RENORMALIZATION OF QCD UNDER LONGITUDINAL RESCALING

by

JING XIAO

A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, the City University of New York.

2009

This manuscript has been read and accepted for the Graduate Faculty in Physics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

Date	Chair of Examining Committee Peter Orland
Date	Executive Officer Steven Greenbaum

Jamal Jalilian-Marian

Peter Orland

Robert Oswald-Pisarski

Alexios Polychronakos

Supervisory Committee

CITY UNIVERSITY OF NEW YORK

ABSTRACT

RENORMALIZATION OF QCD UNDER LONGITUDINAL RESCALING

by

JING XIAO

Under a longitudinal rescaling of coordinates $x^{0,3} \to \lambda x^{0,3}, \lambda \ll 1$, the classical QCD action simplifies dramatically. This is the high-energy limit, as $\lambda \sim s^{-1/2}$ where s is the center-of-mass energy squared of a hadronic collision. We find the quantum corrections to the rescaled action at one loop, in particular finding the anomalous powers of λ in this action, with $\lambda < 1$. The method is an integration over high-momentum components of the gauge field. This is a Wilsonian renormalization procedure, and counterterms are needed to make the sharp-momentum cut-off gauge invariant. Our result for the quantum action is found, assuming $\mid \ln \lambda \mid \ll 1$, which is essential for the validity of perturbation theory. If λ is sufficiently small (so that $\mid \ln \lambda \mid \gg 1$), then the perturbative renormalization group breaks down. This is due to uncontrollable fluctuations of the longitudinal chromomagnetic field.

ACKNOWLEDGMENTS

I would like to express my deep and sincere gratitude to my advisor, Prof. Peter Orland. His understanding, encouragement and personal guidance have provided the basis for the present work. I am also grateful to Profs. Adrian Dumitru and Jamal Jalilian-Marian and Dr. Robert Pisarski, with whom Prof. Orland and I discussed this work, as it developed.

I would also like to thank Profs. Sultan Catto and Ramzi Khuri for their assistance and guidance putting my graduate career on the right track in the City University of New York. I would also like to thank all the faculty and staff members in the Department of Natural Sciences of Baruch College, for support and help, while I was working as a teaching adjunct there.

Finally, I would like to give my special thanks to my wife Liping, for her patience, encouragement and love.

Contents

Chapter 1

Introduction

Finding a consistent and complete theory behind the strong interaction was a monumental task. The simple ideas proposed by Heisenberg and Yukawa to describe the nucleon were known to be inadequate by the 1950's. By then it was clear that there are an unlimited number of hadrons and their scattering amplitudes have a complicated phenomenology. By the 1970's it was generally agreed that the theory of quantum chromodynamics (QCD) was the only sensible candidate to describe the data. Unfortunately, what can be calculated in QCD is limited in certain respects. Perturbation theory has only been successful for large transverse-momentum scattering. The theory is expected to describe nature at large distances and small transverse momenta. There are scenarios to connect the theory to experiment in these regions, but no straightforward analytic methods. Numerical lattice methods appear to account for the low-energy features of hadrons. An important kinematic regime is at very high energies and small transverse momenta, in collisions of hadrons and of nuclei. This kinematic regime is of major importance at RHIC, and will be further explored at the LHC.

One approach to extending the range of analytic tools for QCD was proposed in the 70's by Fadin, Kuraev and Lipatov and by Balitski and Lipatov [1]. They suggested how Regge behavior could take place in the large-s, small-x region of the theory, which could be tested experimentally. An effective vertex, describing emission of gluons from charges (either quarks or gluons), leads rather naturally to Reggeization of color-singlet amplitudes, *i.e.* Pomeron behavior. This vertex is usually called the Lipatov vertex, and the approach to high energy QCD is called the BFKL theory.

Another approach is a QCD-inspired picture of nuclear scattering, called the color-glass condensate [2], [3]. This picture consists of an effective action, consisting of a Yang-Mills action with background color sources, to which the eikonal approximation is be applied. A similar action, without the sources, was proposed by Verlinde and Verlinde [4], who derived it from a simple rescaling of longitudinal coordinates. Verlinde and Verlinde derived the Lipatov vertex from this effective theory and discussed an alternative approach to Reggeization. These developments show a close connection between the color-glass-condensate picture and the BFKL approach.

The rescaling done by Verlinde and Verlinde was classical. In this thesis, we will discuss longitudinal rescaling in quantized gauge theories, based on joint work with P. Orland [5]. We find that there are anomalous dimensions which appear in the rescaled action. In particular, we find that some of the couplings become strong at high energies.

Generally the color-glass condensate is thought of as a weakly-coupled theory, by many people working in the field. It is a theory for which transverse forces are strong and longitudinal forces are weak. We point out that, as a quantum theory, the color-glass condensate is actually strongly-coupled. The motivation for the color-glass condensate is that at high velocities, the electric and magnetic flux of a charge is squeezed toward the plane perpendicular (transverse) to the motion. At ultra-relativistic velocities, this flux is called a Weizsäcker-Williams shock wave (effective actions based on this idea can be found in References [6], [7]). In the color-glass action the longitudinal-magnetic-field-squared term is ignored. This is also true in the Verlindes' approach. By doing this, however, quantum fluctuations of the longitudinal magnetic field become very large. In this sense, such theories are strongly coupled, as was first stated explicitly in Reference [8].

Chapter 2

Classical Longitudinal Rescaling

The gluon field of QCD is an SU(3)-Lie-algebra-valued Yang-Mills field. In this thesis, we will often just consider the gluon field A_μ, $\mu = 0, 1, 2, 3$, to be SU(N)-Lie-algebra-valued, for some integer N greater than or equal to 2.

The Yang-Mills action is (we use the Einstein summation convention and sum over repeated raised and lowered indices)

$$S_{\rm YM} = -\frac{1}{4}\int d^4x \; {\rm Tr} F_{\mu\nu}F^{\mu\nu} \, , \tag{2.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu - {\rm i}g[A_\mu, A_\nu] \, , \tag{2.2}$$

$\partial_\mu = \partial/\partial x^\mu$ and $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$, where $\eta^{\mu\nu}$ is the Lorentz metric tensor, with signature $(+,-,-,-)$. This action is invariant under a gauge transformation $G(x) \in \mathrm{SU}(N)$, under which fields transform as

$$A_\mu(x) \to G(x)A_\mu G(x)^{-1} + \frac{\mathrm{i}}{g}G(x)\partial_\mu G(x)^{-1} \,. \tag{2.3}$$

We will chose a set of generators of $\mathrm{SU}(N)$, t_a, $a = 1,\ldots,N^2-1$, normalized according to $\mathrm{Tr} t_a t_b = \delta_{ab}$, and define structure coefficients by $[t_a, t_b] = \mathrm{i}\sum_c f^c_{ab} t_c$.

Imagine a hadron-hadron collision at very high center of mass energy $\sqrt{s}$, along the direction x^3. We define the longitudinal coordinates to be $x^L = (x^0, x^3)$ and the transverse coordinates to be $x^\perp = (x^1, x^2)$. Verlinde and Verlinde [4] considered the longitudinal rescaling $x^L \to \lambda x^L$, $x^\perp \to x^\perp$. The motivation for this rescaling is that momenta will also be rescaled, according to $p_L \to \lambda^{-1} p_L$, $p_\perp \to p_\perp$. Hence $s \to \lambda^{-2} s$. As we take λ to zero, the center-of-mass energy goes to infinity.

It is convenient to use light-cone coordinates, $x^\pm = (x^0 \pm x^3)/\sqrt{2}$. In such coordinates, the longitudinal derivatives and gauge field components are $\partial_\pm = (\partial_0 \pm \partial_3)/\sqrt{2}$ and $A_\pm = (A_0 \pm A_3)/\sqrt{2}$, respectively. We now write $x^L = (x^+, x^-)$. The metric tensor is given by $\eta_{+-} = \eta_{-+} = 1$, $\eta_{ii} = -1$, for $i = 1, 2$, with all other components zero.

Under a longitudinal rescaling, the longitudinal components of the gauge field are also rescaled, $A_\pm \to \lambda^{-1} A_\pm$. The Yang-Mills action becomes

$$S_{\rm YM} = \frac{1}{2}\int d^4x\, {\rm Tr}\left(\sum_{i=1}^{2} F_{0i}^2 - F_{\perp 3}^2 + \lambda^{-2} F_{03}^2 - \lambda^2 F_{12}^2\right)$$
$$= \frac{1}{2}\int d^4x\, {\rm Tr}\left(\sum_{\pm, i=1}^{2} F_{\pm i}^2 + \lambda^{-2} F_{+-}^2 - \lambda^2 F_{12}^2\right)\,, \qquad (2.4)$$

or

$$S_{\rm YM} = \int d^4x\, {\rm Tr}\left[\frac{1}{2}(E^{+-}F_{+-} + \sum_{\pm}\sum_{i=1}^{2} F_{\pm i}^{\;2}) + \frac{\lambda^2}{2}(E^{+-}E_{+-} - F_{12}^2)\right],\ (2.5)$$

where $E_\pm$ is a Lie-algebra-valued auxilliary field. One of the equations of motion is $E_{+-} = -2\lambda^{-2} F_{+-}$

The extreme high-energy limit is obtained by dropping the second term in (2.4). Physically, this means that the curvature in longitudinal planes F_{+-}, is zero. Following Reference [4], however, we will first consider $\lambda > 0$.

We shall later discuss how the classical rescaling of terms in the actions (2.4) and (2.5) is modified in the quantum theory. There are anomalous powers of λ in all these terms. Calculating these is the main goal of this thesis. In this chapter, however, we will only consider classical rescaling.

In addition to the Yang-Mills field, there are also quark fields $\overline{\psi}$ and ψ in QCD. These 4-component spinor fields appear in color N-plets. The quark action, after rescaling, is

$$S_{\rm Q} = -{\rm i}\int d^4x\, \overline{\psi}\,[\lambda^{-1}\gamma^{\pm}(\partial_\pm - {\rm i}gA_\pm) + \gamma^i(\partial_i - {\rm i}gA_i)]\,\psi\,,$$

where we sum over $i = 1, 2$. If we rescale the spinor fields by $\overline{\psi} \to \lambda^{-1/2}\overline{\psi}$, $\psi \to \lambda^{-1/2}\psi$, this action becomes

$$S_{\mathrm{Q}} = -\mathrm{i}\int d^4x\,\overline{\psi}\,[\gamma^{\pm}(\partial_{\pm} - \mathrm{i}gA_{\pm}) + \lambda\gamma^{i}(\partial_{i} - \mathrm{i}gA_{i})]\,\psi\,, \tag{2.6}$$

and in the classical high-energy limit, the second term can be neglected.

Another motivation for longitudinal rescaling is that transverse transport of glue is suppressed and longitudinal transport is enhanced. This can be most easily seen in the Hamiltonian formalism. If the scale factor λ is small, but not zero, the resulting Hamiltonian has one extremely small coupling and one extremely large coupling. Let us change the normalization of the gauge field by a factor of g_0, to obtain

$$S = \frac{1}{2g_0^2}\int d^4x \mathrm{Tr}\left(\lambda^{-2}F_{03}^2 + \sum_{j=1}^{2} F_{0j}^2 - \sum_{j=1}^{2} F_{j3}^2 - \lambda^2 F_{12}^2\right), \tag{2.7}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}[A_\mu, A_\nu]$. The resulting Hamiltonian in $A_0 = 0$ gauge is therefore

$$H = \int d^3x\left[\frac{g_0^2}{2}\mathcal{E}_\perp^2 + \frac{1}{2g_0^2}\mathcal{B}_\perp^2 + \lambda^2\left(\frac{g_0^2}{2}\mathcal{E}_3^2 + \frac{1}{2g_0^2}\mathcal{B}_3^2\right)\right], \tag{2.8}$$

where the electric and magnetic fields are $\mathcal{E}_i = -\mathrm{i}\delta/\delta A_i$ and $\mathcal{B}_i = \epsilon^{ijk}(\partial_j A_k + A_j \times A_k)$, respectively and $(A_j \times A_k)^a = f^a_{bc}A^b_j A^c_k$. Physical states Ψ must satisfy Gauss's law

$$(\partial_\perp \cdot \mathcal{E}_\perp + \partial_3 \mathcal{E}_3 - \rho)\,\Psi = 0\ , \tag{2.9}$$

where ρ is the quark color-charge density. If the term of order λ^2 is neglected, all the energy is contained in the transverse electric and magnetic fields. Chromo-electromagnetic waves can only move longitudinally. This is most easily seen in an axial gauge $A_3 = 0$, in which case the $\lambda = 0$ Hamiltonian contains no transverse derivatives [8].

