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INTRODUCTION

This research concerns measurability and more generally elementary embeddings, within the framework of ZFC.
The different chapters are relatively independent, but some unity can be found in the use of iterated ultrapowers as the main technical tool.

. The conceptual and technical background is defined in the first chapter. We introduce the notions of skies and constellations (A.Kanamori) which witness the complexity of an elementary embedding, and present (briefly) different variations of the notion of iterated ultrapower (K.Kunen, W.J.Mitchell).

. The second chapter is devoted to the study of non-closure properties of the image model. Let $j:V \longrightarrow M$ be an elementary embedding from the universe V into an inner model M. Large cardinal definitions are often expressed in terms of closure of M : compactness, supercompactness or hugeness. Contrarily, we show that some non-closure of M (for instance ${}^{\omega}M \subset M$ and ${}^{\omega_1}M \not\subset M$) also require large cardinal hypotheses, and give examples of elementary embeddings satisfying this pseudo-closure.

. In the third chapter, we study the connections between large cardinal assumptions and the absence of some fixed points of elementary embeddings. The difficulty to move such cardinals as inaccessible cardinals is related there to the existence of inner models with measurable cardinals.

In these two chapters, the construction of inner models with measurable cardinals is based either on K.Kunen's results about the constructible universe from a sequence of ultrafilters, or on W.J.Mitchell's work on the Core Model for sequences of measures.

. To answer a question of A.Taylor concerning the existence of a non-separative κ-ultrafilter, techniques of Forcing and iterated ultrapowers are used in the fourth chapter.

. The last chapter deals with the constructible universe for the infinitary language $L_{\omega_1\omega_1}$ (C.C.Chang). In analogy with L and $0^{\#}$, we obtain equiconsistency relations between existence of indiscernibles for this model and existence of measurable cardinals.

CONTENTS

NOTATION

This study is developped within ZFC. We use standard notation : κ, λ, μ ... are infinite cardinals, α, β, γ ... ordinals. Let $\alpha < \beta$ be two ordinals. $[\alpha, \beta[$ and $[\alpha, \beta]$ represent respectively the sets $\{\gamma \in \mathrm{Ord} : \alpha \leq \gamma < \beta\}$ and $\{\gamma \in \mathrm{Ord} : \alpha \leq \gamma \leq \beta\}$.

If X and Y are sets, ${}^{X}Y$ denotes the set of functions with domain X and values in Y. Whenever f belongs to ${}^{X}Y$, $f''X$ is $\{f(x) : x \in X\}$. Let $X' \subset X$. $f_{|X'}$ represents the restriction of f on X'.

We note $|X|$ the cardinal number of X and d the identity function.

If U is an ω_1-complete free ultrafilter on I , Ult_U denotes the transitive collapse of the ultrapower of the universe V with respect to U , j_U, the canonical elementary embedding from V into Ult_U, and for any function f with domain I , $[f]_U$ is a representative in Ult_U, of the equivalence class of f modulo U .

The expression κ-ultrafilter means free and κ-complete ultrafilter on κ.

I. PRELIMINARY DEFINITIONS AND PROPOSITIONS.

(A) Complexity of an elementary embedding.

Let us introduce the notions of skies and constellations defined by Puritz to witness the complexity of an elementary embedding. These two concepts have been studied more specifically in the measurable case by Kanamori [Ka.1] .

Given an elementary embedding $j:V\longrightarrow M$ with critical point κ and M a transitive class containing the ordinals, we define on the interval $[\kappa,j(\kappa)[$ two relations:

Definition 1.1: let $\gamma,\gamma'\in[\kappa,j(\kappa)[$. Then,

$\gamma\sim\gamma'$ iff for some $f,g\in{}^{\kappa}\kappa$, $j(f)(\gamma)\geqslant\gamma'$ and $j(g)(\gamma')\geqslant\gamma$.

$\gamma\longleftrightarrow\gamma'$ iff for some $f,g\in{}^{\kappa}\kappa$, $j(f)(\gamma)=\gamma'$ and $j(g)(\gamma')=\gamma$

For $\gamma\in[\kappa,j(\kappa)[$, the sky and constellation of γ are then given by: $\mathrm{Sk}(\gamma)=\{\gamma'<j(\kappa):\gamma'\sim\gamma\}$ and

$\mathrm{Con}(\gamma)=\{\gamma'<j(\kappa):\gamma'\longleftrightarrow\gamma\}$.

We give now some basic results on skies and constellations.

Proposition 1.1 [Ka.1]: (a) $\sim$ and $\longleftrightarrow$ are equivalence relations.

(b) A sky is a subinterval of $[\kappa,j(\kappa)[$ and a constellation is cofinally spread out within its sky.

Proof: (a) let us only check the transitivity of $\sim$. If $\gamma\sim\gamma'$ and

$\chi' \sim \chi''$, then let $\chi' \leqslant j(f)(\chi)$ and $\chi'' \leqslant j(g)(\chi')$. Since κ is regular, we can suppose g is increasing. Therefore $\chi'' \leqslant j(g \circ f)(\chi)$.
In the same way, there would exist $g', f' \in {}^{\kappa}\kappa$ such that $\chi \leqslant j(g' \circ f')(\chi'')$

(b) Let $\chi' \in Sk(\chi)$ and $\chi' \geqslant \chi$. There is f strictly increasing such that $j(f)(\chi) \geqslant \chi'$. Therefore if $\chi \leqslant \Theta \leqslant \chi'$, then $j(f)(\Theta) \geqslant \chi'$ which implies $\Theta \in Sk(\chi)$: a sky is an interval.
Since $j(f)(\chi) \in Con(\chi)$ and $j(f)(\chi) \geqslant \chi'$, this gives also the second assertion of (b). ■

We have thus obtained a partition of $[\kappa, j(\kappa)[$ in subintervals and each subinterval can in turn be decomposed in cofinal subsets.

The following concept was initially introduced by W.Rudin and Choquet in the study of ultrafilters over $\mathbb{N}$.

<u>Definition 1.2</u> : Let κ be regular . $f \in {}^{\kappa}\kappa$ is almost 1-1 iff for every $\alpha < \kappa$ $|f^{-1}\{\alpha\}| < \kappa$

We shall use later the following results:

<u>Proposition 1.2</u> [Ka.1]: Let $\chi \in [\kappa, j(\kappa)[$.

(a) If $f \in {}^{\kappa}\kappa$ is almost 1-1, then $\chi \sim j(f)(\chi)$

(b) If $S \subset Sk(\chi)$ but $|S| \leqslant \kappa$, then S is not cofinal in $Sk(\chi)$

Proof: (a) Set $g(\alpha) = Sup\{\beta : f(\beta) \leqslant \alpha\}$, then $g \in {}^{\kappa}\kappa$ and $\chi \leqslant j(g \circ f)(\chi)$

(b) Suppose $S = \{a_\eta : \eta < \kappa\}$ and $S \subset Sk(\chi)$. For each $\eta < \kappa$, there is f_η so that $j(f_\eta)(\chi) \geqslant a_\eta$. We next consider $f(\alpha) = Sup\{f_\eta(\alpha) : \eta < \alpha\}$
By elementarity of j , we obtain $j(f)(\chi) \geqslant j(f_\eta)(\chi) \geqslant a_\eta$. ■

It is possible to give a characterization of constellations:

<u>Proposition 1.3</u> [Ka.1]: For $\gamma \in [\kappa, j(\kappa)[$,

$$\mathrm{Con}(\gamma) = \{ j(P)(\gamma) \ : \ P \text{ is a permutation of } \kappa \} .$$

Proof: Let $\gamma' \in \mathrm{Con}(\gamma)$ and $j(f)(\gamma)=\gamma'$, $j(g)(\gamma')=\gamma$. If A is the set $\{\alpha<\kappa : g\circ f(\alpha)=\alpha\}$, then f is 1-1 on A and $\gamma \in j(A)$.

Since $\gamma \geqslant \kappa$, we have $|A|=\kappa$. Let $A_0 \cup A_1 = A$, $A_0 \cap A_1 = \emptyset$ and $|A_0| = |A_1| = \kappa$. We assume, for example $\gamma \in j(A_0)$. Let us then define a bijection $h: \kappa \setminus A_0 \longrightarrow \kappa \setminus f''A_0$.

Finally $P = f_{|A_0} \cup h_{|\kappa \setminus A_0}$ is a permutation such that $j(P)(\gamma) = \gamma'$. ∎

We can associate with the two partitions of the interval $[\kappa, j(\kappa)[$ in skies and in constellations, two subsets of $[\kappa, j(\kappa)[$ and two ordinals:

<u>Definition 1.3</u> : $\Gamma(j) = \{ \gamma < j(\kappa) \ : \ \gamma \text{ is the least element of its sky} \}$

$\Lambda(j) = \{ \gamma < j(\kappa) \ : \ \gamma \text{ is the least element of its constellation} \}$

and $\tau(j) =$ order-type of $\Gamma(j)$

$C(j) = |\Lambda(j)|$.

Let us notice that, given a commutative diagram

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "i"'] & M \\ N \arrow[ur, "k"'] & \end{tikzcd}$$

where i,j and k are elementary, we have $\tau(j) \geqslant \tau(i)$ and $C(j) \geqslant C(i)$.

B. Ultrafilters.

We shall deal now with elementary embeddings constructed from ultrafilters.

The following notions have originated in the study of ultrafilters over $\mathbb{N}$, and have been developped in the measurable case by Ketonen [Ke] and Kanamori [Ka.1,2] .

Definition 1.4 : Let U be a κ-ultrafilter.

(a) $f \in {}^{\kappa}\kappa$ is unbounded (modulo U) iff for every $\alpha < \kappa$,

$$\{\eta < \kappa : \alpha < f(\eta)\} \in U .$$

(b) $f \in {}^{\kappa}\kappa$ is almost 1-1 (modulo U) iff there is $X \in U$ such that for all $\alpha < \kappa$,

$$|f^{-1}\{\alpha\} \cap X| < \kappa$$

(c) $f \in {}^{\kappa}\kappa$ is 1-1 (modulo U) iff there is $X \in U$ so that for all $\alpha < \kappa$, $|f^{-1}\{\alpha\} \cap X| \leq 1$.

Definition 1.5 : Let U be a κ-ultrafilter.

(a) U is a p-point iff any unbounded function (modulo U) in is almost 1-1 (modulo U).

(b) U is selective iff any unbounded function (modulo U) in is 1-1 (modulo U).

If U is a free and κ-complete ultrafilter on I where $|I| = \kappa$, then in order to simplify the notation, $\Gamma(U)$, $\Lambda(U)$, $\mathcal{T}(U)$ and $C(U)$ represent respectively $\Gamma(j_U)$, $\Lambda(j_U)$, $\mathcal{T}(j_U)$ and $C(j_U)$. We thus notice:

Proposition 1.4 : Let U be a κ-ultrafilter.

(a) U is a p-point iff $\tau(U) = 1$.

(b) U is selective iff $\tau(U) = C(U) = 1$.

Proof : (a) If U is a p-point, then any $\gamma \in [\kappa, j_U(\kappa)[$ can be represented by $[f]_U$ where f is almost $1-1$. Since $[f]_U = j_U(f)([d]_U)$, Proposition 1.2 (a) implies $[f]_U \in Sk([d]_U)$.

Conversely, let $\tau(U) = 1$ and $[f]_U \in [\kappa, j_U(\kappa)[$. Since $[f]_U$ belongs to $Sk([d]_U)$, there exists $g \in {}^{\kappa}\kappa$ such that $[g \circ f]_U \geqslant [d]_U$. Let $X = \{\alpha < \kappa : g \circ f(\alpha) \geqslant \alpha\}$, $X \in U$ and f is almost $1-1$ on this set.

(b) We use similar arguments and Proposition 1.3 . ■

A particular example of selective κ-ultrafilter is a normal κ-ultrafilter.

Let us present an usual partial ordering on ultrafilters:

Definition 1.7 : The Rudin-Keisler ordering (R-K) is defined as follows:

let U and D be ultrafilters respectively on I and J.

$D \leqslant_{R-K} U$ iff there exists $f : I \longrightarrow J$ such that $D = f^*U$ where

$$f^*U = \{X \subset J : f^{-1}(X) \in U\}.$$

Let $D \simeq_{R-K} U$ iff $D \leqslant_{R-K} U$ and $U \leqslant_{R-K} D$, and let $D <_{R-K} U$ iff $D \leqslant_{R-K} U$ and $U \not\leqslant_{R-K} D$.

If D and U are ω_1-complete and $D = f^*U$, we can consider the following diagram :

$$\begin{tikzcd} V \arrow[r, "j_U"] \arrow[d, "j_D"'] & Ult_U \\ Ult_D \arrow[ur, "k"'] & \end{tikzcd}$$

where $k([g]_D) = [g \circ f]_U$

Therefore $D \leqslant_{R-K} U$ implies $\tau(D) \leqslant \tau(U)$ and $C(D) \leqslant C(U)$.

Definition 1.8 : Let U and D be two κ-ultrafilters. The product $U \otimes D$ is the free and κ-complete ultrafilter on ${}^2\kappa$ defined as follows: if $X \subset {}^2\kappa$, $X \in U \otimes D$ iff $\{\alpha < \kappa : \{\beta < \kappa : (\alpha, \beta) \in X\} \in D\} \in U$. If we write $\underset{2}{\otimes} U$ for $U \otimes U$, we can define by induction the product $\underset{n+1}{\otimes} U = (\underset{n}{\otimes} U) \otimes U$.

One can show that if U is a κ-ultrafilter, the product $U \otimes U$ is such that $\tau(U \otimes U) \geq 2$: κ and $j_U(\kappa)$ are in different skies. Hence, if U is a p-point, the situation $D \otimes D \leq_{R-K} U$ cannot occur. This fact will be used in the fourth chapter.

The following ordering on normal κ-ultrafilters was defined by Mitchell [Mi.1]:

Definition 1.9 : Let U and U' be two normal κ-ultrafilters .

$U \lhd U'$ iff $U \in \mathrm{Ult}_{U'}$.

Proposition 1.5 : $\lhd$ is a well-founded partial ordering.

Proof: It suffices to show that $U \lhd U'$ implies $j_U(\kappa) < j_{U'}(\kappa)$. Since $j_{U'}(\kappa)$ is inaccessible in $\mathrm{Ult}_{U'}$, we have $j_U(j_{U'}(\kappa)) = j_{U'}(\kappa) > j_U(\kappa)$. ■

Let U be a normal κ-ultrafilter, By proposition 1.5, we can consider the rank of U with respect to the ordering $\lhd$, we note it $o(U)$ and set:

Definition 1.10 : If κ is a measurable cardinal, then

$o(\kappa) = \{o(U) : U$ is a normal κ-ultrafilter$\}$.

One can show that for all κ , $o(\kappa) \leq (2^{\kappa})^+$ and that $o(\kappa) = (2^{\kappa})^+$

if κ is 2^κ supercompact.

The notion of coherence is defined as follows :

Definition 1.11 : $\vec{F}$ is a coherent sequence if

(a) $\vec{F}$ is a function and domain$(\vec{F}) = \{(\alpha,\beta) : \beta < o^F(\alpha)\}$ for some function $o^F(\alpha)$.

(b) For any $\alpha,\beta \in \mathrm{Dom}\,\vec{F}$, $\vec{F}(\alpha,\beta)$ is a normal α-ultrafilter of order β .

(c) Every normal ultrafilter is equal to $F(\alpha,\beta)$ for some pair (α,β).

C. Iterated ultrapowers.

Iterated ultrapowers have been invented by Gaifman, Kunen [Ku.1] then developped the method for the study of inner models with one or seral measurable cardinals.

For an exposition of classic iterated ultrapowers, we refer to [Je] or [Ku.1].

There are two possible extensions of the notion of iterated ultrapower.

In one case, one does not necessarily iterate with the same ultrafilter or with a measurable cardinal from the ground model. This is Mitchell's method [Mi.1]:

Definition 1.12 : An iterated ultrapower is any member of a sequence $\langle N_\gamma : \gamma < \alpha \rangle$ constructed as follows:

$N_0 = V$

$N_{\gamma+1} = \mathrm{Ult}_{U_\gamma}(N_\gamma)$ for a normal κ_γ-ultrafilter U_γ in N_γ. Set $i_{\gamma,\gamma+1} = j_{U_\gamma}$ the elementary embedding from N_γ into $N_{\gamma+1}$. And if $\gamma' < \gamma$, then $i_{\gamma',\gamma+1} = i_{\gamma,\gamma+1} \circ i_{\gamma'\gamma}$.

If γ is a limit ordinal, then N_γ is the direct limit of the system $\{N_{\gamma'}, i_{\gamma'\gamma''}$ for $\gamma' < \gamma'' < \gamma\}$.

We give without proof the following :

Proposition 1.6 [Mi.1]: Such iterated ultrapowers are well-founded.

In the second case, we do not require the cardinal to be really measurable. For this purpose, Kunen has defined the concept of M-ultrafilter.

Definition 1.13 [Ku.1]: Let M be a transitive model of ZFC and let $U \subset P^M(\kappa)$ where $\kappa > \omega$. U is a normal M-ultrafilter on κ if:

(a) For any $\alpha < \kappa$, $\{\alpha\} \notin U$.

(b) If $X, Y \in P^M(\kappa)$, then $X \subset Y$ and $X \in U$ imply $Y \in U$.

(c) For any $X \in P^M(\kappa)$, either X or $X^c \in U$.

(d) If $\langle X_\xi : \xi < \kappa \rangle \in M$, then $\{\xi : X_\xi \in U\} \in M$

(e) If $\langle X_\xi : \xi < \eta \rangle \in M$ and $\{X_\xi : \xi < \eta\} \subset U$, for some $\eta < \kappa$, then $\bigcap \{X_\xi : \xi < \eta\} \in U$.

(f) Whenever $\langle X_\xi : \xi < \kappa \rangle \in M$ and $\{X_\xi : \xi < \kappa\} \subset U$, then $\{\xi : \xi \in \bigcap_{\eta < \xi} X_\eta\} \in U$.

The clause (d) is devised to allow an iteration.

All the M-ultrafilters we shall use, will be normal (satisfy (f)), hence we shall drop the expression "normal".

Let us notice that if $j : M \longrightarrow N$ is an elementary embedding with critical point κ, such that $P^M(\kappa) = P^N(\kappa)$, then $U = \{X \in P^M(\kappa) : \kappa \in j(X)\}$ is an M-ultrafilter.

Using the algebra of subsets of ${}^{\alpha}\kappa$ with finite support, Kunen (cf [Ku.1]) defines the α^{th} iterated ultrapower, modulo such ultrafilters.

<u>Proposition 1.7</u> [Ku.1] : Let U be an M-ultrafilter. If arbitrary countable intersections of elements of U are non-empty, then the α^{th} iterated ultrapower N_α is well-founded for all α .

We have the following informations about the computation with M-ultrafilters:

<u>Proposition 1.8</u> [Ku.1] : Let U be an M-ultrafilter on κ, and let $\beta \geqslant 1$ be such that the β^{th} iterated ultrapower N_β modulo U, is well-founded. If β is a cardinal $> 2^\kappa$, then $i_{o\beta}(\kappa) = \beta$. If a cardinal $\delta > \beta$ is such that $cf^M(\delta) > \kappa$ and $|({\xi^\kappa})^M| < \delta$ for $\xi < \delta$, then $i_{o\beta}(\delta) = \delta$.

<u>Remark</u>: We use the same notation $i_{o\gamma} : V \longrightarrow N_\gamma$, for an iteration of length γ , in all the different cases: classical iterated ultrapowers, iterated ultrapowers for Mitchell's definition and iteration with an M-ultrafilter. The context will indicate which definition we will be using.

II. ON THE SIZE OF THE IMAGE MODEL.

Given an elementary embedding $j:V \longrightarrow M$ with critical point κ, the more closure we require for M, the higher κ must be. For example the concepts of compactness, supercompactness, hugeness can be defined from the size of the image model of an elementary embedding.

The surprising fact we shall deal with in this chapter, is that non-closure properties also require large cardinal assumptions. For instance, if there is $j:V \longrightarrow M$ such that ${}^{\omega}M \subset M$ and ${}^{\omega_1}M \not\subset M$, then there is an inner model with ω_1 measurable cardinals.

Let us notice first that it is easy to obtain $j:V \longrightarrow M$ and ${}^{\omega}M \not\subset M$: if $i^U_{o\omega}$ is the usual embedding from V into its ω^{th} iterated ultrapower, modulo a κ-ultrafilter U, then the cofinality in V of $i^U_{o\omega}(\kappa)$ is ω.

On the other hand, when we start with a κ-ultrafilter U, j_U from V into Ult_U is such that ${}^{\kappa}Ult_U \subset Ult_U$.

Hence, what seems more difficult is to require both ${}^{\omega}M \subset M$ and ${}^{\kappa}M \not\subset M$. We obtain the following:

Theorem 2.1 : Let λ regular be such that $\aleph_1 \leqslant \lambda \leqslant \kappa$. If there is an elementary embedding $j:V \longrightarrow M$, with critical point κ such that ${}^{<\lambda}M \subset M$ and ${}^{\kappa}M \not\subset M$, then there is an inner model with λ measurable cardinals.

The notation ${}^{<\lambda}M \subset M$ means that M is closed under sequences of length strictly less than λ.

We shall give two proofs of this result: in the manner of Mitchell and in the manner of Kunen. In the first one, we follow a "black box approach", admitting the properties of the "Core Model for sequences of measures". The second proof is longer but self-contained and uses the "constructible universe for a sequence of ultrafilters"

Hence let us present without proof fundamental results about "the Core Model for a maximal sequence of measures", noted $K(\vec{U}_m)$ (cf [Mi.2]).

<u>Theorem 2.2</u> [Mi.2,3] : Let M be a transitive model of ZFC such that there is no inner model of "$\exists\nu(o(\nu)=\nu^{++})$". One can construct an inner model of M : $(K(\vec{U}_m))^M$. $\vec{U}_m$ is the maximal coherent sequence of measures and depends on M. The maximality gives the following: whenever U is a $(K(\vec{U}_m))^M$-ultrafilter and $U \in M$, then $U \in (K(\vec{U}_m))^M$.

<u>Theorem 2.3</u> [Mi.3] : If $j: M \longrightarrow N$ is elementary and $j \in M$, then $j((K(\vec{U}_m))^M) = (K(\vec{U}_m))^N$ and $j|(K(\vec{U}_m))$ $:(K(\vec{U}_m))^M \longrightarrow (K(\vec{U}_m))^N$ is an iterated ultrapower.

By this expression, we mean that $j|(K(\vec{U}_m))$ is the usual embedding from $(K(\vec{U}_m))^M$ into one of its iterated ultrapowers (for Mitchell's definition).

We present first the proof using the model $K(\vec{U}_m)$.

Proof 1: Let $j: V \longrightarrow M$ with critical point κ , and λ regular be such that $\aleph_1 \leq \lambda \leq \kappa$, ${}^{<\lambda}M \subset M$ and ${}^{\kappa}M \not\subset M$. If there is a model of "$\exists\nu(o(\nu)=\nu^{++})$" we are done. Otherwise we can consider $K(\vec{U}_m)$ and $j: K(\vec{U}_m) \longrightarrow (K(\vec{U}_m))^M$. By the previous theorem, $j|K(\vec{U}_m)$ is an iterated ultrapower $i_{0\delta}$.

Our goal is to show that the length δ of the iteration is $\geq \lambda$.

Lemma 2.1 : If $i_{0\eta} : N_0 \longrightarrow N_\eta$ is an arbitrary iterated ultrapower, then for any $x \in N_\eta$, there exist a function f and a finite subset E of $A = \{i_{\alpha+1,\eta}(\kappa_\alpha) : \alpha+1 \leq \eta\}$ such that $x = i_{0\eta}(f)(E)$.

proof: Let us prove it by induction on η.

. If $\eta = 1$, for any x in N_1, $x = i_{01}(f)(\kappa_0)$.

. Let us assume it is true for η. If $x \in N_{\eta+1}$, then there is $f \in N_\eta$ such that $x = i_{\eta,\eta+1}(f)(\kappa_\eta)$. Therefore by hypothesis, there exist g and E' finite $\subset \{i_{\alpha+1,\eta}(\kappa_\alpha) : \alpha+1 \leq \eta\}$ such that $f = i_{0\eta}(g)(E')$ Henceforth, we get $x = i_{0\eta+1}(g)(i_{\eta\eta+1}(E'))(\kappa_\eta)$. If D is the set $\{(s,\alpha) : s \in \mathrm{Dom}(g) \text{ and } \alpha \in \mathrm{Dom}(g(s))\}$, then we can define on D the function $h(s,\alpha) = g(s)(\alpha)$. We have $i_{0\eta+1}(h)(i_{\eta\eta+1}(E'), \kappa_\eta) = x$. Since $i_{\eta\eta+1}(E') \cup \{\kappa_\eta\} \subset \{i_{\alpha+1,\eta+1}(\kappa_\alpha) : \alpha+1 \leq \eta+1\}$ and it is finite, we we are done

. If η is limit, let $x \in N_\eta$ and $x = i_{\beta\eta}(y)$ for $y \in N_\beta$ $\beta < \eta$. Hence there exist E' finite $\subset \{i_{\alpha+1,\beta}(\kappa_\alpha) : \alpha+1 \leq \beta\}$ and f such that $y = i_{0\beta}(f)(E')$. We thus obtain $x = i_{0\eta}(f)(i_{\beta\eta}(E'))$ and $i_{\beta\eta}(E')$ is included in $\{i_{\alpha+1,\eta}(\kappa_\alpha) : \alpha+1 \leq \eta\}$.

The lemma is thus proved. ■

Let us assume now that the length δ of our iteration is $< \lambda$. If A is the set $\{i_{\alpha+1,\delta}(\kappa_\alpha) : \alpha < \delta\}$ and $s \in {}^\delta A$ is the function defined by $s(\alpha) = i_{\alpha+1,\delta}(\kappa_\alpha)$ for $\alpha < \delta$, then the closure ${}^{<\lambda}M \subset M$ implies $s \in M$. We can thus define $U \subset P(\prod_{\alpha<\delta} s(\alpha))$ by setting for $X \subset \prod_{\alpha<\delta} s(\alpha)$:

$$X \in U \quad \text{iff} \quad M \models s \in j(X).$$

Lemma 2.2 : U is a free κ-complete ultrafilter on $\prod_{\alpha<\delta} s(\alpha)$.

Proof: Since $i_{\alpha+1,\delta}(\kappa_\alpha) \geqslant \kappa_\alpha$, we obtain

$$\begin{aligned} j(i_{\alpha+1,\delta}(\kappa_\alpha)) &\geqslant j(\kappa_\alpha) \\ &\geqslant i_{\alpha\delta}(\kappa_\alpha) \\ &\geqslant i_{\alpha+1,\delta}(i_{\alpha\alpha+1}(\kappa_\alpha)) \\ &> i_{\alpha+1,\delta}(\kappa_\alpha). \end{aligned}$$

δ is $< \lambda$, therefore $j(\prod_{\alpha<\delta} s(\alpha))$ is simply $\prod_{\alpha<\delta} j(s(\alpha))$. With the previous inequalities, this implies $s \in j(\prod_{\alpha<\delta} s(\alpha))$. Henceforth, if $X \subset \prod_{\alpha<\delta} s(\alpha)$, either X or $X^c \in U$. The κ-completeness and the filter condition are straightforward. Let us show that U is a free ultrafilter.

If $\{t\} \in U$, then $s = j(t)$ and $s(0) = j(t(0))$. Hence $i_{1\delta}(\kappa_0)$ is equal to $i_{0\delta}(t(0))$ and $\kappa_0 = i_{01}(t(0))$, which is contradictory since κ_0 is the critical point of i_{01}. ■

Let us then consider j_U and Ult_U, the embedding and ultrapower associated with U. We have the factorization :

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "j_U"'] & M \\ Ult_U \arrow[ur, "k_U"'] & \end{tikzcd}$$

where $k_U([f]_U) = j(f)(s)$
for $f: \prod_{\alpha<\delta} s(\alpha) \longrightarrow V$.

