Points of Canonical Height Zero on Projective Varieties

by

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Abstract

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Let k be an algebraically closed field of characteristic zero, C a smooth connected projective curve defined over k, $K = k(C)$ the function field of C. Let Y be a projective K-variety, L a very ample line bundle on Y and $\psi : Y \to Y$ a K-morphism such that $\psi^*L \cong L^{\otimes d}$. We prove that a projective integral C-scheme Y is isotrivial when it is covered by a projective integral k-scheme $X := X_0 \times_k C$, where X_0 is a k-scheme. This result provides a setup for a conjecture of L. Szpiro on parametrization of points of canonical height zero of the dynamical system (Y, L, ψ).

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Contents

Chapter 1

Introduction

Let k be an algebraically closed field of characteristic zero, C a smooth projective connected curve over k, $K = k(C)$ the function field of C. Let Y be a projective variety over K and L a very ample line bundle on Y. Let $\psi : Y \to Y$ be an endomorphism such that $\psi^* L \cong L^{\otimes d}, d > 1$.

In this thesis we provide the setup to state a conjecture (due to L. Szpiro) on parametrization of points of canonical height zero of the dynamical system (Y, L, ψ). The main tools used are Theorems 3.15, 4.2 and 4.14.

We briefly state the number field analogs of results known in this direction. For Y a projective variety defined over a number field, equipped with L and ψ as above, it is well known that a point $P \in Y$ is preperiodic for ψ if and only if it has canonical height zero [9]. Moreover over number fields, Northcott's Theorem [9] states that the set of preperiodic points (or equivalently, the points of canonical height zero) form a finite set. Over a function field K, the Northcott principle fails [2] : if the ground field k is algebraically closed, then for sufficiently large M, the set of points of bounded canonical height

$$\{P \in Y(K) : h_\psi(P) \leq M\}$$

is infinite.

The layout of the thesis is as follows: Chapter 2 discusses some basic properties of heights. Some of the lemmas are function field analogs of similar facts over number fields in [14]. In Chapter 3 we discuss the Hilbert scheme of graph of sections of $\mathcal{Y} \xrightarrow{q} C$, where $\mathcal{Y}$ is a model of Y over C and in Theorem 3.15 we show that a subscheme of the the universal family of the Hilbert scheme has a product structure. Chapter 4 contains the core for the conjecture on parametrization of points of canonical height zero. We prove that if an integral projective C-scheme X is covered by an integral projective k-scheme $W := W_0 \times_k C$ and $W \xrightarrow{f} X$ satisfies $f_*\mathcal{O}_W \cong \mathcal{O}_X$, then $X_K \to Spec(K)$ is isotrivial (see Definition 4.1). The proof requires, existence of the scheme of isomorphisms and a result of Greenberg [6] on existence of rational points over henselian discrete valuation rings. Finally we put together the results on heights, hilbert scheme and isotriviality in chapter 5 to state the conjecture on parametrization of points of canonical height zero of (Y, L, ψ). The conjecture states:

Conjecture (L. Szpiro): Let k be an algebraically closed field of characteristic zero, C a smooth projective connected curve over k with function field K. Let Y be a projective K-variety, L a very ample line bundle on Y and $\psi : Y \to Y$ a morphism such that $\psi^*L \cong L^{\otimes d}$ for some $d > 1$. Then the points of canonical height zero of the dynamical system (Y, L, ψ) is a finite union of the constant points of closed irreducible preperiodic isotrivial subvarieties X_i, for $1 \leq i \leq n$.

Notation/Conventions: k denotes a algebraically closed field of characteristic zero, C a smooth, projective, connected curve and $K = k(C)$, its function field. For a projective variety equipped with a line bundle, $h_L(P)$ denotes height of P with respect to L. For $\psi : Y \to Y$ a polarized endomorphism of degree at least 2, $h_\psi(P)$ denotes the canonical height of P. We write $p_a(C)$ for the arithmetic genus of C and g_C for the geometric genus.

Chapter 2

Heights

Let k be a field of characteristic zero, C a smooth projective connected curve over k, $K = k(C)$ the function field of C. Let Y be a projective variety over K and L a very ample line bundle on Y. We have an embedding $Y \hookrightarrow \mathbb{P}(H^0(Y, L)) \cong \mathbb{P}^N_K$, where $N = \dim_K H^0(Y, L) - 1$.

Definition 2.1. *Let X be a K-variety and L a line bundle on X. A model of (X, L) over C is a scheme $\mathcal{X} \to C$ and $\mathcal{L}$ a line bundle on $\mathcal{X}$ such that $(\mathcal{X}, \mathcal{L})$ is isomorphic to (X, L) over the generic fiber. We say the model $(\mathcal{X}, \mathcal{L})$ is ample if $\mathcal{L}$ is relatively ample on $\mathcal{X}$.*

Let $\mathcal{Y}$ be the Zariski closure of Y in $\mathbb{P}^N_C$. $\mathcal{Y}$ is a projective model of Y. Let $p_2 : \mathbb{P}^N_C \to C$ be the projection on the second factor. Since $\mathcal{Y}$ is proper, giving $P \in \mathcal{Y}(K)$ is equivalent to having a section s_P of $q := p_2|_{\mathcal{Y}}$.

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{i} & \mathbb{P}^N_C \\ s_P \curvearrowright \ \downarrow q & & \\ C & & \end{array}$$

For $P_0 \in C$, define the line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}^N}(1)|_{\mathcal{Y}} \otimes p_2^*\mathcal{O}_C(P_0)|_{\mathcal{Y}}$ on $\mathcal{Y}$. $\mathcal{L}$ is a model of L and it is ample on $\mathcal{Y}$.

Definition 2.2. *We define the height of $P \in \mathcal{Y}(K)$ relative to $\mathcal{L}$, denoted by $h_{\mathcal{L}}(P)$ as:*

$$h_{\mathcal{L}}(P) = deg(s_P^*\mathcal{L})$$

Let $C' \xrightarrow{f} C$ be a finite covering of C and K' be the function field of C'. Let $\mathcal{Y}' := \mathcal{Y} \times_C C'$, and $\mathcal{L}'$ the pullback of $\mathcal{L}$ to $\mathcal{Y}'$. We have the following diagram:

$$\begin{array}{ccc} \mathcal{Y}' & \longrightarrow & \mathcal{Y} \\ s_{P'} \curvearrowright \ \downarrow q' & & \downarrow q \\ C' & \longrightarrow & C \end{array}$$

For $P' \in \mathcal{Y}(K')$

$$h_{\mathcal{L}}(P') = \frac{deg(s_{P'}^*\mathcal{L}')}{[K' : K]}$$

where $s_{P'}$ is a section of q'.

Definition 2.3. *Let X be a projective scheme over k and $\mathscr{F}$ a coherent sheaf* *on X. We define the Euler characteristic of $\mathscr{F}$ by*

$$\chi(\mathscr{F}) = \sum_{i=0}^{\dim X} (-1)^i \dim_k H^i(X, \mathscr{F})$$

By the Riemann-Roch theorem for curves [8], for a positive integer t, we have

$$\chi(C, s_P^* \mathcal{L}^{\otimes t}) = \deg(s_P^* \mathcal{L}) \cdot t - g_C + 1 \tag{2.4}$$

where χ denotes the Euler characteristic. Fixing $h_{\mathcal{L}}(P)$ yields a unique polynomial (the Hilbert polynomial of C relative to $s_P^* \mathcal{L}$) which we denote by $F(t)$. In this chapter our goal is to describe the set of points of $\mathcal{Y}(K')$ of bounded height for any finite extension K' of K. Before we proceed further, we introduce some properties of heights.

Lemma 2.5. *For any $P \in \mathcal{Y}(\bar{K})$, $h_{\mathcal{L}}(P)$ is invariant under any change of field of definition of P.*

Proof: Let C_1, C_2 be two distinct finite coverings of C with function fields K_1, K_2 respectively and let P be defined over K_1 and K_2. There exists a curve C_3 covering C_1 and C_2. Let K_3 be the function field of C_3. Then we have the following diagram:

$$\begin{array}{ccccc}
\mathcal{Y}_3 & \xrightarrow{f_1} & \mathcal{Y}_1 & \xrightarrow{f_2} & \mathcal{Y} \\
{\scriptstyle s_{P_3}} \uparrow \downarrow {\scriptstyle q_3} & & {\scriptstyle s_{P_1}} \uparrow \downarrow {\scriptstyle q_1} & & {\scriptstyle s_P} \uparrow \downarrow {\scriptstyle q} \\
C_3 & \xrightarrow[g_1]{} & C_1 & \xrightarrow[g_2]{} & C
\end{array}$$

where $\mathcal{Y}_i = \mathcal{Y} \times_C C_i$ for $i = 1, 2, 3$. We have

$$\begin{aligned} h_{\mathcal{L},K_3}(P) &= deg(s_{P_3}^* f_1^* f_2^* L) \cdot [K_3 : K]^{-1} \\ &= deg(g_1^* s_{P_1}^* f_2^* \mathcal{L}) \cdot ([K_3 : K_1][K_1 : K])^{-1} \\ &= deg(s_{P_1}^* f_2^* \mathcal{L}) \cdot [K_1 : K]^{-1} \\ &= h_{L,K_1}(P) \end{aligned}$$

Similarly, $h_{\mathcal{L},K_3}(P) = h_{\mathcal{L},K_2}(P)$. This shows that the height of P is invariant under any change of field of definition of P. ■

Remark: Since the height is independent of the choice of field of definition of the point we do not indicate the field in the subscript.

