Stochastic Completeness of Graphs

by

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A dissertation submitted to the Graduate Faculty in Mathematics in partial
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Abstract

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by

Radosław Krzysztof Wojciechowski

Advisor: Professor Józef Dodziuk

We analyze the stochastic completeness of the heat kernel on infinite, locally finite, connected graphs. For general graphs, a sufficient condition for stochastic completeness is given in terms of the maximum valence on spheres about a fixed vertex. That this result is optimal follows by studying a specific family of trees. We also prove a lower bound for the bottom of the spectrum of the discrete Laplacian and use this lower bound to show that in certain cases the Laplacian has empty essential spectrum.

For my parents

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To my family and friends, I hope that this finds you well. Your support has always meant the most to me and I offer you this work with the highest respect. And to Raha, for putting up with all the meowing.

R.K.W.

April of 2008

New York, NY

Contents

Chapter 1

Introduction

1.1 Introduction and Statement of Results

The purpose of this thesis is to analyze a diffusion process on infinite graphs which is analogous to the flow of heat on open Riemannian manifolds. In particular, we are interested in the stochastic completeness of this process and a precise borderline for when the stochastic completeness breaks down. Stochastic completeness can be formulated in several equivalent ways: as a property of the heat kernel; as the uniqueness of bounded solutions of the heat equation; or as the non-existence of bounded, positive, λ-harmonic (or λ-subharmonic) functions for a negative constant λ. In studying this property we have benefited tremendously from the survey article of Grigor'yan [10] which discusses in great depth stochastic completeness in the case of Riemannian manifolds. For graphs, the starting point for our work is the paper of Dodziuk and Mathai [7] where it is shown that a graph whose valence is uniformly bounded above by a constant is stochastically complete.

In the first part of the thesis we start by giving a construction of the heat kernel on a general graph via an exhaustion argument. This is analogous to the construction on open Riemannian manifolds and we follow the presentation in [2]. One can also construct the heat kernel by utilizing the spectral theorem and the two constructions result in the same kernel [2]. We next introduce the notion of stochastic completeness and recall a proof of the equivalence of the various formulations mentioned above. This material is adapted from [10]. We then turn our focus to a class of trees which we call *model* because their definition is analogous to that of rotationally symmetric or model manifolds. The defining property of these trees is that they contain a vertex x_0, which we call the *root* for the model, such that the valence at every other vertex depends only on the distance from x_0. Let $m(r)$ denote this common number where r denotes the distance from x_0. The main result of this section says that such trees will be stochastically complete if and only if $\sum_{r=0}^{\infty} \frac{1}{m(r)} = \infty$. We note here the similarity between this criterion and the one for the recurrence of the Brownian motion on a geodesically complete model surface [12, 10].

We then consider general trees and prove that, in certain cases, if a tree contains a stochastically incomplete model subtree then it must be stochastically incomplete. We first prove this in case the branching of the general tree is growing more rapidly in all directions from the root then the branching of a stochastically incomplete model subtree then, more generally, in case the branching condition holds in one direction from the root. Also, we show that if a tree is contained in a stochastically complete model tree then it must be stochastically complete. The proof of this fact follows from the more general statement that, for any graph G, if there exists a vertex x_0 such that the maximum valence of vertices on spheres centered at x_0 is not growing too rapidly then G must be stochastically complete.

We next prove two theorems analogous to a result of Cheeger and Yau [1] which compare the heat kernel on a model tree and the heat kernel on a general graph. Let T_n denote a model tree with root vertex x_0 where $n(r)$ is $m(r) - 1$, that is, one less than the common valence of vertices on the sphere of radius r about x_0. We denote the heat kernel on T_n by $\rho_t(x_0, x)$ and first show that, as a function of x, $\rho_t(x_0, x)$ only depends on the distance from x_0. That is, if we let $r(x) = d(x, x_0)$ where $d(x, x_0)$ denotes the distance between x and x_0 then we can write $\rho_t(r) = \rho_t(x_0, r(x))$. Let G denote a general graph with heat kernel $p_t(x_0', x)$ where x_0' is a fixed vertex of G. We show that if the branching on G grows faster in all directions from x_0' then $n(r)$ then $p_t(x_0', x) \leq \rho_t(r)$ for all $x \in S_r(x_0') \subset G$. Similarly, if the branching grows slower on G than on T_n and there are unique geodesics in G connecting each vertex back to a fixed vertex then $\rho_t(r) \leq p_t(x_0', x)$ for all $x \in S_r(x_0') \subset G$.

We finish this chapter by considering an operator related to the Laplacian that we study throughout the rest of the thesis. This operator, in this work referred to as the *bounded Laplacian*, arises when one assigns the standard weights to the edges of a graph. We show that the heat kernel associated to the bounded Laplacian is stochastically complete for every graph G. In particular, bounded solutions of the combinatorial heat equation involving the bounded Laplacian are uniquely determined by initial data.

In the final part of the thesis we study the spectrum of the Laplacian on a general graph. We first introduce $\lambda_0(\Delta)$, the bottom of the spectrum of the Laplacian, and recall a characterization of this number in terms of the existence of positive λ-harmonic functions. Specifically, there always exist positive functions satisfying $\Delta v = \lambda v$ for $\lambda \leq \lambda_0(\Delta)$ whereas such functions never exist for $\lambda > \lambda_0(\Delta)$ [15, 5]. We then prove a lower bound for $\lambda_0(\Delta)$ under a geometric assumption on G. Precisely, fixing a vertex x_0, we assume that, at every vertex, the ratio of the difference between the number of edges leading away from x_0 and the number of edges going towards x_0 divided by the total valence is bounded below by a positive constant. The lower bound is then given in terms of this constant. In the final section, we use this lower bound to prove that, with the additional assumption that the infimum of the valence of vertices outside of a ball of radius r about x_0 is going to infinity as r goes to infinity, the Laplacian on the graph has empty essential spectrum. This is analogous to a result of Donnelly and Li for geodeiscally complete, simply connected, negatively curved Riemannian manifolds [8].

1.2 Notation and Fundamentals

In this section we fix our notation and prove some basic lemmas which will be used throughout. In general, $G = (V, E)$ will denote an infinite, locally finite, connected graph where $V = V(G)$ is the set of vertices of G and $E = E(G)$ the set of edges. At times, we write $x \in G$ when x is a vertex of G. We will use the notation $x \sim y$ to indicate that an edge connects the vertices x and y while $[x, y]$ will denote an *oriented* edge from x to y. In general, to be able to write down certain formulas unambiguously, we will assume that our graphs come with an orientation, that is, that every edge is oriented, but none of our results depend on the choice of this orientation. We do not allow loops or multiple edges between vertices. We use the notation $m(x)$ to indicate the *valence* at a vertex x, that is, the number of edges emanating from x.

For a finite subgraph D of G we let $\mathrm{Vol}(D)$ denote the *volume* of D which we take to be the number of vertices of D. That is,

$$\mathrm{Vol}(D) = \#\{x \mid x \in V(D)\}.$$

We also use the usual notion of distance between two vertices of the graph. Specifically, $d(x, y)$ will denote the number of edges in the shortest path connecting the vertices x and y. We write $r(x) = d(x, x_0)$ to denote the distance between a vertex x and a fixed vertex x_0.

We call f a *function on the graph* G if it is a mapping $f : V \to \mathbf{R}$. The set of all such functions will be denoted by $C(V)$. We will also use the notation $C_0(V)$ for the space of all finitely supported functions on G and $\ell^2(V)$ for the space of all square summable functions. That is, $\ell^2(V)$ consists of all functions on G which satisfy

$$\sum_{x \in V} f^2(x) < \infty$$

and is a Hilbert space with inner product

$$\langle f, g \rangle = \sum_{x \in V} f(x) g(x).$$

Similarly, we let $\ell^2(\tilde{E})$ denote the Hilbert space of all square summable functions on oriented edges with inner product

$$\langle \varphi, \psi \rangle = \sum_{[x,y] \in \tilde{E}} \varphi([x, y]) \psi([x, y])$$

where $\tilde{E}$ denotes the set of all oriented edges of G.

We now recall the definitions of the coboundary and Laplacian operators and state and prove an analogue of Green's Theorem for them. The *coboundary* operator d takes a function on the vertices of G and sends it to a function on the oriented edges of G defined by:

$$df([x,y]) = f(y) - f(x).$$

The *Laplacian* Δ operates on functions on G as prescribed by the formula:

$$\Delta f(x) = \sum_{y \sim x} \big(f(x) - f(y)\big) = m(x)f(x) - \sum_{y \sim x} f(y) \tag{1.1}$$

where the summation is taken over all vertices y such that $y \sim x$ forms an edge in G. If the Laplacian is applied to a function of more than one variable then we will put the variable in which it is applied as a subscript when necessary. For a constant λ, we call a function v on G λ-*harmonic* if $\Delta v(x) = \lambda v(x)$ for all vertices x.

Note that it follows from formula (1.1) that the Laplacian will be bounded if and only if there exists a constant M such that $m(x) \leq M$ for all vertices x. Indeed, letting δ_x denote the delta function at a vertex x so that

$$\delta_x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

it follows that the matrix coefficients of the Laplacian are given by

$$\begin{aligned} \Delta(x,y) &= \langle \Delta\delta_x, \delta_y \rangle = \Delta\delta_x(y) \\ &= \begin{cases} m(x) & \text{if } x = y \\ -1 & \text{if } x \sim y \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

As mentioned in the introduction, under the assumption $m(x) \leq M$, all graphs are stochastically complete [7, Theorem 2.10]. Therefore, for the purposes of our inquiry, we do not impose this restriction on the graph and the Laplacian will be unbounded.

Let D denote a finite, connected subgraph of G. We then have the following analogue of Green's Theorem.

Lemma 1.2.1. *For $D \subset G$ finite and connected*

$$\begin{aligned}\sum_{x \in D} \Delta f(x) g(x) &= \sum_{[x,y] \in \tilde{E}(D)} df([x,y]) dg([x,y]) + \sum_{\substack{x \in D \\ z \sim x, z \notin D}} \big(f(x) - f(z)\big) g(x) \\ &= \sum_{[x,y] \in \tilde{E}(D)} df([x,y]) dg([x,y]) - \sum_{\substack{[x,z] \\ x \in D, z \notin D}} df([x,z]) g(x).\end{aligned}$$

Proof: Every oriented edge $[x, y]$ with $x, y \in V(D)$ contributes two terms to the sum on the left hand side: $\big(f(x) - f(y)\big)g(x)$ from $\Delta f(x)g(x)$ and $\big(f(y) - f(x)\big)g(y)$ from $\Delta f(y)g(y)$. Their sum is

$$\big(f(y) - f(x)\big)\big(g(y) - g(x)\big) = df([x,y]) dg([x,y]).$$

The remaining contributions come from a vertex x in D that is connected to a neighbor z which is not in D and these give $\big(f(x) - f(z)\big)g(x) = -df([x,z])g(x)$. $\square$

We say that a vertex x is in the *boundary* of D, and denote this $x \in \partial D$, if x is a vertex of D and is connected to a vertex which is not in D. Otherwise, a vertex x in D is said to be in the *interior* of D, denoted $x \in \text{int}(D)$. We then see that, if either f or g are supported on the interior of D, then the second term on the right hand side of the equation above is zero and we can write Lemma 1.2.1 as

$$\langle \Delta f, g\rangle_{V(D)} = \langle df, dg\rangle_{\tilde{E}(D)} = \langle f, \Delta g\rangle_{V(D)}.$$

Also, if f and g are functions on the graph and one of them is finitely supported, it is true that

$$\langle \Delta f, g\rangle = \langle df, dg\rangle = \langle f, \Delta g\rangle$$

where now the inner products are taken over $V(G)$ and $\tilde{E}(G)$.

Throughout, we wish to study solutions of the *combinatorial heat equation*. These will be functions on G and a time parameter in which they are differentiable and which satisfy the equation

$$\Delta u(x,t) + \frac{\partial u}{\partial t}(x,t) = 0$$

for every vertex x and $t \geq 0$. We start by recalling a proof of analogues for the weak and strong maximum principles for the heat equation [11, 7].

Lemma 1.2.2. *Suppose that D is a finite, connected subgraph of G and*

$$u : D \times [0,T] \to \mathbf{R}$$

is continuous for $t \in [0,T]$, C^1 for $t \in (0,T)$, and satisfies the combinatorial heat equation

$$\Delta u + \frac{\partial u}{\partial t} = 0 \text{ on } \operatorname{int} D \times [0,T].$$

Then, if there exists $(x_0,t_0) \in \operatorname{int} D \times (0,T)$ such that (x_0,t_0) is a maximum (or a minimum) for u on $D \times [0,T]$, then $u(x,t_0) = u(x_0,t_0)$ for all $x \in D$.

Proof: At either a maximum or minimum $\frac{\partial u}{\partial t}(x_0, t_0) = 0$ implying that

$$\Delta u(x_0, t_0) = \sum_{x \sim x_0} \Big(u(x_0, t_0) - u(x, t_0) \Big) = 0.$$

In either case, this implies that $u(x, t_0) = u(x_0, t_0)$ for all $x \sim x_0$. Iterating the argument and using the assumption that D is connected gives the statement of the lemma. □

Lemma 1.2.3. *Under the same hypotheses as above*

$$\max_{D \times [0,T]} u = \max_{\substack{D \times \{0\} \cup \\ \partial D \times [0,T]}} u$$

and

$$\min_{D \times [0,T]} u = \min_{\substack{D \times \{0\} \cup \\ \partial D \times [0,T]}} u.$$

Proof: Let $w = u - \epsilon t$ for $\epsilon > 0$. Then $\Delta w + \frac{\partial w}{\partial t} = -\epsilon < 0$. If w has a maximum at $(x_0, t_0) \in \text{int } D \times (0, T]$ then

$$\frac{\partial w}{\partial t}(x_0, t_0) \geq 0 \quad \text{and} \quad \Delta w(x_0, t_0) \geq 0$$

yielding a contradiction. Therefore,

$$\max_{D \times [0,T]} w = \max_{\substack{D \times \{0\} \cup \\ \partial D \times [0,T]}} w.$$

Then

$$\begin{aligned} \max_{D \times [0,T]} u &= \max_{D \times [0,T]} w + \epsilon t \\ &\leq \max_{D \times [0,T]} w + \epsilon T \\ &= \max_{\substack{D \times \{0\} \cup \\ \partial D \times [0,T]}} w + \epsilon T \\ &\leq \max_{\substack{D \times \{0\} \cup \\ \partial D \times [0,T]}} u + \epsilon T. \end{aligned}$$

Letting $\epsilon \to 0$ it follows that

$$\max_{D\times[0,T]} u = \max_{\substack{D\times\{0\}\,\cup \\ \partial D\times[0,T]}} u.$$

The statement about the minimum follows by considering $-u$. $\square$

Remark 1.2.1. Using the same techniques as above it follows that, if u satisfies

$$\Delta u + \frac{\partial u}{\partial t} \geq 0 \text{ } on \text{ int } D \times [0,T]$$

then

$$\min_{D\times[0,T]} u = \min_{\substack{D\times\{0\}\,\cup \\ \partial D\times[0,T]}} u$$

while if u satisfies

$$\Delta u + \frac{\partial u}{\partial t} \leq 0 \text{ } on \text{ int } D \times [0,T]$$

then

$$\max_{D\times[0,T]} u = \max_{\substack{D\times\{0\}\,\cup \\ \partial D\times[0,T]}} u.$$

1.3 Essential Self-Adjointness of the Laplacian

As in the case of the Laplacian on a Riemannian manifold the discrete Laplacian with domain $C_0(V)$, the set of all finitely supported functions on the graph G, is a symmetric but not self-adjoint operator. It is; however, essentially self-adjoint by which we mean that it has a unique self-adjoint extension $\tilde{\Delta}$, a fact whose proof we now recall [4, Theorem 1.2]. Consider Δ with domain $C_0(V)$ and let Δ^* denote its adjoint.