Since U is κ-complete, we can use the classic proof to show ${}^{\kappa}Ult_U \subset Ult_U$
Let $\{[f_\alpha]_U : \alpha < \kappa\} \subset Ult_U$ and $[h]_U = \kappa$. Thus f defined by setting $f(t) = \langle f_\alpha(t) : \alpha < h(t)\rangle$ is such that $[f]_U = \langle [f_\alpha]_U : \alpha < \kappa\rangle$.

From the first lemma, we know that for any ordinal θ, there exists $f \in K(\vec{U}_m)$ such that $i_{0\delta}(f)(E) = \theta$, where E finite $\subset A$. Hence there is $g \in V$ so that $j(g)(s) = \theta$. This implies that k_U is surjective on the ordinals and therefore is the identity. We thus obtain $M = Ult_U$, but ${}^{\kappa}Ult_U \subset Ult_U$ and ${}^{\kappa}M \not\subset M$.

Henceforth the hypothesis $\delta < \lambda$ is contradictory. We shall conclude the proof with the following :

Lemma 2.3 : If the length δ of the iteration is $\geqslant \lambda$, then N satisfies " there are at least λ measurable cardinals "

Proof: Let us now consider the set $B = \{ i_{\alpha\delta}(\kappa_\alpha) : \alpha < \delta \}$. Since each iterated ultrapower N_α satisfies " κ_α is measurable ", all the elements of B are measurable in N_δ which is $(K(\vec{U}_m))^M$.

If $|B| \geqslant \lambda$, we are done. Otherwise we define $F : B \longrightarrow P(\delta)$

$\theta \longrightarrow X_\theta$ where

$X_\theta = \{ \alpha < \delta : i_{\alpha\delta}(\kappa_\alpha) = \theta \}$. We thus obtain a partition of δ in strictly less than λ subsets. Therefore, since $|\delta| \geqslant \lambda$ and λ is regular, there exists $\theta \in B$ such that $|X_\theta| \geqslant \lambda$.

Let $\langle \alpha_\beta : \beta < \lambda \rangle$ be an increasing enumeration of the first λ elements of X_θ. We can assume $o(\kappa_{\alpha_0}) = 1$ in N_{α_0}, otherwise we would be done. Let us notice that if $\beta < \beta' < \lambda$, then $\kappa_{\alpha_{\beta'}} = i_{\alpha_\beta \alpha_{\beta'}}(\kappa_{\alpha_\beta})$.
For β limit $< \lambda$, we define $\mu_\beta = \sup\limits_{\gamma < \beta} \alpha_\gamma$

Claim 2.4 : For any $\beta < \beta'$ limit ordinals, $i_{\alpha_0 \mu_\beta}(\kappa_{\alpha_0}) < i_{\alpha_0 \mu_{\beta'}}(\kappa_{\alpha_0})$ and $N_\delta \models i_{\alpha_0 \mu_\beta}(\kappa_{\alpha_0})$ is measurable .

The previous lemma will follow.

Proof: Let D be the κ_{α_0}-ultrafilter in N_{α_0} and $\beta < \lambda$. We shall show as in the classic case: if $X \subset i_{\alpha_0 \mu_\beta}(\kappa_{\alpha_0})$ and $X \in N_{\mu_\beta}$, then

$$X \in i_{\alpha_0 \mu_\beta}(D) \quad \text{iff} \quad \exists \eta < \beta \ \forall \eta \leqslant \gamma < \beta \ \ i_{\alpha_{\gamma+1}, \mu_\beta}(\kappa_{\alpha_\gamma}) \in X .$$

. If $X \in N_{\mu_\beta}$ and $X \in i_{\alpha_0, \mu_\beta}(D)$, there exist Y and $\eta < \beta$ so that $X = i_{\alpha_\eta \mu_\beta}(Y)$. Since $X \in i_{\alpha_0 \mu_\beta}(D)$, $Y \in i_{\alpha_0 \alpha_\eta}(D)$. For $\gamma > \eta$, let $Z = i_{\alpha_\eta \alpha_\gamma}(Y)$ $Z \in i_{\alpha_0 \alpha_\gamma}(D)$. Since $\kappa_{\alpha_\gamma} = i_{\alpha_0 \alpha_\gamma}(\kappa_{\alpha_0})$, $i_{\alpha_0 \alpha_\gamma}(D)$ is the unique normal κ_{α_γ}-ultrafilter (because we have supposed $o(\kappa_{\alpha_0}) = 1$, and by coherence of the maximal sequence). We thus have $\kappa_{\alpha_\gamma} \in i_{\alpha_\gamma\, \alpha_{\gamma+1}}(Z)$

$$i_{\alpha_{\gamma+1},\mu_\beta}(\kappa_{\alpha_\gamma}) \in i_{\alpha_\gamma \mu_\beta}(Z)$$

$$i_{\alpha_{\gamma+1},\mu_\beta}(\kappa_{\alpha_\gamma}) \in X .$$

. Let us assume there is $\eta_1 < \beta$ such that, for $\gamma \geq \eta_1$, $i_{\alpha_{\gamma+1},\mu_\beta}(\kappa_{\alpha_\gamma})$ belongs to X and $X \notin i_{\alpha_0 \mu_\beta}(D)$. Hence $X^c \in i_{\alpha_0 \mu_\beta}(D)$ and there exists by the previous proof $\eta_2 < \beta$ so that for $\gamma \geq \eta_2$, $i_{\alpha_{\gamma+1},\mu_\beta}(\kappa_{\alpha_\gamma}) \in X^c$. For $\gamma \geq \operatorname{Sup}(\eta_1, \eta_2)$, we would obtain a contradiction.

Let F_{μ_β} be the filter on $i_{\alpha_0 \mu_\beta}(\kappa_{\alpha_0})$ defined from the sequence $\langle i_{\alpha_{\gamma+1},\mu_\beta}(\kappa_{\alpha_\gamma}) : \gamma < \beta \rangle$: $X \in F_{\mu_\beta}$ iff $\exists \eta < \beta. \forall \eta \leq \gamma < \beta,\ i_{\alpha_{\gamma+1},\mu_\beta}(\kappa_{\alpha_\gamma}) \in X$. Since $N_{\mu_\beta} \models F_{\mu_\beta} \cap N_{\mu_\beta}$ is an $i_{\alpha_0 \mu_\beta}(\kappa_{\alpha_0})$-ultrafilter , $F_{\mu_\beta} \cap N_\delta$ is a N_δ-ultrafilter on $i_{\alpha_0 \mu_\beta}(\kappa_{\alpha_0})$. The closure ${}^{<\lambda}M \subset M$ and $(K(\vec{U}_m))^M = N_\delta$ imply $F_{\mu_\beta} \cap N_\delta \in M$. By theorem 2.2, we thus have $F_{\mu_\beta} \cap N_\delta \in (K(\vec{U}_m))^M$. Hence $i_{\alpha_0 \mu_\beta}(\kappa_{\alpha_0})$ is measurable in $(K(\vec{U}_m))^M$.

It remains to check : for β limit $< \beta'$ limit $< \lambda$,

$$i_{\alpha_0 \mu_\beta}(\kappa_{\alpha_0}) < i_{\alpha_0 \mu_{\beta'}}(\kappa_{\alpha_0}) .$$

We have $\mu_\beta \leq \alpha_\beta < \mu_{\beta'}$ and $i_{\alpha_0 \alpha_\beta}(\kappa_{\alpha_0}) = \kappa_{\alpha_\beta}$. Using these elements, we obtain : $i_{\alpha_0 \mu_{\beta'}}(\kappa_{\alpha_0}) = i_{\alpha_\beta \mu_{\beta'}}(i_{\alpha_0 \alpha_\beta}(\kappa_{\alpha_0})) = i_{\alpha_\beta \mu_{\beta'}}(\kappa_{\alpha_\beta})$

$$\geq i_{\alpha_\beta\, \alpha_\beta+1}(\kappa_{\alpha_\beta}) > \kappa_{\alpha_\beta} \geq i_{\alpha_0 \alpha_\beta}(\kappa_{\alpha_0}) \geq i_{\alpha_0 \mu_\beta}(\kappa_{\alpha_0}).$$

This completes the first proof of theorem 2.1 . ■

Let us present now the proof in the manner of Kunen.

Proof 2 : Let $j : V \longrightarrow M$, λ and κ satisfy the hypotheses of theorem 2.1 . We define by induction a strictly increasing sequence $\langle \gamma_\alpha : \alpha < \lambda \rangle$ as follows : . $\gamma_0 = \kappa$

. If $\alpha < \lambda$, let us assume the sequence $s_\alpha = \langle \gamma_\beta : \beta < \alpha \rangle$ has been constructed such that for any $\beta < \alpha$, $\kappa \leq \gamma_\beta < j(\gamma_\beta)$. As in the proof in the manner of Mitchell, we consider the ultrafilter $U_\alpha \subset P(\prod_{\beta<\alpha} s_\alpha(\beta))$ such that, for $X \subset \prod_{\beta<\alpha} s_\alpha(\beta)$, $X \in U_\alpha$ iff $M \models s_\alpha \in j(X)$.

We obtain, in the same way, a commutative diagram :

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "j_{U_\alpha}"'] & M \\ Ult_U \arrow[ur, "k_\alpha"'] & \end{tikzcd}$$

where $k_\alpha([f]_{U_\alpha}) = j(f)(s_\alpha)$

for $f : \prod_{\beta<\alpha} s_\alpha(\beta) \longrightarrow V$.

Again , we have the closure ${}^{\kappa}Ult_{U_\alpha} \subset Ult_{U_\alpha}$. k_α cannot be the identity because it would imply ${}^{\kappa}M \subset M$. Therefore we define γ_α as being the critical point of k_α . γ_α satisfies $\gamma_\alpha < j(\gamma_\alpha)$.

This process can be iterated λ times because of the closure ${}^{<\lambda}M \subset M$. Let us check that the sequence $\langle \gamma_\alpha : \alpha < \lambda \rangle$ is strictly increasing. Let $\alpha < \alpha' < \lambda$. If $\theta < \gamma_\alpha$, since k_α is 1-1 onto $[\kappa, \gamma_\alpha[$, there exists f such that $j(f)(s_\alpha) = \theta$. We have $s_{\alpha'}|_\alpha = s_\alpha$. Hence if we define $g(t) = f(t|_\alpha)$ for $t \in \prod_{\beta<\alpha'} s_{\alpha'}(\beta)$, we get $j(g)(s_{\alpha'}) = \theta$.

Let $P_\alpha : \prod_{\beta<\alpha'} s_{\alpha'}(\beta) \longrightarrow Ord$ be the projection onto the α^{th} coordinate.

$$t \longmapsto t(\alpha)$$

Thus $j(P_\alpha)(s_{\alpha'}) = \gamma_\alpha$. Therefore $k_{\alpha'}$ is 1-1 onto $[\kappa, \gamma_\alpha]$, this implies $\gamma_{\alpha'} > \gamma_\alpha$.

We have thus in our possession λ embeddings, $k_\alpha : Ult_{U_\alpha} \longrightarrow M$, for $\alpha < \lambda$, with increasing critical points.

Let μ be a regular cardinal such that $\mu \geqslant \operatorname{Sup}\{\gamma_\alpha : \alpha < \lambda\}$.

Following Kunen [Ku.2] , we then define by induction, the classes:

$K_1 = \{\nu : \nu \text{ strong limit cardinal, } \nu > 2^\mu \text{ and } cf(\nu) > \mu\}$.

$K_{\alpha+1} = \{\nu \in K_\alpha : |K_\alpha \cap \nu| = \nu\}$.

$K_\beta = \bigcap_{\alpha<\beta} K_\alpha$ if β limit.

Now, for $\alpha < \lambda$, we let

γ_α = the least element of K_α

$A_\alpha = \{\gamma_{\omega\alpha+n} : 1 \leq n < \omega\}$ and

$\lambda_\alpha = \mathrm{Sup}(A_\alpha)$

If $A = \{\gamma_{\eta+1} : \eta < \lambda\}$, we consider in $L[A]$, the usual filter F_α on λ_α, for $\alpha < \lambda$: let $X \subset \lambda_\alpha$. $X \in F_\alpha$ iff $\exists n \forall p \geqslant n\ \gamma_{\omega\alpha+p} \in X$.
If $\vec{F}$ denotes $\langle F_\alpha : \alpha < \lambda \rangle$, our aim is to prove :

$$L[\vec{F}] \models F_\alpha \cap L[\vec{F}] \text{ is a } \lambda_\alpha\text{-ultrafilter.}$$

For $\alpha < \lambda$, we now use the canonical well-ordering of $L[A]$ to define the Skolem Hull H_α of $K_\alpha \cup \{A\}$ in $L[A]$. That is to say :

H_α = the class of all $x \in L[A]$ such that

$$L[A] \models x = t(\nu_0, ---, \nu_k, A)$$

where t is a Skolem term and $\nu_0, ----, \nu_k \in K_\alpha$.

If $\alpha < \lambda$, H_α is an elementary submodel of $L[A]$ and $A \in H_\alpha$, therefore H_α is isomorphic to $L[\pi_\alpha(A)]$ where π_α is the transitive collapsing isomorphism of H_α (cf [Dev]). $\pi_\alpha^{-1} : L[\pi_\alpha(A)] \longrightarrow L[A]$ is thus an elementary embedding.

We now modify Kunen's key-lemma [Ku.2] :

<u>Lemma 2.4</u> : For any $\alpha < \lambda$, $\pi_{\alpha+1}(\gamma_{\alpha+1}) < \not\gamma_{\alpha+1}$.

Proof: . Let us show first $\pi_1(\gamma_1) < \not\gamma_1$.
If $\delta < \gamma_1$ is in H_1, there is a Skolem term t such that (in $L[A]$), $\delta = t(\nu_0, ---, \nu_k, A)$, for $\nu_0, ---, \nu_k$ in K_1.

Let U be a κ-ultrafilter and let $i_{0\delta} : V \longrightarrow N_\delta$ be the usual embedding from V into its δ^{th} iterated ultrapower. By the choice of γ_1, we have $\delta < \nu$ for all $\nu \in K_1$. Henceforth, we deduce from Proposition 1.8

that $i_{0\delta}(\nu) = \nu$, this implies $i_{0\delta}(A) = A$. Hence the well-ordering of $L[A]$ is preserved by $i_{0\delta}$, we thus obtain $i_{0\delta}(\delta) = t(\nu_0, ---, \nu_k, A) = \delta$ and δ must be $< \kappa$. Since $\pi_1(\gamma_1) = \operatorname{Sup}\{\pi_1(\delta) : \delta \in H_1, \delta < \gamma_1\}$, we get $\pi_1(\gamma_1) \leq \kappa < \chi_1$.

. Let us assume now that for all $\eta < \alpha$, $\pi_{\eta+1}(\gamma_{\eta+1}) < \chi_{\eta+1}$. We want to show $\pi_{\alpha+1}(\gamma_{\alpha+1}) < \chi_{\alpha+1}$.

- If $\eta < \alpha$, then $\eta + 1 \leq \alpha$. By hypothesis, we thus get :
$\pi_\alpha(\gamma_{\eta+1}) \leq \pi_{\eta+1}(\gamma_{\eta+1}) < \chi_{\eta+1} \leq \chi_\alpha$.
- If $\eta \geq \alpha$, then $\gamma_{\eta+1} \in K_{\alpha+1}$. This implies, by definition of $K_{\alpha+1}$, $\pi_\alpha(\gamma_{\eta+1}) = \gamma_{\eta+1}$.

Let us consider the embedding $k_\alpha : \mathrm{Ult}_{U_\alpha} \longrightarrow M$. Since $\pi_\alpha(A)$ belongs to Ult_{U_α} , let $k_\alpha : L[\pi_\alpha(A)] \longrightarrow L[k_\alpha(\pi_\alpha(A))]$ be the restriction of k_α on $L[\pi_\alpha(A)]$. Let $D_\alpha = \{X \subset \chi_\alpha : X \in L[\pi_\alpha(A)] \text{ and } \chi_\alpha \in k_\alpha(X)\}$
D_α is not necessarily a $L[\pi_\alpha(A)]$-ultrafilter because we do not know whether $P^{L[k_\alpha(\pi_\alpha(A))]}(\chi_\alpha) = P^{L[\pi_\alpha(A)]}(\chi_\alpha)$
But we can construct the ultrapower of $L[\pi_\alpha(A)]$ modulo this ultrafilter, it is well-founded because of the diagram :

$$\begin{tikzcd}
L[\pi_\alpha(A)] \arrow[r, "k_\alpha"] \arrow[d, "j_{D_\alpha}"'] & L[k_\alpha(\pi_\alpha(A))] \\
\mathrm{Ult}_{D_\alpha}(L[\pi_\alpha(A)]) \arrow[ur, "k'_\alpha"'] &
\end{tikzcd}$$

where $k'_\alpha([f]_{D_\alpha}) = k_\alpha(f)(\chi_\alpha)$, for $f \in L[\pi_\alpha(A)]$ with domain χ_α .

Moreover, one can show by usual arguments that $j_{D_\alpha}(\nu) = \nu$, for all $\nu \in K_1$
Since $\pi_\alpha(\gamma_{\eta+1}) < \chi_\alpha$, for $\eta < \alpha$, and $\pi_\alpha(\gamma_{\eta+1}) = \gamma_{\eta+1}$, for $\eta \geq \alpha$, we obtain that $j_{D_\alpha}(\pi_\alpha(A)) = \pi_\alpha(A)$ and that $\mathrm{Ult}_{D_\alpha}(L[\pi_\alpha(A)]) = L[\pi_\alpha(A)]$.

Therefore D_α is a $L[\pi_\alpha(A)]$-ultrafilter. Since ${}^\omega\mathrm{Ult}_{U_\alpha} \subset \mathrm{Ult}_{U_\alpha}$, arbitrary countable intersections of elements of D_α are non-empty.

Hence, by proposition 1.7 , all the iterated ultrapowers are well-founded.

Since $\pi_{\alpha+1}(\gamma_{\alpha+1}) \leqslant \pi_\alpha(\gamma_{\alpha+1})$ and $\pi_\alpha(\gamma_{\alpha+1}) = \mathrm{Sup}\,\{\pi_\alpha(\delta) : \delta \in H_{\alpha+1} , \delta < \gamma_{\alpha+1}\}$ to obtain the expected inequality $\pi_{\alpha+1}(\gamma_{\alpha+1}) < \chi_{\alpha+1}$, it suffices to show that , for $\delta < \gamma_{\alpha+1}$ in $H_{\alpha+1}$, $\pi_\alpha(\delta) \leqslant \chi_\alpha$.

Hence let $\delta < \gamma_{\alpha+1}$ in $H_{\alpha+1}$ and let t be a Skolem term such that $L[A] \models \delta = t(\nu_1 , - - - , \nu_k , A)$, where $\nu_1 , - - - , \nu_k$ are in $K_{\alpha+1}$. We deduce $L[\pi_\alpha(A)] \models \pi_\alpha(\delta) = t(\pi_\alpha(\nu_1), - - - , \pi_\alpha(\nu_k), \pi_\alpha(A))$, and $\pi_\alpha(\nu_i) = \nu_i$ for each $i \leqslant k$.

We have already seen that $\pi_\alpha(A)$ is left fixed by j_{D_α} . Let us show that an iteration of D_α of length δ does not move it either.

. If $\eta < \alpha$, then obviously $i_{0\delta}(\pi_\alpha(\gamma_{\eta+1})) = \pi_\alpha(\gamma_{\eta+1})$.

. If $\eta \geqslant \alpha$, then $\delta < \gamma_{\eta+1}$ and $\pi_\alpha(\gamma_{\eta+1}) = \gamma_{\eta+1}$. $\gamma_{\eta+1}$ is a strong limit cardinal in the real world such that $\mathrm{cf}(\gamma_{\eta+1}) > \mu$ and $2^\mu < \gamma_{\eta+1}$. Since $\chi_\alpha < \mu$, by applying Proposition 1.8, we obtain $i_{0\delta}(\gamma_{\eta+1}) = \gamma_{\eta+1}$:

Therefore we have $i_{0\delta} : L[\pi_\alpha(A)] \longrightarrow L[\pi_\alpha(A)]$ and (in $L[\pi_\alpha(A)]$) $i_{0\delta}(\pi_\alpha(\delta)) = t(i_{0\delta}(\nu_1), \quad , i_{0\delta}(\nu_k), \pi_\alpha(A))$. For each $i \leqslant k$, $\nu_i \in K_{\alpha+1}$, hence ν_i is $> \delta$ and $i_{0\delta}(\nu_i) = \nu_i$. This gives $i_{0\delta}(\pi_\alpha(\delta)) = \pi_\alpha(\delta)$ and $\pi_\alpha(\delta) < \chi_\alpha$. Using the remark in the beginning, we are thus able to conclude the proof of lemma 2.4 . ∎

Let us recall that, for $\eta < \lambda$, F_η is the filter defined from the set $A_\eta = \{\gamma_{\omega\eta+n} : 1 \leqslant n < \omega\}$ on $\lambda_\eta = \mathrm{Sup}(A_\eta)$.
If we fix $\alpha < \lambda$, the filters $\pi_{\omega\alpha}(F_\eta)$, for $\eta < \lambda$, will be defined , in $L[\pi_{\omega\alpha}(A)]$, from $\pi_{\omega\alpha}(A_\eta)$.

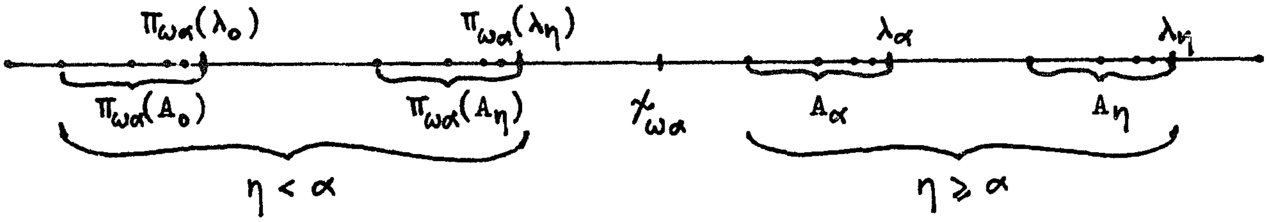
By elementarity of $\pi_{\omega\alpha}{}^{-1}$, we have the following equivalences :

$L[\vec{F}] \models F_\alpha \cap L[\vec{F}]$ is a λ_α-ultrafilter .

$$\Updownarrow\begin{array}{l} L[A] \models (\ L[\vec{F}] \models F_\alpha \cap L[\vec{F}] \text{ is a } \lambda_\alpha\text{-ultrafilter }) \\ L[\pi_{\omega\alpha}(A)] \models (\ L[\pi_{\omega\alpha}(\vec{F})] \models \pi_{\omega\alpha}(F_\alpha) \cap L[\pi_{\omega\alpha}(\vec{F})] \text{ is a } \pi_{\omega\alpha}(\lambda_\alpha)\text{-ultrafilter }) \\ L\, \pi_{\omega\alpha}(\vec{F}) \models \pi_{\omega\alpha}(F_\alpha) \cap L[\pi_{\omega\alpha}(\vec{F})] \text{ is a } \pi_{\omega\alpha}(\lambda_\alpha)\text{-ultrafilter .} \end{array}$$

- All the elements $\gamma \in A$, such that $\gamma > \gamma_{\omega\alpha}$ satisfy $\pi_{\omega\alpha}(\gamma) = \gamma$. Therefore $\pi_{\omega\alpha}(F_\eta) = F_\eta$ if $\eta \geq \alpha$ and $\pi_{\omega\alpha}(\lambda_\alpha) = \lambda_\alpha$.
- By lemma 2.4 , if $\gamma < \gamma_{\omega\alpha}$ and $\gamma \in A$, then $\pi_{\omega\alpha}(\gamma) < \gamma_{\omega\alpha}$.

We can thus represent the situation graphically :



As in [Ku.I] (lemma 10.10), the different ingredients will be:

- the $L[\pi_{\omega\alpha}(A)]$-ultrafilter $D_{\omega\alpha}$ on $\gamma_{\omega\alpha}$,
- the fact that iterations of $D_{\omega\alpha}$, of length $< \lambda_\alpha$, do not change F_α and $\pi_{\omega\alpha}(A_\eta)$, for $\eta \neq \alpha$.
- the canonical well-ordering of $L[\pi_{\omega\alpha}(\vec{F})]$.

Let us assume $F_\alpha \cap L[\pi_{\omega\alpha}(\vec{F})]$ is not a λ_α-ultrafilter:

If there is $X \subset \lambda_\alpha$ such that X and $X^c \notin F_\alpha$, let X_0 be the least such subset in the canonical $L[\pi_{\omega\alpha}(\vec{F})]$- well ordering. All the following iterations $i_{0\xi}$, for $\xi \in \mathrm{Ord}$, use the $L[\pi_{\omega\alpha}(A)]$-ultrafilter $D_{\omega\alpha}$ on $\gamma_{\omega\alpha}$. Since $i_{0\gamma_{\omega\alpha+n}}(\pi_{\omega\alpha}(\vec{F})) = \pi_{\omega\alpha}(\vec{F})$, we obtain $i_{0\gamma_{\omega\alpha+n}}(X_0) = X_0$, for any $n < \omega$. By proposition 1.8 , we know $i_{0\gamma_{\omega\alpha+n}}(\gamma_{\omega\alpha}) = \gamma_{\omega\alpha+n}$. Therefore $\gamma_{\omega\alpha} \in X \iff \gamma_{\omega\alpha+n} \in X$, for all $n < \omega$. This contradicts X and $X^c \notin F_\alpha$.

If $F_\alpha \cap L[\pi_{\omega\alpha}(\vec{F})]$ is not λ_α-complete, let $\langle X_\xi : \xi < \rho \rangle$, for $\rho < \lambda_\alpha$, be the least counterexample : $X_\xi \in F_\alpha$ for all $\xi < \rho$ and $\bigcap_{\xi<\rho} X_\xi \notin F_\alpha$.

As previously $i_{0\gamma_{\omega\alpha+n}}(\langle X_\xi : \xi<\rho\rangle) = \langle X_\xi : \xi<\rho\rangle$ for all $n<\omega$. Let $n_0<\omega$ be such that $\gamma_{\omega\alpha+n_0} > \rho$. If $\xi<\rho$, then for $n \geqslant n_0$, we have $i_{\gamma_{\omega\alpha+n_0},\gamma_{\omega\alpha+n}}(\xi) = \xi$ and $i_{\gamma_{\omega\alpha+n_0},\gamma_{\omega\alpha+n}}(X_\xi) = X_\xi$.
Since $i_{\gamma_{\omega\alpha+n_0},\gamma_{\omega\alpha+n}}(\gamma_{\omega\alpha+n_0}) = \gamma_{\omega\alpha+n}$, for $n \geqslant n_0$, we obtain :

$$\gamma_{\omega\alpha+n_0} \in X_\xi \Leftrightarrow \forall n \geqslant n_0 \ \ \gamma_{\omega\alpha+n} \in X_\xi .$$

We have assumed $X_\xi \in F_\alpha$ for all $\xi<\rho$, therefore we deduce :

$$\{\gamma_{\omega\alpha+n} : n \geqslant n_0\} \subset \bigcap_{\xi<\rho} X_\xi$$

which is contradictory.

The second proof of Theorem 2.1 is thus complete. ■

Let us give now some examples of elementary embeddings $j : V \longrightarrow M$ with critical point κ such that ${}^{<\lambda}M \subset M$ and ${}^{\kappa}M \not\subset M$, for λ regular and $\aleph_1 \leqslant \lambda \leqslant \kappa$. All these examples are constructed from special ultrafilters, noted $U_\lambda(\beta)$, for $\beta \in \mathrm{Ord}$. Hence let us define and present the main properties of these ultrafilters.

<u>Definition 2.1</u> : Let λ be a regular cardinal. $\mathcal{S}_\lambda(\beta)$ denotes the collection of subsets of ${}^{\beta}\kappa$ with support of cardinality strictly less than λ : $X \subset {}^{\beta}\kappa$ is such that $X = in_{E,\beta}(Y)$, where $E \subset \beta$, $|E|<\lambda$ and $Y \subset {}^{E}\kappa$ if $X = \{t \in {}^{\beta}\kappa : t_{|E} \in Y\}$.

