On the Homotopy Groups of Toric Spaces with Applications to the Homotopy Groups of Certain Manifolds.

by David Allen

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Abstract

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by

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Given an n-dimensional, q-neighborly simple convex polytope P one has the associated borel space, moment angle complex and family of toric manifolds that sit over P. Recently there has been much focus on the homotopy groups and homotopy type of the moment angle complex. Buchstaber and Panov determined the first non-trivial homotopy group of the borel space using a particular cellular structure. In this thesis the notion of relations among relations is put in the unstable BP context. This allows for the determination of $R^1PBP_*(BTP)$ through a range. The stable and unstable co-action on $R^1PBP_*(BTP)$ is computed and shown to coincide with the co-action on a product of spheres whose dimensions depend on the combinatorics of P. As a result, the higher homotopy groups of the borel space can be determined through a range that was previously unknown. Applications to the homotopy type of a family of moment angle complexes will be given.

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Contents

Chapter 1

Combinatorics

1.1 Introduction

The material in this chapter contains much of the standard combinatorics needed in the sequel. For a more detailed exposition the reader should refer to [Bron], [BP1], [BP2], [BP3], [BP4], [BP5],[BR2] and [DJ].

1.2 Simplicial Complexes

Definition 1.2.1. *A simplicial complex* K *on* $[m]$ *is a collection of subsets* $\{I\}$ *of* $[m]$ *such that if* $I \in K$ *and* $I' \subset I$ *then* $I' \in K$

We call $\sigma \in K$ a simplex of K.

Definition 1.2.2. *Let* K *be a simplicial complex on a set* $\underline{S}$ *a missing face of* K *is a subset* $\sigma \subset \underline{S}$ *such that* $\sigma \notin K$ *but every proper subset of* σ *is a simplex of* K.

Definition 1.2.3. *A vertex of* K *is a one element subset of* $[m]$*. Let* $Ver(K)$ *denote the vertex set of* K

Definition 1.2.4. *If* $\sigma \in K$ *then the dimension of* σ $= |\sigma|$ *- 1.*

Let $\sigma_m \in K$ denote the simplex of maximal dimension in K. That is, for any other simplex $\sigma' \in K$, the dimension $\sigma' \leq$ dimension σ.

Definition 1.2.5. *The dimension of* K *is the dimension of* σ_m*.*

Definition 1.2.6. *A sub-collection of simplicies* $K' \subset K$ *which is also a simplcial complex is called a sub-complex of* K*.*

The notion of *geometric simplicial complex* or Polyhedron will be defined later. It should be clear from the context what type of simplicial complex is being discussed.

Definition 1.2.7. *Let* $\sigma \in K$ *be a simplex. A face of* σ' *of* σ *is a subset* $\sigma' \subset [m]$ *such that* $Ver(\sigma') \subset Ver(\sigma)$

Definition 1.2.8. *The q-skeleton,* K^q *of* K *is the sub-complex of* K *consisting of all simplicies* σ *such that the dimension* $\sigma \leq q$

Definition 1.2.9. *For* $\sigma \in K$*, the sub-complex* $Link_K\ \sigma = \{\tau \in K | \tau \cup \sigma \in K, \tau \cap \sigma = \phi\}$

Definition 1.2.10. *Let* K *and* L *be two simplicial complexes. We say* $f : K \rightarrow L$ *is a simplicial map if for any* $\sigma \in K$*,* $f(\sigma) \in L$

If we assume that $f : K \rightarrow L$ is a simplicial map then we have the

Definition 1.2.11. *We say that f is non-degenerate if for each simplex $\sigma \in K$ f restricted to σ is a bijection of sets.*

Definition 1.2.12. *A Flag complex K_F is a simplicial complex whose missing faces ω satisfy the condition: $|\omega| = 2$.*

Neighborliness is a key concept that is related to various homotopy theoretic computations. We will define it for both polytope and simplcial complexes. First,

Definition 1.2.13. *A simplicial complex K is called k-neighborly if for $\sigma \subseteq [m]$ with $|\sigma| = k$ then $\sigma \in K$.*

1.3 Operations on Simplicial Complexes

Throughout this section we asssume that K is an abstract simplicial complex on the set $[m]$. All of the constructions in this section can be found in [BP1] and [BP4]. Proofs of these facts can be found in [BP4].

Given K as above one can construct the underyling polyhedron associated to K called the geometric realization of K. This polyhedron is denoted by the symbol $|K|$. let e_i be the vector $\langle 0, .., 0, \underbrace{1}_{ith}, 0, .., 0\rangle$ in $\mathbb{R}^m$. Let Δ_σ denote the convex hull of e_j for $j \in \sigma$. We define the geometric realization in the following way:

$$P = \bigcup_{\sigma \in K} \Delta_\sigma$$

Let K and L be two simplicial complexes on $[m_1]$ and $[m_2]$. Let $S = [m_1] \cup [m_2]$ where $[m_1] \bigcap [m_2]$ is not necessarily empty. There is an operation on them called the join denoted by $*$.

Construction 1.3.1. *The join of K and L is the simplicial complex given by*

$$K * L = \{\sigma \subset S | \sigma = \sigma_K \cup \sigma_L, \sigma_K \in K, \sigma_L \in L\}$$

Remark 1.3.2. *It is important to note that $\phi \in K$. When one performs the join operation on K with a vertex say v, all of the simplicies $\sigma \in K$, σ are included as well as the unions of the form $\sigma \cup v$. We call such an operation the cone over K.*

The cone over K is given by $\triangle^0 * K$.

Definition 1.3.3. *The barycentric subdivision of K is the simplicial complex $\overline{K}$ whose vertex set is $\{\sigma \in K\}$. A k simplex, $\overline{\sigma_k} \in \overline{K}$ is a chain of the form $\sigma_1 \subseteq \sigma_2 \subseteq \cdot\cdot \subseteq \sigma_k = \overline{\sigma_k}$ where $\sigma_i \in K$.*

Later on we will form the cone over the barycentric subdivision of the boundary complex. We will also need to define maps between simplcial complexes and their barycentric subdivisions.

1.4 Simple Convex Polytopes

We begin by giving some preliminary definitions taken from [BP1], [BP2], [BP3], [BP4].

Definition 1.4.1. *A convex polytope is the convex hull of finitely many points in $\mathbb{R}^n$.*

More generally we have the

Definition 1.4.2. *A convex polyhedron P is an intersection of finitely many half-spaces in $\mathbb{R}^n$.*

$$P = \{\overrightarrow{x} \in \mathbb{R}^n | \langle \overrightarrow{l_i}, \overrightarrow{x} \rangle \geq -a_i, i = 1, .., m\}$$

Definition 1.4.3. *The affine hull of the points $x_1, .., x_n$ where $x_i \in \mathbb{R}^n$ is the set $\{\sum_{i=1}^n a_i x_i | a_i \in \mathbb{R}, a_1 + \cdots + a_n = 1\}$.*

The dimension of P is the dimension of its affine hull.

Definition 1.4.4. *A supporting hyperplane H of P is an affine hyperplane which intersects P such that P is contained in one the closed half-spaces determined by H.*

This allows for us to make the following

Definition 1.4.5. *A face of P is $P \cap H$ where H is a supporting hyperplane.*

Definition 1.4.6. *The boundary of P, denoted ∂P is the union of all the proper faces of P.*

The vertices of P are the 0-dimensional faces. The facets are the $(n-1)$-dimensional faces. That is, the co-dimension 1-faces.

Definition 1.4.7. *An n-dimensional simplex, $\triangle^n$ is the convex hull of $(n+1)$ independent points in $\mathbb{R}^n$.*

Definition 1.4.8. *The standard n-simplex $\triangle^n$ is the convex hull of the unit vectors $\overrightarrow{e_i}$ in $\mathbb{R}^n$.*

The definition of $\triangle^n$ should be clear from the context.

Definition 1.4.9. *An n-dimensional convex polytope is simple if the number of facets meeting at each vertex is exactly n.*

Definition 1.4.10. *An n-dimensional convex polytope is called simplicial if each facet contains at least n vertices.*

The previous definition implies that the boundary complex of a simplicial polytope is a simplicial complex. Suppose the polyhedra P and Q' are the geometric realizations of the complexes K and K' we must understand when $P \cong Q'$ as geometric polyhedra. In the definitions that follow we interchange function and map.

Definition 1.4.11. *A function $f : P \to Q$ is called simplicial if f takes simplicies of K to simplicies of K'.*

If there exists a simplicial map from K to K' then we can extend by linearity on the simplices to give a map between the polyhedra P and Q'. Let $\sigma = \{v_0, .., v_k\} \in K$ and let $\lambda_0, .., \lambda_k \in \mathbb{R}$ such that $\lambda_0 + \cdot \cdot + \lambda_k = 1$. Let $x = \lambda_0 v_0 + \cdot \cdot \lambda_k v_k$ then $f(x) = \lambda_0 f(v_0) + \cdot \cdot + \lambda_k f(v_k)$.

Definition 1.4.12. *We say a polyhedron $\overline{P}$ is a subdivision of P if for each simplex $\overline{\tau} \in \overline{P}$ there exists a simplex $\kappa \in P$ such that $\overline{\tau} \subseteq \kappa$ and each simplex in P can be written as a finite union of simplicies in $\overline{P}$.*

In this context, simplex refers to a simplex in the underlying simplicial complex. Let $\overline{P}$ and $\overline{Q}$ be subdivisions of P and Q.

Definition 1.4.13. *A function $\phi : P \to Q$ is piecewise linear (PL) if there exists a simplcial map between the subdivisions $\overline{P}$ and $\overline{Q}$.*

Definition 1.4.14. *Suppose $f : P \to Q$ is a PL map of polyhedra. We say f is a PL homeomorphism if there exists an inverse f^{-1} which is a PL map of polyhedra.*

Definition 1.4.15. *Two polyhedra P and P' are homeomorphic if there exists a PL-homeomorphism between them.*

It follows from he definitions above that for any simplcial complex K, $|K| \approx |bs(K)|$. This will be applied to the boundary complex of K.

Definition 1.4.16. *A simple convex polytope is k-neighborly if the intersection of any k facets is non-empty.*

For example, m-gons are 1-neighborly. In the next section the relationship between the neighborliness of a simplicial complex K and its dual polyhedron P will be developed. The following polytope exhibits some interesting properties.

Definition 1.4.17. *For each integer $n \geq 4$ and $k \geq n+1$ there is a n-dimensional convex polytope with k vertices denoted by C^n_k defined to be the convex hull of k-points on the curve:*

$$\gamma(t) = (t, t^2, .., t^{n-1})$$

This polytope is $m = [n/2]$ neighborly

It was shown in [DJ] that for $n \geq 4$ and $k \geq 2n$ cyclic polytopes can not be the base space for a family of quasi-toric manifolds. For a more complete discussion on cyclic polytopes the reader should refer to [Stanley] and [Bron]. For their relation to the non-existence of quasi-toric manifolds over C^n_k the reader should refer to [DJ] and

[Civ]. The way that the neighborliness of cyclic polytopes is independent of the number of vertices is exhibited by the relation: $k \geq n+1$. We call such polytopes *neighborly*. The reader should refer to [BP4] for other examples and more information.

The next construction is vital. An n-dimensional simple convex polytope P can be viewed as a manifold with corners. Specifically,

Construction 1.4.18. *Let P be an n-dimensional simple convex polytope and $v \in P$. Let U_v be the open subset of P obtained by deleting all of the edges of P not containing v. It is clear that U_v is diffeomorphic to $\mathbb{R}^n$. Performing this procedure for each vertex gives rise to an atlas $\{U_v\}_{v \in P}$*

1.5 Examples of Simplicial Complexes

The purpose of this section is to describe the simplicial complexes that may not be well known to the non-expert. First, we describe a simplcial complex that [DJ] call K^n that dualizes to U-a simple polyhedral complex which behaves like a classifying space in the category of toric spaces. Moreover, the complex K^n has some very nice algebraic properties.

Construction 1.5.1. *The vertex set of K^n is the set of lines in $\mathbb{Z}^n$. A k-simplex $\sigma \in K^n$ is a collection of lines $\{l_0, .., l_k\}$ which span a $(k+1)$-dimensional unimodular subspace of $\mathbb{Z}^n$.*

Another very important simplicial complex that is related to K^n is $X(\mathbb{Z}^n)$, whose construction we give below. This gives a direct relation between toric spaces and algebraic K-theory. [DJ] use the complex $X(\mathbb{Z}^n)$ to prove that K^n is Cohen-Macaulay.

Construction 1.5.2. *A sequence of vectors* $(v_1, .., v_k)$ *in* $\mathbb{Z}^n$ *is unimodular if it spans a* k *dimensional direct summand of* $\mathbb{Z}^n$*. Define a partial ordering on the set of all such sequences by requiring the relation to be a sub-sequence. The simplicial complex associated to this poset is* $X(\mathbb{Z}^n)$*.* $X(\mathbb{Z}^n)$ *is the barycentric subdivision of the simplicial complex associated to the vectors* $(v_1, .., v_k)$ *where the condition for such a vector to be a simplex in this complex is completely analogous to the condition given in 1.5.1.*

Definition 1.5.3. *A simplicial complex* K *is called a simplicial sphere if* $|K| \approx S^t$ *for some* t*.*

m-gons are simplicial spheres. The simplicial complex dual to the boundary complex of such a polytope is a 1-dimensional simplicial sphere. For such a K, $|K| \approx S^1$. The geometric realization defines a triangulation of S^1. The facets in P, dualize to the vertices in K such that the geometric realization of K defines the triangulation of a sphere [DJ]. In the sections that follow a procedure will be described that will allow for one to construct an an n-dimensional simple polyhedral complex P_K from an $(n-1)$-dimensional simplicial complex K. It was shown in [BP4] that the moment angle complexes associated to simplicial spheres are $m+(t+1)$-dimensional manifolds. There is a heirarchy of combinatorial objects for which a similiar statement holds. First we need some preliminary definitions.

Definition 1.5.4. *A simplicial complex* K *is called a PL-sphere if* $|K|$ *is PL homeomorphic to the boundary of a simplex.*

A *PL* sphere is a simplicial sphere which has a subdivision which is combinatorially equivalent to a subdivision of the boundary of a simplex.

Definition 1.5.5. *A polytopal sphere is a PL-sphere which is isomorphic to a simplicial polytope.*

It is well known in combinatorics that:

polytopal spheres $\subset$ PL spheres $\subset$ simplicial spheres.

It was shown in [BP1] that the moment angle complex associated to a n- dimensional simple convex polytope P (which dualizes to a $(n-1)$ simplcial sphere) is a manifold. This implies that the moment angle complex associated to a simplicial PL sphere or polytopal sphere is also a manifold [BP4].

1.6 Operations on Simple Convex Polytopes

This material can be found in [BP4] and [BR2]. Let P_1 and P_2 be two simple convex polytopes. In [BP4] it is shown that the product of two simple convex polytopes P_1 and P_2 of dimensions n_1 and n_2 is another simple convex polytope of dimension n_1+n_2. In terms of neighborliness we have the

Lemma 1.6.1. *Let P be a k-neighborly simple convex polytope and Q an l-neighborly simple convex polytope, then $P \times Q$ is $min(k,l)$ neighborly.*

The proof of this fact follows easily by examining the face ring of the product.

Construction 1.6.2. *Let P and Q be two n-dimensional simple convex polytopes with distinguished vertices v and w. The connected sum $P\sharp_{v,w}Q$ is formed by removing v and w by a hyperplane cut and gluing P and Q along the facets that have non-empty intersection with the hyperplanes that isolate v and w.*

A detailed description of the connected sum as well as there interpretation as a sequence of pruning operations can be found in [BR2].

1.7 The Dual of Simple Convex Polytopes and Generalizations

When dealing with simple convex polytopes [DJ] outline a procedure that allows one to construct the simplcial complex K_P which is dual to the boundary complex of P. An i dimensional simplex of K dualizes to a codimension $i+1$ face of the dual polytope P_K. The facets (the codimension 1 faces) of P are dual to the vertices of K. This dualization process works in both directions. Given an n-dimensional simple convex polytope P we can construct the $(n-1)$-dimensional simplcial complex K_P which is dual to the boundary complex of P. On the other hand, given K we can construct the n-dimensional simple convex polytope P_K by following the same procedure. It is important to note that not every simplicial complex K dualizes to a simple convex polytope. However, if we allow for a more general object to be the dual of K then we have a dualization procedure [BP1], [DJ]. Given an $(n-1)$-dimensional simplicial complex K on $[m]$ it is possible to construct a polyhedron P_K which is not a simple convex polytope. Although P_K is simplcial. [DJ] call such combinatorial objects simple polyhedral complexes. We describe their construction below. The following construction is from [DJ].

Construction 1.7.1. *Let K be a $(n-1)$ dimensional simplicial complex. Let K'*

denote its barycentric subdivision. For each simplex $\sigma \in K$, let F_σ denote the geometric realization of the poset $K_{\geq\sigma} = \{\tau \in K | \sigma \leq \tau\}$. It is clear that F_σ is the subcomplex of K' which consists of all simplicies of the form $\sigma = \sigma_0 < \sigma_1 < \cdots < \sigma_k$. F_σ is a cone on the geometric realization of $K_{>\sigma}$. If σ is a $(k-1)$ simplex then we say that F_σ is a face of co-dimension k. We let P_K denote the cone on K. The polyhedron P_K together with its decomposition of faces $\{F_\sigma\}_{\sigma\in K}$ is a simple polyhedral complex. It is important to note that the dual polyhedron is n dimensional.

Let P be an n-dimensional simple convex polytope. The construction above reduces to a very easy case for such P. The k simplicies of K_P (which dualizes to P) are identified with the co-dimension *k+1*-faces of P. For example, if we let P be an m-gon then the vertices of P dualize to edges in K_P and the edges dualize to vertices. One has to be careful when identifying the vertices and edges in K_P. This correspondence also holds in the general case as long as P is simplicial.

Definition 1.7.2. *We say that a simple polyhedral complex P_K is q-neighborly if the $(q-1)$ skeleton of K_P coincides with the $(q-1)$ skeleton of a $(m-1)$ simplex.*

We list some important definitions that will be used later to link the combinatorics of P_K to computable algebraic invariants. It can be shown that the previous definition is equivalent to:

Definition 1.7.3. *We say that P_K is q-neighborly if the intersection of any q co-dimension one faces is nonempty.*

Suppose P is k-neighborly. This implies that $F_{\iota_1} \bigcap \cdot\cdot \bigcap F_{\iota_{k+1}} = \phi$ for some subset $\{\iota_1, .., \iota_{k+1}\}$ of $\{\iota_1, .., \iota_n\}$, but $F_{\iota_1} \bigcap \cdot\cdot \bigcap F_{\iota_k} \neq \phi$ for any k-element subset of $\{\iota_1, .., \iota_{k+1}\}$.

We can easily relate the notion of neighborliness of P and K_P by using construction 1.7.1 without reference to the face ring. Although analyzing the face ring would provide a much more efficient method for understanding the relation between the neighborliness of P and K_P. It is important to note that when K is a simplicial sphere P_K enjoy the following property: Any co-dimension k face, $F_k = F_{\iota_1} \bigcap \cdot \cdot \bigcap F_{\iota_k}$ where each F_{ι_j} is a facet. If P_K is simplicial then 0-faces may be the intersection of at least n facets.

1.8 Combinatorial Classes of Polyhedron and the Dehn-Sommerville relations

Definition 1.8.1. *For $i > 0$, h_i is defined to be the coefficient of t^{n-i} when we equate the coefficients of*

$$(t-1)^n + \sum_{i=0}^{n-1} f_i (t-1)^{n-1-i} = \sum_{i=o}^{n} h_i t^{n-i}$$

This polynomial determines the h_i as a polynomial in the f_i. By the binomial theorem we have

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{n-i}{n-k} f_{i-1}$$

If we replace t with $t+i$ and expand we have

$$f_{n-1-k} = \sum_{q=k}^{n} \binom{q}{k} h_{n-q} \text{ for } k = 0, .., n$$

We associate the vector $\overrightarrow{h} = (h_0, .., h_n)$ to the polyhedron P_K which is dual to K and $\overrightarrow{h}$ is an invariant of P_K. This vector defines a combinatorial class of polyhedron i.e.,the face lattices of two polyhedra in a class are isomorphic as posets. Given P the

f_i are the number of i-simplicies in K_P. If P is a simple convex polytope then we having the following

Theorem 1.8.2. *The* $\overrightarrow{h}$ *of any simple polytope satisfies the relation*

$$h_i = h_{n-i} \text{ for } i \in \{0, .., n\}$$

Proof. See [Bron] □

This theorem will help in computing the homology groups of quasi-toric manifolds without determining the higher coefficients of the $\overrightarrow{h}$. Of course this theorem does not necessarily hold for a P_K when K is not a simplicial sphere.

