \times f\colon (I^2, \partial I^2, *) \to (I\underline{\times}A, \partial I\underline{\times}A, \underline{*}).$$

For $n \geq 2$, let $\Phi_n(X,A)(\alpha) \in \pi_{n+1}(I\underline{\times}X, \partial I\underline{\times}X \cup I\underline{\times}A, \underline{*})$ be the element represented by

$$I \times f\colon (I^{n+1}, \partial I^{n+1}, L^n) \to (I\underline{\times}X, \partial I\underline{\times}X \cup I\underline{\times}A, \underline{*}).$$

(We are writing $I \times f$ here for the composition $I \times I^n \to I \times X \to I\underline{\times}X$.) This is well-defined on homotopy classes and so defines a function

$$\Phi_n(X,A)\colon \pi_n(X,A) \to \pi_{n+1}(I\underline{\times}X, \partial I\underline{\times}X \cup I\underline{\times}A, \underline{*}).$$

Theorem 3.3. *Φ has the following properties:*

(1) *Naturality: If $g\colon (X,A) \to (Y,B)$ is a map of pairs, then the diagram*

$$\begin{tikzcd}
\pi_n(X,A) \arrow[r, "\Phi_n(X,A)"] \arrow[d, "g_*"'] & \pi_{n+1}(I\underline{\times}X, \partial I\underline{\times}X \cup I\underline{\times}A) \arrow[d, "(I\underline{\times}g)_*"] \\
\pi_n(Y,B) \arrow[r, "\Phi_n(Y,B)"'] & \pi_{n+1}(I\underline{\times}Y, \partial I\underline{\times}Y \cup I\underline{\times}B)
\end{tikzcd}$$

commutes.

(2) *Φ_n is a crossed homomorphism for $n = 1$ and a $\pi_1 A$-homomorphism for $n \geq 2$.*

(3) *For $n = 1$, let $\alpha \in \pi_1(A,*)$ be represented by $f\colon (I,\partial I) \to (A,*)$. For*

$$\partial_2'\colon \pi_2(I\underline{\times}A, \partial I\underline{\times}A) \to \pi_1(\partial I\underline{\times}A, \underline{*})$$

we have the formula

$$\partial_2'\Phi_1(A,*)(\alpha) = [I \times f|1 \times I][I \times f|0 \times I]^{-1}. \tag{3.1}$$

For $n \geq 2$, let $\alpha \in \pi_n(X, A)$ be represented by $f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, *)$. For the maps in the following non-commutative diagram,

$$\begin{tikzcd}
\pi_n(X,A) \arrow[r, "\partial_n"] \arrow[dd, "\Phi_n(X,A)"'] & \pi_{n-1}(A,*) \arrow[d, "\Phi_{n-1}(A,*)"] \\
& \pi_n(I\underline{\times}A, \partial I\underline{\times}A) \arrow[d, "i_*"] \\
\pi_{n+1}(I\underline{\times}X, \partial I\underline{\times}X \cup I\underline{\times}A) \arrow[r, "\partial'_{n+1}"'] & \pi_n(\partial I\underline{\times}X \cup I\underline{\times}A, \partial I\underline{\times}A, \partial I\underline{\times}A)
\end{tikzcd}$$

we have the formula

$$\partial'_{n+1}\Phi_n(X,A)(\alpha) = -[I \times f|0 \times I^n] - i_*\Phi_{n-1}(A,*)(\partial_n\alpha) + [I \times f|1 \times I^n]. \tag{3.2}$$

Proof: Item (1): The naturality condition is immediate from the formula $(I \times f) \circ (I \times g) = I \times (f \circ g)$.

Item (2): We first check that Φ_1 is a crossed homomorphism. We have

$$\Phi_1\colon \pi_1(A,*) \to \pi_2(I\underline{\times}A, \partial I\underline{\times}A) = \pi_2(I\underline{\times}A, A_0 \vee A_1).$$

$\pi_2(I\underline{\times}A, A_0 \vee A_1)$ is a crossed module over $\pi_1(A_0 \vee A_1)$, and the inclusion of A at zero induces a map $(j_0)_*\colon \pi_1 A \to \pi_1(A_0 \vee A_1)$. If $\alpha \in \pi_1 A$, we will denote the image of α under this inclusion by α_0. Inclusion of α at 1 induces another such map $(j_1)_*$ and the image of α under this map will be called α_1. Via the inclusion at zero, $\pi_1 A$ acts on $\pi_2(I\underline{\times}A, A_0 \vee A_1)$ and the formula we wish to verify is

$$\Phi_1(\alpha\beta) = \Phi_1\alpha + {}^{\alpha_0}\Phi_1\beta. \tag{3.3}$$

We verify this formula in three stages. First we check that for $\partial'_2\colon \pi_2(I\underline{\times}A, \partial I\underline{\times}A) \to \pi_1(\partial I\underline{\times}A, \underline{*})$, we have

$$\partial'_2\Phi_1(\alpha\beta) = \partial'_2(\Phi_1\alpha + {}^{\alpha_0}\Phi_1\beta). \tag{3.4}$$

Next we observe that for $A = S^1 \vee S^1$, ∂_2' is injective and so for this particular A, (3.3) follows from (3.4). Finally, we argue that the truth of (3.3) for $A = S^1 \vee S^1$ implies its truth for all A.

Stage 1: Let $f, g\colon (I, \partial I) \to (A, *)$ represent α and β, respectively. The proof of the formula

$$\partial_2'\Phi_1(A,*)(\alpha) = [I \times f|1 \times I][I \times f|0 \times I]^{-1} \tag{3.5}$$

from part (3) of Theorem 3.3 makes no use of the fact that Φ_1 is a crossed homomorphism, so we can assume (3.5) here. Simplify the notation in (3.5) by letting $F_i = [I \times f|i \times I]$ and $G_i = [I \times g|i \times I]$ for $i = 0, 1$. Then, using the rewritten version of (3.5), we can compute

$$\begin{aligned} \partial_2'\Phi_1(\alpha\beta) &= F_1G_1(F_0G_0)^{-1} \\ &= F_1G_1G_0^{-1}F_0^{-1} \\ &= (\partial_2'\Phi_1\alpha)F_0(\partial_2'\Phi_1\beta)F_0^{-1} \\ &= \partial_2'(\Phi_1\alpha)\partial_2'({}^{\alpha_0}\Phi_1\beta) \\ &= \partial_2'(\Phi_1\alpha + {}^{\alpha_0}\Phi_1\beta), \end{aligned}$$

and this was the goal of stage 1.

Stage 2: Since $\pi_2(I\underline{\times}(S^1 \vee S^1)) \cong \pi_2(S^1 \vee S^1) = 0$, it follows that

$$\partial_2'\colon \pi_2(I\underline{\times}(S^1 \vee S^1), (S^1 \vee S^1)_0 \vee (S^1 \vee S^1)_1) \to \pi_2((S^1 \vee S^1)_0 \vee (S^1 \vee S^1)_1)$$

is injective, so that (3.3) follows from (3.4) for $A = S^1 \vee S^1$, completing stage 2.

Stage 3: Now we check that the truth of (3.3) for $A = S^1 \vee S^1$ implies its truth for all A. For this, let $\sigma \in \pi_1 S^1$ be the generator in the class of the identity map of S^1, and let $\mu\colon S^1 \to S^1 \vee S^1$ be the multiplication on S^1. Let $\sigma_l, \sigma_r \in \pi_1(S^1 \vee S^1)$ be the images of σ under the maps $(i_l)_*, (i_r)_*$ induced by including S^1 as the left (respectively right) summand of $S^1 \vee S^1$. We have $\mu_*(\sigma) = \sigma_l\sigma_r$ and, by virtue of the results of stage 2, $\Phi_1(\sigma_l\sigma_r) = \Phi_1\sigma_l + {}^{(\sigma_l)_0}\Phi_1\sigma_r$.

If $\nabla \colon A \vee A \to A$ is the folding map, then in $\pi_1 A$ we have

$$\alpha\beta = \nabla_*(f \vee g)_*\mu_*(\sigma) = \nabla_*(f \vee g)_*(\sigma_l\sigma_r).$$

The computations that follow use the maps and groups in the following commutative diagram:

$$\begin{tikzcd}
\sigma \in \arrow[d] & \pi_1 S^1 \arrow[d, "\mu_*"'] & & \\
\sigma_l\sigma_r \in \arrow[dd] & \pi_1(S^1 \vee S^1) \arrow[r, "\Phi_1"] \arrow[d, "(f\vee g)_*"'] & \pi_2(I\underline{\times}(S^1 \vee S^1), (S^1 \vee S^1)_0 \vee (S^1 \vee S^1)_1) \arrow[r, tail, "\partial_2'"] \arrow[d, "(I\underline{\times}(f\vee g))_*"'] & \pi_2((S^1 \vee S^1)_0 \vee (S^1 \vee S^1)_1) \arrow[d, "((f\vee g)_0\vee(f\vee g)_1)_*"] \\
& \pi_1(A \vee A) \arrow[r, "\Phi_1"'] \arrow[d, "\nabla_*"'] & \pi_2(I\underline{\times}(A \vee A), (A \vee A)_0 \vee (A \vee A)_1) \arrow[r, "\partial_2'"'] \arrow[d, "(I\underline{\times}\nabla)_*"'] & \pi_1((A \vee A)_0 \vee (A \vee A)_1) \arrow[d, "(\nabla_0\vee\nabla_1)_*"] \\
\alpha\beta \in & \pi_1 A \arrow[r, "\Phi_1"'] & \pi_2(I\underline{\times}A, A \vee A) \arrow[r, "\partial_2'"'] & \pi_1(A \vee A)
\end{tikzcd}$$

We begin with

$$\begin{aligned}
\Phi_1(\alpha\beta) &= \Phi_1\nabla_*(f \vee g)_*(\sigma_l\sigma_r) \\
&= (I\underline{\times}\nabla)_*(I\underline{\times}(f \vee g)_*)\Phi_1(\sigma_l\sigma_r) \\
&= (I\underline{\times}\nabla)_*(I\underline{\times}(f \vee g)_*)(\Phi_1\sigma_l + {}^{(\sigma_l)_0}\Phi_1\sigma_r) \\
&= (I\underline{\times}\nabla)_*(I\underline{\times}(f \vee g)_*)(\Phi_1\sigma_l) + (I\underline{\times}\nabla)_*(I\underline{\times}(f \vee g)_*)({}^{(\sigma_l)_0}\Phi_1\sigma_r), \qquad (3.6)
\end{aligned}$$

using the naturality of Φ_1 in the second line.

Since $\sigma_l = [i_l]$, we have $\Phi_1\sigma_l = [I \times i_l]$. Furthermore, $\nabla(f \vee g)i_l = f$, so that

$$[I\underline{\times}\nabla(f \vee g)i_l] = [I \times f] = \Phi_1\alpha.$$

We use these facts to continue the above computation:

$$\begin{aligned}
(I\underline{\times}\nabla)_*(I\underline{\times}(f \vee g)_*)(\Phi_1\sigma_l) &= (I\underline{\times}\nabla)_*(I\underline{\times}(f \vee g)_*)[I \times i_l] \\
&= [I\underline{\times}\nabla(f \vee g)i_l] \\
&= \Phi_1\alpha. \qquad (3.7)
\end{aligned}$$

An analogous calculation shows that

$$(3.8) \qquad (I\underline{\times}\nabla)_*(I\underline{\times}(f \vee g)_*)(\Phi_1 \sigma_l) = \Phi_1 \beta,$$

but before we can use this we have to see how to express the action of $(\sigma_l)_0$ after commuting it with $(I\underline{\times}\nabla)_*(I\underline{\times}(f \vee g)_*)$. First note that for any

$$x \in \pi_2(I\underline{\times}(S^1 \vee S^1), (S^1 \vee S^1)_0 \vee (S^1 \vee S^1)_1),$$

we have

$$(3.9) \quad (I\underline{\times}\nabla)_*(I\underline{\times}(f \vee g))_*({}^{(\sigma_l)_0}x) = {}^{(\nabla_0 \vee \nabla_1)_*((f\vee g)_0 \vee (f \vee g)_1))_*(\sigma_l)_0}(I\underline{\times}\nabla)_*(I\underline{\times}(f \vee g))_*(x).$$

The commutativity of the following diagram

$$\begin{tikzcd}
\pi_1 S^1 \arrow[r, "(i_l)_*"] \arrow[d, "f_*"'] & \pi_1(S^1 \vee S^1) \arrow[r, "(j_0)_*"] \arrow[d, "(f \vee g)_*"] & \pi_1((S^1 \vee S^1)_0, (S^1 \vee S^1)_1) \arrow[d, "((f \vee g)_0 \vee (f \vee g)_1)_*"] \\
\pi_1 A \arrow[r, "(i_l)_*"] \arrow[dr, equal] & \pi_1(A \vee A) \arrow[r, "(j_0)_*"] \arrow[d, "\nabla_*"] & \pi_1((A \vee A)_0 \vee (A \vee A)_1) \arrow[d, "(\nabla_0 \vee \nabla_1)_*"] \\
& \pi_1 A \arrow[r, "(j_0)_*"] & \pi_1(A \vee A)
\end{tikzcd}$$

shows that

$$(\nabla_0 \vee \nabla_1)_*((f \vee g)_0 \vee (f \vee g)_1)_*(\sigma_l)_0 = \alpha_0 \in \pi_1(A \vee A).$$

Using this result in (3.9), we get

$$(3.10) \qquad (I\underline{\times}\nabla)_*(I\underline{\times}(f \vee g))_*({}^{(\sigma_l)_0}x) = {}^{\alpha_0}(I\underline{\times}\nabla)_*(I\underline{\times}(f \vee g))_*(x).$$

Assembling the results of (3.6), (3.7), (3.8), and (3.10) we see that (3.3) has been verified and so Φ_1 is a crossed homomorphism as claimed.

The case for $n \geq 2$ contains two verifications: First, we verify that Φ respects the $\pi_1 A$-action. Then we check that Φ is an additive homomorphism. To begin the first verification, let $\alpha \in \pi_1(A, *)$ be represented by $a \colon (I, \partial I) \to (A, *)$, let $\xi \in \pi_n(X, A, *)$ be represented by

$f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, *)$, and let ${}^{\alpha}\xi$ be represented by $g\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, *)$. From the definition of the $\pi_1 A$-action we know that f is freely homotopic to g by a homotopy

$$H\colon I \times (I^n, \partial I^n, J^{n-1}) \to (X, A, A)$$

in which, for each $\vec{x} \in J^{n-1}$, $H|(I \times \vec{x}) = a$. Define a homotopy

$$G\colon I \times (I^{n+1}, \partial I^{n+1}, L^n) \to (I\underline{\times}X, \partial I\underline{\times}X \cup I\underline{\times}A, I\underline{\times}A)$$

by using the identification $(I^{n+1}, \partial I^{n+1}, L^n) \equiv (I \times I^n, \partial I \times I^n \cup I \times \partial I^n, I \times J^{n-1})$ and setting $G(s,t,\vec{x}) = (t, H(s,\vec{x}))$ for $\vec{x} \in I^n$. G is a free homotopy of $I \times f$ with $I \times g$. The image of the basepoint $(0,\vec{0}) \in I \times I^n$ traces a loop in $0 \times A$ that represents the image of α under the map $\iota_*\colon \pi_1 A \to \pi_1(\partial I\underline{\times}A) = \pi_1(A \vee A)$ induced by inclusion of A at zero. Consequently, in $\pi_{n+1}(I\underline{\times}X, \partial I\underline{\times}X \cup I\underline{\times}A)$, we have ${}^{\iota_*(\alpha)}[I \times f] = [I \times g]$; in other words ${}^{\iota_*(\alpha)}\Phi_n(\xi) = \Phi_n({}^{\alpha}\xi)$, and so Φ_n is a $\pi_1 A$- homomorphism.

The second part of the verification for $n \geq 2$ is that Φ is an additive homomorphism. Let $f, g\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, *)$ represent a pair of elements of $\pi_n(X, A, *)$. Then $\Phi_n[f]$, $\Phi_n[g]$ are represented by maps

$$I \times f,\ I \times g\colon (I^{n+1}\partial I^{n+1}, L^n) \to (I\underline{\times}X, \partial I\underline{\times}X \cup I\underline{\times}A, \underline{*}).$$

Strictly speaking, addition in $\pi_{n+1}(I\underline{\times}X, \partial I\underline{\times}X \cup I\underline{\times}A)$ is defined using representatives defined on $(I^{n+1}, \partial I^{n+1}, J^n)$. However, the definition of addition of representatives using the last coordinate given in the conventions section can also be used to combine $I \times f$ and $I \times g$. Let us temporarily call the result $(I \times f) +_1 (I \times g)$. Since the definitions of $+$ in $\pi_n(X, A)$ and $+_1$ in $\pi_{n+1}(I\underline{\times}X, \partial I\underline{\times}X \cup I\underline{\times}A)$ both use the last coordinate, it is immediate that

$$(I \times f) +_1 (I \times g) = I \times (f + g).$$

The only question is when $I \times f$ and $I \times g$ are deformed to representatives defined on $(I^{n+1}, \partial I^{n+1}, J^n)$, whether the sum $(I \times f) +_1 (I \times g)$ deforms to the sum $(I \times f) + (I \times g)$. Now $+$ and $+_1$ are defined in the same way on the last coordinate—they differ only in

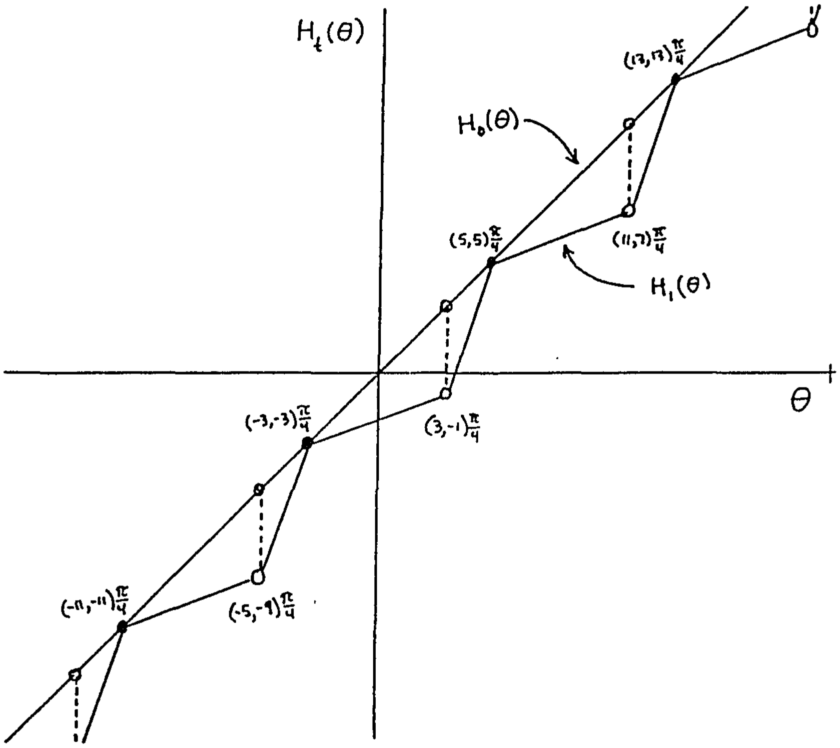

Figure 3.2. The homotopy H

the type of representative they apply to. Thus we can affirmatively settle the question by showing that, for $n \geq 2$, a map $f\colon (I^{n+1}, \partial I^{n+1}, L^n) \to (Y, B, *)$ can be deformed to a representative $f'\colon (I^{n+1}, \partial I^{n+1}, J^n) \to (Y, B, *)$ without altering the last coordinate.