(we write $in_{E,\beta}$ for inclusion map from $P({}^{E}\kappa)$ into $P({}^{\beta}\kappa)$)
<u>Remark</u> : If $\beta \leqslant \lambda$, then $\mathcal{S}_\lambda(\beta)$ is simply $P({}^{\beta}\kappa)$

Let $j : V \longrightarrow M$ be an elementary embedding with critical point κ and λ be a regular cardinal such that $\aleph_0 \leqslant \lambda \leqslant \kappa$ and ${}^{<\lambda}M \subset M$. If β is an ordinal, then σ represents an element of ${}^{j''\beta}[\kappa, j(\kappa)[$.

Definition 2.2 : Let $U_\lambda(\beta) \subset \mathcal{S}_\lambda(\beta)$ be defined as follows:

if $X = in_{E,\beta}(Y)$, then $X \in U_\lambda(\beta)$ iff $M \models \sigma|_{j(E)} \in j(Y)$.

(Since $|E| < \lambda$, $j(E) = j''E$ and $\sigma|_{j(E)} \in M$)

Lemma 2.1 : $U_\lambda(\beta)$ is a λ-complete ultrafilter.

Proof: Let us check first that the definition of $U_\lambda(\beta)$ is coherent : if $X \in \mathcal{S}_\lambda(\beta)$ and $X = in_{E,\beta}(Y) = in_{F,\beta}(Z)$, do we always have

$$\sigma|_{j(E)} \in j(Y) \quad \text{iff} \quad \sigma|_{j(F)} \in j(Z) \quad ?$$

We can show that in this case $Y|_{E \cap F} = Z|_{E \cap F}$. Let us note this set H, thus $X = \{ s \in {}^\beta\kappa : s|_{E \cap F} \in H \}$. We obtain for any $s \in {}^\beta\kappa$ the equivalences : $s \in X \iff s|_{E \cap F} \in H \iff s|_E \in Y \iff s|_F \in Z$, the result follows.

If $X = in_{E,\beta}(Y)$ where $Y \subset {}^E\kappa$, then $X^c = in_{E,\beta}(Y^c)$. Therefore either X or $X^c \in U_\lambda(\beta)$

$U_\lambda(\beta)$ is λ-complete : let $\rho < \lambda$, $\{X_\alpha : \alpha < \rho\} \subset U_\lambda(\beta)$ and for each $\alpha < \rho$, $X_\alpha = in_{E_\alpha,\beta}(Y_\alpha)$. If we set $E = \bigcup_{\alpha<\rho} E_\alpha$, then by regularity of λ , $|E|$ is $< \lambda$. We can define $Y \subset {}^E\kappa$ as follows:

$$t \in Y \quad \text{iff} \quad \text{for all } \alpha < \rho \quad t|_{E_\alpha} \in Y_\alpha \ .$$

Therefore $(\bigcap_{\alpha<\rho} X_\alpha) = in_{E,\beta}(Y)$, and we have :

$$\begin{aligned} (\bigcap_{\alpha<\rho} X_\alpha) \in U_\lambda(\beta) \quad & \text{iff} \quad M \models \sigma|_{j(E)} \in j(Y) \\ & \text{iff} \quad \text{for each } \alpha < \rho , \ \sigma|_{j(E_\alpha)} \in j(Y_\alpha) \\ & \text{iff} \quad \text{for each } \alpha < \rho , \ X_\alpha \in U_\lambda(\beta) \end{aligned}$$

Let us assume now $X \subset Y$ and $X \in U_\lambda(\beta)$. We know from the previous equivalence : $(X \cap Y) \in U_\lambda(\beta)$ iff X and $Y \in U_\lambda(\beta)$. ■

Let us consider now $F_\lambda(\beta)$ the collection of functions with support of cardinality strictly less than λ .

Definition 2.3 : Let f be a function with domain ${}^{\beta}\kappa$.

$f \in F_\lambda(\beta)$ iff there exist $E \subset \beta$, $|E| < \lambda$ and $g : {}^{E}\kappa \longrightarrow V$ such that for $s \in {}^{\beta}\kappa$, $f(s) = g(s_{|E})$.

Notation : The ultrapower obtained from $F_\lambda(\beta)$, modulo the ultrafilter $U_\lambda(\beta)$ is well-founded because of the λ-completeness if $\lambda \geqslant \aleph_1$ and because of the following result if $\lambda = \aleph_0$. Let us note it $M_\lambda(\beta)$.

Lemma 2.2 : (a) The following diagram is commutative :

$$\begin{tikzcd}
V \arrow[r, "j"] \arrow[d, "j_{U_\lambda(\beta)}"'] & M \\
M_\lambda(\beta) \arrow[ur, "k_\beta"'] &
\end{tikzcd}$$

where k_β is defined as follows:
if $f \in F_\lambda(\beta)$ is such that $f(s) = g(s_{|E})$
then $k_\beta([f]_{U_\lambda(\beta)}) = j(g)(\sigma_{|jE})$

(b) $M_\lambda(\beta)$ is closed under sequences of length strictly less than λ .

Proof: (a) is a direct consequence of the definition of $U_\lambda(\beta)$.

(b) Let $\rho < \lambda$ and $\{x_\alpha : \alpha < \rho\} \subset M_\lambda(\beta)$ where $x_\alpha = [s \in {}^{\beta}\kappa \longmapsto g_\alpha(s_{|E_\alpha})]_{U_\lambda(\beta)}$ and $E_\alpha \subset \beta$, $|E| < \lambda$. Let us set $E = \bigcup_{\alpha<\rho} E_\alpha$ we thus have $|E| < \lambda$. If $f : {}^{\beta}\kappa \longrightarrow V$ is the function so that $f(s) = \langle g_\alpha(s_{|E_\alpha}) : \alpha < \rho \rangle$, f has support E . Let us check that $[f]_{U_\lambda(\beta)}$ is the sequence $\langle x_\alpha : \alpha < \rho \rangle$. Because of the previous diagram, the critical point of $j_{U_\lambda(\beta)}$ is $\geqslant \kappa$, hence we obtain :

$$M_\lambda(\beta) \models \ [f]_{U_\lambda(\beta)} \text{ is a sequence of length } \rho \ .$$

And for $\alpha < \rho$, $M_\lambda(\beta) \models$ the α^{th} term of $[f]_{U_\lambda(\beta)}$ is $[s \in {}^{\beta}\kappa \mapsto g_\alpha(s_{|E_\alpha})]_{U_\lambda(\beta)}$
This gives $[f]_{U_\lambda(\beta)} = \langle x_\alpha : \alpha < \rho \rangle$. ■

The last property of $U_\lambda(\beta)$ is the following :

Lemma 2.3 : The critical point of $j_{U_\lambda(\beta)}$ is κ.

Proof: We already know that for any $\gamma<\kappa$, $j_{U_\lambda(\beta)}(\gamma)=\gamma$. Let us show that $\kappa\leqslant[s\in{}^{\beta}\kappa\longmapsto s(0)]_{U_\lambda(\beta)}<j_{U_\lambda(\beta)}(\kappa)$.
If $\gamma<\kappa$, $\gamma<\sigma(0)$ by the choice of σ. Henceforth, by definition of $U_\lambda(\beta)$, we obtain : $\gamma<[s\in{}^{\beta}\kappa\longmapsto s(0)]_{U_\lambda(\beta)}$ and $\kappa\leqslant[s\in{}^{\beta}\kappa\longmapsto s(0)]_{U_\lambda(\beta)}$.
In the same way, $\sigma(0)<j(\kappa)$ implies $[s\in{}^{\beta}\kappa\mapsto s(0)]_{U_\lambda(\beta)}<j_{U_\lambda(\beta)}(\kappa)$. ∎

Remark : These ultrapowers are a generalization of Kunen's iterated ultrapowers. We can show that, for an accurate choice of σ, $U_\omega(\beta)$ can be Kunen's ultrafilter U_β on the subsets of ${}^{\beta}\kappa$ with finite support.

All the different applications of the ultrafilters $U_\lambda(\beta)$ will be based on the choice of the sequence $\sigma\in{}^{j''\beta}j(\kappa)$.

We have seen that the existence of an elementary embedding $j:V\longrightarrow M$ with critical point κ, such that ${}^{<\lambda}M\subset M$ and ${}^{\kappa}M\not\subset M$ for λ regular and $\aleph_1\leqslant\lambda\leqslant\kappa$, implies large cardinal hypotheses. Henceforth, to construct such embeddings, we shall start with strong assumptions on κ, in terms of skies and constellations.

We have studied in a previous work [Su], the relation between skies and constellations and large cardinal axioms. W. Mitchell [Mi.3] has then obtained more powerful results.

Let us recall that for a κ-ultrafilter U, $\tau(U)$ represents the order-type of skies and $C(U)$ the cardinality of constellations.

Theorem 2.4 [Mi.3]: (a) If there is a κ-ultrafilter U such that $\tau(U)$ is $\geqslant\lambda$, for $\aleph_0\leqslant\lambda$ regular $\leqslant\kappa$, then there is an inner model which satisfies "$\exists\nu(o(\nu)\geqslant\lambda+1)$" if $\lambda>\aleph_0$, and "$\exists\nu(o(\nu)\geqslant 2)$" if $\lambda=\aleph_0$.

(b) Conversely, if $o(\kappa) \geqslant \mu+1$, for μ regular $< \kappa$, then there is an extension $V[G]$ in which there is a κ-ultrafilter U such that $\tau(U) \geqslant \mu$ if $\mu > \aleph_0$, or $\tau(U) \geqslant \omega$ if $\mu = 1$.

<u>Theorem 2.5</u> [Mi.3]: If there is a κ-ultrafilter U such that $C(U) \geqslant \aleph_0$, then, in an inner model, there is a measurable cardinal limit of measurable cardinals.

Therefore the hypotheses in terms of skies and constellations are much stronger than the mere existence of λ measurable cardinals we have obtained previously. But the embeddings $j : V \longrightarrow M$ derived from these assumptions are such that ${}^{\kappa}j(\kappa) \not\subset M$, and the non-closure ${}^{\kappa}M \not\subset M$ seems to be much weaker than ${}^{\kappa}j(\kappa) \not\subset M$.

Let us deal first with the constellations:

<u>Proposition 2.1</u> : Let $\lambda \geqslant \aleph_0$ be regular and μ be inaccessible. If κ is the critical point of the following embeddings and $\lambda \leqslant \mu \leqslant \kappa$, then we have the implications:

(a) there is $j : V \longrightarrow M$ such that $C(j) \geqslant \mu$ and ${}^{<\lambda}M \subset M$

(b) there is $j : V \longrightarrow M$ such that ${}^{<\lambda}M \subset M$ and ${}^{\mu}j(\kappa) \not\subset M$

(c) there exists $j : V \longrightarrow M$ such that ${}^{<\lambda}M \subset M$ and $C(j) \geqslant \lambda$.

Proof: (a) $\Longrightarrow$ (b).

Let $j : V \longrightarrow M$ be such that $C(j) \geqslant \mu$ and ${}^{<\lambda}M \subset M$. If $\langle \gamma_\alpha : \alpha < \delta \rangle$ is the increasing enumeration of $\Lambda(j)$, then we set $\sigma = \langle \gamma_\alpha : \alpha < \mu \rangle$.

Let us consider the ultrafilter $U_\lambda(\mu)$ obtained from j and σ :

if $X = in_{E,\mu}(Y)$ where $Y \subset {}^{E}\kappa$, $E \subset \mu$ and $|E| < \lambda$, then

$$X \in U_\lambda(\mu) \quad \text{iff} \quad \sigma|_E \in j(Y)$$

($\mu \leq \kappa$, hence $j(E) = E$)

Our goal is to show that $j_{U_\lambda(\mu)} : V \longrightarrow M_\lambda(\mu)$ is the expected embedding. We already know, by lemma 2.2 (b), that $M_\lambda(\mu)$ is closed under sequences of length strictly less than λ. The next step is to prove that $\sigma = \langle \gamma_\alpha : \alpha < \mu \rangle$ does not belong to $M_\lambda(\mu)$.

Let us consider the factorization :

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "j_{U_\lambda(\mu)}"'] & M \\ M_\lambda(\mu) \arrow[ur, "k_\mu"'] & \end{tikzcd}$$

Since, for $\alpha < \mu$, $k_\mu([s \in {}^\mu\kappa \longmapsto s(\alpha)]_{U_\lambda(\mu)}) = \sigma(\alpha) = \gamma_\alpha$, we deduce from proposition 1.3 that k_μ is surjective onto $[\kappa, \mathrm{Sup}\{\gamma_\alpha : \alpha < \mu\}$. Therefore $j_{U_\lambda(\mu)}(\kappa) \geq \mathrm{Sup}\{\gamma_\alpha : \alpha < \mu\}$. Let us assume $\sigma \in M_\lambda(\mu)$. This implies $k_\mu(\sigma) = \sigma$. If $\sigma = [s \in {}^\mu\kappa \longmapsto g(s_{|E})]_{U_\lambda(\mu)}$, for $E \subset \mu$ and $|E| < \lambda$, then $k_\mu(\sigma) = j(g)(\sigma_{|E}) = \sigma$.

<u>Claim 2.1</u> : Let $E \subset \mu$ such that $|E| < \lambda$ be fixed. If X is the set $\{\gamma \in \Lambda(j) : \exists f : {}^E\kappa \longrightarrow \kappa$ such that $j(f)(\sigma_{|E}) = \gamma\}$, then $|X| \leq |\mathrm{Sup}(E)|^{|E|}$

Let us admit the claim for a moment and let E, g be such that $\sigma = j(g)(\sigma_{|E})$. Since $|\mathrm{Sup}(E)|^{|E|} < \mu$, there exists $\alpha < \mu$ such that for any function $f : {}^E\kappa \longrightarrow \kappa$, $j(f)(\sigma_{|E}) \neq \gamma_\alpha$. Let α_0 be such an ordinal. We then define the projection $P_{\alpha_0} : \begin{array}{l} {}^\mu\kappa \longrightarrow \kappa \\ s \longmapsto s(\alpha_0) \end{array}$. We thus obtain $j(P_{\alpha_0} \circ g)(\sigma_{|E}) = \gamma_{\alpha_0}$, which is contradictory. Therefore $\sigma \notin M_\lambda(\mu)$ and we can conclude.

It thus remains to prove the claim. Let us show first the following result: Given E, if $f : {}^E\kappa \longrightarrow \kappa$ is such that $j(f)(\sigma_{|E}) \geq \kappa$, then there exists $E' \subset \mathrm{Sup}(E)$, $|E'| \leq |E|$ and a bijection $b_f : {}^{E'}\kappa \longrightarrow \kappa$ such that $j(b_f)(\sigma_{|E'}) = j(f)(\sigma_{|E})$

Let $\theta = j(f)(\sigma\restriction_{E})$. We define $A: \kappa \longrightarrow P({}^{E}\kappa)$ and $h: \kappa \longrightarrow {}^{E}\kappa$ by setting, for $\alpha < \kappa$, $A(\alpha) = \left\{ s \in {}^{E}\kappa : f(s) = \alpha \right\}$ and

$$h(\alpha) = \begin{cases} \text{- the null sequence if } A(\alpha) = \emptyset \\ \text{- an element } s \text{ of } A(\alpha) \text{ such that } \mathrm{Sup}(s) \text{ is minimal, otherwise.} \end{cases}$$

Since $j(f)(\sigma\restriction_{E}) = \theta$, $\sigma\restriction_{E} \in j(A)(\theta)$. Henceforth, if $j(h)(\theta) = t$, for t in ${}^{E}j(\kappa)$, we must have $\mathrm{Sup}(t) \leqslant \mathrm{Sup}(\sigma\restriction_{E})$. This implies that for any $x \in E$, $t(x) \in \mathrm{Con}(\chi_{\alpha})$ where $\alpha < \mathrm{Sup}(E)$. We then deduce from proposition 1.3 , the existence of a set $E' \subset \mathrm{Sup}(E)$, $|E'| \leqslant |E|$ and of a bijection $\pi_f: {}^{E'}\kappa \longrightarrow {}^{E}\kappa$ such that $j(\pi_f)(\sigma\restriction_{E'}) = t$.

We finally obtain $j(\pi_f^{-1} \circ h)(\theta) = \sigma\restriction_{E'}$ and $j(f \circ \pi_f)(\sigma\restriction_{E'}) = \theta$, by using an auxilliary bijection from ${}^{E'}\kappa$ onto κ and proposition 1.3 we get the expected result.

Now, let $X = \left\{ \chi \in \Lambda(j) : \exists f: {}^{E}\kappa \longrightarrow \kappa \text{ such that } j(f)(\sigma\restriction_{E}) = \chi \right\}$.
We consider the map $\varphi : X \longrightarrow [\mathrm{Sup}(E)]^{\leqslant |E|}$
$\chi \longmapsto E_{\chi}$ where E_{χ} is costructed from the previous study, and such that, for a bijection $b_{\chi} : {}^{E_{\chi}}\kappa \longrightarrow \kappa$,
$\chi = j(b_{\chi})(\sigma\restriction_{E_{\chi}})$.

We claim that φ is 1-1 : if $\varphi(\chi) = \varphi(\chi') = F$, for $\chi, \chi' \in \Lambda(j)$, there are bijections b_{χ} and $b_{\chi'}$ such that :

$$\chi = j(b_{\chi})(\sigma\restriction_{F}) \quad \text{and} \quad \chi' = j(b_{\chi'})(\sigma\restriction_{F}) .$$

Therefore χ and χ' are in the same constellation, which implies $\chi = \chi'$
The proof of the claim is thus complete. ■

(b) $\Longrightarrow$ (c) :

Let $j: V \longrightarrow M$ be such that ${}^{<\lambda}M \subset M$ and ${}^{\mu}j(\kappa) \not\subset M$. We want to show that this embedding has at least λ constellations.

Let us assume the converse is true and let $\sigma \in {}^{\delta}j(\kappa)$, for $\delta < \lambda$, be an increasing enumeration of $\Lambda(j)$.

Because of ${}^{<\lambda}M \subset M$, we can define the ultrafilter U_σ on ${}^{\delta}\kappa$ by setting : $X \subset {}^{\delta}\kappa$, $X \in U_\sigma$ iff $M \models \sigma \in j(X)$. This ultrafilter is free and κ-complete, we obtain the commutative diagram:

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "j_{U_\sigma}"'] & M \\ Ult_{U_\sigma} \arrow[ur, "k_\sigma"'] & \end{tikzcd}$$

where $k_\sigma([f]_{U_\sigma}) = j(f)(\sigma)$ for $f : {}^{\delta}\kappa \longrightarrow V$.

By definition of σ , k_σ is $1-1$ onto $[\kappa, j(\kappa)[$. Since ${}^{\kappa}Ult_{U_\sigma} \subset Ult_{U_\sigma}$, $s \in {}^{\mu}j(\kappa)$ implies $s \in Ult_{U_\sigma}$ and $k_\sigma(s) = s$. We thus get ${}^{\mu}j(\kappa) \subset M$, which contradicts the hypothesis. ■

We deduce from the previous result:

<u>Proposition 2.2</u> : Let $\mu > \omega$ be inaccessible and $\mu \leq \kappa$. The following assertions are equivalent:

(a) there exists $j : V \longrightarrow M$ with critical point κ such that $C(j) \geq \mu$ and ${}^{<\mu}M \subset M$.

(b) there exists $j : V \longrightarrow M$ with critical point κ such that ${}^{\kappa}j(\kappa) \not\subset M$ and ${}^{<\mu}M \subset M$.

(c) there is an ascending R-K chain of κ-ultrafilters $\langle D_\alpha : \alpha < \mu \rangle$ and a set of functions $\{ f_{\alpha\beta} : \alpha < \beta < \mu \}$ such that for $\alpha < \beta < \gamma$ $D_\alpha = f_{\alpha\beta}{}^{*} D_\beta$ and $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$.

Proof: (a) $\Longleftrightarrow$ (b) is a consequence of the preceding proposition for $\lambda = \mu$.

(a) $\Longmapsto$ (c)

Let $j : V \longrightarrow M$ be such that $C(j) \geq \mu$ and ${}^{<\mu}M \subset M$, and let $\sigma = \langle \gamma_\alpha : \alpha < \eta \rangle$ be an increasing enumeration of $\Lambda(j)$.

We define as previously the ultrafilters on ${}^{\alpha}\kappa$, for $\alpha < \mu$, noted here U_α, by setting: $X \subset {}^\alpha\kappa$, $X \in U_\alpha$ iff $\sigma\restriction_\alpha \in j(X)$.

Let $P_{\alpha\beta}$, for $\alpha<\beta<\mu$, be the functions: $P_{\alpha\beta}: {}^\beta\kappa \longrightarrow {}^\alpha\kappa$.
$$s \longmapsto s\restriction_\alpha$$

For $\alpha<\beta<\gamma$, we have $P_{\alpha\gamma} = P_{\alpha\beta} \circ P_{\beta\gamma}$ and $U_\alpha = P_{\alpha\beta}{}^* U_\beta$.

Let us define inductively, for $\alpha<\mu$:
$$\delta(0) = 0$$
$$\delta(\alpha+1) = (2^{\delta(\alpha)})^+$$
$$\delta(\beta) = \bigcup_{\alpha<\beta}\delta(\alpha) \text{ if } \beta \text{ limit.}$$

Let $D_\alpha = U_{\delta(\alpha)}$, for $\alpha<\mu$. We claim that the system $\{D_\alpha, P_{\delta(\alpha),\delta(\beta)} : \alpha<\beta<\mu\}$ has the required properties. It suffices to show $D_\alpha \not\simeq D_\beta$ if $\alpha<\beta$.

Let us apply the claim 2.1 for $E = \alpha$:

the set $X = \{\chi \in \Lambda(j) : \exists f: {}^\alpha\kappa \longrightarrow \kappa$ such that $j(f)(\sigma\restriction_\alpha) = \chi\}$ is of cardinality $\leq |\alpha|^{|\alpha|} = 2^{|\alpha|}$.

The embedding $k_\alpha : [f]_{U_\alpha} \longmapsto j(f)(\sigma\restriction_\alpha)$ is elementary, therefore $\chi \in \Lambda(U_\alpha)$ iff $k_\alpha(\chi) \in \Lambda(j)$. We thus obtain $C(U_\alpha) \leq 2^{|\alpha|}$. On the other hand, $\{\chi_\beta : \beta<\alpha\} \subset \text{Range}(k_\alpha)$, hence $C(U_\alpha) \geq |\alpha|$.

We thus get $|\delta(\alpha)| \leq C(U_\alpha) \leq 2^{|\delta(\alpha)|}$ and
$$C(D_\alpha) \leq 2^{|\delta(\alpha)|} < (2^{|\delta(\alpha)|})^+ \leq C(D_{\alpha+1}).$$

Henceforth D_α and $D_{\alpha+1}$ cannot be isomorphic. ■

(c) $\Longmapsto$ (a) :

Let $\{D_\alpha, f_{\alpha\beta} : \alpha<\beta<\mu\}$ be a system which satisfies (c).

For $\alpha<\beta<\gamma$, we have the commutative diagram :

$$\begin{tikzcd}
V \arrow[r, "j_{D_\gamma}"] \arrow[d, "j_{D_\alpha}"'] & \text{Ult}_{D_\gamma} \\
\text{Ult}_{D_\alpha} \arrow[r, "i_{\alpha\beta}"] \arrow[ur, "i_{\alpha\gamma}"] & \text{Ult}_{D_\beta} \arrow[u, "i_{\beta\gamma}"']
\end{tikzcd}$$

where $i_{\alpha\beta}([f]_{D_\alpha}) = [f \circ f_{\alpha\beta}]_{D_\beta}$.

We can define the direct limit M of the system $\{\text{Ult}_{D_\alpha}, i_{\alpha\beta} : \alpha<\beta<\mu\}$

and we obtain the diagrams:

$$\begin{tikzcd} V \arrow[r, "j_\mu"] \arrow[d, "j_{D_\alpha}"'] & M \\ \mathrm{Ult}_{D_\alpha} \arrow[ur, "i_{\alpha\mu}"'] & \end{tikzcd} \qquad \text{for } \alpha<\mu .$$

We claim that $j_\mu : V \longrightarrow M$ is an elementary embedding which satisfies (a)

Let us show first that M is well-founded:

If we assume the converse is true, then there is a sequence of ordinals $\langle x_n : n<\omega \rangle$ such that, for each $n<\omega$, $x_{n+1} \mathrel{E} x_n$. If $n<\omega$, let $\alpha_n<\mu$ and y_n be such that $x_n = i_{\alpha_n\mu}(y_n)$. Since μ is regular, $\alpha = \mathrm{Sup}\{\alpha_n : n<\omega\}$ is $<\mu$.

Thus let us consider the sequence $\langle i_{\alpha_n\alpha}(y_n) : n<\omega \rangle$. This sequence belongs to Ult_{D_α} and we have $i_{\alpha_{n+1}\alpha}(y_{n+1}) \in i_{\alpha_n\alpha}(y_n)$. This yields a contradiction.

One can show by induction that j_μ is the identity on κ: let $\xi<\kappa$ and $\xi = i_{\alpha\mu}(\eta)$. If we suppose, for any $\gamma<\xi$, $j_\mu(\gamma)=\gamma$, then the case $\eta<\xi$ cannot occur. Since $i_{0\alpha}(\xi)=\xi$, we are done.

We want to prove now ${}^{<\mu}M \subset M$. Let $\lambda<\mu$ and $f \in {}^{\lambda}M$. Using again the regularity of μ and the fact that $j_\mu(\lambda)=\lambda$, we can show that there exists $\alpha<\mu$ such that $f = i_{\alpha\mu}(g)$. Henceforth $f \in M$.

Finally, let us check $C(j_\mu) \geqslant \mu$. We shall verify that the elements $\{ i_{\alpha\mu}([d]_{D_\alpha}) : \alpha<\mu \}$ are in different constellations.

If we had $i_{\alpha\mu}([d]_{D_\alpha}) \longleftrightarrow i_{\beta\mu}([d]_{D_\beta})$, where $\alpha<\beta$, relatively to the embedding j_μ. This would imply $i_{\alpha\beta}([d]_{D_\alpha}) \longleftrightarrow [d]_{D_\beta}$ relatively to j_{D_β}, that is $[f_{\alpha\beta}]_{D_\beta} \longleftrightarrow [d]_{D_\alpha}$, and we would finally obtain $f_{\alpha\beta}{}^{*}D_\beta \simeq D_\beta$, which contradicts the hypotheses.

The proof of proposition 2.2 is thus complete. ■

A consequence of Theorem 2.5 and Proposition 2.1 is the following:

Proposition 2.3 : If there exists $j:V \longrightarrow M$ with critical point κ such that ${}^{\omega}M \subset M$ and ${}^{\kappa}j(\kappa) \not\subset M$, then there is an inner model with a measurable cardinal limit of measurable cardinals.

Proof: Let $j:V \longrightarrow M$ be such that ${}^{\omega}M \subset M$ and ${}^{\kappa}j(\kappa) \not\subset M$. By proposition 2.1 (b)$\models\Rightarrow$(c), for $\lambda = \aleph_1$ and $\mu = \kappa$, we obtain $C(j) \geqslant \aleph_1$.
Let $\sigma = \langle \gamma_\alpha : \alpha < \omega \rangle$ be an increasing enumeration of the first ω elements of $\Lambda(j)$. Since ${}^{\omega}M \subset M$, we can consider the ultrafilter U_σ defined by setting: $X \subset {}^{\omega}\kappa$, $X \in U_\sigma$ iff $M \models \sigma \in j(X)$.
U_σ is isomorphic to a κ-ultrafilter and $C(U_\sigma) \geqslant \omega$, it suffices now to apply Theorem 2.5 to obtain the proposition. ■

Before dealing with the skies question, let us give another example derived from the same methods.

Proposition 2.4 : If κ is 2^κ- supercompact, for any regular cardinals $\omega \leqslant \lambda \leqslant \mu \leqslant \kappa$, there exists $j:V \longrightarrow M$ with critical point κ , such that ${}^{<\lambda}M \subset M$ and $cf(((2^\kappa)^+)^M) = \mu$.