Theorem 1.8.3. *The Dehn-Sommerville relations hold for any simplicial sphere.*

Proof. See [Stanley] □

Remark 1.8.4. *If P is a simple convex polytope then $h_0 = h_n = 1$.*

Using the formulas above one may reformulate the Dehn-Sommerville relations in terms of the f-vector. We have

$$f_{k-1} = \textstyle\sum_{j=k}^{n}(-1)^{n-j}\binom{j}{k} f_{j-1} \text{ for } k = 0, .., n$$

It is possible for two non-equivalent polytopes to have the same $\overrightarrow{f}$. For a particular example the reader should refer to [BP4] page 12, fig. 1.1. It is important to note that two simplicial complexes can have the same $\overrightarrow{f}$ however the polyhedron dual to them may not be of the same combinatorial type. When P is a simple convex polytope both the $\overrightarrow{h}$ and $\overrightarrow{f}$ behave well with respect to products and connected sums. If P_1 and P_2 are an n_1 and n_2 dimensional simple convex polytopes we have:

$$f_k(P_1 \times P_2) = \textstyle\sum_{i=-1}^{n_1-1} f_i(P_1) f_{k-i-1}(P_2) \text{ for } k = -1, .., n_1 + n_2 + 1$$

as well as

$$h(P \times Q) = h(P)h(Q)$$

For the connected sums we have the following relations:

$f_i(P^n \sharp Q^n) = f_i(P^n) + f_i(Q^n) - \binom{n}{i+1}$ for $i = 0, .., n-2$ and

$$f_{n-1}(P^n \sharp Q^n) = f_{n-1}(P^n) + f_{n-1}(Q^n) - 2$$

The reader should refer to [BP4](pg. 9) for more details.

Chapter 2

Face Rings and Quasi-Toric Manifolds

2.1 Introduction

Given any simple polyhedral complex P_K one can associate a computable algebraic invariant called the face ring. We will define the face ring for a simplicial complex K with $Ver(K)$ =$\{v_1, .., v_m\}$. Let R be a commutative ring with unit and regard the v_i as indeterminants of degree 2. Let I be the homogeneous ideal generated by all square free monomials $v_1 \cdots v_k$ such that $\{v_1, .., v_k\}$ does not span a simplex in K. We have the

Definition 2.1.1. *For a simplicial complex K the face ring $k(K) = R[v_1, .., v_m]/\langle I\rangle$*

Recall that an $(n-1)$ dimensional simplicial complex K dualizes to a n-dimensional simple polyhedral complex P_K. If we assume that K is k-neighborly then the ideal I_K

is generated by the monomials which come from the missing faces. Any missing face of K contains $(k+1)$ vertices. Dually the ideal I_{P_K} which we will call I' is generated by codimension $(k+1)$ faces that are dual to the missing faces in K. Using the construction from chapter 1 it is easy to see that I' is generated by various intersections of facets each of which is dual to a monomial in I. We can define the face ring for P_K

Definition 2.1.2. *For the simple polyhedral complex P_K the face ring*

$$k(P_K) = R[v_1, .., v_m]/I' = \langle v_{i_1} \cdots v_{i_k} | \bigcap_{j=1}^{k} F_{i_j} = \emptyset \rangle$$

The F_{ι_j} are the facets under the correspondence given in §6. They are dual to the vertices in K. The missing faces in K correspond to those facets in P_K whose intersection is empty. To illustrate how the ideals I and I' are dual we consider the following

Example 2.1.3. *Let $\sigma = \{1, 2\} \in K$ where K is dual to the boundary of a simple polyhedral complex. In this case edges dualize to vertices. Clearly, σ dualizes to $\widehat{v} = F_{\iota_1} \bigcap F_{\iota_2}$ which must belong to P_K. Now suppose $\overline{\sigma} = \{1, 2, 3\}$ is a missing face of K which shows up in the face ring, $k(K)$ as the monomial $v_1v_2v_3 = 0$ then the dual of $\overline{\sigma}$ would be a co-dimension 3-face, $F_3 = F_{\iota_1} \bigcap F_{\iota_2} \bigcap F_{\iota_3}$. However, $F_3 \notin P_K$ by the correspondence. Therefore, $F_{\iota_1} \bigcap F_{\iota_2} \bigcap F_{\iota_3} = \phi$. Hence, the monomial $v_1v_2v_3 = 0$ shows up in $k(P_K)$.*

Since the simple polyhedral complex P_K and the simplicial complex K are dual we immediately have the following

Proposition 2.1.4. $k(P_K) \cong k(K)$

Proof. observe that the facets of P_K dualize to the vertices of K. If $\bigcap_{j=1}^{k} F_{i_j} = \emptyset$ then $\{v_1, .., v_k\}$ does not span a simplex in K. □

One observes that the length of the monomials in the ideal I increase as neighborliness increases. Clearly, the face ring depends on the notion of neighborliness.

2.2 Quasi-Toric Manifolds

Much of the exposition of this section can be found in [BP2], [BP4] and [DJ].

Remark 2.2.1. *We will assume that P is a n-dimensional simple convex polytope. We will use P_K to denote the n-dimensional simple polyhedral complex dual to a $(n-1)$-dimensional simplicial complex K on $[m]$.*

There is a slight conflict in the literature between the notion toric manifold and quasi-toric manifolds. In algebraic geometry, toric manifolds are reserved for smooth (non-singular) toric varieties. Quasitoric-manifolds include projective and smooth toric varieties as well as those toric varieties that are singular, for example $CP^2 \sharp CP^2$. Using orientations on various bundles associated to quasi-toric manifolds it has been shown by [Civ] that the class of Quasi-toric manifolds includes the class of toric manifolds.

Definition 2.2.2. *The positive cone* $R^n_+ = \{(x_1, .., x_n) \in \mathbb{R}^n | x_i \geq 0 \ \forall \ i\}$.

Definition 2.2.3. *The topological torus* $T^n = \{(e^{2\pi i \varphi_1}, .., e^{2\pi i \varphi_n}) \in C^n\}$ *where* $\{\varphi_1, .., \varphi_n\} \in R^n$.

The torus can be viewed as the standard subgroup in $(C^*)^n$ which defines an orientation of each coordinate subgroup $T_i \cong S^1$ for $i = 1, .., n$.

Definition 2.2.4. *The standard action on C^n is the representation of T^n by diagonal matrices in $U(n)$*

It follows immediately that the orbit space for the standard action is the positive cone R^n_+. If T^n acts on a manifold we arrive at the following

Definition 2.2.5. *A $2n$-dimensional manifold-M^{2n} endowed with a T^n-action will be called a T^n-manifold.*

Definition 2.2.6. *A standard chart on a T^n-manifold-M^{2n} is a triple (U, f, ψ) where U is stable under the T^n-action, $\psi \in Aut(T^n)$, $f : U \to W$ is a homeomorphism which satisfies the following compatibility condition for some T^n-invariant open subset $W \subset \mathbb{C}^n$*

$$f(t \cdot y) = \psi(t) f(y)$$

$\forall\, y \in U, t \in T^n$

We say M^{2n} has a standard atlas if every point of M^{2n} lies in a standard chart.

Definition 2.2.7. *A T^n-action on M^{2n} is locally standard if M^{2n} has a standard atlas.*

In other words, every point in the manifold lies in a neighborhood which is invariant under the torus action and is homeomorphic to an invariant open subset W where the homeomorphism is ψ-equivariant.

We can state the definition of Quasi-toric manifold:

Definition 2.2.8. *A T^n-manifold M^{2n} is called a Quasi-toric manifold if the following two conditions hold:*

1) The T^n-action is locally standard.

2) There is a projection $\pi : M^{2n} \to P$ constant on T^n-orbits which maps every k-dimensional orbit to a point in the interior of a co-dimension k face of P for $k = 0, .., n$.

The fixed point set of the action consist of the vertices of P. The action is free over the interior. It is a direct consequence of the definitions above that the orbit space of a Quasi-toric manifold is a manifold with corners. Specifically, the orbit space is diffeomorphic to P as a manifold with corners. Let $\Im$ denote the set of facets of P. Given the torus T^m the coordinate torus $T^k_{\iota_1,..,\iota_k}$ is a product of Torii that come from the factors $\iota_1, .., \iota_k$ in T^m.

Definition 2.2.9. *Let $\lambda : \Im \to \mathbb{Z}^n$ be a function which assigns to each coordinate torus a primitive vector in the integer lattice $\mathbb{Z}^n$. We call such a function the characteristic function of M^{2n}.*

Since P is simple then any co-dimension k-face F can be written as the intersection: $F_k = \bigcap_{i=1}^k F_{i_j}$ where $F_{i_j} \in \Im$. The λ map determines a subgroup G_F of the lattice $\mathbb{Z}^n$.

It was shown in [DJ] that a Quasi-toric manifold can be constructed from a certain quotient space $T^n \times P^n / \sim$. First, one needs a particular pair (P^n, λ) where λ is a specific linear map defined below. Given an n dimensional simple convex polytope P and a co-dimension k-face, F of P we let G_F denote the stabilizer of F under the torus action for any $x \in F$. For any $F \in \Im$, G_F is a rank-one subgroup (coordinate torus). Coordinate torii are determined by primitive vectors in the integer lattice $\mathbb{Z}^n$ up to sign. This gives a prescription for defining a function. In the next definition let $n = dim\ P$.

Definition 2.2.10. *Let F_k be a co-dimension k-face of P. We define $\lambda(F_k)$ as*

$$\mathbb{Z} - span\ \langle \lambda(F_{i_1}), .., \lambda(F_{i_k}) \rangle \subset \mathbb{Z}^n$$

The subgroup G_F is determined by the sub-lattice that is generated by the image of λ defined above. It follows immediately that for any vertex $v \in P$ the subspace $\lambda(v)$ is a basis for $\mathbb{Z}^n$. If one puts a rather restricted condition on the image of λ then it is possible to construct families of Quasi-toric manifolds over P. [DJ] refer to this restriction as condition $(*)$. Let F_k be a co-dimension k-face of P. Let $\lambda(F_k)$ be given as above. We say λ satisfies condition $(*)$ if $\lambda(F_k)$ spans a k-dimensional uni-modular subspace of $\mathbb{Z}^n$. In general, let K be an $(n-1)$ dimensional simplicial complex and $\lambda : Ver(K) \to \mathbb{Z}^n$ be a function. For each $(k-1)$ simplex $\sigma \in K$ let E_σ be the $\mathbb{Z}$-span of $\lambda(v)$ where $v \in \sigma$. In this context, condition $(*)$ translates as follows: for each σ as above, E_σ is a k dimensional uni-modular subspace of $\mathbb{Z}^n$ [DJ]. the resulting space is a toric space and not necessarily a Quasi-toric manifold. We are ready to describe the construction of Quasi-toric manifolds from pairs of the form (P, λ) where λ satisfies condition $(*)$. [BP4] call such pairs *characteristic pairs.*

Construction 2.2.11. *Consider the charactersitic pair (P, λ), the following construction will produce a family of quasi-toric manifolds $M^{2n}(\lambda)$ dependent on λ that sit over P [DJ]. The relation between this construction and the construction of toric varieties from the normal fan $\Sigma(P)$ associated to P can be found in [BP4] 5.1.3. Let $F(p)$ be the* *unique face of* P *which contains the point* p *in its interior. Suppose we have an action of the torus on* P*. Define an equivalence relation* $\sim$ *on* $T^n \times P^n$ *by* $(g,p) \sim (h,q) \Leftrightarrow p = q$ *and* $g^{-1}h \in G_{F(p)}$*. Denote the resulting quotient space by* $M^{2n}(\lambda)$*. It is shown in [DJ] that this space is a Quasi-toric manifold.*

As λ varies so does the manifold that sits over P. It has been shown that the homology groups of $M^{2n}(\lambda)$ are independent of λ and in fact are a function of the $\overrightarrow{h}$ associated to P [DJ]. Later, we will show that the homotopy groups are independent of λ leading to many interesting observations.

We assume that $M^{2n}(\lambda)$ is a Quasi-toric manifold over the simple convex polytope P. For a set $X \neq \phi$ let $F_{ab}(X)$ denote the free abelian group on X. Let X_j be a set such that $|X_j| = h_j(P)$ for $0 \leq j \leq n$

Theorem 2.2.12. $H_{2j}(M^{2n}(\lambda)) = F_{ab}\langle X_j\rangle$

Proof. See [DJ]. □

This gives the

Corollary 2.2.13. $H_{2j+1}(M^{2n}(\lambda)) = 0$

The components of the $\overrightarrow{h}$ associated to P satisfy the Dehn-Sommerville relations. This duality manifests itself in the computation above. Once the $\overrightarrow{h}$ of P is determined the homology groups of $M^{2n}(\lambda)$ can be determined. In order to describe the co-homology ring of $M^{2n}(\lambda)$ we need to define an ideal $< J >$ that comes from the λ map. Let $\lambda : \mathbb{Z}^m \to \mathbb{Z}^n$. The matrix of this map produces an $(n \times m)$ matrix with integer coefficients.

$$M = \begin{pmatrix} a_{11} & & \cdots & a_{1m} \\ & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & & \cdots & a_{nm} \end{pmatrix}$$

Let V be the matrix of inderminants of the face ring of P.

$$V = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$$

Let $\lambda_i = \lambda_{i1}v_1 + \cdots + \lambda_{im}v_m$ be the linear forms that come from the multiplication: $M \cdot V = 0$.

Definition 2.2.14. *Let J be the homogeneous ideal in $\mathbb{Z}[v_1, .., v_m]$ generated by the λ_i for $1 \leq i \leq n$.*

If P_K is a simple polyhedral complex then the ideal J is defined in the same way as long as a toric space (or perhaps) manifold exists.

[DJ] determine the cohomology ring of $M^{2n}(\lambda)$ without reference to the algebraic geometry. It happens that this ring is isomorphic to the Chow ring.

Theorem 2.2.15. *Let $M^{2n}(\lambda)$ be a Quasi-toric manifold over the simple convex polytope P. Then*

$$H^*(M^{2n}(\lambda)) \cong \mathbb{Z}[v_1, .., v_m]/(I + J)$$

Proof. See [DJ]

□

This theorem does not necessarily hold for general P_K without imposing some combinatorial restrictions on K, specifically K must be Cohen-Macaulay. This is in addition to the existence of a λ that satisfies condition $(*)$ [DJ].

2.3 Examples of Quasi-Toric Manifolds

The following examples can be found in [OR] and [BR2].

$\mathbb{C}P^n$ is a quasi-toric manifold over Δ^n. $\mathbb{C}P^i \times \mathbb{C}P^j$ is a quasi-torc manifold over $\Delta^i \times \Delta^j$.

$CP^n \sharp CP^n$ is a quasi-toric manifold over $\Delta^1 \times \Delta^{n-1}$.

$S^2 \times S^2$ is a quasi-toric manifold over $\Delta^1 \times \Delta^1$.

Definition 2.3.1. *A bounded flag in* $\mathbb{C}^{n+1}$ *is a complete flag* $U = \{U_1 \subset U_2 \subset \cdots U_{n+1} = \mathbb{C}^{n+1}\}$ *for which* U_k *for* $2 \leq k \leq n$ *contains the coordinate subspaces* $\mathbb{C}^{k-1}$ *spanned by the first* $k-1$ *basis vectors.*

We have the following theorem from [BR2]

Theorem 2.3.2. *The* $2n$*-dimensional manifold,* B_n *of all bounded flags in* $\mathbb{C}^{n+1}$ *is a Quasi-toric manifold over* I^n*.*

Buchstaber and Ray construct another interesting family of quasi-toric manifolds.

Definition 2.3.3. *The manifold* $B_{i,j}$ *consists of pairs* (U, W) *where* U *is a bounded flag in* $\mathbb{C}^{n+1}$ *and* W *is a line in* $U_1^{\perp} \oplus \mathbb{C}^{j-1}$*.*

Theorem 2.3.4. *$B_{i,j}$ is a quasi-toric manifold over $I^i \times \Delta^{j-1}$.*

Proof. See [BR2]

□

2.4 Classification of Quasi-Toric manifolds

In [Dob] Quasi-toric manifolds that sit over products of simplexes are classified. [KL] classifies smooth projective varieties that have few generators. In this section we list for the convenience of the reader the known classifications of Quasi-toric manifolds. In [OR] the classification of 4 dimensional Quasi-toric manifolds that sit over m-gons is given by the following theorem which we do not prove.

Theorem 2.4.1. *A four dimensional Quasi-toric manifold, M^4 is diffeomorphic to the connected sum of copies of CP^2, $\overline{CP^2}$ and $S^2 \times S^2$*

To illustrate we have the following example of 4 dimensional Quasi-toric manifolds that sit over the square. They are:

Example 2.4.2.

$$M^4(\lambda) = \begin{cases} S^2 \times S^2 \\ CP^2 \sharp CP^2 \\ \overline{CP^2} \sharp \overline{CP^2} \\ CP^2 \sharp \overline{CP^2} \end{cases}$$

Example 2.4.3. *The Quasi-toric manifolds that sit over the pentagon are connected sums of the Quasi-toric manifolds that sit over the square with the Quasi-toric manifold that sits over the simplex. They take the following form*

$$M^4(\lambda) = \begin{cases} (S^2 \times S^2)\sharp CP^2 \\ (CP^2\sharp CP^2)\sharp CP^2 \\ (\overline{CP^2}\sharp\overline{CP^2})\sharp CP^2 \\ (CP^2\sharp\overline{CP^2})\sharp CP^2 \end{cases}$$

2.5 Toric Spaces and the Universal Toric Space

The proofs of the theorems in this section can be found in [DJ]. Let K be an $(n-1)$-dimensional simplicial complex and P_K its dual. Toric spaces exist over P_K if a map $\lambda : Ver(K) \to \mathbb{Z}^n$ which satisfies condition $(*)$ can be constructed. However, the resulting space is not necessarily a manifold. The existence of a Toric space gives the existence of a non-degenerate simplical map from K to the simplicial complex which [DJ] call K^n. This simplicial complex dualizes to a simple polyhedral complex U^n such that the toric space that sits over it is universal in a way that is analogous to the bundle γ over BU being universal. Many of the theorems that hold for Quasi-toric manifolds do not necessarily hold for toric spaces unless one puts certain resrtictions on the simplicial complex K (Cohen-Macaulyness). For example, the determination of the homology groups of M^{2n} given by [DJ] depended on a morse theoretic argument which relied heavily on P being a simple convex polytope. Following [DJ], let U^n be the simple polyhedral complex dual to K^n defined in section 4. [DJ] prove that there exists a toric space-Y^n over U. The next theorem describes how the T^n-space Y^n is universal.

Theorem 2.5.1. *Let P_K be the n-dimensional simple polyhedral complex dual to K. Any T^n-space $Y \longrightarrow P_K$ is equivalent to the pullback of the universal T^n-space $Y^n \longrightarrow U^n$ via some non-degenerate simplicial map $f : K \longrightarrow K^n$. In fact, there is a bijection between the set of equivalence classes of T^n-spaces over P_K and the set of non-degenerate simplicial maps $f : K \longrightarrow K^n$ modulo the natural action of $Aut(T^n)$.*

If $f : K \longrightarrow K^n$ is a non-degenerate simplcial map then the previous theorem can be summarized by the following commutative diagram.

$$\begin{tikzcd}
\overline{f}^*(Y^n) \cong Y \arrow[r] \arrow[d] & Y^n \arrow[d] \\
P_K \arrow[r, "\overline{f}"'] & U^n
\end{tikzcd}$$

The next Theorem from [DJ] shows that the complex K^n is Cohen-Macaulay.

Theorem 2.5.2. *The complex K^n is $(n-1)$-dimensional and $(n-2)$-connected. For each i simplex $\sigma \in K^n$ the $Link(\sigma, K^n)$ is $(n-i-2)$-dimensional and $(n-i-3)$ connected*

Proof. See [DJ] □

We also have the

Theorem 2.5.3. *Let* $k = Q$ *and let* K *be an* $(n-1)$*-dimensional Cohen-Macaulay complex over* K*,* P_K *its dual and* $Y \to P_K$ *a* T*-space. The co-homology ring of* Y *is given by*

$$H^*(Y;k) \cong k[v_1, .., v_m]/\langle I + J\rangle$$

Proof. [DJ]. $\square$

Chapter 3

Moment Angle Complexes

3.1 Introduction

In this section we define moment angle complexes Z_P for simple convex polytopes and more general combinatorial objects, simple polyhedral complexes. Results on the cohomology of these spaces will be listed. For more the reader should refer to [BP1], [BP3], [BP4] and [DJ].

Remark 3.1.1. *We assume that* P *is a simple polyhedral complex.*

The coordinate torus $T^k_{\iota_1,..,\iota_k}$ is a product of Torii that come from the factors $\iota_1, .., \iota_k$ in T^m. In [DJ] and [BP1] it was shown that in the case of a general simple polyhedral complex P there is a canonical way to construct Z_P. Define an equivalence relation on $T^m \times P^n$. $(g, p) \sim (h, q)$ if and only if $gh^{-1} \in T^k_{\iota_1,..,\iota_k}$ where $p \in int(F_{\iota_1} \bigcap \cdots F_{\iota_k})$.