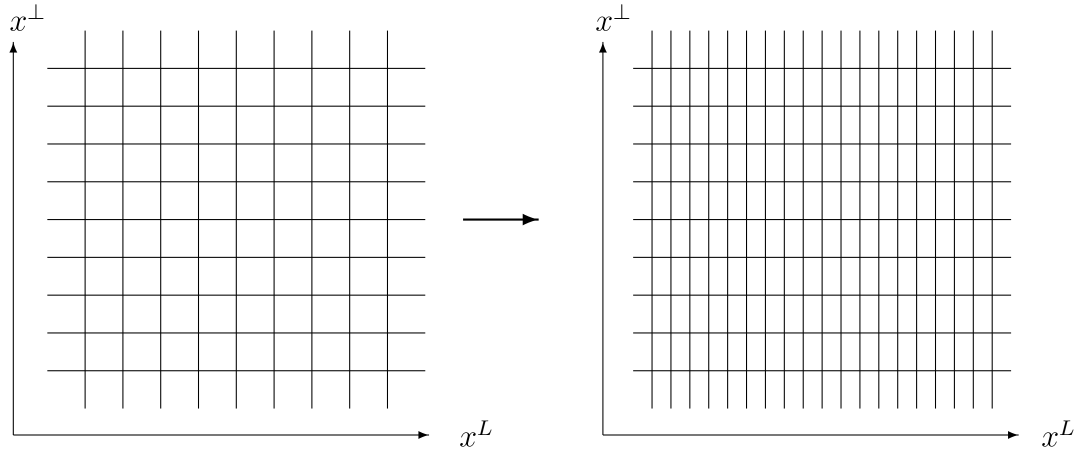
What does not often seem to be stressed in the literature is that (2.7) is a theory with a large coupling - namely the inverse coefficient of the longitudinal magnetic field $F_{12} = \mathcal{B}_3$. This is also apparent in the Hamiltonian formulation (2.8). This field may be classically small, but will have large quantum fluctuations [8].

Chapter 3

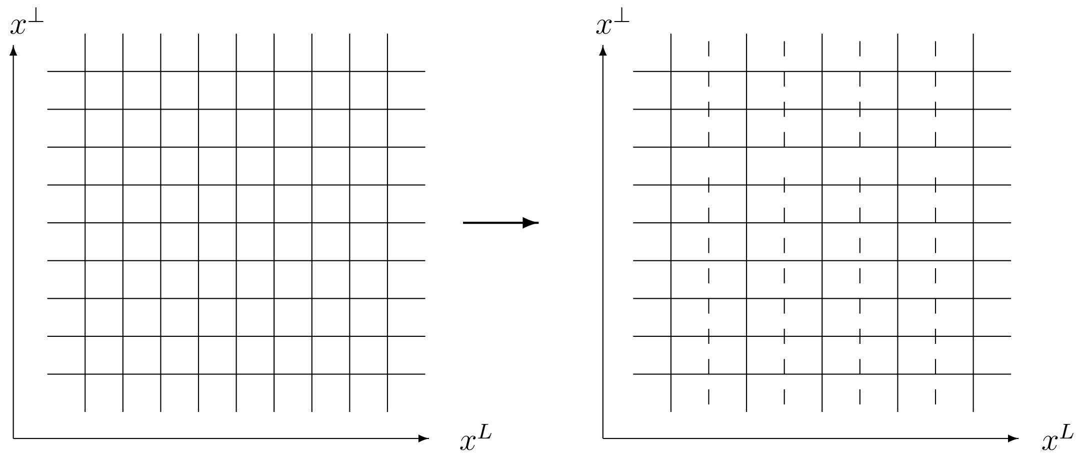
Quantum Longitudinal Rescaling

As we have remarked in the previous chapter, the longitudinal rescaling in Reference [4] is classical. How does such a rescaling change the action of a quantum field theory?

Imagine regularizing QCD on a cubic lattice; the details of the particular lattice cut-off are not important. We want to find a new lattice action whose Green's functions have been longitudinally-rescaled. If we just carry out the rescaling, the lattice spacing a is rescaled to λa in the longitudinal directions. The lattice spacing is not affected in the transverse directions. Thus, the effect of rescaling looks like the following (with $\lambda = 0.5$):



Thus the effect of a simple rescaling changes the ultraviolet cut-off, as well as the action. Clearly, this is not what should be done. The cut-off after rescaling should not be changed. Unless we can modify the procedure to keep the cut-off invariant, the continuum limit of the rescaling procedure will make no sense. Therefore, what we must actually do is a two-step process; we must integrate out some degrees of freedom to restore the isotropic cut-off. The "integrating-out" proceedure can be done either before or after the rescaling; but it must be done. The integrating-out procedure is just a renormalization-group operation, otherwise known as a Kadanoff transformation or block-spin transformation. Our procedure is now a two-step process. First we integrate over some degrees of freedom to increase the size of the lattice spacing in the longitudinal direction to $\lambda^{-1}a$ (as in our previous picture, $\lambda = 0.5$):



The dashed lines on the right indicate where degrees of freedom have been integrated out. Once the block-spin transformation is done, we perform the longitudinal rescaling. Now our lattice has its original dimensions. The action on the blocked, rescaled action is the effective action we seek.

In practice, lattice real-space renormalization is very difficult for gauge theories. It is more straightforward to begin with some other cut-off and renormalize using perturbation theory. This can be done using Wilson's renormalization procedure [9], instead of a Kadanoff transformation. We briefly review this procedure here, providing a more complete discussion in the next two chapters. We start with a momentum cut-off Λ, and restrict our gauge fields to have no Fourier components larger than Λ:

$$A_\mu(x) = \int_{p^2<\Lambda^2} \frac{d^4p}{(2\pi)^4} e^{-\mathrm{i}p\cdot x} A_\mu(p)\,, \tag{3.1}$$

in Euclidean four-dimensional space. In the standard Wilsonian approach, we would introduce a new cut-off $\tilde{\Lambda} < \Lambda$, then split $A_\mu(x)$ into a "fast" field $a_\mu(x)$ and a "slow" field $\tilde{A}_\mu(x)$:

$$A_\mu(x) = \tilde{A}_\mu(x) + a_\mu(x)\,, \tag{3.2}$$

and

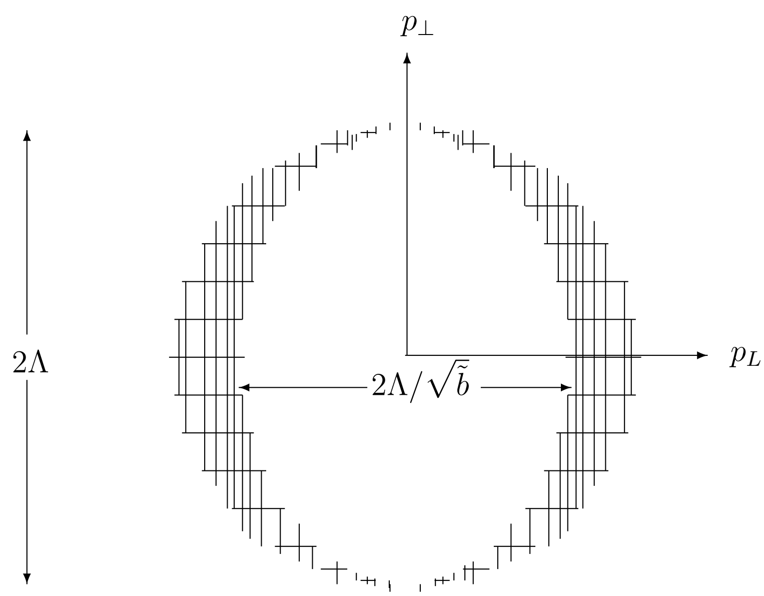
$$\tilde{A}_\mu(x) = \int_{p^2<\tilde{\Lambda}^2} \frac{d^4p}{(2\pi)^4} e^{-\mathrm{i}p\cdot x} A_\mu(p),\; a_\mu(x) = \int_{\tilde{\Lambda}^2<p^2<\Lambda^2} \frac{d^4p}{(2\pi)^4} e^{-\mathrm{i}p\cdot x} A_\mu(p). \tag{3.3}$$

Then the fast field a_μ is integrated out of the functional integral, leaving a new effective theory with a smaller cut-off $\tilde{\Lambda}$. Physical quantities in the effective theory are the same as those of the original theory, provided that they are defined so that no fluctuations with Fourier components with $|p| > \tilde{\Lambda}$ are included.

Sharp-momentum cut-offs violate gauge invariance, unlike lattice or dimensional regularization methods. Counterterms restoring gauge invariance must therefore be included in both the original action (with cut-off Λ) and the effective action (with cut-off $\tilde{\Lambda}$).

For longitudinal renormalization, we should not simply follow the standard Wilsonian procedure. In particular, we do not just want to integrate out the degrees of freedom with Fourier components in a spherical shell between radii Λ and $\tilde{\Lambda}$. Instead we want to integrate from a sphere of radius Λ to an ellipsoid. This ellipsoid has major axes 2Λ, in the transverse (p^1

and p^2) directions, and minor axes $2\Lambda/\sqrt{\tilde{b}}$, in the longitudinal ($p^0$ and p^1) directions, for some number $\tilde{b} > 1$. Thus the longitudinal momenta will be cut-off at a smaller scale than transverse momenta. This is similar to our lattice discussion, in which the longitudinal lattice spacing increases, but the transverse lattice spacing is unaffected, after integrating out some degrees of freedom. We can see that the constant $\tilde{b}$ should be interpreted as $\tilde{b} = \lambda^{-2}$. We integrate over the hatched region in the following picture:



This region is the "onion skin" of Wilson. The outer boundary of the onion skin is the original sphere of radius Λ and the inner boundary is the new ellipsoidal momentum cut-off. After the renormalization-group transformation (which we call a "renormalization", as the term is used in condensed-matter physics), the fields have Fourier components in the interior of the ellipsoid.

After the renormalization, we must carry out a longitudinal scale transformation, $x^L \to \lambda x^L$, $x^\perp \to x^\perp$. This rescales the longitudinal components of momenta $p_L = (p_0, p_3)$, by $p_L \to \lambda^{-1} p_L$, leaving transverse components of momenta $p_\perp = (p_0, p_1)$, unaffected. As a result of this rescaling, the cut-off has been restored to a sphere of radius $\lambda\Lambda/\sqrt{\tilde{b}} = \Lambda$.

The anisotropic renomalization group was discussed long ago in References [10]. These authors were motivated, to some extent by Verlinde and Verlinde's ideas, but did not actually perform the calculation for Yang-Mills theories.

In the next chapter, we will outline the how the renormalization will be carried out. The details of the integrations are provided in Chapter 5 for the spherical case and Chapter 6 for the ellipsoidal case.