Proposition 1.3.1. *The domain of* Δ^* *is*

$$dom(\Delta^*) = \{f \in \ell^2(V) \mid \Delta f \in \ell^2(V)\}.$$

Proof: By definition,

$$\mathrm{dom}(\Delta^*) = \left\{ f \in \ell^2(V) \;\middle|\; \begin{array}{c} \text{there exists a unique } h \in \ell^2(V) \text{ satisfying} \\ \langle \Delta g, f\rangle = \langle g, h\rangle \text{ for all } g \in C_0(V) \end{array} \right\}$$

and then, if $f \in \mathrm{dom}(\Delta^*)$, $\Delta^* f = h$. If g is finitely supported as above then it follows from Green's Theorem, Lemma 1.2.1, that for $f \in \mathrm{dom}(\Delta^*)$

$$\langle \Delta g, f\rangle = \langle g, \Delta f\rangle = \langle g, h\rangle.$$

Letting $g = \delta_x$ it follows that $\Delta f(x) = h(x)$ for all vertices x so that $\Delta f \in \ell^2(V)$. □

Theorem 1.3.1. Δ *with domain* $C_0(V)$ *is essentially self-adjoint.*

Proof: From the criterion stated in [14, Theorem X.26] applied to the operator $(\Delta + I)$ it suffices to show that -1 is not an eigenvalue of Δ^*. In other words, if f satisfies $\Delta^* f = -f$ then f cannot be in $\ell^2(V)$ unless it is exactly 0. As can be seen by applying the analogue of Green's Theorem, Lemma 1.2.1, pointwise, it is true that $\Delta^* f(x) = \Delta f(x)$. Therefore, if f

satisfies $\Delta^* f(x) = -f(x)$ for every vertex x, it follows that

$$\big(m(x)+1\big)f(x) = \sum_{y\sim x} f(y).$$

This implies that there must exist a neighbor $y \sim x$ such that $f(y) > f(x)$. By repeating this argument the conclusion follows. $\square$

Chapter 2

The Heat Kernel

2.1 Construction of the Heat Kernel

We now give a construction of the heat kernel $p = p_t(x, y)$ for an infinite, locally finite, connected graph G. By *heat kernel* we mean that $p_t(x, y)$ will be the smallest non-negative function

$$p : V \times V \times \ [0, \infty) \to \mathbf{R}_+$$

which is smooth in t, satisfies the heat equation $\Delta p + \frac{\partial p}{\partial t} = 0$ where the Laplacian is applied in either x or y and satisfies $p_0(x, y) = \delta_x(y)$. The heat kernel will generate a bounded solution of the heat equation on G for any bounded initial condition. That is, for any bounded function u_0, $u(x, t) = \sum_{y \in V} p_t(x, y) u_0(y)$ will give a bounded solution of

$$\begin{cases} \Delta u(x,t) + \frac{\partial u}{\partial t}(x,t) = 0 & \text{for } x \in V, t \geq 0 \\ u(x,0) = u_0(x) & \text{for } x \in V. \end{cases} \tag{2.1}$$

The construction given here follows the approach of [2, Section 3] and the formalism of [3].

Starting with an exhaustion sequence of the graph by finite, connected subgraphs, we construct heat kernels with Dirichlet boundary conditions for each graph in the exhaustion. Although we will exhaust the graph by balls of increasing radii it will be shown later that the resulting heat kernel is independent of the choice of subgraphs in the exhaustion. Let $x_0 \in V(G)$ be a fixed vertex. We will let $B_r = B_r(x_0)$ denote the ball of radius r about x_0, $\partial B_r = \partial B_r(x_0)$ its boundary, and int B_r its interior. In particular, if $d(x, x_0)$ denotes the standard metric on graphs,

$$\begin{aligned} V(B_r) &= \{x \in V(G) \mid d(x, x_0) \leq r\} \\ E(B_r) &= \{x \sim y \mid x, y \in V(B_r) \text{ and } x \sim y \in E(G)\}. \end{aligned}$$

We let $C(B_r, \partial B_r)$ denote functions on B_r which vanish on the boundary ∂B_r and let Δ_r denote the *reduced Laplacian* which acts on these spaces. That is,

$$C(B_r, \partial B_r) = \{f \in C(B_r) \mid f_{|\partial B_r} = 0\}$$

and

$$\Delta_r f(x) = \begin{cases} \Delta f(x) & \text{for } x \in \text{int } B_r \\ 0 & \text{otherwise} \end{cases}$$

for all $f \in C(B_r, \partial B_r)$.

With these definitions it follows that

Lemma 2.1.1. Δ_r *is a self-adjoint, non-negative operator on* $C(B_r, \partial B_r)$.

Proof: This follows from the analogue of Green's Theorem, Lemma 1.2.1, since

$$\langle \Delta_r f, g \rangle_{V(B_r)} = \langle df, dg \rangle_{\tilde{E}(B_r)} = \langle f, \Delta_r g \rangle_{B(B_r)}.$$

□

From Lemma 2.1.1 it follows that all eigenvalues λ_i^r of Δ_r are real and non-negative. In fact, as mentioned in [3] and shown later, $\lambda_0(\Delta_r) = \lambda_0^r$, the smallest eigenvalue of Δ_r, is given by the Rayleigh-Ritz quotient. That is,

$$\lambda_0^r = \min_{\substack{f \in C(B_r, \partial B_r) \\ f \not\equiv 0}} \frac{\langle df, df \rangle}{\langle f, f \rangle}$$

so that all of the eigenvalues of Δ_r are positive. Denote by $\{\lambda_i^r\}_{i=0}^{k(r)}$ the set of all eigenvalues of Δ_r listed in increasing order and choose a set $\{\phi_i^r\}_{i=0}^{k(r)}$ of corresponding eigenfunctions which are an orthonormal basis for $C(B_r, \partial B_r)$ with respect to the ℓ^2 inner product. That is, $\{\phi_i^r\}_{i=0}^{k(r)}$ satisfy

$$\Delta_r \phi_i^r = \lambda_i^r \phi_i^r \qquad \text{for } i = 0, \ldots, k(r) \tag{2.2}$$

and

$$\langle \phi_i^r, \phi_j^r \rangle = \sum_{x \in B_r} \phi_i^r(x) \phi_j^r(x) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

We now define the heat kernels $p_t^r(x, y)$ for each subgraph in the exhaustion.

Definition 2.1.1. For $x, y \in B_r$ and $t \geq 0$

$$p_t^r(x, y) = \sum_{i=0}^{k(r)} e^{-\lambda_i^r t} \phi_i^r(x) \phi_i^r(y). \tag{2.4}$$

Theorem 2.1.2. $p_t^r(x, y)$ *satisfies the following properties:*

1) $p_t^r(x, y) = p_t^r(y, x)$, $p_t^r(x, y) = 0$ *if* $x \in \partial B_r$ *or* $y \in \partial B_r$.

2) $\Delta_r p_t^r(x,y) + \frac{\partial}{\partial t} p_t^r(x,y) = 0$ *where* Δ_r *denotes the reduced Laplacian applied in either* x *or* y.

3) $p_{s+t}^r(x,y) = \sum_{z \in B_r} p_s^r(x,z) p_t^r(z,y)$.

4) $p_0^r(x,y) = \delta_x(y)$ *for* $x, y \in \text{int } B_r$.

5) $p_t^r(x,y) > 0 \quad$ *for* $t > 0,\ x, y \in \text{int } B_r$.

6) $\sum_{y \in B_r} p_t^r(x,y) < 1$ *for* $t > 0,\ x \in B_r$.

Remark 2.1.3. We could also define the heat operator semigroup as the convergent power series

$$Q_t^r = e^{-t\Delta_r} = I - t\Delta_r + \frac{t^2}{2}\Delta_r^2 - \frac{t^3}{6}\Delta_r^3 + \dots$$

and then take its kernel given by $q_t^r(x,y) = \langle Q_t^r \delta_x, \delta_y \rangle = Q_t^r \delta_x(y)$. The equivalence of these two approaches can be seen by applying the maximum principle, Lemma 1.2.3, to the difference of the two kernels.

Proof: 1), 2) Clear from the definition of $p_t^r(x,y)$, from $\phi^r_{i|\partial B_r} = 0$, and from (2.2).

3) Using the orthonormality of $\{\phi_i^r\}_{i=1}^{k(r)}$ we compute

$$\begin{aligned}
\sum_{z\in B_r} p_s^r(x,z)p_t^r(z,y) &= \sum_{z\in B_r}\sum_{i=0}^{k(r)} e^{-\lambda_i^r s}\phi_i^r(x)\phi_i^r(z)\sum_{j=0}^{k(r)} e^{-\lambda_j^r t}\phi_j^r(z)\phi_j^r(y) \\
&= \sum_{i,j=0}^{k(r)} e^{-\lambda_i^r s}e^{-\lambda_j^r t}\phi_i^r(x)\phi_j^r(y)\sum_{z\in B_r}\phi_i^r(z)\phi_j^r(z) \\
&= \sum_{i=0}^{k(r)} e^{-\lambda_i^r(s+t)}\phi_i^r(x)\phi_i^r(y) \qquad (2.3)\\
&= p_{s+t}^r(x,y).
\end{aligned}$$

4) By definition,

$$p_0^r(x,y) = \sum_{i=0}^{k(r)} \phi_i^r(x)\phi_i^r(y).$$

Since $\{\phi_i^r\}_{i=0}^{k(r)}$ forms an orthonormal basis it follows that

$$\delta_x(y) = \sum_{i=0}^{k(r)}\langle\delta_x,\phi_i^r\rangle\phi_i^r(y) = \sum_{i=0}^{k(r)}\phi_i^r(x)\phi_i^r(y).$$

Therefore, $p_0(x,y) = \delta_x(y)$.

5) The maximum principle, Lemma 1.2.3, applied in each of the variables separately to $p_t^r(x,y)$ over the set $B_r\times B_r\times[0,T]$ implies that $0\le p_t^r(x,y)\le 1$ since $p_0^r(x,y)=\delta_x(y)$ and $p_t^r(x,y)=0$ if either x or y is in the boundary of B_r.

Now, assume that there exists $t_0>0$ and $\hat{x},\hat{y}\in \text{int } B_r$ such that $p_{t_0}^r(\hat{x},\hat{y})=0$. It follows that $(\hat{x},\hat{y},t_0)$ is a minimum for $p_t^r(x,y)$ on $B_r\times B_r\times[0,t_0]$. Using that $p_t^r(x,y)$ satisfies the heat equation in both variables and B_r is connected and applying the argument used in the proof of Lemma

1.2.2 over the set $B_r \times B_r \times [0, t_0]$ gives that

$$p^r_{t_0}(x, y) = p^r_{t_0}(\hat{x}, \hat{y}) = 0 \quad \text{for all } x, y \in \text{int } B_r.$$

In particular,

$$p^r_{t_0}(x, x) = \sum_{i=0}^{k(r)} e^{-\lambda^r_i t_0} (\phi^r_i(x))^2 = 0 \quad \text{for all } x \in \text{int } B_r.$$

This implies that $\phi^r_i(x) = 0$ for all i and all $x \in \text{int } B_r$ contradicting the fact that $\{\phi^r_i\}_{i=0}^{k(r)}$ forms an orthonormal basis for $C(B_r, \partial B_r)$.

6) We may assume that $x, y \in \text{int } B_r$ since $p^r_t(x, y) = 0$ otherwise. Then, for $x \in \text{int } B_r$ and $t = 0$, it follows from property 4) that

$$\sum_{y \in \text{int } B_r} p^r_0(x, y) = \sum_{y \in \text{int } B_r} \delta_x(y) = 1.$$

By using the analogue of Green's Theorem, Lemma 1.2.1, over the interior of B_r, whose boundary consists of vertices in the interior which have a neighbor in the boundary ∂B_r, we show that the expression $\sum_{y \in \text{int } B_r} p^r_t(x, y)$ is decreasing as a function of t. That is,

$$\begin{aligned}
\frac{\partial}{\partial t} \sum_{y \in \text{int } B_r} p^r_t(x, y) &= \sum_{y \in \text{int } B_r} \frac{\partial}{\partial t} p^r_t(x, y) \\
&= \sum_{y \in \text{int } B_r} -\Delta_y p^r_t(x, y) \\
&= \sum_{\substack{y \in \text{int } B_r \\ z \sim y,\ z \in \partial B_r}} \Big(p^r_t(x, z) - p^r_t(x, y) \Big) \\
&= \sum_{\substack{y \in \text{int } B_r \\ z \sim y,\ z \in \partial B_r}} -p^r_t(x, y) < 0.
\end{aligned}$$

Therefore,

$$\sum_{y \in B_r} p_t^r(x,y) < 1 \quad \text{for } t > 0, x \in B_r.$$

$\square$

Remark 2.1.4. The proof of Part 6) is identical to the proof of the corresponding property for the Dirichlet heat kernels on a Riemannian manifold [2, Lemma 3.3, Part (i)] and shows that a finite subgraph with Dirichlet boundary conditions is *not* stochastically complete.

We now wish to show that $p_t^r(x,y)$ converge to the heat kernel $p_t(x,y)$ mentioned at the beginning of this section. For this purpose the following lemma is instrumental.

Lemma 2.1.2.

$$p_t^r(x,y) \le p_t^{r+1}(x,y) \quad \textit{for } t \ge 0, x, y \in B_r.$$

Proof: This is clear for x or y in ∂B_r. Fix $y \in \text{int } B_r$ and let

$$u(x,t) = p_t^{r+1}(x,y) - p_t^r(x,y).$$

It follows that $\Delta u + \frac{\partial u}{\partial t} = 0$ on int $B_r \times [0,T]$ which implies that the minimum of u is attained on the set $(B_r \times \{0\}) \cup (\partial B_r \times [0,T])$. Since, $u(x,0) = 0$ by Part 4) of Theorem 2.1.2 while, on $\partial B_r \times [0,T]$,

$$u(x,t) = p_t^{r+1}(x,y) \ge 0$$

it follows that

$$\min_{B_r \times [0,T]} u \ge 0.$$

Therefore, $p_t^{r+1}(x,y) \geq p_t^r(x,y)$ for $x, y \in B_r$ and $t \geq 0$. □

By extending $p_t^r(x,y)$ to be 0 outside of B_r and using $0 \leq p_t^r(x,y) \leq 1$ and Lemma 2.1.2 we see that the sequence $p_t^r(x,y)$ converges pointwise as $r \to \infty$ for $x, y \in V$ and $t \geq 0$.