Definition 3.1.2. *Given P, define the moment angle complex*

$$Z_P = T^m \times P / \sim$$

where $(g,p) \sim (h,q) \Leftrightarrow p = q$ *and* $g^{-1}h \in G_{F(p)}$

In other words the following diagram commutes

$$\begin{tikzcd}
T^m \times P^n \arrow[r, "\varphi"] \arrow[d] & P^n \arrow[d] \\
T^m \times P^n / \sim \arrow[r, "\overline{\varphi}"] & P^n
\end{tikzcd}$$

where $\overline{\varphi}((\overline{p,q})) = \varphi((p,q))$ where $(p,q) \in \overline{(p,q)}$. It can be shown that $\overline{\varphi}$ is well defined. The reader is urged to compare this definition with the construction of Quasi-toric manifolds (or spaces) via a characteristic pair given in the previous chapter. The construction of moment angle complexes is independent of the existence of a λ that satisfies condition $(*)$. If P is a simple convex polytope then Z_P is a $(m+n)$-dimensional manifold. In fact [BP1] prove the following statement.

Proposition 3.1.3. *If P is an n dimensional simple convex polytope then Z_P is a smooth $m+n$-dimensional manifold.*

We have the following important

Proposition 3.1.4. *If* $P = Q \times W$ *then* $Z_P \cong Z_Q \times Z_W$

This result is particularly important when P is a product of simplicies.

It is shown in [BP4] that Z_K is a manifold when K is a simplicial sphere i.e, $|K| \approx S^{n-1}$. Characterizing the class of simplicial complexes whose associated moment angle complexes are manifolds is still an open question.

Remark 3.1.5. *Given a simplicial complex K there exists a moment angle complex Z_K [DJ], [BP4].*

3.2 Operations on Moment Angle Complexes

Given two simplicial complexes K_1 and K_2 on $[m_1]$ and $[m_2]$ we can form another simplicial complex K_J by performing the join operation on K_1 and K_2. When one of the simplicial complexes is Δ^0 we called this operation the cone over the simplicial complex, denoted $con(K_2)$. Operations on this level correspond to specific operations on the moment angle complexes which sit over these complexes. The next construction will describe how a moment angle complex behaves with respect to simplicial complexes that can be decomposed as the join of two simplicial complexes. Specifically we have:

Proposition 3.2.1. *Given K_1 and K_2 as above we have $Z_{K_1 * K_2} \approx Z_{K_1} \times Z_{K_2}$*

Proof. See [BP4] □

The interesting case arises when $K_2 \approx \Delta^0$. In this case we have

$$Z_{K_1 * \Delta^0} \approx Z_{K_1} \times S^1$$

There are other operations such as connected sum and bi-stellar moves on the level of simplicial complexes that correspond to equivariant surgery operations on the resulting moment angle complexes. For more on this see [BP4] page 93.

3.3 The Co-homology Ring of Moment Angle Complexes and Resolutions of the Face Ring

Throughout this section we assume R is a commutative ring with unit. We will give a comprehensive listing of all the theorems and machinery used to describe the cohomology ring of the moment angle complex Z_P when P is a simple convex polytope and $H^*(Z_K)$ when K is a simplicial complex on $[m]$. All of this material can be found in [BP4]. We list the material here for the convenience of the reader. We start with the Koszul resolution.

Let $\bigwedge = \bigwedge[u_1,..,u_m]$ be an exterior algebra and let $P = k[v_1,..,v_m]$ where k is a field. We can replace k by $\mathbb{Z}$ for more detail see [BBP]. Let $R = \bigwedge \otimes P$. R becomes a differential bi-graded algebra if we make the following agreements:

bi-degree $u_i = (-1, 2)$ and bi-degree $v_i = (0, 2)$

We will define the differential d in the following way, $d(v_i) = 0$ and $d(u_i) = v_i$ requiring d to be a derivation of algebras. Let $\bigwedge^i[u_1,..,u_m]$ be the submodule of $\bigwedge$ generated by monomials of length i. Let $R^{-i} = \bigwedge^i[u_1,..,u_m] \otimes P$. Clearly, R is a P module. We obtain the following resolution of k by free P- modules.

$$0 \longrightarrow R^{-m} \longrightarrow R^{-(m-1)} \longrightarrow ...... \longrightarrow R^{-1} \longrightarrow P \xrightarrow{\epsilon} k \longrightarrow 0$$

The augmentation $\epsilon : P \to k$ is defined by $\epsilon(v_i) = 0$. If N is a P module we obtain a complex of graded modules by tensoring the complex above by $- \otimes_P N$. It is important to note that the complex is a cochain complex by the indexing.

$$0 \longrightarrow R^{-m} \otimes_P N \longrightarrow ..... \longrightarrow R^{-1} \otimes_P N \longrightarrow N \longrightarrow 0$$

taking homology we have $Tor_P^{-i}(M, N)$ Specifically,

$$Tor_P^{-i}(M, N) = \frac{Ker[d{:}R^{-i} \otimes N \to R^{-i+1} \otimes N]}{d(R^{-i-1} \otimes N)}$$

Since N and R^{-i} are graded modules we have:

$Tor_P^{-i}(M, N) = \bigoplus_j Tor_P^{-i,j}(M, N)$ where

$$Tor_P^{-i,j}(M, N) = \frac{Ker[d : (R^{-i} \otimes N)^j \to (R^{-i+1} \otimes N)^j]}{d(R^{-i-1} \otimes N)^j}$$

To get a handle on the multiplicative structure of $H^*(Z_P)$ the resolution above must be used. When P is a simple convex polytope one can use poincare duality to derive some interesting combinatorial results see [BP4] Thm 7.17 as well as pages 112 and 113. However, if one were interested in knowing the co-homology additively then the minimal resolution of the face ring would be sufficient. It is also true that the bigraded betti-numbers of Z_K denoted by $b^{-i,2j}$ correspond to the bi-graded betti-numbers of the face ring $k(K)$ which is the dimension of Tor as a free P-module [BP4]. In fact we have the

Definition 3.3.1. *For any simplicial complex K, the bi-graded betti numbers of $k(K)$,*

$\beta^{-i,2j}(k(K)) = dim_P \, Tor_P^{-i,2j}(k(K), k)$

In [BP1] pg. 32 a sub-complex A of the co-chain complex $[k(P) \otimes \bigwedge, d]$ was defined. As a k-module it is generated by the monomials $v_{\iota_1} \cdots v_{\iota_p} \otimes u_{j_1} \cdots u_{j_q}$ such that $\{v_{\iota_1} \cdots v_{\iota_p}\}$ span a simplex in K_P and $\{\iota_1, .. \iota_p\} \bigcap \{j_1, .., j_q\} = \emptyset$. It was shown that A inherits a bi-graded module structure from $k(P) \otimes \bigwedge$. In [BP4] another co-chain complex $C^*(Z_K)$ was constructed and shown to be isomorphic to A as differential graded algebras. See [BP4] page 105. Specifically, we have

Theorem 3.3.2. *For any simplcial complex K, $b_{-q,2p}(Z_K) = \beta^{-q,2p}(k(K))$*

Proof. [BP4] □

The ordinary betti numbers of Z_K are given by $b_k(Z_K) = \sum_{-q+2p=k} b_{-q,2p}(Z_K)$ for $k = 0, .., (m + n)$. We need the

Lemma 3.3.3. *Let K be any $(n-1)$-dimensional simplicial complex on $[m]$ and let Z_K be the corresponding $(m+n)$-dimensional moment angle complex associated to K. Then*

1. $b_{0,0}(Z_K) = b_0(Z_K) = 1$

2. $b_{0,2p}(Z_K) = 0$ *for* $p > 0$

3. $b_{-q,2p} = 0$ *for* $p > m$ *or* $q > p$

4. $b_1(Z_K) = b_2(Z_K) = 0$

5. $b_3(Z_K) = b_{-1,4}(Z_K) = \binom{f_0}{2} - f_1$

6. $b_{-q,2p}(Z_K) = 0$ *for* $q \geq p > 0$ *or* $p - q > n$

7. $b_{m+n}(Z_K) = b_{-(m-n),2m}(Z_K)$

Proof. See [BP4] page 107. □

To understand how the notion of relations among relations fits into the resolution of the face ring we need to review free resolutions of a module of the sort M/I. This material is standard and we take it from [Sp] and [BP4]. Let M be a module on $\{x_1, .., x_n\}$. The free resolution of M as a P-module is the following chain complex:

$$0 \longrightarrow R^{-n} \longrightarrow R^{-n+1} \longrightarrow \cdots \longrightarrow R^{-1} \longrightarrow R^{0} \longrightarrow 0$$

The differentials d are degree preserving. We should note that for each i, R^{-i} is the free P-module on *ker*: $R^{-i+1} \to R^{-i+2}$. We know that Tor is independent of the resolution. If we specialize to the minimal resolution of the face ring the exact sequence above takes the following form.

$$0 \longrightarrow R^{-k} \longrightarrow \cdots \longrightarrow R^{-1} \longrightarrow \mathbb{Z}[x_1, .., x_m] \longrightarrow \mathbb{Z}[x_1, .., x_m]/I \longrightarrow 0$$

As finitely generated P-modules we have R^{-1} is generated by $Ker : \mathbb{Z}[x_1, .., x_m] \to \mathbb{Z}[x_1, .., x_m]/I$

R^{-2} is generated by $Ker : R^{-1} \to \mathbb{Z}[x_1, .., x_m]$ and so on...

It turns out that R^{-1} has generators whose image in $\mathbb{Z}[x_1, .., x_m]$ are the monomials that generate I. This resolution will be important in defining the notion of relation among relations.

In the case that P is a simple convex polytope, $k(P)$ is a Cohen-Macaulay complex and the Hilbert-Syzygy Theorem provides an upper bound for the homological dimension of $k(P)$. In this case $-k = m - n$.

For the moment we will assume that P is an dimensional simple convex polytope and Z_P is the associated $(m + n)$ dimensional manifold. In this case we the following from [BP4].

Theorem 3.3.4. *The following isomorphism of algebras holds:*

$$H^*(Z_K) \approx Tor_{k[v_1,..,v_m]}(k(K), k)$$

Proof. See [BP4] □

Remark 3.3.5. *The theorem above tells us that we have the following isomorphism additively:*

$$H^p(Z_K) \approx \textstyle\sum_{-i+2j=p} Tor^{-i,2j}_{k[v_1,..,v_m]}(k(K), k)$$

By using some homological algebra they obtain the following

Theorem 3.3.6. *The following isomorphism of bi-graded algebras holds:*

$$H^{*,*}(Z_K) \approx H[\bigwedge[u_1, .., u_m] \otimes k(K), d]$$

where $d(u_i) = v_i$, $d(v_i) = 0$ *and bi-degree* $v_i = (0, 2)$, *bi-degree* $u_i = (-1, 2)$

Proof. See [BP4]

□

When P is an m-gon such that $m \geq 4$ the co-homology ring of Z_P is given by

Theorem 3.3.7. *The only non-zero bi-graded co-homology groups of* Z_P *are*

1. $H^{0,0}$, $H^{-p,2(p+1)}$, *for* p = *1,..,*$m - 3$ *and* $H^{-m+2,2m}$
2. $H^{-p,2(p+1)}$ *is free and is generated by the classes* $[v_i u_\tau]$ *such that* $|\tau| = p$, $i \notin \tau$ *and* $i + 1 \notin \tau$. *These classes satisfy the relation:* $d(u_{\tau'})$ *=0 for* $|\tau'| = p + 1$
3. $H^{-m+2,2m}$ *is generated by the class* $[v_1 v_2 u_3 \cdots u_m]$
4. *The product of two classes* $[v_{i_1} u_{\tau_1}] \in H^{-p_1,2(p_1+1)}$ *and* $[v_{i_2} u_{\tau_2}] \in H^{-p_2,2(p_2+1)}$ *equals* $[v_1 v_2 u_3 \cdots u_m]$ *up to sign if* $\{i_1, i_2, \tau_1, \tau_2\}$ *is a partition of* $[m]$ *and zero otherwise.*

Proof. See [BP4] □

The co-homology of Quasi-toric manifolds and moment angle complexes are related by the following

Theorem 3.3.8. *Let* K *be a Cohen-Macaulay complex and* J *an ideal in* $k(K)$ *generated by the regular sequence* $(\lambda_1, .., \lambda_n)$. *Then we have the following isomorphism of* *algebras:*

$$H^*(Z_K) \approx Tor_{k[v_1,..,v_m]/J}(k(K)/J,k)$$

The reader should keep in mind that $k(K)/(J) \cong H^*(M^{2n}(\lambda))$. The next theorem describes how the fundamental class of the $(m+n)$-dimensional manifold Z_K is represented when K is an $(n-1)$- dimensional simplicial sphere. For an explicit description of the module $A^*(K)$ the reader should refer to [BP1] and [BP4].

Theorem 3.3.9. *The fundamental class of Z_K is represented by any monomial $\pm v_\sigma u_\tau \in A^*(K)$ of bi-degree $(-(m-n),2m)$ such that σ is an $(n-1)$ simplex of K and $\sigma \cap \tau = \phi$. The sign is depends on a choice of orientation for Z_K*

Proof. See [BP4] □

This gives the

Corollary 3.3.10. *Poincare duality for the moment angle manifold Z_K corresponding to a simplicial sphere K^{n-1} respects the bigraded structure in the co-homology*

$$H^{-q,2p}(Z_K) \cong H_{-(m-n)+q,2(m-p)}(Z_K)$$

In particular we have the following:

$$b_{-q,2p} = b_{-(m-n)+q,2(m-p)}$$

Chapter 4

The Borel Space

4.1 Introduction

In this chapter the borel space B_TP will be defined as well as a decomposition of Z_P into T^m invariant blocks. A unified combinatorial definition of B_TP and Z_P due to Strickland will be discussed.

4.2 The Block Decomposition and the Strickland Construction

The manifolds Z_P enjoy some very nice geometrical descriptions. [BP1] describe a block decomposition of Z_P that depends on P being a simple convex polytope. In simple cases this allows for Z_P to be explicitly determined for certain P. Neil Strickland has formulated a very general combinatorial description which describes various spaces related to both quasi-toric manifolds and Moment angle complexes by glueing together disks and circles. His construction describes a wider class of moment angle complexes. (Those associated to a simplicial complex K). Historically, the block decomposition of Z_P was developed by [BP1]. So we list this proposition first. It depended on viewing P as a manifold with corners. In such a case it is possible to decompose Z_P into T^m invariant blocks. We have the

Proposition 4.2.1. *$Z_P = \bigcup_{v \in P} B_v$ where $B_v = (D^2)^m \times T^{m-n}$*

Proof. See [BP1]

□

Using this construction it was shown in [BP1] that $Z_{\triangle^n} \cong S^{2n+1}$. If P is a simple convex polytope and K_P its dual simplicial complex then Z_P can be identified with Z_{K_P} see [BP4]. Let K be a simplicial complex on $[m]$. Suppose X is a space and W a subspace of X. The Strickland construction is as follows.

$$K_\bullet(X,W)= \bigcup_{\sigma \in K} (\prod_{i \in \sigma} X \times \prod_{i \notin \sigma} W)$$

For any simplicial complex K we have

$$Z_K = K_\bullet(D^2, S^1)$$

This description of the moment angle complex associated to K works regardless of the combinatorics of K. One may decompose Z_K into a certain union of disks and circles eventhough K may not be a simplicial sphere. It should be noted that Z_P is not a manifold for any simple polyhedral complex P. Eventhough there is a decomposition of Z_K into unions of circles and disks it is not at all clear what the homotopy type of Z_K is. One can not determine whether or not Z_K is a manifold just by applying the Strickland construction.

4.3 The Borel Construction and DJ(K)

The purpose of this section is to describe a topological space which can always be associated to a simple polyhedral complex independent of the existence of a quasi-toric manifold. This space B_TP depends only on the combinatorial structure of P and gives rise to various interlocking fibrations. When P is dual to a simplicial sphere these fibrations are particularly important in relating the homotopy groups of quasi-toric manifolds to Z_P. The geometry of the Borel space is interesting in its own right and will play an important role in explicitly writing down some non-intuitive fibrations. We will also define the Davis-Januszkiewicz space.

Definition 4.3.1. *Let γ be the universal bundle over the classifying space BT^m*

The space BT^m is equivalent to $\prod_m \mathbb{CP}^\infty$ in the homotopy category. We form the associated T^m-bundle of γ. This space is contractible and we call it ET^m. We say a space X is a T^m-space if we have a map $T^m \times X \to X$.

Definition 4.3.2. *Given a T^m-space X the Borel space is the identification space*

$$ET^m \times X/\sim\; = ET^m \times_{T^m} X$$

where the equivalence relation is defined by: $(e,x) \sim (eg, g^{-1}x)$ for any $e \in ET^m$ and $x \in X$, $g \in T^m$

Remark 4.3.3. *There is no ambiguity in writing B_TP instead of B_TX-the borel construction applied to a space. B_TP refers to $ET^m \times X/\sim$ which has the homotopy type of P when P is a simple convex polytope [DJ].*

Sometimes one refers to this as applying the borel construction to X. One nice property of this definition is that it allows for one to move the torus action into the quotient. Another very important characteristic of the borel space is the existence of the fibration:

$$X \longrightarrow ET^m \times_{T^m} X \longrightarrow BT^m$$

If K is an $(n-1)$-dimensional simplicial complex on $[m]$ then the borel construction applied to the corresponding moment angle complex gives the space $B_TZ_K = ET^m \times_{T^m} Z_K$.

Remark 4.3.4. *T^m acts on itself diagonally. Therefore, by 4.3.2 this construction can be made.*

If one were to apply the Strickland construction then

$$B_T Z_K = K_{\bullet}(ES^1 \times_{S^1} D^2, ES^1 \times_{S^1} S^1)$$

These observations allow for the borel construction to be applied to moment angle complexes associated to a simple polyhedral complex P as well quasi-toric manifolds if P is a simple convex polytope. The space $B_T P$ is not an H-space. This will cause come complication in the bookkeeping in the sections that follow. For P a simple polyhedral complex one can construct a canonical topological space Z_P. Clearly, it is endowed with a torus action. Here the orbit space of the action is P. Clearly, by the definition of $B_T X$ it is easy to see that the Borel construction only depends on the combinatorial type of P. More importantly we have the following fibration:

$$Z_P \longrightarrow B_T Z_P \longrightarrow BT^m$$

In the case that K is a general $(n-1)$ dimensional simplicial complex on $[m]$ and assuming that a T^n-space exists; the fibration above takes the form:

$$M_{P_K} \longrightarrow B_T M_{P_K} \longrightarrow BT^n$$

Notice that the number of factors of $\mathbb{CP}^\infty$ space depends on the dimension of the torus acting on the space. Eventhough toric spaces and their counterparts quasi-toric manifolds depend on the construction of a particular λ map the Borel space is independent of the choice of λ and only depends on the combinatorics of P. In the case that P is a simple convex polytope, both Z_P and M^{2n} are manifolds. In this case we have the following collection of interlocking fibrations:

$$T^{m-n} \longrightarrow Z_P \longrightarrow M^{2n}$$

$$M^{2n} \longrightarrow B_T M^{2n} \longrightarrow BT^n$$

and

$$Z_P \longrightarrow B_T Z_P \longrightarrow BT^m$$

The next space plays an important role in augmenting the last fibration. From [BP4] we have the following notation. If $\sigma \in K$ and $|\sigma| = k$ let $BT_\sigma \cong BT^k$. We are associating a particular coordinate Torii to a simplex σ such that the coordinates depend on the sub-scripts of the vertices in σ.

Definition 4.3.5. *Let K be an $(n-1)$-dimensional simplicial complex on $[m]$. Denote by $DJ(K)$ the following space*

$$\bigcup_{\sigma\in K} BT_\sigma \subset BT^m$$

In terms of the Strickland construction we have

$$DJ(K) = K_\bullet(\mathbb{C}P^\infty, *)$$

The space $DJ(K)$ can be viewed as a certain homotopy co-limit of diagrams. The Davis-Janususzkiewicz space fits into the language of model categories. This approach is examined in [PRV].

For any simplicial complex K, Z_K is the fiber of the map $p : B_T Z_K \to BT^m$. The following theorem describes how the moment angle complex and the space $DJ(K)$ are related.

Theorem 4.3.6. *For any simplcial complex* K *there is a deformation retraction* $B_T Z_K \to DJ(K)$ *such that the following diagram commutes*

$$\begin{tikzcd}
B_T Z_K \arrow[r, "p"] \arrow[d] & BT^m \arrow[d, "="'] \\
DJ(K) \arrow[r, "\iota"] & BT^m
\end{tikzcd}$$

Proof. See [BP4]

□

4.4 The Co-homology Ring of the Borel Construction

Davis and Januszkiewicz proved that the co-homology ring of the Borel space associated to any simple polyhedral complex is the face ring associated to P. It is not always the case that the fibrations above exist. In fact, if P is not a simple convex polytope a toric manifold or quasi-toric space may not exist. However, for any simplcial complex

K the fibration

$$Z_K \longrightarrow B_T Z_K \xrightarrow{p} BT^m$$

exits. We have the following

Theorem 4.4.1. *The projection* $p : B_T Z_K \to BT^m$ *induces the quotient epimorphism* $p^* : \mathbb{Z}[v_1, .., v_m] \to k(K)$.