Definition 2.6. *The support of a Cartier divisor D is the closed subset consisting of those $x \in X$ at which $1 \in \Gamma(X, \mathcal{O}_X)$ is not a local equation. We denote the support of D by $Supp(D)$.*

Definition 2.7. *Let $\mathcal{Y} \xrightarrow{q} C$ be a model of Y over C. A Cartier divisor D on $\mathcal{Y}$ is vertical if $q(Supp(D))$ is a finite set of points in C.*

We need the following theorem for the next lemma:

Theorem 2.8. *Let $f : X \to Y$ be a proper morphism of noetherian schemes with $Y = Spec(A)$ affine, and $\mathcal{F}$ a coherent sheaf on X, flat over Y. There is a finite complex $K^\cdot : 0 \to K^0 \to K^1 \to \ldots \to K^n \to 0$ of finitely generated projective A-modules and an isomorphism of functors*

$$H^p(X \times_Y Spec(B), \mathcal{F} \otimes_A B) \cong H^p(K^\cdot \otimes_A B), \quad p \geq 0$$

on the category of A-algebras B.

Proof: [12] ■

Lemma 2.9. *For $q : \mathcal{Y} \to C$ as above, if $\mathcal{L}$ and $\mathcal{M}$ are ample models of L over $\mathcal{Y}$, then there exists a positive constant B such that*

$$\left|h_{\mathcal{L}}(P) - h_{\mathcal{M}}(P)\right| \leq B \text{ for all } P \in \mathcal{Y}(\bar{K})$$

Proof: Without loss of generality we may assume that $\mathcal{L}, \mathcal{M}$ are effective. If required replace $\mathcal{L}, \mathcal{M}$ by $\mathcal{L} \otimes nH, \mathcal{M} \otimes nH$ repectively, where H is a hyperplane on $\mathcal{Y}$ and $n \gg 0$.

<u>Claim</u>: If $\mathcal{L}, \mathcal{M}$ are generically isomorphic i.e. $\mathcal{L}_K \cong \mathcal{M}_K$, then there exists Cartier divisors $D_1 \in |\mathcal{L}|$, $D_2 \in |\mathcal{M}|$ such that $D_{1,K} = D_{2,K}$.

<u>Proof of Claim</u>: Choose an affine open cover $C_1 \cup C_2$ of C such that $\mathcal{L}$ is flat over C_1 and $\mathcal{M}$ is flat over C_2. Let $C_i = Spec(A_i), \mathcal{Y}_i = q^{-1}(C_i)$, for $i = 1, 2$. Using the previous theorem let $R^{\cdot}$, $S^{\cdot}$ be finite complexes of finitely generated projective modules corresponding to $\mathcal{L}$ and $\mathcal{M}$ such that

$$H^p(\mathcal{Y}_i \times_{C_i} B, \mathcal{L} \otimes_A B) \cong H^p(R^{\cdot} \otimes_A B) \;\; p \geq 0, i = 1, 2$$

and

$$H^p(\mathcal{Y}_i \times_{C_i} B, \mathcal{M} \otimes_A B) \cong H^p(S^{\cdot} \otimes_A B) \;\; p \geq 0, i = 1, 2$$

Since $\mathcal{L}_K \cong \mathcal{M}_K$, we have $H^0(Y, \mathcal{L}_K) = H^0(Y, \mathcal{M}_K)$. Observe that $H^0(\mathcal{Y}_1, \mathcal{L})_K \xrightarrow{\alpha_1} H^0(Y, \mathcal{L}_K)$ is an isomorphism since

$$H^0(\mathcal{Y}_1, \mathcal{L})_K \cong H^0(C_1, q_*\mathcal{L})_K \cong H^0(Y, \mathcal{L}_K)$$

Similarly, $\alpha_2 : H^0(\mathcal{Y}_2, \mathcal{L})_K \to H^0(Y, \mathcal{L}_K)$, $\beta_1 : H^0(\mathcal{Y}_1, \mathcal{M})_K \to H^0(Y, \mathcal{M}_K)$ and $\beta_2 : H^0(\mathcal{Y}_2, \mathcal{M})_K \to H^0(Y, \mathcal{M}_K)$ are isomorphisms. Let t be a nonzero element of $H^0(Y, \mathcal{L}_K)$, and denote the associated divisor by D. Patching the local isomorphisms α_i, β_i, we get $s_1 \in H^0(\mathcal{Y}, \mathcal{L})$, $s_2 \in H^0(\mathcal{Y}, \mathcal{M})$ such that $(s_1)_K = (s_2)_K = D$, where (s_i) denotes the associated divisor. Set $D_1 = (s_1), D_2 = (s_2)$. This proves our claim.

Let $\bar{D}$ be the Zariski closure of D in $\mathcal{Y}$. Denote the horizontal and vertical components of D_i by D_i^h, D_i^v respectively. Note that $\bar{D} = D_1^h = D_2^h$. Now $D_1 - D_2 = D_1^v - D_2^v$ is a difference of two vertical divisors. In other words,

$$\mathcal{L} \otimes \mathcal{M}^{-1} \cong \mathcal{O}(D_1^v - D_2^v)$$

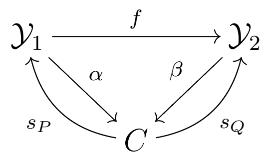
Pulling back the line bundles to C along s_P we get

$$deg(s_P^*\mathcal{L}) - deg(s_P^*\mathcal{M}) = deg(s_P^* D_1^v) - deg(s_P^* D_2^v)$$

The right side is bounded above by $B := deg(q(Supp(D_1^v - D_2^v))$ and B is independent of the section along which the divisors are pulled back to C. We now have the desired inequality. ■

Lemma 2.10. *Let Y be a projective K-variety, $\mathcal{Y}_1, \mathcal{Y}_2$ two models of Y over C, $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ a finite morphism and $\mathcal{L}_1, \mathcal{L}_2$ ample line bundles on $\mathcal{Y}_1, \mathcal{Y}_2$ respectively such that $f^*\mathcal{L}_2 \cong \mathcal{L}_1$. Let $P \in \mathcal{Y}_1(\bar{K}), Q \in \mathcal{Y}_2(\bar{K})$ such that $f(P) = Q$. Then $h_{\mathcal{L}_1}(P) = h_{\mathcal{L}_2}(Q)$.*

Proof: We have the following diagram



where s_P and s_Q are sections of α and β respectively. Then

$$h_{\mathcal{L}_1}(P) = \frac{deg(s_P^*\mathcal{L}_1)}{[K(P):K]} = \frac{deg(s_P^* f^*\mathcal{L}_2)}{deg(f)\cdot[K(P):K]} = \frac{deg(s_Q^*\mathcal{L}_2)}{[K(Q):K]} = h_{\mathcal{L}_2}(Q)$$

■

Lemma 2.11. *Let Y be a projective K-variety, L a very ample line bundle on Y and $(\mathcal{Y}_1, \mathcal{L}_1)$, $(\mathcal{Y}_2, \mathcal{L}_2)$ be two ample models of (Y, L) over C. Then there exists a positive constant B such that*

$$\left| h_{\mathcal{L}_1}(P) - h_{\mathcal{L}_2}(P) \right| \leq B \textit{ for all } P \in \mathcal{Y}(\bar{K})$$

Proof: Let $\mathcal{X}$ be the Zariski closure of the graph of the diagonal embedding $Y \xrightarrow{\Delta} Y \times_K Y$ in $\mathcal{Y}_1 \times_C \mathcal{Y}_2$. Note that $\mathcal{X}$ is a model of Y dominating $\mathcal{Y}_1$ and $\mathcal{Y}_2$. Let $p_i : \mathcal{Y}_1 \times_C \mathcal{Y}_2 \to \mathcal{Y}_i$, $i = 1, 2$ be the projections, $\mathcal{M}' := p_1^*\mathcal{L}_1$ and