Chapter 4

Wilsonian Renormalization

It is interesting to consider integrating over momenta from one ellipsoidal cut-off to another. We choose Λ and $\tilde{\Lambda}$ to be real positive numbers with units of cm^{-1} and b and $\tilde{b}$ to be two dimensionless real numbers, such that $b \geq 1$ and $\tilde{b} \geq 1$. We require furthermore that $\Lambda > \tilde{\Lambda}$ and that $\Lambda^2/b \geq \tilde{\Lambda}^2/\tilde{b}$. We define the region of momentum space $\mathbb{P}$ to be the set of points p, such that $bp_L^2 + p_\perp^2 < \Lambda^2$. We define the region $\tilde{\mathbb{P}}$ to be the set of points p, such that $\tilde{b}p_L^2 + p_\perp^2 < \tilde{\Lambda}^2$. The Wilsonian onion skin $\mathbb{S}$ is $\mathbb{S} = \mathbb{P} - \tilde{\mathbb{P}}$.

The basic cut-off functional integral is

$$Z_\Lambda = \int \left[\prod_{p \in \mathbb{P}} dA(p)\right] \exp -S, \quad S = \int d^4x \frac{1}{4g_0^2} \mathrm{Tr}\ F_{\mu\nu}F^{\mu\nu} + S_{c.t.,\Lambda,b} \qquad (4.1)$$

where $S_{c.t.,\Lambda,b}$ contains counterterms needed to maintain gauge invariance with the sharp-momentum cut-off Λ and anisotropy parameter b. One can view $S_{c.t.,\Lambda,b}$ as simply an ingredient of the regularization scheme; its inclusion is needed to make the cut-off action gauge invariant.

The cut-off is implemented in the measure of integration in (4.1). We can write the Fourier transform of the gauge field as

$$A_\mu(x) = \int_{\mathbb{P}} \frac{d^4p}{(2\pi)^4}\, A_\mu(p)\, e^{-\mathrm{i}p\cdot x}\ .$$

Following Wilson's procedure, we split the field A_μ into slow parts $\tilde{A}_\mu$, and fast parts a_μ, defined by

$$\tilde{A}_\mu(x) = \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4}\, A_\mu(p)\, e^{-\mathrm{i}p\cdot x}\ ,\quad a_\mu(x) = \int_{\mathbb{S}} \frac{d^4p}{(2\pi)^4}\, A_\mu(p)\, e^{-\mathrm{i}p\cdot x}\ ,$$

so that $A_\mu(x) = \tilde{A}_\mu(x) + a_\mu(x)$. We may also write in momentum space: $A_\mu(p) = \tilde{A}_\mu(p) + a_\mu(p)$, by defining

$$\tilde{A}_\mu(p) = \begin{cases} A_\mu(p), & p \in \tilde{\mathbb{P}}, \\ 0, & p \in \mathbb{S} \end{cases}\ ,\quad a_\mu(p) = \begin{cases} 0, & p \in \tilde{\mathbb{P}}, \\ A_\mu(p), & p \in \mathbb{S} \end{cases}\ . \tag{4.2}$$

Our goal in this chapter is to integrate out the fast components a_μ, of the field to obtain

$$\begin{aligned} Z_\Lambda &= e^{-f} Z_{\tilde{\Lambda}}\ ,\quad Z_{\tilde{\Lambda}} = \int \left[\prod_{p\in\tilde{\mathbb{P}}} dA(p)\right] \exp -\tilde{S}, \\ \tilde{S} &= \int d^4x\, \frac{1}{4\tilde{g}_0^2}\ \mathrm{Tr}\ \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + S_{c.t.,\tilde{\Lambda},\tilde{b}}\ , \end{aligned} \tag{4.3}$$

where f is an unimportant ground-state-energy renormalization, $\tilde{g}_0$ is the coupling at the new cut-off $\tilde{\Lambda}$, $\tilde{b}$, $\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu - \mathrm{i}[\tilde{A}_\mu, \tilde{A}_\nu]$, and $S_{c.t.,\tilde{\Lambda},\tilde{b}}$ contains the counterterms needed to restore gauge invariance with the new cut-off. We will find the form of both $S_{c.t.,\Lambda,b}$ and $S_{c.t.,\tilde{\Lambda},\tilde{b}}$.

Before we integrate over the fast gauge field, yielding the new action in (4.3), we need to expand the original action in terms of this field to quadratic order:

$$\begin{aligned} S &= \frac{1}{4g_0^2}\int d^4x \ \mathrm{Tr}\left\{\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - 4[\tilde{D}_\mu, \tilde{F}^{\mu\nu}]a_\nu \right. \\ &+ \left. ([\tilde{D}_\mu, a_\nu] - [\tilde{D}_\nu, a_\mu])([\tilde{D}^\mu, a^\nu] - [\tilde{D}^\nu, a^\mu]) - 2\mathrm{i}\tilde{F}^{\mu\nu}[a_\mu, a_\nu]\right\}, \quad (4.4) \end{aligned}$$

where $\tilde{D}_\mu = \partial_\mu - \mathrm{i}\tilde{A}_\mu$ is the covariant derivative determined by the slow gauge field.

The action is invariant under the gauge transformation of the fast field:

$$\tilde{A}_\mu \to \tilde{A}_\mu \ , \quad a_\mu \to a_\mu + [\tilde{D}_\mu - \mathrm{i}a_\mu, \omega] \ .$$

A variation δa_μ orthogonal to these gauge transformation satisfies $[\tilde{D}_\mu, \delta a_\mu] = 0$. We can add with impunity the term $\frac{1}{2g_0^2}\int d^4x \mathrm{Tr}[\tilde{D}_\mu, a_\mu]^2$ to the action.

There is a linear term in a_μ in the action (4.4). After we integrate out the fast field, the only result of this term will be to induce terms of order $[\tilde{D}_\mu, \tilde{F}^{\mu\nu}]^2$ in $\tilde{S}$. These terms are of dimension greater than four or nonlocal, so we ignore them, as they will be irrelevant. We therefore replace (4.4) with

$$S = \frac{1}{4g_0^2}\int d^4x\,\mathrm{Tr}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{2g_0^2}\int d^4x\left([\tilde{D}_\mu, a_\nu][\tilde{D}^\mu, a^\nu] - 2\mathrm{i}\tilde{F}^{\mu\nu}[a_\mu, a_\nu]\right),$$

In terms of coefficients of the generators t_b, $b = 1, \ldots, N^2 - 1$, this expression may be written as

$$S = \frac{1}{4g_0^2}\int d^4x\ \tilde{F}^b_{\mu\nu}\tilde{F}^{\mu\nu}_b + S_{\mathrm{O}} + S_{\mathrm{I}} + S_{\mathrm{II}}\ ,$$

where

$$S_{\mathrm{O}} = \frac{1}{2g_0^2}\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\ q^2\ a^b_\mu(-q)a^\mu_b(q)\ , \tag{4.5}$$

$$\begin{aligned} S_{\mathrm{I}} \ &= \ \frac{\mathrm{i}}{g_0^2}\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4}q^\mu f_{bcd}a^b_\nu(q)\,\tilde{A}^c_\mu(p)\,a^d_\nu(-q-p) \\ &+ \frac{1}{2g_0^2}\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4}\int_{\tilde{\mathbb{P}}}\frac{d^4l}{(2\pi)^4}f_{bcd}f_{bfg}\,a^d_\nu(q) \\ &\times\ \tilde{A}^c_\mu(p)\tilde{A}^f_\mu(l)\,a^g_\nu(-q-p), \end{aligned} \tag{4.6}$$

and

$$S_{\mathrm{II}} = \frac{1}{2g_0^2}\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4}f_{bcd}\,a^b_\mu(q)\tilde{F}^c(p)a^d_\nu(-p-q)\ . \tag{4.7}$$

The gluon propagator can be read of from the expression for $S_{\rm O}$ in (4.5):

$$\langle a_\mu^b(q) a_\nu^c(p)\rangle = g_0^2 \delta^{bc}\delta_{\mu\nu}\delta^4(q+p)q^{-2}\ . \tag{4.8}$$

We define the brackets $\langle W\rangle$, around any quantity W to be the expectation value of W with respect to the measure $\mathcal{N}\exp -S_{\rm O}$, where $\mathcal{N}$ is chosen so that $\langle 1\rangle = 1$.

One more term must be included in the action. This term depends on the anticommuting ghost fields $G_\mu^b(x)$, $H_\mu^b(x)$, associated with the gauge fixing of $a_\mu^b(x)$. The ghost action is

$$\begin{aligned} S_{\rm ghost} = & \frac{\rm i}{g_0^2}\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4} q^\mu f_{bcd} G^b(q)\, \tilde{A}_\mu^c(p)\, H^d(-q-p) \\ & + \frac{1}{2g_0^2}\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4}\int_{\tilde{\mathbb{P}}}\frac{d^4l}{(2\pi)^4} f_{bcd}f_{bfg}\, G^d(q)\, \tilde{A}_\mu^c(p)\tilde{A}_\mu^f(l)\, H^g(-q-p), \end{aligned}$$

which is similar to $S_{\rm I}$, except that the fast vector gauge field has been replaced by the scalar ghost fields. Integration over the ghost fields eliminates two of the four spin degrees of freedom of the fast gauge field.

To integrate out the fast gauge field and its associated ghost fields, we use the connected-graph expansion for the expectation value of the exponential of minus a quantity R:

$$\begin{aligned} \langle e^{-R}\rangle \ = \ & \exp\Bigg[-\langle R\rangle + \frac{1}{2!}(\langle R^2\rangle - \langle R\rangle^2) \\ & - \ \frac{1}{3!}\left(\langle R^3\rangle - 3\langle R^3\rangle\langle R\rangle + 2\langle R\rangle^3\right) + \cdots\Bigg]\,. \end{aligned} \tag{4.9}$$

When evaluating a functional integral, each of the terms of a given order is can be represented as a sum of connected Feynman diagrams. We now briefly discuss the derivation of this expansion. The expansion for the left-hand side of (4.9) begins

$$\langle e^{-R}\rangle = 1 - \langle R\rangle + O(R^2)\,,$$

so we write

$$\begin{aligned}
\langle e^{-R}\rangle &= e^{-\langle R\rangle}(e^{\langle R\rangle}\langle e^{-R}\rangle) \\
&= e^{-\langle R\rangle}\left[1 + \langle R\rangle + \frac{1}{2!}\langle R\rangle^2 + O(R^3)\right] \\
&\times \left[1 - \langle R\rangle + \frac{1}{2!}\langle R^2\rangle + O(R^3)\right] \\
&= e^{-\langle R\rangle}\left[1 + \frac{1}{2}(\langle R^2\rangle - \langle R\rangle^2) + O(R^3)\right]\,.
\end{aligned}$$

Having found this result, we write

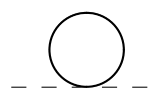
$$\langle e^{-R}\rangle = e^{-\langle R\rangle} e^{\frac{1}{2!}(\langle R^2\rangle - \langle R\rangle^2)}[e^{\langle R\rangle} e^{-\frac{1}{2!}(\langle R^2\rangle - \langle R\rangle^2)}\langle e^{-R}\rangle]\,,$$

and expand the factors in square brackets on the right in powers of R, to find the term of third order. Continuing this procedure yields (4.9).

The connected-graph expansion gives to second order

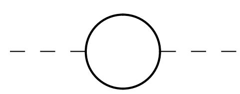
$$\begin{aligned}\exp -\tilde{S} &= \exp\left(-\frac{1}{4g_0^2}\int d^4x\ \tilde{F}^b_{\mu\nu}\tilde{F}^{\mu\nu}_b\right)\left\langle \exp\left(-\frac{1}{2}S_{\rm I} - S_{\rm II}\right)\right\rangle \\ &\approx \exp\left[-\frac{1}{4g_0^2}\int d^4x\ \tilde{F}^b_{\mu\nu}\tilde{F}^{\mu\nu}_b\right]\exp\left[-\frac{1}{2}\langle S_{\rm I}\rangle\right. \\ &+ \left.\frac{1}{4}(\langle S_{\rm I}^2\rangle - \langle S_{\rm I}\rangle^2) + \frac{1}{2}(\langle S_{\rm II}^2\rangle - \langle S_{\rm II}\rangle^2)\right]\ . \qquad (4.10)\end{aligned}$$

We remark briefly on the coefficients in the last exponential in (4.10). We can represent these by Feynman diagram with slow fields as dashed external lines and fast propagators as solid internal lines. The coefficient of $\langle S_{\rm I}\rangle$ has a contribution -1 from a fast gluon loop and $1/2$ from a fast ghost loop. This contribution corresponds to the diagram:



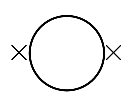
.