Proof: Let us take Kunen's example of p-points (cf [Ka.1]) :
If $i:V \longrightarrow N$ is associated with the supercompact cardinal, then $(2^\kappa)^+$ is $< j(\kappa)$. By proposition 1.3 , each constellation has at most 2^κ elements, therefore there are at least $(2^\kappa)^+$ constellations below $(2^\kappa)^+$.

Let $\sigma = \langle \gamma_\alpha : \alpha \leqslant \mu \rangle$ be an increasing enumeration of the first $\mu+1$ elements of $\Lambda(i)$. We then consider the ultrafilter $U_\lambda(\mu)$ and the ultrapower $M_\lambda(\mu)$ defined from i and σ .
This yields the following diagram :

$$\begin{tikzcd}
V \arrow[r, "i"] \arrow[d, "j_{U_\lambda(\mu)}"'] & N \\
M_\lambda(\mu) \arrow[ur, "k_\mu"'] &
\end{tikzcd}$$

We claim that the critical point θ of k_μ is $((2^\kappa)^+)^{M_\lambda(\mu)}$. θ is also γ_μ which can be shown to be $\bigcup_{\alpha<\mu} \gamma_\alpha$.

To prove this, we shall apply:

<u>Lemma 2.4</u> (Kunen) : If $\alpha < \beta < (2^\kappa)^+$, there exists $f \in {}^\kappa\kappa$ such that

$i(f)(\beta) = \alpha$.

For a proof of this result, we refer to [Ka.1].

Let us show first, by classic ways, $\theta = ((2^\kappa)^+)^{M_\lambda(\mu)}$.

We have $k_\mu(\kappa) = \kappa$, which yields the inequalities :

$2^\kappa \leqslant (2^\kappa)^{M_\lambda(\mu)} \leqslant k_\mu((2^\kappa)^{M_\lambda(\mu)}) \leqslant (2^\kappa)^N = 2^\kappa$. By lemma 2.4, from 2^κ, we can reach any element of $[\kappa, 2^\kappa[$. Since $2^\kappa \in \mathrm{Con}(\kappa)$, we obtain that $[\kappa, 2^\kappa] \subset \mathrm{Con}(\kappa)$ and $\gamma_\mu > 2^\kappa$.

k_μ is the identity on $[\kappa, \gamma_\mu[$ and $\gamma_\mu > 2^\kappa$, hence θ is necessarily $> (2^\kappa)^{M_\lambda(\mu)}$. This gives us $\theta \geqslant ((2^\kappa)^+)^{M_\lambda(\mu)}$.

Let us show $((2^\kappa)^+)^{M_\lambda(\mu)} < (2^\kappa)^+$, thus we would be able to conclude. If we want to compute $j_{U_\lambda(\mu)}(\kappa)$, it suffices to count the number of functions $\{g : {}^E\kappa \longrightarrow \kappa$ for $E \subset \mu$ and $|E| < \lambda\}$.

Henceforth $|j_{U_\lambda(\mu)}(\kappa)| \leqslant \mu^{<\lambda} . \kappa^{(\kappa^{<\lambda})} \leqslant 2^\kappa$, and we obtain :

$$((2^\kappa)^+)^{M_\lambda(\mu)} < j_{U_\lambda(\mu)}(\kappa) < (2^\kappa)^+ .$$

Let us prove now $\gamma_\mu = \bigcup_{\alpha<\mu} \gamma_\alpha$ and $\theta = \gamma_\mu$.

We first notice that, because of lemma 2.4, if $\alpha < \beta \leqslant \mu$, then there cannot exist $f \in {}^\kappa\kappa$ such that $i(f)(\gamma_\alpha) = \gamma_\beta$.

Let $\gamma = \bigcup_{\alpha<\mu} \gamma_\alpha$. If we assume $\gamma \neq \gamma_\mu$, then there exists $\alpha_0 < \mu$ so that $\gamma \in \mathrm{Con}(\gamma_{\alpha_0})$. Again by the lemma, there is $f \in {}^{\kappa}\kappa$ such that $i(f)(\gamma) = \gamma_{\alpha_0+1}$, and therefore there is $g \in {}^{\kappa}\kappa$ such that $i(g)(\gamma_{\alpha_0}) = \gamma_{\alpha_0+1}$ We have obtained a contradiction.

Let us assume $\gamma_\mu \in \mathrm{Range}(k_\mu)$. Hence there exists $E \subset \mu$, $|E| < \lambda$ and $g: {}^{E}\kappa \longrightarrow \kappa$ such that $\gamma_\mu = k_\mu([s \in {}^{\mu}\kappa \longmapsto g(s_{|E})]_{U_\lambda(\mu)}) = i(g)(\sigma_{|E})$. If $\alpha = \mathrm{Sup}(E)$, the lemma implies the existence of $f: \kappa \longrightarrow {}^{E}\kappa$ such that $i(f)(\gamma_\alpha) = \sigma_{|E}$. This gives also $i(g \circ f)(\gamma_\alpha) = \gamma_\mu$, which is impossible.

Since k_μ is surjective onto $[\kappa, \gamma_\mu[$ and $\gamma_\mu \notin \mathrm{Range}(k_\mu)$, γ_μ must be the critical point θ of k_μ and we can conclude. ■

Concerning the skies, we obtain:

Proposition 2.5 : For λ, μ regular cardinals such that $\lambda \leq \mu \leq \kappa$, we have the equivalence:

(a) There exists $j: V \longrightarrow M$ with critical point κ such that $\tau(j) \geq \mu$ and ${}^{<\lambda}M \subset M$.

$\Updownarrow$

(b) There exists $j: V \longrightarrow M$ with critical point κ such that $\mathrm{cf}(j(\kappa)) = \mu$ and ${}^{<\lambda}M \subset M$.

Proof: (a) $\Longrightarrow$ (b).

Let $j: V \longrightarrow M$ be such that $\tau(j) \geq \mu$ and ${}^{<\lambda}M \subset M$. If $\sigma = \langle \gamma_\alpha : \alpha < \mu \rangle$ is an increasing enumeration of $\Gamma(j)$, we define from j and σ, the ultrafilter $U_\lambda(\mu)$ and the ultrapower $M_\lambda(\mu)$.

Let us show $\mathrm{cf}(j_{U_\lambda(\mu)}(\kappa)) = \mu$.

For $\alpha < \mu$, let $\kappa_\alpha = [s \in {}^{\mu}\kappa \longmapsto s(\alpha)]_{U_\lambda(\mu)}$. We want to prove that $j_{U_\lambda(\mu)}(\kappa) = \bigcup_{\alpha<\mu} \kappa_\alpha$.

If $\gamma < j_{U_\lambda(\mu)}(\kappa)$, then there are $E \subset \mu$, $|E| < \lambda$ and $g: {}^{E}\kappa \longrightarrow \kappa$ such that $\gamma = [s \in {}^{\mu}\kappa \longmapsto g(s_{|E})]_{U_\lambda(\mu)}$. Hence $k_\mu(\gamma) = j(g)(\sigma_{|E})$.

Let E be fixed, we define $f \in {}^{\kappa}\kappa$ by setting: for $\alpha < \kappa$,

$$f(\alpha) = \operatorname{Sup}\left\{ g(s) : s \in {}^{E}\kappa \text{ and } \operatorname{Sup}(s''E) \leqslant \alpha \right\} .$$

If $\delta > \operatorname{Sup}(E)$, we have $\chi_\delta \geqslant \operatorname{Sup}(\sigma''E)$. This implies:

$$j(f)(\chi_\delta) \geqslant j(g)(\sigma_{|E}) \geqslant k_\mu(\chi) .$$

Therefore $k_\mu(\chi)$ is at most in $Sk(\chi_\delta)$. By definition of σ , $k_\mu(\chi) < \chi_{\delta+1} = k_\mu(\kappa_{\delta+1})$ and we are done.

$(b) \Longrightarrow (a)$:

Let $j : V \longrightarrow M$ be so that $cf(j(\kappa)) = \mu$. If $\tau(j)$ were a successor ordinal, in the highest sky, there would be a cofinal set of cardinality μ. This is impossible by proposition 1.2 (b). Henceforth $\tau(j)$ is a limit ordinal and $cf(j(\kappa)) = cf(\tau(j))$.

Therefore, we have at least μ skies. ■

Let us assume κ is 2^κ-compact. There exists a fine measure U on $P_\kappa(2^\kappa)$ (cf [Je]). Let $j_U : V \longrightarrow Ult_U$ be the corresponding embedding. The model Ult_U satisfies the following (cf [So.Re.Ka]):

$$(*) : \begin{cases} \text{whenever } X \subset Ult_U \text{ and } |X| \leqslant 2^\kappa, \text{ then there exists } Y \in Ult_U \\ \text{such that } X \subset Y \text{ and } Ult_U \models |Y| < j_U(\kappa). \end{cases}$$

Proof: if $X = \left\{ [f_\alpha]_U : \alpha < 2^\kappa \right\}$, then we define $f : P_\kappa(2^\kappa) \longrightarrow V$ as follows: $f(P) = \left\{ f_\alpha(P) : \alpha \in P \right\}$. We obtain $X \subset [f]_U$ and

$$Ult_U \models \quad \left| [f]_U \right| \leqslant \left| [d]_U \right| < j_U(\kappa) .$$

$(*)$ implies $cf(j_U(\kappa)) > 2^\kappa$. This will give:

<u>Proposition 2.6</u> : If κ is 2^κ-compact, then for any regular cardinal $\lambda \leqslant \kappa$, there exists $j : V \longrightarrow M$ such that ${}^{<\lambda}M \subset M$ and $cf(j(\kappa)) = \lambda$.

Proof: Let us consider $j_U : V \longrightarrow Ult_U$ where U is a fine measure on

$P_\kappa(2^\kappa)$. We have seen $cf(j_U(\kappa)) > 2^\kappa$. If $\tau(j_U)$ were a successor ordinal, then by proposition 1.1, any constellation in the highest sky would be cofinal under $j_U(\kappa)$. But all constellations have cardinality 2^κ, therefore $\tau(j_U)$ is limit and $cf(\tau(j_U)) = cf(j_U(\kappa)) \geqslant (2^\kappa)^+$.

U is κ-complete, hence ${}^\kappa Ult_U \subset Ult_U$. We can thus apply proposition 2.5 to the embedding j_U. ■

Let C_κ represent the closed unbounded filter over a cardinal κ so that $cf(\kappa) > \omega$. We quote without proof the following :

<u>Theorem 2.6</u> (Ketonen) [Ke], [Ka.1] : Let $\lambda < \kappa$ be regular. If U is a κ-ultrafilter, we have the equivalence:

$\tau(U) \geqslant \lambda$ iff there exists $D \leqslant_{R-K} U$ such that

$$D \supset C_\kappa \cup \{\{\alpha<\kappa : cf(\alpha) = \lambda\}\}$$

The preceding proposition, for $\lambda < \kappa$, could be deduced from this theorem. An other consequence is the following:

<u>Corollary</u>: Let λ regular $< \kappa$, if there exists a κ-ultrafilter U which contains $C_\kappa \cup \{\{\alpha < \kappa : cf(\alpha) = \lambda\}\}$, then there is $j : V \longrightarrow M$ with critical point κ such that ${}^{<\lambda} M \subset M$ and $cf(j(\kappa)) = \lambda$.

Mitchell, starting from a κ-ultrafilter with λ skies, for $\omega<\lambda$ regular $\leqslant \kappa$, has shown the existence of an inner model of "$\exists\nu(o(\nu) = \lambda+1)$" (cf [Mi.3]). By using the same arguments, one could show that if there is $j : V \longrightarrow M$ such that ${}^\omega M \subset M$ and $\tau(j) \geqslant \lambda$, then there is an inner model of "$\exists\nu(o(\nu) = \lambda)$".

The difference, $\lambda+1$ in one case, λ in the other one, comes from the closure under λ-sequences which is always satisfied for a κ-ultrafilter.

Let us enounce now two more results of Mitchell :

Theorem 2.7 [Mi.2]: - If there exists $j:V \longrightarrow M$ such that ${}^{\omega}M \subset M$ and $j(\kappa) \geqslant \kappa^{++}$ for some cardinal κ, then there is an inner model of "$\exists\nu(o(\nu) = \nu^{++})$" .

- Hence the same conclusion follows from the existence of a measurable cardinal κ such that $2^{\kappa} > \kappa^{+}$.

Proposition 2.7 : Let $\omega < \lambda$ regular $\leqslant \kappa$. The following assertions imply the existence of an inner model of "$\exists\nu(o(\nu) = \lambda)$" :

(a) there is $j:V \longrightarrow M$ with critical point κ such that ${}^{\omega}M \subset M$ and $cf(j(\kappa)) = \lambda$.

(b) there is $j:V \longrightarrow M$ with critical point κ such that ${}^{<\lambda}M \subset M$ and $cf(j(\kappa)) \neq 2^{\kappa}$.

Proof: (a) is a direct consequence of proposition 2.5 and of the previous remark.

(b) By theorem 2.7, we can assume $2^{\kappa} = \kappa^{+}$. If $cf(j(\kappa)) \leqslant \kappa$, we conclude with (a). If $cf(j(\kappa)) \geqslant \kappa^{++}$, we apply theorem 2.7 . ∎

Let us end the chapter with a few open questions :

- Is there a converse to theorem 2.1 ? Do we have the following ? If there exist λ measurable cardinals $\langle \kappa_{\alpha} : \alpha < \lambda \rangle$, for a regular cardinal $\lambda < \mathrm{Sup}\{\kappa_{\alpha} : \alpha < \lambda\}$, then there is an extension $V[G]$ with an embedding $j:V[G] \longrightarrow M$ of critical point $\kappa \geqslant \lambda$ such that ${}^{<\lambda}M \subset M$ and ${}^{\kappa}M \not\subset M$.

- Let λ inaccessible $\leqslant \kappa$. The following assertions are equivalent:

- There is a R-K chain of κ-ultrafilters of length $\lambda+1$.

$\Updownarrow$

- There is a κ-ultrafilter with $\lambda+1$ constellations.

In view of this result, can we reduce proposition 2.2 to the equivalence:

$\Updownarrow$

(a) there is $j: V \longrightarrow M$ with critical point κ such that ${}^{<\lambda}M \subset M$ and $C(j) \geqslant \lambda$.

(b) there is $j: V \longrightarrow M$ with critical point κ such that ${}^{<\lambda}M \subset M$ and ${}^{\lambda}j(\kappa) \not\subset M$.

(c) there is an ascending R-K chain of κ-ultrafilters of length λ.

– And finally, can we obtain ? :

Let $\omega < \lambda$ regular $\leqslant \kappa$. If $o(\kappa) = \lambda$, then there is an extension $V[G]$ with an embedding $j: V[G] \longrightarrow M$ of critical point κ such that ${}^{\omega}M \subset M$ and $cf(j(\kappa)) = \lambda$.

III. FIXED POINTS

Let us consider the elementary embedding associated with a κ-ultrafilter. Many ordinals are not moved: for example, it is well known that an inaccessible cardinal $> \kappa$ is left fixed, one can prove also that $(2^{\lambda})^{+}$ for λ inaccessible $\geq \kappa$, has the same property.

On the other hand, if κ is supercompact, then any cardinal can be moved by an elementary embedding .

In this chapter, we study some connections between large cardinal assumptions and the absence of fixed points of elementary embeddings.

Let us first remark that, given two measurable cardinals, there exists $j: V \longrightarrow M$ which moves two inaccessible cardinals and such that ${}^{\omega}M \subset M$: if U_0 and U_1 are respectively a κ_0-ultrafilter and a κ_1-ultrafilter for $\kappa_0 < \kappa_1$, it suffices to set $j = j_{j_{U_0}(U_1)} \circ j_{U_0}$. The following proposition is a kind of converse:

<u>Proposition 3.1</u> : Let λ be a strong limit cardinal such that $\lambda > \kappa$ and $\mathrm{cf}(\lambda) > \kappa$. If there exists $j: V \longrightarrow M$ with critical point κ such that $j(\lambda) > \lambda$ and ${}^{\omega}M \subset M$, then there is an inner model with two measurable cardinals.

We have more information in two cases:

<u>Proposition 3.2</u> : With the same hypotheses as in proposition 3.1 :

(a) if there is no inner model of "$\exists \nu (o(\nu) = \nu^{++})$", then there is an inner model in which κ and $\mathrm{cf}^{K(\vec{U}_m)}(\lambda)$ are measurable.

(b) If λ is inaccessible and $j''\lambda \subset \lambda$, then κ and λ are measurable in an inner model.

We present two proofs of proposition 3.1 :

Proof 1 (in the manner of Mitchell) :

If there is an inner model of "$\exists\nu(o(\nu)=\nu^{++})$", we are done. Otherwise, by theorems 2.2 and 2.3, we can consider $K(\vec{U}_m)$ and the iterated ultrapower $j|_{K(\vec{U}_m)}$. Let $j|_{K(\vec{U}_m)} = i_{o\gamma}$, for $\gamma \in \mathrm{Ord}$.

Let α be the least ordinal δ such that $i_{o\delta}(\lambda) > \lambda$.

. We assume first that α is a successor ordinal: $\alpha = \beta+1$.

By hypothesis, $i_{o\beta}(\lambda) = \lambda$. Let κ_β be the measurable cardinal in the β^{th} iterated ultrapower N_β which is used in the embedding $i_{\beta\beta+1}$. Since λ is a strong limit cardinal and λ is moved by $i_{\beta\beta+1}$, we must have : $\mathrm{cf}^{N_\beta}(\lambda) \leq \kappa_\beta$ and $\kappa_\beta \leq \lambda$.

Hence $N_\beta \models$ there is a measurable cardinal μ such that $\mathrm{cf}(\lambda) \leq \mu \leq \lambda$ and $N_o \models$ there is a measurable cardinal θ such that $\mathrm{cf}(\lambda) \leq \theta \leq \lambda$ because $i_{o\beta}(\lambda) = \lambda$.

Therefore, we are done, since κ is measurable in $N_o = K(\vec{U}_m)$.

. If α is a limit ordinal, then let us show $i_{o\alpha}(\lambda) = \bigcup_{\delta<\lambda} i_{o\alpha}(\delta)$ (contrary to the classic iterated ultrapowers, it is not always true for λ limit cardinal such that $\mathrm{cf}(\lambda) \neq \kappa$)

Let $\xi < i_{o\alpha}(\lambda)$. There is $\beta < \alpha$ such that $\xi = i_{\beta\alpha}(\eta)$ where $\eta < i_{o\beta}(\lambda)$. By hypothesis, $i_{o\beta}(\lambda) = \lambda = \bigcup_{\delta<\lambda} i_{o\beta}(\delta)$. Therefore, there exists $\delta < \lambda$ such that $\eta < i_{o\beta}(\delta)$. This gives $\xi < i_{o\alpha}(\eta)$ for $\eta < \lambda$. Since $\bigcup_{\delta<\lambda} i_{o\alpha}(\delta) > \lambda$, there is $\delta < \lambda$ such that $i_{o\alpha}(\delta) > \lambda > \delta^{++}$.

We can apply now theorem 2.7 to obtain a contradiction. ■

Proof 2 of Proposition 3.1 (in the manner of Kunen) :

Let $j : V \longrightarrow M$ be such that $j(\lambda) > \lambda$. We define the sequence $\langle \theta_\alpha : \alpha < \beta \rangle$ for $\beta < \omega_1$, as follows :

- $\theta_0 = \kappa$
- Let us assume $T_\alpha = \langle \theta_\gamma : \gamma < \alpha \rangle$ has been

constructed such that $\theta_\gamma < j(\kappa)$, for each $\gamma < \alpha$. Since $T_\alpha \in M$, we can consider the ultrafilter U_{T_α} on ${}^\alpha\kappa$: let $X \subset {}^\alpha\kappa$.

$$X \in U_{T_\alpha} \quad \text{iff} \quad T_\alpha \in j(X)$$

We again obtain the diagram:

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "j_{U_{T_\alpha}}"'] & M \\ \mathrm{Ult}_{U_{T_\alpha}} \arrow[ur, "k_\alpha"'] & \end{tikzcd}$$

where $\mathrm{Ult}_{U_{T_\alpha}}$ is closed under κ-sequences.

If the critical point θ of k_α is $> j(\kappa)$, we stop the process. Otherwise, we set $\theta_\alpha = \theta$, and iterate the process.

First case : The iteration of the process is of length $> \omega$. Hence we shall argue as in theorem 2.1 .

Let $\overline{K}_1 = \{ \nu : \nu$ strong limit, $\nu > 2^\kappa$ and $cf(\nu) > \kappa \}$

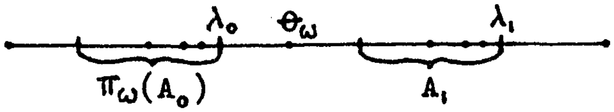
$\overline{K}_{\alpha+1} = \{ \nu \in \overline{K}_\alpha : |\nu \cap \overline{K}_\alpha| = \nu \}$

$\overline{K}_\beta = \bigcap_{\alpha<\beta} \overline{K}_\alpha$ if β limit

and let γ_η = the least element of K_η.

We define $A_0 = \{ \gamma_n : 1 \leq n < \omega \}$ and $A_1 = \{ \gamma_{\omega+n} : 1 \leq n < \omega \}$ and the usual filters F_0 and F on $\lambda_0 = \mathrm{Sup}(A_0)$ and $\lambda_1 = \mathrm{Sup}(A_1)$.

- With a κ-ultrafilter, we can show: $L[F_0, F_1] \models F_0$ is a λ_0-ultrafilter .
- For F_1, by using the embeddings k_n, $n < \omega$, we obtain the same graphic as in theorem 2.1 (with corresponding notations) :



the $L[\pi_\omega(F_0), F_1]$-ultrafilter on θ_ω is the last tool necessary to obtain

$L[F_0,F_1] \models$ F_1 is a λ_1-ultrafilter.

Second case: there exists $\alpha \leqslant \omega$ such that, in the diagram

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "j_{U_{T_\alpha}}"'] & M \\ Ult_{U_{T_\alpha}} \arrow[ur, "k_\alpha"'] & \end{tikzcd}$$

, k_α is I-I onto $j(\kappa)$.

U_{T_α} is isomorphic to a κ-ultrafilter, therefore $j_{U_{T_\alpha}}(\lambda) = \lambda$. This implies that the critical point θ of k_α is so that $j(\kappa) < \theta \leqslant \lambda$. We note $\rho_0 = \theta$ and $\rho_{p+1} = k_\alpha(\rho_p)$. Let us consider $A = \{\rho_p : p < \omega\}$ and let D be a $j_{U_{T_\alpha}}(\kappa)$-ultrafilter in $Ult_{U_{T_\alpha}}$.
Since $A \in Ult_{U_{T_\alpha}}$, we can define in $Ult_{U_{T_\alpha}}$ the usual filter F on $\rho = Sup(A)$: let $X \subset \rho$. $X \in F$ iff $\exists n \ \forall p \geqslant n \ \ \rho_p \in X$.
Obviously, $L[F,D] \models$ $D \cap L[D,F]$ is a $j_{U_{T_\alpha}}(\kappa)$-ultrafilter.
The critical point of k_α is $> j_{U_{T_\alpha}}(\kappa)$, hence $k_\alpha(D) \cap Ult_{U_{T_\alpha}} = D$.
This implies $L[k_\alpha(D),F] = L[D,F]$, and $k_\alpha : L[D,F] \longrightarrow L[D,F]$ is elementary. In this case, it is a known fact (Kunen) used in [Ka.2] that $F \cap L[D,F]$ is, in $L[D,F]$, a ρ-ultrafilter.

Let us give a proof inspired by lemma IO.IO of [Ku.I] :
Let us assume $F \cap L[D,F]$ is not a ρ-ultrafilter.

If X is the least subset of ρ in the well-ordering of $L[D,F]$ such that X and $X^c \notin F$, then as previously, by using the fact that $k_\alpha(X) = X$, one shows : $\rho_0 \in X$ iff $\forall p < \omega, \ \rho_p \in X$, which is contradictory.

Let $\langle X_\xi : \xi < \delta \rangle$, for $\delta < \rho$, be the least counterexample to the ρ-completeness. In the same way, $k_\alpha(\langle X_\xi : \xi < \delta \rangle) = \langle X_\xi : \xi < \delta \rangle$.
We distinguish two cases: – if $\delta \leqslant \rho_0$, then $k_\alpha(\xi) = \xi$; for any $\xi < \delta$,

and $k_\alpha(X_\xi) = X_\xi$. We obtain $\{\rho_p : p < \omega\} \subset \bigcap_{\xi<\delta} X_\xi$.

- Let $\rho_0 \leq \delta < \rho$.

Since $k_\alpha(\langle X_\xi : \xi < \delta \rangle) = \langle X_\xi : \xi < \delta \rangle$, $k_\alpha(\delta)$ must be equal to δ . But there is $p < \omega$ such that $\rho_p \leq \delta < \rho_{p+1}$, this implies $k_\alpha(\rho_p) \leq k_\alpha(\delta)$ and $\rho_{p+1} \leq k_\alpha(\delta)$. We have obtained a contradiction.
Therefore $L[D,F] \models F \cap L[D,F]$ is a ρ-ultrafilter . ■

Proof of Proposition 3.2 :

(a) Let us return to the proof in the manner of Mitchell of the previous proposition.
If there is no model of "$\exists v(o(v)=v^{++})$", the first case must occur: the least α such that $i_{0\alpha}(\lambda) > \lambda$ is a successor ordinal . If $\alpha = \beta+1$, we have seen $cf^{N_\beta}(\lambda) \leq \kappa_\beta \leq \lambda$.
Let us assume $cf^{N_\beta}(\lambda) < \kappa_\beta$. This implies $i_{\beta\beta+1}(\lambda) = \bigcup_{\xi<\lambda} i_{\beta\beta+1}(\xi)$. Since $i_{\beta\beta+1}(\lambda) > \lambda$, there is $\xi < \lambda$ such that $i_{\beta\beta+1}(\xi) > \lambda > (2^{|\xi|})^+$, and it is impossible, with a single ultrafilter to obtain such a result.
Therefore $cf^{N_\beta}(\lambda) = \kappa_\beta$ and $N_0 \models cf(\lambda)$ is measurable .

(b) We shall need the following lemma which will be useful later.

<u>Lemma 3.I</u> : Let $j : V \longrightarrow M$ with critical point κ be such that ${}^\omega M \subset M$ and $j(\lambda) > \lambda$, for $\lambda > \kappa$ inaccessible. One of the following cases occurs:

(I) there is $\gamma < \lambda$ such that $j(\gamma) > \lambda$.

(2) there exists a commutative diagram:

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "i"'] & M \\ N \arrow[ur, "k"'] & \end{tikzcd}$$

such that ${}^\omega N \subset N$ and $i(\lambda) = \lambda$. Moreover the critical point of k is λ, and if ν is a strong limit cardinal such that

$v > 2^\lambda$ and $cf(v) > \lambda$, then $i(v) = v$.

Proof: Let us define by induction the sequence $\langle \gamma_\beta : \beta < \delta \rangle$ as follows:

- $\gamma_0 = \kappa$
- If $\langle \gamma_\beta : \beta < \alpha \rangle$ has already been defined such that $j(\gamma_\beta) > \gamma_\beta$, for $\beta < \alpha$, then in analogy with with previous definitions, $S_{\omega_1}(\prod_{\beta<\alpha} \gamma_\beta)$ denotes the collection of subsets of $\prod_{\beta<\alpha} \gamma_\beta$ with countable support:

$X = in_{E,\alpha}(Y)$ if there are $E \subset \alpha$, $|E| \leqslant \aleph_0$ and $Y \subset \prod_{\beta \in E} \gamma_\beta$ such that $X = \{ s \in \prod_{\beta<\alpha} \gamma_\beta : s|_E \in Y \}$.

We next define $\sigma : j''\alpha \longrightarrow Ord$ by setting $\sigma(j(\beta)) = \gamma_\beta$ for $\beta < \alpha$.

The ultrafilter U_α is thus : let $X \subset \prod_{\beta<\alpha} \gamma_\beta$ and $X = in_{E,\alpha}(Y)$.

$$X \in U_\alpha \qquad \text{iff} \qquad \sigma|_{j(E)} \in j(Y) \ .$$

We obtain the diagram :

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "j_{U_\alpha}"'] & M \\ Ult_{U_\alpha} \arrow[ur, "k_\alpha"'] & \end{tikzcd}$$

where $k_\alpha([s \longmapsto g(s|_E)]_{U_\alpha})$ is $j(g)(\sigma|_{jE})$.