$\mathcal{N}' := p_2^*\mathcal{L}_2$ be line bundles on $\mathcal{Y}_1 \times_C \mathcal{Y}_2$ and $\mathcal{M} = \mathcal{M}'|_X, \mathcal{N} = \mathcal{N}'|_X$. By the previous Lemma $h_{\mathcal{L}_1}(P) = h_{\mathcal{M}}(P)$ and $h_{\mathcal{L}_2}(P) = h_{\mathcal{N}}(P)$. Applying Lemma 2.9 to $\mathcal{M}$ and $\mathcal{N}$ we conclude that the height is independent of the model of Y and L up to a constant. ■

Theorem 2.12. *Let Y be a projective K-variety, L a very ample line bundle on Y, $\psi : Y \to Y$ a morphism satisfying $\psi^* L \cong L^{\otimes d}, d > 1$ and $(\mathcal{Y}, \mathcal{L})$ an ample model of (Y, L) over C. Then there exists a positive constant B such that*

$$\left| h_{\mathcal{L}}(\psi(P)) - d \cdot h_{\mathcal{L}}(P) \right| \leq B \text{ for all } P \in \mathcal{Y}(\bar{K})$$

Proof: Let $\mathcal{X}$ be the Zariski closure of the graph of ψ in $\mathcal{Y} \times_C \mathcal{Y}$. Let $p_1 : \mathcal{X} \to \mathcal{Y}, p_2 : \mathcal{X} \to \mathcal{Y}$ be the projections on the first and second factor respectively. Then generically $p_2^*\mathcal{L} \cong p_1^*\mathcal{L}^{\otimes d}$ i.e. $(p_2^*\mathcal{L})_K \cong (p_1^*\mathcal{L}^{\otimes d})_K$. Applying Lemma 2.9, to $p_2^*\mathcal{L}$ and $p_1^*\mathcal{L}^{\otimes d}$ we get the desired result. ■

Canonical Height

Theorem 2.13. *Let Y be a projective K-variety, L a very ample line bundle on Y and $\psi : Y \to Y$ a morphism satisfying $\psi^* L \cong L^{\otimes d}$, $d > 1$. Then there is a unique function*

$$h_\psi : Y(\bar{K}) \to \mathbb{R}$$

called the canonical height on Y, relative to L and ψ. It is defined as:

$$h_{\psi}(P) := \lim_{n\to\infty} \frac{h_L(\psi^n(P))}{d^n}$$

and satisfies the following properties:

1. $h_{\psi}(P) = h_L(P) + O(1)$ *for all* $P \in Y(\bar{K})$

2. $h_{\psi}(\psi(P)) = d \cdot h_{\psi}(P)$ *for all* $P \in Y(\bar{K})$

3. $h_{\psi}(P) \geq 0$ *for all* $P \in Y(\bar{K})$

Proof: [3] ■

To show that the limit exists we prove that the sequence $\left\{\frac{h_L(\psi^n(P))}{d^n}\right\}$ converges by verifying that it is a Cauchy sequence. The following computation is due to John Tate. Take $n > m$ and compute

$$\left|\frac{h_L(\psi^n(P))}{d^n} - \frac{h_L(\psi^m(P))}{d^m}\right| = \left|\sum_{i=m+1}^{n} \frac{1}{d^i}\Big\{h_L(\psi^i(P)) - d \cdot h_L(\psi^{i-1}(P))\Big\}\right| \tag{2.14}$$

$$\leq \sum_{i=m+1}^{n} \frac{1}{d^i}\Big|h_L(\psi^i(P)) - d \cdot h_L(\psi^{i-1}(P))\Big|$$

(by triangle inequality)

$$\leq \sum_{i=m+1}^{n} \frac{B}{d^i} \qquad \text{(by Theorem 2.12)}$$

$$= \Big(\frac{d^{-m} - d^{-n}}{d-1}\Big)B \tag{2.15}$$

The last term tends to 0 as $m, n \to \infty$, which proves that the sequence is Cauchy.

Definition 2.16. *We say a point $P \in Y$ is preperiodic for ψ if its orbit $\{P, \psi(P), \psi^2(P), \ldots\}$ is finite.*

Corollary 2.17. *Let Y be a projective K-variety, L a very ample line bundle on Y and $\psi : Y \to Y$ a morphism satisfying $\psi^* L \cong L^{\otimes d}$, $d > 1$. If $P \in Y$ is preperiodic, then $h_\psi(P) = 0$.*

Proof: Since P is preperiodic, its orbit under ψ is finite. Let

$$M = \max_n \{h_L(\psi^n(P))\}$$

Then

$$h_\psi(P) = \lim_{n\to\infty} \frac{h_L(\psi^n(P))}{d^n} \le \lim_{n\to\infty} \frac{M}{d^n} = 0$$

■

Proposition 2.18. *Let Y be a projective K-variety, L a very ample line bundle on Y and $\psi : Y \to Y$ a morphism satisfying $\psi^* L \cong L^{\otimes d}, d > 1$. Let $(\mathcal{Y}, \mathcal{L})$ be an ample model of (Y, L) over C. Then*

$$\left|h_\psi(P) - h_{\mathcal{L}}(P)\right| \le \frac{B}{d-1} \quad \textit{for all } P \in \mathcal{Y}(\bar{K})$$

Proof: In 2.14 taking $m = 0, n \to \infty$ gives us the desired inequality. ■

Observe that in contrast with the number field case the converse of Corollary 2.17 does not hold true [2]. If k is infinite, then for M sufficiently large the set $\{P \in Y(K) : h_\psi(P) \leq M\}$ is infinite. Indeed, if $P \in Y(k)$, then $h_{\mathcal{L}}(P) = 0$ and $h_\psi(P) \leq B$ for all $P \in Y(k)$ by Proposition 2.18. In particular, the set $\{P \in Y(K) : h_\psi(P) \leq B\}$ is infinite.

Corollary 2.19. *Let Y be a projective K-variety, L a very ample line bundle on Y and $\psi : Y \to Y$ a morphism satisfying $\psi^* L \cong L^{\otimes d}, d > 1$. Let $(\mathcal{Y}, \mathcal{L})$ be an ample model of (Y, L) over C. Then $h_\psi(P) = 0$ if and only if $|h_{\mathcal{L}}(\psi^n(P))|$ is uniformly bounded for every $n \geq 0$.*

Proof: ($\Leftarrow$) Assume $|h_{\mathcal{L}}(\psi^n(P))|$ is uniformly bounded by some constant (say M) for every $n \geq 0$. Then,

$$0 = \lim_{n\to\infty} \frac{-M}{d^n} \leq h_\psi(P) := \lim_{n\to\infty} \frac{h_{\mathcal{L}}(\psi^n(P))}{d^n} \leq \lim_{n\to\infty} \frac{M}{d^n} = 0$$

($\Rightarrow$) By proposition 2.18 for any $P \in \mathcal{Y}(\bar{K})$ we have

$$|h_\psi(P) - h_{\mathcal{L}}(P)| \leq \frac{B}{d-1}$$

For any $n \geq 1$, we have

$$|h_\psi(\psi^n(P)) - h_{\mathcal{L}}(\psi^n(P))| \leq \frac{B}{d-1}$$

By Thm 2.13, $h_\psi(\psi^n(P)) = d^n h_\psi(P)$. Since $h_\psi(P) = 0$, $|h_{\mathcal{L}}(\psi^n(P))|$ is uniformly bounded for every $n \geq 0$. ■

Chapter 3

Hilbert Scheme

Let $k, C, K, Y, L, \mathcal{Y}, \mathcal{L}, F(t)$ be defined as in chapter 2. Let $Hilb_{\mathcal{Y}/k}^{F(t)}$ be the Hilbert scheme of $\mathcal{Y}$ over k relative to $F(t)$. Let X be universal subscheme of $Hilb_{\mathcal{Y}/k}^{F(t)} \times_k \mathcal{Y}$. In this chapter we show that a subscheme X' of the universal family X is isomorphic to $H' \times_k C$ where H' is an open subscheme of $Hilb_{\mathcal{Y}/k}^{F(t)}$.

Definition 3.1. *Let S be a graded ring. A graded S-module is a S-module M, together with a decomposition $M = \oplus_{d \in \mathbb{Z}} M_d$, such that $S_d \cdot M_e \subseteq M_{d+e}$.*

Definition 3.2. *We define the Hilbert function ϕ_M of M as*

$$\phi_M(l) = \dim_k M_l$$

for each $l \in \mathbb{Z}$.