Proof. [BP4]

□

In case that K_P is dual to a simple polyhedral complex P we have

Theorem 4.4.2. $p^* : H^*(BT^m) \to H^*(B_T P)$ *is a surjection and* $H^*(B_T P) \approx k(P)$ *as graded rings.*

Proof. See [DJ]

□

In the case that P is a simple convex polytope and k is a field we have the fibration $M^{2n} \to B_T P \to BT^n$ and the theorem

Theorem 4.4.3. *The map* $p_0^* : H^*(BT^n) \to H^*(B_T P)$ *is a monomorphism and* $p_0 : H^2(BT^n) \to H^2(B_T P)$ *coincides with* $\lambda^* : k^n \to k^n$. *After the identification* $H^*(BT^n) \cong k[t_1, ..t_n]$, *the elements* $\lambda_i = p_0^*(t_i) \in H^*(B_T P) \cong k(P)$ *form a regular sequence of degree two elements of* $k(P)$.

Proof. See [BP1]

□

Chapter 5

Subspace Arrangement Complements

In this chapter we list the most important definitions and constructions found in [BP4], [BP5] concerning complex coordinate subspace arrangement complements.

5.1 Definition and relation to the Moment Angle Complex

From a homotopy theoretic point of view the key fact in this section describes $U(K)$ as a deformation retract of Z_K.

Definition 5.1.1. *An arrangement* $A = \{L_1, .., L_j\}$ *is coordinate if each* L_i *is coordinate.*

Let $\sigma = \{\iota_1, .., \iota_k\} \subset [m]$

Definition 5.1.2. *A coordinate subspace of* $\mathbb{C}^m$ *is* $L_\sigma = \{(z_1, .., z_m) \in \mathbb{C}^m | z_{\iota_1} = \cdots = z_{\iota_k} = 0\}$

The dimension of $L_\sigma = m - |\sigma|$. Suppose K is a simplicial complex on $[m]$. We now have

Definition 5.1.3. *The complex coordinate subspace arrangement is*

$$CA(K) = \{L_\sigma | \sigma \notin K\}$$

leading to the

Definition 5.1.4. *The complex coordinate subspace arrangement complement is*

$$U(K) = \mathbb{C}^m \backslash \bigcup_{\sigma \notin K} L_\sigma$$

In [BP4] it is shown that there is a bi-jection bewteen the set of simplcial complexes on $[m]$ and the set of coordinate subspace arrangement complements in $\mathbb{C}^m$. Let $\mathbb{C}^*$ denote $\mathbb{C}\backslash\{0\}$. In terms of the Strickland construction we have $U(K) = K_\bullet(\mathbb{C}, \mathbb{C}^*)$. We list two examples from [BP4].

Example 5.1.5. *If* $K = \partial\Delta^{m-1}$ *then* $U(K) = \mathbb{C}^m \backslash \{0\}$

and

Example 5.1.6. *If* K_P *is dual to an* m*-gon then*

$$U(K) = \mathbb{C}^m \backslash \bigcup_{i-j \neq 0,1 \; mod \; (m)} \{z_i = z_j = 0\}$$

The following results are important.

Theorem 5.1.7. *$Z_K \subset U(K)$.*

Proof. See [BP4]

□

and

Theorem 5.1.8. *There is an equivariant deformation retraction $U(K) \to Z_K$.*

Proof. See [BP4] □

In particular, any information concerning the homotopy of Z_K gives information about the homotopy of $U(K)$.We also have the following result on the co-homology ring of $U(K)$

Theorem 5.1.9. *For any simplicial complex K on $[m]$ the following isomorphism of graded algebras holds:*

$$H^*(U(K)) \cong Tor_{k[v_1,.,v_m]}(k(K), k)$$

Proof. See [BP4] □

The Tor-algebra $Tor_{k[v_1,.,v_m]}(k(K), k)$ is the co-homology ring of the moment angle complex associated to K. For more information and examples see [BP4] chapter 8 and [BP5].

Chapter 6

Unstable Homotopy Theory

The purpose of this chapter is to list the relevant theorems and constructions from unstable homotopy theory that will be used in the sequel. The proofs can be found in the papers of [BCR], [BCM] and [Bous]. The interested reader should refer to them for a more complete exposition and reference list.

6.1 Unstable G-Coalgebras

Our primary focus in this section will be a rather comprehensive discussion on co-triples in their most general sense and their role in defining right derived functors. This will allow for a general description of non-abelian right derived functors i.e., those right derived functors which come from the models in the category *M(G)*. First, the generalities.

We assume that Co-algebras are over the ring BP_*

Let C be a category.

Definition 6.1.1. *A cotriple* (G, δ, ϵ) *consists of a covariant functor* $G : C \to C$ *and natural transformations* $\delta : G \to G^2$ *and* $\epsilon : G \to I$ *such that the following diagrams commute:*

$$\begin{tikzcd}
G & G^2 \arrow[l, "G\epsilon"'] & G \arrow[l, "\epsilon G"'] \\
G \arrow[u, equal] \arrow[r, equal] & G \arrow[u, "\delta"'] \arrow[r, equal] & G \arrow[u, equal]
\end{tikzcd}$$

and

$$\begin{tikzcd}
G^3 & G^2 \arrow[l, "G\delta"'] \\
G^2 \arrow[u, "\delta G"'] & G \arrow[l, "\delta"'] \arrow[u, "\delta"']
\end{tikzcd}$$

Given a co-triple G we can define a G-co-algebra

Definition 6.1.2. *A* G*-co-algebra is an object* $C \in C$ *endowed with a map* $\psi : C \to G(C)$ *such that the following diagrams commute.*

$$\begin{tikzcd}
C \arrow[r, "\psi"] \arrow[d, equal] & G(C) \arrow[d, "\epsilon"] \\
C \arrow[r, equal] & C
\end{tikzcd}$$

and

$$\begin{tikzcd}
G^2(C) & G(C) \arrow[l, "\delta"'] \\
G(C) \arrow[u, "G(\psi)"'] & C \arrow[l, "\psi"'] \arrow[u, "\epsilon"']
\end{tikzcd}$$

In other words, a map of G-co-algebras is a map $f : C \to D$ such that the following diagram commutes.

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "\psi"] & D \arrow[d, "\psi"] \\
G(C) \arrow[r, "G(f)"] & G(D)
\end{tikzcd}$$

A map of G-co-algebras is a map which is compatible with the G co-algebra structure map. The category *M(G)* consists of pairs (C, ψ) where $C \in C$ and ψ is the G-Co-algebra structure map. It is important to note that objects of the form $G(C)$ have a canonical G-structure defined on them. Specifically, $\delta : G(C) \to G^2(C)$.

Definition 6.1.3. *Let $C \in M(G)$. We call objects of the form $(G(C), \delta)$ the models in M(G).*

Later, we will form unstable resolutions of G co-algebras. We will use the fact that the adjoint of the co-triple (G, δ, ϵ) gives rise to a triple (G, μ, η). Specifically, we have the functor $G : C \to C$ and natural transformations $\mu : G^2 \to G$ and $\eta : I \to G$ such that following diagrams commute:

$$\begin{tikzcd}
G \arrow[r, "G\eta"] \arrow[d, equal] & G^2 \arrow[d, "\delta"] & G \arrow[l, "\eta G"'] \arrow[d, equal] \\
G \arrow[r, equal] & G \arrow[r, equal] & G
\end{tikzcd}$$

and

$$\begin{tikzcd}
G^3 \arrow[r, "G\eta"] \arrow[d, "\mu G"] & G^2 \arrow[d, "\mu"] \\
G^2 \arrow[r, "\mu"] & G
\end{tikzcd}$$

Definition 6.1.4. *Let D be a category. A co-simplicial object $\mathbf{X}$ over D consists of*

1. *for every integer $n \geq 0$ an object $\mathbf{X}^n \in D$*
2. *for every pair of integers (i, n) with $0 \leq i \leq n$ coface and co-degeneracy maps $d^i : \mathbf{X}^{n-1} \to \mathbf{X}^n$ and $s^i : \mathbf{X}^{n+1} \to \mathbf{X}^n$ satisfying the co-simplicial identities:*

$$\begin{aligned}
s^j s^i &= s^{i-1} s^j, i > j \\
d^j d^i &= d^i d^{j-1}, i < j \\
s^j d^i &= d^i s^{j-1}, i < j \\
&= id, i = j, i = j+1 \\
&= d^{i-1} s^j, i > j+1
\end{aligned}$$

Definition 6.1.5. *An augmentation for $\mathbf{X}$ consists of a map $d^0 : \mathbf{X}^{-1} \to \mathbf{X}^0$ such that the following relation holds:*

$$d^1 d^0 = d^0 d^0 : \mathbf{X}^{-1} \to \mathbf{X}^0$$

Let A be an abelian category, $CO - A$ the category of co-simplcial objects over A and **Ch**(A) the category of normalized co-chain complexes over A. We list the Dold-Kan correspondence theorem:

Theorem 6.1.6. *For any abelian category A there is an equivalence $CO - A \cong$ **Ch**(A)*

Proof. See [W]

$\square$

It follows that co-simplical objects over a category are in a one to one correspondence with chain complexes over the same category. Given a co-triple C we now define a co-simplicial object $K_G(C)$ by $K_G(C) = \mathbf{G}C^n$. For each $(C, \psi) \in M(G)$ we make the following

Definition 6.1.7. **GC** *is the co-simplicial object over the category C defined by $\mathbf{G}C^n = G^{n+1}(C)$.*

The cosimplcial object $\mathbf{G}C^n = G^{n+1}(C)$ gives rise to a diagram of the form:

$$\begin{tikzcd}
C \arrow[r] & G(C) \arrow[r, shift left=2] \arrow[r, shift right=2] & G^2(C) \arrow[r, shift left=4] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=4] & \cdots
\end{tikzcd}$$

We call such a diagram the *G-resolution* of C. The augmentation $C \to G(C)$ is just the G co-algebra structure map $\psi : C \to G(C)$. Let T be the functor: $T : C \to A$ where A is an abelian category. For $n \geq 0$ we apply T to the co-simplicial object **GC**. Here the category C consists of those objects C with a G Co-algebra structure map.

$T\mathbf{GC}$ is a cosimplicial object over the abelian category A. All that is needed to obtain the chain complex which results from $T\mathbf{GC}$ is to define the boundary map.

Definition 6.1.8. *The co-chain complex over A associated to the cosimplicial object $T\mathbf{GC}$ has boundary map*

$$\partial = \sum(-1)^i T(d^i)$$

We can now define the right G- derived functors of the functor T with respect to objects in C (keep in mind each object C has a G Co-algebra structure map)

Definition 6.1.9. *The right G-derived functors of T of a G-co-algebra C are defined by:*

$$R^i T_G(C) = H^i(T\mathbf{GC})$$

As an example we will consider $R^i P_G(M)$, where $M \in S$, S is the category of co-associative, co-commutative, co-free co-algebras without co-unit over BP_*, and P is the primitives functor. A more detailed description can be found in [BCM] and [BDM]. Let M be a BP_*- module free on the generators $\{x_i\}$. Let $BP(M)$ be the zeroth space of the Ω-spectrum associated to the homology theory $BP_*(-) \otimes M$.

Definition 6.1.10. *Given M as above we define $G(M) = BP_*(BP(M))$*

Let $\mathcal{G}$ denote the category of unstable G co-algebras. In the case that X is a topological space such that the BP_*-homology is a free BP_*-module we make the following change in notation:

$$G(BP_*(X)) = G(X) = BP_*(BP(X))$$

It is possible to give a space X whose BP_*-homology satisfies the condition above a G Co-algebra structure. Let $\eta : X \to BP(X)$ be the unstable Hurewicz map. Apply $BP_*(-)$ to the unstable Hurewicz map to give:

$$BP_*(X) \to BP_*(BP(X))$$

Definition 6.1.11. *For $M \in \mathcal{G}$ we define $U(M) = P \circ G$.*

Recall, $\Gamma = BP_*(\underline{BP})$, so h^I is defined. To define the category of unstable Γ-comodules, *A(U)* we list the

Proposition 6.1.12. *If M is a free left BP_*-module then*

1. $U(M) = BP_*\{h^I \otimes m | 2l(I) < |m|\} \subset \Gamma \otimes_{BP_*} M$

2. *If M is an unstable Γ-co-module, free as a BP_*-module with coaction $\psi : M \to U(M)$. Then the unstable cobar complex is the chain complex $C^{s,t}(M) = U^s(M)_t$ with differential given by*

   $d([\gamma_1|\gamma_2| \cdots |\gamma_s]m) = [1|\gamma_1| \cdots |\gamma_s]m + \sum_{j=1}^{s}(-1)^j[\gamma_1| \cdots |\gamma_j'|\gamma_j''| \cdots |\gamma_s]m + (-1)^{s+1}\sum[\gamma_1| \cdots |\gamma_s|\gamma']m''$

where $\gamma_j \in \Gamma$, $\psi(\gamma_j) = \sum \gamma_j' \otimes \gamma_j''$ *and* $\psi(m) = \sum \gamma' \otimes m''$

Proof. See [BT]

□

[BCM] has shown that a BP_*-module M equipped with a U structure map ψ_M satisfies the conditions of a cotriple on M-the category of positively graded BP_*-modules which are free of finite type. *A(U)* is an abelian category and $Ext^s(BP_*(S^t), -)$ can be computed as the homology of the unstable cobar complex described above. If Δ is the reduced diagonal then we have the

Definition 6.1.13. *For a co-algebra* $M \in S$ *we define the group of primitives* $P(M)$ *in the following way:*

$$P(M) = ker\Delta : M \to M \otimes_{BP_*} M$$

One may think of P as a functor from S to the category of abelian groups. The category *A(U)* consists of those modules M with a U-structure map. It was shown in [BCM] that the functor U also extends to a functor of a cotriple on the category of all positively graded BP_*-modules(free or not) We also call this functor U. The category of unstable G-Coalgebras $\mathcal{G}$ is non-abelian and difficult to work with. However, the primitive element functor P satisfies the relation $P \circ G = U$ which implies that P induces a functor. In other words the primitive element functor is the functor $P : \mathcal{G} \to U$. For more on the structure of $U(M)$ see [RW] and [BH].

6.2 Injective Extension Sequences

In an attempt to keep the notation stable we adopt the following from [BCR]. All of the details of this section can be found there. Let S denote the category of co-associative, co-commutative, co-free co-algebras without co-unit over BP_*

Definition 6.2.1. *An injective extension sequence is a sequence of maps in S*

$$C' \xrightarrow{f} C \xrightarrow{g} C''$$

such that

1. *g is an epimorphism*
2. *f is an inclusion*
3. *C is injective as a C''-co-module*
4. *$C' = C\Box_{C''}BP_*$*

In other words

$$C' \xrightarrow{f} C \xrightarrow{g} C''$$

is a short exact sequence in the category S. Dually,

Definition 6.2.2. *A projective extension sequence is a sequence of maps in S^**

$$\overline{C''} \xrightarrow{\overline{g}} \overline{C} \xrightarrow{\overline{f}} \overline{C'}$$

such that

1. $\overline{g}$ *is an injection*

2. $\overline{f}$ *is a surjection*

3. $\overline{C}$ *is projective as a* $\overline{C''}$*-module*

4. $\overline{C'} = \overline{C} \otimes_{\overline{C''}} BP_*$

In particular a sequence is an injective extension sequence iff its dual ia a projective extension sequence.

Definition 6.2.3. *Let A and B be BP_*-algebras. The range where A is a free B-module is the range where no relations among relations occur.*

A strict upper bound on the range for which A is a free B-module is given by $|\alpha|$ where α is a relation among relations of minimal degree.

Remark 6.2.4. *The theorems that follow hold in the range where $\overline{C}$ is projective as a $\overline{C''}$-module or dually where C is injective as a C'' co-module.*

Injective extension sequences give rise to long exact sequences of higher derived functors. We have

Theorem 6.2.5. *Let*

$$C' \xrightarrow{f} C \xrightarrow{g} C''$$

be an injective extension sequence in S. There is a long exact sequence of abelian groups:

$$0 \to P(C') \to P(C) \to P(C'') \to \cdots$$

$$\cdots \to R^iP(C') \to R^iP(C) \to R^iP(C'') \to \cdots$$

Proof. See [BCR] □

Another useful property of injective extension sequences is the fact that they give rise to a long exact sequence of *Ext* terms in the category $\mathcal{G}$.

Theorem 6.2.6. *Let*

$$C' \xrightarrow{f} C \xrightarrow{g} C''$$

be an injective extension sequence in S. There is a long exact sequence of Ext-terms

$$\cdots \longrightarrow Ext_{\mathcal{G}}^{s,t}(C') \xrightarrow{f_*} Ext_{\mathcal{G}}^{s,t}(C) \xrightarrow{g_*} Ext_{\mathcal{G}}^{s,t}(C'') \xrightarrow{\delta} \cdots$$

Where the differential δ has bi-degree $= (1,0)$.

Proof. See [BCR]

□

We take the following definition from [Bous]. It was shown in [BCR] that the theory of homology co-algebras over a field can be generalized to homology co-algebras over a commutative ring. Let $S(V)$ be the universal co-free, co-commutative, co-algebra functor on V. i.e., if C is any BP_* co-algebra and $f : C \to V$ is a map of graded BP_* modules then there is an f^p making the following diagram commute.

$$\begin{tikzcd}
S(V) \arrow[r, "f^p"] & V \\
C \arrow[u] \arrow[r, "="] & C \arrow[u, "f"']
\end{tikzcd}$$

Definition 6.2.7. *We say a homology BP_* co-algebra M is co-free if $M \cong S(V)$ for some positively graded BP_* -module V.*

We need the following from [Bous]

Definition 6.2.8. *A homology BP_*-co-algebra C is nice if there exists an injective extension sequence*

$$C' \xrightarrow{f} C \xrightarrow{f''} C''$$

such that C and C'' are co-free.

For equivalent formulations of the previous definition the reader should refer to [Bous] pg 474. The primitives of a co-algebra M are dual to the indecomposables in the dual algebra F which lie in a quotient of F and are isomorphic to the module of algebra generators. [Bous] gives a brief description of how one may think about $R^iP(M)$ as measuring the failure of the dual algebra to be free. For example, $R^1P(M)$ comes from the relations in the algebra. We have the following from [Bous]

Proposition 6.2.9. *A homology BP_*-co-algebra M is co-free if and only if $R^iPM = 0$ for $i \geq 1$ and M is nice if $R^iPM = 0$ for $i > 1$.*

Remark 6.2.10. *From this point forward we will refer to a homology BP_*-co-algebra M as a co-algebra over BP_*.*

Examples of nice co-algebras would include $BP_*(\Omega S^{2n+1})$ as well $BP_*(\widehat{S}^{2n})$ where $\widehat{S}^{2n}$ is the Toda sphere. The BP homology of odd dimensional spheres also falls into this group. In the case that M is co-free as a co-algebra we have the following from [BCR].

Theorem 6.2.11. *Let $M \in \mathcal{G}$ and suppose that M is co-free as a co-algebra then*

$$Ext^{s,t}_{\mathcal{G}}(M) \cong Ext^{s,t}_{A(U)}(PM)$$

Proof. See [BCR]

□

The following result is also proven in [BCR]

Theorem 6.2.12. *Let $M \in \mathcal{G}$ and suppose that M is nice. Then there is a long exact sequence of Ext terms:*

$$\cdots \longrightarrow Ext^{s,t}_{A(U)}(PM) \longrightarrow Ext^{s,t}_{\mathcal{G}}(M) \longrightarrow Ext^{s,t}_{A(U)}(R^1P(M)) \xrightarrow{\partial} \cdots$$

where ∂ has bi-degree $(2,0)$

In the case that M is not nice as a co-algebra the long exact sequence is replaced by a composite functor spectral sequence.

Theorem 6.2.13. *For each $M \in \mathcal{G}$ there is a CFSS and an UANSS such that the following holds:*

$$Ext^r_{A(U)}(BP_*(S^t), R^s_G P(M)) \Rightarrow Ext^{r+s}_{\mathcal{G}}(BP_*(S^t), BP_*(M))$$

Proof. See [BCM] □

6.3 Hopf Rings

The purpose of this section is to define a Hopf Ring. For a more complete description of Hopf Rings the reader should refer to [RW] and [BH]. We assume throughout this section that the category **C** has finite products as well as a terminal object. Below we define abelian group objects as well as ring objects following [RW], Later we will describe the basic properties of the $\circ$ and $*$ products. As a matter of convenience we will keep the notation developed in [RW]. For two elements $C, D \in \mathbf{C}$ let $C \prod D$ denote their product in **C**

Definition 6.3.1. *An abelian group object of* **C** *is the quadruple* $(C, \eta, *, \chi)$ *such that the following diagrams commute:*

$$\begin{tikzcd}
C \prod D \arrow[r, "p_2"] \arrow[d, "\eta \Pi 1_D"] & D \arrow[d, "1_D"] \\
D \prod D \arrow[r, "*"] & D
\end{tikzcd}$$

corresponding to addition and

$$\begin{tikzcd}
D\prod D \arrow[r, "*"] \arrow[d, "{(p_2,p_1)}"] & D \arrow[d, "="] \\
D\prod D \arrow[r, "*"] & D
\end{tikzcd}$$

corresponding to commutativity of the $*$ product and

$$\begin{tikzcd}
D\prod D\prod D \arrow[r, "1_X\Pi_*"] \arrow[d, "*\Pi 1_X"] & D\prod D \arrow[d, "*"] \\
D\prod D \arrow[r, "*"] & D
\end{tikzcd}$$

corresponding to associativity and

$$\begin{tikzcd}
D \arrow[r, "{(1_X,\chi)}"] \arrow[d, "\epsilon"] & D\prod D \arrow[d, "*"] \\
C \arrow[r, "\eta"] & D
\end{tikzcd}$$

corresponding to inverses.