The coefficient of $\langle S_{\rm I}^2\rangle - \langle S_{\rm I}\rangle^2$ has a contribution $1/2$ from a fast gluon loop and $-1/4$ from a fast ghost loop. This corresponds to the diagram:



.

The coefficient of $\langle S_{\rm II}^2\rangle - \langle S_{\rm II}\rangle^2$ has no ghost contribution. This has external slow field strengths, represented as crosses in the diagram:



.

Other terms in the exponential, of the same order, vanish upon contraction of group indices.

The terms in the new action (4.10) are given by

$$\frac{1}{2}\langle S_{\rm I}\rangle - \frac{1}{4}(\langle S_{\rm I}^2\rangle - \langle S_{\rm I}\rangle^2) = \frac{C_N}{4}\int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4}\tilde{A}^b_\mu(-p)\tilde{A}^b_\nu(p)P_{\mu\nu}(p) ,$$
$$P_{\mu\nu}(p) = \int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\left[-\frac{q_\mu(p_\nu+2q_\nu)}{4q^2(q+p)^2}+\frac{\delta_{\mu\nu}}{4q^2}\right] , \tag{4.11}$$

where C_N is the Casimir of SU(N), defined by $f^{bcd}f^{hcd} = C_N\delta^{bh}$, and

$$-\frac{1}{2}(\langle S_{\rm II}^2\rangle - \langle S_{\rm II}\rangle^2) = -\frac{C_N}{2}\int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4}\tilde{F}^b_{\mu\nu}(-p)\tilde{F}^b_{\mu\nu}(p) \times \int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\frac{1}{q^2(p+q)^2} . \tag{4.12}$$

Next we will evaluate the integrals in (4.11) and (4.12).

Consider the integral $I(p)$, defined as

$$I(p) = \int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\frac{p_\alpha+2q_\alpha}{q^2(q+p)^2} .$$

Then $I(p)+I(-p) = 0$. We can see this by changing the sign of q in the integration. We can replace the polarization tensor $P_{\mu\nu}(p)$ in (4.11) by the manifestly symmetric form $\Pi_{\mu\nu}(p)$:

$$\frac{1}{2}\langle S_{\rm I}\rangle - \frac{1}{4}(\langle S_{\rm I}^2\rangle - \langle S_{\rm I}\rangle^2) = C_N\int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4}\ \tilde{A}^b_\mu(-p)\tilde{A}^b_\nu(p)\ \Pi_{\mu\nu}(p) ,$$
$$\Pi_{\mu\nu}(p) = \int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\left[-\frac{(p_\mu+2q_\mu)(p_\nu+2q_\nu)}{8q^2(q+p)^2}+\frac{\delta_{\mu\nu}}{4q^2}\right] . \tag{4.13}$$

The polarization tensor is symmetric, but breaks gauge invariance. This is because at this order in the loop expansion, $p_\mu \Pi_{\mu\nu}(p) \neq 0$. The reason for this is clear; gauge symmetry is explicitly broken by sharp-momentum cutoffs. The purpose of the counterterms $S_{c.t.,\Lambda,b}$ and $S_{c.t.,\tilde{\Lambda},\tilde{b}}$ in (4.1) and (4.3), respectively, is to restore this symmetry.

There are other pieces of the renormalized action which are the contributions to the cubic and quartic Yang-Mills vertices, consisting of three and four external lines, respectively. These are completely determined by the Slavnov-Taylor identities of the Yang-Mills theory, so we do not have to calculate them separately.

Chapter 5

Spherical Cut-offs

In this chapter we will carry out Wilson's renormalization for pure Yang-Mills theory from a spherical cut-off of radius Λ to a smaller spherical cut-off of radius $\tilde{\Lambda}$. This calculation is neither novel nor original, though we provide more details in Chapter 4 and this chapter than appear elsewhere, *e.g.* in Polyakov's book [11]. The calculation may be regarded as a warm-up exercise for the anisotropic renormalization group of the next chapter, which is considerably more tedious.

We first evaluate $\Pi_{\mu\nu}(p)$ in (4.13), splitting it into a gauge-invariant part and a non-gauge-invariant part. At $p = 0$,

$$\Pi_{\mu\nu}(0) = \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4} \left[-\frac{q_\mu q_\nu}{2(q^2)^2} + \frac{\delta_{\mu\nu}}{4q^2} \right] \ .$$

If we change the sign of one component only of q, *e.g.* $q_0 \to -q_0$, $q_{\mathrm{i}} \to q_{\mathrm{i}}$, i$=1,2,3$, the first term of the integrand changes sign for $\mu=0$ and $\nu=$i. Thus $\Pi_{\mu\nu}(0)$ vanishes for $\mu\neq\nu$. Hence

$$\Pi_{\mu\nu}(0)=\frac{1}{8}\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\frac{\delta_{\mu\nu}}{q^2}=\frac{1}{128\pi^2}(\Lambda^2-\tilde{\Lambda}^2)\delta_{\mu\nu}\ .$$

If we write $\Pi_{\mu\nu}(p)=\hat{\Pi}_{\mu\nu}(p)+\Pi_{\mu\nu}(0)$, we find

$$\hat{\Pi}_{\mu\nu}(p)=\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\left[-\frac{(p_\mu+2q_\mu)(p_\nu+2q_\nu)}{8q^2(q+p)^2}+\frac{\delta_{\mu\nu}}{8q^2}\right]\ .$$

If we subtract the polarization tensor at zero momentum by a counterterms of identical form at each scale, or in other words

$$S_{c.t.,\Lambda}=-\frac{\Lambda^2}{128\pi^2}\int d^4x\ \ A^2\ ,\ \ S_{c.t.,\tilde{\Lambda}}=-\frac{\tilde{\Lambda}^2}{128\pi^2}\int d^4x\ \ \tilde{A}^2\ , \tag{5.1}$$

the result is gauge invariant, as we show below.

Next we expand the polarization tensor $\hat{\Pi}_{\mu\nu}(p)$ in powers of p. The terms of more than quadratic order in p have canonical dimension greater than four, so they can be ignored in the new action. To this order,

$$\hat{\Pi}_{\mu\nu}(p)=\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\left[\frac{p_\mu p_\nu+\delta_{\mu\nu}p^2}{8(q^2)^2}-\frac{2p_\alpha p_\beta q_\alpha q_\beta q_\mu q_\nu}{(q^2)^4}\right]+\cdots \tag{5.2}$$

The right-hand side of (5.2) is evaluated using Euclidean O(4) symmetry: we emphasize this point, because in the aspherical case, we do not have invariance under O(4), but only under its subgroup O(2) $\times$ O(2). Exploiting this symmetry, we write the nontrivial tensor integral in (5.2) in terms of a scalar integral:

$$\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4} \frac{q_\alpha q_\beta q_\mu q_\nu}{(q^2)^4} = \frac{1}{24} \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4} \frac{1}{q^2} \left(\delta_{\alpha\beta}\delta_{\mu\nu} + \delta_{\alpha\nu}\delta_{\mu\beta} + \delta_{\alpha\mu}\delta_{\beta\nu}\right) \ .$$

Hence the polarization tensor is

$$\hat{\Pi}_{\mu\nu}(p) = \frac{1}{192\pi^2} \ln \frac{\Lambda}{\tilde{\Lambda}} \left(\delta_{\mu\nu} - p_\mu p_\nu\right) + \cdots \ . \tag{5.3}$$

Gauge invariance is satisfied to this order of p, *i.e.* $p^\mu \hat{\Pi}_{\mu\nu}(p) = 0$.

We also need to evaluate (4.12). Once again, the terms of dimension higher than four can be dropped, by expanding the integral over $\mathbb{S}$ in powers of p:

$$\begin{aligned} -\frac{1}{2}(\langle S_{\rm II}^2\rangle - \langle S_{\rm II}\rangle^2) &= -\frac{C_N}{2} \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} \tilde{F}^b_{\mu\nu}(-p)\tilde{F}^b_{\mu\nu}(p) \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2)^2} + \cdots \\ &= -\frac{C_N}{16\pi^2} \ \ln \frac{\Lambda}{\tilde{\Lambda}} \ \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} \tilde{F}^b_{\mu\nu}(-p)\tilde{F}^b_{\mu\nu}(p) + \cdots \ . \end{aligned} \tag{5.4}$$

Combining (4.13), (5.1), (5.3) and (5.4) gives the standard result for the new coupling $\tilde{g}_0$ in (4.3):

$$\frac{1}{\tilde{g}_0^2} = \frac{1}{g_0^2} - \frac{C_N}{4\pi^2} \ \ln \frac{\Lambda}{\tilde{\Lambda}} + \frac{1}{12}\frac{C_N}{4\pi^2} \ \ln \frac{\Lambda}{\tilde{\Lambda}} = \frac{1}{g_0^2} - \frac{11\, C_N}{48\pi^2} \ln \frac{\Lambda}{\tilde{\Lambda}} \ . \tag{5.5}$$

Equation (5.5) is the well-known statement of asymptotic freedom [12]. If we start with a very small coupling, at a very large cut-off, such as some unification scale or the Planck scale, then the effective coupling at low energies becomes large. This is encoded in the beta function:

$$\beta(\tilde{g}_0) = \frac{\partial \tilde{g}_0}{\partial \ln \tilde{\Lambda}} = -\frac{11 C_N}{48\pi^2} \tilde{g}_0^3 \,,$$

or, dropping the tildes,

$$\beta(g_0) = \frac{\partial g_0}{\partial \ln \Lambda} = -\frac{11 C_N}{48\pi^2} g_0^3 \,. \tag{5.6}$$

In the next chapter, we repeat this calculation with ellipsoidal cut-offs. The results of this chapter are recovered, as isotropy is restored.

Chapter 6

Ellipsoidal Cut-offs

Integration over the region $\mathbb{S}$ is much more work with ellipsoidal cut-offs than spherical cut-offs, because we have less symmetry to exploit. We take advantage of the $O(2)\times O(2)$ symmetry by making a change of variables, from q_μ to two angles θ and ϕ, and two variables with dimensions of momentum squared, u and w. The relation between the old and new variables is

$$q_1 = \sqrt{u}\,\cos\theta,\ q_2 = \sqrt{u}\,\sin\theta,\ q_3 = \sqrt{w-u}\,\cos\phi,\ q_0 = \sqrt{w-u}\,\sin\phi \quad (6.1)$$

(note that $u = q_\perp^2$ and $w - u = q_L^2$), which gives

$$\begin{aligned}\int_{\mathbb{S}} d^4q &= \frac{1}{4}\int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \left[\int_0^{\tilde{\Lambda}^2} du \int_{\tilde{b}^{-1}\tilde{\Lambda}^2+(1-\tilde{b}^{-1})u}^{b^{-1}\Lambda^2+(1-b^{-1})u} dw \right. \\ &+ \left. \int_{\tilde{\Lambda}^2}^{\Lambda^2} du \int_u^{b^{-1}\Lambda^2+(1-b^{-1})u} dw \right] . \qquad (6.2)\end{aligned}$$

The O(2) × O(2) symmetry group is generated by translations of the angles $\theta \to \theta + d\theta$ and $\phi \to \phi + d\phi$.