Exactly with the same arguments as in the case of the ultrafilters $U_\lambda(\beta)$ one shows that Ult_{U_α} is closed under ω-sequences.

- If $j_{U_\alpha}(\lambda) > \lambda$, then we stop the process.
- Otherwise, we have $k_\alpha(\lambda) > \lambda$ and we define γ_α as being the critical point of k_α . We then iterate the construction with $\{ \gamma_\beta : \beta \leqslant \alpha \}$.

If $\gamma_\alpha \geqslant \lambda$, then $k_{\alpha+1}$ is $1-1$ onto $(\lambda+1)$ and hence $j_{U_{\alpha+1}}(\lambda) > \lambda$ Since the function $\gamma : \alpha \longmapsto \gamma_\alpha$ is strictly increasing, we know $\alpha \leqslant \gamma_\alpha$ Therefore the iteration must end up, at most, at the $\lambda+1$ st step.

Let $\alpha \leqslant \lambda+1$ be the least ordinal δ such that $j_{U_\delta}(\lambda) > \lambda$.

<u>First case</u>: $\alpha = \lambda + 1$.

In this case $j_{U_\lambda}(\lambda) = \lambda$ and $\gamma_\lambda \geqslant \lambda$. Since $k_\lambda(\lambda) > \lambda$, necessarily

$\kappa_\lambda = \lambda$. Let us check that if ν is strong limit $> 2^\lambda$ and $cf(\nu) > \lambda$, then $j_{U_\lambda}(\nu) = \nu$. Thus we shall be done by setting $i = j_{U_\lambda}$ and $k = k_\lambda$.

As for the usual iterated ultrapowers,
- $j_{U_\lambda}(\nu) = \bigcup_{\xi<\nu} j_{U_\lambda}(\xi)$ is a consequence of $cf(\nu) > \lambda$, and
- $|j_{U_\lambda}(\xi)| \leqslant |\xi|^{\lambda^\omega} . |\lambda|^\omega$ entails $|j_{U_\lambda}(\xi)| < \nu$.

We thus obtain $j_{U_\lambda}(\nu) = \nu$.

<u>Second case:</u> $\alpha = \beta + 1$ for $\beta < \lambda$.

We shall show this time that $i = j_{U_\beta}$ and $k = k_\beta$.

Since $j_{U_\beta}(\lambda) = \lambda$, we have $k_\beta(\lambda) > \lambda$ and hence $\kappa_\beta \leqslant \lambda$. Let us assume $\kappa_\beta < \lambda$. The two points:
- $j_{U_{\beta+1}}(\lambda) = \bigcup_{\xi<\lambda} j_{U_{\beta+1}}(\xi)$ because $|\kappa_\beta|^\omega < cf(\lambda)$, and
- $|j_{U_{\beta+1}}(\xi)| < |\xi|^{|\kappa_\beta|^\omega} . |\beta|^\omega$.

give $j_{U_{\beta+1}}(\lambda) = \lambda$.

Therefore λ must be the critical point of k_β and the last part of the lemma is proved as previously.

<u>Third case:</u> α is a limit ordinal.

Because of the remark before the different cases, we must have $\kappa_\beta < \lambda$, for all $\beta < \alpha$. Hence $\alpha \leqslant \lambda$. If we assume $\alpha < \lambda$, then by regularity of λ, $\sup\{\kappa_\beta : \beta < \alpha\} < \lambda$. We would thus obtain $j_{U_\alpha}(\lambda) = \lambda$.

Therefore α is λ. We then consider the diagrams : for $\beta < \beta' < \lambda$,

$$\begin{tikzcd}
V \arrow[r, "j_{U_{\beta'}}"] \arrow[d, "j_{U_\beta}"'] & \mathrm{Ult}_{U_{\beta'}} \\
\mathrm{Ult}_{U_\beta} \arrow[ur, "k_{\beta\beta'}"'] &
\end{tikzcd}$$

where $k_{\beta\beta'}$ is defined as follows:

$$k_{\beta\beta'}\left(\left[s \in \prod_{\gamma<\beta} \kappa_\gamma \longmapsto g(s|_E)\right]_{U_\beta}\right) = \left[s \in \prod_{\gamma<\beta'} \kappa_\gamma \mapsto g(s|_E)\right]_{U_{\beta'}}$$

We can consider the direct limit of the system $\{\mathrm{Ult}_{U_\beta}, k_{\beta\beta'}, \beta < \beta' < \lambda\}$.

Since λ is regular $> \omega$, this is precisely Ult_{U_λ}.

To show that $j_{U_\lambda}(\lambda) = \mathrm{Sup}\left\{ j_{U_\lambda}(\xi) : \xi<\lambda \right\}$, we argue as in the limit case in the first proof of Proposition 3.I . The two elements of the proof are the fact that $j_{U_\beta}(\lambda) = \lambda = \mathrm{Sup}\left\{ j_{U_\beta}(\xi) : \xi<\lambda \right\}$ for $\beta<\lambda$, and that Ult_{U_λ} is the direct limit of $\left\{ \mathrm{Ult}_{U_\beta} : \beta<\lambda \right\}$. Since $j_{U_\lambda}(\lambda) > \lambda$, there is $\xi<\lambda$ such that $j_{U_\lambda}(\xi) > \lambda$ and (I) of the lemma must occur. ■

Let us prove now (b) of Proposition 3.2 :

If $j''\lambda \subset \lambda$, then case (2) of the previous lemma must occur. We have the diagram:

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "i"'] & M \\ N \arrow[ur, "k"'] & \end{tikzcd}$$

where $i(\lambda) = \lambda$, ${}^{\omega}N \subset N$ and $i(\nu) = \nu$, for ν strong limit cardinal such that $\nu > 2^{\lambda}$ and $\mathrm{cf}(\nu) > \lambda$

the critical point of k is λ.

If $A = \left\{ k^n(\lambda) : n<\omega \right\}$, $\rho = \mathrm{Sup}(A)$, then we define in N the usual filter F on ρ . Let D be a κ-ultrafilter. Since $\lambda > i(\kappa)$, we have already seen that in these conditions: (cf proof of Proposition 3.1)

$$L[i(D),F] \models \quad i(\kappa) \text{ and } \rho \text{ are measurable}$$

We then apply Kunen's methods [Ku.1] to our special context.

By iterations of F, we can replace ρ and F by $\bar{\rho}$ and $\bar{F}$, where $\bar{\rho}$ is strong limit $>2^{|\rho|}$, $\mathrm{cf}(\bar{\rho}) > \lambda$, and $\bar{F}$ is the closed unbounded filter on $\bar{\rho}$.

Let us consider the ultrafilter $U = \left\{ X \subset \lambda : X \in L[i(D),\bar{F}] \text{ and } \lambda \in k(X) \right\}$
The properties of $\bar{\rho}$ imply that j_U is an embedding from $L[i(D),\bar{F}]$ into itself. Thus U is an iterable $L[i(D),\bar{F}]$-ultrafilter, and since ${}^{\omega}N \subset N$, all the iterated ultrapowers are well-founded.

Let K be (in V) the class $\left\{ \nu : \nu \text{ strong limit cardinal, } \mathrm{cf}(\nu) > \lambda \text{ , } \bar{\rho} < \nu \right\}$
We consider the Skolem Hull $S = H^{L[i(D),\bar{F}]}(K \cup \lambda \cup \{\bar{\rho}\})$.

If $\alpha \in S$ and $\alpha < \bar{\rho}$, then $i^{U}_{0\alpha}(\alpha) = \alpha$, this implies $\alpha < \lambda$. Hence there are no ordinals α in S such that $\lambda \leqslant \alpha < \bar{\rho}$ and the transitive collapse of S satisfies " $i(\kappa)$ and λ are measurable ". But S has not been constructed in N, hence we cannot apply i to obtain the expected result.

Let us consider the Skolem Hull $S_0 = H^{L[D,\bar{F}]}(K \cup \lambda \cup \{\bar{\rho}\})$.
We define $\bar{i}: S_0 \longrightarrow S$, as follows: if $x = t^{L[D,\bar{F}]}((\nu_l)_{l \leqslant k}, (\alpha_j)_{j \leqslant n}, \bar{\rho})$, where $\nu_l \in K$ for $l \leqslant k$, $\alpha_j < \lambda$ for $j \leqslant n$, then we set

$$\bar{i}(x) = t^{L[i(D),\bar{F}]}((\nu_l)_{l \leqslant k}, (i(\alpha_j))_{j \leqslant n}, \bar{\rho}) .$$

Since $\alpha_j < \lambda$, for $j \leqslant n$, we have $i(\alpha_j) < \lambda$. Hence if $x \in S_0$, then $\bar{i}(x) \in S$. (Remark: S is not $i(S_0)$ though $i(x) = \bar{i}(x)$ for $x \in S_0$).
One can see that $\bar{i}$ is an elementary embedding because $i(L[D,\bar{F}])$ is $L[i(D),\bar{F}]$.

Let us assume there exists α in S_0 such that $\lambda \leqslant \alpha \leqslant \bar{\rho}$. Since $i(\lambda) = \lambda$ and $i(\bar{\rho}) = \bar{\rho}$, this would imply $\lambda \leqslant i(\alpha) < \bar{\rho}$ and $i(\alpha) \in S$, which is impossible.

Furthermore, by applying the embedding $\bar{i}$, we obtain that κ and $\bar{\rho}$ are measurable in S_0 . Therefore the transitive collapse of S_0 satisfies " κ and λ are measurable " .
The proof of Proposition 3.2 is thus complete. ■

Remark : Whereas part (a) of the proposition was not provable by Kunen's methods, we do not see how to prove part (b) by Mitchell's methods: the hypothesis "there is no inner model of "$\exists \nu (o(\nu) = \nu^{++})$" seems to be needed in the arguments.

Extending the previous results, we obtain:

Proposition 3.3 : Let $j: V \longrightarrow M$ be such that ${}^{\omega}M \subset M$. If there are α inaccessible cardinals moved by j, then there is an inner model with α measurable cardinals.

Proof: Let $\langle \lambda_\beta : \beta < \alpha \rangle$ be the sequence of inaccessible cardinals moved by j. If there is a model of "$\exists \nu (o(\nu) = \nu^{++})$", we are done. Otherwise, let $j|_{K(\vec{U}_m)} = i_{o\gamma}$ for $\gamma \in \mathrm{Ord}$.
From the previous study, we know (proof 1 of Proposition 3.1) that, for each $\beta < \alpha$, there exists $\delta(\beta) < \gamma$ such that $\lambda_\beta = \kappa_{\delta(\beta)}$ and $i_{o\delta(\beta)}(\lambda_\beta) = \lambda_\beta$.
Hence for any $\beta < \alpha$, $N_o \models \lambda_\beta$ is measurable. The proof is thus complete.■

With Kunen's methods, we could obtain only the following:

Proposition 3.4 : Let $j: V \longrightarrow M$ be an elementary embedding which moves α inaccessible cardinals. If ${}^{|\alpha|}M \subset M$, then there is an inner model $L[\vec{F}]$ with α measurable cardinals.

Proof: Let λ_o be the critical point of j and $\langle \lambda_\beta : \beta < \alpha \rangle$ be the increasing enumeration of the inaccessible cardinals moved by j.
If the proposition is true for all $\alpha < \lambda_o$, then the case $\alpha \geqslant \lambda_o$ can be deduced by using iterated ultrapowers.

On the assumption that $\alpha < \lambda_o$, our aim is to construct $\omega\alpha$ embeddings $i_\eta : S_\eta \longrightarrow N_\eta$ such that ${}^{\alpha}S_\eta \subset S_\eta$, for $\eta < \omega\alpha$, and such that the sequence of critical points is strictly increasing. Thus we will apply the same method as in the second proof of Theorem 2.1.

First case: There exist $\lambda_\beta < \lambda_\gamma$ and $n < \omega$ such that $j^n(\lambda_\beta) > \lambda_\beta$.
We thus consider $i = j^n$, $i : V \longrightarrow M_n$, where $M = M_1$ and $j^p(V) = M_p$

for $p<\omega$. One proves by induction on $p<\omega$, that ${}^{\alpha}M_p \subset M_p$: let us assume it is true for p , since $V \cap {}^{\alpha}M \subset M$, we obtain $j^p(V) \cap {}^{\alpha}j^p(M) \subset j^p(M)$, that is $M_p \cap {}^{\alpha}M_{p+1} \subset M_{p+1}$.
$M_{p+1} \subset M_p$ and ${}^{\alpha}M_p \subset M_p$ lead to ${}^{\alpha}M_{p+1} \subset M_{p+1}$.

We construct by induction the sequence $\langle \kappa_\eta : \eta<\delta \rangle$ as follows:

- $\kappa_0 = \lambda_0$
- if $T_\eta = \langle \kappa_\varepsilon : \varepsilon<\eta \rangle$ is already defined for $\eta<\omega\alpha$, $\sigma \in {}^{i''\eta}\mathrm{Sup}(T_\eta)$ is the sequence such that $\sigma(i(\varepsilon)) = \kappa_\varepsilon$ for $\varepsilon<\eta$.

U_η is the ultrafilter on $S_{|\alpha|^+}(\prod_{\varepsilon<\eta} \kappa_\varepsilon)$ = the collection of subsets of $\prod_{\varepsilon<\eta} \kappa_\varepsilon$ with support of cardinality $\leq |\alpha|$. If $X = in_{E,\eta}(Y)$, then

$$X \in U_\eta \quad \text{iff} \quad \sigma_{|E} \in i(Y) \ .$$

In the diagram
$$\begin{tikzcd} V \arrow[r, "i"] \arrow[d, "j_{U_\eta}"'] & M_n \\ \mathrm{Ult}_{U_\eta} \arrow[ur, "k_\eta"'] & \end{tikzcd}$$
, if k_η is the identity, we stop the process. Otherwise we set κ_η = the critical point of k_η , and iterate the construction.

We claim that the length of the iteration is $\geq \omega\alpha$.

Assume it is not the case: there exists $\eta<\omega\alpha$ such that k_η is the identity. An usual computation gives :

$$|j_{U_\eta}(\lambda_\beta)| \leq \lambda_\beta^{|\mathrm{Sup}(T_\eta)|^{|\alpha|}} \cdot |\eta|^{|\alpha|} \text{ and } j_{U_\eta}(\lambda_\beta) < \lambda_\gamma \ .$$

Since $\lambda_\gamma < (k_\eta \circ j_{U_\eta})(\lambda_\beta) < k_\eta(\lambda_\gamma)$, λ_γ is moved by k_η , which is contradictory.

The embeddings $k_\eta : \mathrm{Ult}_{U_\eta} \longrightarrow M_n$, for $\eta<\omega\alpha$, are the expected embeddings.

<u>Second case:</u> For all $\beta<\alpha$ and $n<\omega$, $j^n(\lambda_\beta) < \lambda_{\beta+1}$.

Once again we construct from j , a similar sequence $\langle \kappa_\eta : \eta<\delta \rangle$ and consider the ultrafilter U_η on $S_{|\alpha|^+}(\prod_{\varepsilon<\eta} \varkappa_\varepsilon)$.

If the length of the iteration is $\geqslant \omega\alpha$, we are done. Otherwise we claim that the length of the construction is at least α and that for any $\beta<\alpha$ there is $\eta(\beta)<\delta$ such that $\varkappa_{\eta(\beta)} = \lambda_\beta$.

By definition k_δ is the identity. If we assume that for all $\eta<\delta$, $\varkappa_\eta<\lambda_\beta$, then we must have $j_{U_\delta}(\lambda_\beta) = \lambda_\beta$. This leads to $k_\delta(\lambda_\beta)>\lambda_\beta$ and hence to a contradiction.

Let $\eta(\beta)$ be the least $\eta<\delta$ such that $\varkappa_\eta \geqslant \lambda_\beta$. For any $\eta<\eta(\beta)$, $\varkappa_\eta<\lambda_\beta$, therefore $j_{U_{\eta(\beta)}}(\lambda_\beta) = \lambda_\beta$ and $k_{\eta(\beta)}(\lambda_\beta)>\lambda_\beta$. We thus obtain $\varkappa_{\eta(\beta)} \leqslant \lambda_\beta$ and finally $\varkappa_{\eta(\beta)} = \lambda_\beta$.

Let us write N_β for $\mathrm{Ult}_{U_{\eta(\beta)}}$ and let us consider $k_{\eta(\beta)}: N_\beta \longrightarrow M$ with critical point λ_β.

We can study $j^n(k_{\eta(\beta)}) : j^n(N_\beta) \longrightarrow j^n(M)$, for $n<\omega$ and $\beta<\alpha$. ${}^\alpha N_\beta \subset N_\beta$ entails $j^n(V) \cap {}^\alpha j^n(N_\beta) \subset j^n(N_\beta)$. Since we have seen previously ${}^\alpha \mathrm{Ord} \subset j^n(V)$, we obtain ${}^\alpha \mathrm{Ord} \subset j^n(N_\beta)$ and ${}^\alpha j^n(N_\beta) \subset j^n(N_\beta)$.

Since $j^n(\lambda_\beta) < \lambda_{\beta+1}$, for any $n<\omega$, we have now in our possession $\omega\alpha$ elementary embeddings : $j_{\omega\beta+n} : j^n(N_\beta) \longrightarrow M_{n+1}$, for $\beta<\alpha$ and $n<\omega$, moreover ${}^\alpha j^n(N_\beta)\subset j^n(N_\beta)$ and the sequence of the critical points is strictly increasing.

The proof of Proposition 3.4 is thus complete. ■

Let us check now that if U is a κ-ultrafilter and if λ is inaccessible, then $j_U((2^\lambda)^+) = (2^\lambda)^+$.

- If $\lambda>\kappa$, we already know $j_U(\lambda) = \lambda$. Hence $j_U((2^\lambda)^+) = ((2^\lambda)^+)^{\mathrm{Ult}_U} \leqslant (2^\lambda)^+$.
- If $\lambda = \kappa$, then $j_U((2^\kappa)^+) = ((2^{j_U(\kappa)})^+)^{\mathrm{Ult}_U}$.

Since $|P^{\mathrm{Ult}_U}(j_U(\kappa))| \leqslant |{}^{\kappa}P(\kappa)| \leqslant 2^{\kappa}$, we are done.

As it is shown in the following result, large cardinal assumptions are necessary to move $(2^{\lambda})^{+}$.

<u>Proposition 3.5</u> : Let $j : V \longrightarrow M$ be such that ${}^{\omega}M \subset M$. If there is an inaccessible cardinal λ such that $j((2^{\lambda})^{+}) > (2^{\lambda})^{+}$, then there is an inner model of " $\exists \nu (o(\nu) = \nu^{++})$ ".

Proof: Let us show first that $j((2^{\lambda})^{+}) > (2^{\lambda})^{+}$ implies $j(\lambda) > \lambda$.
If $j(\lambda) = \lambda$, then $j((2^{\lambda})^{+}) = ((2^{j(\lambda)})^{+})^{M} = ((2^{\lambda})^{+})^{M} \leqslant (2^{\lambda})^{+}$, and this contradicts the hypothesis.
Hence we can apply Lemma 3.1:
(1) either there is $\gamma < \lambda$ such that $j(\gamma) > \lambda$.
(2) or there exists a commutative diagram

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "i"'] & M \\ N \arrow[ur, "k"'] & \end{tikzcd}$$

such that $i(\lambda) = \lambda$ and the critical point of k is λ .
If λ is the critical point κ of j , we simply set i = identity and $k = j$.
In case (1), since ${}^{\omega}M \subset M$, the proposition is a consequence of Theorem 2.7 . In case (2), let us check the following:

<u>Claim</u> : $(2^{\lambda})^{M} \geqslant 2^{\lambda}$.

Proof: Since $i(\lambda) = \lambda$, the embedding i is injective from $P(\lambda)$ into $P^{N}(\lambda)$. Therefore $|P^{N}(\lambda)| = 2^{\lambda}$.
We have also $P^{N}(\lambda) \subset P^{M}(\lambda)$ because the critical point of k is λ , consequently $|P^{M}(\lambda)| = 2^{\lambda}$. Finally this gives us $(2^{\lambda})^{M} \geqslant 2^{\lambda}$.

We next consider the model $H = j(M)$ and the embedding $e: V \longrightarrow H$, defined by $e = j \circ j = j(j) \circ j$.

<u>Claim</u> : $e(\lambda) \geqslant (2^{\lambda})^{+}$.

Since ${}^{\omega}H \subset H$, we will be able to conclude by Theorem 2.7.

Proof: $e(\lambda)$ is inaccessible in H and $\lambda < j(\lambda)$ entails $j(\lambda) < e(\lambda)$.
Hence $(2^{j(\lambda)})^{H} < e(\lambda)$. We have seen in the first claim $(2^{\lambda})^{M} \geqslant 2^{\lambda}$.
If we apply j, we get $(2^{j(\lambda)})^{H} \geqslant (2^{j(\lambda)})^{M}$.
Let us show now $(2^{j(\lambda)})^{M} \geqslant (2^{\lambda})^{+}$.
If the converse were true: $(2^{j(\lambda)})^{M} < (2^{\lambda})^{+}$, then $j((2^{\lambda})^{+}) = ((2^{j(\lambda)})^{+})^{M} \leqslant (2^{\lambda})^{+}$

and we would get a contradiction.
Therefore we obtain: $e(\lambda) > (2^{j(\lambda)})^{H} \geqslant (2^{j(\lambda)})^{M} \geqslant (2^{\lambda})^{+}$ and the proof is is complete. ■

IV. NON-SEPARATIVE P-POINTS

We introduce now the notion of separative κ-ultrafilters.

Definition 4.1 (cf Kanamori and Taylor): A κ-ultrafilter U is termed separative if it satisfies the following :

Whenever $f,g \in {}^{\kappa}\kappa$ are unbounded (mod U) and such that $[f]_U \neq [g]_U$ there exists X in U such that $f''X \cap g''X = \emptyset$.

Let us notice first that this can be formulated with the Rudin-Keisler ordering :

Fact: Let U be a κ-ultrafilter and $f,g \in {}^{\kappa}\kappa$ be unbounded (mod U), the two following assertions are equivalent:

- there is $X \in U$ such that $f''X \cap g''X = \emptyset$
- $f^*U \neq g^*U$.

Proof: Let $X \in U$ be so that $f''X \cap g''X = \emptyset$. Since $f''X \in f^*U$ and $g''X \in g^*U$, we necessarily have $f^*U \neq g^*U$.

Conversely, if $f^*U \neq g^*U$, then there is $Y \subset \kappa$ such that Y is in f^*U and not in g^*U . Let us take $X = f^{-1}(Y) \cap g^{-1}(Y^c)$. We obtain $X \in U$ and $f''X \cap g''X = \emptyset$. ■

A selective κ-ultrafilter is separative: for any unbounded function (mod U) f in ${}^{\kappa}\kappa$, $[d]_{f^*U}$ is simply $[f]_U$. Therefore all the f^*U, with $[f]_U$ in $[\kappa, j_U(\kappa)[$ are different.

To find examples of non-separative κ-ultrafilters, let us consider the product $U \otimes U$ where U is an arbitrary κ-ultrafilter. If P_1 and

P_2 denote the projections onto the first and the second coordinates, then we have: $[P_1]_{U\otimes U} \neq [P_2]_{U\otimes U}$ and $P_1{*}(U\otimes U) = U = P_2{*}(U\otimes U)$.

But are there other examples of non-separative κ-ultrafilters ? We answer in this chapter a question of A. Taylor :
" let U be a non-separative κ-ultrafilter. Does there exist a κ-ultrafilter N such that $N\otimes N \leqslant_{R-K} U$? "

If $N\otimes N \leqslant_{R-K} U$, then the number of skies of U must be $\geqslant 2$. Therefore, we shall answer in the negative by proving the following result:

<u>Proposition 4.1</u> : (a) if κ is 2^{2^κ}-supercompact, then there is a non-separative p-point.

(b) Con(ZFC + there is a measurable cardinal) implies Con(ZFC + there is a non-separative p-point)

Proof: (a):
Let $j: V \longrightarrow M$ with critical point κ be such that ${}^{2^{2^\kappa}}M \subset M$. We thus have $((2^{2^\kappa})^+)^M = (2^{2^\kappa})^+$. Hence $(2^{2^\kappa})^+$ belongs to the first sky because $(2^{2^\kappa})^+ = j(f)(\kappa)$, where $f(\alpha) = (2^{2^\alpha})^+$ for $\alpha < \kappa$.

For each $\gamma \in [\kappa, (2^{2^\kappa})^+[$, we define the κ-ultrafilter U_γ : $U_\gamma = \{X \subset \kappa : \gamma \in j(X)\}$. The number of κ-ultrafilters is 2^{2^κ} , hence there must exist $\gamma_1 \neq \gamma_2$ in $[\kappa, (2^{2^\kappa})^+[$ such that $U_{\gamma_1} = U_{\gamma_2}$.
We next consider the ultrafilter U_{γ_1,γ_2} on $\kappa \times \kappa$ defined as follows: let $X \subset \kappa\times\kappa$. $X \in U_{\gamma_1,\gamma_2}$ iff $(\gamma_1,\gamma_2) \in j(X)$.

Let P_1 and P_2 be the projection maps. We obtain:
$U_{\gamma_1} = U_{j(P_1)(\gamma_1,\gamma_2)} = P_1{*}U_{\gamma_1,\gamma_2}$ and $U_{\gamma_2} = U_{j(P_2)(\gamma_1,\gamma_2)} = P_2{*}U_{\gamma_1,\gamma_2}$.
By definition of U_{γ_1,γ_2} , $[P_1]_{U_{\gamma_1,\gamma_2}} \neq [P_2]_{U_{\gamma_1,\gamma_2}}$ is equivalent to $\gamma_1 \neq \gamma_2$
Finally we claim that U_{γ_1,γ_2} is isomorphic to a p-point.

Let us consider the commutative diagram:

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "j_{U_{\gamma_1,\gamma_2}}"'] & M \\ \mathrm{Ult}_{U_{\gamma_1,\gamma_2}} \arrow[ur, "k"'] & \end{tikzcd}$$

where $k([(\alpha,\beta)\in {}^2\kappa \longmapsto f(\alpha,\beta)]_{U_{\gamma_1,\gamma_2}}) = j(f)(\gamma_1,\gamma_2)$.

By elementarity of k, it would suffice to show that $k''j_{U_{\gamma_1,\gamma_2}}(\kappa)$ is included in one sky (relatively to j).

If $[f]_{U_{\gamma_1,\gamma_2}} < j_{U_{\gamma_1,\gamma_2}}(\kappa)$, we define $g \in {}^\kappa\kappa$: let $\beta<\kappa$.

$$g(\beta) = \mathrm{Sup}\left\{f(\alpha,\beta) : \alpha<\beta\right\}.$$

This gives us $j(g)(\gamma_2) \geqslant j(f)(\gamma_1,\gamma_2)$. Since $\gamma_2 \in Sk(\kappa)$, $j(f)(\gamma_1,\gamma_2)\in Sk(\kappa)$ and we are done. ■

The fact that existence of non-selective p-points and measurability are equiconsistent has been proved by Gitik in [Gi]. We shall use his technique in the proof of (b).

(b) The method consists of adding a function f in ${}^\kappa\kappa$, while preserving the measurability of κ. Let $V[G]$ be the extension and $\alpha = ((2^{2^\kappa})^+)^{V[G]}$. This function f will link the first and the highest sky of the embedding $i_{0\alpha+1}$ so that the extended embedding $i^*_{0\alpha+1}$ will have only one sky. Since $i_{0\alpha+1}(\kappa) \geqslant \alpha$, by using the same arguments as in part (a) of the proposition, we shall be able to conclude.

Following Gitik, we use Silver's Forcing to add a function in ${}^\kappa\kappa$. Let us define, by induction on $\alpha \leqslant \kappa$, the α^{th} iterate P_α (cf [Je])

– For $\alpha\leqslant\kappa$, C_α is the Cohen Forcing in V^{P_α} for adding a generic function from α to α:

$$C_\alpha = \left\{f \in V^{P_\alpha} : f \text{ partial function from } \alpha \text{ to } \alpha \text{ and } |f|<\alpha\right\}.$$

. If α is regular and $\alpha \neq \beta^+$ for β limit cardinal, then we set

$P_{\alpha+1} = P_\alpha * C_\alpha$.