Definition 2.1.5. For $x, y \in V$ and $t \geq 0$

$$p_t(x,y) = \lim_{r \to \infty} p_t^r(x,y).$$

We show that the convergence is uniform in t on every compact interval $[0,T]$. To this end, we fix x and y in V, let $f_r(t) = p_t^r(x,y)$ and $f(t) = p_t(x,y)$. Then, from the definition and properties of each of the heat kernels p^r it follows that each $f_r : [0,\infty) \to \mathbf{R}$ is C^∞ and satisfies

1) $f_r \leq f_{r+1}$
2) $f_r(t) \to f(t)$ pointwise for all t
3) $f_r(t) \leq 1$.

Dini's Theorem then implies that $f_r \to f$ uniformly on all compact subsets $[0,T] \subset [0,\infty)$.

We now show that $p_t(x,y)$ satisfies the heat equation. This will follow if we are able to show that $\frac{\partial}{\partial t} p_t^r(x,y)$ converges uniformly in t on compact intervals as $r \to \infty$. But

$$\begin{aligned} \frac{\partial}{\partial t} p_t^r(x,y) &= -\Delta_{r,x} p_t^r(x,y) \\ &= \sum_{z \sim x} \big(p_t^r(z,y) - p_t^r(x,y) \big) \end{aligned}$$

and since both $p_t^r(z,y)$ and $p_t^r(x,y)$ converge uniformly in t on $[0,T]$ it follows that $\frac{\partial}{\partial t} p_t(x,y)$ exists and is continuous. In fact, iterating this argument and using the fact that $\frac{\partial^i}{\partial t^i} p_t^r(x,y)$ also satisfy the heat equation and are continuous for all i, we obtain that $p_t(x,y)$ is C^∞ in t. Then, from the pointwise convergence of $p_t^r(x,y)$, it follows that

$$\begin{aligned}\frac{\partial}{\partial t} p_t(x,y) &= \frac{\partial}{\partial t} \lim_{r\to\infty} p_t^r(x,y) \\ &= \lim_{r\to\infty} \frac{\partial}{\partial t} p_t^r(x,y) \\ &= \lim_{r\to\infty} -\Delta_x p_t^r(x,y) \\ &= -\Delta_x p_t(x,y)\end{aligned}$$

implying $\Delta_x p_t(x,y) + \frac{\partial}{\partial t} p_t(x,y) = 0$. The same argument applied in y then implies $\Delta_y p_t(x,y) + \frac{\partial}{\partial t} p_t(x,y) = 0$. In summary, using the corresponding properties of $p_t^r(x,y)$, Lemma 2.1.2, and what was just shown, we have proved statements 1), 2), 3), 4), 5), and 6) of the following theorem.

Theorem 2.1.6. $p : V \times V \times [0,\infty) \to \mathbf{R}_+$ *satisfies*

1) $p_t(x,y) = p_t(y,x)$ *and* $p_t(x,y) > 0$ *for* $t > 0$, $x, y \in V$.

2) p *is* C^∞ *in* t *on compact subsets of* $[0,\infty]$.

3) $\Delta p_t(x,y) + \frac{\partial}{\partial t} p_t(x,y) = 0$ *where* Δ *denotes the Laplacian applied in either* x *or* y.

4) $p_0(x,y) = \delta_x(y)$.

5) $p_{s+t}(x,y) = \sum_{z\in V} p_s(x,z) p_t(z,y)$.

6) $\sum_{y\in V} p_t(x,y) \leq 1$.

7) p *is independent of the exhaustion used to define it.*

8) *p is the smallest non-negative function satisfying Properties 3) and 4).*

Proof: 7) Assume that D_i is another exhaustion of G. That is, each D_i is a finite and connected subgraph, $D_i \subseteq D_{i+1}$ for all i, and $G = \bigcup_{i=0}^{\infty} D_i$. Let $q_t^{D_i}(x,y)$ denote the Dirichlet heat kernels for this exhaustion and suppose that $q_t^{D_i}(x,y) \to q_t(x,y)$. Then, for every D_i, there exists R large enough so that $D_i \subseteq B_R$. By the maximum principle, Lemma 1.2.3, since $q_t^{D_i}(x,y)$ vanishes on ∂D_i, it follows that $q_t^{D_i}(x,y) \leq p_t^R(x,y)$. Because $p_t^R(x,y) \leq p_t^{R+1}(x,y) \leq \ldots$ and $p_t^R(x,y) \to p_t(x,y)$ this implies $q_t^{D_i}(x,y) \leq p_t(x,y)$. Letting $i \to \infty$ gives

$$q_t(x,y) \leq p_t(x,y).$$

Interchanging the roles of q^{D_i} and p^r in the preceding argument implies $q_t(x,y) \geq p_t(x,y)$ and, therefore, $p_t(x,y) = q_t(x,y)$.

8) Assume that $q_t(x,y)$ is another non-negative function that satisfies Properties 3) and 4). In particular, both q and p^r satisfy the heat equation on int $B_r \times [0,T]$. Since p^r vanishes on ∂B_r while q is non-negative it follows, by applying the maximum principle, Lemma 1.2.3, to the difference of q and p^r, that $q_t(x,y) \geq p_t^r(x,y)$ on $B_r \times [0,T]$ and, therefore, for all $x, y \in V$ and $t \geq 0$. Letting $r \to \infty$ it follows that $q_t(x,y) \geq p_t(x,y)$. □

Remark 2.1.7. A stronger statement concerning $p_t(x,y)$ at $t = 0$ than the one given in 4) is true. Precisely, it is true that $\lim_{t\to 0} p_t(x,y) = \delta_x(y)$ and $\lim_{t\to 0} \sum_{\substack{y\in G \\ y\neq x}} p_t(x,y) = 0$. This follows from $p_t^r(x,x) \leq p_t(x,x) \leq 1$ and $\sum_{y\in G} p_t(x,y) = p_t(x,x) + \sum_{\substack{y\in G \\ y\neq x}} p_t(x,y) \leq 1$ by letting $t \to 0$. This fact is needed to show that, for any bounded function u_0 on G, $u(x,t) = \sum_{y\in V} p_t(x,y)u_0(y)$ is continuous at $t=0$. Therefore, $u(x,t)$ as defined above is bounded, continuous for $t \geq 0$, smooth for $t > 0$, and satisfies (2.1).

2.2 The Spectral Theorem Construction

As mentioned previously, an alternate way of defining the heat kernels $p_t^r(x,y)$ on B_r with Dirichlet boundary conditions is by considering

$$e^{-t\Delta_r} = I - t\Delta_r + \frac{t^2}{2}\Delta_r^2 - \frac{t^3}{6}\Delta_r^3 + \dots$$

and letting $p_t^r(x,y) = e^{-t\Delta_r}\delta_x(y)$. On the entire graph, since the Laplacian is not bounded on $\ell^2(V)$, one cannot use the power series definition. One can still construct $e^{-t\tilde{\Delta}}$ for, $\tilde{\Delta}$, the unique self-adjoint extension of Δ with domain $C_0(V)$ in $\ell^2(V)$ by using the functional calculus developed through the spectral theorem [13, Chapter VIII]. The purpose of this section is to show that the construction given in the previous section via exhaustion and this approach result in the same kernel [2, Proposition 4.5]. Let $P_t v(x) = \sum_{y\in V} p_t(x,y)v(y)$ for any bounded function v, and $P_t^r v(x) = \sum_{y\in B_r} p_t^r(x,y)v(y)$. The following theorem states that P_t and $e^{-t\tilde{\Delta}}$ agree on a dense subset of $\ell^2(V)$ and, as such, have the same kernel.

Theorem 2.2.1.

$$P_t v = e^{-t\tilde{\Delta}}v \quad \textit{for all } v \in C_0(V).$$

Proof: We begin by showing that if $v \in C_0(V)$ then $P_t v \in \ell^2(V)$ and $\Delta P_t v \in \ell^2(V)$. Since v is finitely supported there exists a ball of large radius

R which contains its support. Therefore,

$$
\begin{aligned}
\|P_t^R v\|_{\ell^2(V)}^2 &= \sum_{x\in V}(P_t^R v(x))^2 \\
&= \sum_{x\in V}\left(\sum_{y\in B_R} p_t^R(x,y)v(y)\right)^2 \\
&\leq \sum_{y\in B_R}\left(\sum_{x\in B_R} p_t^R(x,y)^2\right)v(y)^2 \\
&\leq \sum_{y\in B_R} v(y)^2 = \|v\|_{\ell^2(V)}^2.
\end{aligned}
$$

By letting $R \to \infty$ and using the dominated convergence theorem it follows that $P_t v \in \ell^2(V)$. In fact, this actually proves that P_t is a bounded operator on $\ell^2(V)$ with $\|P_t\| \leq 1$.

We now show that $\Delta P_t v = P_t \Delta v$ and, since if v is finitely supported then so is Δv, it will follows that $\Delta P_t v \in \ell^2(V)$. To show that $\Delta P_t v = P_t \Delta v$ we calculate

$$
\begin{aligned}
\Delta(P_t v)(x) &= \sum_{y\sim x}\big((P_t v)(x) - (P_t v)(y)\big) \\
&= \sum_{y\sim x}\left(\sum_{z\in V}\big(p_t(x,z) - p_t(y,z)\big)v(z)\right).
\end{aligned}
$$

Meanwhile, by using the analogue of Green's Theorem, Lemma 1.2.1, and the fact that the heat kernel satisfies the heat equation in both variables, it follows that

$$\begin{aligned}
P_t(\Delta v)(x) &= \sum_{z \in V} p_t(x,z)\Delta v(z) \\
&= \sum_{z \in V} \Delta_z p_t(x,z) v(z) \\
&= \sum_{z \in V} \Delta_x p_t(x,z) v(z) \\
&= \sum_{z \in V} \left(\sum_{y \sim x} \big(p_t(x,z) - p_t(y,z)\big) \right) v(z).
\end{aligned}$$

We now prove the theorem. Let

$$u(x,t) = \left(P_t - e^{-t\tilde{\Delta}}\right) v(x).$$

Then $u(x,0) = 0$ and since $\Delta P_t v$ is in $\ell^2(V)$ we can apply Green's Theorem once more to obtain

$$\begin{aligned}
\frac{\partial}{\partial t} \sum_{x \in V} u^2(x,t) &= 2 \sum_{x \in V} u(x,t) \frac{\partial}{\partial t} u(x,t) \\
&= -2 \sum_{x \in V} u(x,t) \Delta u(x,t) \\
&= -2 \sum_{[x,y] \in \tilde{E}} \big(u(x,t) - u(y,t)\big)^2 \leq 0
\end{aligned}$$

from which it now follows that $P_t v(x) = e^{-t\tilde{\Delta}} v(x)$ for all finitely supported v. Since both P_t and $e^{-t\tilde{\Delta}}$ are bounded it follows that they are equal on $\ell^2(V)$. □

Chapter 3

Stochastic Incompleteness

3.1 Stochastic Incompleteness

We now define the notion of stochastic incompleteness and give several equivalent conditions. The material here is adapted from [10, p. 170-172]. We recall the definition of P_t. For any bounded function u_0, $P_t u_0$ is defined by

$$P_t u_0(x) = \sum_{y \in V} p_t(x, y) u_0(y).$$

This sum converges from Part 6) of Theorem 2.1.6 and from Part 5), P_t satisfies the semigroup property

$$P_s(P_t u_0) = P_{s+t} u_0.$$

We now apply P_t to $\mathbf{1}$, the function which is 1 on every vertex of G

$$P_t \mathbf{1}(x) = \sum_{y \in V} p_t(x, y)$$

and note that this sum is less than or equal to 1 from Part 6) of Theorem 2.1.6.

Definition 3.1.1. A graph G is called *stochastically incomplete* if for some vertex x_0 of G and some $t_0 > 0$

$$P_{t_0}\mathbf{1}(x_0) = \sum_{y \in V} p_{t_0}(x_0, y) < 1.$$

Remark 3.1.2. Although this is a property of the heat kernel, or of the diffusion process which is modeled by the heat kernel, it is customary to say, as above, that it is a property of the underlying space.

Theorem 3.1.3. *The following statements are equivalent:*

1) *For some* $t_0 > 0$, *some* $x_0 \in V$, $P_{t_0}\mathbf{1}(x_0) < 1$.

1′) *For all* $t > 0$, *all* $x \in V$, $P_t\mathbf{1}(x) < 1$.

2) *For every* $\lambda < 0$ *there exists a positive (equivalently, non-zero) bounded function* v *on* G *satisfying* $\Delta v = \lambda v$.

2′) *For every* $\lambda < 0$ *there exists a non-negative, non-zero, bounded function* v *on* G *satisfying* $\Delta v \leq \lambda v$.

3) *There exists a nonzero, bounded solution of*

$$\begin{cases} \Delta u(x,t) + \frac{\partial u}{\partial t}(x,t) = 0 & \text{for } x \in V,\ t \geq 0 \\ u(x,0) = 0 & \text{for } x \in V. \end{cases}$$

Definition 3.1.4. A function v on G satisfying $\Delta v = \lambda v$ is called λ-*harmonic* whereas, if $\Delta v \leq \lambda v$, v is called λ-*subharmonic.*

Therefore, stochastic incompleteness is equivalent to the existence of a positive, bounded λ-harmonic (or λ-subharmonic) function for negative λ and to the non-uniqueness of bounded solutions for the heat equation on G.

Proof:

1′) ⇒ 1) Obvious.

1) ⇒ 1′) If there exist $x_0 \in V$ and $t_0 > 0$ such that $P_{t_0}\mathbf{1}(x_0) = 1$ then by the argument in the proof of the strong maximum principle for the heat equation, Lemma 1.2.2, applied to the function $P_t\mathbf{1}$ it follows that

$$P_{t_0}\mathbf{1}(x) = 1 \text{ for all } x.$$

Now, if $s < t_0$ then, from the semigroup property,

$$P_{t_0}\mathbf{1} = P_{t_0-s}(P_s\mathbf{1}) \leq P_{t_0-s}\mathbf{1} \leq \mathbf{1}.$$

If $t_0 > 0$ is such that $P_{t_0}\mathbf{1} = \mathbf{1}$ it follows that the inequalities become equalities and, in particular, $P_s\mathbf{1} = \mathbf{1}$ for all $s < t_0$. If $s > t_0$ then there exists a k such that $s < kt_0$ and from the semigroup property

$$P_{kt_0}\mathbf{1} = (\underbrace{P_{t_0}\dots P_{t_0}}_{k})\mathbf{1} = \mathbf{1}$$

provided that $P_{t_0}\mathbf{1} = \mathbf{1}$ giving $P_s\mathbf{1} = \mathbf{1}$ from the argument above. Therefore, if $P_t\mathbf{1} = \mathbf{1}$ for some t then $P_t\mathbf{1} = \mathbf{1}$ for all t.

1′) ⇒ 2) For $\lambda < 0$, let $w(x) = \int_0^\infty e^{\lambda t}u(x,t)dt$ where $u(x,t) = P_t\mathbf{1}(x) < 1$ by assumption. Then

$$\begin{aligned} 0 < w &< \int_0^\infty e^{\lambda t}dt \\ &= \frac{1}{\lambda}\left(e^{\lambda t}\Big|_0^\infty\right) \\ &= \frac{1}{\lambda}(0-1) = -\frac{1}{\lambda}. \end{aligned}$$

Integration by parts yields

$$\begin{aligned}\Delta w = \int_0^\infty e^{\lambda t}\Delta u(x,t)dt &= -\int_0^\infty e^{\lambda t}\frac{\partial u}{\partial t}(x,t)dt \\ &= -e^{\lambda t}u(x,t)\Big|_0^\infty + \int_0^\infty \lambda e^{\lambda t}u(x,t)dt \\ &= 1+\lambda w.\end{aligned}$$

If we let $v = 1+\lambda w$ then v satisfies

$$\Delta v = \lambda \Delta w = \lambda(1+\lambda w) = \lambda v$$

which shows that v is λ-harmonic. Since $0 < w < -\frac{1}{\lambda}$ it follows that $0 < v < 1$ so that v is positive and bounded.