We write the polarization tensor $\Pi_{\mu\nu}(p)$ in (4.13), expanded to second order in p_α as the sum of six terms:

$$\Pi_{\mu\nu}(p) = \Pi^1_{\mu\nu}(p) + \Pi^2_{\mu\nu}(p) + \Pi^3_{\mu\nu}(p) + \Pi^4_{\mu\nu}(p) + \Pi^5_{\mu\nu}(p) + \Pi^6_{\mu\nu}(p) \ ,$$

where

$$\begin{aligned}
\Pi^1_{\mu\nu}(p) &= \frac{\delta_{\mu\nu}}{4}\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\frac{1}{q^2} \ , \ \ \Pi^2_{\mu\nu}(p) \ = \ -\frac{1}{2}\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\frac{q_\mu q_\nu}{(q^2)^2}, \\
\Pi^3_{\mu\nu}(p) &= \frac{p_\mu p_\alpha}{2}\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\frac{q_\nu q_\alpha}{(q^2)^3} + \frac{p_\nu p_\alpha}{2}\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\frac{q_\mu q_\alpha}{(q^2)^3} \ , \\
\Pi^4_{\mu\nu}(p) &= -\frac{p_\mu p_\nu}{8}\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\frac{1}{(q^2)^2} \ , \ \ \Pi^5_{\mu\nu}(p) \ = \ \frac{p^2}{2}\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\frac{q_\mu q_\nu}{(q^2)^3} \ , \\
\Pi^6_{\mu\nu}(p) &= -2p_\alpha p_\beta I^6_{\alpha\beta\mu\nu}(p), \text{ where} \\
I^6_{\alpha\beta\mu\nu}(p) &= \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\frac{q_\alpha q_\beta q_\mu q_\nu}{(q^2)^4} \ .
\end{aligned} \tag{6.3}$$

We will evaluate each of these six terms of the polarization tensor (6.3), by using the integration (6.2) over the variables (6.1). This is very tedious, though straightforward. The details of the integration are given in the appendix to this chapter. Since the integrals are invariant under O(2) × O(2), but not O(4), we introduce some notation. We assume the indices C and D take only the values 1 and 2, and the indices Ω and Ξ take only the values 3 and 0. As is standard, the indices μ, ν, etc., can take any of the four values

1, 2, 3 and 0. Here is a summary of the results:

$$\Pi^1_{\mu\nu}(p) = \frac{\delta_{\mu\nu}}{64\pi^2}\left(\frac{\Lambda^2 \ln b}{b-1} - \frac{\tilde{\Lambda}^2 \ln \tilde{b}}{\tilde{b}-1}\right) , \tag{6.4}$$

$$\begin{aligned}
\Pi^2_{CD}(p) &= -\frac{\Lambda^2\delta_{CD}}{64\pi^2}\left[1 + \frac{b}{(b-1)^2}(1-b+\ln b)\right] \\
&+ \frac{\tilde{\Lambda}^2\delta_{CD}}{64\pi^2}\left[1 + \frac{\tilde{b}}{(\tilde{b}-1)^2}(1-\tilde{b}+\ln \tilde{b})\right] , \\
\Pi^2_{\Omega\Xi}(p) &= -\frac{\Lambda^2\delta_{\Omega\Xi}}{64\pi^2}\left[\frac{1}{b-1} - \frac{\ln b}{(b-1)^2}\right] + \frac{\tilde{\Lambda}^2\delta_{\Omega\Xi}}{64\pi^2}\left[\frac{1}{\tilde{b}-1} - \frac{\ln \tilde{b}}{(\tilde{b}-1)^2}\right] , \\
\Pi^2_{C\Omega}(p) &= \Pi^2_{\Omega C}(p) = 0 ,
\end{aligned} \tag{6.5}$$

$$\begin{aligned}
\Pi^3_{CD}(p) &= \frac{p_C p_D}{32\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{p_C p_D}{64\pi^2}\left[\frac{b\ln b}{(b-1)^2} - \frac{b}{b-1}\right] \\
&+ \frac{p_C p_D}{64\pi^2}\left[\frac{\tilde{b}\ln \tilde{b}}{(\tilde{b}-1)^2} - \frac{\tilde{b}}{\tilde{b}-1}\right] , \\
\Pi^3_{\Omega\Xi}(p) &= \frac{p_\Omega p_\Xi}{32\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{p_\Omega p_\Xi}{64\pi^2}\left[\frac{2b\ln b}{b-1} - \frac{b\ln b}{(b-1)^2} + \frac{b}{b-1}\right] \\
&+ \frac{p_\Omega p_\Xi}{64\pi^2}\left[\frac{2\tilde{b}\ln \tilde{b}}{\tilde{b}-1} - \frac{\tilde{b}\ln \tilde{b}}{(\tilde{b}-1)^2} + \frac{\tilde{b}}{\tilde{b}-1}\right] , \\
\Pi^3_{C\Omega}(p) &= \Pi^3_{\Omega C}(p) = \frac{p_C p_\Omega}{32\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{p_C p_\Omega}{64\pi^2}\frac{b\ln b}{b-1} \\
&+ \frac{p_C p_\Omega}{64\pi^2}\frac{\tilde{b}\ln \tilde{b}}{\tilde{b}-1} ,
\end{aligned} \tag{6.6}$$

$$\Pi^4_{\mu\nu}(p) = -\frac{p_\mu p_\nu}{64\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} + \frac{p_\mu p_\nu}{128\pi^2}\left(\frac{b\ln b}{b-1} - \frac{\tilde{b}\ln\tilde{b}}{\tilde{b}-1}\right) , \tag{6.7}$$

$$\begin{aligned}
\Pi^5_{CD}(p) &= \frac{p^2\delta_{CD}}{64\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{p^2\delta_{CD}}{128\pi^2}\left[\frac{b\ln b}{(b-1)^2} - \frac{b}{b-1}\right] \\
&+ \frac{p^2\delta_{CD}}{128\pi^2}\left[\frac{\tilde{b}\ln\tilde{b}}{(\tilde{b}-1)^2} - \frac{\tilde{b}}{\tilde{b}-1}\right] , \\
\Pi^5_{\Omega\Xi}(p) &= \frac{p^2\delta_{\Omega\Xi}}{64\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{p^2\delta_{\Omega\Xi}}{128\pi^2}\left[\frac{b(2b-3)\ln b}{(b-1)^2} + \frac{b}{b-1}\right] \\
&+ \frac{p^2\delta_{\Omega\Xi}}{128\pi^2}\left[\frac{\tilde{b}(2\tilde{b}-3)\ln\tilde{b}}{(\tilde{b}-1)^2} + \frac{\tilde{b}}{\tilde{b}-1}\right] , \\
\Pi^5_{C\Omega}(p) &= \Pi^5_{\Omega C}(p) = 0 ,
\end{aligned} \tag{6.8}$$

and finally, we present the components of the tensor $I^6_{\alpha\beta\mu\nu}(p)$ (from which the components of $\Pi^6_{\mu\nu}(p)$ can be obtained)

$$\begin{aligned}
I^6_{CCCC}(p) &= \frac{1}{64\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{b^3}{128\pi^2(b-1)^3}\left[\ln b - \frac{2(b-1)}{b} + \frac{b^2-1}{2b^2}\right] \\
&+ \frac{\tilde{b}^3}{128\pi^2(\tilde{b}-1)^3}\left[\ln\tilde{b} - \frac{2(\tilde{b}-1)}{\tilde{b}} + \frac{\tilde{b}^2-1}{2\tilde{b}^2}\right] , \\
I^6_{1122}(p) &= \frac{1}{3}I^6_{CCCC}(p) , \\
I^6_{\Omega\Omega\Omega\Omega}(p) &= \frac{1}{64\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{1}{64\pi^2(b-1)^3}\left[\ln b - 2(b-1) + \frac{b^2-1}{2}\right] \\
&+ \frac{1}{64\pi^2(\tilde{b}-1)^3}\left[\ln\tilde{b} - 2(\tilde{b}-1) + \frac{\tilde{b}^2-1}{2}\right] , \\
I^6_{0033}(p) &= \frac{1}{3}I^6_{\Omega\Omega\Omega\Omega}(p) ,
\end{aligned}$$

$$
\begin{aligned}
I^{\,6}_{CC\Omega\Omega} &= \frac{1}{192\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} \\
&- \frac{1}{384\pi^2}\left[\frac{3b(2b-3)\ln b}{(b-1)^2} - \frac{2b^3\ln b}{(b-1)^3} + \frac{3b}{b-1} + \frac{2b-1}{b} + \frac{b^2-1}{2b^2}\right] \\
&+ \frac{1}{384\pi^2}\left[\frac{3\tilde{b}(2\tilde{b}-3)\ln \tilde{b}}{(\tilde{b}-1)^2} - \frac{2\tilde{b}^3\ln \tilde{b}}{(\tilde{b}-1)^3} + \frac{3\tilde{b}}{\tilde{b}-1} + \frac{2\tilde{b}-1}{\tilde{b}} + \frac{\tilde{b}^2-1}{2\tilde{b}^2}\right]. \quad (6.9)
\end{aligned}
$$

All other nonvanishing components of $I^6_{\alpha\beta\mu\nu}(p)$ can be obtained by permuting indices of those shown in (6.9). See the appendix for further discussion.

Note that $\Pi^j_{\mu\nu}(p)$, $j = 1, \ldots, 6$ changes sign under the interchange of Λ and b with $\tilde{\Lambda}$ and $\tilde{b}$, respectively. We can eliminate $\Pi^1_{\mu\nu}(p)$ and $\Pi^2_{\mu\nu}(p)$ by a mass counterterm. The sum of the other pieces of the polarization tensor, $\sum_{j=3}^{6}\Pi^j_{\mu\nu}(p)$, reduces to the expression in (5.3) if $b = \tilde{b}$; integrating degrees of freedom with momenta between two similar ellipsoids yields the same result as integrating degrees of freedom with momenta between two spheres.

Next we set $b = 1$ and expand $\tilde{b} = 1 + \ln\tilde{b} + \cdots$. We drop the part of the polarization tensor of order $(\ln\tilde{b})^2$. We write the polarization tensor as matrix whose rows and columns are ordered by $1, 2, 3, 0$. Expanding to leading order in $\ln \tilde{b}$, we obtain

$$\sum_{j=3}^{6} \Pi^j(p) = \frac{1}{192\pi^2} \ln \frac{\Lambda}{\tilde{\Lambda}} \left(\mathbf{1} - pp^T\right) + \frac{\ln \tilde{b}}{64\pi^2} \begin{pmatrix} -\frac{3}{4}p_1^2 - \frac{1}{6}p_2^2 - \frac{13}{12}p_L^2 & -\frac{7}{12}p_1p_2 & -\frac{7}{4}p_1p_3 & -\frac{7}{4}p_1p_0 \\ -\frac{7}{12}p_1p_2 & -\frac{3}{4}p_2^2 - \frac{1}{6}p_1^2 - \frac{13}{12}p_L^2 & -\frac{7}{4}p_2p_3 & -\frac{7}{4}p_2p_0 \\ -\frac{7}{4}p_1p_3 & -\frac{7}{4}p_2p_3 & \frac{7}{4}p_3^2 + \frac{2}{3}p_0^2 + \frac{1}{3}p_\perp^2 & \frac{13}{12}p_3p_0 \\ -\frac{7}{4}p_1p_0 & -\frac{7}{4}p_2p_0 & \frac{13}{12}p_3p_0 & \frac{2}{3}p_3^2 + \frac{7}{4}p_0^2 + \frac{1}{3}p_\perp^2 \end{pmatrix}, \tag{6.10}$$

where $\mathbf{1}$ is the four-by-four identity matrix and the superscript T denotes the transpose. The first term on the right-hand side of (6.10) is the polarization tensor found in the previous section (5.3). The second term does not depend on Λ or $\tilde{\Lambda}$. Had we taken $b > 1$, and expanded in $b = 1 + \ln b + \cdots$, the quantity $\ln \tilde{b}$ in (6.10) would have been $\ln(\tilde{b}/b)$.