. Otherwise $P_{\alpha+1} = P_\alpha * \emptyset$

– At limit steps of the iteration, we use direct limits when the ordinal is inaccessible, and inverse limits otherwise.

We thus consider the iterated Forcing P of length $\kappa+1$.

Let U be a κ-ultrafilter. Given an ordinal α , $i_{0\alpha}$ denotes the usual embedding from V into its α^{th} iterated ultrapower N_α , with respect to U . U_α represents the ultrafilter, obtained from U , on the algebra of subsets of ${}^\alpha\kappa$ with finite support. If $n<\omega$, then U_n is simply $\bigotimes_n U$.

We always assume GCH holds in V .

Let $i_{02}: V \longrightarrow N_2$ and $\kappa_1 = i_{01}(\kappa)$, $\kappa_2 = i_{02}(\kappa)$. It is possible by Silver's method to find G_2 in $V[G]$ so that G_2 is N_2-generic over $i_{02}(P)$ and $i_{02}{}''G \subset G_2$. Therefore one can extend the previous embedding i_{02} to $i^*_{02}: V[G] \longrightarrow N_2[G_2]$.

<u>Lemma 4.1</u> (Gitik): For an accurate choice of G_2 , there exists a function $f \in {}^\kappa\kappa$ in $V[G]$ such that $i^*_{02}(f)(\kappa) = \kappa_1$.

Proof: We have $i_{02}(P) = P_{\kappa+1} * P_{\kappa+1,\kappa_2+1}$ (cf factor lemma in [Je]). G is N_2-generic over $P_{\kappa+1}$, it thus remains to choose G' in $V[G]$ which would be $N_2[G]$-generic over $P_{\kappa+1,\kappa_2+1}$ (we use the same notation for the name $P_{\kappa+1,\kappa_2+1}$ and its interpretation $K_G(P_{\kappa+1,\kappa_2+1})$).

. The Forcing $P_{\kappa+1,\kappa_2+1}$ is κ-closed .

. There are κ^+ dense subsets of $P_{\kappa+1,\kappa_2+1}$ because $|(2^{\kappa_2})^{N_2[G]}| \leqslant |(2^{\kappa_2})^{N_2}|$ and $|(2^{\kappa_2})^{N_2}| \leqslant 2^\kappa$.

. ${}^\kappa N_2[G] \subset N_2[G]$.

From these three elements, one can deduce the existence, in $V[G]$ of a $N_2[G]$-generic G' over $P_{\kappa+1,\kappa_2+1}$. Moreover if $\pi = \bigcup\{p(\kappa) : p \in G\}$, we can require that $\langle 1,1,---,1,\pi \cup (\kappa,\kappa_1)\rangle$ belongs to G'. We finally set $G_2 = G * G'$. If $p = p_{|_\kappa}{}^\wedge p(\kappa)$, then $i_{o2}(p) = p_{|_\kappa}{}^\wedge\langle 1,---,1,p(\kappa)\rangle$ by definition of C_κ. Hence if $p \in G$, then $p(\kappa) \geqslant \pi \cup (\kappa,\kappa_1)$ and $i_{o2}(p) \in G_2$

Let us note $G = \overline{G}_\kappa * \overline{H}_\kappa$ where $\overline{G}_\kappa$ is generic over P_κ and $\overline{H}_\kappa$ is $V[G_\kappa]$ - generic over C_κ. In the same way, $G_2 = \overline{G}_{\kappa_2} * \overline{H}_{\kappa_2}$.

Let $f = \bigcup \overline{H}_\kappa$. f belongs to ${}^\kappa\kappa$. Since $i^*_{o2}(\overline{H}_\kappa) = \overline{H}_{\kappa_2}$ and $(\kappa,\kappa_1) \in \bigcup \overline{H}_{\kappa_2}$, we obtain $i^*_{o2}(f)(\kappa) = \kappa_1$. ■

We would like to obtain a similar result for an arbitrary successor ordinal $\alpha+1$: $i^*_{o\alpha+1}(f)(\kappa) = \kappa_\alpha$ where $\kappa_\alpha = i_{o\alpha}(\kappa)$

$$= [s \in {}^{\alpha+1}\kappa \longmapsto s(\alpha)]_{U_{\alpha+1}}$$

(for the last equalities, cf [Ku.1])

This can be done for $\alpha < \omega$ because we only need the closure of N_α under κ-sequences to carry out the proof.

Let us make now a digression to explain why a natural attempt to prove (b) fails. By the previous remark, we can consider $i^*_{o4} : V[G] \longrightarrow N_4[G_4]$ and $U_{\kappa_2,\kappa_3} = \{X \in V[G] : X \subset \kappa \times \kappa \text{ and } (\kappa_2,\kappa_3) \in i^*_{o4}(X)\}$.

U_{κ_2,κ_3} could be a good example of p-point , but nothing proves that:
$U_{\kappa_2} = \{X \in V[G] : \kappa_2 \in i^*_{o4}(X)\}$ and $U_{\kappa_3} = \{X \in V[G] : \kappa_3 \in i^*_{o4}(X)\}$ are the same κ-ultrafilter.

In fact one can show that, with a similar notation, U_κ , U_{κ_1} and U_{κ_3} are 3 different κ-ultrafilters though $U_\kappa \cap V = U_{\kappa_1} \cap V = U_{\kappa_3} \cap V = U$.

Let us return to the extension of the embedding $i_{o\alpha+1}$. If $\alpha \geqslant \omega$, we shall need a little more work.

Lemma 4.2 : The following diagram is commutative: let $\alpha<\beta<\gamma$.

$$\begin{tikzcd}
V \arrow[r, "i_{0\beta+1}"] \arrow[d, "i_{0\alpha+1}"'] & N_{\beta+1} \\
N_{\alpha+1} \arrow[ru, "i_{\alpha\beta}"] \arrow[r, "i_{\alpha\gamma}"] & N_{\gamma+1} \arrow[u, "i_{\beta\gamma}"']
\end{tikzcd}$$

If γ is a limit ordinal, then $N_{\gamma+1}$ is the direct limit of the system $\{N_{\alpha+1}, i_{\alpha\beta} : \alpha<\beta<\gamma\}$ and $i_{\alpha\gamma}$, for $\alpha<\gamma$, are the corresponding embeddings.

Proof: Let us show first that $i_{\alpha\beta}$ is an elementary embedding from $N_{\alpha+1}$ into $N_{\beta+1}$. Let $x\in N_{\alpha+1}$ and φ be a formula such that $N_{\alpha+1}\models\varphi(x)$.

We have the equivalences : $x\in N_{\alpha+1}$ and $N_{\alpha+1}\models\varphi(x)$

$\Updownarrow$ $N_\alpha\models$ $x\in$ ultrapower of the universe modulo $i_{0\alpha}(U)$ and this ultrapower $\models\varphi(x)$

$\Updownarrow$ $N_\beta\models$ $i_{\alpha\beta}(x)\in$ ultrapower of the universe modulo $i_{0\beta}(U)$ and this ultrapower $\models\varphi(i_{\alpha\beta}(x))$

$\Updownarrow$ $i_{\alpha\beta}(x)\in N_{\beta+1}$ and $N_{\beta+1}\models\varphi(i_{\alpha\beta}(x))$.

Let us check $i_{0\beta+1} = i_{\alpha\beta}\circ i_{0\alpha+1}$.

Let $x\in V$. $i_{0\alpha+1}(x) = i_{\alpha\alpha+1}(i_{0\alpha}(x)) = j_{i_{0\alpha}(U)}(i_{0\alpha}(x))$, if we apply $i_{\alpha\beta}$, we obtain:

$$\begin{aligned} i_{\alpha\beta}(i_{0\alpha+1}(x)) &= j_{i_{\alpha\beta}\circ i_{0\alpha}(U)}(i_{\alpha\beta}(i_{0\alpha}(x))) \\ &= j_{i_{0\beta}(U)}(i_{0\beta}(x)) \\ &= i_{0\beta+1}(x) \ . \end{aligned}$$

Hence the diagram is commutative.

Let M be the direct limit of the system $\{N_{\alpha+1}, i_{\alpha\beta} : \alpha<\beta<\gamma\}$ and let $k_{\alpha\gamma}$, for $\alpha<\gamma$, be the corresponding embeddings :

$$\begin{tikzcd}
N_{\alpha+1} \arrow[r, "k_{\alpha\gamma}"] \arrow[d, "i_{\alpha\beta}"'] & M \\
N_{\beta+1} \arrow[ru, "k_{\beta\gamma}"'] &
\end{tikzcd}$$

We consider the function $\theta : M \longrightarrow N_{\gamma+1}$ defined as follows: let $x \in M$. There exists $\alpha < \gamma$ such that $x = k_{\alpha\gamma}(y)$, with $y \in N_{\alpha+1}$, we thus set $\theta(x) = i_{\alpha\gamma}(y)$.

θ is an elementary embedding: for any formula φ,

$$\begin{tikzcd}[row sep=small] M \models \varphi(x) \arrow[d, Leftrightarrow] \\ N_{\alpha+1} \models \varphi(y) \arrow[d, Leftrightarrow] \\ N_{\gamma+1} \models \varphi(i_{\alpha\gamma}(y)) \end{tikzcd}$$

(one checks easily that $\theta(x)$ does not depend on the choice of α and y)

θ is surjective onto $N_{\gamma+1}$: let $\bar{x} \in N_{\gamma+1}$. Since $\bar{x} \in N_\gamma$, there is $\alpha < \gamma$, $y \in N_\alpha$ such that $\bar{x} = i_{\alpha\gamma}(y)$. Let us show $y \in N_{\alpha+1}$.

$$\begin{tikzcd}[row sep=small] N_\gamma \models \bar{x} \in \text{ultrapower of the universe modulo } i_{0\gamma}(U) \arrow[d, Leftrightarrow] \\ N_\alpha \models y \in \text{ultrapower of the universe modulo } i_{0\alpha}(U) \arrow[d, Leftrightarrow] \\ y \in N_{\alpha+1} . \end{tikzcd}$$

We finally obtain $\bar{x} = \theta(k_{\alpha\gamma}(y))$.

If we consider the transitive collapse of M, then we must have $\theta = \mathrm{Id}$, and since $\theta \circ k_{\alpha\gamma} = i_{\alpha\gamma}$, we obtain $i_{\alpha\gamma} = k_{\alpha\gamma}$. ■

We are now ready to prove:

<u>Lemma 4.3</u> : For any ordinal α, there exists $G_{\alpha+1}$, $N_{\alpha+1}$-generic on $i_{0\alpha+1}(P)$ (in $V[G]$) such that the following diagram is commutative:

for $1 \leq \beta < \gamma$

$$\begin{tikzcd} V[G] \arrow[r, "i^*_{0\gamma+1}"] \arrow[d, "i^*_{0\beta+1}"'] & N_{\gamma+1}[G_{\gamma+1}] \\ N_{\beta+1}[G_{\beta+1}] \arrow[ur, "i^*_{\beta\gamma}"'] & \end{tikzcd}$$

where $i_{0\beta+1} \subset i^*_{0\beta+1}$, $i_{0\gamma+1} \subset i^*_{0\gamma+1}$ and $i_{\beta\gamma} \subset i^*_{\beta\gamma}$.

Proof: We shall proceed inductively. Let β be an ordinal. We assume that for any $1 \leq \alpha \leq \gamma < \beta$ the following diagram is commutative:

$$\begin{tikzcd}
V[G] \arrow[r, "i^*_{o\gamma+1}"] \arrow[d, "i^*_{o\alpha+1}"'] & N_{\gamma+1}[G_{\gamma+1}] \\
N_{\alpha+1}[G_{\alpha+1}] \arrow[ur, "i^*_{\alpha\gamma}"'] &
\end{tikzcd}$$

and that if $\alpha<\beta$, then $G_{\alpha+1}|_{\kappa_\alpha+1} = \{p|_{\kappa_\alpha+1} : p\in G_{\alpha+1}\}$ is N_α-generic over $i_{o\alpha}(P)$.

Let us check first that there is G_2 which has the property of Lemma 4.1 and such that $G_2|_{\kappa_1+1}$ is N_1-generic over $i_{o1}(P)$.

We had $i_{o2}(P) = P_{\kappa+1} * P_{\kappa+1,\kappa_1+1} * P_{\kappa_1+1,\kappa_2+1}$. Instead of choosing G' $N_2[G]$ generic over $(P_{\kappa+1,\kappa_1+1} * P_{\kappa_1+1,\kappa_2+1})$ in one stroke as before, we will take first H, $N_1[G]$ generic over $P_{\kappa+1,\kappa_1+1}$. If we argue in $V[G]$ the same arguments work. At the next step, we argue in $N_1[G][H]$ to find H', $N_2[G][H]$ generic over $P_{\kappa_1+1,\kappa_2+1}$ such that $\langle 1,---,1,\pi\cup(\kappa,\kappa_1)\rangle\in H'$. Thus $G_2 = G*H*H'$ is the required generic.

<u>Successor case</u> : There is δ such that $\beta = \delta+1$. We start from $G_{\delta+1}$, $N_{\delta+1}$-generic over $i_{o\delta+1}(P)$ such that $G_{\delta+1}|_{\kappa_\delta+1}$ is N_δ-generic over $i_{o\delta}(P)$.

We have $i_{o\delta+1}(P) = P_{\kappa_\delta+1} * P_{\kappa_\delta+1,\kappa_{\delta+1}+1}$ and the corresponding generics $G_{\delta+1} = H_o * H_1$ where H_o is $N_{\delta+1}$-generic over $P_{\kappa_\delta+1}$ and H_1 is $N_{\delta+1}[H_o]$ generic over $P_{\kappa_\delta+1,\kappa_{\delta+1}+1}$.

We have also $i_{o\delta+2}(P) = P_{\kappa_{\delta+1}+1} * P_{\kappa_{\delta+1}+1,\kappa_{\delta+2}+1}$ and

$i_{\delta\delta+1}(P_{\kappa_\delta+1}) = P_{\kappa_{\delta+1}+1}$

$i_{\delta\delta+1}(P_{\kappa_\delta+1,\kappa_{\delta+1}+1}) = P_{\kappa_{\delta+1}+1,\kappa_{\delta+2}+1}$.

Hence the problem is to find generics $\overline{H}_o$ and $\overline{H}_1$ respectively $N_{\delta+2}$-generic over $P_{\kappa_{\delta+1}+1}$ and $N_{\delta+2}[\overline{H}_o]$-generic over $(P_{\kappa_{\delta+1}+1,\kappa_{\delta+2}+1})$ such that: $i_{\delta\delta+1}{}''H_o\subset\overline{H}_o$ and $i_{\delta\delta+1}{}''H_1\subset\overline{H}_1$.

- Let us deal first with $\overline{H}_0$:

$P_{\kappa_{\delta+1}+1} = P_{\kappa_\delta+1} * P_{\kappa_\delta+1,\kappa_{\delta+1}+1}$. H_0 is also $N_{\delta+2}$-generic over $P_{\kappa_\delta+1}$, it remains to choose H_0' , $N_{\delta+2}[H_0]$-generic over $P_{\kappa_\delta+1,\kappa_{\delta+1}+1}$.

If $p \in P_{\kappa_\delta+1}$, then $i_{\delta\delta+1}(p) = p|_{\kappa_\delta}{}^\frown\langle 1,----,1,p(\kappa_\delta)\rangle$. Hence we set $e = \bigcup\{p(\kappa_\delta) : p \in G_{\delta+1}\}$.

By hypothesis, H_0 is N_δ-generic, hence let us argue in $N_\delta[H_0]$:

- there are $(\kappa_\delta)^+$ dense subsets of $P_{\kappa_\delta+1,\kappa_{\delta+1}+1}$ in $N_{\delta+1}[H_0]$
- ${}^{\kappa_\delta}N_{\delta+1}[H_0] \subset N_{\delta+1}[H_0]$
- $P_{\kappa_\delta+1,\kappa_{\delta+1}+1}$ is κ_δ-closed .

We thus deduce the existence, in $N_\delta[H_0]$ of a $N_{\delta+1}[H_0]$-generic H_0' over $P_{\kappa_\delta+1,\kappa_{\delta+1}+1}$ such that $\langle 1,---,1,e\rangle \in H_0'$.

We then set $\overline{H}_0 = H_0 * H_0'$, and the induction hypothesis is satisfied since H_0 is $N_{\delta+1}$-generic.

We are also able to extend the embedding $\overline{i}_{\delta\delta+1} : N_{\delta+1}[H_0] \longrightarrow N_{\delta+2}[\overline{H}_0]$.

If $\dot{x}$ is a name for $x = K_{H_0}(\dot{x})$, then $\overline{i}_{\delta\delta+1}(x) = K_{H_0}(i_{\delta\delta+1}(\dot{x}))$

- Choice of $\overline{H}_1$:

We use here Gitik's idea [Gi] . Let D be in $N_{\delta+2}[\overline{H}_0]$ an open dense subset of $P_{\kappa_{\delta+1}+1,\kappa_{\delta+2}+1}$. We want to prove that there exists D' in $N_{\delta+1}[H_0]$ which is an open dense subset of $P_{\kappa_\delta+1,\kappa_{\delta+1}+1}$ such that $i_{\delta\delta+1}(D') \subset D$.

If we assume it is true, then the set :

$$\{p \in P_{\kappa_{\delta+1}+1,\kappa_{\delta+2}+1} : \exists q \in H_1 \text{ such that } i_{\delta\delta+1}(q) \leq p\}$$

is the expected generic $\overline{H}_1$.

If $D = K_{\overline{H}_0}(\dot{D})$, there is $p \in \overline{H}_0$ such that :

$p \Vdash$ ($\dot{D}$ is an open dense subset of $P_{\kappa_{\delta+1}+1,\kappa_{\delta+2}+1}$ and $P_{\kappa_{\delta+1}+1,\kappa_{\delta+2}+1}$ is $\kappa_{\delta+1}$-closed)

Since $N_{\delta+2} = \mathrm{Ult}_{\underset{\ell}{\otimes} i_{0\delta}(U)}(N_\delta)$, there exist $p : {}^2\kappa_\delta \longrightarrow N_\delta$ and $f : {}^2\kappa_\delta \longrightarrow N_\delta$ such that

$$p = [(\eta,\xi) \longmapsto p(\eta,\xi)]_{\underset{\ell}{\otimes} i_{0\delta}(U)} \text{ and}$$

$$\dot{D} = [(\eta,\xi) \longmapsto f(\eta,\xi)]_{\underset{\ell}{\otimes} i_{0\delta}(U)} .$$

Using $\kappa_{\delta+1} = [(\eta,\xi) \longmapsto \xi]_{\underset{\ell}{\otimes} i_{0\delta}(U)}$, we obtain that the following set

$X = \{(\eta,\xi) : p(\eta,\xi) \Vdash$ ($f(\eta,\xi)$ is an open dense subset of $P_{\xi+1,\kappa+1}$ and

$P_{\xi+1,\kappa+1}$ is ξ-closed$)\}$

belongs to $\underset{\ell}{\otimes} i_{0\delta}(U)$.

Let us define in N_δ , $g(\xi) = \bigcap_{\eta<\xi} f(\eta,\xi)$. On $X \cap \{(\eta,\xi) : \eta<\xi\}$, $g(\xi)$ is an open dense subset of $P_{\xi+1,\kappa+1}$ and $f(\eta,\xi) \subset g(\xi)$. Thus, we have:

$p \Vdash [(\eta,\xi) \longmapsto g(\xi)]_{\underset{\ell}{\otimes} i_{0\delta}(U)}$ is an open dense subset of $P_{\kappa_{\delta+1}+1,\kappa_{\delta+2}+1}$

and finally:

$N_{\delta+2}[\overline{H}_0] \models K_{\overline{H}_0}([(\eta,\xi) \longmapsto g(\xi)]_{\underset{\ell}{\otimes} i_{0\delta}(U)})$ is an open dense subset of

$P_{\kappa_{\delta+1}+1,\kappa_{\delta+2}+1}$.

If N is an arbitrary κ-ultrafilter, then we have : for $h \in {}^\kappa\kappa$

$$j_N([\alpha<\kappa \longmapsto h(\alpha)]_N) = [(\alpha,\beta) \in {}^2\kappa \longmapsto h(\beta)]_{N \times N} .$$

Therefore in our case, taking $i_{0\delta}(U)$ instead of N , we obtain :

$$i_{\delta\delta+1}([\eta \longmapsto g(\eta)]_{i_{0\delta}(U)}) = [(\eta,\xi) \longmapsto g(\xi)]_{\underset{\ell}{\otimes} i_{0\delta}(U)} .$$

And by applying $\overline{i}_{\delta\delta+1}^{-1}$

$N_{\delta+1}[H_0] \models K_{H_0}([\eta \longmapsto g(\eta)]_{i_{0\delta}(U)})$ is an open dense subset of $P_{\kappa_\delta+1,\kappa_{\delta+1}+1}$.

Since $K_{\overline{H}_0}([(\eta,\xi) \longmapsto g(\xi)]_{\underset{\ell}{\otimes} i_{0\delta}(U)}) \subset D$, it suffices to set $D' = K_{H_0}([\eta \longmapsto g(\eta)]_{i_{0\delta}(U)})$ and the proof of the successor case is complete. $G_{\delta+2}$ is $\overline{H}_0 * \overline{H}_1$ and we can define:

$$i^*_{\delta\delta+1} : N_{\delta+1}[G_{\delta+1}] \longrightarrow N_{\delta+2}[G_{\delta+2}] .$$

<u>Limit case</u> :

β is a limit ordinal, and we assume that we have a sequence of generics $\langle G_{\alpha+1} : \alpha<\beta \rangle$ such that, if $\alpha<\gamma<\beta$, $p \in G_{\alpha+1}$ iff $i_{\alpha\gamma}(p) \in G_{\gamma+1}$, and such that $G_{\alpha+1|K_\alpha+1}$ is N_α-generic on $i_{o\alpha}(P)$.

We know from Lemma 4.2 that $N_{\beta+1}$ is the direct limit of the system $\{N_{\alpha+1}, i_{\alpha\gamma} \text{ for } \alpha<\gamma<\beta\}$. Hence we define $G_{\beta+1}$ on $i_{o\beta+1}(P)$ as follows: $p \in G_{\beta+1}$ iff there is $\alpha<\beta$ such that $p = i_{\alpha\beta}(q)$ and $q \in G_{\alpha+1}$.

This is a filter, let us check the denseness condition:

If D is a dense subset of $i_{o\beta+1}(P)$, there is $\alpha<\beta$ such that $D = i_{\alpha\beta}(D')$ and D' is dense in $N_{\alpha+1}$. Hence let $p_{\alpha+1} \in D' \cap G_{\alpha+1}$. This implies $i_{\alpha\beta}(p_{\alpha+1}) \in G_{\beta+1} \cap D$.

The definition of $G_{\beta+1}$ yields the following diagram: for $\alpha<\gamma<\beta$

$$\begin{tikzcd}
N_{\alpha+1}[G_{\alpha+1}] \arrow[r, "i^*_{\alpha\beta}"] \arrow[d, "i^*_{\alpha\gamma}"'] & N_{\beta+1}[G_{\beta+1}] \\
N_{\gamma+1}[G_{\gamma+1}] \arrow[ur, "i^*_{\gamma\beta}"'] &
\end{tikzcd}$$

where $i^*_{\alpha\beta}$, $i^*_{\alpha\gamma}$, $i^*_{\gamma\beta}$ are defined from $i_{\alpha\beta}$, $i_{\alpha\gamma}$, $i_{\gamma\beta}$ in the usual way.

Let us verify that $G_{\beta+1|K_\beta+1}$ is N_β-generic on $i_{o\beta}(P)$.

If Δ is a dense subset of $i_{o\beta}(P)$ in N_β, there is $\alpha<\beta$ such that $\Delta = i_{\alpha\beta}(\Delta')$ where Δ' is a dense subset of $i_{o\alpha}(P)$ in N_α.

By hypothesis, $G_{\alpha+1|K_\alpha+1}$ is N_α-generic over $i_{o\alpha}(P)$. Hence there is $p \in G_{\alpha+1}$ such that $p_{|K_\alpha+1} \in \Delta'$. $i_{\alpha\beta}(p) \in G_{\beta+1}$ and since $i_{\alpha\beta}(p_{|K_\alpha+1})$ is $(i_{\alpha\beta}(p))_{|K_\beta+1}$, we are done.

To define the embedding $i^*_{o\beta+1}$, we set:

- for $\beta = \delta+1$, $i^*_{o\beta+1} = i^*_{\delta\delta+1} \circ i^*_{o\delta+1}$.
- for β limit, $i^*_{o\beta+1} = i^*_{\alpha\beta} \circ i^*_{o\alpha+1}$ where α is an arbitrary ordinal $<\beta$. ■

Finally, we obtain :

Lemma 4.4 : Let $\alpha \geqslant 1$. The extended embedding $i^*_{0\alpha+1} : V[G] \longrightarrow N_{\alpha+1}[G_{\alpha+1}]$ has only one sky and at least $|\alpha|$ constellations.

Remark : the last fact does not play any role in the proof but shows that we do not introduce too many functions in ${}^{\kappa}\kappa$.

Proof : We have seen that we can choose G_2 so that, for some $f \in {}^{\kappa}\kappa$, $i^*_{02}(f)(\kappa) = \kappa_1$, and then carry out the inductive construction. Since $i^*_{0\alpha+1} = i^*_{1\alpha} \circ i^*_{02}$, we obtain :

$$i^*_{0\alpha+1}(f)(\kappa) = (i^*_{1\alpha} \circ i^*_{02})(f)(\kappa) = i^*_{1\alpha}(i^*_{02}(f)(\kappa)) = i^*_{1\alpha}(\kappa_1) = \kappa_\alpha .$$

Let us show that κ_α is in the highest sky of $i_{0\alpha+1}$.

If $\xi < i_{0\alpha+1}(\kappa)$, there exist E finite $\subset \alpha$ and $g : {}^{E}\kappa \times \kappa \longrightarrow \kappa$ such that

$$\xi = [s \in {}^{\alpha+1}\kappa \longmapsto g(s_{|E}, s(\alpha))]_{U_{\alpha+1}} .$$

We thus define $h \in {}^{\kappa}\kappa$: for $\gamma < \kappa$,

$h(\gamma) = \operatorname{Sup} \{ g(t,\gamma) : t \in {}^{E}\kappa$ and $\operatorname{Sup}(t''E) \leqslant \gamma \}$.

The set $X = \{ s \in {}^{\alpha+1}\kappa : \operatorname{Sup}(s''E) \leqslant s(\alpha) \}$ belongs to $U_{\alpha+1}$, and if $s \in X$ then $h(s(\alpha)) \geqslant g(s_{|E}, s(\alpha))$.

Since $\kappa_\alpha = [s \in {}^{\alpha+1}\kappa \longmapsto s(\alpha)]_{U_{\alpha+1}}$, this implies :

$$i_{0\alpha+1}(h)(\kappa_\alpha) \geqslant [s \in {}^{\alpha+1}\kappa \longmapsto g(s_{|E}, s(\alpha))]_{U_{\alpha+1}} \geqslant \xi .$$

Let us check now that the κ_γ , for $\gamma \leqslant \alpha$, are in different constellations. We assume the converse is true : there exists $g \in {}^{\kappa}\kappa$ such that

$i^*_{0\alpha+1}(g)(\kappa_\gamma) = \kappa_{\gamma'}$ for $\gamma < \gamma' \leqslant \alpha$,

We thus obtain $i^*_{\gamma'\alpha}(i^*_{0\gamma'+1}(g)(\kappa_\gamma)) = \kappa_{\gamma'}$, which is impossible since $\kappa_{\gamma'}$ is the critical point of $i^*_{\gamma'\alpha}$. ■

To conclude the proof of the proposition, it suffices to take $\alpha = ((2^{2^\kappa})^+)^{V[G]} = ((2^{2^\kappa})^+)^V$, and then to apply to $i^*_{0\alpha+1}$ the same method as in part (a). ■

V. INDISCERNIBLES FOR C^{ω_1}.

In analogy with Gödel's model L, Chang [Ch] introduced the model C^{ω_1} constructed by using the infinitary language $L_{\omega_1\omega_1}$.