Theorem 3.3. *(Hilbert-Serre) Let $S = k[x_0, \ldots, x_n]$, M a finitely generated graded S-module. Then there is a unique polynomimal $P_M(z) \in \mathbb{Q}[z]$ such that $\phi_M(l) = P_M(l)$ for all $l \gg 0$.*

Proof: [8] ■

Definition 3.4. *The polynomial P_M of the previous theorem is called the Hilbert polynomial of M. If $Y \subset \mathbb{P}^n$ is an algebraic set of dimension r we define the Hilbert polynomial of Y to be the Hilbert polynomial P_Y of its homogeneous coordinate ring.*

Definition 3.5. *For any k-scheme T, let*

$$\underline{Hilb}^{P(t)}_{Y/k}(T) = \left\{ \begin{array}{c} \textit{flat families } X \subset Y \times_k T \textit{ of closed subschemes} \\ \textit{of } Y \textit{ parametrized by } T \textit{ with fibers} \\ \textit{having Hilbert polynomial } P(t) \end{array} \right\}$$

Since flatness is preserved under base change [8] this defines a functor:

$$\underline{Hilb}^{P(t)}_{Y/k} : (k - Schemes)^{\circ} \rightarrow (Sets)$$

called the Hilbert functor of Y over k relative to $P(t)$.

We recall the theorem connecting flat families and Hilbert polynomials.

Theorem 3.6. *Let T be an integral noetherian scheme. Let $X \subseteq \mathbb{P}^n_T$ be a closed subscheme. For each point $t \in T$, we consider the Hilbert polynomial $P_t \in \mathbb{Q}[z]$ of the fiber X_t considered as a closed subscheme of $\mathbb{P}^n_{k(t)}$. Then X is flat over T if and only if the Hilbert polynomial P_t is independent of t.*

Proof: [8] ■

Definition 3.7. *A numerical polynomial is a polynomial $P(z) \in \mathbb{Q}[z]$ such that $P(n) \in \mathbb{Z}$ for all $n \gg 0, n \in \mathbb{Z}$.*

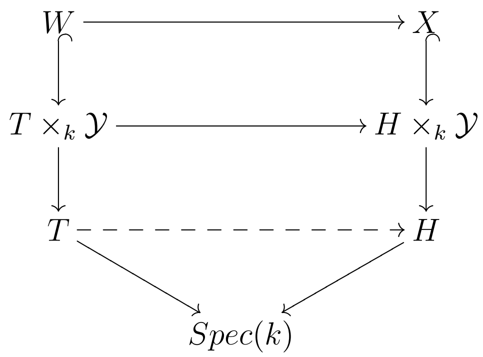
Theorem 3.8. *For every projective scheme $\mathcal{Y}$ and every numerical polynomial $P(t)$, the functor $\underline{Hilb}_{\mathcal{Y}/k}^{P(t)}$ is representable. The scheme representing $\underline{Hilb}_{\mathcal{Y}/k}^{P(t)}$ is a projective k-scheme and is called the Hilbert scheme of $\mathcal{Y}$ over k relative to $P(t)$, denoted by $Hilb_{\mathcal{Y}/k}^{P(t)}$.*

The concept of Hilbert scheme and its representability was introduced in [7]. For a detailed proof of the representability of the Hilbert scheme we refer to [13].

Universal Property of the Hilbert Scheme

$Hilb_{\mathcal{Y}/k}^{P(t)}$ has a universal element (say X) i.e. there is a flat family X of closed subschemes of $\mathcal{Y}$, parametrized by $Hilb_{\mathcal{Y}/k}^{P(t)}$, having Hilbert polynomial $P(t)$ with the following universal property:

for each scheme T and for each flat family $W \subset T \times_k \mathcal{Y}$ of closed subschemes of $\mathcal{Y}$ having the Hilbert polynomial $P(t)$ there is a unique morphism from T to $Hilb_{\mathcal{Y}/k}^{P(t)}$ such that $W = T \times_{Hilb_{\mathcal{Y}/k}^{P(t)}} X \subset T \times_k \mathcal{Y}$. We express the universal property in the following diagram (H denotes $Hilb_{\mathcal{Y}/k}^{P(t)}$):



Lemma 3.9. *Let $f : X \to Y$ be a finite morphism of smooth connected projective curves, then $g_X \geq g_Y$. Moreover if $g_X = g_Y \geq 2$, then $X \cong Y$.*

Proof:[8] Let R be the ramification divisor of f, $\deg(f) = n$ and $\deg(R) = r$. The Riemann-Hurwitz formula states:

$$2g_X - 2 = n(2g_Y - 2) + r$$

If $g_Y = 0$, then $g_X \geq g_Y$ since genus is always nonnegative. If $g_Y = 1$, then $g_X = 1 + r/2 \geq 1$ since $r \geq 0$. We rewrite the Riemann-Hurwitz formula as:

$$g_X = g_Y + (n-1)(g_Y - 1) + \frac{r}{2}$$

If $g_Y \geq 2$, it follows that $g_X \geq g_Y$ since $n \geq 1, r \geq 0$.

If $g_X = g_Y \geq 2$, then the Riemann-Hurwitz formula reduces to

$$(n-1)(g_Y - 1) + \frac{r}{2} = 0$$

Equality holds if and only if $n = 1$ and $r = 0$. In other words, $X \cong Y$. ■

Definition 3.10. *Let V be a discrete valuation ring with fraction field F and algebraically closed residue field k. We say a curve over $Spec(F)$ has semi-stable reduction if it is the general fiber of a regular curve W proper over $Spec(V)$, whose closed fiber is reduced and has only ordinary double points.*

Let $F(t)$ be the Hilbert polynomial of C relative to $s_P^*\mathcal{L}$ (cf.2.4) and denote the Hilbert scheme of $\mathcal{Y}$ over k relative to $F(t)$ by H.

Proposition 3.11. *Let V be a discrete valuation ring with fraction field F and $\theta : Spec(F) \to H$ a morphism. Let W_F be a curve in $\mathcal{Y}$ defined over F with Hilbert polynomial $F(t)$ and let W_V be the regular minimal model of W_F over V. Let X be the universal family in $H \times_k \mathcal{Y}$ and X_V the base change to $Spec(V) \times_k \mathcal{Y}$ i.e. the diagram below is a fiber square*

$$\begin{array}{ccc} X_V & \longrightarrow & Spec(V) \times_k \mathcal{Y} \\ \downarrow & & \downarrow \\ X & \longrightarrow & H \times_k \mathcal{Y} \end{array}$$

Then $X_V \cong W_V$.

<u>Remark</u>: It is well known (cf. [10], [11]) that the regular minimal model for two dimensional schemes exist.

Proof: Since H is projective, $Spec(F) \to H$ extends to $Spec(V) \to H$. Observe that $W_V \to Spec(V)$ is flat and W_V, X_V have the same Hilbert polynomial. Thus $X_V \cong W_V$. ■

To show that the universal family $X \subset H \times_k \mathcal{Y}$ has reduced fibers we need the following result from [1]:

Theorem 3.12. *Let W be a smooth proper cuve of genus g over the fraction field F of a discrete valuation ring V with algebraically closed residue field k. Then there is a finite field extension F' of F such that $W_{F'}$ has semi-stable reduction over the integral closure V' of V in F'.*

Proof: [1] ■

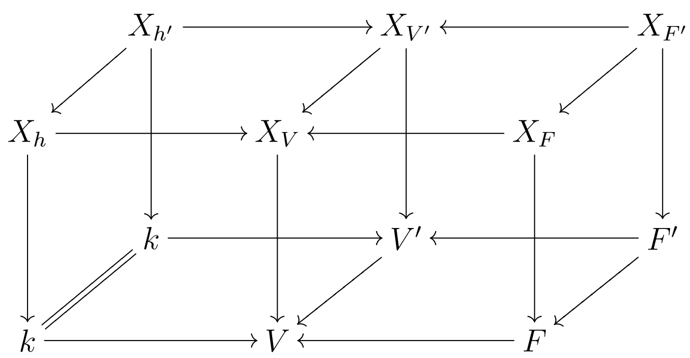
We denote the k-points of H by $H(k)$.

Theorem 3.13. *Let C be a smooth projective connected curve of genus at least 2, X the universal family in $H \times_k \mathcal{Y}$. Let $\phi := ((Id \times q) \circ i)$ be the composition of morphisms*

$$X \xrightarrow{i} H \times_k \mathcal{Y} \xrightarrow{Id \times q} H \times_k C$$

where q is the composition $\mathcal{Y} \overset{i}{\hookrightarrow} \mathbb{P}^n_C \overset{p_2}{\to} C$. For $h \in H(k)$, let X_h be the fiber over $\{h\} \times C$. Then X_h is reduced for every $h \in H(k)$.

Proof: There exists a generically smooth discrete valuation ring V with closed fiber X_h, fraction field F, algebraically closed residue field k and a morphism $Spec(V) \overset{\theta}{\to} H$. Indeed, using Bertini Theorem [8] we get a curve W in H passing through h. If required, normalize W and then localizing at a suitable prime $\mathfrak{p}$ we obtain a d.v.r. V with the required properties.