A specific homotopy can be obtained as follows: Make the identification $I^{n+1} \equiv D^2 \times I^{n-1}$, identifying I^2 with $[-1,1] \times [-1,1]$ and $[-1,1] \times [-1,1]$ with D^2 by radial projection. View D^2 as the unit disk in the complex plane. Define $L\colon I \times D^2 \times I^{n-1} \to D^2 \times I^{n-1}$ by

$$L(t, re^{i\theta}, \vec{v}) = (re^{iH(t,\theta)}, \vec{v}) \quad \text{for } t \in I,\ re^{i\theta} \in D^2,\ \vec{v} \in I^{n-1}$$

where $H\colon I \times \mathbf{R} \to \mathbf{R}$ is the homotopy

$$H(t,\theta) = \begin{cases} \frac{3-2t}{3}\theta + \frac{8n-3}{6}\pi t & -\frac{3\pi}{4} + 2n\pi \leq \theta \leq \frac{3\pi}{4} + 2n\pi \\ (1+2t)\theta - \frac{8n+5}{2}\pi t & \frac{3\pi}{4} + 2n\pi \leq \theta \leq \frac{5\pi}{4} + 2n\pi \end{cases}$$

The 1-parameter family of piecewise linear functions $H_t\colon \mathbf{R} \to \mathbf{R}$ is sketched in figure 3.2. $H_0(\theta) = \theta$, and as t goes from 0 to 1, the open dots move vertically along the lines $\theta =$

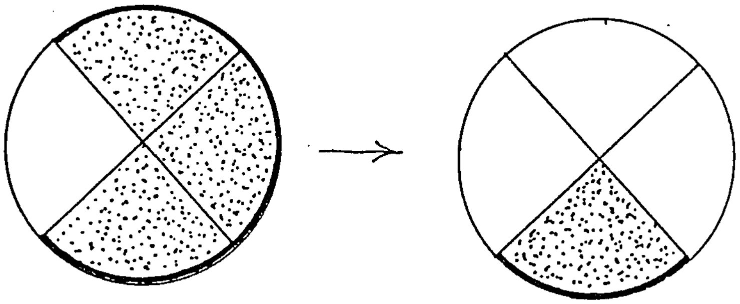

Figure 3.3. The effect of L on D^2 when $t = 1$

$\pm\frac{3\pi}{4} + 2n\pi$, $n \in \mathbf{Z}$. The two cases in the definition of H define straight line segments joining these open dots to points on the graph of the identity map with coordinates $\frac{\pi}{4}(8n-3, 8n-3)$.

H_1 maps the θ-intervals $[-\frac{3\pi}{4} + 2n\pi, \frac{3\pi}{4} + 2n\pi]$ onto the intervals $[-\frac{3\pi}{4} + 2n\pi, -\frac{\pi}{4} + 2n\pi]$ and maps the intervals $[\frac{3\pi}{4} + 2n\pi, \frac{5\pi}{4} + 2n\pi]$ onto $[-\frac{\pi}{4} + 2n\pi, \frac{5\pi}{4} + 2n\pi]$. Thus the effect of L on D^2 when $t = 1$ is the reparametrization sketched in figure 3.3. When D^2 is identified with I^2, this map carries L^n onto J^n in I^{n+1}, yielding the conversion of $f\colon (I^{n+1}, \partial I^{n+1}, L^n) \to (Y, B, *)$ to $f\colon (I^{n+1}, \partial I^{n+1}, J^n) \to (Y, B, *)$ required to finish the verification.

Item (3): In order to establish the boundary formulas we need some additional facts and definitions. View I as a CW-complex with two zero-cells $\{0\}$ and $\{1\}$ and one one-cell I. View I^n as a CW-complex with the product cell structure obtained from $I^n = I \times I \times \cdots \times I$. For fixed n, let $C = I^n$ and then use C^k to denote the k-skeleton of C. The $(n-1)$-cells or faces of C have attaching maps $\partial_i^\epsilon\colon I^{n-1} \to I^n$ for $n \geq 2$ given by

$$\partial_i^\epsilon(t_1, t_2, \ldots, t_{n-1}) = (t_1, \ldots, t_{i-1}, \epsilon, t_i, \ldots, t_{n-1}), \quad i \in \{1, 2, \ldots, n\}, \quad \epsilon \in \{0, 1\}.$$

The homology classes $D_i^\epsilon \in H_{n-1}(C^{n-1}, C^{n-2})$ determined by these attaching maps are the generators of the $(n-1)^{\text{st}}$ cellular chain group of C. If $\iota_{n-1} \in \pi_{n-1}(I^{n-1}, \partial I^{n-1})$ is the class of the identity map of I^{n-1}, then the elements

$$d_i^\epsilon = \partial_i^\epsilon(\iota_{n-1}) \in \pi_{n-1}(C^{n-1}, C^{n-2}) \quad \text{for } n \geq 3$$

are carried by the Hurewicz homomorphism to the corresponding $D_i^\epsilon \in H_{n-1}(C^{n-1}, C^{n-2})$. For $n > 3$, this is a 1-1 correspondance of free abelian group generators. For $n = 3$ it induces abelianization of the free crossed $\pi_1 C^1$-module $\pi_2(C^2, C^1)$.

$H_n(C^n, C^{n-1})$ has a single generator σ_n which is the image under the Hurwicz homomorphism of $\iota_n \in \pi_n(I^n, \partial I^n) = \pi_n(C^n, C^{n-1})$, and we have the homology formula

$$\partial_n \sigma_n = \sum_{i=1}^{n} (-1)^i (D_i^0 - D_i^1).$$

We need the analogous formula for $\partial(\iota_n) \in \pi_{n-1}(C^{n-1}, C^{n-2}, *)$, where $* = \vec{0} \in I^n$.

Let $\vec{e_i}$ be the i^{th} standard unit vector in $\mathbf{R}^n$ and let $m_i \colon I \to I^n$ be the path $m_i(t) = t\vec{e_i}$. Let $\mu_i \colon \pi_k(C^k, C^{k-1}, \vec{e_i}) \to \pi_k(C^k, C^{k-1}, *)$ be the induced isomorphism of homotopy groups. For $n \geq 3$, let

$$\bar{d}_i^\epsilon = \mu_i(d_i^\epsilon) = \mu_i \partial_i^\epsilon \iota_{n-1} \in \pi_{n-1}(C^{n-1}, C^{n-2}, *).$$

Then $\bar{d}_i^\epsilon$ is freely homotopic to d_i^ϵ by a homotopy in which the basepoint traverses the path m_i. Because of this, both d_i^ϵ and $\bar{d}_i^\epsilon$ have the same image D_i^ϵ under the Hurewicz homomorphism.

For $n = 2$, the functions $\partial_i^\epsilon \colon I \to I^2$ describe the edges of I^2 as paths, and we let d_i^ϵ denote the homotopy class of the path ∂_i^ϵ rel. endpoints

LEMMA 3.4. *Cubical Homotopy Addition Formulas.*

For $\partial_{n+1}^C \colon \pi_{n+1}(C^{n+1}, C^n) \to \pi_n(C^n, C^{n-1})$ *the following formula holds:*

$$\partial_{n+1}^C \iota_{n+1} = \begin{cases} d_2^0 d_1^1 (d_2^1)^{-1} (d_1^0)^{-1}, & \text{for } n = 1 \\ -d_1^0 - \bar{d}_2^1 - d_3^0 + \bar{d}_1^1 + d_2^0 + \bar{d}_3^1, & \text{for } n = 2 \\ \sum_{i=1}^{n} (-1)^i (d_i^0 - \bar{d}_i^1), & \text{for } n \geq 3. \end{cases}$$

For $n = 1$ *juxtapostition denotes composition of paths. For* $n = 2$ *the formula consists of a non-abelian sum contained in the center of* $\pi_2(C^2, C^1, *)$ *and so invariant under cyclic permutation of its terms. For* $n \geq 3$ *the formula is an ordinary abelian sum.*

Lemma 3.4 is a cubical version of the homotopy addition lemma usually stated for simplices. We postpone its proof to the end of the section, and apply it now to the verification of the boundary formulas (3.1) and (3.2) in the theorem.

Let $\alpha \in \pi_n(X, A)$ be represented by $f\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, *)$. Then $\Phi_n(X, A)\alpha$ is represented by $(I \times f)_* \iota_{n+1}$. For ∂'_{n+1} as defined in Theorem 3.3 and ∂^C_{n+1} as defined in Lemma 3.4, we have

$$\begin{aligned} \partial'_{n+1}\Phi_n(X, A)\alpha &= \partial'_{n+1}(I \times f)_*\iota_{n+1} \\ &= (I \times f)_*\partial^C_{n+1}\iota_{n+1} \end{aligned} \tag{3.11}$$

If $n = 1$, we continue the above calculation by using the first line of the formula in Lemma 3.4, obtaining

$$\begin{aligned} \partial'_{n+1}\Phi_n(X, A)\alpha &= (I \times f)_*(d^0_2 d^1_1 (d^1_2)^{-1}(d^0_1)^{-1}) \\ &= [I \times f|1 \times I][I \times f|0 \times I]^{-1}. \end{aligned}$$

The final line above is justified by the observations

$$\begin{aligned} &(I \times f)_* d^0_2 = (I \times f)_* d^1_2 = 1 \in \pi_1(A, *), \\ &(I \times f)_* d^1_1 = [I \times f|1 \times I], \\ &(I \times f)_* d^0_1 = [I \times f|0 \times I]. \end{aligned}$$

This verifies (3.1).

For $n \geq 2$, using the fact that $I \times f$ carries the faces of L^n to $\underline{*}$, the two cases for $n \geq 2$ in Lemma 3.4 combine to yield a single continuation of (3.11), namely

$$(I \times f)_*\partial^C_{n+1}\iota_{n+1} = -(I \times f)_* d^0_1 - (I \times f)_* \bar{d}^2_1 - (I \times f)_* \bar{d}^1_1, \tag{3.12}$$

because for all ommitted terms $(I \times f)_* d^\epsilon_i = 0$. But

$$\begin{aligned} &(I \times f)_* d^0_1 = [I \times f|0 \times I^n], \\ &(I \times f)_* \bar{d}^1_1 = [I \times f|1 \times I^n], \\ &(I \times f)_* \bar{d}^2_1 = [I \times f|I \times (1 \times I^{n-1})], \end{aligned}$$

and

$$\begin{aligned} [I \times f|I \times (1 \times I^{n-1})] &= i_*[I \times (f|1 \times I^{n-1})] \\ &= i_*[I \times \partial_n \alpha] \\ &= i_*\Phi_{n-1}(A, *)\alpha. \end{aligned}$$

Using these substitutions in (3.12) and combining the results with (3.11), we end up with (3.2). This finishes the proof of Theorem 3.3. ◇

We now define the mapping Ψ whose existence and properties are the subject of Theorem 3.1, and then finally give the proof of Theorem 3.1.

DEFINITION. Let X_{F} and Y_{G} be filtered spaces. Recall that $I\underline{\times}X_{\mathrm{F}}$ has the P-filtration defined in section 3.2, and let $H : I\underline{\times}X_{\mathrm{F}} \to Y_{\mathrm{G}}$ be a filtered homotopy. Define Φ_n to be the composition

$$\begin{aligned}\pi_n(F^nX, F^{n-1}X) &\xrightarrow{\Phi_n} \pi_{n+1}(I\underline{\times}F^nX, \partial I\underline{\times}F^nX \cup I\underline{\times}F^{n-1}X)\\ &\xrightarrow{i_*} \pi_{n+1}(\partial I\underline{\times}F^{n+1}X \cup I\underline{\times}F^nX, \partial I\underline{\times}F^nX \cup I\underline{\times}F^{n-1}X)\\ &= \pi_{n+1}(P^{n+1}(I\underline{\times}X), P^n(I\underline{\times}X)).\end{aligned}$$

This definition opens the door for the

PROOF OF THEOREM 3.1: Since $\Psi_n = i_*\Phi_n$, the naturality and homomorphism properties of Ψ_n, parts (1) and (2) of Theorem 3.1, are immediate consequences of the corresponding properties of Φ_n, parts (1) and (2) of Theorem 3.3. We thus turn to the boundary formulas for Φ_n. For these we inspect some diagrams, one for $n = 1$ and one for $n \geq 2$.

For $n = 1$ we use the commutative diagram

$$\begin{tikzcd}
\pi_1(F^1X,*) \arrow[d, "\Phi_1"] \arrow[dd, bend right=60, "\Psi_1"'] & \\
\pi_2(I\underline{\times}F^1X, P^1(I\underline{\times}X)) \arrow[d, "i_*"] \arrow[dr, "\partial_2'"] & \\
\pi_2(P^2(I\underline{\times}X), P^1(I\underline{\times}X)) \arrow[r, "\partial_2^{\times}"'] & \pi_1(P^1(I\underline{\times}X))
\end{tikzcd}$$

and compute

$$\begin{aligned}\partial_2^{\times}\Psi_1\alpha &= \partial_2^{\times}i_*\Phi_1\alpha\\ &= \partial_2'\Phi_1\alpha\\ &= [I \times f|1 \times I][I \times f|0 \times I]^{-1},\end{aligned}$$

where $\alpha \in \pi_1(F^1X, *)$ has been represented by $f\colon (I, \partial I) \to (F^1X, *)$, and the last line is justified by (3.1) of Theorem 3.3. This proves the boundary formula for Φ_1.

For $n \geq 2$, we have to consider the following non-commutative diagram: (When $n = 2$, the two right-hand columns are identical and $j_* = 1$.)

$$\begin{tikzcd}
\pi_n(F^nX, F^{n-1}X) \arrow[r, "\partial_n"] \arrow[rr, bend left, "\bar{\partial}_n"] \arrow[d, "\Phi_n"'] & \pi_{n-1}(F^{n-1}X, *) \arrow[r, "j_*"] \arrow[d, "\Phi_{n-1}"] \arrow[dl, phantom, "(\clubsuit)"] & \pi_{n-1}(F^{n-1}X, F^{n-2}X) \arrow[d, "\Phi_{n-1}"] \\
\pi_{n+1}(I\underline{\times}F^nX, P^n(I \times X)) \arrow[d, "i_*"'] \arrow[dr, "\partial'_{n+1}"] & \pi_n(I\underline{\times}F^{n-1}X, \partial I\underline{\times}F^{n-1}X) \arrow[r, "j_*"'] \arrow[d, "i_*"] & \pi_n(I\underline{\times}F^{n-1}X, P^{n-1}(I\underline{\times}X)) \arrow[d, "i_*"] \\
\pi_{n+1}(P^{n+1}(I\underline{\times}X), P^n(I\underline{\times}X)) \arrow[r, "\partial_{n+1}"'] \arrow[rr, bend right, "\partial^{\times}_{n+1}"'] & \pi_n(P^n(I\underline{\times}X), \partial I\underline{\times}F^{n-1}X) \arrow[r, "j_*"'] & \pi_n(P^n(I\underline{\times}X), P^{n-1}(I\underline{\times}X))
\end{tikzcd}$$

Every cell except the one marked (♣) commutes in this diagram. The cell marked (♣) is described in Theorem 3.3 (3). Using (3.2) from that theorem, we let $\alpha \in \pi_n(F^nX, F^{n-1}X)$ be represented by $f\colon (I^n, \partial I^n, J^{n-1}) \to (F^nX, F^{n-1}X, *)$ and compute

$$\begin{aligned}
\partial^{\times}_{n+1}\Psi_n\alpha &= \partial^{\times}_{n+1}i_*\Phi_n\alpha \\
&= j_*\partial'_{n+1}\Phi_n\alpha \\
&= j_*(-[I \times f|0 \times I^n] - i_*\Phi_{n-1}\partial_n\alpha + [I \times f|1 \times I^n]) \\
&= -[I \times f|0 \times I^n] - \Psi_{n-1}\bar{\partial}_n\alpha + [I \times f|1 \times I^n].
\end{aligned}$$

(3.2) is used in the third line of the computation. Note that the brackets denoting homotopy classes refer to different groups in lines 3 and 4. This proves the boundary formula for Φ_n for $n \geq 2$ and so completes the proof of Theorem 3.1. ◇

For completeness, we finish this section with a proof of the cubical homotopy addition theorem, Lemma 3.4.