2′) ⇒ 2) Exhaust the graph G by finite, connected subgraphs D_i. That is, $D_i \subseteq D_{i+1}$ and $G = \bigcup_{i=0}^\infty D_i$ where each D_i is finite and connected. Let Δ_i denote the reduced Laplacian acting on the space $C(D_i, \partial D_i)$ of functions on D_i which vanish on the boundary ∂D_i. Then, for $\lambda < 0$, one can solve

$$\begin{cases} \Delta_i v_i = \lambda v_i & \text{on int } D_i \\ v_{i|\partial D_i} = 1. \end{cases} \tag{3.1}$$

Indeed, letting $\mathbf{1}_{D_i}$ denote the function which is 1 on every vertex of D_i and 0 elsewhere, if v_i is a solution to (3.1) then $w_i = v_i - \mathbf{1}_{D_i}$ would vanish on the boundary of D_i and on the interior would satisfy

$$\Delta_i w_i(x) = \Delta_i v_i(x) = \lambda v_i(x) = \lambda(w_i(x)+1).$$

That is,

$$(\Delta_i - \lambda I)w_i = \lambda_{\text{int } D_i}$$

where $\lambda_{\text{int } D_i}$ denotes the function which is equal to λ on every vertex in the interior of D_i and is 0 on ∂D_i. Since $\lambda < 0$, $\Delta_i - \lambda I$ is invertible on $C(D_i, \partial D_i)$ and, therefore,

$$w_i = (\Delta_i - \lambda I)^{-1}(\lambda_{\text{int } D_i})$$

yielding

$$v_i = (\Delta_i - \lambda I)^{-1}(\lambda_{\text{int } D_i}) + \mathbf{1}_{D_i}$$

as a solution for (3.1).

We now claim that

$$0 < v_i \leq 1 \text{ on } D_i. \tag{3.2}$$

For, if there exists an x_0 in the interior of D_i such that $v_i(x_0) \leq 0$ then we may assume that x_0 is a minimum for v_i implying

$$\Delta v_i(x_0) = \sum_{x \sim x_0} \big(v_i(x_0) - v_i(x)\big) \leq 0.$$

On other hand, $\Delta v_i(x_0) = \lambda v_i(x_0) \geq 0$ so that $\Delta v_i(x_0) = 0$. This implies that $v_i(x) = v_i(x_0)$ for all $x \sim x_0$ and, by repeating the argument, it follows that v_i is a non-positive constant on D_i contradicting the fact that $v_i = 1$ on the boundary of D_i.

Therefore, $v_i > 0$ so that $\Delta v_i < 0$ on the interior which implies that

$$\max_{D_i} v_i = \max_{\partial D_i} v_i = 1.$$

Indeed, at an interior maximum $\Delta v_i(x_0) \geq 0$. This completes the proof of (3.2).

Furthermore, extending each v_i to be 1 outside of D_i, it is true that

$$v_i \geq v_{i+1}.$$

This is clear on ∂D_i since $v_i = 1$ there while $v_{i+1} \leq 1$. On the interior of D_i $\Delta(v_i - v_{i+1}) = \lambda(v_i - v_{i+1})$ from which it follows that $v_i - v_{i+1} > 0$ by the argument that implies $v_i > 0$ above.

Therefore, the functions v_i form a non-increasing, bounded sequence and so

$$v_i \to v \text{ as } i \to \infty$$

where $0 \leq v \leq 1$ and $\Delta v = \lambda v$. It remains to show that v is non-zero and here we use the assumption that there exists a non-negative, non-zero bounded λ-subharmonic function w on G. That is, w is bounded, $w \geq 0$, $w \not\equiv 0$, and w satisfies $\Delta w \leq \lambda w$ on G. Assuming that $w \leq 1$ we show that

$$v_i \geq w \text{ on } D_i \text{ for all } i.$$

This can be seen as follows: on the interior of D_i

$$\Delta(v_i - w) \geq \lambda(v_i - w). \tag{3.3}$$

Therefore, if there exists an x_0 in the interior of D_i such that $(v_i - w)(x_0) < 0$ and x_0 is a minimum for $v_i - w$ then, by computation, $\Delta(v_i - w)(x_0) \leq 0$ while $\Delta(v_i - w)(x_0) \geq \lambda(v_i - w)(x_0) > 0$ from (3.3). The contradiction implies that $v_i \geq w$ on D_i and by passing to the limit it follows that $v \geq w$ so that v is non-zero.

2) $\Rightarrow$ 3) Let $w(x,t) = e^{-\lambda t} v(x)$ where v is a non-zero, bounded λ-harmonic function for $\lambda < 0$. Then w is non-zero and bounded on $V \times [0,T]$ and satisfies $\Delta w + \frac{\partial w}{\partial t} = 0$ with $w(x,0) = v(x)$. The function $P_t v$ also satisfies both equations. Moreover,

$$\sup_{x \in V} P_t |v|(x) \leq \sup_{x \in V} |v(x)| \tag{3.4}$$

while

$$|w(x,t)| > |v(x)| \text{ for } t > 0, \;\; x \in V. \tag{3.5}$$

Therefore, there exist two different bounded solutions of

$$\begin{cases} \Delta u(x,t) + \frac{\partial u}{\partial t}(x,t) = 0 & \text{for } (x,t) \in V \times [0,T] \\ u(x,0) = v(x) & \text{for } x \in V. \end{cases}$$

Taking the difference of the two solutions gives a non-zero, bounded solution of

$$\begin{cases} \Delta u(x,t) + \frac{\partial u}{\partial t}(x,t) = 0 & \text{for } (x,t) \in V \times [0,T] \\ u(x,0) = 0 & \text{for } x \in V. \end{cases} \tag{3.6}$$

We have shown the existence of a non-zero, bounded solution of the heat equation with initial condition equal to 0 for a bounded time interval. In the argument below, we show that this suffices to imply condition 1) which is equivalent to 1′), that is, $P_t\mathbf{1} < \mathbf{1}$. Then, given $P_t\mathbf{1} < \mathbf{1}$, $\mathbf{1} - P_t\mathbf{1}$ will give a nonzero, bounded solution to (3.6) for all time completing the proof.

3) $\Rightarrow$ 1) Suppose that $u(x,t)$ is non-zero, bounded and satisfies

$$\begin{cases} \Delta u(x,t) + \frac{\partial u}{\partial t}(x,t) = 0 & \text{for } (x,t) \in V \times [0,T] \\ u(x,0) = 0 & \text{for } x \in V. \end{cases}$$

Then, by rescaling, we may assume that $|u(x,t)| < 1$ for all x and t, and that there exist a vertex x_0 and $t_0 > 0$ such that $u(x_0,t_0) > 0$. It follows that $w(x,t) = 1 - u(x,t)$ is bounded, positive and satisfies

$$\begin{cases} \Delta w(x,t) + \frac{\partial w}{\partial t}(x,t) = 0 & \text{for } (x,t) \in V \times [0,T] \\ w(x,0) = 1 & \text{for } x \in V. \end{cases} \tag{3.7}$$

Furthermore, $w(x_0, t_0) < 1$. Letting $P_t^r \mathbf{1}(x) = \sum_{y \in B_r} p_t^r(x, y)$ and, applying the maximum principle for the heat equation to $P_t^r \mathbf{1}(x) - w(x, t)$ on $B_r \times [0, T]$, it follows that $P_t^r \mathbf{1}(x) \leq w(x, t)$ for all r. Therefore, letting $r \to \infty$,

$$P_{t_0} \mathbf{1}(x_0) \leq w(x_0, t_0) < 1.$$

2) $\Rightarrow$ 2′) The implication is obvious in case v is positive, bounded, and λ-harmonic. If v is non-zero instead of positive then, as shown above, this suffices to imply condition 3) which implies 1) which is equivalent to 1′). Then from the proof of the implication 1′) $\Rightarrow$ 2) it follows that there exists a positive, bounded, λ-harmonic function from which 2′) follows.

Thus, we have shown the implications **1) $\Leftrightarrow$ 1′) $\Rightarrow$ 2) $\Leftrightarrow$ 2′)** and **2) $\Rightarrow$ 3) $\Rightarrow$ 1)** completing the proof. $\square$

3.2 Model Trees

We now analyze the stochastic completeness of a particular family of trees. A tree will be called *model* if it contains a vertex x_0, henceforth called the *root* for the model, such that the valence $m(x)$ is constant on spheres $S_r(x_0) = S_r$ of radius r about x_0. That is, if

$$S_r = \{x \mid d(x, x_0) = r\}$$

then

$$m(x) = m(r) \text{ for all } x \in S_r.$$

For $r > 0$, we let $n(r) = m(r) - 1$ denote the *branching* of T, that is, the number of edges connecting a vertex in S_r with vertices in S_{r+1}, and let

$n(0) = m(x_0)$. We then denote such trees T_n to indicate that their structure is completely encoded in the branching function $n(r)$.

The main purpose of this section is to prove the following theorem which tells us precisely when such trees are stochastically complete.

Theorem 3.2.1. *T_n is stochastically complete if and only if*

$$\sum_{r=0}^{\infty} \frac{1}{n(r)} = \infty.$$

The idea of the proof is to study positive λ-harmonic functions on T_n for $\lambda < 0$. By averaging over spheres we can reduce to the case of λ-harmonic functions depending only on the distance from the root x_0. It turns out that such a function will be bounded if and only if the series above converges. Since, by Theorem 3.1.3, the existence of a positive, bounded, λ-harmonic function is equivalent to stochastic incompleteness, Theorem 3.2.1 follows.

Let, therefore, $v(r)$ denote a function on the vertices of T_n depending only on $r = r(x) = d(x, x_0)$ the distance from a vertex to the root. When the Laplacian is applied to such a function it follows that

$$\begin{aligned} \Delta v(r) &= \big(n(r)+1\big)v(r) - n(r)v(r+1) - v(r-1) \\ &= n(r)\Big(v(r) - v(r+1)\Big) + \Big(v(r) - v(r-1)\Big). \end{aligned}$$

We will study the existence and boundedness of such functions on T_n when, in addition, they are positive and λ-harmonic for a negative λ, that is, they satisfy $\Delta v(r) = \lambda v(r)$ for $\lambda < 0$. We start by showing that there is no loss in generality in restricting our study of λ-harmonic functions to only functions of this type.

Lemma 3.2.1. *If there exists a positive, bounded λ-harmonic function on T_n then there exists one depending only on the distance from x_0.*

Proof: Let $w(x)$ denote a positive, bounded λ-harmonic function on T_n. If S_r denotes the sphere of radius r about the root x_0 and $\mathrm{Vol}(S_r)$ denotes its volume, that is,

$$\mathrm{Vol}(S_r) = \#\{x \mid x \in S_r\}$$

then we define a function $v(r)$ depending only on the radius r by averaging w over S_r:

$$v(r) = \frac{1}{\mathrm{Vol}(S_r)} \sum_{x \in S_r} w(x).$$

Clearly, v is positive and bounded since w is. We now show that $v(r)$ is also λ-harmonic. At x_0, since $\mathrm{Vol}(S_1) = n(0)$, it follows that

$$\begin{aligned}
\Delta v(0) &= n(0)\Big(v(0) - v(1)\Big) \\
&= n(0)\left(w(x_0) - \frac{1}{n(0)} \sum_{x \in S_1} w(x)\right) \\
&= n(0)w(x_0) - \sum_{x \sim x_0} w(x) \\
&= \Delta w(x_0) = \lambda w(x_0) = \lambda v(0).
\end{aligned}$$

If $x \in S_r$ for $r > 0$ then $\Delta w(x) = \lambda w(x)$ can be written as

$$\begin{aligned}
\Delta w(x) &= \big(n(r)+1\big)w(x) - \sum_{\substack{z \sim x \\ z \in S_{r+1}}} w(z) - w(y) \\
&= \lambda w(x)
\end{aligned}$$

where y is the unique neighbor of x that is in S_{r-1}. Therefore,

$$\big(n(r)+1-\lambda\big)w(x) = \sum_{\substack{z \sim x \\ z \in S_{r+1}}} w(z) + w(y). \tag{3.8}$$

We will average this equation over S_r and use the fact that $n(r) \cdot \mathrm{Vol}(S_r) = \mathrm{Vol}(S_{r+1})$ which we write as

$$\frac{1}{\mathrm{Vol}(S_r)} = \frac{n(r)}{\mathrm{Vol}(S_{r+1})}. \tag{3.9}$$

From the definition of v it follows that

$$\begin{aligned}
\big(n(r)+1-\lambda\big)v(r) &= \frac{(n(r)+1-\lambda)}{\mathrm{Vol}(S_r)} \sum_{x \in S_r} w(x) \\
&= \frac{1}{\mathrm{Vol}(S_r)} \sum_{x \in S_r} \left(\sum_{\substack{z \sim x \\ z \in S_{r+1}}} w(z) + \sum_{\substack{y \sim x \\ y \in S_{r-1}}} w(y) \right) && (3.8) \\
&= \frac{1}{\mathrm{Vol}(S_r)} \sum_{z \in S_{r+1}} w(z) + \frac{n(r-1)}{\mathrm{Vol}(S_r)} \sum_{y \in S_{r-1}} w(y) \\
&= \frac{n(r)}{\mathrm{Vol}(S_{r+1})} \sum_{z \in S_{r+1}} w(z) + \frac{1}{\mathrm{Vol}(S_{r-1})} \sum_{y \in S_{r-1}} w(y) && (3.9) \\
&= n(r)v(r+1) + v(r-1).
\end{aligned}$$

Rewriting this gives $\Delta v(r) = \lambda v(r)$. □

Therefore, on model trees, the existence of a positive, bounded λ-harmonic function is equivalent to the existence of such a function depending only on the distance from the root. The values of such a function are uniquely determined by the value of the function at x_0 and are given by

$$v(1) = \left(1 - \frac{\lambda}{n(0)}\right) v(0) \tag{3.10}$$

$$v(r+1) = \frac{1}{n(r)} \Big(\big(n(r)+1-\lambda\big)v(r) - v(r-1) \Big). \tag{3.11}$$

We now study under what conditions such a function remains bounded. We start by showing that such a function must increase with the radius.

Lemma 3.2.2. *If $v > 0$ satisfies $\Delta v(r) = \lambda v(r)$ for $\lambda < 0$ then*

$$v(r) < v(r+1) \quad \textit{for all } r \geq 0.$$

Proof: The proof is by induction. That $v(0) < v(1)$ follows from (3.10).