In the diagram above we omit the word Spec in the base. Let X_V^m denote the regular minimal model of X_V. By the previous theorem there exists a finite extension F' of F such that for some d.v.r. V' (in the integral closure of V in F'), $X_{V'}^m$ has semi-stable reduction. By Proposition 3.11 we have

$$X_{V'}^m \cong X_V \times_{Spec(V)} Spec(V')$$

and $X_{V'} := X_V \times_{Spec(V)} Spec(V')$ by definition, so $X_{V'} \cong X_{V'}^m$. Thus $X_{V'}$ has semi-stable reduction. In particular, $X_{h'}$ is reduced. By the base change property of the Hilbert scheme [13], pp. 223 we have

$$Hilb_{X_V/V}^{F(t)} \times_{Spec(V)} Spec(V') \cong Hilb_{X_{V'}/V'}^{F(t)}$$

Using the commutativity of the left face of the above diagram and observing that $X_{h'}$ and X_h have the same Hilbert polynomial we conclude that X_h is reduced. ■

Lemma 3.14. *Let X be the universal family in $H \times_k \mathcal{Y}$, $\phi : X \to H \times_k C$ as in the previous theorem and $p_1 : H \times_k C \to H$ the projection on the first factor. Define $H' = \{h \in H : X_h \to C$ is surjective$\}$. Then H' is an open subscheme of H.*

Proof: For any $(h, c) \in H \times_k C$, its preimage $\phi^{-1}(h, c)$ in X is the empty set or a finite set of points or a one dimensional subscheme of X. By the semicontinuity theorem ([8], pp. 288) we know that for $\phi : X \to H \times_k C$, $\dim_{k((h,c))} H^1(X_{h,c}, \mathcal{O}_{X_{h,c}})$ is an upper semicontinuous function on $H \times C$. Equivalently, by [8] (Remark 12.7.1, pp 288) the set

$$V = \{(h, c) \in H \times C : \dim_{k((h,c))} H^1(X_{h,c}, \mathcal{O}_{X_{h,c}}) \geq 1\}$$

is a closed subset of $H \times C$. Note that $H_0 := p_1(V)$ is closed in H. By definition, H' is the complement of H_0 in H, hence it is open and inherits the subscheme structure from H. ■

Theorem 3.15. *Let C be a smooth projective connected curve of genus at least 2, X the universal family in $H \times_k \mathcal{Y}$. Let $\phi : X \to H \times_k C$ be as in Theorem 3.13, H' be the open subscheme of H defined in the previous lemma and $X' = X \times_{(H \times_k C)} (H' \times_k C)$. Then $X'_h \to C$ is an isomorphism for every $h \in H'(k)$. Moreover $X' \cong H' \times_k C$.*

Proof: X' is the family of curves in $\mathcal{Y}$ with Hilbert polynomial $F(t)$. Since X'_h and C have the same Hilbert polynomial, $p_a(X_h) = p_a(C)$ (say g). By the previous theorem we know that X'_h is reduced for every $h \in H'(k)$. We now show that it is irreducible and isomorphic to C. We split the proof into two cases:

Case 1: X'_h is connected

Case 2: X'_h is not connected

Proof of Case 1: Let $\tilde{X}'_h$ be the normalization of X'_h, $p_a(\tilde{X}'_h) = \tilde{g}$. By Lemma 3.9 applied to $\tilde{X}'_h \to C$ we have $\tilde{g} \geq g$. Since $\tilde{X}'_h \xrightarrow{\alpha} X'_h$ is birational, we have the short exact sequence of sheaves on X'_h

$$0 \to \mathcal{O}_{X'_h} \xrightarrow{\alpha^{\#}} \alpha_* \mathcal{O}_{\tilde{X}'_h} \to N \to 0 \tag{3.16}$$

where $N = Coker(\alpha^{\#})$, which induces the long exact sequence on cohomology:

$$0 \to Ker(\beta) \to H^1(X'_h, \mathcal{O}_{X'_h}) \xrightarrow{\beta} H^1(X'_h, \alpha_* \mathcal{O}_{\tilde{X}'_h}) \to H^1(X'_h, N) \to 0 \tag{3.17}$$

Since $\dim[Supp(N)] = 0$, $H^1(X'_h, N) = 0$. $p_a(\tilde{X}'_h) = \dim_k H^1(\tilde{X}'_h, \mathcal{O}_{\tilde{X}'_h})$ by definition. Since α is finite it is an affine morphism. Therefore,

$$H^1(\tilde{X}'_h, \mathcal{O}_{\tilde{X}'_h}) \cong H^1(X'_h, \alpha_* \mathcal{O}_{\tilde{X}'_h})$$

Computing the dimension of the cohomology groups (as k-vector spaces) we get $\tilde{g} \leq g$. So $g = \tilde{g}$. By Lemma 3.9, $\tilde{X}'_h \cong C$. The isomorphism $\tilde{X}'_h \xrightarrow{\sim} C$ factors through X'_h so by verifying at the local rings we get $X'_h \cong C$.

Proof of Case 2: Assume X'_h has two components, say $X'_h = X_1 \cup X_2$. Let $\tilde{X}'_h$ be the normalization of X'_h. Then $\tilde{X}'_h = \tilde{X}_1 \amalg \tilde{X}_2$, the disjoint union of $\tilde{X}_1$ and $\tilde{X}_2$. Note that $\tilde{X}_1 \cap \tilde{X}_2 = \emptyset$. Let $p_a(\tilde{X}'_h) = \tilde{g}, p_a(\tilde{X}_1) = \tilde{g}_1, p_a(\tilde{X}_2) = \tilde{g}_2$. Applying Lemma 3.9 to $\tilde{X}'_h \xrightarrow{f} C$ we have $\tilde{g} \geq g$. Restricting f to $\tilde{X}_1$ and $\tilde{X}_2$, we get $\tilde{g}_1 \geq g$ and $\tilde{g}_2 \geq g$ respectively. As before the sequences (3.16) and (3.17) yield,

$$\dim_k H^1(X'_h, \mathcal{O}_{X'_h}) \geq \dim_k H^1(X'_h, \alpha_* \mathcal{O}_{\tilde{X}'_h})$$

Moreover α being finite it is an affine morphism so

$$H^1(X'_h, \alpha_* \mathcal{O}_{\tilde{X}'_h}) \cong H^1(\tilde{X}'_h, \mathcal{O}_{\tilde{X}'_h})$$

and

$$H^1(\tilde{X}'_h, \mathcal{O}_{\tilde{X}'_h}) \cong H^1(\tilde{X}'_h, \mathcal{O}_{\tilde{X}_1}) \oplus H^1(\tilde{X}'_h, \mathcal{O}_{\tilde{X}_2})$$

Computing the dimensions of the cohomology groups (as k-vector spaces) we have $g \geq \tilde{g}_1 + \tilde{g}_2$. From the inequalities, $\tilde{g}_1 \geq g, \tilde{g}_2 \geq g$ and $g \geq \tilde{g}_1 + \tilde{g}_2$ we conclude that either $\tilde{g}_1 = 0$ or $\tilde{g}_2 = 0$ i.e. $\tilde{X}_1 \cong \mathbb{P}^1$ or $\tilde{X}_2 \cong \mathbb{P}^1$. This cannot be the case, since there does not exist a finite surjective morphism $\mathbb{P}^1 \to C$ when $g_C \geq 2$. Thus we conclude that $\tilde{X}'_h$ has one component. Hence

X'_h has one component. A similar argument works if X'_h has more than two components. Thus we are reduced to Case 1 where X'_h has one component and is isomorphic to C for every $h \in H'(k)$. So $X'(k) \cong (H' \times_k C)(k)$. Since k is algebraically closed, we have $X' \cong H' \times_k C$.

Chapter 4

Isotriviality

Definition 4.1. *Let S be a variety defined over k and $\pi : Y \to S$ a family of varieties. We say π is trivial if $Y \cong Y_0 \times_k S$ for some k-variety Y_0. π is called isotrivial if there exists a finite extension $S' \to S$ such that $Y \cong Y_0 \times_k S'$.*

In this chapter we show that if a projective integral C-scheme X is covered by a trivial projective integral k-scheme W and $f : W \to X$ is a morphism satisfying $f_*\mathcal{O}_W \cong \mathcal{O}_X$, then $X_K \to Spec(K)$ (the generic fiber of $X \to C$) is isotrivial. To achieve the conclusion we need to study the deformations of X which are controlled by the sequence of differentials. We proceed to show that the sequence of differentials on X is split exact.