PROOF OF LEMMA 3.4: For $n = 1$, the result follows from figure 3.4. For $n \geq 3$, the result is immediate form the corresponding homology formula and the fact that the Hurewicz

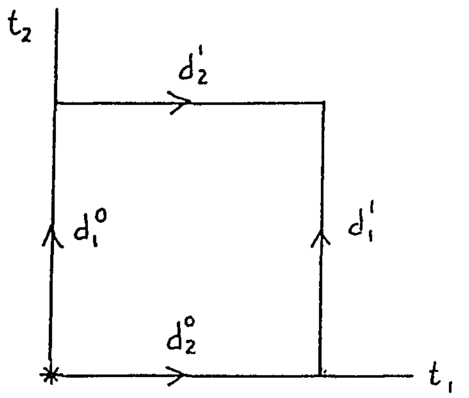

Figure 3.4. $\partial_2^C \iota_2$

homomorphism is an isomorphism for $n \geq 2$. This leaves the case for $n = 2$. For this case, our arguments refer to the labels in figure 3.5: If X, Y are vertex labels, let $\langle X, Y \rangle$ denote the path from X to Y. Faces at 0 are oriented by taking the unit vectors that span them in the order of increasing subscripts, and the same orientation is transferred to the faces at 1.

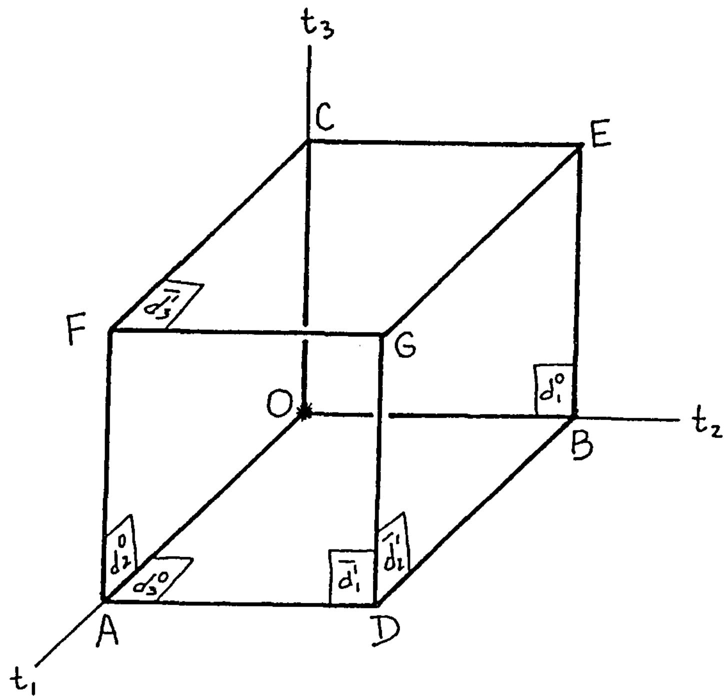

Figure 3.5. $\partial_3^C \iota_3$

The boxed labels on the faces denote the homotopy elements of $\pi_3(C^3, C^2, \vec{0})$ determined by the pointed attaching maps of the faces. For $\partial_2^C : \pi_2(C^2, C^1, *) \to \pi_1(C^1, *)$ we have

$$\partial_2^C d_1^0 = \langle OB\rangle\langle BE\rangle\langle CE\rangle^{-1}\langle OC\rangle^{-1}$$

$$\partial_2^C \bar{d}_1^1 = \langle OA\rangle\langle AD\rangle\langle DG\rangle\langle FG\rangle^{-1}\langle AF\rangle^{-1}\langle OA\rangle^{-1}$$

$$\partial_2^C d_2^0 = \langle OA\rangle\langle AF\rangle\langle CF\rangle^{-1}\langle OC\rangle^{-1}$$

$$\partial_2^C \bar{d}_2^1 = \langle OB\rangle\langle BD\rangle\langle DG\rangle\langle EG\rangle^{-1}\langle BE\rangle^{-1}\langle OB\rangle^{-1}$$

$$\partial_2^C d_3^0 = \langle OA\rangle\langle AD\rangle\langle BD\rangle^{-1}\langle OB\rangle^{-1}$$

$$\partial_2^C \bar{d}_3^1 = \langle OC\rangle\langle CF\rangle\langle FG\rangle\langle EG\rangle^{-1}\langle CE\rangle^{-1}\langle OC\rangle^{-1}$$

Using these results, a straightforward calculation shows that the claimed formula for $\partial_3^C \iota_3$ given in the lemma lies in the kernel of ∂_2^C. Using the following commutative diagram and the fact that C is acyclic, it is easy to see that the Hurewicz homomorphism h_2 is injective on the elements of $\ker \partial_2^C$.

$$\begin{tikzcd}
\pi_3(C^3, C^2) \arrow[r, "\partial_3^C"] \arrow[d, "h_3"', "\cong"] & \pi_2(C^2, C^1) \arrow[r, "\partial_2^C"] \arrow[d, "h_2"', "\text{abelianization}"] & \pi_1(C^1) \\
H_3(C^3, C^2) \arrow[r, tail] & H_2(C^2, C^1) &
\end{tikzcd}$$

Both the claimed formula for $\partial_3^C \iota_3$ and $\partial_3^C \iota_3$ have the same image under h_2 because h_2 is abelianization and the diagram commutes. Since the claimed formula for $\partial_3^C \iota_3$ and $\partial_3^C \iota_3$ belong to a subset on which h_2 is injective, they must be equal, and this finishes the proof. ◊

Chapter 4. CW- and J-Filtered Spaces

In **[CH II]**, Whitehead defines a CW complex X with skeleta $\{X^n\}$ to be a J_m*-complex* if $j_n\colon \pi_n X^n \to \pi_n(X^n, X^{n-1})$ is injective for $2 \leq n \leq m$. In **[CH II]** he proves that if Y is a J_m-complex, then the map of homotopy classes

$$[X_{\mathrm{CW}}, Y_{\mathrm{CW}}] \to [\rho(X_{\mathrm{CW}}), \rho(Y_{\mathrm{CW}})]$$

is a surjection if dim $X \leq m+1$ and a bijection if dim $X \leq m$. Inspection of the proofs reveals that one needs the properties of the CW filtration for X, but that only the J_m property is used for Y.

The decisiveness of the J_m property also appears in Whitehead's account of the realization of homotopy systems C as $\rho(X)$ for some CW complex X. In **[CH II]**, Whitehead proves that any four-dimensional homotopy system is geometrically realizable by observing that the two-dimensional (crossed module) subsystem can be realized by a J_2- complex, and that if C^m can be realized by a J_m-complex, then C^{m+2} is geometrically realizable.

Once again, it is the the J_m property of the filtration, not its CW properties, that allow geometric realization to proceed. These observations suggest the utility of freeing the J_m concept of the requirement that the filtration be a CW filtration. When we do this, we see, in Theorem 5.1 of Chapter 5, that any homotopy system can be realized by a filtered space which has the J_m property for all m, *if one is willing to dispense with the requirement that the filtration be CW.*

The perspectives of the previous paragraphs motivate the study of what we have called J-filtered spaces in section 1 of Chapter 3. The essential result of this chapter is the natural bijection

$$[X_{\mathrm{CW}}, Y_{\mathrm{J}}] \to [\rho(X_{\mathrm{CW}}), \rho(Y_{\mathrm{J}})]$$

of filtered homotopy classes of filtered maps, where Y_J has a J-filtration. This is achieved in Theorem 4.5. The preceeding lemmas, 4.1–4.4, handle various details needed to show the correspondance is 1-1 and onto. In Corollary 4.6 we note that if the J-filtration on Y is sufficiently connected, then the cellular approximation theorem works and we can replace $[X_{\mathrm{CW}}, Y_{\mathrm{J}}]$ by $[X, Y]$ in the above bijection.

In Corollary 4.7, we obtain as an application of these results the fact that the homotopy groups of a space can be computed as the homology groups of the crossed complex obtained from a well-connected J-filtration of the space.

The results of this chapter are written at a level of generality sufficient for use in the next chapter. In particular, in Chapter 5 we have to consider filtered homotopy classes of maps between filtered spaces where the domain filtration is not CW. For this reason, the domain filtrations in this chapter are not assumed CW when it is not necessary to do so, and the peculiar hypotheses of Lemma 4.3 are adapted to the applications in Chapter 5 as well as the more modest demands of this chapter.

DIAGRAM CONVENTION: In this chapter, homomorphisms that have not been geometrically realized will be represented in commutative diagrams by dotted arrows, and homomorphisms that are geometrically realized will be denoted by solid arrows.

4.1. Basic Properties of J-Filtered Spaces.

LEMMA 4.1. *Let X_{F} be a filtered space, Y_{J} a J-filtered space, and $\phi\colon \rho(X_{\mathrm{F}}) \to \rho(Y_{\mathrm{J}})$ a chain map. Let $\psi_1 = \phi_1 \colon \pi_1 F^1 X \to \pi_1 J^1 Y$. Then there are unique homomorphisms $\psi_n \colon \pi_n F^n X \to \pi_n J^n Y$ for $n \geq 2$ that yield commutative diagrams*

$$\begin{tikzcd}
\rho_{n+1}(X_{\mathrm{F}}) \arrow[r] \arrow[d, dashed, "\phi_{n+1}"'] & \pi_n F^n X \arrow[r] \arrow[d, dashed, "\psi_n"] & \rho_n(X_{\mathrm{F}}) \arrow[d, dashed, "\phi_n"] \\
\rho_{n+1}(Y_{\mathrm{J}}) \arrow[r] & \pi_n J^n Y \arrow[r, tail, "j_n"'] & \rho_n(Y_{\mathrm{J}})
\end{tikzcd}$$

PROOF: ψ_1 has already been given. Suppose ψ_{n-1} has been found and is unique. Consider the following commutative diagram, without ψ_n as yet:

$$\begin{tikzcd}
\rho_{n+1}(X_{\mathrm{F}}) \arrow[r] \arrow[d, dashed, "\phi_{n+1}"'] & \pi_n F^n X \arrow[r] \arrow[d, dashed, "\psi_n\,?"] & \rho_n(X_{\mathrm{F}}) \arrow[r] \arrow[d, dashed, "\phi_n"] & \pi_{n-1} F^{n-1} X \arrow[d, dashed, "\psi_{n-1}"] \\
\rho_{n+1}(Y_{\mathrm{J}}) \arrow[r] \arrow[ur, phantom, "\text{(III)}"] & \pi_n J^n Y \arrow[r, tail, "j_n"'] \arrow[ur, phantom, "\text{(II)}"] & \rho_n(Y_{\mathrm{J}}) \arrow[r] \arrow[ur, phantom, "\text{(I)}"] & \pi_{n-1} J^{n-1} X
\end{tikzcd}$$

ψ_n is defined using squares (I) and (II). The exactness of the rows of squares (I) and (II) and the fact that j_n is a monomorphism can be used to verify that ψ_n is unique, a homomorphism, and that square (II) commutes. The commutativity of the rectangle formed by squares (II) and (III) (without ψ_n) and the fact that j_n is monic guarantee that square (III) commutes. ◇

Lemma 4.2. *Let X_{CW} be a strictly pointed CW complex and Y_{J} a J-filtered space. Let $\phi\colon \rho(X_{\mathrm{CW}}) \to \rho(Y_{\mathrm{J}})$ be a map of homotopy systems. Let $A_{\mathrm{CW}} \subset X_{\mathrm{CW}}$ be a subcomplex and suppose $l\colon A_{\mathrm{CW}} \to Y_{\mathrm{J}}$ is a partial realization of ϕ, in the sense that*

$$\begin{tikzcd}
\rho(X_{\mathrm{CW}}) \arrow[rr, dashed, "\phi"] & & \rho(Y_{\mathrm{J}}) \\
& \rho(A_{\mathrm{CW}}) \arrow[ul] \arrow[ur, "l_*"'] &
\end{tikzcd}$$

commutes. Then l can be extended to a map $f\colon X_{\mathrm{CW}} \to Y_{\mathrm{J}}$ that realizes ϕ.

Proof: The proof is given in [**CH II**] in the case that Y is a J_m-complex (for sufficiently large m). We summarize it here for convenience, and to emphasize that the filtration on Y does not have to be CW.

$A \cup X^1$ is obtained from A by wedging on circles, so l_1 can be extended to a map $f_1\colon X^1 \to Y^1$ such that $(f_1)_* = \phi_1$. This gives us the commutative diagram

$$\begin{tikzcd}
\pi_2(X^2, X^1) \arrow[r] \arrow[d, dashed, "\phi_2"'] & \pi_1 X_1 \arrow[d, "\phi_1 ="', "(f_1)_*"] \\
\pi_2(J^2X, J^1X) \arrow[r] & \pi_1 J^1 Y
\end{tikzcd}$$

from which it is easy to conclude ([**CH II**], Lemma 4) that there is a map $f_2\colon X^2 \to J^2Y$ with $(f_2, f_1)_* = \phi_2$ and $f_2|A^2 = l_2$.

Assume that we have realized ϕ up through dimension n by a filtered map f whose components $f_r\colon X^r \to J^rY$ satisfy $f_r|A^r = l_r$, $1 \leq r \leq n$. We now have a diagram in which square $(\clubsuit)$ is not necessarily commutative:

$$\begin{tikzcd}
\rho_{n+1}(X_{\mathrm{CW}}) \arrow[r] \arrow[d, dashed, "\phi_{n+1}"'] \arrow[dr, phantom, "(\clubsuit)"] & \pi_n X_n \arrow[r] \arrow[d, "(f_n)_*"] & \rho_n(X_{\mathrm{CW}}) \arrow[d, "\phi_n = "', "(f_n, f_{n-1})_*"] \\
\rho_{n+1}(Y_{\mathrm{J}}) \arrow[r] & \pi_n J^n Y \arrow[r, tail, "j_n"'] & \rho_n(Y_{\mathrm{J}})
\end{tikzcd}$$

However, square $(\clubsuit)$ must commute, because the outer square commutes and j_n is a monomorphism. Just as in the dimension 2 case above, square $(\clubsuit)$ allows us to extend f to X^{n+1} in a way that is consistent with l_{n+1} on A^{n+1} and such that $(f_{n+1}, f_n)_* = \phi_{n+1}$. $\Diamond$

DEFINITION. Let $f, g\colon X_{\mathrm{F}} \to Y_{\mathrm{G}}$ be filtered maps and let $\Theta\colon \rho(X_{\mathrm{F}}) \to \rho(Y_{\mathrm{G}})$ be a chain homotopy $f_* \simeq g_*$. We say that a filtered map $H : I\underline{\times}X_{\mathrm{F}} \to Y_{\mathrm{G}}$ is a *geometric realization* of Θ if $H_0 = f$, $H_1 = g$, and the diagram

$$\begin{tikzcd}
\rho(X_{\mathrm{F}}) \arrow[rr, dashed, "\Psi"] \arrow[dr, dashed, "\Theta"'] & & \rho(I\underline{\times}X_{\mathrm{F}}) \arrow[dl, "H_*"] \\
& \rho(Y_{\mathrm{G}}) &
\end{tikzcd}$$

commutes as a diagram of graded modules (not as crossed complexes). Ψ is the mapping described in Theorem 3.1.

LEMMA 4.3. *Let X_{F} be a filtered space with free filtration F, and let $\{E_i^n\}$ denote the preferred basis of $\rho_n(X_{\mathrm{F}})$ for each n. Suppose further that the P-filtration on $I\underline{\times}X_{\mathrm{F}}$ is free, and that $\rho_{n+1}(I\underline{\times}X_{\mathrm{F}})$ has the basis $\{0\underline{\times}E_j^{n+1}\} \cup \{1\underline{\times}E_j^{n+1}\} \cup \{\Psi_n(E_i^n)\}$.*

With these assumptions, let D be a crossed complex and $\Theta\colon \rho(X_{\mathrm{F}}) \to D$ a chain homotopy $\alpha \simeq \beta$. Then there is a unique chain map $\nu\colon \rho(I\underline{\times}X_{\mathrm{F}}) \to D$ such that

(a) *$\nu(i_0)_* = \alpha$, $\nu(i_1)_* = \beta$, where $i_0, i_1\colon X \to I\underline{\times}X$ are the inclusions at either end of the cylinder.*

(b) *The following diagram commutes as a diagram of graded modules:*

$$\begin{tikzcd}
\rho(X_{\mathrm{F}}) \arrow[rr, dashed, "\Psi"] \arrow[dr, dashed, "\Theta"'] & & \rho(I\underline{\times}X_{\mathrm{F}}) \arrow[dl, dashed, "\nu"] \\
& \rho(Y_{\mathrm{G}}) &
\end{tikzcd}$$

REMARK: The hypotheses of the lemma are satisfied when the filtration F is a CW filtration, in which case the basis elements E_i^n correspond to n-cells. We shall later have occasion (Theorem 5.3) to consider a cellular filtration which is not the CW filtration but which nonetheless satisfies the hypotheses of the lemma.

PROOF: If $f\colon (I^{n+1}, \partial I^{n+1}, J^n) \to (F^{n+1}X, F^nX, *)$ represents E_j^{n+1}, then by $t\underline{\times}E^{n+1}$, $t = 0, 1$, we mean the element represented by $[I \times f | t \times I^{n+1}] \in \pi_{n+1}(P^{n+1}X, P^nX)$.

The conditions of the lemma force the definition of ν. In order to satisfy (a) we must have

$$\nu_{n+1}(0\underline{\times}E_j^{n+1}) = \alpha_{n+1}(E_j^{n+1})$$
$$\nu_{n+1}(1\underline{\times}E_j^{n+1}) = \beta_{n+1}(E_j^{n+1})$$

while condition (b) forces us to define $\nu_{n+1}\Psi_n(E_i^n) = \Theta_n(E_i^n)$. The only question is whether these definitions produce a chain map.

For $n \geq 2$ we have

$$\begin{aligned}
\partial\nu_{n+1}(0\underline{\times}E_j^{n+1}) &= \partial\alpha_{n+1}(E_j^n) \\
&= \alpha_n\partial(E_j^n) \\
&= \nu_n(i_0)_*\partial(E_j^n) \\
&= \nu_n\partial(i_0)_*(E_j^n) \\
&= \nu_n\partial(0\underline{\times}E_j^{n+1}).
\end{aligned}$$

There is an analogous verification for $1\underline{\times}E_j^{n+1}$. Furthermore,

$$\begin{aligned}
\partial\nu_{n+1}\Psi_n(E_i^n) &= \partial\Theta_n(E_i^n)\\
&= -\alpha_n(E_i^n) - \Theta_{n-1}\partial(E_i^n) + \beta_n(E_i^n)\\
&= -\nu_n(i_0)_*(E_i^n) - \nu_n\Psi_{n-1}\partial(E_i^n) + \nu_n(i_1)_*(E_i^n)\\
&= \nu_n\partial\Psi_n(E_i^n).
\end{aligned}$$

The third line uses the fact that $\nu_{n+1}\Psi_n(z) = \Theta_n(z)$ for all $z \in \rho_n(X_{\mathrm{F}})$, which follows from the definition of ν_{n+1} on generators and the fact that both Ψ_n and Θ_n are $\pi_1 X$-homomorphisms. The fourth line uses the formula for $\partial\Psi_n$ given in Theorem 3.1.