Looking at the diagrams above one can see that $*$ corresponds to addition, χ corresponds to the inverse and η corresponds to the zero. From now on we will let C_* denote an abelian group object over **C** To define the category of graded objects over **C** we need a graded object **GC** and graded morphisms f_* such that for each n, f_n respects the diagrams above. We list the following important definition from [RW].

Definition 6.3.2. *A graded ring object with unit over **C** is an abelian group object C_* such that each C_n is an abelian group object of **C** endowed with two maps: e called the multiplicative unit and the multiplication map: $\circ_{i,j}$ defined by $D_i \prod D_j \to D_{i+j}$ such* *that the following diagrams commute:*

$$\begin{tikzcd}
D_i \prod D_j \prod D_k \arrow[r, "1_{D_i} \prod \circ"] \arrow[d, "\circ \prod 1_{X_k}"] & D_i \prod D_{j+k} \arrow[d, "\circ"] \\
D_{i+j} \prod D_k \arrow[r, "\circ"] & D_{i+j+k}
\end{tikzcd}$$

corresponding to associativity of the $\circ$ *product and*

$$\begin{tikzcd}
D_i \prod D_j \arrow[r, "\circ"] \arrow[d, "(p_2, p_1)"] & D_{i+j} \arrow[d, "\chi^{ij}"] \\
D_j \prod D_i \arrow[r, "\circ"] & D_{i+j}
\end{tikzcd}$$

corresponding to commutativity of $\circ$ *product and*

$$\begin{tikzcd}
D_i \prod D_j \prod D_j \arrow[r, "1_\psi \prod 1_{X_j} \prod X_j"] \arrow[d, "1_{X_k} \prod *"] & D_i \prod D_i \prod D_j \prod D_j \arrow[r, "(p_1, p_2, p_3, p_4)"] \arrow[d, "\circ"] & D_i \prod D_j \prod D_i \prod D_j \arrow[d, "\circ \prod \circ"] \\
D_i \prod D_j \arrow[r, "\circ"] & D_{i+j} & D_{i+j} \prod D_{i+j} \arrow[l, "\circ"']
\end{tikzcd}$$

corresponding to the distributive law and

$$\begin{tikzcd}
C \prod D_i \arrow[r, "p_2"] \arrow[d, "e \prod 1_{X_i}"] & D_i \arrow[d, "1_{X_i}"] \\
D_0 \prod D_i \arrow[r, "\circ"] & D_i
\end{tikzcd}$$

corresponding to multiplication by the unit and

$$\begin{tikzcd}
C\prod D_i \arrow[r, "\varepsilon"] \arrow[d, "\eta\prod 1_{X_i}"] & C \arrow[d, "\eta"] \\
D_i\prod D_j \arrow[r, "\circ"] & D_{i+j}
\end{tikzcd}$$

corresponding to multiplication by 0.

The map $\psi : D \to D\prod D$ will turn out to be the co-algebra structure map. Let $C0 - Alg$ be the category of graded co-associative, co-commutative C0-algebras with co-unit over a graded commutative, associative ring R. We are now in a position to make the following

Definition 6.3.3. *A Hopf Ring is a ring object over the category* $Co - Alg$

It turns out that the product $C\prod D$ in $Co - Alg$ is given by $C\bigotimes_R D$. The co-product map $\psi_C : C \to C\bigotimes_R C$ and the co-unit $\epsilon : C \to R$ fit into the diagrams above.

Example 6.3.4. $BP_*(\underline{BP})$ *is a hopf ring. For more the reader should refer to [RW]*

Chapter 7

Homotopy Groups of Toric Manifolds

In this chapter the E_2 term of the UNSS is set up through a range allowing for a generalization of all known homotopy group theoretic calculations of toric manifolds.

7.1 Generalities

$BP_*(-)$ denotes unreduced BP homology theory. Let K be an $(n-1)$ dimensional simplicial complex on $[m]$ and P_K the simple polyhedral complex which is dual to K. If K is a simplicial sphere then P_K is an n dimensional simple convex polytope. Fix K and write P for P_K. Recall from chapter 2 that $k(K)$ is the algebra

$$k(P) \cong \mathbb{Z}[v_1, .., v_m]/\langle I \rangle$$

where the v_i are dual to the facets in P and $\langle I \rangle$ is generated by square free monomials r_i of degree $|r_i|$. In what follows B_TP will denote the Borel space associated to P.

Proposition 7.1.1. *If K is an $(n-1)$-dimensional simplicial complex on $[m]$ then*

$$BP^*(B_TP) \cong BP_*[v_1, .., v_m]/\langle I\rangle$$

Proof. The proof given in [DJ] carries over without change if one replaces the ordinary chern classes with the Conner-Floyd classes.

□

We will call the algebra $BP^*(B_TP)) \cong BP_*[v_1, .., v_m]/\langle I\rangle$ the BP-Face ring. Resolve the algebra in the category of BP_*-algebras to give the following exact sequence where $|x_j| = |r_j|$

$$BP^*[x_1, .., x_k] \xrightarrow{\iota^*} BP^*[v_1, .., v_m] \longrightarrow BP^*(B_TP)$$

where

$$\iota^*(x_i) \quad = \quad r_i$$

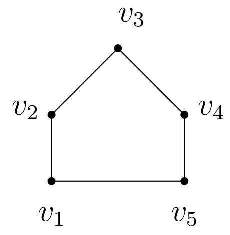
Definition 7.1.2. *Suppose $i \neq j$. Let I and I' be two multi-indexes such that $I \neq I'$ and $I, I' \subset S = \{\iota_1, .., \iota_m\}$. We call a relation of the form:*

$$\iota^*(x_i)\textstyle\prod_{k\in I} v_k - \iota^*(x_j)\prod_{k'\in I'} v_{k'} = 0$$

in $BP^*[v_1, .., v_m]$ a relation among relations. We denote such a relation by $\overline{R}$.

Example 7.1.3. *Consider the following 2 dimensional simple convex polytope.*

$P \quad =$



$F^* \cong BP_*[v_1, .., v_5]/I = \langle v_1v_3, v_2v_4, v_3v_5, v_4v_1, v_5v_2\rangle$. *The resolution above is*

$$BP^*[x_1, .., x_5] \xrightarrow{\iota^*} BP^*[v_1, .., v_5] \longrightarrow BP^*(B_TP)$$

$$\begin{aligned}
\iota^*(x_1) &= v_1v_3 \\
\iota^*(x_2) &= v_2v_4 \\
\iota^*(x_3) &= v_3v_5 \\
\iota^*(x_4) &= v_4v_1 \\
\iota^*(x_5) &= v_5v_2
\end{aligned}$$

A relation among relations is $\overline{R} = \iota^*(x_1)v_2v_4 - \iota^*(x_2)v_1v_3$. *Clearly,* $|\overline{R}| = 8$

Definition 7.1.4. *Let* $\Re$ *denote the set of all possible relations among relations. Define* $\Re_{min}$ *to be the relation among relations such that* $|\,\Re_{min}| \leq |\,r| \;\forall\; r \in \Re$

$\Re_{min}$ is a relation among relations of the smallest degree. In general $\Re_{min}$ is not unique.

Example 7.1.5. *Given F^* from the previous example pick x_1 and x_3. A relation among relations $\Re_{min}$ is:*

$$\iota^*(x_1)v_5 - \iota^*(x_3)v_1$$

it follows that $|\Re_{min}| = 6$

When given a specific simplicial complex K and an ideal I explicitly the degree of the smallest relation among relations can be determined.

$BP^*[v_1, .., v_m]$ is a $BP^*[x_1, .., x_k]$-module induced by the map ι^*. The action is given by

$$x_i \cdot v = \iota^*(x_i)v$$

Lemma 7.1.6. *$BP^*[v_1, .., v_m]$ is a free $BP^*[x_1, .., x_k]$-module on $\Re_{min}$ up to dimension $|\Re_{min}| - 1$*

Proof. It is clear that $BP^*[v_1, .., v_m]$ is a free $BP^*[x_1, .., x_k]$-module in the range where the relation among relations does not occur. $\Re_{min}$ appears in degree $|\Re_{min}|$. Hence, $BP^*[v_1, .., v_m]$ is a free $BP^*[x_1, .., x_k]$-module in degrees $< |\Re_{min}|$.

□

We now have the following

Theorem 7.1.7.

$$BP^*[x_1,..,x_k] \xrightarrow{\iota^*} BP^*[v_1,..,v_m] \longrightarrow BP^*(B_TP)$$

is a projective extension sequence up to dimension: $|\Re_{min}| - 1$

Proof. Resolve the BP face ring in the category of BP_* algebras to give the following diagram

$$BP^*[x_1,..,x_k] \xrightarrow{\iota^*} BP^*[v_1,..,v_m] \longrightarrow BP^*(B_TP)$$

The result follows from 7.1.6, 6.2.2, 6.2.3 and 6.2.4. □

We apply the functor $Hom(-, BP_*)$ to the projective extension sequence to obtain the following injective extension sequence

$$BP^*(B_TP)^\sharp \longrightarrow BP^*[v_1,..,v_m]^\sharp \xrightarrow{\iota_*} BP^*[x_1,..,x_k]^\sharp$$

up to dimension $|\Re_{min}| - 1$ For convenience we make the following changes in notation:

$$\begin{aligned} C^* &= BP^*(\prod_m CP^\infty) \\ F^* &= BP^*(B_TP) \\ R^* &= BP^*[x_1,..,x_k] \end{aligned}$$

We write bases for these algebras over BP_* we have the monomial basis for C^*:

$$\underline{B}_{C^*} = \{v^I |\ I = (i_1, .., i_m)\}$$

and a basis for F^*

$$\underline{B}_{F^*} = \{v^I |\ im\ \iota^* \nmid v^I\}$$

and the monomial basis for R^*

$$\underline{B}_{R^*} = \{x^I | I = (i_1, .., i_k)\}$$

These algebras dualize to co-algebras C, F and R. To determine the bases of these co-algebras we consider the canonical pairings

$$BP^*(-) \otimes Hom(BP^*(-), BP_*) \to BP_*$$

For example,

$$BP^*[v_1, .., v_m] \otimes Hom(BP^*[v_1, .., v_m], BP_*) \to BP_*$$

is the canonical pairing for C^*. Define $\{\beta_J\}$ to be the basis dual to $\underline{B}_{C^*}$. i.e.,

$$\langle v^I, \beta_J \rangle = \delta_{I,J}$$

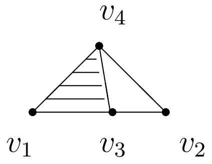
$\{\lambda_J\}$ the basis dual to $\underline{B}_{F^*}$. i.e.,

$$\langle v^I, \lambda_J \rangle = \delta_{I,J}$$

where I and J are subject to the condition defining the basis for F^* Finally, $\{z_J\}$ the basis dual to $\underline{B}_{R^*}$ i.e.,

$$\langle x^I, z_J \rangle = \delta_{I,J}$$

Example 7.1.8. *To illustrate how the algebras and co-algebras are dual we suppose K is the simplicial complex:*



The BP^* face ring is $BP_*[v_1, v_2, v_3, v_4]/I =< v_1v_2, v_2v_3v_4 >$. Clearly, $C^* \cong BP^*\{v^I|\iota_j \geq 0\}$ and $F^* \cong C^*/I$. R^* is a polynomial algebra whose generators are in a one to one correspondence with the relations. $R^* \cong BP^*\{x_1^{\iota_1}x_2^{\iota_2}|\iota_1, \iota_2 \geq 0\}$ The Kronecker pairing $<,>$ is used to write down the dual basis elements for the co-algebras. For example, $C \cong BP_{-*}\{\beta_J|J = (\iota_1, .., \iota_4)\}$. For the co-algebra F we have for example the element $\lambda_{(1,0,1,0)}$ which is dual to $v_1v_3 \in F^*$. The bases in F and R can be written down in a similar way as well as any dual elements.

Let $l(J) = \sum_{i=1}^{l} j_i$ for $j_i \in J$. The following example will illustrate how one computes the map $F \to C$ in terms of the dual basis.

Example 7.1.9. *Suppose $M = \{1, .., q\}$ is a missing face of K. The element $v_1 \cdots v_q$ = 0 in F^*. In particular $v_1 \cdots v_q$ is not a basis element in F^* which implies that $\iota^*(x) = v_1 \cdots v_q$ for a basis element $x \in R^*$. It follows that*

$$\lambda_{(\underbrace{1, .., 1}_{q-terms},0,..,0)}$$

does not exist in the co-algebra F.

The degrees of the elements in the dual bases are as follows:

$$|\beta_J| = \begin{cases} 2 \cdot l(J) & \text{if } \; im\, \iota^* \nmid v^I \\ \\ |\lambda_J| & \text{otherwise.} \end{cases}$$

and

$$|z_J| = |\iota^*(x^I)|$$

Example 7.1.10. *In 7.1.8 the element $z_{(1,0)} \in R$ is dual to $x_1 \in R^*$ and $z_{(0,1)} \in R$ is dual to $x_2 \in R^*$ by the kronecker pairing. Therefore,*

$$|z_{(1,0)}| = |\; x_1| = |\iota^*(x_1)| = |v_1 v_2| = 4$$

and similarly we have

$$|z_{(0,1)}| = |\; x_2| = |\iota^*(x_2)| = |v_2 v_3 v_4| = 6$$

The co-algebra structure maps are defined as follows.

$$C \xrightarrow{\psi} C \otimes_{BP_*} C$$

$$\psi(\beta_I) = \Sigma_{j_p+k_p=i_p}\beta_{j_1,..,j_m} \otimes \beta_{k_1,..,k_m}$$

The co-algebra map R is defined similarly. The co-algebra map for F can be defined with restrictions on the multi-indicies as illustrated by the following

Remark 7.1.11. *Given F we have*

$$\lambda_I \to \sum a_{KJ}\lambda_K \otimes \lambda_J$$

The coefficients are given by

$$\begin{aligned} a_{KJ} &= \langle \psi(\lambda_I), v^K \otimes v^J \rangle \\ &= \langle \lambda_I, v^{K+J} \rangle \end{aligned}$$

Hence,

$$a_{KJ} = \begin{cases} 1 & \textit{if } I \ = \ K + J \\ 0 & \textit{otherwise.} \end{cases}$$

Theorem 7.1.12. *If P is an n-dimensional, $q \geq 1$ neighborly simple polyhedral complex then up to dimension $|\Re_{min}| - 1$ the BP_*- module structure of $R^iPBP_*(B_TP)^\sharp$ is as follows*

$$R^iPBP_*(B_TP)^\sharp \cong \begin{cases} BP_*\{a_1, .., a_m\} & i = 0,\ |\ a_i|\ = \mathit{2} \\ BP_*\{e_1, .., e_k\} & \textit{if}\ i = 1\ ,\ |\ e_j| = |\ r_j| \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. It is easier to work with algebras then dualizing to the co-algebras. The projective extension

$$\begin{tikzcd} R^* \arrow[r, "\iota^*"] & C^* \arrow[r] & F^* \end{tikzcd}$$

holds through the range $|\Re_{min}| - 1$. Dually,

$$\begin{tikzcd} F \arrow[r] & C \arrow[r, "\iota*"] & R \end{tikzcd}$$

is an injective extension sequence through the range $|\Re_{min}| - 1$ giving rise to the following long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & PF \arrow[r] & PC \arrow[r] & PR \arrow[r] & R^1PF \arrow[r] & R^1PC \arrow[r] & \cdots \end{tikzcd}$$

Since C is co-free as a co-algebra we obtain the exact sequence

$$0 \longrightarrow PF \longrightarrow PC \longrightarrow PR \longrightarrow R^1PF \longrightarrow 0$$

through the range $|\Re_{min}| - 1$. Clearly, $QR \to QC$ factors through zero which implies that dually the map $PC \to PR$ factors through zero. Observe that R^1PF is the co-kernel of the map $PC \to PR$ and the result follows.

□

Let X and Y be topological spaces with free BP_* homology. Let M and N be the BP-homology of X and Y respectively where $M = BP_*\{x_1,,..,x_{2k}\}$ and $N = \{y_1,..,y_{2k}\}$ and the generators are even dimensional. For each generator of M and N there is a copy of $\underline{BP}_{|x_{2k}|}$ and $\underline{BP}_{|y_{2j}|}$ in $G(M)$ and $G(N)$. Recall, the lowest dimensional generator $b^i_{(0)} \in BP_{2i}(\underline{BP}_{2i})$. Let $\overline{x} = b^{|x|/2}_{(0)} \in BP_*(\underline{BP}_{|x|})$ which is the bottom dimensional generator for the copy of $BP_*(\underline{BP}_{|x|})$ corresponding to x in $G(M)$. For each module we can define a linear map on the generators $\rho : M \to G(M)$ by $\rho(x) = \overline{x}$. Similarly, there is a class $b^{|y|/2}_{(0)} \in BP_*(\underline{BP}_{|y|})$ for each generator of N and a map can be defined from $N \to G(N)$. To state the next theorem we need to set up some notation. Given modules M and N as above we have $G(M) = BP_*(\prod_{x_i \in M} \underline{BP}_{|x_i|})$ and similarly $G(N) = BP_*(\prod_{y_i \in N} \underline{BP}_{|y_i|})$. We have $G(M) \otimes G(N) = BP_*(\prod_{x_i \in M} \underline{BP}_{|x_i|}) \otimes BP_*(\prod_{y_i \in N} \underline{BP}_{|y_i|})$ which is equal to $BP_*(\prod \underline{BP}_{|x_i|} \times \underline{BP}_{|y_i|})$. Note that $G(M \otimes N) = G(BP_*(X \times Y))$

Theorem 7.1.13. *On the generators of M and N the map $\kappa_{\mathcal{G}} : G(M) \otimes G(N) \to G(M \otimes N)$ sends $\overline{x} \otimes \overline{y}$ to $\overline{x} * \overline{y}$*

Proof. Given M and N as above let $\{x\}$ and $\{y\}$ be families of generators. We can define a map

$$BP_*(X) \otimes BP_*(Y) \xrightarrow{\rho\otimes\rho} G(M) \otimes G(N)$$

by $(\rho \otimes \rho)(x \otimes y) = \overline{x} \otimes \overline{y}$ on the generators. By the Kunneth theorem we have the isomorphism

$$BP_*(X) \otimes BP_*(Y) \xrightarrow{\kappa} BP_*(X \times Y)$$

Define a map

$$BP_*(X \times Y) \xrightarrow{\overline{\rho}} G(M \otimes N)$$

by

$$\overline{\rho}(x\widehat{\otimes}y) = \rho(x)\widehat{\otimes}\rho(y) = \overline{x}\widehat{\otimes}\overline{y}$$

where $\overline{x}\widehat{\otimes}\overline{y}$ is the bottom generator of the factor $BP_{|x\otimes y|}$. Note that $\widehat{\otimes}$ is the completed tensor product and $x \in BP_*(X)$, $y \in BP_*(Y)$. The loop space product (Pontryagin product) will be denoted by the $*$ product. Hence, $\overline{x}\widehat{\otimes}\overline{y} = \overline{x} * \overline{y}$. Define $\kappa_{\mathcal{G}}(\overline{x} \otimes \overline{y}) = \overline{\rho}(x\widehat{\otimes}y)$ giving the following commutative diagram.

$$\begin{tikzcd}
BP_*(X) \otimes BP_*(Y) \arrow[r, "\rho\otimes\rho"] \arrow[d, "\kappa"'] & G(M) \otimes G(N) \arrow[d, "\kappa_{\mathcal{G}}"'] \\
BP_*(X \times Y) \arrow[r, "\overline{\rho}"] & G(M \otimes N)
\end{tikzcd}$$

$\kappa_{\mathcal{G}}$ depends on the basis and maps the product of the generators to the generator of the factor of the product corresponding to the tensor product of the generators. □

Remark 7.1.14. *By* $x \in BP_*(X \times Y)$ *we mean* $x\widehat{\otimes}1$ *and similarly* y *means* $1\widehat{\otimes}y$*. In*

$G(M) \otimes G(N)$ $\overline{x}$ corresponds to $\overline{x} \otimes 1$. In $G(M \otimes N)$ $\overline{x}$ corresponds to $\overline{x} * 1$ So,

$$\begin{aligned} \overline{x} * \overline{y} &= (\overline{x}\widehat{\otimes}1) * (1\widehat{\otimes}\overline{y}) \\ &= (\overline{x} * 1)\widehat{\otimes}(1 * \overline{y}) \\ &= \overline{x}\widehat{\otimes}\overline{y} \end{aligned}$$

Remark 7.1.15. *The class $\overline{x} \otimes \overline{y} \neq \overline{x \otimes y}$. This is the difference between the class in $BP_*(\underline{BP}_{|x|} \times \underline{BP}_{|y|})$ and $BP_*(\underline{BP}_{|x\otimes y|})$*

Remark 7.1.16. *The previous theorem does not say anything about the higher dimensional classes.*

We summarize the above with the

Remark 7.1.17. *If M and N are free BP_*-modules on even dimensional generators then the element $\overline{x} \otimes \overline{y} \in G(M) \otimes G(N)$ corresponds to the element $\overline{x} * \overline{y} \in G(M \otimes N)$. This is an external star product. Hence, $\overline{x} * \overline{y}$ is given by the previous diagram.*

We want to apply this to C. Since $BP_*(\mathbb{C}P^\infty)$ is torsion free we have $BP_*(\prod_m CP^\infty) \cong \bigotimes_m BP_*(CP^\infty)$ by the Kunneth theorem. Suppose for each j, X_j is a space such that $BP_*(X_j)$ is torsion free and is generated by even dimensional generators then we have the following lemma by induction over m and

Lemma 7.1.18. *The following diagram commutes.*

$$\begin{tikzcd} BP_*(X_1) \otimes ... \otimes BP_*(X_m) \arrow[r, "\otimes_m \rho"] \arrow[d, "\otimes_m \kappa"'] & G(M_1) \otimes ... \otimes G(M_m) \arrow[d, "\otimes_m \kappa_G"] \\ BP_*(X_1 \times ... \times X_m) \arrow[r, "\otimes_m \overline{\rho}"] & G(M_1 \otimes ... \otimes M_m) \end{tikzcd}$$

The element $\overline{x}_1 \otimes ... \otimes \overline{x}_m \in G(M_1) \otimes ... \otimes G(M_m)$ corresponds to the element $\overline{x}_1 * ... * \overline{x}_m \in G(M_1 \otimes \cdots \otimes M_m)$. Since the tensor product respects the kronecker pairing we easily determine that for $J = (j_1, .., j_m)$

$$\beta_J = \beta_{j_1} \otimes \beta_{j_2} \otimes \cdot \cdot \otimes \beta_{j_m}$$

We introduce the following notation

Definition 7.1.19. $\beta_{1,j} = \beta_{(0,..,0,\underbrace{1}_{j},0,..,0)}$

Remark 7.1.20. $\beta_{1,j}$ *is not to be confused with a multi-index which is enclosed in paranthesis.*

$\beta_{1,j}$ generates the jth factor of $BP_*(CP^\infty)$ in the tensor decomposition of C.