Theorem 4.2. *Let C be a smooth connected projective curve over k, K the function field of C. Let W_0 be a projective integral k-scheme and $W = W_0 \times_k C$ so that $\pi : W \to C$ is a trivial family and X a projective integral*

C-scheme. Let U be a nonempty open subset of C such that X is flat over U. For $Q \in U$, let R be the local ring at Q and $f : W \to X$ a morphism defined over C such that $f_*\mathcal{O}_W \cong \mathcal{O}_X$. Consider the diagram

$$W \xrightarrow{f} X \xrightarrow{p} Spec(R) \to Spec(k) \tag{4.3}$$

Then the associated sequence of differentials

$$p^*\Omega_{R/k} \xrightarrow{\phi} \Omega_{X/k} \to \Omega_{X/R} \to 0 \tag{4.4}$$

is split exact.

Remark: We write R instead of $Spec(R)$ in the subscript of the sheaves.

Proof: Since W is trivial over C (hence also over R) the sequence

$$0 \to f^*p^*\Omega_{R/k} \to \Omega_{W/k} \to \Omega_{W/R} \to 0$$

is split exact. Pulling back the sequence of differentials on X to W along f we get the exact sequence

$$f^*p^*\Omega_{R/k} \xrightarrow{\theta} f^*\Omega_{X/k} \to f^*\Omega_{X/R} \to 0 \tag{4.5}$$

$W \xrightarrow{f} X$ induces morphisms between the above two exact sequences, yielding the following commutative diagram:

$$\begin{array}{ccccccccc} & & f^*p^*\Omega_{R/k} & \xrightarrow{\theta} & f^*\Omega_{X/k} & \longrightarrow & f^*\Omega_{X/R} & \longrightarrow & 0 \\ & & \downarrow \rho & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & f^*p^*\Omega_{R/k} & \longrightarrow & \Omega_{W/k} & \longrightarrow & \Omega_{W/R} & \longrightarrow & 0 \end{array} \tag{4.6}$$

We claim that $\rho : f^*p^*\Omega_{R/k} \to f^*p^*\Omega_{R/k}$ is the identity map. Since differentials contain local information it suffices to check the commutativity locally. Let $S \subset W, T \subset X$ be open affine subsets and $A = \Gamma(S, \mathcal{O}_W)$ and $B = \Gamma(T, \mathcal{O}_X)$.

From the k-algebra homomorphisms $k \to \mathcal{O}_R \to A$ and $k \to \mathcal{O}_R \to B$ we get a A-module homomorphism $\alpha : \Omega_{\mathcal{O}_R/k} \otimes_{\mathcal{O}_R} A \to \Omega_{A/k}$ and a B-module homomorphism $\beta : \Omega_{\mathcal{O}_R/k} \otimes_{\mathcal{O}_R} B \to \Omega_{B/k}$. Commutativity of the square

$$\begin{array}{ccc} f^*p^*\Omega_{R/k} & \xrightarrow{\theta} & f^*\Omega_{X/k} \\ \downarrow{\scriptstyle\rho} & & \downarrow \\ f^*p^*\Omega_{R/k} & \longrightarrow & \Omega_{W/k} \end{array}$$

is equivalent to commutativity of the square (of A-modules) below

$$\begin{array}{ccc} \Omega_{\mathcal{O}_R/k} \otimes_{\mathcal{O}_R} B \otimes_B A & \longrightarrow & \Omega_{B/k} \otimes_B A \\ \downarrow{\scriptstyle\rho} & & \downarrow \\ \Omega_{\mathcal{O}_R/k} \otimes_{\mathcal{O}_R} B \otimes_B A & \longrightarrow & \Omega_{A/k} \end{array}$$

which in turn is equivalent to the commutativity of the diagram below (as $\mathcal{O}_R$-modules).

$$\begin{array}{ccc} \Omega_{\mathcal{O}_R/k} & \longrightarrow & \Omega_{B/k} \\ & \searrow & \downarrow \\ & & \Omega_{A/k} \end{array}$$

Commutativity of the triangle is clear. Thus ρ is the identity map and we obtain a section of θ by going down, across (to the left) and up (along ρ).

Hence

$$f^*\Omega_{X/k} \cong f^*\Omega_{X/R} \oplus f^*p^*\Omega_{R/k}$$

Since the isomorphism is preserved under the push forward f_* we have

$$f_*f^*\Omega_{X/k} \cong f_*f^*\Omega_{X/R} \oplus f_*f^*p^*\Omega_{R/k}$$

We have the natural map $\Omega_{X/k} \to f_*f^*\Omega_{X/k}$ inducing the following diagram:

$$\begin{array}{ccccccccc} & & p^*\Omega_{R/k} & \xrightarrow{\phi} & \Omega_{X/k} & \longrightarrow & \Omega_{X/R} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f^\#} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & f_*f^*p^*\Omega_{R/k} & \longrightarrow & f_*f^*\Omega_{X/k} & \longrightarrow & f_*f^*\Omega_{X/R} & \longrightarrow & 0 \end{array}$$

Observe that $p^*\Omega_{R/k} \cong \mathcal{O}_X$ and $f_*f^*p^*\Omega_{R/k} \cong f_*\mathcal{O}_W$. Since $f^\#$ is an isomorphism by assumption, we obtain a section of ϕ by going down, across to the left and up via $f^\#$. Thus (4.4) is split exact. ■

We now consider an infinitesimal deformation of X over a henselian discrete valuation ring and proceed to show that $X_K \to Spec(K)$ is isotrivial. Before we proceed we need the following definitions:

Definition 4.7. *We say a local ring A is henselian if every finite A-algebra B is a product of local rings.*

Definition 4.8. *Let A be a local ring. We define the henselization of A to be a pair $(\tilde{A}, i)$, where $\tilde{A}$ is a local henselian ring and $i : A \to \tilde{A}$ is* *a local homomorphism such that: for any local henselian ring B and any local homomorphism $u : A \to B$ there exists a unique local homomorphism $\tilde{u} : \tilde{A} \to B$ such that $u = \tilde{u} \circ i$. We have the following commutative diagram:*

$$\begin{array}{ccc} A & \xrightarrow{i} & \tilde{A} \\ & {\scriptstyle u}\searrow & \downarrow{\scriptstyle \tilde{u}} \\ & & B \end{array}$$

Let $R = \mathcal{O}_{U,Q}$, the local ring at $Q, \mathfrak{m}$ its maximal ideal and let $\tilde{R}$ denote the henselization of R. Define $\tilde{R}_n := \tilde{R}/\tilde{\mathfrak{m}}^{n+1} = R/\mathfrak{m}^{n+1}$ for each $n \geq 0$. We have the natural maps $Spec(\tilde{R}) \to Spec(R)$ and $Spec(\tilde{R}_n) \to Spec(\tilde{R}_{n-1})$ for each $n \geq 1$ induced by the projections $\tilde{R}_n \to \tilde{R}_{n-1}$. Defining $\tilde{X} := X \times_R Spec(\tilde{R})$, $\tilde{X}_n := \tilde{X} \times_R Spec(\tilde{R}_n)$, for each $n \geq 0$ yields the following commutative diagram (we omit the word Spec in the bottom row):

$$\begin{array}{ccccccccc} X_K & \longrightarrow & X & \longleftarrow & \tilde{X} & \longleftarrow \cdots \longleftarrow & \tilde{X}_n & \longleftarrow \cdots \longleftarrow & \tilde{X}_0 \\ \downarrow & & \downarrow{\scriptstyle p} & & \downarrow & & \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle p_0} \\ K & \longrightarrow & R & \longleftarrow & \tilde{R} & \longleftarrow \cdots \longleftarrow & \tilde{R}_n & \longleftarrow \cdots \longleftarrow & k \end{array} \tag{4.9}$$

Lemma 4.10. *In the preceding diagram we have, $\tilde{X}_n \cong \tilde{X}_0 \times_k Spec(\tilde{R}_n)$, for every $n \geq 0$.*

Proof: To prove we do induction on n. The case $n = 0$ is clear. Suppose it holds true for $n \leq m - 1$. Then pulling back (4.4) along the natural map $Spec(\tilde{R}_m) \to Spec(R)$ and using the characterization of differentials of a closed embedding, we see that the sequence:

$$0 \to p_m^* \Omega_{\tilde{R}_m/k} \to \Omega_{\tilde{X}_m/k} \to \Omega_{\tilde{X}_m/\tilde{R}_m} \to 0 \tag{4.11}$$

is split exact. By Theorem A.3 (Appendix A), we see that

$$\tilde{X}_m \cong \tilde{X}_{m-1} \times_{\tilde{R}_{m-1}} Spec(\tilde{R}_m) \cong \tilde{X}_0 \times_k Spec(\tilde{R}_m)$$

Thus, $\tilde{X}_n \cong \tilde{X}_0 \times_k Spec(\tilde{R}_n)$ for each $n \geq 0$. ■