The verification for $n = 1$ is analogous but simpler because the boundary formulas for Ψ_1 and Θ_1 are less complex. ◊

LEMMA 4.4. *Let X_{CW} be a strictly pointed CW complex, Y_{J} a J-filtered space, and $f, g\colon X_{\mathrm{CW}} \to Y_{\mathrm{J}}$ filtered maps. Then any chain homotopy $\Theta\colon \rho(X_{\mathrm{CW}}) \to \rho(Y_{\mathrm{J}})$ of f_* with g_* has a geometric realization.*

PROOF: Consider the following diagram:

$$\begin{tikzcd}
\rho(X_{\mathrm{CW}}) \arrow[rr, dashed, "\Psi"] \arrow[dr, dashed, "\Theta"'] & & \rho(I\underline{\times}X_{\mathrm{CW}}) \arrow[dl, dashed, "\nu"] & \\
& \rho(Y_{\mathrm{J}}) & & \rho(X_{\mathrm{CW}} \vee X_{\mathrm{CW}}) \arrow[ul, "(i_0 \vee i_1)_*"'] \arrow[ll, "(f \vee g)_*"]
\end{tikzcd}$$

Lemma 4.3 provides the left triangle, which commutes only as a diagram of graded modules. ν is a chain map and the right triangle commutes as a diagram of homotopy systems. Applying Lemma 4.2 to the right triangle, the partial realization $f \vee g$ can be extended to a map $H\colon I\underline{\times}X_{\mathrm{CW}} \to Y_{\mathrm{J}}$ that geometrically realizes ν. Hence H geometrically realizes Θ. ◊

THEOREM 4.5. *Let X_{CW} be a strictly pointed CW complex and Y_{J} a J-filtered space. Let $[X_{\mathrm{CW}}, Y_{\mathrm{J}}]$ denote pointed filtered homotopy classes of pointed filtered maps $f\colon X_{\mathrm{CW}} \to Y_{\mathrm{J}}$,* *and let* $[\rho(X_{\mathrm{CW}}), \rho(Y_{\mathrm{J}})]$ *denote chain homotopy classes of chain maps* $\phi\colon \rho(X_{\mathrm{CW}}) \to \rho(Y_{\mathrm{J}})$. *Then there is a natural bijection*

$$[X_{\mathrm{CW}}, Y_{\mathrm{J}}] \longleftrightarrow [\rho(X_{\mathrm{CW}}), \rho(Y_{\mathrm{J}})].$$

Proof: Let us use $\hom(X_{\mathrm{CW}}, Y_{\mathrm{J}})$ to denote the set of pointed filtered maps, and let us use $\hom(\rho(X_{\mathrm{CW}}), \rho(Y_{\mathrm{J}}))$ to denote the set of chain maps. ρ induces a natural function

$$\hom(X_{\mathrm{CW}}, Y_{\mathrm{J}}) \to \hom(\rho(X_{\mathrm{CW}}), \rho(Y_{\mathrm{J}})),$$

and Lemma 4.2 with $A = *$ indicates that this function is surjective. Theorem 3.2 guarantees that this function respects homotopy classes, so we have a natural surjection

$$[X_{\mathrm{CW}}, Y_{\mathrm{J}}] \longrightarrow [\rho(X_{\mathrm{CW}}), \rho(Y_{\mathrm{J}})].$$

Lemma 4.4 indicates that this function is also injective. ◇

Corollary 4.6. *If* X *is as in Theorem 4.5 and* Y *has as well-connected J-filtration, then there is a natural bijection*

$$[X, Y] \longleftrightarrow [\rho(X_{\mathrm{CW}}), \rho(Y_{\mathrm{J}})]$$

where $[X, Y]$ *denotes pointed homotopy classes of maps* $f\colon X \to Y$.

Proof: An inspection of the proof of the cellular approximation theorem for CW complexes reveals that the only necessary condition on the target space is that its filtration be well-connected. Hence the map $[X_{\mathrm{CW}}, Y_{\mathrm{J}}] \to [X, Y]$ is a bijection. ◇

Corollary 4.7. *If* Y *has a well-connected J-filtration, there are natural bijections*

$$\pi_n(Y) \longleftrightarrow H_n(\rho(Y_{\mathrm{J}})) \quad n \geq 1$$

which are isomorphisms of $\pi_1 Y$*-modules for* $n \geq 2$ *and an isomorphism of groups for* $n = 1$.

Proof: When $n = 1$, the requirement that the low-dimensional terms of a filtration be CW guarantees that $\pi_1 Y \cong H_1(\rho(Y_{\mathrm{J}}))$, so the remainder of our argument assumes $n \geq 2$.

In Corollary 4.6 take $X_{\mathrm{CW}} = S^n$ with the CW structure of a 0-cell and an n-cell. We get a bijection

$$\pi_n(Y) = [S^n_{CW}, Y_J] \longleftrightarrow [\rho(S^n_{CW}), \rho(Y_J)]. \tag{4.1}$$

For $\phi\colon \rho(S^n_{CW}) \to \rho(Y_J)$ we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[rr] \arrow[d, dashed, "\phi_{n+1}"'] & & \mathbf{Z} \arrow[rr] \arrow[d, dashed, "\phi_n"] & & 0 \arrow[d, dashed, "\phi_{n-1}"] \\
\rho_{n+1}(Y_J) \arrow[rr, "\partial_{n+1}"] \arrow[dr, "\bar{\partial}_{n+1}"'] & & \rho_n(Y_J) \arrow[rr, "\partial_n"] \arrow[dr, "\bar{\partial}_n"'] & & \rho_{n-1}(Y_J) \\
& \pi_n J^n Y \arrow[ur, tail, "j_n"'] \arrow[dr] & & \pi_{n-1} J^{n-1} Y \arrow[ur, tail, "j_{n-1}"'] & \\
& & H_n(\rho_n(Y_J)) & &
\end{tikzcd}$$

Associated with this diagram is the exact sequence

$$0 \to \pi_{n+1} J^{n+1} Y \xrightarrow{j_{n+1}} \rho_{n+1}(Y_J) \xrightarrow{\bar{\partial}_{n+1}} \pi_n J^n Y \to H_n \rho_n Y_J \to 0.$$

Thus there are functions

$$\begin{aligned}
\hom(\rho(S^n_{CW}), \rho(Y_J)) &\longleftrightarrow \{\, \phi_n\colon \mathbf{Z} \to \rho_n(Y_J) \mid \partial_n \phi_n = 0 \,\} \\
&\longleftrightarrow \hom(\mathbf{Z}, \pi_n J^n Y) \\
&\xrightarrow[\text{onto}]{} \hom(\mathbf{Z}, H_n(\rho_n(Y_J))) \\
&\cong H_n \rho_n(Y_J).
\end{aligned} \tag{4.2}$$

The composition of the four maps displayed above is a surjection. Passing to homotopy classes creates a bijection

$$[\rho(S^n_{CW}), \rho(Y_J)] \longleftrightarrow H_n \rho(Y_J). \tag{4.3}$$

This is because we can write the composition as $\phi \mapsto \mu\bar{\phi}_n(t)$ where t generates $\mathbf{Z}$, ϕ_n is the component of ϕ in dimension n (the only non-zero component of ϕ), $\bar{\phi}_n\colon \mathbf{Z} \to \pi_n J^n Y$ satisfies $j_n \bar{\phi}_n = \phi_n$, and $\mu\colon \pi_n J^n Y \to H_n(\rho_n Y_J)$ is the quotient map.

If $\phi \simeq \psi$, then there is is a chain homotopy Λ whose only non-zero component Λ_n satisfies $\partial_{n+1}\Lambda_n(t) = -\phi(t) + \psi(t)$. But then $j_n\bar{\partial}_{n+1}\Lambda_n(t) = -j_n\bar{\phi}_n(t) + j_n\bar{\psi}_n(t)$, and since j_n is injective, $\bar{\partial}_n\Lambda_n(t) = -\bar{\phi}(t) + \bar{\psi}(t)$, which means that $\mu\bar{\phi}_n(t) = \mu\bar{\psi}_n(t)$. This indicates that there is a well- defined map, (necessarily a surjection)

$$[\rho(S^n_{CW}), \rho(Y_J)] \longrightarrow H_n\rho_n Y_J.$$

The argument is, in this case, reversible. If $\mu\bar{\phi}_n(t) = \mu\bar{\psi}_n(t)$, we can create a chain homotopy $\Lambda\colon \phi \simeq \psi$. Define $\Lambda_n\colon \mathbf{Z} \to \rho_{n+1}(Y_J)$ by requiring $\Lambda_n(t) \in \bar{\partial}^{-1}_{n+1}(-\bar{\phi}_n(t) + \bar{\psi}_n(t))$, and let all the other components of Λ to be 0. Thus the surjection above is a bijection.

The composition of the bijections (4.1) and (4.3) will yield the theorem for $n \geq 2$, once we have checked that both of these bijections are isomorphisms.

In the first case, we can choose a filtered representative $f\colon (S^n, *) \to (J^n, Y)$ for $\alpha \in \pi_n Y$. The correspondance (4.1) carries this to $f_*\colon \rho(S^n_{CW}) \to \rho(Y_J)$, whose only non-zero component is $j_n f_*\colon \pi_n S^n \to \pi_n J^n Y \to \pi_n(J^n Y, J^{n-1}Y)$. Thus the correspondance may be viewed as a map $\pi_n Y \to \hom(\pi_n S^n, \pi_n J^n Y)$. The group operation in $\hom(\pi_n S^n, \pi_n J^n Y)$ is given by $(f+g)(\iota_n) = f_*(\iota_n) + g_*(\iota_n)$, where ι_n is the generator of $\pi_n S^n$ determined by the identity map of S^n. Thus, the verification that the bijection in (4.1) is a homomorphism reduces to the easy verification that $(f+g)_*(\iota_n) = f_*(\iota_n) + g_*(\iota_n)$ where the first addition is in $\pi_n Y$ and the second is in $\pi_n J^n Y$.

In the second case, $\hom(\rho(S^n_{CW}), \rho(Y_J))$ gets its group structure from its identification with $\hom(\mathbf{Z}, \pi_n J^n Y)$ given in (4.2), and the remaining maps in that seqence are clearly homomorphisms. ◊

Chapter 5. The Functor K

In Theorem 5.1 of this chapter, we perform the construction that motivates the paper. Given a homotopy system C, we construct an iterated cellular complex X with a J-filtration $\{ J^n X \}$ such that $\rho(X_{\mathrm{J}}) \cong C$. Although X is not, in general, a CW complex because the attaching maps do not necessarily have images in the proper skeleta, X is homotopic to a CW complex with interleaved J- and CW filtrations.

In Theorems 5.3, 5.5, and 5.6 we show that the choices made in constructing X from C do not affect the homotopy type of X, thereby justifying the notation $X \simeq KC$ and allowing us to view K as a functor on the appropriate homotopy categories. Our strategy is to take one particular KC and prove that there is a natural bijection of homotopy classes

$$[\,KC_{\mathrm{J}},\, KD_{\mathrm{J}}\,] \longleftrightarrow [\,C,\, D\,].$$

Such a bijection provides homotopy equivalences between any two versions of KC and defines $K\phi$ for any $\phi\colon C \to D$ in a functorial manner.

In Theorem 5.6 we also apply the results of Chapter 5 in order to show that K is right adjoint to ρ by concatenating the natural bijections

$$[\,X,\, KC\,] \xrightarrow[\text{(Cor 4.6)}]{} [\,\rho(X_{\mathrm{CW}}),\, \rho(KC_{\mathrm{J}})\,] \xrightarrow[\text{(Thm 5.1)}]{} [\,\rho(X_{\mathrm{CW}}),\, C\,].$$

In Theorem 5.7, we apply Corollary 4.7 to KC to conclude that the homotopy of KC is the homotopy of C.

5.1. The Construction of a KC Space.

THEOREM 5.1. *Let C be a homotopy system. There is a strictly pointed iterated cellular complex X with a J-filtration J such that*

(a) $X^{n-1} \subset J^{n-1}X \subset X^n \subset J^n X$ *for* $n \geq 1$.

(b) $\rho(X_{\mathrm{J}}) \cong C$.

Proof: Let C^n denote the *n-skeleton* of C, namely the crossed complex that agrees with C up to and including dimension n and is zero in all dimensions above n.

Our characterization of X as an *iterated cellular complex* means that there is a filtration $\{X^n\}$ of X such that X^{n+1} is obtained from X^n by attaching $(n+1)$- cells. The attaching process allows the boundary of a particular $(n+1)$-cell to be mapped into the union of X^n and the other $(n+1)$-cells, so X may not be a CW complex, although it has the homotopy type of a CW complex with the same skeleta, obtained by deforming the attaching maps out of any higher dimensional cells.

We prove the theorem by induction on n. Begin by realizing C^2 with a CW complex X^2, yielding a commutative diagram

$$\begin{tikzcd}
& C^2 \arrow[r, dashed] \arrow[d, dashed, "\cong"'] & C^1 \arrow[d, dashed, "\cong"] \\
& \pi_2(X^2,X^1) \arrow[r] & \pi_1 X^1 \\
\pi_2 X^2 \arrow[ur, tail, "j_1"'] & &
\end{tikzcd}$$

in which j_1 is a monomorphism. Since $J^1X = X^1$, $J^2X = X^2$, (a) and (b) as they apply to C^2, are trivially satisfied.

Suppose we have constructed X with n-skeleta $\{X^n\}$ and a J-filtration J satisfying (a) and (b) as they apply to C^n. This means that we have

(a) $X^1 \subset X^2 \subset X^3 \subset J^3X \subset \cdots \subset X^{n-1} \subset J^{n-1}X \subset X^n \subset J^nX$.

(b) For $2 \leq m \leq n$ the commutative diagram

$$\begin{tikzcd}
& C^m \arrow[r, dashed] \arrow[d, dashed, "\cong"'] & C^{m-1} \arrow[d, dashed, "\cong"] \\
& \pi_m(J^mX, J^{m-1}X) \arrow[r] & \pi_{m-1}(J^{m-1}X, J^{m-2}X) \\
\pi_m J^m X \arrow[ur, tail, "j_m"'] & &
\end{tikzcd}$$

in which j_m is a monomorphism.

Let $\{ b_i \mid i \in S \}$ be a π_1-basis for C_{n+1}, where $\pi_1 = C_1/\operatorname{im}\partial_2$. Since j_{n-1} and j_n are monomorphisms, each b_i determines a unique element in $\pi_n J^n X$, which can be used to attach an $(n+1)$-cell to $J^n X$. Let X^{n+1} denote the resulting space, which geometrically realizes C^{n+1} and yields a commutative diagram

$$\begin{tikzcd}
 & C^{n+1} \arrow[r, dashed] \arrow[d, dashed, "\cong"'] & C^n \arrow[d, dashed, "\cong"] \\
 & \pi_{n+1}(X^{n+1}, J^n X) \arrow[r] & \pi_n(J^n X, J^{n-1} X) \\
\pi_{n+1} X^{n+1} \arrow[ur, "k_{n+1}"'] & &
\end{tikzcd}$$

Since k_{n+1} is not necessarily a monomorphism, (b) has not yet been satisfied. In order to satisfy (b), we construct the space $J^{n+1}X$ by attaching $(n+2)$-cells to X^{n+1} so as to kill the kernel of k_{n+1}. This gives us the commutative diagram

$$\begin{tikzcd}
 & \pi_{n+1}(X^{n+1}, J^n X) \arrow[rr] \arrow[dr] \arrow[dd, "\cong"', "l_{n+1}"] & & \pi_n(J^n X, J^{n-1} X) \\
\pi_{n+1} X^{n+1} \arrow[ur, "k_{n+1}"] \arrow[dd] & & \pi_n J^n X \arrow[ur, "j_n"'] & \\
 & \pi_{n+1}(J^{n+1} X, J^n X) \arrow[ur] & & \\
\pi_{n+1} J^{n+1} X \arrow[ur, "j_{n+1}"'] & & &
\end{tikzcd}$$

with j_{n+1} now a monomorphism by construction. The crucial observation, already indicated in the diagram above, is that l_{n+1} is an isomorphism, so that we have not destroyed the geometric realization in order to make j_{n+1} injective.

To see that l_{n+1} is an isomorphism, consider the following portion of the long exact sequence of the triple $(J^{n+1}X, X^{n+1}, J^nX)$:

$$\begin{tikzcd}
\pi_{n+2}(J^{n+1}X, X^{n+1}) \arrow[rr, "\partial"] \arrow[dr, "\partial'"'] & & \pi_{n+1}(X^{n+1}, J^nX) \arrow[r, "l_{n+1}"] & \pi_{n+1}(J^{n+1}X, J^nX) \arrow[r] & 0 \\
& \pi_{n+1}X^{n+1} \arrow[ur, "k_{n+1}"'] & & &
\end{tikzcd}$$

The fact that $(n+2)$-cells were attached to kill $\ker k_{n+1}$ means that $k_{n+1}\partial' = 0$, hence $\partial = 0$ and so l_{n+1} is an isomorphism.

The space $J^{n+1}X$ now satisfies both (a) and (b) as they apply to C^{n+1}, completing the induction. ◊

REMARKS:

(1) If $C_1 = 0$, let C_m be the first non-zero group in C. C_m will be a free abelian group. Let $J^kX = *$ for $k < m$ and let $J^mX = \vee S^m$. Then j_m is an isomorphism and the inductive step can proceed.

(2) If, in the inductive step, $C_{n+1} = 0$, then $X^{n+1} = J^nX$, $k_{n+1} = 0$, and so $J^{n+1}X$ is obtained from J^nX by attaching $(n+2)$-cells to kill $\pi_{n+1}J^{n+1}X$. Thus, if C is n-dimensional, the construction produces J^nX and then kills all its homotopy groups above dimension n.