The second term on the right-hand side of (6.10) violates gauge invariance (multiplying the vector p by the matrix in this term does not yield zero). This means that an additional counterterm is needed. The most general local action of dimension 4, which is quadratic in $\tilde{A}_\mu$ and which does not change under $\mathrm{O}(2) \times \mathrm{O}(2)$ transformations, and is gauge invariant to linear order is

$$S_{\rm quad} = \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} \, \mathrm{Tr}\, \tilde{A}(-p)^T [a_1 M_1(p) + a_2 M_2(p) + a_3 M_3(p)] \tilde{A}(p) \ ,$$

where a_1, a_2 and a_3 are real coefficients and

$$M_1(p) \ = \ \begin{pmatrix} p_2^2 & -p_1p_2 & 0 & 0 \\ -p_1p_2 & p_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \ M_2(p) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p_3^2 & -p_3p_0 \\ 0 & 0 & -p_3p_0 & p_0^2 \end{pmatrix} ,$$

$$M_3(p) \ = \ \begin{pmatrix} p_L^2 & 0 & -p_1p_3 & -p_1p_0 \\ 0 & p_L^2 & -p_2p_3 & -p_2p_0 \\ -p_1p_3 & -p_2p_3 & p_\perp^2 & 0 \\ -p_1p_0 & -p_2p_0 & 0 & p_\perp^2 \end{pmatrix} .$$

We must now determine a_1, a_2 and a_3 such that the difference

$$\begin{aligned} S_{\text{diff}} &= \int_{\tilde{\mathbb{P}}} \frac{d^4q}{(2\pi)^4} \operatorname{Tr} \tilde{A}(-p)^T M_{\text{diff}}(p) \tilde{A}(p) \\ &= \int_{\tilde{\mathbb{P}}} \frac{d^4q}{(2\pi)^4} \operatorname{Tr} \tilde{A}(-p)^T \sum_{j=3}^{6} \Pi^j(p) \tilde{A}(p) \;-\; S_{\text{quad}} \end{aligned} \tag{6.11}$$

is maximally non-gauge invariant. By this we mean that the projection of tensor $M_{\text{diff}}(p)$ to a gauge-invariant expression:

$$\left(\mathbf{1} - \frac{p\,p^T}{p^T p}\right) M_{\text{diff}}(p) \left(\mathbf{1} - \frac{p\,p^T}{p^T p}\right) ,$$

has no local part. This gives a precise determination of S_{diff}, which is proportional to the counterterm to be subtracted. To carry out this procedure, we break up the second term of (6.10) into a linear combination of M_1, M_2 and M_3 and a diagonal matrix:

$$\begin{aligned} \sum_{j=3}^{6} \Pi^j(p) &= \frac{1}{192\pi^2} \ln \frac{\Lambda}{\tilde{\Lambda}} \left(\mathbf{1} - pp^T\right) \\ &+ \frac{\ln \tilde{b}}{64\pi^2} \left[\frac{7}{12} M_1(p) - \frac{13}{12} M_2(p) + \frac{7}{4} M_3(p) \right] \\ &+ \frac{\ln \tilde{b}}{64\pi^2} \begin{pmatrix} -(\frac{3}{4}p_\perp^2 + \frac{17}{6}p_L^2) I & 0 \\ 0 & -(\frac{17}{12}p_\perp^2 - \frac{7}{4}p_L^2) I \end{pmatrix} , \end{aligned} \tag{6.12}$$

where I denotes the 2×2 identity matrix. The diagonal matrix is maximally non-gauge-invariant. It is local, $\mathrm{O}(2) \times \mathrm{O}(2)$ invariant and of dimension four;

we remove it with local counterterms, rendering our ellipsoidal cut-offs gauge invariant, to one loop. We have thereby found

$$a_1 = \frac{\ln \tilde{b}}{64\pi^2} \cdot \frac{7}{12} \ , \quad a_2 = -\frac{\ln \tilde{b}}{64\pi^2} \cdot \frac{13}{12} \ , \quad a_3 = \frac{\ln \tilde{b}}{64\pi^2} \cdot \frac{7}{4} \ .$$

Removing the last term from (6.12) leaves us with our final result for the polarization tensor

$$\begin{aligned} \hat{\Pi}(p) &= \sum_{j=3}^{6} \Pi^j(p) - \frac{\ln \tilde{b}}{64\pi^2} \begin{pmatrix} -(\frac{3}{4}p_\perp^2 + \frac{17}{6}p_L^2)I & 0 \\ 0 & -(\frac{17}{12}p_\perp^2 - \frac{7}{4}p_L^2)I \end{pmatrix} \\ &= \frac{1}{192\pi^2} \ln \frac{\Lambda}{\tilde{\Lambda}} \, (\mathbf{1} - pp^T) + \frac{\ln \tilde{b}}{64\pi^2} \left[\frac{7}{12} M_1(p) - \frac{13}{12} M_2(p) + \frac{7}{4} M_3(p) \right] \ . \end{aligned}$$

One of the terms to be induced in the renormalized action by integrating out fast degrees of freedom is

$$\begin{aligned} \frac{1}{2}\langle S_{\rm I}\rangle \ - \ \frac{1}{4}(\langle S_{\rm I}^2\rangle - \langle S_{\rm I}\rangle^2) &= C_N \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} \, \tilde{A}_\mu^b(-p)\tilde{A}_\nu^b(p) \, \hat{\Pi}_{\mu\nu}(p) \\ &= \ C_N \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} \, \tilde{A}_\mu^b(-p)\tilde{A}_\nu^b(p) \left\{ \frac{1}{192\pi^2} \ln \frac{\Lambda}{\tilde{\Lambda}} \, (\mathbf{1} - pp^T) \right. \\ &\quad + \ \left. \frac{\ln \tilde{b}}{64\pi^2} \left[\frac{7}{12} M_1(p) - \frac{13}{12} M_2(p) + \frac{7}{4} M_3(p) \right] \right\} \ . \qquad (6.13) \end{aligned}$$

The other term induced by this integration, namely $-(\langle S_{\rm II}^2\rangle - \langle S_{\rm II}\rangle^2)/2$, will be discussed next.

We showed in Chapter 4, that the term $-(\langle S_{\rm II}^2\rangle - \langle S_{\rm II}\rangle^2)/2$ is given by (4.12). We may expand this term in powers of p, as we did for the spherical case in (5.4). The result is

$$-\frac{1}{2}(\langle S_{\rm II}^2\rangle - \langle S_{\rm II}\rangle^2) = -\frac{C_N}{2}\int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4}\tilde{F}^b_{\mu\nu}(-p)\tilde{F}^b_{\mu\nu}(p)\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\frac{1}{(q^2)^2}+\cdots$$
$$= -C_N\left[\frac{1}{16\pi^2}\ \ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{b\ln b}{32\pi^2(b-1)} + \frac{\tilde{b}\ln\tilde{b}}{32\pi^2(\tilde{b}-1)}\right]$$
$$\times \int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4}\tilde{F}^b_{\mu\nu}(-p)\tilde{F}^b_{\mu\nu}(p)+\cdots\ . \qquad (6.14)$$

For $b=1$, to leading order in $\ln\tilde{b}$, (6.14) becomes

$$-\frac{1}{2}(\langle S_{\rm II}^2\rangle - \langle S_{\rm II}\rangle^2) = -C_N\left(\frac{1}{16\pi^2}\ \ln\frac{\Lambda}{\tilde{\Lambda}} + \frac{\ln\tilde{b}}{32\pi^2}\right)\int_{\tilde{\mathbb{P}}}\frac{d^4p}{(2\pi)^4}\tilde{F}^b_{\mu\nu}(-p)\tilde{F}^b_{\mu\nu}(p)$$
$$+ \ \cdots\ . \qquad (6.15)$$

Our final expression for the new action $\tilde{S}=\int d^4x\tilde{\mathcal{L}}$, is obtained by putting together (6.13) and (6.15):

$$\tilde{\mathcal{L}} = \frac{1}{4}\left(\frac{1}{g_0^2} - \frac{11C_N}{48\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{C_N\ln\tilde{b}}{64\pi^2}\right)\left(\tilde{F}^2_{01}+\tilde{F}^2_{02}+\tilde{F}^2_{13}+\tilde{F}^2_{23}\right)$$
$$+\frac{1}{4}\left(\frac{1}{g_0^2} - \frac{11C_N}{48\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{37C_N\ln\tilde{b}}{192\pi^2}\right)\tilde{F}^2_{03}$$
$$+\frac{1}{4}\left(\frac{1}{g_0^2} - \frac{11C_N}{48\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} - \frac{17C_N\ln\tilde{b}}{192\pi^2}\right)\tilde{F}^2_{12}+\cdots\ . \qquad (6.16)$$

In the next chapter, we will discuss the implications of (6.16).

6.1 Appendix: Integrals Between Ellipsoids

In this appendix, we explain how to evaluate Feynman integrals between ellipsoids. Let us define $u = q_\perp^2$ and $v = q_L^2$. The restriction within the outer ellipsoid is $u + bv < \Lambda^2$, and the restriction outside the inner ellipsoid is $u + \tilde{b}v > \tilde{\Lambda}^2$. We can split this region $\mathbb{S}$ into two regions:

$$\begin{aligned}
\mathbb{S}_{\mathrm{I}} :&\quad 0 < u < \tilde{\Lambda}^2,\quad \frac{\tilde{\Lambda}^2 - u}{\tilde{b}} < v < \frac{\Lambda^2 - u}{b} \\
\mathbb{S}_{\mathrm{II}} :&\quad \Lambda^2 < u < \Lambda^2,\ \ 0 < v < \frac{\Lambda^2 - u}{b}\,,
\end{aligned}$$

or, replacing v with $w = u + v$,

$$\begin{aligned}
\mathbb{S}_{\mathrm{I}} :&\quad 0 < u < \tilde{\Lambda}^2,\quad \frac{\tilde{\Lambda}^2}{\tilde{b}} + \left(1 - \frac{1}{\tilde{b}}\right) < w < \frac{\Lambda^2}{b} + \left(1 - \frac{1}{b}\right) u \\
\mathbb{S}_{\mathrm{II}} :&\quad \Lambda^2 < u < \Lambda^2,\ \ u < w < \frac{\Lambda^2}{b} + \left(1 - \frac{1}{b}\right) u\,,
\end{aligned}$$

An integral over $\mathbb{S}$ is the sum of the integral over $\mathbb{S}_{\mathrm{I}}$ and $\mathbb{S}_{\mathrm{II}}$, giving (6.2). We will use this to evaluate the expressions in (6.3).

First we turn to the quadratically-divergent parts of the polarization tensor, $\Pi^1_{\mu\nu}(p)$ and $\Pi^2_{\mu\nu}(p)$. These terms will eventually be removed with counterterms, but their evaluation is useful as preparation for the other integrals to be determined. We find

$$\begin{aligned}\Pi^1_{\mu\nu}(p) &= \frac{\Lambda\delta_{\mu\nu}}{256\pi^4}\int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \left[\int_0^{\omega^2} dU \int_{\omega^2\tilde{b}^{-2}+(1-\tilde{b}^{-1})U}^{b^{-1}+(1-b^{-1})U} dW \right. \\ &+ \left. \int_{\omega^2}^1 \int_U^{b^{-1}+(1-b^{-1})U} dW\right] \frac{1}{W}\,, \qquad (6.A.1)\end{aligned}$$

where $U = u/\Lambda^2$, $W = w/\Lambda^2$ and $\omega = \tilde{\Lambda}/\Lambda$. Performing the integrals over the angles and W yields

$$\begin{aligned}\Pi^1_{\mu\nu}(p) &= \frac{\Lambda^2\delta_{\mu\nu}}{64\pi^2}\left\{\int_0^1 dU \ln[b^{-1} + (1-b^{-1})U] \right. \\ &- \left. \int_0^{\omega^2} dU \ln[\omega^2\tilde{b}^{-1} + (1-\tilde{b}^{-1})u] - \int_{\omega^2}^1 dU \ln U\right\}.\end{aligned}$$

The remaining integration is done by changing variables to $r = b^{-1} + (1 - b^{-1})U$ in the first term and $\tilde{r} = \omega^2\tilde{b}^{-1} + (1-\tilde{b}^{-1})U$ in the second term, giving the result (6.4).