Definition 4.12. *If X, Y and T are S-schemes, an S-isomorphism from X to Y parametrized by T will mean a T-isomorphism from $X \times_S T \to Y \times_S T$. The set of all such isomorphisms will be denoted by $\underline{Isom}_S(X,Y)(T)$.*

The association $T \mapsto \underline{Isom}_S(X,Y)(T)$ defines a contravariant functor

$$\underline{Isom}_S(X,Y) : (S - schemes)^\circ \to (Sets)$$

The functor $\underline{Isom}_S(X,Y)$ is representable whenever X, Y are flat and projective over S. We present a proof of this fact in Appendix B. Denote the scheme representing the functor $\underline{Isom}_S(X,Y)$ by $Isom_S(X,Y)$. To conclude that $X_K \to Spec(K)$ is isotrivial we need the following result of Greenberg:

Theorem 4.13. *Let $\tilde{R}$ be a henselian discrete valuation ring, with t the generator of the maximal ideal. Let $\tilde{Z}$ be a scheme of finite type over $\tilde{R}$. Then $\tilde{Z}$ has a point in $\tilde{R}$ if and only if $\tilde{Z}$ has a point in $\tilde{R}/t^n$ for every $n \geq 1$.*

Proof: [6]

Theorem 4.14. *With the hypothesis and assumptions as in Proposition 4.3 and* $\tilde{X}_0$ *as in diagram 4.10, we have* $X_K \to Spec(K)$ *is isotrivial i.e.*

$$X_K \times_{Spec(K)} Spec(K') \cong \tilde{X}_0 \times_k Spec(K')$$

where K' *is some finite extension of* K.

Proof: Observe that $\tilde{X}$ and $\tilde{X}_0 \times_k \tilde{R}$ are flat, projective over $\tilde{R}$. Let $\underline{Isom}_{\tilde{R}}(\tilde{X}, \tilde{X}_0 \times_k \tilde{R})(T)$ be the set of isomorphisms from

$$\tilde{X} \times_{\tilde{R}} T \to (\tilde{X}_0 \times_k \tilde{R}) \times_{\tilde{R}} T$$

Let $\tilde{Z} = Isom_{\tilde{R}}(\tilde{X}, \tilde{X}_0 \times_k \tilde{R})$ be the scheme representing the functor $\underline{Isom}_{\tilde{R}}(\tilde{X}, \tilde{X}_0 \times_k \tilde{R})$. Note that $\tilde{Z}$ is of finite type over $\tilde{R}$ and $\tilde{Z} = Z \times_R \tilde{R}$, where $Z = Isom_R(X, \tilde{X}_0)$. By Lemma 4.10 $\tilde{X}_n \cong \tilde{X}_0 \times_{Spec(k)} Spec(\tilde{R}_n)$ for each $n \geq 0$, thus $\tilde{Z}$ has a $\tilde{R}_n$-point for every $n \geq 0$. Applying Theorem 4.13 to $\tilde{Z}$ we see that $\tilde{Z}$ has a $\tilde{R}$-point i.e. $\tilde{X} \cong \tilde{X}_0 \times_k Spec(\tilde{R})$. Note that $\tilde{R} = \varinjlim R_\alpha$ and $\tilde{Z}(\tilde{R}) = \varprojlim Z(R_\alpha)$. Hence, there exists an etale cover R' of R such that $X_{R'} \cong \tilde{X}_0 \times_k R'$, thus K' (the quotient field of R') satisfies the requirements of the theorem. ■

Using model theoretic techniques Chatzidakis-Hrushovski [4] have recently obtained a result similar to Theorem 4.14. The result states:

Theorem 4.15. *[4] Let* $K_1 \subset K_2$ *be fields, with* K_2/K_1 *regular and let* (V_2, ϕ_2) *be a dynamical system defined over* K_2*. Assume that* (V_2, ϕ_2) *is primitive and* $\deg(\phi_2) > 1$*. Assume furthermore that for some* $n \geq 1$, (V_2, ϕ_2^n) *is dominated by a dynamical system* (V_1, ϕ_1) *defined over* K_1*.*

1. *There is some variety* V_3 *defined over* K_1*, a dominant constructible map* $\phi_3 : V_3 \to V_3$ *also defined over* K_1*, and a constructible isomorphism* $h : (V_2, \phi_2) \to (V_3, \phi_3)$*.*

2. *Assume that the characteristic is* 0 *or* K_1 *is perfect, and* $\dim(V_2) = 1$*. Then* (V_2, ϕ_2) *is rationally isotrivial i.e. there is some dynamical system* (V_3, ϕ_3) *defined over* K_1 *which is isomorphic to* (V_2, ϕ_2)*.*

Chapter 5

Applications

Let k, C, K be as in Chapter 2. We assume that $g_C \geq 2$. Let Y be a projective K-variety with a very ample line bundle L, $\psi : Y \to Y$ a morphism satisfying $\psi^* L \cong L^{\otimes d}$, $d > 1$. As in chapter 2, let $\mathcal{Y}$ be the Zariski closure of Y in $\mathbb{P}^N_C$ and for $P_0 \in C$, $\mathcal{L} := \mathcal{O}_{\mathbb{P}^N}(1)|_{\mathcal{Y}} \otimes p_2^* \mathcal{O}_C(P_0)|_{\mathcal{Y}}$.

Theorem 4.14 provides us the setup to state a conjecture on parametrization of points of canonical height zero of (Y, L, ψ), due to L. Szpiro.

Let $F(t)$ be the Hilbert polynomial of C relative to $s_P^* \mathcal{L}$ (cf. 2.4). Following the notation in chapter 3, we denote $Hilb^{F(t)}_{\mathcal{Y}/k}$, the Hilbert scheme of $\mathcal{Y}$ over k relative to $F(t)$ by H, let X be the universal element in $H \times_k \mathcal{Y}$, H' the open subscheme of H as defined in Lemma 3.14 and $X' = X \times_{H \times_k C} (H' \times_k C)$. From Theorem 3.15 we know $X' \overset{\theta}{\cong} H' \times_k C$. Let f be the composition $X' \xrightarrow{i} H' \times_k \mathcal{Y} \xrightarrow{p_2} \mathcal{Y}$. Using the isomorphism θ we get the following commutative diagram

$$\begin{CD}
H'_K @>>> H' \times_k C \\
@V{f_K}VV @VV{f}V \\
Y @>>> \mathcal{Y} \\
@VVV @VV{q}V \\
Spec(K) @>>> C
\end{CD}$$

where $H'_K = H' \times_{Spec(k)} Spec(K)$. Let $Z := Image(f_K) \subset Y$. Denote by $H(k)$ the k-points of H and let T be the set of points of canonical height zero of (Y, L, ψ).

Since ψ acts on Z the intersection $S := \cap_{n=0}^{\infty}(\psi^{-n}(Z))$ is a well defined subscheme of Y. Note that the set of points of S is equivalent to the set of points whose height (w.r.t. $\mathcal{L}$) remains uniformly bounded under any forward iterate of ψ, which in light of corollary 2.19 is equivalent to the set of points of Y with canonical height zero (w.r.t. ψ).

Define $W := H'(k) \cap S$ and assume that W is a k-scheme. Then, with the notation above, we have the following diagram:

$$\begin{array}{ccc}
W \times_k C & \hookrightarrow & H' \times_k C \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
T & \hookrightarrow & \mathcal{Y}
\end{array}$$

where $g = f|_{W \times C}$. We need a few definitions to state the conjecture.

Definition 5.1. *Let Y be a isotrivial variety defined over K i.e. for some*

k-variety Y_0, $Y_{K'} \overset{\theta}{\cong} Y_0 \times_k Spec(K')$. We say $P \in Y(K)$ is a constant point if $P \in \theta(Y_0(k)) \cap \theta(Y_0 \times_k Spec(K')(K'))$.

Definition 5.2. *Let X be a projective variety and $\psi : X \to X$ an endomorphism of X. We say the subvariety $Y \subset X$ is preperiodic for ψ if for some integers $m > n \geq 1$, $\psi^m(Y) = \psi^n(Y)$.*

Conjecture (L. Szpiro): Let k be an algebraically closed field of characteristic zero, C a smooth projective connected curve over k with function field K. Let Y be a projective K-variety, L a very ample line bundle on Y and $\psi : Y \to Y$ a morphism such that $\psi^* L \cong L^{\otimes d}$ for some $d > 1$. Then the points of canonical height zero of the dynamical system (Y, L, ψ) is a finite union of the constant points of closed irreducible preperiodic isotrivial subvarieties X_i, for $1 \leq i \leq n$.