(3) From the previous two remarks we see that if C is two-dimensional, the space X we construct is a $K(\pi, 1)$ and if C has the form $C_{n+1} \to C_n$ $(n \geq 2)$ with all other groups zero, we construct a $K(\pi, n)$ with $\pi = C_n/\operatorname{im}(C_{n+1} \to C_n)$. In this case the J-filtration on X is related to the CW filtration by the formula

$$J^kX = \begin{cases} X^k, & \text{for } k = n \\ X^{k+1}, & \text{for } k \geq n+1. \end{cases}$$

(4) If we deform X to a CW complex (still called X) by pushing all attaching maps off higher dimensional cells, then we have a diagram of inclusions

$$* \xhookrightarrow{E^1} X^1 \xhookrightarrow{E^2} X^2 \xhookrightarrow{E^3} X^3 \xhookrightarrow{e^4} J^3X \xhookrightarrow{E^4} X^4 \xhookrightarrow{e^5} \cdots \xhookrightarrow{E^{n-1}} X^{n-1} \xhookrightarrow{e^n} J^{n-1}X \xhookrightarrow{E^n} X^n \xhookrightarrow{e^{n+1}} \cdots$$

where the inclusions labelled with E^i result from the adjunction of i-cells to realize basis elements of C_i, and the inclusions labelled with e^i result from the adjunction of i-cells to create the monomorphisms j_{i-1}.

DEFINITION. The process in Theorem 5.1 for producing the space X from the homotopy system C involves choices of bases for C and generators for the various submodules to be annihilated. Call a space a *KC space* if it is obtained from C by the process of Theorem 5.1 for some particular choice of bases and generators. We shall misuse the symbol KC to denote a particular KC space.

Any particular KC space will be viewed as a CW complex with a J-filtration that is related to the CW filtration as described in Remark (4) above.

Since the choice of bases does not affect hom sets. there is a natural bijection

$$[C, D] \longleftrightarrow [\rho(KC_{\mathrm{J}}), \rho(KD_{\mathrm{J}})].$$

5.2. Properties of KC spaces.

LEMMA 5.2. *If $\phi\colon C \to D$ is a map of homotopy systems, and if KC and KD are particular realizations, then there is a map of spaces $f\colon KC \to KD$ that respects both the CW and the J-filtrations and induces ϕ.*

PROOF: Identify C with $\rho(KC_{\mathrm{J}})$ and D with $\rho(KD_{\mathrm{J}})$. The J-filtration coincides with the CW-filtration up through dimension 2, and so by Lemma 4.2 there is a map $f_2\colon KC^2 \to KD^2$ partially realizing ϕ.

Suppose that we have obtained a map $f_n\colon KC^n \to KD^n$ partially realizing ϕ. This means that we have the following commuting diagrams:

(A_n)

$$\begin{tikzcd}
J^nKC & & J^nKD \\
KC^n \arrow[u, "c"'] \arrow[rr, "f_n"] & & KD^n \arrow[u, "d"] \\
 & (\mathbf{I_n}) & \\
J^{n-1}KC \arrow[uu, "c'"'] \arrow[rr, "J^{n-1}f"] & & J^{n-1}KD \arrow[uu, "d'"] \\
KC^{n-1} \arrow[u] \arrow[rr, "f_{n-1}"] & & KD^{n-1} \arrow[u] \\
\vdots \arrow[u] & & \vdots \arrow[u]
\end{tikzcd}$$

(B_n)

$$\begin{tikzcd}
\pi_n J^n KC \arrow[rrrrr, tail, "j_n^C"] \arrow[dddd, dashed, "\psi_n"'] & & & & & \pi_n(J^n KC, J^{n-1}KC) \arrow[r] \arrow[dddd, dashed, "\phi_n"'] & \pi_{n-1}J^{n-1}KC \arrow[dddd, "(J^{n-1}f)_* ="', "\phi_{n-1}"] \\
& & \pi_n KC^n \arrow[r] \arrow[dd, "(f_n)_*"] \arrow[ull] & \pi_n(KC^n, J^{n-1}KC) \arrow[dd, "(f_n{,}J^{n-1}f)_*"] \arrow[urr, "\cong"] & & & \\
& (\mathbf{II_n}) & & & (\mathbf{III_n}) & & \\
& & \pi_n KD^n \arrow[r] \arrow[dll] & \pi_n(KD^n, J^{n-1}KD) \arrow[drr, "\cong"] & & & \\
\pi_n J^n KD \arrow[rrrrr, tail, "j_n^D"'] & & & & & \pi_n(J^n KD, J^{n-1}KD) \arrow[r] & \pi_{n-1}J^{n-1}KD
\end{tikzcd}$$

In diagram (B_n), the maps ϕ_n, ϕ_{n-1} are the ones determined by Lemma 4.1 and, following the diagram convention, the dotted arrows identify algebraic homomorphisms not yet geometrically realized. The goal of the inductive argument is to extend f_n to a map $J^n f\colon J^n KC \to J^n KD$ and then to further extend $J^n f$ to a map $f_{n+1}\colon KC^{n+1} \to KD^{n+1}$ so that $(J^n f, J^{n-1} f)_* = \phi_n$, $(J^n f)_* = \psi_n$ and diagram (B_n) is reproduced one dimension higher as diagram (B_{n+1}).

To perform the first extension, recall from the proof of Theorem 5.1 that $J^n KD$ is obtained from KD^n by attaching $(n+1)$-cells to kill $\ker k_n = \operatorname{im}(d'_*\colon \pi_n J^{n-1}KD \to \pi_n KD^n)$. In light of this fact, square $(\mathbf{I_n})$ in diagram (A_n) indicates that $\pi_n(d \circ f_n)$ annihilates $\operatorname{im}\pi_n c'$, so $d \circ f_n$ extends over $J^n KC$, yielding $J^n f$. $(J^n f)_*$ in place of ψ_n makes square $(\mathbf{II_n})$ in diagram (B_n) commute, and so makes the outer left square of diagram (B_n) commute. By Lemma 4.1, $(J^n f)_* = \psi_n$. Furthermore, $(J^n f, J^{n-1} f)_*$ in place of ϕ_n makes square $(\mathbf{III_n})$ commute and so $(J^n f, J^{n-1} f)_* = \phi_n$.

To extend $j^n f$ to f_{n+1}, consider the following commutative diagram:

$$\begin{tikzcd}
\pi_{n+1}(KC^{n+1}, J^nKC) \arrow[r] \arrow[d, "\cong"'] & \pi_n(J^nKC) \arrow[ddd, "\psi_n ="', "(J^nf)_*"] \\
\pi_{n+1}(J^{n+1}KC, J^nKC) \arrow[d, dashed, "\phi_{n+1}"'] & \\
\pi_{n+1}(J^{n+1}KD, J^nKD) & \\
\pi_{n+1}(KD^{n+1}, J^nKD) \arrow[r] \arrow[u, "\cong"] & \pi_nJ^nKD
\end{tikzcd}$$

The composition of the three left-hand vertical maps is an algebraic homomorphism that, by virtue of the commutativity of the diagram, can be geometrically realized. The resulting map $f_{n+1}\colon KC^{n+1} \to KD^{n+1}$ has the property that $(f_{n+1}, J^nf)_*$ makes square $(\mathbf{III}_{n+1})$ commute. Finally, the commutativity of diagram (B_{n+1}) without square $(\mathbf{II}_{n+1})$ and the injectivity of j_{n+1} forces square $(\mathbf{II}_{n+1})$ to commute and so establishes the commutativity of (B_{n+1}). ◊

THEOREM 5.3. *ρ induces a natural bijection*

$$[KC_{\mathrm{J}}, KD_{\mathrm{J}}] \longleftrightarrow [\rho(KC_{\mathrm{J}}), \rho(KD_{\mathrm{J}})] = [C, D]$$

PROOF: ρ induces a natural function

$$[KC_{\mathrm{J}}, KD_{\mathrm{J}}] \longrightarrow [\rho(KC_{\mathrm{J}}), \rho(KD_{\mathrm{J}})]$$

and this map is surjective by Lemma 5.2. It remains to be seen that it is also injective. In order to show this, we have to inspect the cellular decomposition of the product $I\underline{\times}KC_{\mathrm{J}}$. Recall that $J^rKC = KC^r \cup \bigcup e^{r+1}$ where the cells $\{e^{r+1}\}$ are attached to KC^r so as to kill $\mathrm{im}(\pi_rJ^{r-1}KC \to \pi_rKC_r)$, and that $KC^{r+1} = J^rKC \cup \bigcup E^{r+1}$ where the cells $\{E^{r+1}\}$ are attached to realize the basis elements of C_{r+1}. The definition of the product filtration given in Chapter 3 indicates that

$$\begin{aligned} P^{r+1}(I\underline{\times}KC_{\mathrm{J}}) &= \partial I\underline{\times}J^{r+1}KC \cup I\underline{\times}J^{r}KC \\ &= (I\underline{\times}KC)^{r+1} \cup \bigcup 0 \times e^{r+2} \cup \bigcup 1 \times e^{r+2} \cup \bigcup I \times e^{r+1}, \\ (I\underline{\times}KC)^{r+1} &= \partial I\underline{\times}KC^{r+1} \cup I\underline{\times}KC^{r} \\ &= P^{r}(I\underline{\times}KC_{\mathrm{J}}) \cup \bigcup 0 \times E^{r+1} \cup \bigcup 1 \times E^{r+1} \cup \bigcup I \times E^{r}. \end{aligned}$$

Thus $P^{r+1}(I\underline{\times}KC_{\mathrm{J}})$ is obtained from $P^{r}(I\underline{\times}KC_{\mathrm{J}})$ by attaching the $(r+1)$-cells $0 \times E^{r+1}$, $1 \times E^{r+1}$, $I \times E^{r}$ and the $(r+2)$-cells $0 \times e^{r+2}$, $1 \times e^{r+2}$, $I \times e^{r+1}$. In order to continue the argument, we need

LEMMA 5.4. *$\pi_{r+1}(P^{r+1}(I\underline{\times}KC_{\mathrm{J}}), P^{r}(I\underline{\times}KC_{\mathrm{J}}))$ is freely generated by the basis elements corresponding to $\{ I \times E^{r} \} \cup \{ 0 \times E^{r+1} \} \cup \{ 1 \times E^{r+1} \}$.*

We defer the proof of this lemma and continue with the main argument. Let $f, g\colon KC_{\mathrm{J}} \to KD_{\mathrm{J}}$ be filtered maps and let $\Theta\colon \rho(KC_{\mathrm{J}}) \to \rho(KD_{\mathrm{J}})$ be a chain homotopy $f_* \simeq g_*$. Lemma 5.4 allows us to apply Lemma 4.3 which provides a chain map $\nu\colon \rho(I\underline{\times}KC_{\mathrm{J}}) \to \rho(KD_{\mathrm{J}})$ such that $\nu(i_0)_* = f_*, \nu(i_1)_* = g_*$, and there is a commutative diagram of graded modules

$$\begin{tikzcd} \rho(KC_{\mathrm{J}}) \arrow[rr, dashed, "\Psi"] \arrow[dr, dashed, "\Theta"'] & & \rho(I\underline{\times}KC_{\mathrm{J}}) \arrow[dl, dashed, "\nu"] \\ & \rho(KD_{\mathrm{J}}) & \end{tikzcd}$$

Let $\mu\colon \rho(I\underline{\times}KC_{\mathrm{CW}}) \to \rho(I\underline{\times}KC_{\mathrm{J}})$ be the epimorphism induced by the identity map $KC_{\mathrm{CW}} \to KC_{\mathrm{J}}$. The map $\eta = \nu \circ \mu\colon \rho(I\underline{\times}KC_{\mathrm{CW}}) \to \rho(KC_{\mathrm{J}})$ is partially realized on the subcomplex $(KC \vee KC)_{\mathrm{CW}}$ by $f \vee g$. Lemma 4.2 indicates that $f \vee g$ can be extended to a map $h\colon I\underline{\times}KC_{\mathrm{CW}} \to KD_{\mathrm{J}}$ that realizes ν. Furthermore, μ_{r+1}, and consequently η_{r+1}, is trivial on the basis elements $\{ 0 \times e^{r+1} \}$, $\{ 1 \times e^{r+1} \}$, and $\{ I \times e^{r} \}$, so that h can be constructed to map $P^{r}(I\underline{\times}KC_{\mathrm{J}})$ into $J^{r}(KD_{\mathrm{J}})$. Thus h can be constructed to respect the filtration of $I\underline{\times}KC_{\mathrm{J}}$ and so is a geometrical realization of Θ. ◇

PROOF OF LEMMA 5.4: $P^{r+1}(I\underline{\times}KC_J)$ is obtained from $P^r(I\underline{\times}KC_J)$ by attaching $(r+1)$- and $(r+2)$-cells. In general, if X is obtained from A by attaching $(r+1)$-cells and Y is obtained from X by attaching $(r+2)$-cells, then the homotopy exact sequence of the triple (Y, X, A) includes

$$\pi_{r+2}(Y,X) \xrightarrow{\partial_{r+2}} \pi_{r+1}(X,A) \to \pi_{r+1}(Y,A) \to 0.$$

In the case at hand, $Y = P^{r+1}(I\underline{\times}KC_J)$, $X = (I\underline{\times}KC)^{r+1}$, $A = P^r(I\underline{\times}KC_J)$, and it will suffice to prove that $\partial_{r+2} = 0$. To do this, observe that the boundaries of the basis elements of $\pi_{r+2}(Y,X)$ can all be deformed into $A = I\underline{\times}J^{r-1}KC \cup \partial I\underline{\times}J^rKC$. More specifically, any one of these basis elements corresponds to one of $\{\,0 \times e^{r+2}\,\}$, $\{\,1 \times e^{r+2}\,\}$, $\{\,I \times e^{r+1}\,\}$. Since the boundary of e^{r+1} can be deformed into $J^{r-1}KC$, the boundary of $I\underline{\times}e^{r+1}$ can be deformed into A. It is immediate that the boundaries of $0 \times e^{r+2}$ and $1 \times e^{r+2}$ can be deformed into A, finishing the proof. ◇

Recall that we have been using the notation KC to denote one particular space obtained from C by the process of Theorem 5.1. This process involves choosing a basis $\mathcal{B}$ for C and then making various choices of generators of submodules to be annihilated by attaching cells. The result is a space KC with a filtration $J(\mathcal{B})$ such that the basis for $\rho(KC_{J(\mathcal{B})})$ obtained from the cells is in 1–1 correspondance with $\mathcal{B}$. We want to justify the notation KC by observing that the homotopy type of KC is determined by C and is independent of the various choices mentioned above.

Let $C_{\mathcal{A}}$, $D_{\mathcal{B}}$ denote homotopy systems C and D with selected bases $\mathcal{A}$ and $\mathcal{B}$. Let $K_1C_{J(\mathcal{A})}$ denote a particular space constructed as in Theorem 5.1. The subscript attached to K reflects a particular choice of generators of submodules to be annihilated, and $J(\mathcal{A})$ denotes a J-filtration of K_1C such that $\rho(K_1C_{J(\mathcal{A})})$ has a basis corresponding to A. Theorem 5.3 provides a natural bijection

$$[\,C_{\mathcal{A}},\,D_{\mathcal{B}}\,] \longleftrightarrow [\,K_1C_{J(\mathcal{A})},\,K_2D_{J(\mathcal{B})}\,].$$

THEOREM 5.5. *Let $\mathcal{A}$, $\mathcal{B}$ be bases for C and let $K_1C_{J(\mathcal{A})}$, $K_2C_{J(\mathcal{B})}$ be KC spaces constructed from C as in Theorem 5.1. Then there is a homotopy equivalence $K_1C_{J(\mathcal{A})} \simeq$*

$K_2C_{J(\mathcal{B})}$.

PROOF: Let $g\colon K_2C_{J(\mathcal{B})} \to K_2C_{J(\mathcal{B})}$ be a map whose homotopy class corresponds to 1_C under the natural bijection

$$[\,K_2C_{J(\mathcal{B})},\, K_1D_{J(\mathcal{A})}\,] \longleftrightarrow [\,C_{\mathcal{B}},\, C_{\mathcal{A}}\,]$$

and let $f\colon K_1C_{J(\mathcal{A})} \to K_2C_{J(\mathcal{B})}$ be chosen analogously. The commutativity of

$$\begin{tikzcd}
{[\,K_1C_{J(\mathcal{A})},\, K_2C_{J(\mathcal{B})}\,]} \arrow[r, "\cong"] \arrow[d, "g_*"'] & {[\,C_{\mathcal{A}},\, C_{\mathcal{B}}\,]} \arrow[d, equal] \\
{[\,K_1C_{J(\mathcal{A})},\, K_1C_{J(\mathcal{A})}\,]} \arrow[r, "\cong"'] & {[\,C_{\mathcal{A}},\, C_{\mathcal{A}}\,]}
\end{tikzcd}$$

Shows that $g \circ f \simeq 1_{K_1C_{J(\mathcal{A})}}$. Similarly, $f \circ g \simeq 1_{K_2C_{J(\mathcal{B})}}$. ◇

THEOREM 5.6. *K is a functor from the homotopy category of homotopy systems to the homotopy category of filtered spaces, and is right adjoint to ρ.*

PROOF: Theorem 5.5 indicates that KC_J is the appropriate object. Use the natural bijection $[\,C, D\,] \longleftrightarrow [\,KC_J,\, KD_J\,]$ of Theorem 5.3 to define $K\phi$ for $\phi\colon C \to D$. The naturality of the bijection makes K functorial.