7.2 Determination of the Stable and Unstable Coaction

Lemma 7.2.1. *ψ: $C \to \Gamma \bigotimes_{BP_*} C$ sends $\beta_{1,j}$ to $1 \otimes \beta_{1,j}$.*

and if I is a multi-index of zeros and ones then the map

$$\psi : C \to \Gamma \bigotimes_{BP_*} C$$

sends $\beta_I \mapsto 1 \otimes \beta_I$.

Proof. Since $\beta_{1,j}$ is a generator of the jth factor of C it is a bottom dimensional class and the result follows. The proof follows from the fact that ψ is multiplicative.

□

The next lemma will compute the unstable co-action on the classes β_J which is a product of $\beta_{1,j}$.

Lemma 7.2.2. *For* $\psi_{G(C)} : C \to G(C)$, $\psi_{G(C)}(\beta_J) = *_{j\in J}\overline{\beta_{1,j}} + 1 \otimes \beta_J$.

Proof. Let J be a multi-index of zeros and ones. Since $G(C)$ is a co-algebra we use the co-algebra map in addition to the the following commutative diagrams

$$\begin{tikzcd}
C \arrow[r, "\psi_{G(C)}"] \arrow[d, equal] & G(C) \arrow[d, "\sigma_*"] \\
C \arrow[r, "\psi"] & \Gamma \otimes_{BP_*} C
\end{tikzcd}$$

and

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & G(C) \arrow[d] \\
C \otimes C \arrow[r] \arrow[d, equal] & G(C) \otimes G(C) \arrow[d, "\epsilon\otimes\epsilon"'] \\
C \otimes C \arrow[r, "="] & C \otimes C
\end{tikzcd}$$

where σ_* is the stablization map. This is used to show that the im $\psi_{G(C)}$ is given by

$$*_{j\in J}\overline{\beta_{1,j}} + \{\text{a stable class.}\}$$

Note that the $*$-product term follows from the fact that the co-action map is a map of co-algebras. It follows from 7.2.1 and the commutative diagrams above that the stable class must be of the form $1 \otimes \beta_J$.

$\square$

Remark 7.2.3. *The co-triple structure is used to give C a canonical co-algebra structure.*

Given any Co-algebra C we form the unstable G resolution:

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r] & G(C) \arrow[r, shift left=2] \arrow[r, shift right=2] & G^2(C) \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & \cdots
\end{tikzcd}$$

We apply the primitive element functor $P(-)$ to the previous chain complex to obtain the following chain complex.

$$\begin{tikzcd}
(*)\ 0 \arrow[r] & U(C) \arrow[r, shift left=2] \arrow[r, shift right=2] & U(G(C)) \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & \cdots
\end{tikzcd}$$

Since $G(N)$ is co-free as a co-algebra the homology of (*) is the derived functors of the primitives i.e., $H^i(*) = R^iP(C)$. We obtain the following complexes:

$$\mathtt{G} = \{0 \to G(C) \to G^2(C) \to \cdot\cdot\}$$

$$\mathtt{U} = \{0 \rightarrow U(C) \rightarrow U(G(C)) \rightarrow \cdot\cdot\}$$

Since $U(C) \hookrightarrow G(C)$ we obtain the following short exact sequence of chain complexes:

$$0 \longrightarrow \mathtt{U} \longrightarrow \mathtt{G} \longrightarrow \mathtt{G/U} \longrightarrow 0$$

Taking homology we obtain the following exact sequence

$$0 \longrightarrow H^0\mathtt{U} \longrightarrow H^0\mathtt{G} \longrightarrow H^0\mathtt{G/U} \longrightarrow H^1\mathtt{U} \longrightarrow H^1\mathtt{G} \longrightarrow 0$$

Since $\mathtt{G}$ is acyclic we obtain the exact sequence

$$0 \longrightarrow H^0\mathtt{U} \longrightarrow H^0\mathtt{G} \longrightarrow H^0\mathtt{G/U} \longrightarrow H^1\mathtt{U} \longrightarrow 0$$

Referring to the $\mathtt{U}$ and $\mathtt{G}$ complexes of C we use exactness to immediately determine that

$$H^0\mathtt{G} = ker : G(C) \longrightarrow G^2(C) = C$$

and

$$H^0\mathtt{U} = ker : U(C) \longrightarrow U(G(C)) = PC$$

As a matter of notational convenience let $\phi : [\frac{G(C)}{U(C)} \rightarrow \frac{G^2(C)}{U(G(C))}]$

$$H^0\mathtt{G/U} = Ker\ \phi$$

`U` is a chain complex whose homology groups are the derived functors of the coalgebra C. We immediately obtain the following long exact sequence:

$$0 \longrightarrow PC \longrightarrow C \xrightarrow{\theta} H^0\mathtt{G/U} \xrightarrow{h} R^1PC \longrightarrow 0$$

The map θ factors in the following way

$$\begin{tikzcd}
 & C \arrow[r] \arrow[d, "\pi"] & H^0(G/U) \arrow[d, "="] \\
0 \arrow[r] & C/PC \arrow[r] \arrow[d] & H^0(G/U) \\
 & 0 &
\end{tikzcd}$$

Lemma 7.2.4. *$R^1P(C) \cong ker\phi$ modulo im $\frac{C}{PC}$*

Proof. The map

$$H^0\mathtt{G/U} \xrightarrow{h} R^1PC$$

is surjective by the previous long exact sequence. The first isomorphism theorem gives

$$Ker\ \phi/Ker\ h \cong R^1P(C)$$

By the factorization of θ we obtain the following long exact sequence

$$0 \longrightarrow C/PC \xrightarrow{\tau} H^0\mathtt{G/U} \longrightarrow \frac{G(C)}{U(C)} \to \frac{G^2(C)}{U(G(C))}$$

By exactness, the result follows from the determination of im τ. □

Remark 7.2.5. *There is a shift that comes from the chain complex* $(*)$ *and the chain complexes that follow. For example,*

$$\frac{G^2(C)}{U(G(C))} = \frac{G^1(C)}{U^1(C)}$$

and

Remark 7.2.6. *If* C *were* $\prod_m \mathbb{C}P^\infty$ *then* $R^1PC = 0$ *and the previous constructions would not yield any new information. However,* $R^1PF \neq 0$. *The constructions above would be applied to the co-algebra* F. *Elements in* R^1PF *must be represented in a way that will allow for the determination of the unstable* U*- co-module structure map.*

Recall that $F = BP_*(B_TP)^\sharp$. In the previous constructions replace the co-algebra C by F. The map we wish to compute is $G(F) \to U(G(F))$. The next theorem shows that the generators of R^1PF can be represented as star products of classes in $G(F)$. It follows that the map $G(F) \to U(G(F))$ must be computed if we are to work with the CFSS.

Theorem 7.2.7. *d:* $G(F) \to U(G(F))$ *is given by* $d(*_{j\in J}\overline{\beta_{1,j}}) = 1 \otimes (*_{j\in J}\overline{\beta_{1,j}})$

Proof. Consider the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r, "="] \arrow[d] & 0 \arrow[r, "="] \arrow[d] & 0 \arrow[d] \\
F/PF \arrow[r] \arrow[d] & G(F)/U(F) \arrow[r, "\mathrm{d}"] \arrow[d] & G^2(F)/U(G(F)) \arrow[d, "\iota"] \\
C/PC \arrow[r, "\psi_{\mathrm{GC}}"] & G(C)/U(C) \arrow[r, "\mathrm{d}"] & G^2(C)/U(G(C))
\end{tikzcd}$$

$\psi_{G(C)}(\beta_J) = *_{j\in J}\overline{\beta_{1,j}} + 1 \otimes \beta_J$ by 7.2.2. Observe that $d \circ \psi_{G(C)} = 0$. Since d is a map of spaces we can write $d(*_{j\in J}\overline{\beta_{1,j}})$ = -$d(1 \otimes \beta_J)$. By exactness there exists a class $\overline{\alpha} \in G(F)/U(F)$ which hits $*_{j\in J}\overline{\beta_{1,j}} + 1 \otimes \beta_J$. Let $\overline{\alpha} = *_{j\in J}\overline{\beta_{1,j}}$. Notice the element $1 \otimes \beta_J$ can not pullback since F does not have a product structure map. Using commutativity we observe that $(\iota \circ d)(*_{j\in J}\overline{\beta_{1,j}}) = 0$. Since ι is an injection, $d(*_{j\in J}\overline{\beta_{1,j}})$ = 0. That is, $*_{j\in J}\overline{\beta_{1,j}}$ is a non-trivial cycle which represents an element in R^1PF which can not be pulled back to F/PF. We immediately have the following commutative diagram:

$$\begin{tikzcd}
G(F) \arrow[r, "\tau"] \arrow[d, equal] & U(G(F)) \arrow[d, "\gamma"'] \\
G(F) \arrow[r] & G^2(F)
\end{tikzcd}$$

where τ is the lift one obtains from $*_{j\in J}\overline{\beta_{1,j}}$ being a cycle. The map $d(1 \otimes \beta_J)$ must be determined. In filtration zero in the unstable cobar complex of $G^s(\Gamma \otimes C)$ we have $d(1 \otimes \beta_J) = 1 \otimes 1 \otimes \beta_J$ - $1 \otimes \psi_{GC}(\beta_J)$. Hence, by proposition 7.2.2 we have $d(1 \otimes \beta_J)$ = -1 $\otimes(*_{j\in J}\overline{\beta_{1,j}})$ giving the result.

□

Remark 7.2.8. *It follows that the map* $G(F) \to U(G(F))$ *fits into a co-simplicial object if we let* $D^{p,q}(F) = U^pG^q(F)$ *with* $q = 1, p \geq 0$.

Remark 7.2.9. *The algebra of relations* R^* *is not realized as the* BP *co-homology of a space* X. *If all the primitives are in the same degree then the triviality of the coaction follows from degree reasons. The content of the earlier part of this section is to deal with mixed degrees.*

Remark 7.2.10. *The map between* F *and* C *is induced from a map of spaces. This allows for the co-action on* F *to be deduced from the co-action on* C.

Corollary 7.2.11. *The* Γ *co-action on* F *is trivial.*

Proof. The map of co-algebras $F \to C$ is injective. The image of F in C is a co-algebra with basis consisting of those $\{\beta_J\}$ such that the dual basis elements $\{v^I\}$ are not divisble by im ι^*. The result follows from proposition 7.2.1.

$\square$

Theorem 7.2.12. *The* Γ *co-action on* R *is trivial.*

Proof. There are two cases to analyze. First, suppose that $\forall\ i \neq j\ |x_i| = |x_j|$ in R. The co-action must be trivial by dimensional reasons since $\psi(x) = 1 \otimes x + \sum \gamma_i \otimes x_i$ where $deg(\gamma_i) > 0$ and $deg(x_i) < deg(x)$. Now suppose the degrees of the relations are not the same. Hence, there are multi-indexes J and J_i consisting of zeros and ones such that for each i, $|z_{J_i}| < |z_J|$ where z_{J_i} and z_J are primitives in R. Let p be a prime. Suppose there is a possible co-action

$$\psi(z_J) = 1 \otimes z_J + \epsilon_1 \cdot h^I \otimes z_{J_1} + \cdots + \{Other\ Terms\}$$

By Theorem 7.1.12 PR is isomorphic to R^1PF as BP_* modules within a specified range. It was shown in Theorem 7.2.7 that the generators of the first derived functor of the primitive element functor of F can be represented by star products of various $\overline{\beta_{i,j}}$ in $G(F)$ that can be identified as a $\otimes$ product since these are the bottom dimensional classes coming from various factors of $\mathbb{C}P^\infty$. Hence, the co-action is trivial . A similar argument holds for the stable part. Therefore, $\epsilon_i = 0$ for each i.

$\square$

Remark 7.2.13. *The co-action on R is computable since the relations come from the square free monomials that generate the ideal in the face ring. The co-action on the $\beta_{i,j}$ is trivial by the geometry since they come from the v_i which come from $\mathbb{C}P^\infty$. If the relations did not come from square free monomials then β_J is an element of C whose multi-index does not consist of zeros and ones. Without the hypothesis of square free, 7.2.1 does not necessarily hold.*

This can be illustrated by the following

Example 7.2.14. *Consider the element $v_1^2 v_2$ which is represented in C by the element $\beta_{(2,1,0,0)}$. At the prime $p = 2$ we have the possible non-trivial co-action*

$$\psi(\beta_{(2,1,0,0)}) = 1 \otimes \beta_{(2,1,0,0)} + \epsilon \cdot h_1 \otimes \beta_{(1,1,0,0)}$$

where $\epsilon = 1$ or 0. We are not able to determine whether the coefficient ϵ is zero.

Remark 7.2.15. *If there is a possible non-trivial co-action it is possible to pick a prime*

p that will force the co-action to be trivial for dimensional reasons. One readily sees that the co-action in the previous example is trivial for dimensional reasons when the prime $p > 2$. However, this is much weaker than the argument given in the previous theorem.

The following example should illuminate the previous proof.

Example 7.2.16. *Let K be the simplicial complex given in 7.1.8. Recall, the BP^* face ring is the algebra $F^* = BP^*[v_1, .., v_4]/I = \langle v_1v_2, v_2v_3v_4\rangle$. As a $BP_* -$ module, the co-algebra of relations R has the following presentation.*

$$R = BP_{-*}\{z_J | J = (j_1, j_2)\}$$

To illustrate how the Kronecker pairing works we determine the coefficients c_J in

$$i_*(\beta_I) = \sum c_J z_J$$

$$\begin{aligned} c_J &= \langle i_*(\beta_I), z_J\rangle \\ &= \langle \beta_I, i^*(x_1^{j_1} \cdot x_2^{j_2})\rangle \\ &= \langle \beta_J, r_1^{j_1} \cdot r_2^{j_2}\rangle \\ &= \langle \beta_J, v_1^{j_1} v_2^{j_1+j_2} v_3^{j_2} v_4^{j_2}\rangle \end{aligned}$$

this gives

$$c_J = \begin{cases} 1 & \text{if I} = (j_1, j_1 + j_2, j_2, j_2) \\ \\ 0 & \text{otherwise.} \end{cases}$$

For $p > 2$ prime there can not be a stable co-action for dimensional reasons. However at the prime $p = 2$ there is a possible co-action :

$$\psi(z_{(0,1)}) = 1 \otimes z_{(0,1)} + \epsilon \cdot h_1 \otimes z_{(1,0)}$$

The primitives in R can be represented by the following products in $G(F)$:

$$z_{(1,0)} = \overline{\beta_{1,1}} * \overline{\beta_{1,2}}$$

and

$$z_{(0,1)} = \overline{\beta_{1,2}} * \overline{\beta_{1,3}} * \overline{\beta_{1,4}}$$

Clearly, we have $\psi(z_{(0,1)}) = \psi(\overline{\beta_{1,2}} \otimes \overline{\beta_{1,3}} \otimes \overline{\beta_{1,4}}) = 1 \otimes (\overline{\beta_{1,2}} \otimes \overline{\beta_{1,3}} \otimes \overline{\beta_{1,4}})$. Suppose the stable co-action were $\psi(z_{(0,1)}) = 1 \otimes z_{(0,1)} + \epsilon \cdot h_1 \otimes z_{(1,0)}$. Since the $\overline{\beta_{i,j}}$ are bottom dimensional classes we have the relation

$$1 \otimes (\overline{\beta_{1,2}} \otimes \overline{\beta_{1,3}} \otimes \overline{\beta_{1,4}}) = 1 \otimes (\overline{\beta_{1,2}} \otimes \overline{\beta_{1,3}} \otimes \overline{\beta_{1,4}}) + \epsilon \cdot h_1 \otimes \overline{\beta_{1,1}} \otimes \overline{\beta_{1,2}}$$

Hence,

$$\epsilon \cdot h_1 \otimes \overline{\beta_{1,1}} \otimes \overline{\beta_{1,2}} = 0$$

Hence, $\epsilon = 0$. Therefore, the stable co-action is trivial. In lieu of the previous remark the co-action is trivial for a prime $p > 2$ for dimensional reasons.

7.3 The Borel Space and the UNSS

Let $M(n)$ denote a free $BP_* - module$ on an element of degree n.

Lemma 7.3.1. *The following isomorphism holds*

$$Ext^{s,t}_{A(U)}(M(2n)) \cong Ext^{s,t-1}_{A(U)}(M(2n-1))$$

Proof. Follows immediately from the definition of $U(M)$. □

Let X be a space such that $R^i PBP_*(X)^\sharp \cong M$ where $M \cong BP_*\{x_{2k} | k > 0\}$ as a BP_* module and the generators are even dimensional.

Lemma 7.3.2. *$R^i PM$ are generated by even dimensional classes.*

Proof. Take the G resolution of M

$$0 \to M \to G(M) \to G^2(M) \to \cdot\cdot$$

Take the augmented chain complex to give:

$$G(M) \to G^2(M) \to \cdot\cdot$$

Apply the primitive element functor $P(-)$ to give the complex

$$U(M) \to U(G(M)) \to U(G^2(M)) \to \cdot\cdot$$

Recall, $U(M) = BP_* - span\{h^I \otimes x | 2 \cdot l(I) \leq |x|\}$. Clearly, the degree of $h^I \otimes x$ is even. Since $M \cong BP_*\{x_{2k} | k \geq 0\}$ as a BP_* module $G(M) = BP_*(\prod \underline{BP}_{|x_i|})$ which is $\bigotimes_i BP_*(\underline{BP}_{|x_i|})$. Hence, $U(G(M))$ is generated by classes of the form $h^I \otimes (\otimes_i x_i)$ which are clearly even dimensional. We will prove by induction over k that $U(G^k(M))$ is generated by even dimensional classes. Suppose that $U(G^{k-1}(M))$ is generated by even dimensional classes. $U(G^k(M)) = U(G^{k-1}(G(M)))$. By the statements above $G(M)$ is even and the proof follows by the inductive step.

□

This immediately gives the

Corollary 7.3.3. *If K is a $(n-1)$ dimensional simplicial complex on $[m]$ and F its BP_* face ring then R^iPF are generated by even dimensional classes.*

We have the following

Theorem 7.3.4. *If K is a $(n-1)$ dimensional simplicial complex on $[m]$ and F is the BP_* face ring associated to P_K then in filtration $s > 0$ and p a prime number and total degree $t \leq s(2p-2) + |\Re_{min}| - 1$*

$$Ext^{s,t}_{A(U)}(BP_*(\textstyle\prod_n S^j)) \Longrightarrow Ext^{s+1,\,t}_{\mathcal{G}}(F)$$

where j is equal to the degree of the primitives in R and $n = |I|$.