Applying Stein factorization [8] to $W \times C \xrightarrow{g} T$ factors g as $W \times C \xrightarrow{\alpha} T' \xrightarrow{\beta} T$ where α satisfies $\alpha_* \mathcal{O}_{W \times C} \cong \mathcal{O}_{T'}$ and β is a finite morphism from T' to T. The condition $\alpha_* \mathcal{O}_{W \times C} \cong \mathcal{O}_{T'}$ is required in Theorem 4.14. Applying Theorem 4.14 to $W \times C \xrightarrow{\alpha} T'$ we have the existence of the closed irreducible isotrivial subvarieties stated in the conjecture up to a finite cover of the points of canonical height zero.

Appendix A

Extensions of Schemes

Let $X \to S$ be a morphism of schemes. The conormal extensions control the square zero deformations of the scheme. We show that if the extensions are trivial then there are no non-trivial deformations.

Definition A.1. *An extension of X/S is a closed immersion $X \subset X'$, where X' is a S-scheme, defined by a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ such that $\mathcal{I}^2 = 0$.*

To give an extension $X \subset X'$ of X/S is equivalent to giving an exact sequence on X:

$$\mathcal{E} : 0 \to \mathcal{I} \to \mathcal{O}_{X'} \xrightarrow{\phi} \mathcal{O}_X \to 0$$

where $\mathcal{I}$ is an $\mathcal{O}_X$-module, ϕ is a homomorphism of $\mathcal{O}_S$-algebras and $\mathcal{I}^2 = 0$ in $\mathcal{O}_{X'}$. We call $\mathcal{E}$ an extension of X/S with kernel $\mathcal{I}$.

Definition A.2. *Two extensions $\mathcal{O}_{X'}$ and $\mathcal{O}_{X''}$ are called isomorphic if there is an $\mathcal{O}_S$-homomorphism $\alpha : \mathcal{O}_{X'} \to \mathcal{O}_{X''}$ inducing the identity both on $\mathcal{I}$ and*

$\mathcal{O}_X$.

It follows that α must be an isomorphism. We denote by $Ex(X/S, \mathcal{I})$ the set of isomorphism classes of extensions of X/S with kernel $\mathcal{I}$.

Definition A.3. *Let $X' \to S$ be a morphism of schemes. For $X \subset X'$, $V(\mathcal{I}) = X$, we define the relative conormal sequence of $X \subset X'$ to be:*

$$\mathcal{I}/\mathcal{I}^2 \to (\Omega_{X'/S})|_X \to \Omega_{X/S} \to 0$$

Since isomorphic extensions have isomorphic relative cotangent sequences we have a well defined map:

$$c_{\_} : Ex(X/S, \mathcal{I}) \to Ext^1_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{I})$$

Let

$$\xi : 0 \to \mathcal{I} \to \mathcal{A} \xrightarrow{p} \Omega_{X/S} \to 0$$

define an element of $Ext^1_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{I})$. Letting $d : \mathcal{O}_X \to \Omega_{X/S}$ be the canonical derivation, consider the sheaf of $\mathcal{O}_S$-algebras $\mathcal{B} = \mathcal{A} \times_{\Omega_{X/S}} \mathcal{O}_X$: for any open set $U \subset X$ we have $\Gamma(U, \mathcal{B}) = \{(a, f) : p(a) = d(f)\}$ and the multiplication rule is

$$(a, f)(a', f') = (af' + a'f, ff')$$

Then we have an exact commutative diagram

$$\begin{array}{ccccccccc} e_\xi : 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \bar{d}} & & \downarrow{\scriptstyle d} & & \\ \xi : 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{A} & \longrightarrow & \Omega_{X/S} & \longrightarrow & 0 \end{array}$$

where $\bar{d} : \mathcal{B} \to \mathcal{A}$ is the projection. This gives us a well defined map:

$$e_{\_} : Ext^1_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{I}) \to Ex(X/S, \mathcal{I})$$

Theorem A.4. *Let $X \to S$ be a morphism of finite type of algebraic schemes. Given an extension η of X/S with kernel $\mathcal{O}_X$ such that the isomorphism class of the relative conormal sequence of $X \subset X'$ is trivial (i.e. the conormal sequence is split exact), then the extension η is trivial.*

Proof: Let

$$\eta : 0 \to \mathcal{O}_X \cdot \epsilon \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

be an extension of X/S with kernel $\mathcal{O}_X$ such that $[c_\eta] \in Ext^1_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{O}_X)$ is trivial i.e.

$$c_\eta : 0 \to \mathcal{O}_X \to \Omega_{X'/S}|_X \to \Omega_{X/S} \to 0$$

is split exact. In other words, $\Omega_{X'/S}|_X \cong \mathcal{O}_X \oplus \Omega_{X/S}$. Set $\mathcal{A} := \Omega_{X'/S}|_X$. Letting $d : \mathcal{O}_X \to \Omega_{X/S}$ be the canonical derivation, consider the sheaf of $\mathcal{O}_S$-algebras $\mathcal{B} = \mathcal{A} \times_{\Omega_{X/S}} \mathcal{O}_X$ as above. Note that

$$\mathcal{B} = \mathcal{A} \times_{\Omega_{X/S}} \mathcal{O}_X \cong (\mathcal{O}_X \oplus \Omega_{X/S}) \times_{\Omega_{X/S}} \mathcal{O}_X \cong \mathcal{O}_X \oplus \mathcal{O}_X$$

Then we have an exact commutative diagram:

$$\begin{array}{ccccccccc}
\eta : 0 & \longrightarrow & \mathcal{O}_X \cdot \epsilon & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \alpha} & & {\scriptstyle d'} \swarrow \ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \beta} & & \\
e_{c_\eta} : 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
& & \| & & \downarrow{\scriptstyle \bar{d}} & & \downarrow{\scriptstyle d} & & \\
c_\eta : 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{A} & \longrightarrow & \Omega_{X/S} & \longrightarrow & 0
\end{array}$$

where $\bar{d} : \mathcal{B} \to \mathcal{A}$ is the projection and $\mathcal{O}_{X'} \xrightarrow{d'} \mathcal{A}$ is the canonical derivation restricted to X. d' along with the natural map $\mathcal{O}_{X'} \to \mathcal{O}_X$ gives us a map from $\mathcal{O}_{X'} \to \Omega_{X'/S} \times_{\Omega_{X/S}} \mathcal{O}_X$. Restricting $\Omega_{X'/S}$ to X and by the universality of fiber product we obtain $f : \mathcal{O}_{X'} \to \mathcal{B}$. Note that the bottom two rows are split exact. Since α and β are identity, it follows that f is an isomorphism. Thus $\eta \in Ex(X/S, \mathcal{O}_X)$ is trivial. ■

Appendix B

Scheme of Isomorphisms

We present a proof of the representability of the $\underline{Isom}_S(X,Y)$ functor.

Definition B.1. *If X and Y are schemes over S, then for any S-scheme T, a S-morphism from X to Y parametrized by T will mean a T- morphism from $X \times_S T \to Y \times_S T$. The set of all such morphisms will be denoted by $\underline{Mor}_S(X,Y)$. The association $T \to \underline{Mor}_S(X,Y)(T)$ defines a contravariant functor:*

$$\underline{Mor}_S(X,Y) : (S - Schemes)^\circ \to (Sets)$$

Similarly, for any S-scheme T, a S-isomorphism from X to Y parametrized by T will mean a T-isomorphism from $X \times_S T \to Y \times_S T$. The set of all such isomorphisms will be denoted by $\underline{Isom}_S(X,Y)$. The association $T \to \underline{Isom}_S(X,Y)(T)$ defines a contravariant functor:

$$\underline{Isom}_S(X,Y) : (S - Schemes)^\circ \to (Sets)$$

Lemma B.2. *Let S be a noetherian scheme, $f : X \to S$ and $g : Y \to S$ be proper flat morphisms. Let $\pi : Y \to X$ be any projective morphism with $g = f \circ \pi$. Then S has open subschemes $S_2 \subset S_1 \subset S$ with the following universal property:*

1. *For any locally noetherian S-scheme T, the base change $\pi_T : Y_T \to X_T$ is a flat morphism if and only if the structure morphism $T \to S$ factors via S_1.*

2. *For any locally noetherian S-scheme T, the base change $\pi_T : Y_T \to X_T$ is an isomorphism if and only if the structure morphism $T \to S$ factors via S_2.*

Proof: [5], pp. 132.

Proposition B.3. *Let S be a notherian scheme, X a projective scheme over S and Y a quasi-projective scheme S. Assume that X is flat over S. Then the functor $\underline{Mor}_S(X, Y)$ is representable by an open subscheme $Mor_S(X, Y)$ of $Hilb_{X \times_S Y/S}$.*

Proof: [5], pp.133

Theorem B.4. *Let S be a notherian scheme, X, Y flat, projective schemes over S. Then the functor $\underline{Isom}_S(X, Y)$ is representable by an open subscheme $Isom_S(X, Y)$ of $Mor_S(X, Y)$.*

Proof: This follows from Lemma A.2 ■

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