For the adjointness claim, we need a natural bijection $[\,X, KC\,] \leftrightarrow [\,\rho(X_{\mathrm{CW}}), C\,]$. The J-filtration on KC is well-connected, so by Corollary 4.6 there is a natural bijection $[\,X, KC\,] \leftrightarrow [\,\rho(X_{\mathrm{CW}}), \rho(KC_J)\,]$. By Theorem 5.1, $\rho(KC_J) \cong C$. ◇

REMARK: We now have the following commutative diagram: (In this diagram, we denote bijections by arrows with a tail and double head rather than using arrows with heads at both ends. With this convention, we retain the information about the direction of the function that induced the bijection.)

$$\begin{tikzcd}
{[\,KC, KD\,]} & \\
{[\,KC_{\mathrm{CW}},\, KD_{\mathrm{CW}}\,]} \arrow[u, tail, two heads, "\text{(cellular approximation)}"] \arrow[r, tail] \arrow[d, tail, two heads, "\text{(cellular approximation)}"'] & {[\,\rho(KC_{\mathrm{CW}}),\, \rho(KD_{\mathrm{CW}})\,]} \arrow[d, two heads] \\
{[\,KC_{\mathrm{CW}},\, KD_{\mathrm{J}}\,]} \arrow[r, tail, two heads, "\text{(Thm 4.5)}"] & {[\,\rho(KC_{\mathrm{CW}}),\, D\,]} \\
{[\,KC_{\mathrm{J}},\, KD_{\mathrm{J}}\,]} \arrow[u] \arrow[r, tail, two heads, "\text{(Thm 5.3)}"] & {[\,C,\, D\,]} \arrow[u]
\end{tikzcd}$$

The composition of the maps around the left side and bottom of the diagram yields a map $[\,C, D\,] \to [\,KC, KD\,]$ whose properties are essentially determined by the map $[\,KC_{\mathrm{J}},\, KD_{\mathrm{J}}\,] \to [\,KC_{\mathrm{CW}},\, KD_{\mathrm{J}}\,]$. Note that these last two filtered homotopy sets can be very different: For example, construct a $K(\mathbf{Z}, n)$ as the CW complex

$$X = S^n \cup \bigcup e^{n+2} \cup \bigcup e^{n+3} \cup \ldots$$

and define a J-filtration on X by

$$J^k X = \begin{cases} X^k & \text{for } k \leq n \\ X^{k+1} & \text{for } k \geq n+1 \end{cases}$$

(This is the J-filtration we would get by applying the construction of Theorem 5.1 to the homotopy system $(\mathbf{Z}, n)$ that is $\mathbf{Z}$ in dimension n and 0 elswhere.) Let Y be a $K(\mathbf{Z}, m)$, $m \neq n$, constructed as was X. Since $[\,(\mathbf{Z}, n), (\mathbf{Z}, m)\,] = 0$, the bijection of Theorem 5.3 indicates that $[\,X_{\mathrm{J}},\, Y_{\mathrm{J}}\,]$ has a single element. However,

$$\begin{aligned} [\,X_{\mathrm{CW}},\, Y_{\mathrm{J}}\,] &= [\,K(\mathbf{Z}, n)_{\mathrm{CW}},\, K(\mathbf{Z}, m)_{\mathrm{J}})\,] \\ &\cong [\,K(\mathbf{Z}, n)_{\mathrm{CW}},\, K(\mathbf{Z}, m)_{\mathrm{CW}}\,] \\ &\cong H^m(\mathbf{Z}, n; \mathbf{Z}) \end{aligned}$$

which shows that $[\,X_{\mathrm{CW}},\, Y_{\mathrm{J}}\,]$ need not consist of a single element.

THEOREM 5.7. *Let C be a homotopy system. Then for any space of type KC, $\pi_n KC \cong H_n C$, $n \geq 1$.*

PROOF: $\pi_n KC \cong H_n(\rho KC_J)$, $n \geq 1$, by Corollary 4.7, and $H_n(\rho KC_J) \cong H_n C$ by Theorem 5.1. ◇

In Theorem 6.2 in the next chapter, we describe the isomorphism $\pi_n KC \to H_n C$ more explicitly.

Chapter 6. The Universal Covering Space of KC

This chapter has the single goal of showing that $\widetilde{KC}$, the universal covering space of KC, has the homotopy type of a product of $K(H_nC, n)$'s. This is accomplished by noting that the Hurewicz homomorphism is a split monomorphism in all dimensions. Byproducts of the argument are a more explicit description of the isomorphism $\pi_n KC \cong H_n C$ of Theorem 5.7 and a second verification of the homotopy invariance of spaces of type KC proved in Theorem 5.5.

The arguments we make could, in the abelian case, be justified by using the concept of additive relations. Since our crossed complexes have non-abelian components, we need the concept of additive relations tailored to groups with operators. We call the obvious generalizations *homomorphic relations.* In the appendix, we check that there are no surprises and that homomorphic relations behave in essentially the same way as additive relations are known to behave.

6.1. The Basic Theorem.

LEMMA 6.1. *Let C be a homotopy system and KC a space constructed as in the proof of Theorem 5.1. Then for each $n \geq 2$, there is a commutative diagram of $\pi_1 KC$-module homomorphisms*

$$\begin{tikzcd}
\pi_n KC \arrow[r, "h"] \arrow[dr, "m"', "\cong"] & H_n\widetilde{KC} \arrow[d, "H_n p"] \\
& H_n C
\end{tikzcd}$$

where h is the Hurewicz homomorphism and p is the chain map $\rho(KC_{\mathrm{CW}}) \to \rho(KC_{\mathrm{J}})$ induced by 1_{KC}.

We postpone the proof of the lemma until after the statement and proof of the theorem it justifies:

THEOREM 6.2.

$$\widetilde{KC} \simeq \prod_{n \geq 2} K(H_nC, n).$$

PROOF: Lemma 6.1 indicates that in each dimension, the Hurewicz homomorphism is a split monomorphism. This implies that

$$\widetilde{KC} \simeq \prod_{n \geq 2} K(\pi_n\widetilde{KC}, n).$$

Since $\pi_n\widetilde{KC} \cong \pi_nKC \cong H_nC$, the theorem is proved. ◇

PROOF OF LEMMA 6.1: Identify C with $\rho(KC_{\mathrm{J}})$ and let D denote $\rho(KC_{\mathrm{CW}})$. We will exploit the fact that KC has a J-filtration. In computing H_nC we consider

$$\begin{tikzcd}
C_{n+1} \arrow[rr, "\partial^C_{n+1}"] \arrow[dr, "\beta_{n+1}"'] & & C_n \arrow[rr, "\partial^C_n"] \arrow[dr, "\beta_n"'] & & C_{n-1} \\
& \pi_nJ^nKC \arrow[ur, "j^C_n"'] & & \pi_{n-1}J^{n-1}KC \arrow[ur, "j^C_{n-1}"'] &
\end{tikzcd}$$

In order to define m and compare it to the Hurewicz homomorphism, we need explicit labels for a number of maps and subspaces. These are contained in the list below:

DEFINITION. Let E denote either C or D, and let p: $\rho(KC_{\mathrm{CW}}) \to \rho(KC_{\mathrm{J}})$ continue to represent the chain map induced by 1_{KC}. Adopt the following notation:

$Z_n(E) = \ker \partial^E_n$

$B_n(E) = \operatorname{im} \partial^E_{n+1}$

$i^E_n\colon Z_n(E) \hookrightarrow E_n$

$r^C_n\colon \pi_nJ^nKC \xrightarrow[\cong]{j^C_n} Z_n(C) \to H_nC$

$r^D_n\colon Z_n(D) \to H_n(D)$

$\hat{p}_n\colon \pi_nKC^n \to \pi_nJ^nKC$ induced by the inclusion $KC^n \hookrightarrow J^nKC$

$q_n\colon \pi_nJ^nKC \xrightarrow{\text{epi}} \pi_nKC^{n+1}$

$k_n\colon \pi_nKC^{n+1} \xrightarrow{\cong} \pi_nKC$

Note that $q_n\hat{p}_n$ is induced by the inclusion $KC^n \hookrightarrow KC^{n+1}$.

Since j_n^C and j_{n-1}^C are monomorphisms, j_n^C maps $\pi_n J^n KC$ isomorphically onto $Z_n(C)$ and $\operatorname{im}\beta_{n+1}$ isomorphically onto $B_n(C)$. Consequently, there is an exact sequence

$$C_{n+1} \xrightarrow{\beta_{n+1}} \pi_n(J^n KC) \xrightarrow{r_n^C} H_n C \to 0. \tag{6.1}$$

m is defined in a commutative diagram that incorporates the exact sequence (6.1) and the isomorphism l_{n+1} from the proof of Theorem 5.1:

$$\begin{tikzcd}
C_{n+1} \arrow[dr, "\beta_{n+1}"] & & \pi_n KC^{n+1} \arrow[r, "k_n", "\cong"'] \arrow[dd, dashed, "m'"', "\cong"] & \pi_n KC \arrow[ddl, dashed, "\cong"', "m"] \\
& \pi_n J^n KC \arrow[ur, "q_n"] \arrow[dr, "r_n^C"'] & & \\
\pi_{n+1}(KC^{n+1}, J^n KC) \arrow[uu, "\cong", "l_{n+1}"'] \arrow[ur] & & H_n C &
\end{tikzcd}$$

The diagram determines the map m' which then determines m. To see how m is related to the Hurewicz homomorphism, again let E denote either C or D and recall the following facts (see appendix):

(1) Let $U_n^E\colon E_n \rightharpoonup H_n E$ be the homomorphic relation $r_n^E(i_n^E)^{-1}$. Then $H_n(p) = U_n^C p_n (U_n^D)^{-1}$.

(2) Let $R_n\colon \pi_n KC^{n+1} \rightharpoonup D_n$ be the additive relation defined by

$$\pi_n KC^{n+1} \xleftarrow{q_n} \pi_n J^n KC \xleftarrow{\hat{p}_n} \pi_n KC^n \xrightarrow{j_n^D} D_n$$

Then the Hurewicz homomorphism $h\colon \pi_n KC \to H_n\widetilde{KC}$ is given by $U_n^D R_n k_n^{-1}$.

More details on homomorphic relations are provided in the appendix that follows this section. The relations between the various quantities named above are included in the following diagram, in which every cell except possibly the rightmost triangle commutes:

$$\begin{tikzcd}
\pi_n K C^{n+1} \arrow[rrrr, "k_n", "\cong"'] \arrow[drr, "R_n"] \arrow[ddrrr, bend right=60] & & & & \pi_n K C \arrow[dl, "h"'] \arrow[ddl, "m"] \\
& \pi_n K D^n \arrow[r, "j_n^D"'] \arrow[d, "\hat{p}_n"] & D_n \arrow[r, "U_n^D"] \arrow[d, "p_n"'] & H_n D \arrow[d, "H_n(p)"'] & \\
& \pi_n J^n K C \arrow[r, tail, "j_n^C"'] \arrow[uul, "q_n"] & C_n \arrow[r, "U_n^C"'] & H_n C &
\end{tikzcd}$$

The commutativity of all the other cells forces the commutativity of the rightmost triangle, and this is the assertion to be proved. More explicitly, we have the following homomorphic relation calculations:

$$\begin{aligned}
H_n(p)h &= U_n^C p_n (U_n^D)^{-1} h && \text{(def. of } H_n(p)\text{)} \\
&= U_n^C p_n (U_n^D)^{-1} U_n^D R_n k^{-1} && \text{(def. of } h\text{)} \\
&= U_n^C p_n R_n {k_n}^{-1} && \\
&= U_n^C p_n (j_n^D \hat{p}_n^{-1} {q_n}^{-1}) {k_n}^{-1} && \text{(def. of } R_n\text{)} \\
&= U_n^C j_n^C \hat{p}_n \hat{p}_n^{-1} {q_n}^{-1} {k_n}^{-1} && (p_n j_n^D = j_n^C \hat{p}_n) \\
&= U_n^C j_n^C {q_n}^{-1} {k_n}^{-1} && \\
&= r_n^C {q_n}^{-1} {k_n}^{-1} && (U_n^C j_n^C = r_n^C) \\
&= m' {k_n}^{-1} && \\
&= m &&
\end{aligned}$$

The passage from the second to third lines above requires comment, since it is not always possible to "cancel" a homomorphic relation and its converse. Their composition may turn out to be a proper subset of the diagonal of the relevant set rather than the entire diagonal. The fact that r_n^D is surjective means that $(U_n^D)^{-1} U_n^D = 1_{H_n D}$ and so the cancellation is indeed valid in this case. ◇

6.2. Homomorphic Relations.

Additive relations are discussed in [**MacL**] and [**GWW**] in the context of R- modules. However, the assumption that the modules involved must be abelian groups is not necessary, and the theory is equally valid for non-abelian groups with operators. Here we shall only be interested in the results that justify the calculations of the previous section.

We begin with a summary of the necessary definitions and notation. First assume that A and B are sets. A *relation R from A to B* is a subset of $A \times B$. In place of the conventional notation $R \subset A \times B$, we shall use $R\colon A \rightharpoonup B$. The *converse of a relation R*, denoted by R^{-1}, is the subset of $B \times A$ defined by

$$R^{-1} = \{\, (b,a) \in B \times A \mid (a,b) \in R \,\}.$$

If $R_1\colon A \rightharpoonup B$ and $R_2\colon B \rightharpoonup C$ are relations, then their *composition* $R_2R_1\colon A \rightharpoonup C$ is the (possibly empty) relation

$$R_2R_1 = \{\, (a,c) \in A \times C \mid \exists b \in B \text{ such that } (a,b) \in R_1 \text{ and } (b,c) \in R_2 \,\}.$$

Commutative diagrams can be used as a device for indicating the equality of compositions of relations just as such diagrams are used for functions.

Let $p_1\colon A \times B \to A$ and $p_2\colon A \times B \to B$ be the projection functions onto the first and second factors respectively. Define the *domain of R*, denoted by $\operatorname{dom} R$, to be $p_1(R) \subset A$, and the *image of R*, denoted by $\operatorname{im} R$, to be $p_2(R) \subset B$. These definitions obviously reduce to the usual definitions of domain and image for functions when R is the graph of a function.

DEFINITION. Let A and B be groups with operators from a group G. View $A \times B$ as an operator group with G operating diagonally. Call a relation R a *homomorphic relation* if it is an operator subgroup of $A \times B$.

Note that the converse of a homomorphic relation is a homomorphic relation, the composition of homomorphic relations is a homomorphic relation, and the domain and image of a homomorphic relation are operator subgroups of A and B respectively. Furthermore, in the context of groups we have canonical monomorphisms $i_1\colon A \to A \times B$ and $i_2\colon B \to A \times B$, allowing us to define the *kernel of* R, denoted by $\ker R$, to be ${i_1}^{-1}(R) \subset A$ and the *indeterminacy of* R, denoted by $\operatorname{ind} R$, to be ${i_2}^{-1}(R) \subset B$. Both $\ker R$ and $\operatorname{ind} R$ are operator subgroups of A and B respectively.

LEMMA 6.3. *If R is a homomorphic relation, then* $\ker R \triangleleft \operatorname{dom} R$ *and* $\operatorname{ind} R \triangleleft \operatorname{im} R$.

PROOF: We can prove both statements at once by proving the following fact about groups: Let P (for product) and F (for factor) be a groups, $i\colon F \to P$, $p\colon P \to F$ homomorphisms such that $i(F) \triangleleft P$ and $pi = 1_F$. Then for any subgroup $R < P$, $i^{-1}(R) \triangleleft p(R)$. To prove this, note that since $pi = 1_F$, it suffices to prove that $i(i^{-1}(R)) \triangleleft R$. But $i(i^{-1}(R)) = i(F) \cap R$, and $i(F) \cap R \triangleleft R$ by the second homomorphism theorem. ◇

Much more is true. If R is a homomorphic relation, then it determines an operator homomorphism

$$\frac{\operatorname{dom} R}{\ker R} \xrightarrow{\cong} \frac{\operatorname{im} R}{\operatorname{ind} R}$$

and, in fact,

THEOREM 6.4. *There is a 1-1 correspondance between the set of homomorphic relations from A to B and the set of operator isomorphisms of operator subquotients of A with operator subquotients of B*

PROOF: Let $R\colon A \rightharpoonup B$, let i_k, p_k $(k = 1, 2)$ be the inclusion and projection maps associated with the product $A \times B$, and let $q_1\colon \operatorname{dom} R \to \operatorname{dom} R/\ker R$, $q_2\colon \operatorname{im} R \to \operatorname{im} R/\operatorname{ind} R$ be quotient maps. Associated with R is the operator isomorphism

$$f_R\colon \frac{\operatorname{dom} R}{\ker R} \xrightarrow{\cong} \frac{\operatorname{im} R}{\operatorname{ind} R}$$

which is obtained by filling in the following diagram so that it commutes:

$$\begin{tikzcd}
R \arrow[r, two heads, "p_1|R"] \arrow[dd, two heads, "p_2|R"'] & \operatorname{dom} R \arrow[d, two heads, "q_1"] \\
 & \dfrac{\operatorname{dom} R}{\ker R} \arrow[d, dashed, "f_R", "\cong"'] \\
\operatorname{im} R \arrow[r, two heads, "q_2"'] & \dfrac{\operatorname{im} R}{\operatorname{ind} R}
\end{tikzcd}$$

The existence (and uniqueness) of such an isomorphism is equivalent to the equality

$$\ker q_1(p_1|R) = \ker q_2(p_2|R). \tag{6.2}$$

For $k = 1, 2$ and $r \in R$,

$$q_k p_k(r) = 1 \Longleftrightarrow p_k(r) \in {i_k}^{-1}(R) \Longleftrightarrow i_k p_k(r) \in R.$$

Hence, (6.2) is equivalent to the statement

$$i_1 p_1(r) \in R \Longleftrightarrow i_2 p_2(r) \in R, \tag{6.3}$$

and since $r = (i_1 p_1(r))(i_2 p_2(r))$, (6.3) is clear.

Conversely, suppose we begin with operator subgroups $K \triangleleft D < A$, $Y \triangleleft M < B$ with quotient maps $q_1 \colon D \to D/K$, $q_2 \colon M \to M/Y$ and an operator isomorphism $f \colon D/K \xrightarrow{\cong} M/Y$. Let $R_f < A \times B$ be the pullback

$$\begin{tikzcd}
R_f \arrow[r, two heads] \arrow[dd, two heads] & D \arrow[d, two heads, "q_1"] \\
 & \dfrac{D}{K} \arrow[d, "f", "\cong"'] \\
M \arrow[r, two heads, "q_2"'] & \dfrac{M}{Y}
\end{tikzcd}$$

i.e. $R_f = \{(a,b) \in D \times M \mid fq_1a = q_2b\}$. It is easy to check that R_f is an operator subgroup of $A \times B$ with $\operatorname{dom} R_f = D$, $\ker R_f = K$, $\operatorname{im} R_f = M$, and $\operatorname{ind} R_f = Y$.