This language consists of :
- ω_1 variables
- the predicates $\in$ and $=$
- the connectives $\neg$, $\bigwedge_{\alpha<\beta}\varphi_\alpha$, $\bigvee_{\alpha<\beta}\varphi_\alpha$ for $\beta<\omega_1$
- the quantifiers $\exists_{\alpha<\beta} v_\alpha$, $\forall_{\alpha<\beta} v_\alpha$ for $\beta<\omega_1$

An element x is definable in $L_{\omega_1\omega_1}$ over a model A from a countable sequence $s \in {}^{\omega}A$ if there is a formula $\varphi \in L_{\omega_1\omega_1}$ such that $x = \{u \in A : A \models \varphi(s^\frown u)\}$.

The cumulative hierarchy $\langle C_\alpha : \alpha \in \mathrm{Ord}\rangle$ is defined as follows :

- $C_0 = \emptyset$
- $C_{\alpha+1}$ is the set of all subsets of C_α , definable in $L_{\omega_1\omega_1}$ over C_α from a countable sequence of elements of C_α .
- $C_\gamma = \bigcup_{\alpha<\gamma} C_\alpha$ if γ is a limit ordinal .

C^{ω_1} is thus $\bigcup_{\alpha\in \mathrm{Ord}} C_\alpha$.

From now on, we write C instead of C^{ω_1} . The definition of C is "pseudo-absolute" : if M is an inner model such that ${}^{\omega}M \subset M$, then $(C)^M = C$. Hence C is the least inner model such that ${}^{\omega}C \subset C$.

Many properties of L can be extended to C (see [Ch]). A great difference between the two models concerns the axiom of choice : by Forcing or under the assumption of ω_1 measurable cardinals [Ku.3], one can obtain a model C which does not satisfy the axiom of choice.

In this chapter, we shall deal with the notion of indiscernibles. Silver has studied the relation between indiscernibles for L and $0^{\#}$, similar results can be obtained for $L[U]$ and $0^{\dagger}$ (Solovay) or the Core Model K and measurability (Dodd and Jensen). The existence of "indiscernibles" for C is also linked to large cardinal assumptions.

Definition 5.1 : Given two sequences s and s' of length β, the notation $s =_f s'$ means that there exists a finite subset X of β such that, for any $\gamma \in \beta \setminus X$, $s(\gamma) = s'(\gamma)$.

Let $\langle \gamma_\delta : \delta < \omega\alpha \rangle$ be an increasing enumeration of a set of ordinals A of order-type $\omega\alpha$. For $\beta < \alpha$, we define $A_\beta = \{\gamma_{\omega\beta+n} : n < \omega\}$ and if $s \in {}^{\omega\alpha}A$ is strictly increasing, we note s_β = "the restriction of s onto A_β", that is :

$$s_\beta(n) = \text{the } (n+1)^{st} \text{ element of } \mathrm{Range}(s) \cap A_\beta \text{ .}$$

Given a formula $\varphi(v_{i_0}, v_{i_1}, ---, v_{i_\beta}, ---)$ in $L_{\omega_1\omega_1}$, where $i : \delta \longrightarrow \omega\alpha$ and $|\delta| \leqslant \omega$, we write $C \models \varphi(s)$ for $C \models \varphi(s(i_0), ---, s(i_\beta),)$ Hence the expression $C \models \varphi(s)$ is coherent even if $\alpha \geqslant \omega_1$.

Definition 5.2 : A set of ordinals A of order-type $\omega\alpha$, is a set of indiscernibles for C, if A satisfies the following :

whenever $s, s' \in {}^{\omega\alpha}A$, strictly increasing are such that $s =_f s'$ and $s_\beta =_f s'_\beta$ for all $\beta < \alpha$, then $C \models \varphi(s) \longleftrightarrow \varphi(s')$, for any formula φ in L_{ω_1,ω_1}.

Remark: If $o.t(A) = \omega$, the definition becomes : for any $s, s' \in {}^{\omega}A$, strictly increasing, ($s =_f s'$) implies ($C \models \varphi(s) \longleftrightarrow \varphi(s')$, for $\varphi \in L_{\omega_1\omega_1}$) The elements of A are, in particular, indiscernibles in the usual meaning.

Proposition 5.1 : Let $\alpha \leq \omega_1$. If there exist α measurable cardinals, then there is a set of indiscernibles for C , of order-type $\omega\alpha$.

Proof: Let $\langle \kappa^\beta : \beta < \alpha \rangle$ be an increasing sequence of measurable cardinals. If U_β is a κ^β-ultrafilter, for $\beta < \alpha$, we consider the set $A_\beta = \left\{ i_{on}^{U_\beta}(\kappa^\beta) : n < \omega \right\}$ and we claim that $A = \bigcup_{\beta<\alpha} A_\beta$ is the expected set of indiscernibles.

For each $\beta < \alpha$, let $s_\beta, s'_\beta \in {}^\omega A_\beta$, strictly increasing be such that $s_\beta =_f s'_\beta$. We assume moreover that $\left\{ \beta_i : i \leq p \right\}$ is the set of ordinals β such that $s_\beta \neq s'_\beta$.

If $C \models \varphi(\langle s_\beta : \beta < \alpha \rangle)$ (there is again an abuse of notation, we should write $\varphi(\langle s_\beta : \beta \in \Delta \rangle)$, for $\Delta \subset \alpha$ and $|\Delta| \leq \omega$ in the case $\alpha = \omega_1$) let us show $C \models \varphi(\langle s_\beta : \beta < \beta_0 \rangle {}^\frown s'_{\beta_0} {}^\frown \langle s_\beta : \beta_0 < \beta < \alpha \rangle)$. If this is true, we can repeat the process, for all β_i , $i \leq p$, and after a finite number of steps, we would obtain the expected result : $C \models \varphi(\langle s'_\beta : \beta < \alpha \rangle)$.

We write κ_n^β for $i_{on}^{U_\beta}(\kappa^\beta)$ where $\beta < \alpha$ and $n < \omega$.
If $s_{\beta_0} = \langle \kappa_{p_n}^{\beta_0} : n < \omega \rangle$ and $s'_{\beta_0} = \langle \kappa_{q_n}^{\beta_0} : n < \omega \rangle$, let n_0 be such that $\kappa_{p_n}^{\beta_0} = \kappa_{q_n}^{\beta_0}$ for $n \geq n_0$.
Let $m < \omega$ and $\beta < \alpha$. i_{om}^β represents the usual embedding from V into its m^{th} iterated ultrapower N_m^β , modulo U_β .

Let r be the integer p_{n_0} . Since ${}^\omega N_r^{\beta_0} \subset N_r^{\beta_0}$, $i_{or}^{\beta_0}$ is an elementary embedding from C into itself.
Let us check $i_{or}^{\beta_0}(s_\beta) = s_\beta$ for $\beta \neq \beta_0$, or equivalently that $i_{or}^{\beta_0}(\kappa_n^\beta)$ is equal to κ_n^β for $\beta \neq \beta_0$ and $n < \omega$.

- If $\beta < \beta_0$, it is straightforward.
- If $\beta > \beta_0$, κ_n^β is inaccessible in the model N_n^β . Since U_{β_0}

belongs to this model, we are done.

There exists $\sigma : [n_0,\omega[\longrightarrow A_{\beta_0}$ such that $s_{\beta_0}|[n_0,\omega[= i^{\beta_0}_{or}(\sigma)$: it suffices to take $\sigma(n) = \kappa^{\beta_0}_{p_n - r}$ for $n \geqslant n_0$. We use here the equality $i_{ok} \circ i_{om} = i_{om+k}$, for any $k,m < \omega$. This is a consequence of the following result: if U and D are two κ-ultrafilters, then $j_{U\times D} = j_U \circ j_D$ (cf. [Ka. Ma.]).

Henceforth, $C \models \varphi(\langle s_\beta : \beta<\alpha\rangle)$ can be written :

$$C \models \varphi(i^{\beta_0}_{or}(\langle s_\beta : \beta<\beta_0\rangle)^\frown s_{\beta_0}|n_0{}^\frown i^{\beta_0}_{or}(\sigma)^\frown i^{\beta_0}_{or}(\langle s_\beta : \beta_0<\beta<\alpha\rangle))$$

and finally :

$C \models \psi(i^{\beta_0}_{or}(\tau), s_{\beta_0}|n_0)$ where τ is defined from $\langle s_\beta : \beta \neq \beta_0\rangle$ and σ.

We have to apply now a classic result of indiscernibility, in our finite case, let us only check the following result:

If U is a κ-ultrafilter and $\kappa_n = i^U_{on}(\kappa)$, for $n<\omega$, then for any increasing sequence $i_0 < i_1 < \dots < i_q < r$,

$$U_{\kappa_{i_0},\dots,\kappa_{i_q}} = \{X \subset {}^{q+1}\kappa : \langle \kappa_{i_0},\dots,\kappa_{i_q}\rangle \in i^U_{or}(X)\} = \underset{q+1}{\otimes} U .$$

It suffices to see that, by definition of $\underset{q+1}{\otimes} U$, for $X \subset {}^{q+1}\kappa$, we have the equivalences:

$$\left\Updownarrow \begin{array}{l} X \in U_{\kappa_{i_0},\dots,\kappa_{i_q}} \\ \{\alpha_0<\kappa : \{\alpha_1<\kappa : \{\dots\dots \{\alpha_{r-1}<\kappa : \langle \alpha_{i_0},\dots,\alpha_{i_q}\rangle \in X\} \in U\} \dots\} \in U \\ \{\alpha_{i_0}<\kappa : \{\alpha_{i_1}<\kappa : \{\dots\dots \{\alpha_{i_q}<\kappa : \langle \alpha_{i_0},\dots,\alpha_{i_q}\rangle \in X\} \in U\} \dots\} \in U \\ X \in \underset{q+1}{\otimes} U . \end{array}\right.$$

Hence if we define the ultrafilters $U(s)$ and $U(s')$:

$X \in U(s)$ iff $N^{\beta_0}_r \models s_{\beta_0}|n_0 \in i_{or}(X)$

$X \in U(s')$ iff $N^{\beta_0}_r \models s'_{\beta_0}|n_0 \in i_{or}(X)$

then we obtain $U(s) = U(s') = \underset{n_0}{\otimes} U$.

Let X be the set $\{t \in {}^{n_0}\kappa^{\beta_0} ; C \models \psi(\tau, t)\}$.

Since $s_{\beta_0 | n_0} \in i^{\beta_0}_{or}(X) \longleftrightarrow s'_{\beta_0 | n_0} \in i^{\beta_0}_{or}(X)$, we get $C \models \psi(i^{\beta_0}_{or}(\tau), s'_{\beta_0 | n_0})$ $i^{\beta_0}_{or}(\tau)$ is also $s'_{\beta_0 | [n_0, \omega[}$, hence the proof is complete :

$$C \models \varphi(\langle s_\beta : \beta < \beta_0 \rangle {}^\wedge s'_{\beta_0} {}^\wedge \langle s_\beta : \beta_0 < \beta < \alpha \rangle) \qquad \blacksquare$$

There is a partial converse.

Theorem 5.1 : (a) Let α be a natural integer or ω. If there exists a set of indiscernibles for C , of order-type $\omega\alpha$, then there is an inner model with α measurable cardinals.

(b) The existence of a set of indiscernibles of order-type ω_1 implies the existence , for any $\alpha < \omega_1$, of an inner model with α measurable cardinals.

Proof: The proof of (a) and (b) is common to a large extent.

Let $\alpha < \omega_1$ and $A = \{ \gamma_\delta : \delta < \omega\alpha \}$ be a set of indiscernibles.

If $\beta < \alpha$, we define the usual filters F_β on $\lambda_\beta = \mathrm{Sup}(A_\beta)$ (where A_β was the set $\{\gamma_{\omega\beta+n} : n < \omega\}$), let $X \subset \lambda_\beta$.

$$X \in F_\beta \qquad \text{iff} \qquad \exists n \, \forall p \geqslant n \ \gamma_{\omega\beta+p} \in X .$$

Next we consider the model $L[\langle F_\beta : \beta < \alpha \rangle]$. If $\bar{F}_\beta = F_\beta \cap L[\langle F_\beta : \beta < \alpha \rangle]$, then we shall write $\vec{\bar{F}}$ for $\langle \bar{F}_\beta : \beta < \alpha \rangle$. If $\beta < \alpha$, σ_β will denote the sequence $\langle \gamma_{\omega\beta+n} : n < \omega \rangle$.

Lemma 5.1 : For any $\beta < \alpha$, $L[\vec{\bar{F}}] \models \bar{F}_\beta$ is a σ-complete free ultrafilter on λ_β.

Proof: Let H_β , for $\beta < \alpha$, be the Skolem Hull of $A_\beta \cup \{\vec{\bar{F}}\}$, in $L[\vec{\bar{F}}]$:
$H_\beta = H^{L[\vec{\bar{F}}]}(A_\beta \cup \{\vec{\bar{F}}\})$.

Let us show that $H_\beta \models \bar{F}_\beta$ is a σ-complete ultrafilter on λ_β.

Since $H_\beta < L[\vec{F}]$, we shall be done.

$\overline{F}_\beta$ is an ultrafilter :

Let $X \in H_\beta$, $X \subset \lambda_\beta$. There exists a Skolem term t and $n < \omega$ such that $X = t(\gamma_{\omega\beta}, ---, \gamma_{\omega\beta+n}, \vec{F})$. The well-ordering of $L[\vec{F}]$ can be expressed by a $\Sigma_1(\vec{F})$ formula (cf. [Dev.]) and ${}^{\omega}C \subset C$, hence there is a formula ψ in $L_{\omega\omega}$ such that:

$C \models X$ is the unique x such that $\psi(x, \gamma_{\omega\beta}, ----, \gamma_{\omega\beta+n}, \vec{F})$.

There exists a formula which defines $\vec{F}$ from $\langle \sigma_\delta : \delta \neq \beta \rangle$ and from $\sigma_\beta|_{[p+2,\omega[}$, for any $p > n$.

We shall write $\sigma_{\geq q}$ for $\sigma|_{[q,\omega[}$. Therefore the expression $\gamma_{\omega\beta+p} \in X$ can be written:

$C \models \varphi(X, \gamma_{\omega\beta}, ----, \gamma_{\omega\beta+n}, \gamma_{\omega\beta+p}, \sigma_{\beta \geq p+2}, \langle \sigma_\delta : \delta \neq \beta \rangle)$, for some formula φ in $L_{\omega\omega}$.

By indiscernibility, we obtain:

$C \models \varphi(X, \gamma_{\omega\beta}, ---, \gamma_{\omega\beta+n}, \gamma_{\omega\beta+p+1}, \sigma_{\beta \geq p+2}, \langle \sigma_\delta : \delta \neq \beta \rangle)$,

and $\gamma_{\omega\beta+(p+1)} \in X$.

Hence for any $p > n$, $\gamma_{\omega\beta+p} \in X \longleftrightarrow \gamma_{\omega\beta+n+1} \in X$. This implies either X or X^c belongs to $\overline{F}_\beta$.

σ-completeness :

Let $T = \langle X_n : n < \omega \rangle$ be an ω-sequence of elements of $\overline{F}_\beta$ and $T = t(\gamma_{\omega\beta}, ---, \gamma_{\omega\beta+n}, \vec{F})$.

If m, q are such that $n < q < m$, $\vec{F}$ can be defined from $\langle \sigma_\delta : \delta \neq \beta \rangle$ and $\sigma_{\beta \geq m+1}$. Again by indiscernibility, we obtain :

$$\Updownarrow \begin{array}{l} C \models \gamma_{\omega\beta+m} \in \text{the } p^{th} \text{ term of } t(\gamma_{\omega\beta}, ----, \gamma_{\omega\beta+n}, \vec{F}) \\ C \models \gamma_{\omega\beta+q} \in \text{the } q^{th} \text{ term of } t(\gamma_{\omega\beta}, ----, \gamma_{\omega\beta+n}, \vec{F}) \end{array}$$

We have assumed that each X_p , $p < \omega$, belongs to $\overline{F}_\beta$, therefore it must be the case that $\{\gamma_{\omega\beta+q} : q > n\}$ is included in any term of $t(\gamma_{\omega\beta}, --, \vec{F})$ and finally $\bigcap_{n<\omega} X_n \in \bar{F}_\beta$. ■

The existence of a σ-complete free ultrafilter on λ_β implies the existence of a measurable cardinal $\kappa \leq \lambda_\beta$ (cf. Lemma 5.4 below). Therefore if $o.t(A) = \omega$, we are done. Otherwise we know that there is a measurable cardinal below each λ_β , for $\beta < \alpha$, but nothing proves that it is not the same one for all $\beta < \alpha$.
To avoid this situation, we shall use the notion of minimal set of indiscernibles, inspired from Silver's notion of remarkability (cf [Je]).

For $\beta < \alpha$, we note $\vec{\gamma}_\beta$ any tuple of the form $\langle \gamma_{\omega\beta+r}, ---, \gamma_{\omega\beta+r+q-1} \rangle$ and we write $|\vec{\gamma}_\beta|$ for $r+q$. The translated of $\vec{\gamma}_\beta$ is thus the tuple $tr(\vec{\gamma}_\beta) = \langle \gamma_{\omega\beta+r+1}, ----, \gamma_{\omega\beta+r+q} \rangle$ and $tr^{k+1}(\vec{\gamma}_\beta) = tr(tr^k(\vec{\gamma}_\beta))$.
If $x \in H_\beta$ and $x = t(\vec{\gamma}_\beta, \vec{F})$, there is a formula φ such that for any $n > |\vec{\gamma}_\beta|$, $C \models x$ is the unique z such that $\varphi(z, \vec{\gamma}_\beta, \langle \sigma_{\delta > n} : \delta < \alpha \rangle)$.
Hence for $n > 2|\vec{\gamma}_\beta|$, we have:

$$C \models \exists! y \text{ such that } \varphi(y, tr(\vec{\gamma}_\beta), \langle \sigma_{\delta > n} : \delta < \alpha \rangle) .$$

(the symbol ! means unique). Let us call this element $tr(x)$, one can see by indiscernibility, that $tr(x)$ does not depend on the choice of the formula φ and on the tuple $\vec{\gamma}_\beta$.

Lemma 5.2 : If for any ordinal $x < \gamma_{\omega\beta}$, in H_β, $tr(x) = x$, then

$H_\beta \models \bar{F}_\beta$ is λ_β-complete.

Proof: By indiscernibility, it suffices to show :

$$H_\beta \models \bar{F}_\beta \text{ is } |\gamma_{\omega\beta}|\text{-complete.}$$

Let $T = \langle X_\xi : \xi < \eta \rangle$, for $\eta < \gamma_{\omega\beta}$, be a sequence of elements of $\bar{F}_\beta$ and let $T = t(\vec{\gamma}_\beta, \vec{F})$.

We use now the same idea as previously : if $|\vec{\gamma}_\beta| < r < s$ and $\xi \in H_\beta$, $\xi < \eta$, then we want to prove the equivalence:

$$\begin{array}{l} \gamma_{\omega\beta+r} \in \xi^{\text{th}} \text{ term of } t(\vec{\gamma}_\beta, \vec{F}) \\ \Updownarrow \\ \gamma_{\omega\beta+s} \in \xi^{\text{th}} \text{ term of } t(\vec{\gamma}_\beta, \vec{F}) \text{ .} \end{array}$$

Since $\xi = tr^k(\xi)$, for all $k < \omega$, we have $\xi = t_\xi(\vec{\gamma}, \vec{F})$ where $\vec{\gamma}$ is a tuple $\langle \gamma_{\omega\beta+1}, ----, \gamma_{\omega\beta+m} \rangle$ for $1 > s$.

If we define $\vec{F}$ from $\langle \sigma_{\delta \geqslant m+1} : \delta < \alpha \rangle$, then the equivalence is obtained by indiscernibility.

Since the ξ^{th} term of T belongs to $\bar{F}_\beta$, we must have :

$$\{ \gamma_{\omega\beta+p} : |\vec{\gamma}_\beta| < p < \omega \} \subset \xi^{\text{th}} \text{ term ,}$$

and finally $\bigcap\limits_{\substack{\xi<\eta \\ \text{in } H_\beta}} X_\xi \in \bar{F}_\beta$. ■

Our goal is now to construct a set of indiscernibles which satisfies the hypothesis of the lemma.

We assume that $A = \{ \gamma_\delta : \delta < \omega\mu \}$ is a set of indiscernibles and that μ is a cardinal $\leqslant \aleph_1$.

For $\beta < \mu$, let us consider the set of ordinals $u < \gamma_{\omega(\beta+1)}$ such that there exist $\vec{\gamma}_\beta$ and $\varphi \in L_{\omega\omega}$ which satisfy: for any $n > |\vec{\gamma}_\beta|$,

$C \models$ u is the unique x such that $\varphi(x, \vec{\gamma}_\beta, \langle \sigma_{\delta > n} : \delta \in \Delta \rangle)$, where $\Delta \subset \mu$ and $|\Delta| \leqslant \aleph_0$. Let us notice that $\gamma_{\omega\beta}$ is such an element.

For $n > 2|\vec{\gamma}_\beta|$, we obtain :

$C \models \exists!$ y such that $\varphi(y, tr(\vec{\gamma}_\beta), \langle \sigma_{\delta > n} : \delta \in \Delta \rangle)$. Let us call this element $\theta(u)$. Once again, $\theta(u)$ does not depend on the definition of u .

$u_{\omega\beta}$ is thus defined as the least u such that $u < \theta(u)$. There is such an element since $\gamma_{\omega\beta} < \theta(\gamma_{\omega\beta}) = \gamma_{\omega\beta+1}$.

For $k < \omega$, let $u_{\omega\beta+k} = \theta^{k|\vec{\gamma}_\beta|}(u_{\omega\beta})$.

By indiscernibility, $u_{\omega\beta} < \theta(u_{\omega\beta})$ implies $\theta^{p}(u_{\omega\beta}) < \theta^{p+1}(u_{\omega\beta})$ for $p<\omega$ and $u_{\omega\beta+k} < u_{\omega\beta+k+1}$ for $k<\omega$.

Let us check now $u_{\omega\beta} \neq u_{\omega\beta'}$ for $\beta \neq \beta'$. Assume $u_{\omega\beta} = u_{\omega\beta'}$, if $u_{\omega\beta}, u_{\omega\beta'}$ are respectively defined by the formulas $\varphi_{\beta}, \varphi_{\beta'}$ from the tuples $\vec{\bar{\gamma}}_{\beta}, \vec{\bar{\gamma}}_{\beta'}$ and the sets $\Delta_{\beta}, \Delta_{\beta'}$, then we have: for $n > \mathrm{Sup}(2|\vec{\bar{\gamma}}_{\beta}|, |\vec{\bar{\gamma}}_{\beta'}|)$

$C \models$ the unique x such that $\varphi_{\beta}(x, \vec{\bar{\gamma}}_{\beta}, \langle \sigma_{\delta > n} : \delta \in \Delta_{\beta} \rangle)$ = the unique y such that $\varphi_{\beta'}(y, \vec{\bar{\gamma}}_{\beta'}, \langle \sigma_{\delta > n} : \delta \in \Delta_{\beta'} \rangle)$.

This gives by indiscernibility:

$C \models$ the ! x such that $\varphi_{\beta}(x, \mathrm{tr}(\vec{\bar{\gamma}}_{\beta}), \langle \sigma_{\delta > n} : \delta \in \Delta_{\beta} \rangle)$ = the ! y such that $\varphi_{\beta'}(y, \vec{\bar{\gamma}}_{\beta'}, \langle \sigma_{\delta > n} : \delta \in \Delta_{\beta'} \rangle)$.

And we would obtain $\theta(u_{\omega\beta}) = u_{\omega\beta'}$, since we have supposed $u_{\omega\beta} = u_{\omega\beta'}$, this is a contradiction.

Let us assume now $u_{\omega\beta} < u_{\omega\beta'}$. By taking $n > \mathrm{Sup}(k|\vec{\bar{\gamma}}_{\beta}|, |\vec{\bar{\gamma}}_{\beta'}|)$, we would show $u_{\omega\beta+k} < u_{\omega\beta'}$.

Since the function $u : \beta \longmapsto u_{\omega\beta}$ is injective and we have supposed μ was a cardinal, we have obtained a new set $I = \{u_{\omega\beta+n} : \beta<\mu,\ n<\omega\}$.

<u>Claim</u> : I is a set of indiscernibles for C .

Proof: (we are aware that, may be, $u_{\omega\beta} > u_{\omega\beta'}$ for $\beta < \beta'$)

If $\beta<\mu$, let $I_{\beta} = \{u_{\omega\beta+n} : n<\omega\}$. We consider $\langle s_{\beta} : \beta<\mu \rangle$ and $\langle s'_{\beta} : \beta<\mu \rangle$ such that $\langle s_{\beta} : \beta<\mu \rangle =_{f} \langle s'_{\beta} : \beta<\mu \rangle$, s_{β} , s'_{β} strictly increasing in ${}^{\omega}I_{\beta}$ and $s_{\beta} =_{f} s'_{\beta}$ for all $\beta<\mu$.

We assume $C \models \varphi(\langle s_{\beta} : \beta \in \Delta \rangle)$ for $\Delta \subset \mu$ and $|\Delta| \leqslant \aleph_0$.

Let β_0 be the least ordinal β such that $s_{\beta} \neq s'_{\beta}$. If we could show

$C \models \varphi(\langle s_{\beta} : \beta \in \Delta, \beta<\beta_0 \rangle {}^\frown s'_{\beta_0} {}^\frown \langle s_{\beta} : \beta \in \Delta , \beta > \beta_0 \rangle)$,

then after a finite number of similar steps, we would obtain :

$C \models \varphi(\langle s'_\beta : \beta \in \Delta\rangle)$, and we would be done.

Since $s_{\beta_0} =_f s'_{\beta_0}$, there exists $n < \omega$ such that $s_{\beta_0 \geqslant n} = s'_{\beta_0 \geqslant n}$.
If $s_{\beta_0}(n) = u_{\omega\beta_0 + k_n}$, then we set $r_0 = |\vec{\bar{\gamma}}_{\beta_0}| \cdot k_n$.
For each $\beta \in \Delta$, there is a formula φ_β which defines $u_{\omega\beta}$ from $\vec{\bar{\gamma}}_\beta$ and the set Δ_β . Hence each $s_\beta(p)$, for $p < \omega$, is of the form :

$s_\beta(p) = u_{\omega\beta + q_p}$ = the ! x such that $\varphi_\beta(x, tr^{q_p|\vec{\bar{\gamma}}_\beta|}(\vec{\bar{\gamma}}_\beta), \langle \sigma_{\delta > n_{\beta,p}} : \delta \in \Delta_\beta\rangle)$
and we can take $n_{\beta,p} > \mathrm{Sup}(r_0, q_p|\vec{\bar{\gamma}}_\beta|)$.

Let $s_{\beta_0}|n = \langle u_{\omega\beta_0 + k_0}, ----, u_{\omega\beta_0 + k_{n-1}}\rangle$ and
$s'_{\beta_0}|n = \langle u_{\omega\beta_0 + i_0}, ----, u_{\omega\beta_0 + i_{n-1}}\rangle$ for
$k_0 < --- < k_{n-1} < k_n$ and $i_0 < --- < i_{n-1} < k_n$.

For any formula $\varphi \in L_{\omega_1\omega_1}$:

$$C \models \varphi(u_{\omega\beta_0 + k_0}, ----, u_{\omega\beta_0 + k_{n-1}}) \longleftrightarrow \varphi(u_{\omega\beta_0 + i_0}, ----, u_{\omega\beta_0 + i_{n-1}})$$

because $u_{\omega\beta_0 + j}$ and $u_{\omega\beta_0 + (j+1)}$, for $j < \omega$, use disjoint subsets of $\{\gamma_{\omega\beta_0 + s} : s < r_0\}$.