Proof. Since B_TP is a space there exists a CFSS (for the moment Suppressing the total degree t) $Ext^s_{A(U)}(R^iP(F)) \Longrightarrow Ext^{s+i}_{\mathcal{G}}(F)$. By Theorem 7.1.12 the co-algebra F is nice in degrees less than $|\Re_{min}|$. There is an isomorphism of $BP_* - modules$ between $R^1P(F)$ and $BP_*(\prod_n S^j)$ where n equals the number of primitives in the co-algebra R and the dimensions of the spheres equals the degrees of the primitives in the co-algebra R. Theorem 7.2.7 shows that the U co-module structure on R^1PF coincides with the co-action on a sphere. This gives the the following isomorphism of E_2 terms of the CFSS $Ext^s_{A(U)}(R^1P(F)) \cong Ext^s_{A(U)}((BP_*(\prod_n S^j)))$. To obtain a least upper bound on the total degree t let $s > 0$. The element $\overbrace{h_1 \otimes \cdots \otimes h_1}^{s} \otimes m$ has degree $s(2p-2) + |m|$. However, $|\ \Re_{min}|$ is a strict upper bound on $|\ m|$ giving $|h_1^{\otimes_s} \otimes m| \leq s(2p-2) + |\ \Re_{min}| - 1$. ☐

Remark 7.3.5. *Assuming that* $I =< r_1, .., r_k >$ *the* E_2 *page of the UNSS will consist of a collection of* η *towers (one for each monomial* $r_i \in I$*). Each of these towers will have a homotopy tower corresponding to the elements* $\alpha, \eta\alpha, \eta^2\alpha \cdots$ *with a* d_3 *differential such that* $d_3(\alpha) = \eta^3$ *where* η^3 *represents the element* $h_1 \otimes h_1 \otimes h_1 \otimes \iota_k$*. Hence, the elements* $\eta^3, \eta^4, \eta^5 \cdots$ *in each tower will be killed by the* d_3 *differential (also a derivation) out of the corresponding homotopy tower. It should be noted that there is an abuse of notation in 7.3.4. The element* α *is called* $\widetilde{\alpha}_{4k+2}$ *in [Bend2] and the* d_3 *differential comes out of* $\widetilde{\alpha}_{4k+2}$ *into the element* $(\eta^3\widetilde{\alpha}_{4k+1})$ *where* $\alpha_1 = \eta$ *when* $k = 0$*. For the definition of* $\widetilde{\alpha_k}$ *and the determination of the unstable 1-line for* S^{2n+1} *the reader should refer to [Bend2].*

Remark 7.3.6. *Theorem 6.33 page 96 in [BP4] follows trivially from the previous*

Theorem if one were to analyze the Ext term $Ext^{0,\,|\,r_i\,|}_{A(U)}(R^1PF)$ *where* r_i *is a monomial that generates* I *in the* BP^**-Face ring.*

Example 7.3.7. *Consider the simplicial complex 7.1.8. By 7.3.4*

$$Ext^{s,t}_{A(U)}(BP_*(S^4 \times S^6)) \Longrightarrow Ext^{s+1,t}_{\mathcal{G}}(F)$$

It is easily determined that $|\Re_{min}| = 8$. *Since*

$$Ext^{s,\,t}_{A(U)}(BP_*(S^4 \times S^6)) = Ext^{s,\,t_1}_{A(U)}(BP_*(S^4)) \oplus Ext^{s,\,t_2}_{A(U)}(BP_*(S^6))$$

we write $t \leq s(2p-2)+7$. *In Filtration* $s = 0$ *we have* $Ext^{0,\,t}_{A(U)}(BP_*(S^4)) \oplus Ext^{0,\,t}_{A(U)}(BP_*(S^6))$ *where* $t \leq 7$ *which is clearly satisfied. In* $s = 1$ *we have* $Ext^{1,\,t}_{A(U)}(BP_*(S^4)) \oplus Ext^{1,\,t}_{A(U)}(BP_*(S^6))$.

Example 7.3.8. *Let* P^2 *be the 1-neighborly simple convex polytope given in example 7.1.3. Recall, the ideal in the* BP^* *face ring is* $I = < v_1v_3, v_2v_4, v_3v_5, v_4v_1, v_5v_2 >$. *By theorem 6.33 in [BP4] we have* $\pi_3(Z_P) \cong Z^{\oplus 5}$. *Notice that we also have the isomorphism* $\pi_3(M^4(\lambda)) \cong \pi_3(Z_P)$ *where* $M^4(\lambda)$ *is the family of Quasi-toric manifolds that sit over* P^2. *Clearly,* $|\,\Re_{min}| = 6$. *Let* $p = 2$. *In filtration* $s = 0$ *and* $* \leq 5$ *we have* $Ext^{0,\,*}_{A(U)}(R^1PF)$. *For* $* = 4$ *we have*

$$Ext^{0,4}_{A(U)}(R^1P(F)) \Longrightarrow Ext^{1,4}_{\mathcal{G}}(F)$$

where

$$Ext_{\mathcal{G}}^{1,4}(F) \Longrightarrow \pi_3(B_T P)$$

recovering Buchstaber's theorem.

Notice that the bottom of the η*-towers are at the points* $(3,1)$ *in the* E_2 *page of the UNSS. In filtration* $s = 1$ *and* $* \leq 7$ *we have* $Ext_{A(U)}^{1,*}(R^1 P(F)) \Longrightarrow Ext_{\mathcal{G}}^{2,*}(F)$*. Note that there are five generators of* $R^1 PF$*. By abuse of notation let* ι *denotes any of these generators. The elements* $h_1 \otimes \iota$ *have degree 6 and fall within the range of the total degree giving*

$$Ext_{A(U)}^{1,6}(R^1 P(F)) \Longrightarrow Ext_{\mathcal{G}}^{2,6}(F)$$

where

$$Ext_{\mathcal{G}}^{2,6}(F) \Longrightarrow \pi_4(B_T P)_{(2)}$$

This represents the class η *(the four dimensional generator) similarly, in filtration* $s = 2$ *and* $* \leq 9$ *the elements* $h_1 \otimes h_1 \otimes \iota$ *have degree 8 and fall within the range of the total degree giving*

$$Ext_{A(U)}^{2,8}(R^1 P(F)) \Longrightarrow Ext_{\mathcal{G}}^{3,8}(F)$$

where

$$Ext_{\mathcal{G}}^{3,8}(F) \Longrightarrow \pi_5(B_T P)_{(2)}$$

Therefore the groups through degree five are determined. One has to observe that R^2PF *has not yet appeared.*

Example 7.3.9. *Let* $p = 2$*. Suppose that for* $i \neq j$ *the monomials* r_i *and* r_j *differ by only one* v *and* $\mid r_i| = \mid r_j| = 2q+2$*. This implies that* $\mid \Re_{min}| = 2q+4$*. For each monomial* $r_i \in I$ *we have* $Ext^{0,\,2q+2}_{A(U)}(R^1P(F)) \Longrightarrow Ext^{1,\,2q+2}_{\mathcal{G}}(F)$ *converging to* $\pi_{2q+1}(F)$ *recovering Buchstaber's result. In filtration* $s = 1$, $* \leq 2q+5$*. The degree of* $h_1 \otimes \iota_{2q+2}$ *is* $2q+4$*. Hence,* $Ext^{1,\,2q+4}_{A(U)}(R^1P(F)) \Longrightarrow Ext^{2,\,2q+4}_{\mathcal{G}}(F)$ *converges to* $\pi_{2q+2}(F)_{(2)}$*. In filtration* $s = 2$, $* \leq 2q+7$*. The degree of* $h_1 \otimes h_1 \otimes \iota_{2q+2}$ *is* $2q+6$*. Hence, we have the following sequence of spectral sequences eventually converging to* $\pi_{2q+3}(F)_{(2)}$

$$Ext^{2,\,2q+6}_{A(U)}(R^1P(F)) \Longrightarrow Ext^{3,\,2q+6}_{\mathcal{G}}(F)$$

By theorem 7.3.10 the element $h_1^{\otimes 4} \otimes \iota_{2q+2}$ *is hit by a* d_3 *out of the corresponding homotopy tower. The result follows by the fibrations described in chapter 4 and for dimensional reasons since there can not be a differential out of one tower killing an element in the other tower.*

Theorem 7.3.10. *Let* p *be a prime number and* $m = \sharp$ *of facets of* P *and* $n = |I|$*. We have for* t *satisfying the hypothesis in 7.3.4*

$$\pi_{t-s-1}(B_TP)_{(p)} = \begin{cases} \mathbb{Z}^{\oplus m} & \text{if } s = 0,\ t \leq |\Re_{min}| - 1 \\ \pi_{t-s-1}((\prod_n S^j))_{(p)} & \text{for } t-s \leq |\Re_{min}| - 1 \end{cases}$$

Proof. In filtration $s > 0$ the result follows from 7.3.4. In filtration $s = 0$ and total degree $t = 2$ we have $Ext^{0,2}_{A(U)}(P(F)) \Longrightarrow Ext^{0,2}_{\mathcal{G}}(F)$. By theorem 7.1.12 $PF \cong PC$ as BP_{-*} modules and the U co-modules coincide by theorem 7.2.7 giving $Ext^{0,2}_{A(U)}(P(C)) \Longrightarrow Ext^{0,2}_{\mathcal{G}}(F)$. The CFSS contributes a $\mathbb{Z}^{\oplus m}$ in $Ext^{0,2}_{\mathcal{G}}(F)$. Hence, there is a $\mathbb{Z}^{\oplus m}$ in $\pi_2(B_T P)$

□

This gives the following

Proposition 7.3.11. *Let P be an n dimensional, q-neighborly simple convex polytope. Suppose Z_P, $M^{2n}(\lambda)$ and $B_T P$ are the associated moment angle complex, Quasi-toric manifold and borel space. For $* \geq 3$*

$$\pi_{*-1}(M^{2n}(\lambda))_{(p)} = \begin{cases} \mathbb{Z}^{\oplus m} & \text{if } s = 0,\ * \leq |\Re_{min}| - 1 \\ \pi_{*-1}((\prod_n S^j))_{(p)} & \text{for } * \leq |\Re_{min}| - 1 \end{cases}$$

Proof. By Cohen-Macaulyness $R^i PBP_*(M^{2n}) \cong R^i PBP_*(B_T P)$. The result follows immediately from 7.3.10

□

Notice that the upper bound on the total degree can be extended if a different prime were chosen. As long as the total degree t satisfies the hypothesis of 7.3.10 we have the

Proposition 7.3.12. *Let P^2 be a polygon with $m \geq 5$ vertices and $M^4(\lambda)$ the corre-*

sponding family of Quasi-toric manifolds. At the prime $p = 2$

$$\pi_4(\sharp_{m-2}\mathbb{C}P^2)_{(2)} \cong \pi_4(\textstyle\prod_{|I|} S^4)_{(2)}$$

Proof. [OR] proved that 4 dimensional Quasi-toric manifolds are connected sums of two dimensional complex projective space. Observe $|\Re_{min}| = 6$. The result follows from 7.3.11 and 7.4.3. □

We recall that Quasi-toric manifolds do not necessarily exist over P_K when K is not a simplicial sphere. Moreover, for such K, Z_K exists but it is not necessarily a manifold. However, the borel space always exists.

Corollary 7.3.13. *Let* K *be an* $n-1$ *dimensional,* q *neighborly simplicial complex on* $[m]$ *and* Z_K *the associated moment angle complex, for* p *a prime number we have*

$$\pi_{*-1}(Z_K)_{(p)} = \begin{cases} \mathbb{Z}^{\oplus m} & \text{if } s = 0,\ * \leq |\Re_{min}| - 1 \\ \\ \pi_{*-1}((\prod_n S^j))_{(p)} & \text{for } * \leq |\Re_{min}| - 1 \end{cases}$$

Proof. Follows from the fibration $Z_K \to B_T Z_K \to BT^m$ in combination with 7.3.10. □

Recall, PR are the primitives in the co-algebra R that represent the relations. Using the minimal resolution of $k(K)$ the bottom dimensional cells of Z_K are in degrees $|PR|-1$. By the connectivity of Z_K and the Hurewicz theorem we immediately obtain the

Corollary 7.3.14. $\pi_{*-1}(Z_K)_{(p)} = \pi_{*-1}(\bigvee_{|I|} S^{|PR|-1})_{(p)}$

Remark 7.3.15. *A similar statement does not hold for B_TP since $\pi_2(B_TP) \neq 0$.*

This corollary will be critical in determining the homotopy type of the simplicial complex $K_{[m]}$ (see 7.5.2). The reader should note that The CW structure of the moment angle complex associated to $K_{[m]}$ is given in 7.5.9.

Proposition 7.3.16. *Given the complex $K_{[m]}$ of 7.5.2. Let $Z_{K_{[m]}}$ be the associated moment angle complex. The top dimensional cell e^{2m-2} of $Z_{K_{[m]}}$ gives rise to a $\mathbb{Z}$ in π_{2m-2} detected by the class in the Ext term*

$$Ext^{0,\,|\,\Re_{min}|}_{\mathcal{A}(\mathcal{U})}(R^2PF)$$

Proof. For $m \geq 4$, $|\,\Re_{min}| = 2m$. First, consider

$$Ext^{0,\,|\,\Re_{min}|-1}_{A(U)}(R^1P(F)) \Longrightarrow Ext^{1,\,|\,\Re_{min}|-1}_{\mathcal{G}}(F) \Longrightarrow \pi_{2m-2}(F)$$

$Ext^{0,\,|\,\Re_{min}|-1}_{A(U)}(R^1P(F)) = 0$ by 7.3.2 and 7.3.3. Next consider the E_2 term,

$$Ext^{1,\,|\Re_{min}|}_{A(U)}(R^1P(F))$$

The bottom dimensional cells of $Z_{K_{[m]}}$ are given in 7.5.9. The classes in $Ext^{1,\,|\Re_{min}|}_{A(U)}(R^1P(F))$ contribute to the homotopy of $S^{2r-1}\bigvee S^{2r-1}$ for m odd and $S^{2r-1}\bigvee S^{2m-2r+1}$ for m even localized at p. No new classes that detect the cell e^{2m-2} are produced. The only way the cell e^{2m-2} is detected is by analyzing

$$Ext_{A(U)}^{0,\,|\,\Re_{min}|}(R^2P(F)) \Longrightarrow Ext_{\mathcal{G}}^{2,\,|\,\Re_{min}|}(F) \Longrightarrow \pi_{2m-2}(F)$$

Notice that $|\Re_{min}|$ is outside the range where the spectral sequence is known. We have absolutely no control over R^2PF. However, by 7.5.9 $\pi_{2m-2}(F) \neq 0$. The cell e^{2m-2} must be detected. This can only happen if

$$Ext_{A(U)}^{0,\,|\,\Re_{min}|}(R^2P(F)) \neq 0$$

Therefore, $R^2PF \neq 0$. □

Remark 7.3.17. *It follows from 7.5.9 that the term* $Ext_{A(U)}^{s,\,t}(R^1P(F))$ *contributes to the homotopy of the bottom dimensional cells for any moment angle complex* Z_K*.*

There is no systematic way to construct an unstable $\mathcal{G}$ resolution in such a way as to get a handle on R^2P. However, the previous proposition gives the following

Corollary 7.3.18. *Given the moment angle complex complex* $Z_{K_{[m]}}$ *of 7.5.2.*

$$R^2PBP_*F \cong BP_*\{\iota\}$$

where $|\iota| = |\,\Re_{min}|$

By 7.3.16 we obtain the

Corollary 7.3.19. *Given the moment angle complex complex* $Z_{K_{[m]}}$ *of 7.5.2 we have*

$$\pi_{*-1}(Z_{K_{[m]}})_{(p)} = \begin{cases} \pi_{*-1}(S^{2r-1} \bigvee S^{2r-1})_{(p)} & \textit{for } m \textit{ odd, } * \leq |\Re_{min}| - 1 \\ \\ \pi_{*-1}(S^{2r-1} \bigvee S^{2m-2r+1})_{(p)} & \textit{for } m \textit{ even, } * \leq |\Re_{min}| - 1 \end{cases}$$

Remark 7.3.20. *In lieu of remark 7.3.17 the homotopy of the complex* $Z_{K_{[m]}}$ *looks like a wedge of spheres by the classes contributed by* $Ext^{s,\,t}_{A(U)}(R^1P(F))$*. The connectivity of* $Z_{K_{[m]}}$ *is essential in making this deduction. A similar statement does not hold for* B_TP *since* $\pi_2(B_TP) \neq 0$*.*

7.4 The Homotopy Groups of certain Quasi-Toric Manifolds, Connected Sums and Wedges via Toric Geometry

Recall, for any simple convex polytope P we have a Quasi-toric manifold M^{2n}, the moment angle complex Z_P and the borel space B_TP. The homotopy groups of these spaces are related by the following collection of interlocking fibrations which always exist for such P [BP1].

$$\begin{tikzcd} T^{m-n} \arrow[r] & Z_P \arrow[r] & M^{2n} \end{tikzcd}$$

$$M^{2n} \longrightarrow B_T M^{2n} \longrightarrow BT^n$$

and

$$Z_P \longrightarrow B_T Z_P \longrightarrow BT^m$$

We immediately observe

Proposition 7.4.1. *For $* \geq 3$, $\pi_*(M^{2n}) \cong \pi_*(Z_P) \cong \pi_*(B_T P)$*

Proof. $\prod_j \mathbb{C}P^\infty$ is a $K(\mathbb{Z}^j, 2)$-type space. Examine the LES that is induced by applying $\pi_*(-)$ to the fibrations above.

□

The fibrations above hold as long as the boundary complex of P is dual to a $(n-1)$-sphere. That is, the dual simplicial complex is an $(n-1)$-dimensional simplicial sphere. In general, the first two fibrations may fail to exist since a Quasi-toric manifold (or space) may not exist for general K. We list the following theorem of Buschstaber which computes the first higher homotopy group of some of the spaces above when P is an n dimensional simple polyhedral complex.

Theorem 7.4.2. *For any simple polyhedral complex P with m co-dimension one faces we have*

1. $\pi_1(Z_P) = \pi_1(B_T P) = 0$
2. $\pi_2(Z_P) = 0$
3. $\pi_2(B_T P) = \mathbb{Z}^m$

4. *For* $q \geq 3$, $\pi_q(B_TP) \cong \pi_q(Z_P)$

5. *For* P^n *q-neighborly we have*

$$\pi_i(Z_P) = \begin{cases} 0 & \textit{if } i < \textit{2q+1} \\ F_{ab}\langle m \rangle & \textit{For } m \in I,\ i = \textit{2q+1} \end{cases}$$

Proof. See [BP1] □

This theorem follows trivially from 7.3.10. The following proposition is enormously important and was communicated by A. Bahri. We assume that P is a n dimensional simple convex polytope and that $M^{2n}(\lambda)$ is the family of quasi-toric manifolds that sit over P. Eventhough quasi-toric manifolds depend on the choice of λ their homotopy groups do not as illustrated by the

Proposition 7.4.3. *If* $\lambda_1 \neq \lambda_2$, *then* $\pi_*(M^{2n}(\lambda_1)) \cong \pi_*(M^{2n}(\lambda_2))$ *for* $* \geq 3$.

Proof. Let λ_1 and λ_2 be two maps from $\mathbb{Z}^m \to \mathbb{Z}^n$ satisfying condition $(*)$ such that $\lambda_1 \neq \lambda_2$. Let $M^{2n}(\lambda_1)$ and $M^{2n}(\lambda_2)$ be the corresponding Quasi-toric manifolds. Using the fibration:

$$M^{2n} \to B_TP \to \textstyle\prod_n CP^\infty$$

we obtain

$$M^{2n}(\lambda_1) \to B_TP \to \textstyle\prod_n CP^\infty$$

and

$$M^{2n}(\lambda_2) \to B_T P \to \prod_n CP^\infty$$

The Borel space $B_T P$ is fixed for each λ and only depends on P. Apply $\pi_*(-)$ and the result follows. □

In other words the homotopy groups of a Quasi-toric manifold are independent of λ. Recall from 2.3 that B_n is the toric manifold of all bounded flags in $\mathbb{C}^{n+1}$. Its quotient polytope is the the cube I^n.