The verification that the functions $R \mapsto f_R$, $f_R \mapsto R$ are inverses of each other is elementary and we omit it. ◇

The two examples we use in the previous section arise from the composition of operator homomorphisms and their converses: If H and K are groups with operators in G and $f\colon H \rightarrow K$ is an operator homomorphism, then the graph of f is a homomorphic relation with converse $f^{-1}\colon K \rightarrow H$. If H, K, and L are groups with operators in G and if there are operator homomorphisms $H \xrightarrow{f_1} K \xleftarrow{g_1} L$, $H \xleftarrow{f_2} K \xrightarrow{g_2} L$, then composition of homomorphic relations yields homomorphic relations from H to L denoted by ${g_1}^{-1}f_1$ and $g_2{f_2}^{-1}$. Here are two examples:

Example 1: The homomorphic relation $U_n^C\colon C_n \rightarrow H_nC$ for a crossed complex C. $U_n^C = r_n{i_n}^{-1}$ where r_n and i_n are the maps

$$\begin{tikzcd} C_n & Z_n(C) \arrow[l, hook', "i_n"'] \arrow[r, two heads, "r_n"] & H_nC. \end{tikzcd}$$

If $f\colon C \rightarrow D$ is a morphism of crossed complexes, then the map induced in homology is defined to be $H_nf = U_n^Df_n(U_n^C)^{-1}$. (The definition defines H_nf as a homomorphic relation. To check that it is actually an operator homomorphism from H_nC to H_nD, one must, in view of the proof of Theorem 6.4, verify that $\operatorname{dom} H_nf = H_nC$ and $\operatorname{ind} H_nf = \{0 \in H_nD\}$.) We symbolize the definition of H_nf with the commutative diagram

$$\begin{tikzcd} C_n \arrow[r, "U_n^C"] \arrow[d, "f_n"'] & H_nC \arrow[d, "H_nf"] \\ D_n \arrow[r, "U_n^D"'] & H_nD \end{tikzcd}$$

Example 2: The Hurewicz homomorphism. Let X be a CW complex with skeleta $\{X^n\}$. JHC Whitehead shows in **[JHCW2]** that $U_nR_n{k_n}^{-1}$ is the Hurewiczs homomorphism, where U_n is the homomorphic relation defined in item (1) above, R_n is the homomorphic relation defined by the triangle in the diagram below, and k_n is the isomorphism in the diagram below.

$$\begin{tikzcd}
\pi_n X & \pi_n X^{n+1} \arrow[l, "k_n"', "\cong"] \arrow[rr, "R_n"] & & \pi_n(X^n, X^{n-1}) \arrow[r, "U_n"] & H_n\widetilde{X} \\
& & \pi_n X_n \arrow[ul] \arrow[ur, "j_n"'] & &
\end{tikzcd}$$

When $X = KC$, the inclusion $KC^n \hookrightarrow KC^{n+1}$ induces the map $q_n\hat{p}_n$ so that R_n described here matches the R_n used in the previous section.

Chapter 7. A Necessary and Sufficient Condition for Geometric Realization

Given a homotopy system C, we can produce a CW complex KC that is unique up to homotopy. KC has both the J and CW filtrations, and although $\rho(KC_{\mathrm{J}}) = C$, we get a new homotopy system $\rho(KC_{\mathrm{CW}})$ when we use the CW filtration.

In this chapter, we observe that there is a map $\alpha\colon \rho(KC_{\mathrm{CW}}) \to C$, and a necessary condition for C to be geometrically realizable is that α have a right homotopy inverse $\beta\colon C \to \rho(KC_{\mathrm{CW}})$, i.e. $\alpha\beta \simeq 1_C$. By an appropriate choice of a chain map $\bar{\alpha}$ in the homotopy class of α, we can have $\bar{\alpha}$ an epimorphism that must split if C can be realized geometrically. We then go on to show that the splitting of $\bar{\alpha}$ is also sufficient for geometric realization. Thus, given any homotopy system C, there is a chain epimorphism $\bar{\alpha}\colon \rho(KC_{\mathrm{CW}}) \to C$ that splits if and only if C can be realized geometrically. We observe that there are obstructions to obtaining a splitting of $\bar{\alpha}$ which are cohomology classes in $H^{n+1}(C; H_n(\ker \bar{\alpha}))$.

7.1. The Necessity of the Retraction Condition for Geometric Realization.

DEFINITION. If C, D are homotopy systems with maps $\phi\colon C \to D$, $\psi\colon D \to C$ such that $\psi\phi \simeq 1_C$, then C is called a *deformation retract* of D. If $\psi\phi = 1_C$, C is called a *retract* of D.

LEMMA 7.1. *If the homotopy system C can be geometrically realized, then C is a deformation retract of $\rho(KC_{\mathrm{CW}})$.*

PROOF: First, let C be any homotopy system, realizable or not. For any CW complex Y, we have the natural bijection induced by ρ,

$$[Y_{\mathrm{CW}}, KC_{\mathrm{J}}] \longleftrightarrow [\rho(Y_{\mathrm{CW}}), C]$$

of Theorem 5.6. Applying this to the case $Y = KC$ gives a natural bijection

$$[KC_{\mathrm{CW}}, KC_{\mathrm{J}}] \longleftrightarrow [\rho(KC_{\mathrm{CW}}), C].$$

The map $1_{KC}\colon KC_{\mathrm{CW}} \to KC_{\mathrm{J}}$ is filtration preserving and so lives in $[\,KC_{\mathrm{CW}}, KC_{\mathrm{J}}\,]$. Its homotopy class determines a homotopy class of maps $[\rho(1_{KC})] \in [\,\rho(KC_{\mathrm{CW}}), C\,]$. For any map $\alpha \in [\rho(1_{KC})]$, we have $\alpha\colon \rho(KC_{\mathrm{CW}}) \to C$.

Now suppose C is realizable with $C = \rho(X_{\mathrm{CW}})$. We get a commutative diagram

$$\begin{tikzcd}
{[\,X_{\mathrm{CW}}, KC_{\mathrm{J}}\,]} \arrow[r, leftrightarrow] & {[\,C, C\,]} \\
{[\,X_{\mathrm{CW}}, KC_{\mathrm{CW}}\,]} \arrow[r, leftrightarrow] \arrow[u, "(1_{KC})_*"] & {[\,C, \rho(KC_{\mathrm{CW}})\,]} \arrow[u, "\alpha_*"']
\end{tikzcd}$$

in which $(1_{KC})_*$ is surjective by the cellular approximation theorem. (α_* must then be surjective as well since the horizontal maps are bijections.) Consequently, there is a map $f\colon X_{\mathrm{CW}} \to KC_{\mathrm{CW}}$ such that $\rho(1_{KC})_*[f] = [1_C]$, and so $\alpha_*[\rho(f)] = [1_C]$. Setting $\beta = \rho(f)$, we have $\beta\colon C \to \rho(KC_{\mathrm{CW}})$ and $\alpha\beta \simeq 1_C$. $\Diamond$

REMARK. There is an algebraic construction, the mapping cylinder $\mathrm{Cyl}(\beta)$ of a map of homotopy systems $\beta\colon C \to D$. As in the case of the mapping cylinder of a chain map of abelian chain complexes, $\mathrm{Cyl}(\beta)$ is homotopy equivalent to D and C is a retract of $\mathrm{Cyl}(\beta)$ whenever C is a deformation retract of D. Thus, by modifying $\rho(KC_{\mathrm{CW}})$, we can get a homotopy equivalent homotopy system with C as a retract.

The modifications made on $\rho(KC_{\mathrm{CW}})$ do not affect its geometric realizability, and the geometric realization of the modified homotopy system is still a space of type KC. This is a consequence of a general fact, announced by J.H.C. Whitehead in [**CH II**], that a homotopy system that is homotopy equivalent to a geometrically realizable homotopy system is itself geometrically realizable, and its geometric realization is homotopy equivalent to the original geometric realization. Whitehead's proof involves a detour through simple homotopy theory [**JHCW1**]. In [**CTCW**], C.T.C. Wall provides a simpler proof of this fact, using the homotopy and homology groups of a mapping and the relative Hurewicz theorem for such groups.

In our situation, we can avoid the realization issues arising from a modification of

$\rho(KC_{\mathrm{CW}})$. It is possible to leave $\rho(KC_{\mathrm{CW}})$ as it is and instead alter α by a chain homotopy, with the result that C is seen to be a retract of the original copy of $\rho(KC_{\mathrm{CW}})$.

LEMMA 7.2. *For every homotopy system C, there is a chain epimorphism $\bar{\alpha}\colon \rho(KC_{\mathrm{CW}}) \to C$ with $\bar{\alpha} \simeq \alpha$, where α is the map in the proof of Lemma 7.1.*

PROOF: Choose a basis for C and construct $X = KC$ using this basis as in Theorem 5.1. We shall use the notation and some of the details of the proof of Theorem 5.1. Since J^nX is obtained from X^n by attaching $(n+1)$-cells, $\pi_n(J^nX, X^n) = 0$ and so $\gamma_n\colon \pi_{n+1}(X^{n+1}, X^n) \to \pi_{n+1}(X^{n+1}, J^nX)$ is an epimorphism. This gives us the epimorphism $\bar{\alpha}_n$, defined to be the composition

$$\rho(KC_{\mathrm{CW}})_{n+1} = \pi_{n+1}(X^{n+1}, X^n) \overset{\gamma_n}{\twoheadrightarrow} \pi_{n+1}(X^{n+1}, J^nX) \xrightarrow[\cong]{l_{n+1}} \pi_{n+1}(J^{n+1}X, J^nX) \cong C_{n+1}$$

where l_{n+1} is the isomorphism in the proof of Theorem 5.1. In order to see that the collection $\{\,\bar{\alpha}_n\,\}$ constitutes a chain map, consider the following diagram:

$$\begin{tikzcd}
\pi_{n+1}(X^{n+1}, X^n) \arrow[rr, "\partial"] \arrow[dr] \arrow[dd, two heads, "\gamma_{n+1}"'] & & \pi_n(X^n, X^{n-1}) \arrow[dd, two heads, "\gamma_n"] \\
& \pi_n X^n \arrow[ur] \arrow[dr] \arrow[dd] & \\
\pi_{n+1}(X^{n+1}, J^nX) \arrow[dr] \arrow[dd, "l_{n+1}"', "\cong"] & & \pi_n(X^n, J^{n-1}X) \arrow[dd, "\cong"', "l_n"] \\
& \pi_n J^n X \arrow[dr] & \\
\pi_{n+1}(J^{n+1}X, J^nX) \arrow[ur] \arrow[rr, "\partial"'] & & \pi_n(J^nX, J^{n-1}X)
\end{tikzcd}$$

Every interior cell in the diagram commutes, which means the two paths around the periphery represent the same function, and this makes $\bar{\alpha}$ a chain map.

For each n, $\bar{\alpha}_{n+1}$ is induced by the inclusion $(X^{n+1}, X^n) \hookrightarrow (J^{n+1}X, J^nX)$, which means that $\bar{\alpha}$ is induced by $1_{KC}\colon KC_{\mathrm{CW}} \to KC_{\mathrm{J}}$. Hence $\bar{\alpha}$ as defined here is in the same homotopy class as the α obtained in Lemma 7.1. ◊

THEOREM 7.3. *If the homotopy system C can be geometrically realized, then C is a retract of $\rho(KC_{\mathrm{CW}})$.*

PROOF: Lemma 7.2 gives us the epimorphism $\bar{\alpha}\colon \rho(KC_{\mathrm{CW}}) \to C$. If C has a geometric realization, Lemma 7.1 indicates the existence of a map $\beta\colon C \to \rho(KC_{\mathrm{CW}})$ and a chain homotopy $\alpha\beta \simeq 1_C$. Combining this chain homotopy with the chain homotopy $\bar{\alpha} \simeq \alpha$, we get a chain homotopy $\Lambda\colon \bar{\alpha}\beta \simeq 1_C$. We conclude the proof by using this data to produce an actual right inverse $\bar{\beta}$ for $\bar{\alpha}$. We do this by inductively defining both $\bar{\beta}$ and a chain homotopy $\Theta\colon C \to \rho(KC_{\mathrm{CW}})$ so that $\bar{\alpha}\bar{\beta} = 1_C$ and $\Theta\colon \bar{\beta} \simeq \beta$.

Set $D = \rho(KC_{\mathrm{CW}})$. Recall that in the construction of KC_{CW} in Theorem 5.1, we realize C in dimensions 1 and 2. $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are isomorphisms, and we take $\bar{\beta}_1 = \bar{\alpha}_1^{-1}$, $\bar{\beta}_2 = \bar{\alpha}_2^{-1}$, and $\Theta_1 = \bar{\alpha}_2^{-1}\Lambda_1$. To check that Θ_1 behaves like the first component of the required chain homotopy, let $c \in C$ and compute

$$\begin{aligned}
\bar{\alpha}_1\partial_2^D\Theta_1(c) &= \partial_2^C\bar{\alpha}_2\Theta_1(c)\\
&= \partial_2^C\bar{\alpha}_2\bar{\alpha}_2^{-1}\Lambda_1(c)\\
&= \partial_2^C\Lambda_1(c)\\
&= (\bar{\alpha}_1\bar{\alpha}_1^{-1})(c)(c)^{-1}\\
&= 1.
\end{aligned}$$

Since $\bar{\alpha}_1$ is an isomorphism, the result above implies that $\partial_2^D\Theta_1(c) = 1$, as it should be since $\bar{\alpha}_1\bar{\beta}_1^{-1} = \bar{\alpha}_1\bar{\alpha}_1^{-1} = 1$.

We define Θ_2 to be a lift of Λ_2 to D_3, using the surjectivity of $\bar{\alpha}_3$ and the fact that C_3 is free. Thus $\Lambda_2 = \bar{\alpha}_3\Theta_2$, and for any $c \in C_2$, we have

$$\begin{aligned}
\bar{\alpha}_2\partial_3^D\Theta_2(c) &= \partial_3^C\bar{\alpha}_3\Theta_2(c)\\
&= \partial_3^C\Lambda_2(c)\\
&= -c - \Lambda_1\partial_2(c)\\
&= -\bar{\alpha}_2\bar{\alpha}_2^{-1}(c) - \Lambda_1\partial_2^C(c) + c\\
&= -c - \Lambda_1\partial_2^C(c) + c.
\end{aligned}$$

Hence,

$$\begin{aligned}\partial_3^D\theta_2(c) &= -\bar{\alpha}_2^{-1}(c) - \bar{\alpha}_2^{-1}\lambda_1\partial_2^C(c) + \bar{\alpha}_2^{-1}(c)\\ &= -\bar{\alpha}_2^{-1}(c) - \theta_1\partial_2^C(c) + \bar{\alpha}_2^{-1}(c)\\ &= -\theta_1\partial_2^C(c).\end{aligned}$$

The last line is obtained by cyclically permuting the terms of the previous line; this is allowable because $\operatorname{im}\partial_3^D$ is contained in the center of D_2.

The previous steps and the next step are clarified by the diagram that follows:

$$\begin{tikzcd}
C_4 \arrow[r, "\partial_4^C"] \arrow[d, "\beta_4"'] & C_3 \arrow[r, "\partial_3^C"] \arrow[d, dashed, shift right, "\bar{\beta}_3"'] \arrow[d, shift left, "\beta_3"] \arrow[dl, dashed, "\Theta_3"'] & C_2 \arrow[r, "\partial_2^C"] \arrow[d, "\bar{\beta}_2"', "=\bar{\alpha}_2^{-1}"] \arrow[dl, "\Theta_2"'] & C_1 \arrow[d, "\bar{\beta}_1"', "=\bar{\alpha}_1^{-1}"] \arrow[dl, "\Theta_1"'] \\
D_4 \arrow[r, "\partial_4^D"'] \arrow[d, two heads, "\bar{\alpha}_4"'] & D_3 \arrow[r, "\partial_3^D"'] \arrow[d, two heads, "\bar{\alpha}_3"'] \arrow[dl, "\Lambda_3"] & D_2 \arrow[r, "\partial_2^D"'] \arrow[d, "\cong"', "\bar{\alpha}_2"] \arrow[dl, "\Lambda_2"] & D_1 \arrow[d, "\cong"', "\bar{\alpha}_1"] \arrow[dl, "\Lambda_1"] \\
C_4 \arrow[r, "\partial_4^C"'] & C_3 \arrow[r, "\partial_3^C"'] & C_2 \arrow[r, "\partial_2^C"'] & C_1
\end{tikzcd}$$

First lift Λ_3 to $\Theta_3\colon C_3 \to D_4$. Then, define $\bar{\beta}_3 = \beta_3 - (\partial_4^D\Theta_3 + \Theta_2\partial_3^C)$. In order to verify that $\partial_3^D\bar{\beta}_3 = \bar{\beta}_2\partial_3^C$, let $c \in C_3$ and calculate

$$\begin{aligned}\partial_3^D\bar{\beta}_3(c) &= \partial_3^D\beta_3(c) - \partial_3^D\Theta_2\partial_3^C(c)\\ &= \bar{\alpha}_2^{-1}\partial_3^C(c) - \partial_3^D\Theta_2\partial_3^C(c)\\ &= \bar{\alpha}_2^{-1}\partial_3^C(c) - \left(-\Theta_1\partial_2^C\partial_3^C(c)\right)\\ &= \bar{\alpha}_2^{-1}\partial_3^C(c)\\ &= \bar{\beta}_2\partial_3^C(c).\end{aligned}$$

In the third line, we used our previous calculation of $\partial_3^D\Theta_2$.

We now know that $\bar{\beta}_1$, $\bar{\beta}_2$, and $\bar{\beta}_3$ are components of a chain map. We next verify that

$\bar{\alpha}_3\bar{\beta}_3 = 1_{C_3}$.

$$\begin{aligned}
\bar{\alpha}_3\bar{\beta}_3 &= \bar{\alpha}_3\big(\beta_3 - (\partial_4^D\Theta_3 + \Theta_2\partial_3^C)\big) \\
&= \bar{\alpha}_3\beta_3 - \big(\bar{\alpha}_3\partial_4^D\Theta_3 + \bar{\alpha}_3\Theta_2\partial_3^C\big) \\
&= \bar{\alpha}_3\beta_3 - \big(\partial_4^C\bar{\alpha}_4\Theta_3 + \bar{\alpha}_3\Theta_2\partial_3^C\big) \\
&= \bar{\alpha}_3\beta_3 - \big(\partial_4^C\Lambda_3 + \Lambda_2\partial_3^C\big) \\
&= \bar{\alpha}_3\beta_3 - \big(\bar{\alpha}_3\beta_3 - 1_{C_3}\big) \\
&= 1_{C_3}
\end{aligned}$$

Finally, the definition of $\bar{\beta}_3$ guarantees that $\partial_4^D\Theta_3 + \Theta_2\partial_3^C = \beta_3 - \bar{\beta}_3$.