Assuming $v(r-1) < v(r)$

$$\Delta v(r) = n(r)\Big(v(r) - v(r+1)\Big) + \Big(v(r) - v(r-1)\Big) = \lambda v(r)$$

gives

$$n(r)\Big(v(r) - v(r+1)\Big) = \lambda v(r) - \Big(v(r) - v(r-1)\Big) < 0$$

implying

$$v(r) < v(r+1).$$

□

Lemma 3.2.3. *If $v > 0$ satisfies $\Delta v(r) = \lambda v(r)$ for $\lambda < 0$ then*

$$\prod_{i=0}^{r}\left(1 - \frac{\lambda}{n(i)}\right) v(0) < v(r+1) < \prod_{i=0}^{\infty}\left(1 + \frac{1-\lambda}{n(i)}\right) v(0).$$

Consequently, for $\lambda < 0$, a positive, λ-harmonic function on T_n depending only on the distance from the root remains bounded if and only if $\prod_{i=0}^{\infty}\left(1 + \frac{1}{n(i)}\right) < \infty$ which is equivalent to $\sum_{i=0}^{\infty} \frac{1}{n(i)} < \infty$. That is, the following conditions are equivalent:

1) $v(r)$ is bounded

2) $\prod_{i=0}^{\infty}\left(1 + \frac{1}{n(i)}\right) < \infty$

3) $\sum_{i=0}^{\infty} \frac{1}{n(i)} < \infty$.

Proof: For the upper bound, rewrite the equation $\Delta v(r) = \lambda v(r)$ as

$$\Big(n(r) + 1 - \lambda\Big) v(r) - n(r) v(r+1) = v(r-1) > 0.$$

Therefore,

$$\Big(n(r) + 1 - \lambda\Big) v(r) > n(r) v(r+1)$$

or

$$\begin{aligned} v(r+1) &< \frac{(n(r) + 1 - \lambda)}{n(r)} v(r) \\ &= \left(1 + \frac{1-\lambda}{n(r)}\right) v(r). \end{aligned}$$

Now, iterate this inequality down to $v(0)$:

$$\begin{aligned} v(r+1) &< \left(1 + \frac{1-\lambda}{n(r)}\right) v(r) \\ &< \left(1 + \frac{1-\lambda}{n(r)}\right)\left(1 + \frac{1-\lambda}{n(r-1)}\right) v(r-1) \\ &< \prod_{i=0}^{r} \left(1 + \frac{1-\lambda}{n(i)}\right) v(0) \\ &< \prod_{i=0}^{\infty} \left(1 + \frac{1-\lambda}{n(i)}\right) v(0). \end{aligned}$$

For the lower bound use Lemma 3.2.2, which implies that $v(r) - v(r-1) > 0$, as follows

$$\begin{aligned} \Delta v(r) &= n(r)\Big(v(r) - v(r+1)\Big) + \Big(v(r) - v(r-1)\Big) \\ &> n(r)\Big(v(r) - v(r+1)\Big). \end{aligned}$$

Since $\Delta v(r) = \lambda v(r)$ this yields

$$n(r)\Big(v(r) - v(r+1)\Big) < \lambda v(r)$$

or

$$\left(1 - \frac{\lambda}{n(r)}\right) v(r) < v(r+1).$$

Iterating as before implies

$$\prod_{i=0}^{r} \left(1 - \frac{\lambda}{n(i)}\right) v(0) < v(r+1)$$

completing the proof of the lemma. □

Proof of Theorem 3.2.1: By Theorem 3.1.3 stochastic incompleteness is equivalent to the existence of a positive, bounded, λ-harmonic function for $\lambda < 0$. We can define such a function on T_n depending only on the distance from x_0 by (3.10) and (3.11). If $\sum_{r=0}^{\infty} \frac{1}{n(r)} < \infty$ this function will remain bounded by Lemma 3.2.3.

If $\sum_{r=0}^{\infty} \frac{1}{n(r)} = \infty$ then every positive, λ-harmonic function depending only on the distance from the root will be unbounded. Therefore, every positive, λ-harmonic function on T_n will be unbounded by Lemma 3.2.1 so that T_n is stochastically complete. □

Remark 3.2.2. We would like to point out the relationship between Theorem 3.2.1 and the case of *spherically symmetric* or *model* manifolds on which we base our definition of model trees. M_σ, a Riemannian manifold of dimension d with *pole* $o \in M_\sigma$, is called *model* if

1) Topologically, $M_\sigma \setminus \{o\}$ is the product of an open interval I and the sphere S^{d-1}. Therefore, each point $x \in M_\sigma \setminus \{o\}$ can be identified with a pair (r, θ) where $r \in I$ and $\theta \in S^{d-1}$.

2) The metric on M_σ is given by

$$ds^2 = dr^2 + \sigma^2(r)d\theta^2 \tag{3.12}$$

where $d\theta^2$ denotes the standard Euclidean metric on S^{d-1}. Here, σ is a smooth, positive function on I called the *twisting* or *warping function* [10, p. 145-148].

It follows from (3.12) that the area of a sphere of radius r in M_σ is given by

$$A(S_r) = \omega_d \sigma^{d-1}(r) \tag{3.13}$$

where ω_d is the area of the unit sphere in $\mathbf{R}^d$. Now, it is shown in [10, Corollary 6.8] that a geodesically complete, noncompact, model manifold is stochastically complete if and only if

$$\int^{\infty} \frac{\text{Vol}(B_r)}{A(S_r)} dr = \infty$$

where $\text{Vol}(B_r)$ denotes the Riemannian volume of the geodesic ball in M_σ. For example, if, for large r, $A(S_r) \leq e^{r^2}$ then M_σ is stochastically complete, whereas, if $A(S_r) = e^{r^{2+\epsilon}}$ for any positive ϵ, then M_σ will be stochastically incomplete.

Observe from (3.13) that

$$\begin{aligned} dA(S_r) &= \omega_d(d-1)\sigma^{d-2}(r)\sigma'(r)\ dr \\ &= (d-1)\omega_d(\ln \sigma(r))' A(S_r)\ dr \end{aligned}$$

implying

$$\frac{dA(S_r)}{A(S_r)} = c(\ln \sigma(r))'\ dr \tag{3.14}$$

where $c = (d-1)\, \omega_d$.

Meanwhile, for model trees, temporarily using the notation $A(S_r) = \mathrm{Vol}(S_r)$ and $dA(S_r) = A(S_{r+1}) - A(S_r)$, it follows that

$$dA(S_r) = \big(n(r) - 1\big) A(S_r)$$

implying

$$\frac{dA(S_r)}{A(S_r)} = \big(n(r) - 1\big). \tag{3.15}$$

Therefore, comparing (3.14) and (3.15) we see that $n(r)$ and $(\ln \sigma(r))'$ play an analogous role in the two cases and, with respect to these two functions, the correspondence between the borderlines for stochastic completeness is exact.

3.3 Comparison Theorems

Throughout this section, we assume that T_n denotes a model tree with root vertex x_0 while T denotes a general tree. From Theorem 3.2.1 proved in the last section we know that T_n will be stochastically complete if and only if $\sum_{r=0}^{\infty} \frac{1}{n(r)} = \infty$ and we wish to obtain a similar criterion for T. In order to state our results for T we make the following definitions.

Definition 3.3.1. For a vertex $x_0 \in T$ let

$$\begin{aligned} \underline{m}(r) = \underline{m}_{x_0}(r) &= \min_{x \in S_r(x_0)} m(x) \\ M(r) = M_{x_0}(r) &= \max_{x \in S_r(x_0)} m(x). \end{aligned}$$

The following result is an immediate consequence of our criterion for model trees and the characterization of stochastic incompleteness in terms of λ-subharmonic functions.

Theorem 3.3.2. *Suppose that T_n is a model tree with $T_n \subseteq T$ where T is a general tree and $\underline{m}_{x_0}(r) = \underline{m}(r) = \min_{x \in S_r(x_0)} m(x)$ on T satisfies*

$$n(r) \leq \underline{m}(r) - 1 \text{ for all } r > 0.$$

Then if T_n is stochastically incomplete so is T.

Proof: Since T_n is stochastically incomplete, for every $\lambda < 0$, there exists a bounded, positive function $v(r)$ on T_n such that $v(r) < v(r+1)$ and $\Delta v(r) = \lambda v(r)$. Define a function w on T by letting, for all $x \in S_r(x_0) \subset T$,

$$w(x) = v(r).$$

Clearly $w(x)$ will be bounded and positive since v is. It follows from the inequalities $v(0) - v(1) < 0$ and $n(0) \leq m(x_0)$ that $w(x)$ is λ-subharmonic at x_0:

$$\begin{aligned}
\Delta w(x_0) &= m(x_0)w(x_0) - \sum_{x \sim x_0} w(x) \\
&= m(x_0)\Big(v(0) - v(1)\Big) \\
&\leq n(0)\Big(v(0) - v(1)\Big) \\
&= \lambda v(0) = \lambda w(x_0).
\end{aligned}$$

If $x \in S_r(x_0) \subset T$ where $r > 0$ and y denotes the unique neighbor of x in $S_{r-1}(x_0)$ then, since $n(r) \leq m(x) - 1$,

$$\begin{aligned}
\Delta w(x) &= m(x)w(x) - \sum_{\substack{z \sim x \\ z \in S_{r+1}}} w(z) - w(y) \\
&= (m(x) - 1)\Big(v(r) - v(r+1)\Big) + \Big(v(r) - v(r-1)\Big) \\
&\leq n(r)\Big(v(r) - v(r+1)\Big) + \Big(v(r) - v(r-1)\Big) \\
&= \lambda v(r) = \lambda w(x).
\end{aligned}$$

Thus, w is a positive, bounded λ-subharmonic function on T implying that T is stochastically incomplete. □

This result has the following immediate corollary:

Corollary 3.3.1. *If T is a tree with a vertex x_0 such that $\underline{m}_{x_0}(r) = \underline{m}(r) = \min_{x \in S_r(x_0)} m(x)$ satisfies $\underline{m}(r) > 1$ and*

$$\sum_{r=0}^{\infty} \frac{1}{\underline{m}(r)} < \infty$$

then T is stochastically incomplete.

Proof: From the assumption on T we can embed $T_n \subseteq T$ where T_n is defined by

$$n(r) = \underline{m}(r) - 1 \text{ for } r > 0$$

and $n(0) = m(x_0)$. Then $\sum_{r=0}^{\infty} \frac{1}{n(r)} < \infty$ implying that T_n is stochastically incomplete and so is, by Theorem 3.3.2, T. □

Remark 3.3.3. This theorem and its corollary are unsatisfactory in the sense that they require the tree to grow very rapidly in every direction from x_0 in order to be stochastically incomplete. However, as we will see in Theorem 3.4.1 in the next section, it is sufficient that the tree grows uniformly rapidly in some direction from x_0.

We now present a criterion for the stochastic completeness of a general graph G.

Theorem 3.3.4. *If G is a graph with a vertex x_0 such that $M_{x_0}(r) = M(r) = \max_{x \in S_r(x_0)} m(x)$ satisfies*

$$\sum_{r=0}^{\infty} \frac{1}{M(r)} = \infty$$

then G is stochastically complete.

Proof: Let v be a positive, λ-harmonic function on G for $\lambda < 0$. We will show that, under the assumption on G, v must be unbounded. At x_0, the equation $\Delta v(x_0) = \lambda v(x_0)$ implies that

$$\sum_{x \sim x_0} v(x) = \big(m(x_0) - \lambda\big) v(x_0). \tag{3.16}$$

Thus, there exists $x_1 \sim x_0$ such that

$$v(x_1) \geq \left(1 - \frac{\lambda}{m(x_0)}\right) v(x_0).$$

If not, then for all $x \sim x_0$, $v(x) < \left(1 - \frac{\lambda}{m(x_0)}\right) v(x_0)$, implying

$$\sum_{x \sim x_0} v(x) < m(x_0) \left(1 - \frac{\lambda}{m(x_0)}\right) v(x_0)$$

contradicting (3.16).

Now, by repeating the argument at x_1, it follows that there must exist a neighbor $y \sim x_1$ such that

$$v(y) \geq \left(1 - \frac{\lambda}{m(x_1)}\right) v(x_1).$$

Although y is not necessarily in $S_2(x_0)$ we repeat the argument until we obtain a vertex $x_2 \in S_2(x_0)$ such that

$$\begin{aligned} v(x_2) &\geq \left(1 - \frac{\lambda}{m(x_1)}\right) v(x_1) \\ &\geq \left(1 - \frac{\lambda}{m(x_1)}\right) \left(1 - \frac{\lambda}{m(x_0)}\right) v(x_0). \end{aligned}$$

Iterating this argument, there exists a sequence of distinct vertices $\{x_i\}_{i=0}^{\infty}$ such that $x_i \in S_i(x_0)$ and

$$v(x_r) \geq \prod_{i=0}^{r-1} \left(1 - \frac{\lambda}{m(x_i)}\right) v(x_0).$$

Since $\sum_{i=0}^{\infty} \frac{1}{m(x_i)} \geq \sum_{i=0}^{\infty} \frac{1}{M(i)} = \infty$ implies that $\prod_{i=0}^{\infty} \left(1 - \frac{\lambda}{m(x_i)}\right) = \infty$ it follows that v cannot remain bounded, implying that G must be stochastically complete. □

Remark 3.3.5. Theorem 3.3.4 is a significant improvement over the result mentioned in the introduction which states that a general graph is stochastically complete if the valence is bounded above by a constant [7, Theorem 2.10]. In fact, the proof there can be extended to show that if $M(r)$ is $o(r)$ then the graph is stochastically complete whereas our result says that $M(r)$ can be $O(r)$. The case of model trees shows that the result of Theorem 3.3.4 is optimal.

A corollary of Theorem 3.3.4 for trees is the following:

Corollary 3.3.2. *Suppose that T_n is a model tree with root vertex x_0 and $T \subseteq T_n$ with $x_0 \in T$. If T_n is stochastically complete then so is T.*

Proof: Since T_n is stochastically complete it follows that $\sum_{r=0}^{\infty} \frac{1}{n(r)} = \infty$ implying $\sum_{r=0}^{\infty} \frac{1}{M(r)} = \infty$ so that T is stochastically complete. □

3.4 General Trees

The purpose of this section is to follow-up on the remark following Corollary 3.3.1. The result there states that a general tree T will be stochastically incomplete if, starting out at a fixed vertex, the branching grows rapidly in all directions from that vertex. The next theorem states that the same conclusion holds if the branching grows uniformly rapidly in one direction from the vertex.

We start by slightly altering the notation used in the previous section. If x_0 and x_1 are vertices of T with $x_0 \sim x_1$ then we now denote

$$\underline{m}(r) = \underline{m}_{\{x_0,x_1\}}(r) = \min_{\substack{x \in S_r(x_0) \\ d(x,x_1)=r-1}} m(x) \quad \text{for } r \geq 1$$

so that the minimum is now taken over those x in $S_r(x_0)$ such that $d(x, x_1) = r - 1$.

Theorem 3.4.1. *If T is a tree with a vertex $x_0 \in T$ such that, for some $x_1 \sim x_0$, $\underline{m}(r) = \underline{m}_{\{x_0,x_1\}}(r)$ satisfies $\underline{m}(r) > 1$ and*

$$\sum_{r=1}^{\infty} \frac{1}{\underline{m}(r)} < \infty$$

then T is stochastically incomplete.