When we add the rest of the expression : the terms $\langle s_\beta : \beta \in \Delta, \beta \neq \beta_0\rangle$ and $s_{\beta_0 \geqslant n}$, we still have the equivalence, because these terms do not involve elements in $\{\gamma_{\omega\beta_0 + s} : s < r_0\}$.

We thus obtain the expected result :

$$C \models \varphi(\langle s_\beta : \beta \in \Delta, \beta < \beta_0\rangle \widehat{\ } s'_{\beta_0|n} \widehat{\ } s_{\beta_0 \geqslant n} \widehat{\ } \langle s_\beta : \beta \in \Delta, \beta > \beta_0\rangle). \quad \blacksquare$$

Let us notice that if μ is a natural integer and if we start from a set A of indiscernibles for $L_{\omega\omega}$, then we obtain by this way a new set I of indiscernibles for $L_{\omega\omega}$. By indiscernibility for $L_{\omega\omega}$, we mean that if $s, s' \in {}^{\omega\mu}A$ strictly increasing are such that $s =_f s'$ and $s_\beta =_f s'_\beta$, for $\beta < \mu$, then $C \models \varphi(s) \longleftrightarrow \varphi(s')$, for any formula $\varphi \in L_{\omega\omega}$.

Let $\alpha = \mu$ if $\mu < \aleph_1$ and α be an arbitrary ordinal $< \aleph_1$, otherwise.

The set $B = \{ u_\delta : \delta < \omega\alpha \}$ is also a set of indiscernibles. Let σ'_β, for $\beta < \alpha$, be the sequence $\langle u_{\omega\beta+n} : n < \omega \rangle$.
As previously, we can consider the filters F_β on $\lambda_\beta = \operatorname{Sup}\{u_{\omega\beta+n} : n<\omega\}$ for $\beta<\alpha$, and the sequence $\vec{F}$.
If $\beta<\alpha$, H_β is the Skolem Hull in $L[\vec{F}]$ of $\{u_{\omega\beta+n} : n<\omega\} \cup \{\vec{F}\}$.
By translating the elements $u_{\omega\beta+n}$, $n<\omega$, we had defined in the beginning of the proof, the translated $tr(x)$ of an element x in H_β.

<u>Lemma 5.3</u> : If we start from the set of indiscernibles B, then, for $\beta<\alpha$ and x in H_β such that $x < u_{\omega\beta}$, $tr(x) = x$.

Proof: Let $x = t(\vec{u}_\beta, \vec{F})$ and $x < u_{\omega\beta}$ where $\vec{u}_\beta = \langle u_{\omega\beta}, ----, u_{\omega\beta+p} \rangle$.
There exists a formula $\psi \in L_{\omega\omega}$ such that : for any $n > |\vec{u}_\beta|$,

$$C \models x \text{ is the ! } y \text{ such that } \psi(y, \vec{u}_\beta, \langle \sigma'_{\delta > n} : \delta < \alpha \rangle)$$

Hence for $n > 2|\vec{u}_\beta|$,

$$C \models tr(x) \text{ is the ! } y \text{ such that } \psi(y, tr(\vec{u}_\beta), \langle \sigma'_{\delta > n} : \delta < \alpha \rangle).$$

For $\eta<\alpha$, there exists a definable function f_η such that :

$$u_{\omega\eta} = f_\eta(\vec{\gamma}_\eta, \langle \sigma_{\delta > n} : \delta \in \Delta_\eta \rangle), \text{ for any } n > |\vec{\gamma}_\eta|.$$

Hence $u_{\omega\eta+k} = f_\eta(tr^{k|\vec{\gamma}_\eta|}(\vec{\gamma}_\eta), \langle \sigma_{\delta > n} : \delta \in \Delta_\eta \rangle)$, for $n > (k+1)|\vec{\gamma}_\eta|$.
If we replace the elements u_ρ, $\rho < \omega\alpha$, by their expression in $\{\gamma_\delta : \delta < \omega\mu\}$
We obtain a formula φ, in $L_{\omega_1\omega_1}$ if $\mu \geq \aleph_0$, in $L_{\omega\omega}$ if $\mu < \aleph_0$, such that

$$C \models x \text{ is the ! } y \text{ such that } \varphi(y, \vec{\gamma}_x, \langle \sigma_{\delta > n} : \delta \in \bigcup_{\eta<\alpha} \Delta_\eta \rangle)$$

where $n > |\vec{\gamma}_\beta| \cdot |\vec{u}_\beta|$ and $\vec{\gamma}_x = \langle \gamma_{\omega\beta+q} : q < |\vec{\gamma}_\beta| \cdot |\vec{u}_\beta| \rangle$.
Therefore x is one of the elements u we considered for the definition of the set I. Since $x < u_{\omega\beta}$, by minimality of $u_{\omega\beta}$, we must have $\theta(x) = x$.
θ is defined by translation of the elements $\gamma_{\omega\beta+n}$, $n<\omega$, whereas tr

is obtained from the elements $\{u_{\omega\beta+n} : n<\omega\}$. But by definition of the elements $u_{\omega\beta+n}$, $n<\omega$, $tr(x) = \theta^{|\vec{\gamma}_\beta|}(x)$, hence we obtain $tr(x) = x$. ■

To conclude the proof of the theorem, it suffices to apply Lemmas 5.2 and 5.3 . ■

As a partial consequence, we obtain :

Proposition 5.1 : The following assertions are equiconsistent:

(a) there exists a measurable cardinal

(b) there is a set of indiscernibles for C and $L_{\omega\omega}$ of order-type ω

(c) there exists an elementary embedding from C into itself.

Proof: (a)$\dashleftarrow\dashrightarrow$(b) has already been proved .
More precisely, if A is any set of indiscernibles of order-type ω and F the usual filter on $Sup(A)$, then

$$L[F] \models \text{there exists a measurable cardinal}$$

(a)$\longrightarrow$(c) : if U is a κ-ultrafilter, then the restriction $j_U|_C$ is elementary from C into itself.

(c)$\dashrightarrow$(a) : let $j:C\longrightarrow C$ be elementary. If κ is the critical point of j , and $\kappa_n = j^n(\kappa)$, for $n<\omega$, are the iterates of κ , then we define the filter F on $\lambda = Sup\{\kappa_n : n<\omega\}$ by :

$$\text{if } X\subset\lambda \text{ , } X\in F \quad \text{iff} \quad \exists n \ \forall p\geqslant n \ \kappa_p\in X .$$

Thus $j:L[F]\longrightarrow L[F]$ is elementary, and by usual Kunen's arguments (see Proposition 3.1, 2nd case of proof 2), one can show that $F\cap L[F]$ is a λ-ultrafilter in $L[F]$. ■

Let λ be a regular cardinal. We can consider the model C^λ defined from the language $L_{\lambda\lambda}$. With a natural extension of the notion of indiscernibles, it is possible to prove similar results for C^{λ} :

. Let $\alpha \leq \lambda$. If there are α measurable cardinals above α, then there is a set of indiscernibles of order-type $\omega\alpha$ for C^{λ} and $L_{\lambda\lambda}$.

. Let $\mu < \lambda$ be a cardinal. If there is a set of indiscernibles for C^{λ} and $L_{\lambda\lambda}$ of order-type $\omega\mu$, then there is an inner model with μ measurable cardinals above μ.

All the inner models with measurable cardinals were inner models of C, and C has the following property:

Fact : . If there is an inner model of V with a measurable cardinal, then there is also an inner model of C with a measurable cardinal.

. But there is no measurable cardinal in C if C satisfies AC.

Proof: . If $N \models U$ is a κ-ultrafilter, then $L[F] \models \mathrm{Sup}\{j_{\underset{n}{\otimes} U}(\kappa) : n<\omega\}$ is measurable.

where F is the usual filter on $\mathrm{Sup}\{j_{\underset{n}{\otimes} U}(\kappa) : n<\omega\}$.

. Let us assume now $C \models D$ is a κ-ultrafilter. We always suppose that the axiom of choice holds in V. We can construct the ultrapower $\mathrm{Ult}_D(C)$, given an element x in $\mathrm{Ult}_D(C)$, we are able (in V) to find $f \in {}^{\kappa}C \cap C$ such that $x = [f]_D$.

If $\langle [f_n]_D : n<\omega \rangle$ is an ω-sequence, then since $f_n \in C$, $\langle f_n : n<\omega \rangle \in C$. This implies that $\mathrm{Ult}_D(C)$ is well-founded, another consequence is that $\mathrm{Ult}_D(C)$ is closed under ω-sequences. Hence $\mathrm{Ult}_D(C) = C$.

Let us consider the embedding j_D. If we assume κ is the least measurable cardinal in C, then $\mathrm{Ult}_D(C) \models j_D(\kappa)$ is the least measurable cardinal.

But we have $\kappa \leq [d]_D < j_D(\kappa)$, hence we have got a contradiction. ∎

Let us return to the set of indiscernibles $A = \{\gamma_\delta : \delta < \omega\alpha\}$, for $\alpha < \omega_1$. We have seen that under some conditions of minimality, the associated model $L[\vec{F}]$ has α measurable cardinals.

If $\alpha = 1$, then for any set A, $L[F] \models$ there exists a measurable cardinal

If $\alpha = \omega$, let us show the following:

Proposition 5.2 : If $A = \{\gamma_\delta : \delta < \omega\omega\}$ is a set of indiscernibles for C and $L_{\omega_1\omega_1}$ such that $\mathrm{Sup}(A)$ is minimal, then the model $L[\vec{F}]$ has ω measurable cardinals.

Proof: We remind that $\lambda_p = \mathrm{Sup}\{\gamma_{\omega p+n} : n<\omega\}$, $\bar{F}_p$ is the filter on λ_p for all $p<\omega$, and $\vec{F} = \langle \bar{F}_p : p<\omega \rangle$.

We already know from Lemma 5.1 that for each $p<\omega$:

$L[\vec{F}] \models \bar{F}_p$ is a σ-complete ultrafilter on λ_p.

If there exists an infinite increasing sequence $\langle \lambda_{p(n)} : n<\omega \rangle$ such that $L[\vec{F}] \models$ there is a measurable cardinal μ such that $\lambda_{p(n)} < \mu \leq \lambda_{p(n+1)}$ then we are done.

Hence let us assume there is $n_0 < \omega$ such that, for all $p \geq n_0$

(i) : $L[\vec{F}] \models$ there is no measurable cardinal μ such that $\lambda_{n_0} < \mu \leq \lambda_p$.

Let us show that in this case, we can construct a set of indiscernibles $A' = \{\eta_\delta : \delta < \omega\omega\}$ such that $\mathrm{Sup}(A') \leq \lambda_{n_0}$, contradicting the minimality of $\mathrm{Sup}(A)$. We will need the following:

Lemma 5.4 (in [Je] for $\nu = \omega_1$) : Let ν be a regular cardinal $\geq \omega$. If there is a ν-complete free ultrafilter on κ, then there is a measurable cardinal μ such that $\nu \leq \mu \leq \kappa$.

Proof: Let μ be the least cardinal over which there is a ν-complete free ultrafilter U. If U is not μ-complete, then there exists a partition $\{X_\alpha : \alpha < \gamma\}$ for $\gamma < \mu$ such that $X_\alpha \notin U$, for all $\alpha < \gamma$.
Let $f : \mu \longrightarrow \gamma$ be defined as follows: $f(x) = \alpha$ iff $x \in X_\alpha$.
We thus consider $D = f^{*}U$: if $X \subset \gamma$, $X \in D$ iff $f^{-1}(X) \in U$.
One can show that D is ν-complete on $P(\gamma)$. Let $\alpha < \gamma$. Since $X_\alpha \notin U$ we must have $f^{-1}\{\alpha\} \notin U$ and $\{\alpha\} \notin D$. Hence D is a ν-complete free ultrafilter on γ. Since μ was chosen minimal, we have obtained a contradiction. Therefore μ must be measurable. ■

Lemma 5.4 and (i) imply that, for any $p > n_0$,
$L[\vec{F}] \models \vec{F}_p$ is not $|\lambda_{n_0}|^{+}$-complete.
We consider $H_p = H^{L[\vec{F}]}(\{\gamma_{\omega p+n} : n < \omega\} \cup \{\vec{F}\})$ and the function tr from H_p into itself.
Let $T = \langle X_\xi : \xi < \lambda_{n_0} \rangle$ in H_p, be a sequence of elements of $\vec{F}_p$. Exactly as previously, we can show that if $\mathrm{tr}(\xi) = \xi$ for all ξ in H_p, below λ_{n_0}, then $\bigcap T \in \vec{F}_p$. Hence there exists $\alpha_p < \lambda_{n_0}$ such that $\mathrm{tr}(\alpha_p) > \alpha_p$ (otherwise we would obtain a strictly decreasing sequence). If $\alpha_p = t(\vec{\gamma}_p, \vec{F})$, we set $\eta_{\omega p} = \alpha_p$ and $\eta_{\omega p+k+1} = \mathrm{tr}^{|\vec{\gamma}_p|}(\eta_{\omega p+k})$ for $k < \omega$.

By indiscernibility, one can prove $\eta_{\omega p+k} < \lambda_{n_0}$ for all $k < \omega$.
If we define such elements $\eta_{\omega p+k}$ for all $p < \omega$, we will obtain a new set $I = \{\eta_{\omega p+k} : p < \omega, k < \omega\}$. Same arguments as before would show that I is a set of indiscernibles the order-type of which is at least ω and such that $\mathrm{Sup}(I) \leq \lambda_{n_0}$.
The proof of the proposition is thus complete. ■

The following definition is motivated by the construction of the set of indiscernibles I in the proof of Theorem 5.1.

Definition 5.3 : Let $A = \{\gamma_\delta : \delta < \omega\alpha\}$, for $\alpha < \omega_1$ be a set of indiscernibles for C and let $\vec{F}$ be the usual sequence of filters defined from A . If H_β for $\beta < \alpha$, is the Skolem Hull $H^{L[\vec{F}]}(\{\gamma_{\omega\beta+n} : n < \omega\} \cup \{\vec{F}\})$ then for x in H_β , $tr_\beta(x)$ is the "translated" of x.

A is minimal if for all $\beta < \alpha$, $x \in H_\beta$ and $x < \gamma_{\omega\beta}$ imply $tr_\beta(x) = x$.

If $A = \{\gamma_\delta : \delta < \omega_1\}$ is a set of indiscernibles, then A is minimal if for all $\alpha < \omega_1$, $A_\alpha = \{\gamma_\delta : \delta < \omega\alpha\}$ is minimal for the previous definition.

The indiscernibles constructed in Proposition 5.1 were iterates of measurable cardinals, conversely we are interested now in large cardinal properties of the indiscernibles.

Proposition 5.3 : If A is a minimal set of indiscernibles of order-type $\omega\alpha$ for $\alpha < \omega_1$, then all the elements of A are Mahlo and limits of Mahlo cardinals in $L[\vec{F}]$.

Proof: If $A = \{\gamma_\delta : \delta < \omega\alpha\}$, we want to show first that the set of regular cardinals below $\gamma_{\omega\beta}$, for $\beta < \alpha$, is stationary.

Let $A_\beta = \{\gamma_{\omega\beta+n} : n < \omega\}$ and $H_\beta = H^{L[\vec{F}]}(A_\beta \cup \{\vec{F}\})$, for $\beta < \alpha$.

tr is an elementary embedding from H_β into itself. This can be shown by induction on the complexity of the formulas.

Since A is minimal, the critical point of tr is $\gamma_{\omega\beta}$, therefore $H_\beta \models \gamma_{\omega\beta}$ is regular. Since $tr : H_\beta \longrightarrow H_\beta$, we can now use the classic proof: let X be a closed unbounded subset of $\gamma_{\omega\beta}$ in H_β . tr(X) is also a closed unbounded set in H_β and $tr(X) \cap \gamma_{\omega\beta} = X$. Therefore $\gamma_{\omega\beta} \in tr(X)$. (it is the same as to argue in the transitive collapse where closed unbounded subset has its real meaning).

Let $Y = \{\mu < \gamma_{\omega\beta} : H_\beta \models \mu \text{ is regular}\}$, $\gamma_{\omega\beta} \in tr(X \cap Y)$, therefore Y is stationary. Since $H_\beta < L[\vec{F}]$, we are done.

Let us show now that, for any formula $\varphi \in L_{\omega\omega}$, $\beta < \alpha$ and $n < \omega$, we have $L[\vec{F}] \models \forall \delta < \gamma_{\omega\beta} (\varphi(\delta, \gamma_{\omega\beta}) \longleftrightarrow \varphi(\delta, \gamma_{\omega\beta+n}))$.
It suffices to prove that it is true in H_β.
$tr^n : H_\beta \longrightarrow H_\beta$ is elementary, since the critical point of tr^n is $\gamma_{\omega\beta}$ and $tr^n(\gamma_{\omega\beta}) = \gamma_{\omega\beta+n}$, the result follows.

Let us assume now that $\gamma_{\omega\beta}$ is not inaccessible in $L[\vec{F}]$: there exists $\mu < \gamma_{\omega\beta}$ such that $L[\vec{F}] \models 2^\mu \geqslant \gamma_{\omega\beta}$. If we apply the preceding result to the formula $\varphi(x,y) : 2^{|x|} \geqslant |y|$, we obtain for all $n < \omega$,

$$L[\vec{F}] \models 2^\mu \geqslant \gamma_{\omega\beta+n} .$$

Since $\lambda_\beta = \operatorname{Sup}\{\gamma_{\omega\beta+n} : n < \omega\}$, $L[\vec{F}] \models 2^\mu \geqslant \lambda_\beta$.
A is minimal, therefore λ_β is measurable in $L[\vec{F}]$ and the last assertion is contradictory.

We want to prove moreover that, for $\beta < \alpha$, $\gamma_{\omega\beta}$ is a limit of Mahlo cardinals in $L[\vec{F}]$. Instead of using the embedding tr for this purpose let us notice that for $\delta < \gamma_{\omega\beta}$, if $L[\vec{F}] \models$ there are no Mahlo cardinals between δ and $\gamma_{\omega\beta}$ then $L[\vec{F}] \models$ there are no Mahlo cardinals between δ and $\gamma_{\omega\beta+1}$.
Since $\gamma_{\omega\beta}$ is Mahlo, the first assertion cannot hold.

Finally, by indiscernibility, we conclude that, if every $\gamma_{\omega\beta}$, for $\beta < \alpha$ is Mahlo and limit of Mahlo cardinals, then all the elements of A have the same properties. ■

Concerning measurability, we have the following:

<u>Proposition 5.4</u> : If A is a minimal set of indiscernibles, then there

is a transitive model of ZFC in which γ_0 is measurable.

Proof: Let F in $L[F]$ be the usual filter on $\lambda = \mathrm{Sup}(A)$. Since A is minimal, $L[F] \models \lambda$ is measurable.

If H is the Skolem Hull $H^{L[F]}(A \cup \{F\})$, then we consider the elementary embedding $tr : H \longrightarrow H$ which sends $t(\vec{\gamma},F)$ onto $t(tr(\vec{\gamma}),F)$. The critical point of tr is γ_0.

Let $N \overset{\pi}{\simeq} H$ be the transitive collapse of H and $j : N \longrightarrow N$ be $\pi \circ tr \circ \pi^{-1}$. Thus we have a N-ultrafilter on $\pi(\gamma_0)$ and $\pi(\lambda)$ is measurable in N, but to obtain the measurability of $\pi(\gamma_0)$, we cannot apply directly Kunen's Theorem ([Ku.1] Thm 6.9) because ${}^{\omega}N \not\subset N$ and nothing proves that the iterated ultrapowers, modulo the N-ultrafilter, are well-founded.

Thus we will modify Kunen's arguments.

Since $H < L[F]$, N has the form $L_\beta[\pi''H \cap F]$ (cf. [Dev]). Since $F \in H$, $N = L[\pi(F)]$, j does not move $\pi(F)$. Hence the well ordering of N is not changed by j.

Let us then consider $K = H^N(\pi(\gamma_0) \cup \{\pi(\lambda)\})$. We want to prove that there are no ordinals η in K, such that $\pi(\gamma_0) \leq \eta < \pi(\lambda)$.

If $\eta \in K$ and $\eta = t^N(\vec{\varepsilon},\pi(\lambda))$ for $\vec{\varepsilon} \subset \pi(\gamma_0)$, then since the well ordering of N is preserved, $j(\eta) = t^N(j(\vec{\varepsilon}), j(\pi(\lambda)))$.

$tr(\lambda) = \lambda$, hence $j(\pi(\lambda)) = \pi(\lambda)$. Since $j(\vec{\varepsilon}) = \vec{\varepsilon}$, we obtain $j(\eta) = \eta$.

If η were in the interval $[\pi(\gamma_0), \pi(\lambda)[$, there would exist $n < \omega$ such that $\pi(\gamma_n) \leq \eta < \pi(\gamma_{n+1})$ and hence $\pi(\gamma_{n+1}) \leq j(\eta)$, which is contradictory.

Let us finally consider the elementary embedding $\pi^{-1} : N \longrightarrow L[F]$.

$\pi^{-1}(K) = H^{L[F]}(\gamma_0 \cup \{\lambda\})$ and since $K \models$ there are no ordinals η such

that $\pi(\gamma_0)\leq\eta<\pi(\lambda)$, we obtain :
$H^{L[F]}(\gamma_0\cup\{\lambda\})\models$ there are no ordinals α such that $\gamma_0\leq\alpha<\lambda$.
Therefore the transitive collapse of $H^{L[F]}(\gamma_0\cup\{\lambda\})$ satisfies :

" γ_0 is measurable " . ■

Concerning several measurable cardinals, we could obtain only partial results, for example:

- If $A=\{\gamma_\delta : \delta<\omega 2\}$ is a minimal set of indiscernibles, then there is a transitive model of ZFC in which λ_0 and γ_ω are measurable, or a transitive model in which γ_ω and λ_1 are measurable.
- For a countable number of measurable cardinals: if $A=\{\gamma_\delta : \delta<\omega\omega\}$ is a minimal set of indiscernibles, there is an inner model in which, all $\gamma_{\omega p}$, for $p\geq 1$, are measurable.

By Proposition 5.3, the elements $\gamma_{\omega p}$, $p<\omega$, are inaccessible in $L[\vec{F}]$. hence it suffices to transmit the measurability of λ_p , for $p<\omega$, to $\gamma_{\omega(p+1)}$ by using an iterated ultrapower.

The following result is a straightforward consequence of Kunen's proof concerning the negation of the axiom of choice in C [Ku.3].

Proposition 5.5 : If there exist ω_1 measurable cardinals, then there is a set of ordinals of cardinality $\aleph_1$ which cannot be covered by any set in C of cardinality $\aleph_1$.

Proof: Let $A=\{\kappa_\alpha : \alpha<\omega_1\}$ be the set of measurable cardinals. We shall show that A is the expected set. Let us assume that there exists B in C such that $A\subset B$ and $|B|=\aleph_1$.
We then give without proof the following result (cf.[Ku.3]):

Lemma 5.5 (Kunen) : For each $x \in C$, there are at most countably many measurable cardinals κ such that for a κ-ultrafilter U ; $j_U(x)$ is different from x .

(this result is a consequence of the fact that only a finite number of measurable cardinals can move a given ordinal)

Hence there is $\alpha < \omega_1$ such that, for any $\beta > \alpha$ and any κ_β-ultrafilter U_β , $j_{U_\beta}(B) = B$. Since $|B| = \aleph_1$, $j_{U_\beta}(B)$ is simply $j_{U_\beta}{}''B$ and we have $j_{U_\beta}{}''B = B$. κ_β belongs to B , hence there is x such that $j_{U_\beta}(x) = \kappa_\beta$. But since κ_β is the critical point of j_{U_β}, this yields a contradiction (in fact, if κ_0 is the least measurable cardinal, A cannot be covered by any set of cardinality $\leqslant \kappa_0$). ■

Let us assume now that $V = L[U]$ where U is a κ-ultrafilter. If s is the sequence $\langle j_{\otimes_n U}(\kappa) : n < \omega \rangle$, then since ${}^{\omega}C \subset C$, $L[s]$ is included in C .
If $U^{(\omega)} = i^U_{0\omega}(U)$, we know that $U^{(\omega)}$ is expressible from the sequence s . Therefore the ω^{th} iterated ultrapower $N_\omega = L[U^{(\omega)}]$ is included in $L[s]$.
Dehornoy [De] has shown that s is Prikry generic over N_ω and that $N_\omega[s] = \bigcap_{n<\omega} N_n$ where $N_n = \text{Ult}_{\otimes_n} U$, for $n < \omega$.
Hence $\bigcap_{n<\omega} N_n \subset L[s] \subset C$. Since ${}^{\kappa}N_n \subset N_n$, for all $n < \omega$, we obtain $C = \bigcap_{n<\omega} N_n = L[s]$.
(it is also possible to check by hand that $N_\omega[s]$ is closed under κ-sequences)

If $\langle \kappa_\alpha : \alpha < \beta \rangle$, for $\beta < \omega_1$ is a countable sequence of measurable cardinals and $\langle U_\alpha : \alpha < \beta \rangle$ the associated ultrafilters, then we can obtain similar results for the model $L[\langle U_\alpha : \alpha<\beta\rangle]$: C is always closed under ω_1-sequences.

We say that an inner model M satisfies the $\aleph_1$-covering property if any set of ordinals of cardinality $\aleph_1$ in the real world, can be covered by a set in M of cardinality $\aleph_1$.

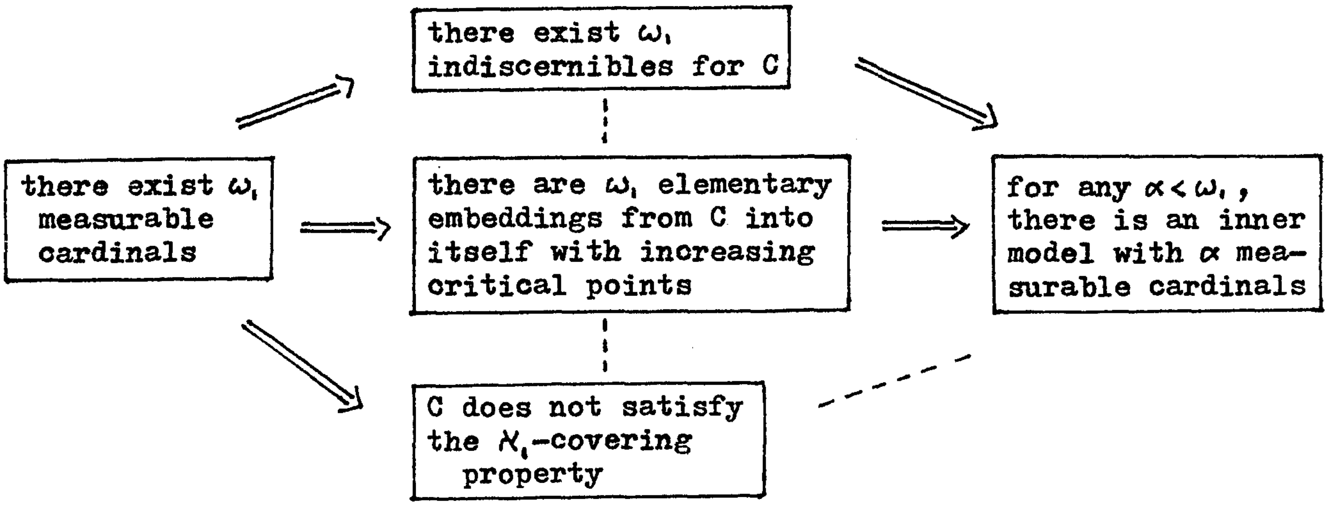
Hence a natural question we plan to study, concerns the possible existence of a covering Theorem for C . A major obstacle to apply known methods as exposed in [Ho], in the case of L , is the absence of a canonical well-ordering of C .

Another convincing element is the following:

Proposition 5.6 : If there exist ω_1 elementary embeddings $i_\alpha : C \longrightarrow C$ with critical point κ_α, for $\alpha<\omega_1$ such that the sequence $\langle \kappa_\alpha : \alpha<\omega_1\rangle$ is strictly increasing, then for any $\beta<\omega_1$, there is an inner model with β measurable cardinals.

Proof: We use the same arguments as in the second proof of Theorem 2.1 . ■

We can summarize the present situation in the following diagram :



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