The general case, in which we first define Θ_n and then $\bar{\beta}_n$, is analogous to the case $n = 3$ but is simpler because all the groups are abelian. We have now replaced the homotopy splitting β with a strict splitting $\bar{\beta}$, completing the proof. ◊

7.2. The Sufficiency of the Retraction Condition for Geometric Realization.

We now consider the converse of Theorem 7.3. The main idea is in Lemma 7.5 below. Its proof involves a standard extension of mappings argument and a new ingredient: Altering the attaching maps of a geometric realization, possibly changing its homotopy type, without changing the fact that it is a geometric realization. The motivating idea is this: Let $f_\sigma \colon S^n \to X^n$ be the attaching map of an $(n+1)$-cell e_σ^{n+1} of a CW complex X. The $(n+1)^{\mathrm{st}}$ boundary operator ∂_{n+1} in the homotopy system $\rho(X_{\mathrm{CW}})$ is the composition $\pi_{n+1}(X^{n+1}, X^n) \to \pi_n X \to \pi_n(X^n, X^{n-1})$. The first map in the composition takes the basis element $x_\sigma \in \pi_{n+1}(X^{n+1}, X^n)$ corresponding to e_σ^{n+1} to $[f_\sigma] \in \pi_n X^n$. Following Whitehead, let $\Gamma_n X = \mathrm{im}(\pi_n X^{n-1} \to \pi_n X^n)$, and let $c \in \Gamma_n X$. If we alter f_σ on the boundary S^n of e_σ^{n+1} in such a way that $[f_n] \in \pi_n X^n$ is replaced by $[f_n] + c$, this change will have no effect on ∂_{n+1} and so will replace one geometric realization with another. In the proof of Lemma 7.4 below, we inductively obtain a partial geometric realization that may not extend to the next dimension, and then use the alteration idea described above to modify the $(n+1)$-skeleton of the realization, obtaining another realization that can be extended to the next dimension.

Definition. Let X be a CW complex, C a homotopy system, and $\alpha\colon \rho(X_{\mathrm{CW}}) \to C$ a map. We say that α has been *partially realized through dimension* n if there is a CW complex Y^n, a map $r_n\colon X^n \to Y^n$, and isomorphisms $\mu_1\colon \pi_1 Y^1 \to C_1$, $\mu_k\colon \pi_k(Y^k, Y^{k-1}) \to C_k$, $2 \le k \le n$, such that $\mu_1(r_1)_* = \alpha_1$ and $\mu_k(r_k, r_{k-1})_* = \alpha_k$ for $2 \le k \le n$.

In what follows, we will identify the domain and codomain of the isomorphisms $\{\mu_k\}$ and just write $(r_1)_* = \alpha_1$, $(r_k, r_{k-1})_* = \alpha_k$.

Lemma 7.4. *Let $\alpha\colon \rho(X_{\mathrm{CW}}) \to C$ be partially realized through dimension n. A necessary and sufficient condition for α to be partially realized through dimension $n+1$ is the existence of a homomorphism $\phi\colon C_{n+1} \to \pi_n Y$ yielding commutativity in the diagram*

$$\begin{tikzcd}
\pi_{n+1}(X^{n+1}, X^n) \arrow[rr, "\partial^X_{n+1}"] \arrow[dr, "\delta^X_{n+1}"'] \arrow[ddd, "\alpha_{n+1}"'] & & \pi_n(X^n, X^{n-1}) \arrow[ddd, "(r_n, r_{n-1})_*"'] \\
& \pi_n X_n \arrow[ur, "j^X_n"'] \arrow[d, "(r_n)_*"'] & \\
& \pi_n Y_n \arrow[dr, "j^Y_n"'] & \\
C_{n+1} \arrow[ur, dashed, "\phi?"] \arrow[rr, "\partial^C_{n+1}"'] & & \pi_n(Y^n, Y^{n-1})
\end{tikzcd}$$

Proof: All the ideas for this lemma occur in **[CH II]**—we simply collect them together here in outline form.

Necessity: If Y^{n+1} and $r_{n+1}\colon X^{n+1} \to Y^{n+1}$ exist realizing α through dimension $n+1$, let $\{\, e^{n+1}_\tau \,\}$ be the set of $(n+1)$-cells of Y^{n+1}, and let $f^Y_\tau\colon S^n \to Y^n$ be an attaching map for e^{n+1}_τ. C_{n+1} is isomorphic to $\pi_{n+1}(Y^{n+1}, Y^n)$ and so has a basis $\{\, c_\tau \,\}$ corresponding to the $(n+1)$-cells $\{\, e^{n+1}_\tau \,\}$. Define ϕ by requiring $\phi(c_\tau) = [f^Y_\tau] \in \pi_n Y^n$. Then under the isomorphism $C_{n+1} \cong \pi_{n+1}(Y^{n+1}, Y^n)$, ϕ corresponds to δ^Y_{n+1}, and this makes the diagram commute.

Sufficiency: Given ϕ making the diagram commute, we can obtain attaching maps for $(n+1)$-cells as above: For each basis element $c_\tau \in C_{n+1}$, choose a representative $f^Y_\tau \in$

$[\phi c_\tau] \in \pi_n Y^n$, and use f_τ^Y to attach an $(n+1)$-cell to Y^n. This gives us $\pi_{n+1}(Y^{n+1}, Y^n) \cong C_{n+1}$. Moreover, under this isomorphism, each basis element $c_\tau \in C_{n+1}$ corresponds to an $(n+1)$-cell basis element $y_\tau \in \pi_{n+1}(Y^{n+1}, Y^n)$, and $\phi c_\tau = [f_\tau^Y] = \delta_{n+1}^Y(y_\tau)$. Consequently, ∂_{n+1}^C is realized as ∂_{n+1}^Y and so Y^{n+1} is a partial realization of C. This much is a consequence of the condition $\partial_{n+1}^C \phi = j_n^Y C$.

The condition $\phi\alpha_{n+1} = (r_n)_*\delta_{n+1}^X$ is needed in order to be able to extend $r_n \colon X^n \to Y^n$ to a map $r^{n+1} \colon X^{n+1} \to Y^{n+1}$. To see how the condition is used, let $i_n^Y \colon Y^n \hookrightarrow Y^{n+1}$, let each $(n+1)$-cell d_σ^{n+1} of X^{n+1} have attaching map $f_\sigma^X \colon S^n \to X^n$, and let $x_\sigma \in \pi_{n+1}(X^{n+1}, X^n)$ denote the basis element corresponding to d_σ^{n+1}. In order to extend r_n over the cell d_σ^{n+1}, we must have $i_n^Y r_n f_\sigma^X \colon S^n \to Y^{n+1}$ null-homotopic, so consider $[i_n^Y r_n f_\sigma^X] \in \pi_n Y^n$:

$$\begin{aligned}
[i_n^Y r_n f_\sigma^X] &= (i_n^Y)_*(r_n)_*[f_\sigma^X] \\
&= (i_n^Y)_*(r_n)_*\delta_{n+1}^X(x_\sigma) \\
&= (i_n^Y)_*\phi\alpha_{n+1}(x_\sigma) \\
&= (i_n^Y)_*\delta_{n+1}^Y\alpha_{n+1}(x_\sigma) \\
&= 0
\end{aligned}$$

(The last line is 0 because $(i_n^Y)_*\delta_{n+1}^Y$ are consecutive mappings in the homotopy exact sequence of the pair (Y^{n+1}, Y^n).) Thus r_n can be extended to X^{n+1}, completing the realization. ◊

LEMMA 7.5. *Let X be a CW complex and let C be a homotopy system that is a retract of $\rho(X_{\mathrm{CW}})$. Then C is geometrically realizable.*

PROOF: Let $\alpha \colon \rho(X_{\mathrm{CW}}) \to C$ be a chain epimorphism and let β be its splitting. (Note that the analogous maps were called $\bar{\alpha}$ and $\bar{\beta}$ in the previous section.) The results of Whitehead in [**CH II**] indicate that, even without the splitting β, we may partially realize α through dimension 3, and so we have the base case for an induction argument. Suppose that α has been partially realized through dimension n, using a CW complex Y^n. This gives us the diagram of unbroken arrows depicted below, which commutes without the arrows β_{n+1} and

β_n. In order to complete the induction step we must, according to Lemma 7.4, find a map ϕ corresponding to the dotted arrow such that $\phi\alpha_{n+1} = (r_n)_*\delta^X_{n+1}$ and $j^Y_n\phi = \partial^C_{n+1}$.

$$\begin{tikzcd}
\pi_{n+1}(X^{n+1},X^n) \arrow[rr, "\partial^X_{n+1}"] \arrow[dr, "\delta_{n+1}X"'] \arrow[ddd, shift right, "\alpha_{n+1}"'] & & \pi_n(X^n,X^{n-1}) \arrow[ddd, shift right, "(r_n,r_{n-1})_*"'] \\
& \pi_nX_n \arrow[ur, "j^X_n"'] \arrow[d, "(r_n)_*"'] & \\
& \pi_nY_n \arrow[dr, "j^Y_n"'] & \\
C_{n+1} \arrow[uuu, shift right, "\beta_{n+1}"'] \arrow[ur, dashed, "\phi?"] \arrow[rr, "\partial^C_{n+1}"'] & & \pi_n(Y^n,Y^{n-1}) \arrow[uuu, shift right, "\beta_n"']
\end{tikzcd}$$

Note that the splitting β_{n+1} provides us with a map $\phi' = (r_n)_*\delta^X_{n+1}\beta_{n+1}$. This makes the lower triangle commute and so would allow us to geometrically realize C_{n+1}. However, it is not necessarily true that $\phi'\alpha_{n+1} = (r_n)_*\delta^X_{n+1}$, so it may not be possible to extend r_n to a map of X^{n+1} into the $(n+1)$-skeleton of the realization. In order to address this problem, we use the relationship between α_{n+1} and β_{n+1} to "correct" ϕ', replacing it with $\phi = \phi' + \xi$. ξ is a map that can be chosen so that $j^Y_n\xi = 0$ and so that the left square commutes using ϕ. Since $j^Y_n\xi = 0$, ϕ and ϕ' both yield (possibly non-homtopic) geometric realizations of C_{n+1}. Since the left square commutes using ϕ, the realization corresponding to ϕ will allow r_n to be extended to a map of the $(n+1)$-skeletons, completing the inductive step. Here are the details:

The composition $\lambda = (r_n)_*\delta^X_{n+1}(1-\beta_{n+1}\alpha_{n+1})$ defines a map on $\pi_{n+1}(X^{n+1},X^n)$ whose image is contained in Γ_nY. This can be verified by computing

$$\begin{aligned}
j^Y_n(r_n)_*\delta^X_{n+1}(1-\beta_{n+1}\alpha_{n+1}) &= (r_n,r_{n-1})_*\partial^X_{n+1}(1-\beta_{n+1}\alpha_{n+1}) \\
&= \partial^C_{n+1}\alpha_{n+1}(1-\beta_{n+1}\alpha_{n+1}) \\
&= \partial^C_{n+1}(\alpha_{n+1}-\alpha_{n+1}) \\
&= 0.
\end{aligned}$$

Thus $\lambda\colon \pi_{n+1}(X^{n+1}, X^n) \to \Gamma_n Y$. Since C_{n+1} is free and α_{n+1} is surjective, we can define a mapping $\xi\colon C_{n+1} \to \Gamma_n Y$ by the equation $\xi\alpha_{k+1} = \lambda$. Now define $\phi\colon C_{n+1} \to \pi_n Y^n$ by $\phi = (r_n)_*\delta^X_{n+1}\beta_{n+1} + \xi$. A completely elementary calculation checks that $\phi\alpha_{n+1} = (r_n)_*\delta^X_{n+1}$. Since $\mathrm{im}(\xi) \subset \Gamma_n Y$, $j^Y_n\xi = 0$. We use this in the second line of the following calculation

$$\begin{aligned} j^Y_n\phi &= j^Y_n\left((r_n)_*\delta^X_{n+1}\beta_{n+1} + \xi\right) \\ &= j^Y_n(r_n)_*\delta^X_{n+1}\beta_{n+1} \\ &= (r_n, r_{n-1})_*\partial^X_{n+1}\beta_{n+1} \\ &= \partial^C_{n+1}\alpha_{n+1}\beta_{n+1} \\ &= \partial^C_{n+1} \end{aligned}$$

which verifies the commutativity of the bottom triangle and so completes the induction. ◇

Remark. Given the realizability of mapping cylinders alluded to in the remark following Lemma 7.1, we can conclude from the above lemma that it is always possible to realize a homotopy system that is a deformation retract of geometrically realizable homotopy system.

As an immediate consequence of Lemma 7.5 we have

Corollary 7.6. *If a homotopy system C is a retract of $\rho(KC_{\mathrm{CW}})$, then it is geometrically realizable.*

Combining Corollary 7.6 with Theorem 7.3 we get our main result, namely

Theorem 7.7. *A homotopy system C is geometrically realizable if and only if it is a retract of $\rho(KC_{\mathrm{CW}})$. If C is identified with $\rho(KC_{\mathrm{J}})$, then the the chain epimorphism in the definition of the retraction is homotopic to the map induced by $1_{KC}\colon \rho(KC_{\mathrm{CW}}) \to \rho(KC_{\mathrm{J}})$, and may be taken to be an isomorphism in dimensions one, two, and three.*

7.3. Obstructions to Geometric Realizability.

We now have the following situation: Given a homotopy system C, there is a chain epimorphism $\bar{\alpha}\colon \rho(KC_{\mathrm{CW}}) \to C$, and this map splits if and only if C has a geometric realization. Whether or not C has a geometric realization, $\bar{\alpha}$, viewed as an epimorphism of graded operator groups, splits, but a particular collection of splittings, one in each dimension, may not be a chain map. This observation gives rise to a sequence of obstructions whose vanishing in each dimension is necessary and sufficient for geometric realization. We give a brief description of these obstructions here:

Let K and C be homotopy systems and $\bar{\alpha}\colon K \to C$ a chain map that is an isomorphism in dimensions one, two, and three and an epimorphism in all higher dimensions. (This corresponds to the situation above with $K = \rho(KC_{\mathrm{CW}})$.) Let $L_i = \ker\bar{\alpha}_i$. Then, using the fact that $K_i \cong C_i$ for $i = 1,2,3$, L is an chain complex of $\mathbf{Z}\pi_1$-modules ($L_i = 0$ for $n = 1,2,3$) with $\pi_1 = C_1/\operatorname{im}\partial_2^C$. Since each C_i is free, there are splittings β_i for $\bar{\alpha}_i$ in each dimension (with $\beta_i = \bar{\alpha}_i^{-1}$ for $i = 1,2,3$). Consequently, L_i is a projective submodule of K_i for all i, and so L, although not a homotopy system, is a projective (trivially crossed) complex. In what follows, we shall use the notation $\ker\bar{\alpha} = L$, but will still refer to the boundary operators of $\ker\bar{\alpha}$ as ∂_n^L.

The collection of maps $\{\beta_i\}$ does not have to be a chain map, although we do have $\beta_{i-1}\partial_i^C = \partial_i^K\beta_i$ for $i = 2,3$. Change the name of β_1, β_2, and β_3 to $\bar{\beta}_1$, $\bar{\beta}_2$, and $\bar{\beta}_3$ respectively and suppose, inductively, that we have replaced $\beta_1, \beta_2, \ldots \beta_n$, $n \geq 3$, with splittings $\bar{\beta}_1, \bar{\beta}_2, \ldots \bar{\beta}_n$, all of which now meet the chain map condition. Although n is required to be at least three, the fact that α_3 is an isomorphism implies that β_4 meets the chain map condition, so we may take $\bar{\beta}_4 = \beta_4$ and assume that n is at least four. Suppose that β_{n+1} cannot be added to the list of splittings meeting the chain map condition. Violating the chain map condition means that $\bar{\beta}_n\partial_{n+1}^C - \partial_{n+1}^K\beta_{n+1}$ will not be the trivial map. The image of this map lies in $\ker(\bar{\alpha})$, thereby yielding a cochain in $C^{n+1}(C; H_n(\ker\bar{\alpha}))$. (Since $n \geq 4$, we are dealing with the abelian portion of C and so this is the ordinary cochain group.) It is easily seen that this cochain is independent of the particular splitting β_{n+1} of $\bar{\alpha}_{n+1}$ used to define it and is, in fact, a cocycle that we denote by $c^{n+1}(\bar{\alpha})$.

The essential property of this cocycle is that $c^{n+1}(\bar{\alpha}) = 0$ if and only if the splitting can be extended to dimension $n+1$, i.e. if and only if β_{n+1} can be replaced by a splitting $\bar{\beta}_{n+1}$ so that $\{\bar{\beta}_1, \bar{\beta}_2, \ldots \bar{\beta}_{n+1}\}$ satisfies the chain map condition. In a manner not simply analogous to but actually contained in the usual topological arguments for obstructions to splittings, we end up with a sequence of obstructions $c^{n+1}(\bar{\alpha}) \in H^{n+1}(C; H_n(\ker \bar{\alpha}))$ for $n \geq 4$ that vanish if and only if $\bar{\alpha}$ splits.

In the case that $K = \rho(KC_{\mathrm{CW}})$, the ability to geometrically realize C is equivalent to the existence of a chain splitting of $\bar{\alpha}$, so the obstructions $c^{n+1}(\bar{\alpha}) \in H^{n+1}(C; H_n(\ker \bar{\alpha}))$ for $n \geq 4$ deserve to be called obstructions to geometrical realization.

The computation of these obstructions requires us to calculate $H_*(\ker \bar{\alpha})$, and this is the task of another paper.

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