The proof of the theorem will use the following general proposition:

Proposition 3.4.1. *Let G denote a graph with $x_0 \in G$. Then, for every $\lambda < 0$, there exists a function v on G such that $v(x_0) = 1$ with $0 < v(x) < 1$ and $\Delta v(x) = \lambda v(x)$ for all vertices $x \neq x_0$.*

Proof of Theorem 3.4.1: Assuming Proposition 3.4.1 we apply it to define a positive, bounded λ-harmonic function v on the part of T below x_0 in Figure 3.1 with $v(0) = v(x_0) = 1$. For the part of T above x_0 in Figure 3.1, the assumption on T implies that we can embed a stochastically incomplete

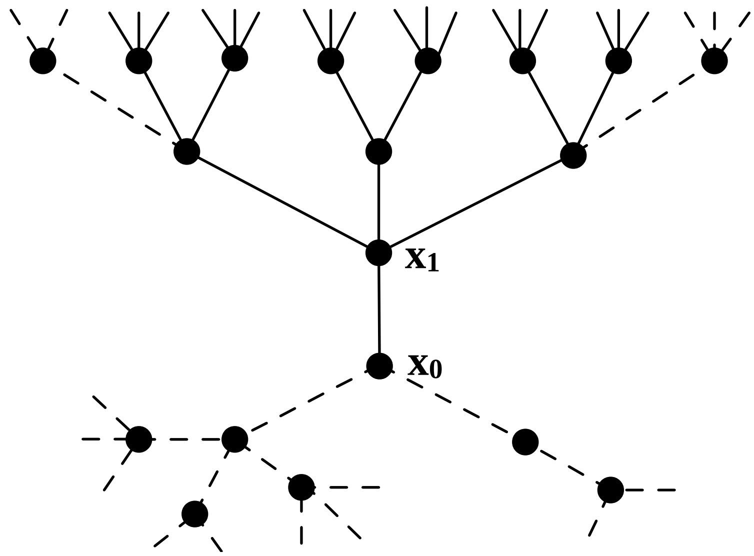

Figure 3.1: T with T_n in solid.

model subtree T_n with root vertex x_0 where $n(0) = 1$ and $n(r) = \underline{m}(r) - 1$ for $r \geq 1$. We extend the function v to be defined on T_n by first making it λ-harmonic at x_0

$$v(1) = v(x_1) = \big(m(x_0) - \lambda\big) - \sum_{\substack{x \sim x_0 \\ x \neq x_1}} v(x).$$

Then, if r denotes the distance to x_0, v can be defined on the rest of T_n as before by

$$v(r+1) = \left(1 + \frac{1-\lambda}{n(r)}\right) v(r) - \frac{1}{n(r)} v(r-1) \qquad \text{for } r \geq 2.$$

This function will remain bounded since $\sum_{r=1}^{\infty} \frac{1}{n(r)} < \infty$. Therefore, by the argument used in the proof of Theorem 3.3.2, there exists a positive, bounded λ-subharmonic function on T. By Theorem 3.1.3 T is stochastically incomplete. $\square$

Proof of Proposition 3.4.1: Let $B_r(x_0)$ denote the ball of radius r about x_0 in G. For $\lambda < 0$, there exists a unique solution to the following system of equations:

$$\begin{cases} \Delta v_r(x) = \lambda v_r(x) & \text{for } x \in \text{int } B_r \setminus \{x_0\} \\ v_r(x_0) = 1 & \\ v_r(x) = 0 & \text{for } x \in \partial B_r. \end{cases} \tag{3.17}$$

Indeed, from basic linear algebra, since the system has the same number of equations as there are values for v_r, there will exist a unique solution if 0 is the only function which satisfies

$$\begin{cases} \Delta w(x) = w(x) & \text{for } x \in \text{int } B_r \setminus \{x_0\} \\ w(x_0) = 0 & \\ w(x) = 0 & \text{for } x \in \partial B_r. \end{cases} \tag{3.18}$$

To show that this is so, suppose that w is a non-zero solution of (3.18). We can then assume that there exists a vertex $\hat{x}$ in the interior such that $w(\hat{x}) > 0$ and $\hat{x}$ is a maximum for w on $B_r(x_0)$. Then, by calculation, $\Delta w(\hat{x}) \geq 0$, while $\Delta w(\hat{x}) = \lambda w(\hat{x}) < 0$ giving a contradiction. The same argument could be used to show that w cannot have a negative minimum. Therefore, any solution to (3.18) must be zero. This gives the existence and uniqueness of a solution to (3.17).

Therefore, for $\lambda < 0$ and each r there exists a unique solution of

$$\begin{cases} \Delta v_r(x) = \lambda v_r(x) & \text{for } x \in \text{int } B_r \setminus \{x_0\} \\ v_r(x_0) = 1 & \\ v_r(x) = 0 & \text{for } x \in \partial B_r. \end{cases}$$

For vertices $x \in$ int $B_r \setminus \{x_0\}$ the solution v_r satisfies

$$0 < v_r(x) < 1.$$

For, supposing that there exists a vertex $\hat{x}$ in the interior such that $v_r(\hat{x}) \leq 0$ and $\hat{x}$ is a minimum for v_r then, by calculation, $\Delta v_r(\hat{x}) \leq 0$ while $\Delta v_r(\hat{x}) = \lambda v_r(\hat{x}) \geq 0$. Therefore, $v_r(x) = v_r(\hat{x})$ for all $x \sim \hat{x}$ and, by repeating the argument, it would follow that v_r is a constant function for a non-positive constant yielding a contradiction since $v_r(x_0) = 1$. Hence, $v_r > 0$ implying $\Delta v_r < 0$ for all vertices in the interior except for x_0 so that $v_r < 1$ since at an interior maximum $\Delta v_r \geq 0$.

Similarly, since, on int $B_r \setminus \{x_0\}$, $\Delta(v_{r+1} - v_r) = \lambda(v_{r+1} - v_r)$, it follows that $v_r \leq v_{r+1}$ on B_r and, by extending each v_r to be 0 outside of B_r, it follows that

$$v_r \leq v_{r+1} \text{ on } G.$$

We define v as the limit

$$v = \lim_{r \to \infty} v_r.$$

It follows that v satisfies $0 < v < 1$, $v(x_0) = 1$, and $\Delta v = \lambda v$ for all vertices of G except for x_0. □

3.5 Heat Kernel Comparison

The purpose of this section is to prove two theorems which compare the heat kernel on a general graph and the heat kernel on a model tree. These theorems were inspired by an analogous result of Cheeger and Yau on model manifolds [1, Theorem 3.1]. Fixing a vertex x_0' in G we denote, for $x \in S_r(x_0')$,

$$\begin{aligned} m_{+1}(x) &= \#\{y \in S_{r+1}(x_0') \mid y \sim x\} \\ m_{-1}(x) &= \#\{y \in S_{r-1}(x_0') \mid y \sim x\} \end{aligned}$$

the number of vertices adjacent to x which are in the next or previous sphere. We use the notation $\rho_t(x, y)$ for the heat kernel on T_n while $p_t(x, y)$ will denote the heat kernel on G. We will first show that, as a function of x, $\rho_t(x_0, x)$ is constant on spheres $S_r(x_0)$ in T_n where x_0 denotes the root of T_n. Let $\rho_t(r) = \rho_t(x_0, r(x))$ denote this common value where $r(x) = d(x, x_0)$. Then the two main theorems of the section can be stated as follows:

Theorem 3.5.1. *If G contains a vertex x_0' such that for all $x \in S_r(x_0')$ $m_{+1}(x) \leq n(r)$ then*

$$\rho_t(r) \leq p_t(x_0', x)$$

for $x \in S_r(x_0') \subset G$.

Theorem 3.5.2. *If G contains a vertex x_0' such that for all $x \in S_r(x_0')$ $n(r) \leq m_{+1}(x)$ and $m_{-1}(x) = 1$ then*

$$p_t(x_0', x) \leq \rho_t(r)$$

for $x \in S_r(x_0') \subset G$.

The proofs will follow easily from the maximum principle for the heat equation once we establish two general lemmas concerning the heat kernel on T_n. We start by proving the property of the heat kernel mentioned at the start of this section.

Lemma 3.5.1. *If T_n is a model tree with root vertex x_0 then*

$$\rho_t(x_0, x) = \rho_t(r)$$

for $x \in S_r(x_0)$.

Proof: This result is a restatement of the fact that the coefficients of the Laplacian depend only on the valence which, for T_n, only depends on the distance from the root.

We establish the result for the heat kernels $\rho_t^R(x_0, x)$ on $B_R(x_0)$ with Dirichlet boundary conditions and pass to the limit. The heat kernel $\rho_t^R(x_0, x)$ is the kernel of the operator semigroup

$$e^{-t\Delta_R} = I - t\Delta_R + \frac{t^2\Delta_R^2}{2} - \frac{t^3\Delta_R^3}{6} + \dots$$

where Δ_R denotes the reduced Laplacian on $B_R(x_0)$. That is,

$$\rho_t^R(x_0, x) = e^{-t\Delta_R}\delta_{x_0}(x) = \langle \delta_{x_0}, \delta_x \rangle - t\Delta_R(x_0, x) + \frac{t^2}{2}\Delta_R^2(x_0, x) - \dots$$

where the coefficients of the Laplacian $\Delta_R(x_0, x)$ are given by

$$\begin{aligned} \Delta_R(x_0, x) &= \Delta_R\delta_{x_0}(x) \\ &= \begin{cases} n(0) & \text{if } x = x_0 \\ -1 & \text{if } x \in S_1(x_0) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\Delta_R^{m+n}(x_0, x) = \sum_{y \in B_R} \Delta_R^m(x_0, y)\Delta_R^n(y, x).$$

Therefore, these only depend on the distance between x_0 and x. □

Lemma 3.5.2. *If T_n is a model tree with root vertex x_0 and heat kernel $\rho_t(r) = \rho_t(x_0, x)$ for $x \in S_r(x_0)$ then*

$$\rho_t(r) \geq \rho_t(r+1) \quad \textit{for } r \geq 0.$$

Proof: We start with the general fact that for any graph

$$\frac{\partial}{\partial t} p_t(x,x) \le 0 \quad \text{for } t > 0.$$

Indeed, considering the heat kernels $p_t^R(x,y)$ on $B_R(x_0)$ with Dirichlet boundary conditions and using the eigenfunction expansion it follows that

$$p_t^R(x,x) = \sum_{i=0}^{k(R)} e^{-\lambda_i^R t} \big(\phi_i^R(x)\big)^2.$$

This implies

$$\frac{\partial}{\partial t} p_t^R(x,x) = \sum_{i=0}^{k(R)} -\lambda_i^R e^{-\lambda_i^R t} \big(\phi_i^R(x)\big)^2 < 0$$

since $\lambda_i^R > 0$ and $\{\phi_i^R\}_{i=0}^{k(R)}$ is an orthonormal basis for $C(B_R, \partial B_R)$ so that $\phi_i^R(x)$ cannot be zero for all i. By passing to the limit it follows that $\frac{\partial}{\partial t} p_t(x,x) \le 0$.

Therefore, in particular, $\frac{\partial}{\partial t}\rho_t^R(0) < 0$ implying that

$$\Delta \rho_t^R(0) = n(0)\Big(\rho_t^R(0) - \rho_t^R(1)\Big) > 0$$

or

$$\rho_t^R(0) > \rho_t^R(1).$$

Furthermore,

$$\rho_t^R(R-1) > \rho_t^R(R) = 0$$

so that, as a function of r, $\rho_t^R(r)$ is decreasing at $r = 0$ as well as at $r = R-1$ for all $t > 0$.

Consider now the function

$$\varphi(t) = \min_{i<j} \big(\rho_t^R(i) - \rho_t^R(j)\big).$$

The claim is that $\varphi(t) > 0$ for $t > 0$. Suppose not, that is, suppose that there exists $t_0 > 0$ such that $\varphi(t_0) \leq 0$. Therefore, there exist $i_0 < j_0$ such that $\rho_{t_0}^R(i_0) \leq \rho_{t_0}^R(j_0)$. We can then assume that $\rho_{t_0}^R(r)$ has a global minimum at $r = i_0$ and a global maximum at $r = j_0$ for $0 < i < j < R$. Then, from $\rho_t^R(0) > \rho_t^R(1)$ and $\rho_t^R(R-1) > \rho_t^R(R)$ shown above, we may assume that

$$\rho_{t_0}^R(i_0 - 1) > \rho_{t_0}^R(i_0) \text{ and } \rho_{t_0}^R(j_0) > \rho_{t_0}^R(j_0 + 1)$$

implying that

$$\Delta\rho_{t_0}^R(i_0) < 0 \text{ and } \Delta\rho_{t_0}^R(j_0) > 0.$$

Therefore,

$$\frac{\partial}{\partial t}\rho_{t_0}^R(i_0) > 0 \text{ and } \frac{\partial}{\partial t}\rho_{t_0}^R(j_0) < 0$$

which implies that $\varphi'(t_0) > 0$. It follows that there exists an $\epsilon > 0$ such that for all $t \in (t_0 - \epsilon, t_0]$, $\varphi(t) < \varphi(t_0) \leq 0$.

Let I denote the maximal interval contained in $[0, t_0]$ which contains t_0 and all $t < t_0$ for which $\varphi(t) \leq 0$. It is clear from the continuity of φ that I is closed. If $I = [a, t_0]$ for some $a > 0$ then the argument above implies that there exists an $\epsilon > 0$ such that all $t \in (a - \epsilon, a]$ are in I contradicting the maximality of I. If $I = [0, t_0]$ then it would follow that $\varphi(0) < 0$ contradicting the fact that $\varphi(0) = 0$. In either case, we obtain a contradiction implying that $\varphi(t) > 0$ for all $t > 0$ and, therefore, that

$$\rho_t^R(r) > \rho_t^R(r + 1).$$

Letting $R \to \infty$ completes the proof of the lemma. □

Proof of Theorems 3.5.1 and 3.5.2: The proofs of the the two Theorems are nearly identical so we give details for the proof of Theorem 3.5.1 and point out the modifications needed for the proof of Theorem 3.5.2. We will denote the Laplacian on T_n by Δ_{T_n} to distinguish it from Δ_G, the Laplacian on G, wherever necessary.