The expression for $\Pi^2_{\mu\nu}(p)$ will vanish if $\mu \neq \nu$. To see this, notice that the measure and limits of the integral do not change, upon changing the sign of q_μ, but not q_ν. After carrying out the angular integrations, $\Pi^2_{CD}(p)$ becomes

$$\begin{aligned}\Pi^2_{CD}(p) &= -\frac{\Lambda^2\delta_{CD}}{64\pi^2}\left[\int_0^{\omega^2} dUU \int_{\omega^2\tilde{b}^{-1}+(1-\tilde{b}^{-1})u}^{b^{-1}+(1-b^{-1})U} dW \right. \\ &+ \left. \int_{\omega^2}^1 dUU \int_U^{b^{-1}+(1-b^{-1})U} dW\right]\frac{1}{W^2}\,. \qquad (6.A.2)\end{aligned}$$

We next carry out the integration over W and define r and $\tilde{r}$, as before, to obtain

$$\begin{aligned}\Pi^2_{CD}(p) &= -\frac{\Lambda^2\delta_{CD}}{64\pi^2}\left[\frac{\tilde{b}^2}{(\tilde{b}-1)^2}\int_{\omega^2\tilde{b}^{-1}}^{\omega^2} dr\,\frac{r-\omega^2\tilde{b}^{-1}}{r}\right.\\ &- \left.\frac{b^2}{(b-1)^2}\int_{\omega^2 b^{-1}}^{1} dr\,\frac{r-\omega^2 b^{-1}}{r} + (1-\omega^2)\right],\end{aligned}$$

which yields the first of (6.5). The other non-vanishing components of $\Pi^2_{\mu\nu}(p)$ are given by

$$\begin{aligned}\Pi^2_{\Omega\Xi}(p) &= -\frac{\Lambda^2\delta_{\Omega\Xi}}{64\pi^2}\left[\int_0^{\omega^2} dU\int_{\omega^2\tilde{b}^{-1}+(1-\tilde{b}^{-1})u}^{b^{-1}+(1-b^{-1})U} dW\right.\\ &+ \left.\int_{\omega^2}^{1} dU\int_{U}^{b^{-1}+(1-b^{-1})U} dW\right]\left(\frac{1}{W}-\frac{U}{W^2}\right).\end{aligned}$$

The first term is proportional to the right-hand side in (6.A.1) and the second term is proportional to the right-hand side in (6.A.2). We can put these results together, to obtain the remainder of (6.5).

Each of the two terms in $\Pi^3_{\mu\nu}(p)$ in (6.3) contain the integral

$$J_{\alpha\beta} = \frac{1}{2}\int_{\mathbb{S}}\frac{d^4q}{(2\pi)^4}\,\frac{q_\alpha q_\beta}{(q^2)^3}\,.$$

As in the case of $\Pi^2_{\mu\nu}(p)$, an examination of how the integral changes under the sign change of one component of momentum shows that it vanishes, unless $\alpha = \beta$. Performing the angular integrations,

$$\begin{aligned} J_{CD} &= \frac{\delta_{CD}}{64\pi^2}\left[\int_0^{\omega^2} dU \int_{\omega^2\tilde{b}^{-1}+(1-\tilde{b}^{-1})u}^{b^{-1}+(1-b^{-1})U} dW \right. \\ &+ \left. \int_{\omega^2}^{1} dU \int_{U}^{b^{-1}+(1-b^{-1})U} dW \right] \frac{U}{W^3}\,, \end{aligned}$$

and

$$\begin{aligned} J_{\Omega\Xi} &= \frac{\delta_{\Omega\Xi}}{64\pi^2}\left[\int_0^{\omega^2} dU \int_{\omega^2\tilde{b}^{-1}+(1-\tilde{b}^{-1})u}^{b^{-1}+(1-b^{-1})U} dW \right. \\ &+ \left. \int_{\omega^2}^{1} dU \int_{U}^{b^{-1}+(1-b^{-1})U} dW \right] \frac{1}{W^2} - \delta_{\Omega\Xi} J_{11}\,, \end{aligned}$$

which reduce to

$$\begin{aligned} J_{CD} &= \frac{\delta_{CD}}{64\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} + \frac{\delta_{CD}}{128\pi^2}\left[\frac{\tilde{b}}{(\tilde{b}-1)^2}\ln\tilde{b} - \frac{\tilde{b}}{\tilde{b}-1}\right] \\ &- \frac{\delta_{CD}}{128\pi^2}\left[\frac{b}{(b-1)^2}\ln\tilde{b} - \frac{b}{b-1}\right]\,, \end{aligned} \qquad (6.A.3)$$

and

$$\begin{aligned} J_{\Omega\Xi} &= \frac{\delta_{\Omega\Xi}}{64\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} + \frac{\delta_{\Omega\Xi}}{128\pi^2}\left\{\left[\frac{2\tilde{b}}{\tilde{b}-1} - \frac{\tilde{b}}{(\tilde{b}-1)^2}\right]\ln\tilde{b} + \frac{\tilde{b}}{\tilde{b}-1}\right\} \\ &- \frac{\delta_{\Omega\Xi}}{128\pi^2}\frac{\delta_{\Omega\Xi}}{128\pi^2}\left\{\left[\frac{2b}{b-1} - \frac{b}{(b-1)^2}\right]\ln b + \frac{b}{b-1}\right\}\,, \end{aligned}$$

which lead to (6.6).

We may write $\Pi^4_{\mu\nu}(p)$

$$\Pi^4_{\mu\nu}(p) = -\frac{p_\mu p_\nu}{4} \sum_\mu J_{\mu\mu} \, ,$$

giving (6.7) and $\Pi^5_{\mu\nu}(p)$ as

$$\Pi^5_{\mu\nu}(p) = p^2 J_{\mu\nu} \, ,$$

giving (6.8).

Finally, to evaluate $\Pi^6_{\mu\nu}(p)$, we need to work out the tensor $I_{\alpha\beta\mu\nu}(p)^6$, defined in (6.3) as

$$I^6_{\alpha\beta\mu\nu}(p) = \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4} \frac{q_\alpha q_\beta q_\mu q_\nu}{(q^2)^4} \, .$$

We discuss below how to evaluate the following special cases of this tensor:

$$\begin{aligned} &I^6_{1111}(p) = I^6_{2222}(p), \quad I^6_{0000}(p) = I^6_{3333}(p), \quad I^6_{1122}(p), \quad I^6_{0033}(p), \\ &I^6_{0011}(p) = I^6_{0022}(p) = I^6_{1133}(p) = I^6_{2233}(p) \, . \end{aligned}$$

All other non-vanishing cases can be obtained by permuting indices of this fully-symmetric tensor. We find

$$I^6_{1111}(p) = \frac{3}{128\pi^2}\left[\int_0^{\omega^2} dU \int_{\omega^2\tilde{b}^{-1}+(1-\tilde{b}^{-1})u}^{b^{-1}+(1-b^{-1})U} dW + \int_{\omega^2}^1 dU \int_U^{b^{-1}+(1-b^{-1})U} dW\right]\frac{U^2}{W^4}$$
$$= \frac{1}{64\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} + \frac{1}{128\pi^2}\frac{\tilde{b}^3}{(1-\tilde{b})^3}\left[\ln\tilde{b} - \frac{2(\tilde{b}-1)}{\tilde{b}} + \frac{(\tilde{b}-1)(\tilde{b}+1)}{2\tilde{b}^2}\right] - \frac{1}{128\pi^2}\frac{b^3}{(1-b)^3}\left[\ln b - \frac{2(b-1)}{b} + \frac{(b-1)(b+1)}{2b^2}\right],$$

$$I^6_{0000}(p) = \frac{3}{128\pi^2}\left[\int_0^{\omega^2} dU \int_{\omega^2\tilde{b}^{-1}+(1-\tilde{b}^{-1})u}^{b^{-1}+(1-b^{-1})U} dW + \int_{\omega^2}^1 dU \int_U^{b^{-1}+(1-b^{-1})U} dW\right]\frac{(W-U)^2}{W^4}$$
$$= \frac{1}{64\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} + \frac{1}{128\pi^2}\frac{1}{(1-\tilde{b})^3}\left[\ln\tilde{b} - 2(\tilde{b}-1) + \frac{(\tilde{b}-1)(\tilde{b}+1)}{2}\right] - \frac{1}{128\pi^2}\frac{1}{(1-b)^3}\left[\ln b - 2(b-1) + \frac{(b-1)(b+1)}{2}\right],$$

$$
\begin{aligned}
I^6_{0011}(p) = {} & \frac{1}{64\pi^2}\left[\int_0^{\omega^2} dU \int_{\omega^2\tilde{b}^{-1}+(1-\tilde{b}^{-1})u}^{b^{-1}+(1-b^{-1})U} dW \right. \\
& \left. + \int_{\omega^2}^1 dU \int_U^{b^{-1}+(1-b^{-1})U} dW \right] \frac{UW-U^2}{W^4} \\
& = J_{11} - I^6_{1111}(p) \\
& = \frac{1}{64\pi^2}\ln\frac{\Lambda}{\tilde{\Lambda}} + \frac{1}{384\pi^2}\left[\frac{3\tilde{b}(2\tilde{b}-3)}{(\tilde{b}-1)^2}\ln\tilde{b} + \frac{3\tilde{b}}{\tilde{b}-1} - \frac{2\tilde{b}^3}{(\tilde{b}-1)^3}\ln\tilde{b}\right. \\
& \left. + \frac{2\tilde{b}-1}{\tilde{b}} + \frac{(\tilde{b}-1)(\tilde{b}+1)}{2\tilde{b}^3}\right] - \frac{1}{384\pi^2}\left[\frac{3b(2b-3)}{(b-1)^2}\ln b + \frac{3b}{b-1}\right. \\
& \left. - \frac{2b^3}{(b-1)^3}\ln b + \frac{2b-1}{b} + \frac{(b-1)(b+1)}{2b^3}\right],
\end{aligned}
$$

and $I^6_{1122}(p) = I^6_{1111}(p)/3$, $I^6_{0033}(p) = I^6_{0000}(p)/3$. This completes the integrals needed in $\Pi^6_{\mu\nu}(p)$.

There is one remaining quantity to consider, namely (6.14). The integral we need to evaluate is

$$
\int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4}\frac{1}{(q^2)^2} = 2\sum_\mu J_{\mu\mu}\,,
$$

which gives the right-hand side of (6.14).

Chapter 7

The rescaled Yang-Mills action

The main result of Chapter 6, equation (6.16), is the action resulting from aspherically integrating out degrees of freedom. In this chapter, we will write this in a way which allows comparison with standard renormalization with an isotropic cut-off, *i.e.* (5.5). We define $\tilde{g}_0$ using (5.5). To leading order in $\ln \tilde{b}$, the effective coupling in the first term of (6.16) is given by

$$\frac{1}{g_{\text{eff}}^2} = \frac{1}{g_0^2} - \frac{11 C_N}{48\pi^2} \ln \frac{\Lambda}{\tilde{\Lambda}} - \frac{C_N \ln \tilde{b}}{64\pi^2} = \frac{1}{\tilde{g}_0^2}\, \tilde{b}^{-\frac{C_N}{64\pi^2}\tilde{g}_0^2} + \cdots \, .$$

Setting $\tilde{b} = \lambda^{-2}$, we find to leading order in $\ln \lambda$

$$g_{\text{eff}}^2 = \tilde{g}_0^2\, \lambda^{-\frac{C_N}{32\pi^2}\tilde{g}_0^2} \, . \tag{7.1}$$

and

$$\tilde{\mathcal{L}} = \frac{1}{4g_{\mathrm{eff}}^2}\,\mathrm{Tr}\,\left(\tilde{F}_{01}^2 + \tilde{F}_{02}^2 + \tilde{F}_{13}^2 + \tilde{F}_{23}^2 + \lambda^{\frac{17C_N}{48\pi^2}\tilde{g}_0^2}\tilde{F}_{03}^2 + \lambda^{\frac{7C_N}{48\pi^2}\tilde{g}_0^2}\tilde{F}_{12}^2\right) + \cdots\ ,$$

where the dots represent corrections of order $(\ln\lambda)^2$. Next we rescale the longitudinal coordinates, $x^L \to \lambda x^L$, drop the tildes on the fields, and Wick-rotate back to Minkowski signature, to find the longitudinally-rescaled effective Lagrangian

$$\begin{aligned}\mathcal{L}_{\mathrm{eff}} = \frac{1}{4g_{\mathrm{eff}}^2}\,\mathrm{Tr}\,\Big(F_{01}^2 + F_{02}^2 - F_{13}^2 - F_{23}^2 \quad &+ \quad \lambda^{-2+\frac{17C_N}{48\pi^2}\tilde{g}_0^2}F_{03}^2 \\ &- \quad \lambda^{2+\frac{7C_N}{48\pi^2}\tilde{g}_0^2}F_{12}^2\Big) + \cdots\ . \end{aligned} \tag{7.2}$$

Once again the corrections are of order $(\ln\lambda)^2$. If we compare (7.2) with the classically-rescaled action (2.1), we see that the field-strength-squared terms are anomalously rescaled.