Proposition 7.4.4. *For* $* \geq 3$, $\pi_*(B_n) \cong \pi_*(\prod_n S^3)$

Proof. I^n is combinatorially equivalent to the simple convex polytope $\prod_n \Delta^1$. By 3.1.4 $Z_{\prod_n \Delta^1} \cong \prod_n S^3$ The result follows by analyzing the fibration:

$$T^{m-n} \to Z_P \to M^{2n}$$

□

Recall from 2.3 the family $B_{i,j}$ of quasi-toric manifolds that sit over $I^i \times \Delta^{j-1}$. This leads to our next observation

Proposition 7.4.5. *For* $* \geq 3$, $\pi_*(B_{i,j}) \cong \pi_*(\prod_i S^3 \times S^{2j-1})$

Proof. Apply the argument given in the proof of 7.4.4. □

There is no known technique that allows for the computation of $\pi_*(\sharp_n \mathbb{C}P^2)$ or $\pi_*(\bigvee_m \mathbb{C}P^\infty)$. It will be shown that knowing the homotopy of a wedge of spheres gives the homotopy of $\bigvee_m \mathbb{C}P^\infty$. Of course, the Milnor-Hilton theorem can be used to make homotopy computations which is interesting since neither space is a suspension. First,

Theorem 7.4.6. *For* $* \geq 3$, $\pi_*(CP^n \sharp CP^n) \cong \pi_*(S^3) \oplus \pi_*(S^{2n-1})$

Proof. [BR2] proved that $\mathbb{C}P^n \sharp \mathbb{C}P^n$ is a Quasi-toric manifold over $\Delta^n \sharp \Delta^n$. This polytope is combinatorially equivalent to the simple convex polytope $\Delta^1 \times \Delta^{n-1}$ [BP4]. We know that $Z_{\Delta^1 \times \Delta^{n-1}} \cong S^3 \times S^{2n-1}$ by 3.1.4. The result follows by applying $\pi_*(-)$ to the fibration

$$T^{m-n} \longrightarrow Z_P \longrightarrow M^{2n}$$

□

Lemma 7.4.7. *If* K *is the disjoint union of* m *vertices then*

$$DJ(K) = \bigvee_m \mathbb{C}P^\infty$$

Proof. Recall from 4.3 that $DJ(K) = K_\bullet(\mathbb{C}P^\infty, *)$ and from 4.2 that

$$K_\bullet(\mathbb{C}P^\infty, *) = \bigcup_{\sigma \in K} \left(\prod_{i \in \sigma} \mathbb{C}P^\infty \times \prod_{i \notin \sigma} \{*\}\right)$$

More explicitly we have

$$K_\bullet(\mathbb{C}P^\infty, *) = \mathbb{C}P^\infty \times \{*\} \times \cdots \times \{*\} \bigcup \cdots \bigcup \{*\} \times \cdots \times \{*\} \times \mathbb{C}P^\infty$$

with the obvious identifications along the boundary.

□

In [GT] it was shown that $U(K)$ is a certain wedge of spheres when K is the disjoint union of m points. This leads to the following

Theorem 7.4.8. *For* $* \geq 3$, $\pi_*(\bigvee_m \mathbb{C}P^\infty) \cong \pi_*(\bigvee(\bigvee_{k=2}^m (k-1)\binom{m}{k} S^{k+1}))$

Proof. The result follows from 7.4.7 and 5.1.8 combined with analyzing the fibration:

$$Z_K \longrightarrow B_T Z_K \longrightarrow BT^m$$

□

The next example will illustrate how the Strickland construction can be used to determine the Borel space or the moment angle complex explicitly when K is the disjoint union of a small number of vertices. The calculation becomes difficult for sufficiently complicated simplicial complexes K.

Example 7.4.9. *For* $* \geq 3$, $\pi_*(S^3) \cong \pi_*(\mathbb{C}P^\infty \bigvee \mathbb{C}P^\infty)$

Let K be the disjoint union of two vertices. We have the following fibration

$$Z_K \longrightarrow B_T Z_K \longrightarrow \mathbb{C}P^\infty \times \mathbb{C}P^\infty$$

Since $B_T Z_K$ is a deformation retract of the Davis-Januszkiewicz space $DJ(K)$ the homotopy groups of the two spaces coincide in dimensions greater than two giving the diagram

$$\pi_*(Z_K) \xrightarrow{\cong} \pi_*(B_T Z_K) \xrightarrow{\cong} \pi_*(DJ(K))$$

Applying Strickland's construction we have

$$Z_K = D^2 \times S^1 \bigcup S^1 \times D^2$$

attached along the boundary and

$$DJ(K) = \mathbb{C}P^\infty \times \{*\} \cup \{*\} \times \mathbb{C}P^\infty$$

with similar attachments along the edges clearly giving $\mathbb{C}P^\infty \bigvee \mathbb{C}P^\infty$. Apply $\pi_*(-)$ to the fibration above and the result follows.

Remark 7.4.10. *Notice that the isomorphism in the previous theorem is induced by a map of spaces. In fact, we have the fibration*

$$S^3 \to \mathbb{C}P^\infty \bigvee \mathbb{C}P^\infty \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$$

$\mathbb{C}P^\infty$ is not a suspension of a space X so, the Hilton-Milnor theorem does not apply. However, 7.4.8 shows that $\bigvee_m \mathbb{C}P^\infty$ can be computed using the Hilton-Milnor theorem.

7.5 The Homotopy Type of the Moment Angle Complex associated to certain simplicial complexes

In this section a simplicial complex which realizes a particular face ring is constructed. The dimension of the relations among relations is maximized when the the monomials that generate I have as few terms in common as possible. The minimal resolution of the face ring $k(K)$ is determined from which it follows that the associated moment angle complex, Z_K will be a wedge of spheres union a cell e^k whose dimensions depends on the degree of the relation among relations. The attaching map can be determined by studying the homotopy groups of the Borel space associated to to the polyhedron P dual to K and using the fibrations described in 4.3. The techniques developed in the previous sections will be used to determine the attaching map by a filtration argument. As a result the homotopy type of Z_K will be determined using methods from unstable homotopy theory. It follows immediately that the homoptopy groups of $B_T K$ which is an infinite dimensional complex will be determined in terms of the homotopy groups of spheres.

Definition 7.5.1. *For each integer $m \geq 4$ define*

$$r = [(m+1)/2]$$

where $[n]$ is the greatest integer value of n

We make the

Definition 7.5.2. *Let* $K_{[m]}$ *be the simplicial complex on* $[m]$ *with the missing faces* $\{1,..,r\}$ *and* $\{r,r+1,..,m\}$

Lemma 7.5.3.

$$dim\ K_{[m]} = \begin{cases} 2r-3 & m\ odd \\ \\ 2r-2 & m\ even \end{cases}$$

Proof. For each $m \geq 4$ an odd integer consider the face ring $k(K_{[m]})$. The ideal $I = \langle m_1, m_2 \rangle$ is generated by two monomials of the following form: $r_1 = v_1 \cdot \cdot v_r$ and $r_2 = v_r \cdot \cdot v_m$. It follows immediately that $|r_1| = |r_2|$. We form the simplex of maximal dimension σ_m in the following way. First, we pick the first $(r-1)$ v's from r_1. Then we pick the last $(r-1)$ v's from r_2. Let σ_m be the union of these two sets. This gives $\sigma_m = \{v_1, .., v_{r-1}, v_{r+1}, .., v_m\}$ Since exactly $(r-1)$ v's were picked from each monomial the cardinality of $\sigma_m = 2r-2$. This simplex is unique. For if there were another set τ whose cardinality were greater than σ_m then τ would have to contain v_r. The even case is similar.

□

For convenience we write $dim\ K_{[m]} = m-2$

Remark 7.5.4. *It is important to keep in mind that the homotopy type* $Z_{K_{[m]}}$ *will be determined. The dimension of the complex* $K_{[m]}$ *is consistent with the dimension of* $Z_{K_{[m]}}$ *since the homological dimension of a finitie CW complex does not determine its homotopy type. Using the block decomposition of* $Z_{K_{[m]}}$ *with a Mayer-Vietoris spectral sequence argument A. Bahri observed that this condition is equivalent to certain trivial intersections among blocks of various sub-complexes of* $Z_{K_{[m]}}$.

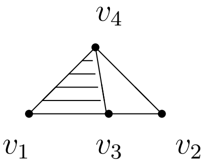
Let I be a square free monomial ideal generated by $r_1 = v_1 \cdots v_r$ and $r_2 = v_r v_{r+1} \cdots v_m$

Proposition 7.5.5. *The face ring of $K_{[m]}$ is the algebra $\mathbb{Z}[v_1, .., v_m]/\langle I \rangle$*

Proof. Follows immediately from 7.5.2 and the definition of face ring. ☐

To show that the face ring is in fact realizable let $m = 4$ then we have the simplicial complex:

$K^2_{[4]} =$



When $m = 5$ we have the simplicial complex: $K^3_{[5]}$ whose simplicies we list below:

0-simplices	1-simplicies	2-simplicies	3-simplex
$\{1\}$	$\sigma_1 = \{1,2\}$	$\overline{\sigma_1} = \{1,2,4\}$	$\sigma_m = \{1,2,4,5\}$
$\{2\}$	$\sigma_2 = \{1,3\}$	$\overline{\sigma_2} = \{1,2,5\}$	
$\{3\}$	$\sigma_3 = \{1,4\}$	$\overline{\sigma_3} = \{1,3,5\}$	
$\{4\}$	$\sigma_4 = \{1,5\}$	$\overline{\sigma_4} = \{1,3,4\}$	
$\{5\}$	$\sigma_5 = \{2,3\}$	$\overline{\sigma_5} = \{1,4,5\}$	
	$\sigma_6 = \{2,4\}$	$\overline{\sigma_6} = \{2,3,4\}$	
	$\sigma_7 = \{2,5\}$	$\overline{\sigma_7} = \{2,3,5\}$	
	$\sigma_8 = \{3,4\}$	$\overline{\sigma_8} = \{2,4,5\}$	
	$\sigma_9 = \{3,5\}$		
	$\sigma_{10} = \{4,5\}$		

It is left it to the reader to to draw a picture of $K^3_{[5]}$. In [DJ] it was shown that a simplicial complex K^{n-1} can be dualized via the cone over the barycentric subdivision of the complex K^{n-1} to give a simple polyhedron P^n see 1.7. Polyhedra of this type can be constructed for any simplicial complex K. It follows immediately that

$$dim\ P_{K_{[m]}} = \begin{cases} 2r-2 & \text{m odd} \\ \\ 2r-1 & \text{m even} \end{cases}$$

It is not clear whether a toric space exists over P_K. It may be interesting to construct T-spaces over $P_{K_{[m]}}$. If one applies the techniques developed in 7.1 it is clear that

$$deg\ (\Re_{min}) = \begin{cases} 4r-2 & \text{m odd} \\ \\ 4r & \text{m even} \end{cases}$$

Remark 7.5.6. *For m odd $|\Re_{min}| = 4r-2$ one can deduce that $K_{[m]}$ produces an injective extension sequence up to dimension $4r-3$. For m even one can deduce that the BP face ring of $K_{[m]}$ produces an injective extension sequence through degree $4r-1$.*

The following lemma will be useful in extracting information concerning the homotopy groups of the associated moment angle complex.

Lemma 7.5.7. *For $m \geq 4$ an integer, $K_{[m]}$ is $(r-1)$-neighborly.*

Proof. By construction $K_{[m]}$ has two missing faces coming from the monomials r_1 and r_2 which generate the ideal I in the face ring. It follows that any $(r-1)$ element subset of $[m]$ is a simplex. □

We will use the minimal resolution of the face ring, $k(K_{[m]})$ to determine the bi-graded betti-numbers of the moment angle complex associated to $K_{[m]}$. As a matter of notational convenience let $P = \mathbb{Z}[v_1, .., v_m]$. Let $p \in P$. To determine $\beta^{-i,2j}$ take the the minimal resolution of $k(K_{[m]})$. This gives the following chain complex: Let $x_{\Re_{min}}$ correspond to the relation among relations where $|x_{\Re_{min}}| = |\Re_{min}|$

$$\begin{tikzcd}
0 \arrow[r, "d_4"] & P \otimes \{x_{\Re_{min}}\} \arrow[r, "d_3"] & P \otimes \{v_{1\cdots r}, v_{r\cdot(r+1)\cdots m}\} \arrow[r, "d_2"] & P \otimes \{\alpha_0\} \arrow[r, "d_1"] & k(K_{[m]}) \arrow[r] & 0
\end{tikzcd}$$

where the maps are defined in the following way:

$$d_1(p \otimes \alpha_0) = p \otimes 1$$

$$d_2(p \otimes v_{1\cdots r}) = p \cdot (v_1 \cdots v_r) \otimes \alpha_0$$

$$d_2(p \otimes v_{r\cdot(r+1)\cdots m}) = p \cdot (v_r v_{r+1} \cdots v_m) \otimes \alpha_0$$

$$d_3(p \otimes x_{\Re_{min}}) = p \cdot (v_{r+1} \cdot \cdot v_m) \otimes v_{1\cdots r} - p \cdot v_1 \cdot \cdot v_r \otimes v_{r+1\cdots m}$$

Clearly d_3 is an injection. In fact, it follows from the fact that there are no relations among relations among relations. An upper bound on the homological dimension of the face ring $k(K_{[m]})$ is given by the chain complex above. This resolution is manageable and will allow for the determination of the bi-graded betti-numbers of the associated moment angle complex. To determine the terms $\beta^{-i,2j}(k(K_{[m]})$ we determine $dim\ _P\, Tor_P^{-i,2j}(P, k(K_{[m]}))$. Tensor the complex above with P and take homology of the complex.

$$0 \longrightarrow \{x_{\Re_{min}}\} \longrightarrow \{v_{1\cdots r}, v_{r\cdot(r+1)\cdots m}\} \longrightarrow \{\alpha_0\} \longrightarrow 0$$

To obtain the following

Proposition 7.5.8. *For $m \geq 5$ an odd integer*

$$\beta^{-i,2j}(k(K_{[m]})) = \begin{cases} 2 & i = 1,\ j = r \\ 1 & i = 2,\ j = m \\ 0 & otherwise \end{cases}$$

and when $m \geq 4$ an even integer

$$\beta^{-i,2j}(k(K_{[m]})) = \begin{cases} 1 & i = 1,\ j = r \\ 1 & i = 1,\ j = m-r+1 \\ 1 & i = 2,\ j = 2m \\ 0 & otherwise \end{cases}$$

This allows us to determine the bi-graded betti numbers of the associated moment angle complex giving the additive structure of the co-homology of the moment angle complex. This gives the

Theorem 7.5.9. *For $m \geq 5$ an odd integer. Let $Z_{K_{[m]}}$ be the associated moment angle complex. The betti numbers*

$$b_k = \begin{cases} 2 & k = 2r-1 \\ 1 & k = 2m-2 \\ 0 & otherwise \end{cases}$$

and when $m \geq 4$ an even integer

$$b_k = \begin{cases} 1 & k = 2r\text{-}1,\ 2m\text{-}2r\text{+}1,\ 2m-2 \\ 0 & otherwise \end{cases}$$

Proof. The result follows from 7.5.8 and 3.3.2. □

We immediately have

Corollary 7.5.10. *For $m \geq 4$ an integer the moment angle complex $Z_{K_{[m]}}$ is $2r-2$ connected.*

We also have the

Corollary 7.5.11. *If $m \geq 5$ an odd integer then the co-homology groups of the moment angle complex associated to $K_{[m]}$ are as follows.*

$$H^p(Z_{K_{[m]}}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & p = 2r-1 \\ \mathbb{Z} & p = 2m-2 \\ 0 & otherwise \end{cases}$$

and when $m \geq 4$ *even*

$$H^p(Z_{K_{[m]}}) = \begin{cases} \mathbb{Z} & p = 2r-1,\ 2m-2r+1,\ 2m-2 \\ 0 & \textit{otherwise} \end{cases}$$

We will now determine the homotopy type of $Z_{K_{[m]}}$.

Theorem 7.5.12. *Let* $m \geq 4$ *be an integer and* $Z_{K_{[m]}}$ *the associated moment angle complex. We have*

$$Z_{K_{[m]}} \simeq \begin{cases} S^{2r-1} \bigvee S^{2m-2r+1} \bigvee S^{2m-2} & m \textit{ even} \\ S^{2r-1} \bigvee S^{2r-1} \bigvee S^{2m-2} & m \textit{ odd} \end{cases}$$

Proof. The dimension of the top dimensional cell is $2m-2$ independent of the parity of m. Suppose $f : S^{2m-3} \to S^i \bigvee S^j$ is a non-trivial attaching map where $i = j = 2r-1$ if m is odd and $i = 2r-1$, $j = 2m-2r+1$ otherwise. Since f is non-trivial there must be a d_2 differential out of $(2, 2m)$ into $(4, 2m+1)$. The classes $\eta^i \cdot \iota$ for $i \leq 3$ are even dimensional. Hence, the smallest possible differential is a d_3 which jumps to filtration 5. Therefore, all the classes survive. Hence, f is trivial. The same argument works for m even. To complete the proof it must be shown that the differential internal to the

CFSS is zero. ie.,

$$Ext^{0,\,|\,\Re_{min}\,|}_{A(U)}(R^2PF) \longrightarrow Ext^{2,\,|\,\Re_{min}\,|}_{A(U)}(R^1PF)$$

We can use the following picture of the E_2 term of the CFSS and the fact that the co-algebra map $G(F) \to G(F) \otimes G(F)$ is multiplicative to show that the differential is zero. The argument is completely analogous to 7.2.7.

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \\
U^2[U(F)] \arrow[r] \arrow[u] & U^2[U(G(F))] \arrow[r] \arrow[u] & U^2[U(G^2(F))] \arrow[r] \arrow[u] & \bullet \\
U[U(F)] \arrow[r] \arrow[u] & U[U(G(F))] \arrow[r] \arrow[u] & U[U(G^2(F))] \arrow[r] \arrow[u] & \bullet \\
{[U(F)]} \arrow[r] \arrow[u] & {[U(G(F))]} \arrow[r] \arrow[u] & {[U(G^2(F))]} \arrow[r] \arrow[u] & \bullet
\end{tikzcd}$$

□

We obtain the

Proposition 7.5.13. *Let $U(K_{[m]})$ be the complex coordinate subspace arrangement complement associated to $K_{[m]}$ then we have*

$$U(K_{[m]}) \simeq \begin{cases} S^{2r-1} \bigvee S^{2m-2r+1} \bigvee S^{2m-2} & m \text{ even} \\ S^{2r-1} \bigvee S^{2r-1} \bigvee S^{2m-2} & m \text{ odd} \end{cases}$$

Proof. Follows from 7.5.12 and 5.1.8. □

The Hilton-Milnor theorem can be used to read off the homotopy groups of $U(K_{[m]})$. Recall, the complex $K_{[m]}$ is $r-1$ neighborly. Buchstaber and Panov's results would be able to determine the first higher non-trivial homotopy group. In this case they would be able to compute $\pi_{2r-1}(Z_{K_{[m]}})$. Using the UNSS it is possible (in theory) to determine the homotopy type of families of $U(K)$ as long as the difference between the dimensions of the top and bottom dimensional cells is reasonably controlled. In otherwords, the attaching map is from a sphere whose dimension is less than $|\Re_{min}|-1$.

Example 7.5.14. *Let $m = 5$. Consider the complex $Z_{K_{[5]}}$ An easy computation shows the betti-numbers of the associated moment angle complex $Z_{K_{[5]}}$ to be $b^5 = 2$ and $b^8 = 1$. It is clear that $Z_{K_{[5]}} = S^5 \bigvee S^5 \cup_f e^8$. Resolving the BP_* face ring of $K_{[5]}$ produces the following 8 dimensional relation among relations: $\iota^*(x_1)v_4v_5 - \iota^*(x_2)v_1v_2$. Suppose the attaching map f is non-trivial then there would have to be a non-trivial differential out of R^2PF that hits either $\eta^2 \cdot \iota_6$ or $\eta^2 \cdot \overline{\iota_6}$ as illustrated by the chart below. This is impossible since the differential sends $(2, 10)$ to $(4, 11)$ and there are no classes of total degree 11. Therefore, the smallest possible differential is a d_3 out of $(2, 10)$ which jumps to filtration 5.*

We obtain the

Proposition 7.5.15. *For an integer $m \geq 4$, $K_{[m]}$ is not a simplicial sphere.*

Proof. We prove the even case. The proof for the odd case is similar. Suppose $K_{[m]}$ were a simplicial sphere. By 3.3.10 the following relation must hold.

$$b_{-q,2p}(Z_{K_{[m]}}) = b_{-(m-n)+q,2(m-p)}(Z_{K_{[m]}})$$

Recall, dim $K_{[m]} = m-2$. We make the following substitutions. Let $q=1$, $p=r$ and $n=m-1$ giving the relation $b_{-1,2r} = b_{0,2(m-r)}$. Therefore, the relation $b_{2r-1} = b_{2(m-r)}$ must hold for the total betti number. By 7.5.9 we have $1 = b_{2r-1} \neq b_{2(m-r)} = 0$. Therefore, $K_{[m]}$ is not a simplicial sphere. □

Remark 7.5.16. *Once the homotopy type of $Z_{K_{[m]}}$ is known one can easily deduce 7.5.15. However, the argument given in the proof only relies on the CW structure of the associated moment angle complex, which can be useful for more general K as outlined in [BP4].*

It is well known that Z_K is a manifold when K is a simplicial sphere. Determining whether or not the moment angle complex $Z_{K_{[m]}}$ is a manifold by using combinatorial arguments such as $\overrightarrow{h}$ manipulations can be tedious. If K were a simplicial sphere then the fundamental class of Z_K must be represented. See 3.3.

Another interesting class of Z_K is produced by considering the simplicial complex $\overline{K_{[m]}}$ with missing faces $\{v_1, v_j\}$ and $\{v_j, v_{j+1}, .., v_m\}$ where $j < m$. It can be shown using the techniques developed in the preceding sections that

$$Z_{\overline{K_{[m]}}} \simeq S^3 \bigvee S^{2m-2j+1} \bigvee S^{2m-2j}$$

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