For both Theorems, for $x \in S_r(x_0') \subset G$, we let $\rho_t(x) = \rho_t(r)$ so that $\rho_t(x)$ is now defined for $x \in G$. For Theorem 3.5.1, by assumption, $m_{+1}(x) \leq n(r)$ and we want to show that $\rho_t(r) \leq p_t(x_0', x)$ for vertices $x \in S_r(x_0') \subset G$. We work with the Dirichlet heat kernels on $B_R(x_0') \subset G$ and consider the function

$$u^R(x,t) = \rho_t^R(x) - p_t^R(x_0', x)$$

on $B_R \times [0, T]$. Then

$$u^R(x,0) = 0 \text{ for } x \in B_R$$

and

$$u^R(x,t) = 0 \text{ for } x \in \partial B_R,\ t \geq 0.$$

Furthermore, it follows from $m_{+1}(x) \leq n(r)$ and from Lemma 3.5.2 that $\rho_t^R(x)$ satisfies the following inequality for $x \in S_r(x_0') \subset G$:

$$\begin{aligned}
\Delta_G \rho_t^R(x) &= m_{+1}(x)\Big(\rho_t^R(r) - \rho_t^R(r+1)\Big) \\
&\quad + m_{-1}(x)\Big(\rho_t^R(r) - \rho_t^R(r-1)\Big) \\
&\leq n(r)\Big(\rho_t^R(r) - \rho_t^R(r+1)\Big) \\
&\quad + \rho_t^R(r) - \rho_t^R(r-1) \\
&= \Delta_{T_n}\rho_t^R(r) = -\frac{\partial}{\partial t}\rho_t^R(r).
\end{aligned}$$

Therefore, $u^R(x,t)$ satisfies

$$\Delta_G u^R(x,t) + \frac{\partial}{\partial t} u^R(x,t) \leq 0.$$

By applying the maximum principle for the heat equation (see Remark 1.2.1) it follows that

$$\max_{B_R\times[0,T]} u^R(x,t) = \max_{\substack{B_R\times\{0\}\ \cup \\ \partial B_R\times[0,T]}} u^R(x,t) = 0.$$

Therefore, $\rho_t^R(r) - p_t^R(x_0', x) \leq 0$ so that $\rho_t^R(r) \leq p_t^R(x_0', x)$ for all $x \in S_r(x_0') \subset G$. Since this holds for every R, by letting $R \to \infty$, it follows that

$$\rho_t(r) \leq p_t(x_0', x).$$

For Theorem 3.5.2, from the assumption that $n(r) \leq m_{+1}(x)$ and $m_{-1}(x) = 1$ for all $x \in S_r(x_0')$, it follows that

$$u^R(x,t) = \rho_t^R(r) - p_t^R(x_0, x)$$

now satisfies

$$\Delta_G u^R(x,t) + \frac{\partial}{\partial t} u_t^R(x,t) \geq 0.$$

This implies that

$$\min_{B_R\times[0,S]} u^R(x,t) = \min_{\substack{B_R\times\{0\}\ \cup \\ \partial B_R\times[0,S]}} u^R(x,t) = 0$$

yielding

$$p_t^R(x_0, x) \leq \rho_t^R(r).$$

□

3.6 Bounded Laplacian

In this section, we introduce the bounded Laplacian Δ_{bd} and prove that, for every graph, the heat kernel associated to Δ_{bd} is stochastically complete. In particular, bounded solutions of the heat equation involving Δ_{bd} with bounded initial conditions are unique. We refer to [5, 6] for the definitions involved.

We define the bounded Laplacian as the operator

$$\begin{aligned} \Delta_{bd} f(x) &= f(x) - \frac{1}{m(x)} \sum_{y \sim x} f(y) \\ &= \frac{1}{m(x)} \Delta f(x). \end{aligned}$$

In order for the analogue of Green's Theorem to hold we alter the inner product on the space of functions on the graph. We now let

$$\langle f, g \rangle_{bd} = \sum_{x \in V} f(x) g(x) m(x)$$

while keeping the inner product on edges the same as before. It is now true that

$$\langle \Delta_{bd} f, g \rangle_{bd} = \langle df, dg \rangle$$

for all f such that

$$\sum_{x \in V} f^2(x) m(x) < \infty$$

that is, for $f \in \ell^2_{bd}(V)$.

What distinguishes Δ_{bd} from Δ is that it is a bounded operator without the assumption $m(x) \leq M$ necessary to imply that Δ is bounded. This follows from the calculation

$$\begin{aligned}\langle df, df\rangle &= \sum_{[x,y]\in\tilde{E}} \big(f(y)-f(x)\big)^2 \\ &\leq 2\sum_{[x,y]\in\tilde{E}} \big(f^2(y)+f^2(x)\big) \\ &= 2\sum_{x\in V} f^2(x)m(x) \;=\; 2\langle f,f\rangle_{bd}.\end{aligned}$$

Therefore, $\|d\|\leq \sqrt{2}$ implying $\|\Delta_{bd}\|\leq 2$.

We next prove that any graph is stochastically complete with respect to this operator by studying λ-harmonic functions for Δ_{bd}.

Theorem 3.6.1. *If v is a positive function on G satisfying*

$$\Delta_{bd}v(x) = \lambda v(x)$$

for $\lambda < 0$ then v will be unbounded.

In particular, since the proof of the equivalence of the various formulations of stochastic incompleteness for Δ holds for Δ_{bd}, it follows that

Corollary 3.6.1. *For any bounded function u_0 there exists a unique bounded solution of*

$$\begin{cases} \Delta_{bd}u(x,t) + \frac{\partial u}{\partial t}(x,t) = 0 & \text{for } x\in V, t\geq 0 \\ u(x,0) = u_0(x) & \text{for } x \in V. \end{cases}$$

Proof of Theorem 3.6.1: The proof is the proof of Theorem 3.3.4 rewritten for the bounded Laplacian. Fix a vertex x_0 of G. We will show that there exists a sequence of distinct vertices

$$x_0 \sim x_1 \sim x_2 \sim \dots$$

such that

$$v(x_i) \to \infty \text{ as } i \to \infty.$$

At x_0,

$$\Delta_{bd} v(x_0) = v(x_0) - \frac{1}{m(x_0)} \sum_{x \sim x_0} v(x) = \lambda v(x_0)$$

implies that

$$\sum_{x \sim x_0} v(x) = m(x_0)(1 - \lambda) v(x_0). \tag{3.19}$$

Therefore, there exists a neighbor $x_1 \sim x_0$ such that

$$v(x_1) \geq (1 - \lambda) v(x_0).$$

If not, if $v(x) < (1 - \lambda) v(x_0)$ for all $x \sim x_0$, then

$$\sum_{x \sim x_0} v(x) < m(x_0)(1 - \lambda) v(x_0)$$

contradicting (3.19). Applying the argument now at x_1 there exists a neighbor $x_2 \sim x_1$ such that

$$v(x_2) \geq (1 - \lambda) v(x_1) \geq (1 - \lambda)^2 v(x_0).$$

In general, there exists a sequence of distinct vertices $x_0 \sim x_1 \sim x_2 \sim \ldots$ such that

$$v(x_i) \geq (1 - \lambda)^i v(x_0)$$

implying that

$$v(x_i) \to \infty \text{ as } i \to \infty.$$

$\square$

Chapter 4

Spectral Analysis

4.1 Bottom of the Spectrum

We recall a definition of $\lambda_0(\Delta)$, the bottom of the spectrum of the Laplacian on a general graph G, and prove a characterization of it in terms of λ-harmonic functions. This result was inspired by an analogous result in [15, Theorem 2.1] and was proven for the bounded Laplacian in [5].

Fix a vertex x_0 in G and let $B_r = B_r(x_0)$ denote the ball of radius r about x_0 with boundary ∂B_r as before. Let Δ_r denote the reduced Laplacian acting on the space $C(B_r, \partial B_r)$ of functions on B_r that vanish on the boundary ∂B_r. We then define $\lambda_0^r = \lambda_0(\Delta_r)$ by

$$\lambda_0^r = \lambda_0(\Delta_r) = \min_{\substack{f \in C(B_r, \partial B_r) \\ f \not\equiv 0}} \frac{\langle df, df \rangle}{\langle f, f \rangle}$$

and show, as in [3, Lemma 1.9], that

Lemma 4.1.1. *λ_0^r is the smallest eigenvalue of Δ_r. Furthermore, if f_0 is a* *function in* $C(B_r, \partial B_r)$ *such that*

$$\lambda_0^r = \frac{\langle df_0, df_0\rangle}{\langle f_0, f_0\rangle} \tag{4.1}$$

then $\Delta_r f_0 = \lambda_0^r f_0$ *and* f_0 *can be chosen so that* $f_0 > 0$ *on the interior of* B_r*.*

Proof: If λ is any eigenvalue of Δ_r with eigenfunction f then

$$\frac{\langle df, df\rangle}{\langle f, f\rangle} = \frac{\langle \Delta_r f, f\rangle}{\langle f, f\rangle} = \lambda$$

implies that $\lambda \geq \lambda_0^r$.

Now, if f_0 satisfies (4.1) above and $\{\lambda_i^r\}_{i=0}^{k(r)}$ are the eigenvalues of Δ_r with $\{\phi_i^r\}_{i=0}^{k(r)}$ a set of corresponding eigenfunctions which form an orthonormal basis for $C(B_r, \partial B_r)$ then

$$f_0 = \sum_{i=0}^{k(r)} a_i \phi_i^r$$

where $a_i = \langle f_0, \phi_i^r\rangle$. We wish to show that $a_i = 0$ if $\lambda_i^r \neq \lambda_0^r$. This can be seen as follows:

$$\begin{aligned}
0 \leq & \left\langle d\Big(f_0 - \sum_{i=0}^{k(r)} a_i \phi_i^r\Big), d\Big(f_0 - \sum_{j=0}^{k(r)} a_j \phi_j^r\Big) \right\rangle \\
= & \langle df_0, df_0\rangle - 2\sum_{i=0}^{k(r)} a_i \langle f_0, \Delta_r \phi_i^r\rangle + \sum_{i,j=0}^{k(r)} a_i a_j \langle \phi_i^r, \Delta_r \phi_j^r\rangle \\
= & \langle df_0, df_0\rangle - 2\sum_{i=0}^{k(r)} a_i^2 \lambda_i^r + \sum_{i,j=0}^{k(r)} a_i a_j \lambda_j^r \langle \phi_i^r, \phi_j^r\rangle \\
= & \langle df_0, df_0\rangle - \sum_{i=0}^{k(r)} a_i^2 \lambda_i^r
\end{aligned}$$

implies that

$$\langle df_0, df_0\rangle \ \geq \ \sum_{i=0}^{k(r)} a_i^2 \lambda_i^r.$$

While (4.1) yields

$$\langle df_0, df_0\rangle = \lambda_0^r \langle f_0, f_0\rangle = \lambda_0^r \sum_{i=0}^{k(r)} a_i^2.$$

Therefore, $a_i = 0$ if $\lambda_i^r \neq \lambda_0^r$.

Now, noting that

$$\langle f_0, f_0\rangle = \langle |f_0|, |f_0|\rangle$$

while

$$\langle df_0, df_0\rangle \;\geq\; \langle d|f_0|, d|f_0|\rangle$$

it is clear that (4.1) can only be decreased by replacing f_0 by $|f_0|$ and we may assume at the onset that $f_0 \geq 0$. Then, if there exists a vertex $\hat{x}$ in the interior of B_r where $f_0(\hat{x}) = 0$, it follows from $\Delta_r f_0 = \lambda_0^r f_0$ that,

$$\Delta_r f_0(\hat{x}) = -\sum_{x\sim\hat{x}} f_0(x) = 0.$$

Therefore, $f_0(x) = 0$ for all $x \sim \hat{x}$. Repeating this argument would imply that $f_0 = 0$ on the interior of B_r yielding a contradiction. □

It follows from Lemma 4.1.1 that

$$\lambda_0^r \geq \lambda_0^{r+1} > 0$$

and we define

$$\lambda_0 = \lambda_0(\Delta) = \lim_{r\to\infty} \lambda_0(\Delta_r).$$

Remark 4.1.1. It is clear that this number is independent of the choice of exhaustion sequence for the graph G. This follows since if $\{D_i\}_{i=0}^{\infty}$ is any other exhaustion sequence then for each R there exists I_R large enough such that $B_R \subseteq D_{I_R}$. Therefore, by Lemma 4.1.1, $\lambda_0^R \geq \lambda_0^{D_{I_R}}$. Reversing the roles of B_r and D_i, it follows that $\lambda_0^{B_r}$ and $\lambda_0^{D_i}$ converge to the same number. For future reference, we point out that $\lambda_0(\Delta)$ can also be defined as

$$\lambda_0(\Delta) = \inf_{\substack{f \in C_0(V) \\ f \not\equiv 0}} \frac{\langle df, df \rangle}{\langle f, f \rangle}$$

where $C_0(V)$ denotes the set of finitely supported functions on the graph G.

We now state and prove the following characterization of $\lambda_0(\Delta)$ in terms of λ-harmonic functions [15, Theorem 2.1].

Theorem 4.1.2. *For every $\lambda \leq \lambda_0(\Delta)$ there exists a positive function v satisfying $\Delta v = \lambda v$. For every $\lambda > \lambda_0(\Delta)$ there is no such function.*

Proof: The proof of the first part is a variation of the argument given in [3, Theorem 2.4]. See also [5, Proposition 1.5] for the case of the bounded Laplacian and [9, Lemma 1] for manifolds.

We start with the case of $\lambda = \lambda_0(\Delta)$. From Lemma 4.1.1, for each r, there exists a positive function v_r satisfying

$$\Delta_r v_r = \lambda_0^r v_r > 0 \text{ on int } B_r.$$

We normalize this function so that $v_r(x_0) = 1$ and extend it to be 0 outside of B_r. We will show that this function is bounded at each vertex x. Let $M(i) = M_{x_0}(i) = \max_{x \in S_i(x_0)} m(x)$ as before. Then if $x_1 \in S_1(x_0) \subseteq \text{int } B_r$ it follows from $\Delta v_r(x_0) > 0$ that

$$m(x_0) v_r(x_0) > \sum_{x \sim x_0} v_r(x) \geq v_r(x_1)$$

implying

$$v_r(x_1) < M(0).$$

By repeating the same argument it follows that if $x_i \in S_i$ where $i < r$ then

$$v_r(x_i) < M(i-1)M(i-2)\dots M(0). \tag{4.2}$$

Using the diagonal process we can find a subsequence of $\{v_r\}_{r=1}^{\infty}$ which converges for all vertices x. Denote this by

$$v_{r_k}(x) \to v(x) \text{ as } k \to \infty \text{ for each } x.$$

It follows that $\Delta v(x) = \lambda_0 v(x)$ for all vertices x, $v \geq 0$ and $v(x_0) = 1$. By using the same argument as in the proof of Lemma 4.1.1 if there exists an x where $v(x) = 0$ then v would have to be constantly 0 yielding a contradiction since $v(x_0) = 1$. This completes the proof for the case of $\lambda = \lambda_0$.

For the case of $\lambda < \lambda_0(\Delta)$ we modify the argument above as follows. First, as noted in the proof of the implication $2') \Rightarrow 2)$ in Theorem 3.1.3, since $\lambda < \lambda_0(\Delta) \leq \lambda_0(\Delta_r)$ the operator $(\Delta_r - \lambda I)$ is positive and hence invertible on $C(B_r, \partial B_r)$, the space of all functions on B_r which vanish on ∂B_r. Therefore, there exists a function v_r which satisfies

$$\begin{cases} \Delta_r v_r = \lambda v_r & \text{on int } B_r \\ v_{r|\partial B_r} = 1. \end{cases}$$

Indeed, as before, let $v_r = (\Delta_r - \lambda I)^{-1}(\lambda_{\text{int } B_r}) + \mathbf{1}$, where $\lambda_{\text{int } B_r}$ is the function equal to λ on every vertex in the interior of B_r and 0 elsewhere and $\mathbf{1}$ is equal to 1 on every vertex of B_r. We renormalize v_r so that it is equal to 1 at x_0 and call it u_r, that is, let

$$u_r = \frac{1}{v_r(x_0)} v_r.$$

Now, if $u_r \geq 0$ on the interior of B_r then $u_r > 0$ on the interior of B_r by the same argument used in the case of $\lambda = \lambda_0$ above. To show that $u_r \geq 0$ we assume that there exists a vertex $\hat{x}$ in the interior of B_r where $u_r(\hat{x}) < 0$ and let w be the function defined by

$$w(x) = \begin{cases} u_r(x) & \text{for } x \text{ such that } u_r(x) < 0 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $w(x) = 0$ if x is in the boundary of B_r, so the error term vanishes when we apply the analogue of Green's Theorem to w below. If x is a vertex in the interior of B_r such that $u_r < 0$ for x and all neighbors of x then $\Delta w(x) = \Delta u_r(x)$. If x is a vertex where $u_r(x) < 0$ and x has a neighbor y for which $u_r(y) \geq 0$ then $\Delta w(x) \geq \Delta u_r(x)$. Combining these, it follows that

$$\begin{aligned} \langle dw, dw \rangle &= \langle \Delta w, w \rangle \\ &= \sum_{\substack{x \in \text{int } B_r \\ u_r(x) < 0}} \Delta w(x) w(x) \\ &\leq \sum_{\substack{x \in \text{int } B_r \\ u_r(x) < 0}} \Delta u_r(x) u_r(x) \\ &= \lambda \sum_{\substack{x \in \text{int } B_r \\ u_r(x) < 0}} u_r^2(x) \\ &= \lambda \langle w, w \rangle \end{aligned}$$

implying

$$\frac{\langle dw, dw \rangle}{\langle w, w \rangle} \leq \lambda.$$

From Lemma 4.1.1 it follows that $\lambda_0(\Delta_r) \leq \lambda$ contradicting the assumption that $\lambda < \lambda_0(\Delta)$. Therefore, $u_r \geq 0$ on the interior of B_r and so $u_r > 0$ there as well.