If we simply take the $\lambda \to 0$ limit of (7.2), all couplings become zero or infinite, except g_{eff} [4]. For very high energy, that is for small λ, this effective coupling becomes strong, as can immediately be seen from (7.1). We are fortunate, however, that the energy where this happens is far larger than what is experimentally accessible. If we take $\tilde{g}_0$ of order one, then

$$g_{\mathrm{eff}}^2 \sim \lambda^{-\frac{1}{100}}\ . \tag{7.3}$$

This tells us that g_{eff}^2 is less than a number of order ten, unless λ is roughly less than an inverse googol, $\lambda \sim 10^{-100}$. Thus the experimentally accessible value of $g_{\rm eff}$ is small. We still have a problem, nonetheless, because the coefficient of F_{12}^2 in the effective Lagrangian is very small as $\lambda \to 0$. This is also for the classically rescaled theory (2.1) [8]. This tiny coefficient means that there is very little energy in longitudinal magnetic flux. Hence the longitudinal magnetic flux fluctuates wildly. If we denote the coefficient of this term in the Lagrangian as $1/(4g_L^2)$, then

$$g_L^2 = g_{\rm eff}^2 \lambda^{-2-\frac{7C_N}{48\pi^2}\tilde{g}_0^2} \; . \qquad (7.4)$$

This coupling explodes for small λ, even if $g_{\rm eff}$ is small.

Chapter 8

Extrapolating to High Energy

We have determined how a quantized non-Abelian gauge action changes under a longitudinal rescaling $\lambda < 1$, but $\lambda \approx 1$. Our analysis suggests the form of the effective action for the high-energy limit, $\lambda \ll 1$, but this effective action cannot be derived perturbatively. The main problem is how the Yang-Mills action changes as λ is decreased. The coefficient of the longitudinal magnetic field squared, in the action, decreases, as λ is decreased. Eventually, we can no longer compute how couplings will run.

Our difficulty is very similar to that of finding the spectrum of a non-Abelian gauge theory. Assuming that there is no infrared-stable fixed point at non-zero bare coupling, a guess for the long-distance effective theory is a strongly-coupled cut-off action. The regulator can be a lattice, for example. One can then use strong-coupling expansions to find the spectrum. The problem is that no one knows how to specify the true cut-off theory (which presumably has many terms, produced by integrating over all the short-distance degrees of freedom). The best we can do is guess the regularized strongly-coupled action. Such strong-coupling theories are not (yet) derivable from QCD, but are best thought of as models of the strong interaction at large distances.

Similarly, we believe that (2.7) for $\lambda \ll 1$, and variants we discuss below, cannot be proved to describe the strong interaction at high energies. Thus it appears that the same statement applies to the the BFKL/BK theory (designed to describe the region where Mandelstam variables satisfy $s \gg t \gg \Lambda_{QCD}$) [1], [13]. Two closely-related problems in this theory are non-unitarity and infrared diffusion of gluon virtualities. These problems indicate that the BFKL theory breaks down at large length scales. There is numerical evidence [14] that unitarizing using the BK evolution equation [13] suppresses diffusion into the infrared and leads to saturation, at least for fixed small impact parameters. This BK equation is a non-linear generalization of the BFKL evolution equation. The non-linearity only becomes important at small x, at large longitudinal distances, where perturbation theory is not trustworthy.

In the color-glass-condensate picture [2], [3], the Yang-Mills action with $\ln \lambda = 0$ is coupled to sources. The classical field strength is purely transverse. If this action is quantized, however, this is no longer the case. The fluctuations of the longitudinal magnetic field $\mathcal{B}_3$ will become extremely large (this can be seen by inspecting (2.7) and (2.8)). In principle, we would hope to derive the color-glass condensate by a longitudinal renormalization-group transformation, with background sources. The obstacle to doing this is precisely the problem of large fluctuations of $\mathcal{B}_3$.

Finally we wish to comment on an approach to soft-scattering and total cross sections. In Reference [8] an effective lattice SU(N) gauge theory was proposed. This gauge theory is a regularization of (2.8) and (2.9). This gauge theory can be formulated as coupled (1+1)-dimensional $\mathrm{SU}(N)\times\mathrm{SU}(N)$ nonlinear sigma models and reduces to a lattice Yang-Mills theory at $\lambda = 1$ (in which case, it is equivalent to the light-cone lattice theory of Bardeen et. al. [15]). The nonlinear sigma model is asymptotically free and has a mass gap. These facts together with the assumption that the terms proportional to λ^2 are a weak perturbation leads to confinement and diffraction in the gauge theory. Similar gauge models in $(2+1)$ dimensions were proposed as laboratories of color confinement [16], and string tensions for different representations [17], the low-lying glueball spectrum [18], and corrections of higher order in order λ to the string tension [19] were found (these calculations were performed using the exact S-matrix [20] and form factors [21] of the $(1+1)$-dimensional nonlinear sigma model). In such theories (whether in $(2+1)$ or $(3+1)$ dimensions), transverse electric flux is built of massive partons (made entirely of glue, but not conventional gluons). These partons can only move and scatter longitudinally, to leading order in λ. The picture which arises from such gauge-theory models is very close to the that of the forward-scattering amplitude suggested by Kovner [22].

The effective gauge theory of Reference [8] has a small value of g_{eff}, as well as a small value of λ, in the Hamiltonian (2.8). We have found in Section 5 that g_{eff} grows extremely slowly, as the energy is increased. If we can naively extrapolate our results to extremely high energies, this effective gauge theory appears correct. We should not, however, regard this as proof that the effective theory is valid, since the perturbative calculation of Chapter 6 breaks down at such energies.

Chapter 9

Discussion

In this thesis, we determined how the action of an SU(N) gauge changes under longitudinal rescaling λ, at one loop. We found, in particular, the anomalous dependence of the coefficients in this action on λ. The technical tool we used was Wilson's formulation of renormalization generalized to a more general cut-off. As the energy increases, the coefficient of F_{12}^2 in the action eventually becomes too small to trust the method further. Therefore, neither classical nor perturbative methods may be entirely trusted beyond a certain energy. The breakdown of these methods at high energies is similar to the breakdown of perturbation theory to compute the force between charges at large distances, in an asymptotically-free theory. Nonetheless, high-energy effective theories, inspired by the longitudinal-rescaling idea, may be phenomenologically useful.

There are two obvious further projects to be done. Our calculation should be redone including Fermions. Aside from the importance of considering QCD with quarks, it would be interesting to study how longitudinal rescaling affects the QED action.

The second project would be to determine how the action changes under a longitudinal rescaling by a different method. The idea would be to study Green's functions of the operator

$$\mathcal{D}(x) = x^0 \mathcal{T}_{00}(x) + x^3 \mathcal{T}_{03}(x) \ , \tag{9.1}$$

where $\mathcal{T}_{\mu\nu}(x)$ is the stress-energy-momentum tensor. The spacial integral of this operator generates longitudinal rescalings on states. Correlators of products of $\mathcal{D}(x)$ and other operators could be studied with simpler regularization methods (such as dimensional regularization) instead of our sharp momentum cut-off. The commutator of $\mathcal{D}(x)$ and an operator $\mathcal{O}(y)$ will reveal how $\mathcal{O}(y)$ behaves under longitudinal rescaling. Such an analysis should be easier than the method we have used here. In particular, we expect calculations beyond one loop should be feasible.

Bibliography

[1] V. Fadin, E. Kuraev and L.N. Lipatov, Sov. Phys. J.E.T.P **44** (1976) 443; I.I.. Balitsky and L.N. Lipatov, Sov. Nucl. Phys. **28** (1978) 822.

[2] L. McLerran and R. Venugopalan, Phys. Rev. **D49** (1994) 2233; **D49** (1994) 3352; **D50** (1994) 2225; **D59** (1999) 094002.

[3] E. Iancu, A. Leonidov and L. D. McLerran, Nucl. Phys. **A692** (2001) 583; Phys. Lett. **B510** (2001)133.

[4] H. Verlinde and E. Verlinde, Princeton University Preprint **PUPT-1319**, **hep-th/9302104** (1993).

[5] P. Orland and J. Xiao, Baruch College Preprint, BCCUNY-HEP/09-02 (2009), **arXiv:0901.2955** [hep-ph], submitted to Physical Review **D**.

[6] L.N. Lipatov, Nucl. Phys. **B452** (1995) 369.

[7] Yu. V. Kovchegov, Phys. Rev. **D54** (1996) 5463; **D55** (1997) 5445.

[8] P. Orland, Phys. Rev. **D77** (2008) 056004.

[9] K.G. Wilson and J.B. Kogut, Phys. Rept. **12** (1974) 75.

[10] I.Ya. Aref'eva and I.V. Volovich, Steklov Math. Inst. Preprint **SMI-15-94**, **hep-th/9412155** (1994); V. Periwal, Phys. Rev. **D52** (1995) 7328.

[11] A.M. Polyakov, **Gauge Fields and Strings**, Harwood Academic Publishers, Chur Switzerland (1987).

[12] D.J. Gross and F. Wilczek, Phys. Rev. Lett. **30** (1973) 1343; H.D. Politzer, Phys. Rev. Lett. **30** (1973) 1343.

[13] I. Balitsky, Nucl. Phys. **B463** (1996) 99; Phys. Rev. Lett. **81** (1998) 2024; Phys. Rev. **D60** (1999) 014020; Phys. Lett. **B518** (2001) 235; Yu. V. Kovchegov, Phys. Rev. **D60** (1999) 034008.

[14] K. Golec-Biernat, L. Motyka and A.M. Stasto, Phys. Rev. **D65** (2002) 074037; K. Golec-Biernat, and A.M. Stasto, Nucl. Phys. **B668** 345.

[15] W.A. Bardeen and R.B. Pearson, Phys. Rev. **D14** (1976) 547; W.A. Bardeen, R.B. Pearson and E. Rabinovici, Phys. Rev. **D21** (1980) 1037.

[16] P. Orland, Phys. Rev. **D71** (2005) 054503.

[17] P. Orland, Phys. Rev. **D75** (2007) 025001.

[18] P. Orland, Phys. Rev. **D75** (2007) 101702(R).

[19] P. Orland, Phys. Rev. **D74** (2006) 085001; Phys. Rev. **D77** (2008) 025035.

[20] E. Abdalla, M.C.B. Abdalla and A. Lima-Santos, Phys. Lett. **B140** (1984) 71; P.B. Wiegmann; Phys. Lett. **B142** (1984) 173; A.M. Polyakov and P.B. Wiegmann, Phys. Lett. **B131** (1983) 121; P.B. Wiegmann, Phys. Lett. **B141** (1984) 217.

[21] M. Karowski and P. Weisz, Nucl. Phys. **B139** (1978) 455.

[22] A. Kovner, Acta Phys.Polon. **B36** (2005) 3551.