Now, if $0 < \lambda < \lambda_0(\Delta)$ then we can use the same argument as above to show that $u_r(x)$ is bounded for all vertices x as in (4.2). If $\lambda \leq 0$ then, from $\Delta u_r = \lambda u_r$, the bound becomes

$$u_r(x_1) \leq M(0) - \lambda$$

for all $x_1 \in S_1$ and

$$u_r(x_i) \leq (M(i-1) - \lambda)(M(i-2) - \lambda) \dots (M(0) - \lambda)$$

for all $x_i \in S_i$. The remainder of the argument is the same as before. This completes the proof of the first part of Theorem 4.1.2.

The proof of the second part of Theorem 4.1.2 is adapted from [15, p. 761]. See [5] for a different proof involving the use of Green's Theorem. Suppose that there exists a positive function v such that $\Delta v = \lambda v$. Then, letting

$$u(x,t) = e^{-\lambda t} v(x)$$

and

$$w(x,t) = \sum_{y \in B_r} p_t^r(x,y) v(y)$$

it follows that both u and w satisfy the heat equation on int $B_r \times [0,T]$ and $u(x,0) = w(x,0) = v(x)$ on int B_r with $u(x,0) = v(x) > 0$ and $w(x,0) = 0$ on ∂B_r. By applying the maximum principle for the heat equation to the difference of the two functions it follows that

$$\min_{B_r \times [0,T]} (u - w) = \min_{\substack{B_r \times \{0\} \cup \\ \partial B_r \times [0,T]}} (u - w) \geq 0$$

by what was noted above and since $w(x,t)$ vanishes on ∂B_r while $u(x,t)$ is positive there. Therefore, $u(x,t) \geq w(x,t)$ or

$$e^{-\lambda t} v(x) \geq \sum_{y \in B_r} p_t^r(x,y) v(y) \text{ on } B_r \times [0,T].$$

Therefore,

$$v(x) \geq \sum_{y \in B_r} e^{\lambda t} p_t^r(x,y) v(y) = e^{(\lambda - \lambda_0^r)t} e^{\lambda_0^r t} \sum_{y \in B_r} p_t^r(x,y) v(y). \tag{4.3}$$

From $p_t^r(x,y) = \sum_{i=0}^{k(r)} e^{-\lambda_i^r t} \phi_i^r(x) \phi_i^r(y)$ it follows that

$$\lim_{t \to \infty} e^{\lambda_0^r t} p_t^r(x,y) = \phi_0^r(x) \phi_0^r(y)$$

and, since ϕ_0^r can be chosen so that $\phi_0^r > 0$, if $\lambda > \lambda_0^r$ then the right hand side of (4.3) would tend to ∞ as $t \to \infty$. Therefore, $\lambda \leq \lambda_0^r$ for all r implying $\lambda \leq \lambda_0(\Delta)$. □

4.1.1 Relationship to Stochastic Incompleteness

In light of the large volume growth that is required for a graph to be stochastically incomplete and the well-known relationship between the bottom of the spectrum and Cheeger's constant which is, at least partially, outlined in this subsection and the next section, it might seem plausible to conjecture that stochastic incompleteness would imply that $\lambda_0(\Delta) > 0$. The purpose of this subsection is to give an example where this is not the case.

The example is constructed as follows: we start with a model tree T_n which is stochastically incomplete, that is, such that $\sum_{r=0}^{\infty} \frac{1}{n(r)} < \infty$, and attach to the root vertex x_0 an infinitely long path $x_0 \sim x_1 \sim x_2 \sim \ldots$. The resulting tree is stochastically incomplete by Theorem 3.4.1.

We now show that $\lambda_0(\Delta) = 0$. As noted before

$$\lambda_0 \leq \frac{\langle df, df \rangle}{\langle f, f \rangle} \tag{4.4}$$

for every nonzero, finitely supported function f. By taking a finite subgraph D and letting $f = 1_D$ in (4.4) it follows that

$$\lambda_0 \leq \frac{L(\partial D)}{\mathrm{Vol}(D)}$$

where

$$L(\partial D) = \#\{y \sim x \mid x \in D \text{ and } y \notin D\}$$

that is, the number of edges with one vertex in D and one not in D, and $\mathrm{Vol}(D)$ denotes the number of vertices in D. By taking increasingly larger connected subgraphs of the path that was attached at the root vertex x_0 of T_n, it is clear that this ratio goes to 0 since $L(\partial D) = 2$ for all such subgraphs.

4.2 Lower Bounds

In this section we prove some estimates of $\lambda_0(\Delta)$, the bottom of the spectrum of Δ, introduced in the last section. In order to prove our results we work with the bounded Laplacian Δ_{bd} and use the characterization of $\lambda_0(\Delta_{bd})$ in terms of Cheeger's constant proved in [6] to obtain a lower bound for $\lambda_0(\Delta_{bd})$. Comparing inner products, we then transfer this result to obtain a lower bound for $\lambda_0(\Delta)$.

Recall that Δ_{bd} is given by

$$\Delta_{bd} f(x) = f(x) - \frac{1}{m(x)} \sum_{y \sim x} f(y) = \frac{1}{m(x)} \Delta f(x).$$

The bottom of the spectrum is, as in the case of Δ, given by an exhaustion argument or, equivalently, as

$$\lambda_0(\Delta_{bd}) = \inf_{\substack{f \in C_0(V) \\ f \not\equiv 0}} \frac{\langle \Delta_{bd} f, f \rangle_{bd}}{\langle f, f \rangle_{bd}}$$

where the infimum is taken over all nonzero, finitely supported functions f and the inner product is given by

$$\langle f, f\rangle_{bd} = \sum_{x\in V} f^2(x)m(x).$$

For a finite subgraph D we define $A(D)$, the *area of* D, to be

$$A(D) = \sum_{x\in D} m(x)$$

and $L(\partial D)$, the *length of the boundary*, to be

$$L(\partial D) = \#\{y \sim x \mid x \in D \text{ and } y \notin D\}$$

as in the previous subsection. If we let

$$\alpha = \inf_{\substack{D\subset G \\ D \text{ finite, connected}}} \frac{L(\partial D)}{A(D)}$$

then the main result in Section 2 of [6] states that

Theorem 4.2.1.

$$\lambda_0(\Delta_{bd}) \geq \frac{\alpha^2}{2}.$$

In fact, the proof in [6] applies in the following more general context. Let A denote a finite subgraph of G and let $\Delta_{bd,G\setminus A}$ denote the reduced bounded Laplacian which is equal to Δ_{bd} on the complement of A and 0 on A. $\Delta_{bd,G\setminus A}$ acts on the space of ℓ^2_{bd} functions which vanish on A. If

$$\alpha(G\setminus A) = \inf_{\substack{D \text{ finite} \\ D\cap A=\emptyset}} \frac{L(\partial D)}{A(D)}$$

then the proof in [6] implies

$$\lambda_0(\Delta_{bd,G\setminus A}) \geq \frac{1}{2}\alpha^2(G\setminus A). \tag{4.5}$$

We fix a vertex x_0 of G and let $r(x) = d(x, x_0)$. Then for a vertex $x \in G$ we let

$$\begin{aligned} m_0(x) &= \#\{y \mid y \sim x \text{ and } r(y) = r(x)\} \\ m_{+1}(x) &= \#\{y \mid y \sim x \text{ and } r(y) = r(x) + 1\} \\ m_{-1}(x) &= \#\{y \mid y \sim x \text{ and } r(y) = r(x) - 1\} \end{aligned}$$

as before. The main theorem of this section states:

Theorem 4.2.2. *If A denote a finite subgraph of G and for all vertices $x \in G \setminus A$*

$$\frac{m_{+1}(x) - m_{-1}(x)}{m(x)} \geq c > 0$$

then

$$\lambda_0(\Delta_{bd, G\setminus A}) \geq \frac{c^2}{2}.$$

If, in addition, $m(x) \geq m$ for $x \in G \setminus A$ then

$$\lambda_0(\Delta_{G\setminus A}) \geq \frac{mc^2}{2}. \tag{4.6}$$

Example 4.2.3. For a tree $m_0(x) = 0$ and $m_{-1}(x) = 1$ for all vertices x so that

$$\frac{m_{+1}(x) - m_{-1}(x)}{m(x)} = \frac{m(x) - 2}{m(x)} = 1 - \frac{2}{m(x)}.$$

Therefore, if $m(x) \geq m > 2$ then $\frac{m_{+1}(x) - m_{-1}(x)}{m(x)} \geq \left(1 - \frac{2}{m}\right)$. Hence $c = \frac{m-2}{m}$, for such a tree, implying

$$\lambda_0(\Delta_{bd}) \geq \frac{(m-2)^2}{2m^2}$$

and

$$\lambda_0(\Delta) \geq \frac{(m-2)^2}{2m}.$$

Proof: Let $D \subset G$ be a finite, connected subgraph disjoint from A. Let $r(x) = d(x, x_0)$ where x_0 is a fixed vertex of G. Then, by applying the analogue of Green's Theorem, it follows that

$$\begin{aligned} \left|\sum_{x\in D} \Delta_{bd} r(x) m(x)\right| &= \left|\sum_{x\in D} \Delta r(x)\right| \\ &= \left|\sum_{\substack{y\sim x \\ x\in D, y\notin D}} \big(r(x) - r(y)\big)\right| \\ &\leq \sum_{\substack{y\sim x \\ x\in D, y\notin D}} |r(x) - r(y)| \\ &\leq L(\partial D) \end{aligned} \tag{4.7}$$

since $r(x) - r(y)$ can only be ± 1 or 0 if $y \sim x$.

On the other hand,

$$\begin{aligned} \Delta_{bd} r(x) &= r(x) - \frac{1}{m(x)}\Big(m_0(x) r(x) + m_{+1}(x)\big(r(x)+1\big) + m_{-1}(x)\big(r(x)-1\big)\Big) \\ &= \frac{m_{-1}(x) - m_{+1}(x)}{m(x)} \end{aligned}$$

since $m(x) = m_0(x) + m_{+1}(x) + m_{-1}(x)$. Therefore, it follows from the assumption $\frac{m_{+1}(x) - m_{-1}(x)}{m(x)} \geq c > 0$ that $\Delta_{bd} r(x) < 0$ and

$$|\Delta_{bd} r(x)| \geq c$$

for all vertices $x \in G \setminus A$. Therefore,

$$\begin{aligned} \left|\sum_{x\in D} \Delta_{bd} r(x) m(x)\right| &= \sum_{x\in D} |\Delta_{bd} r(x) m(x)| \\ &\geq c \sum_{x\in D} m(x) = cA(D). \end{aligned} \tag{4.8}$$

Combining inequalities (4.7) and (4.8) it follows that

$$cA(D) \leq L(\partial D)$$

or

$$c \leq \frac{L(\partial D)}{A(D)}$$

for all finite, connected subgraphs D disjoint from A. Applying Theorem 4.2.1 as formulated in (4.5) it follows that

$$\lambda_0(\Delta_{bd,G\setminus A}) \geq \frac{c^2}{2}. \tag{4.9}$$

This gives the first part of Theorem 4.2.2.

For the second part, we proceed as follows. By using the Rayleigh-Ritz characterization of $\lambda_0(\Delta_{bd})$ inequality (4.9) implies

$$\langle \Delta_{bd} f, f \rangle_{bd} \ \geq \frac{c^2}{2} \langle f, f \rangle_{bd}$$

for every nonzero function on the graph G finitely supported disjoint from A. Now,

$$\begin{aligned} \langle \Delta_{bd} f, f \rangle_{bd} &= \sum_{x \in V} \Delta_{bd} f(x) f(x) m(x) \\ &= \sum_{x \in V} \Delta f(x) f(x) \ = \ \langle \Delta f, f \rangle \end{aligned}$$

while, if $m(x) \geq m$, then

$$\begin{aligned} \frac{c^2}{2} \langle f, f \rangle_{bd} &= \frac{c^2}{2} \sum_{x \in V} f^2(x) m(x) \\ &\geq \frac{mc^2}{2} \sum_{x \in V} f^2(x) \ = \ \frac{mc^2}{2} \langle f, f \rangle. \end{aligned}$$

Therefore, it follows that

$$\frac{\langle \Delta f, f\rangle}{\langle f, f\rangle} \geq \frac{mc^2}{2}$$

and taking the infimum over the set of all finitely supported, nonzero functions f on $G \setminus A$ it follows that

$$\lambda_0(\Delta_{G\setminus A}) \geq \frac{mc^2}{2}.$$

□

4.3 Essential Spectrum

We now use Theorem 4.2.2 to prove that, under certain assumptions on the graph, $\tilde{\Delta}$, the unique self-adjoint extension of Δ with domain $C_0(V)$ in $\ell^2(V)$, has empty essential spectrum as in [8, Theorem 1.1].

By definition, the essential spectrum is the complement in the spectrum of the set of isolated eigenvalues of finite multiplicity. We use the notation $\sigma(\tilde{\Delta})$ and $\sigma_{ess}(\tilde{\Delta})$ for the spectrum and essential spectrum of $\tilde{\Delta}$ respectively. Now, as pointed out in [13, Theorem VII.12 and remarks following Theorem VIII.6], the essential spectrum of a self-adjoint operator can be characterized as follows

Theorem 4.3.1. *$\lambda \in \sigma_{ess}(\tilde{\Delta})$ if and only if there exists an sequence of orthonormal function $\{f_i\}_{i=0}^{\infty}$ in the domain of $\tilde{\Delta}$ such that*

$$\lim_{i\to\infty} \|\tilde{\Delta} f_i - \lambda f_i\|_{\ell^2} = 0.$$

In fact, it is sufficient that the sequence is noncompact, that is, have no convergent subsequence, and this will be the characterization of the essential spectrum that we use to prove the lemma below.

Fix a vertex x_0 and let $\underline{m}_c(r)$ denote the infimum of the valence of vertices outside of $B_r(x_0)$

$$\underline{m}_c(r) = \inf_{x \in G \setminus B_r(x_0)} m(x).$$

We can then state

Theorem 4.3.2. *If, for all vertices x of G,*

$$\frac{m_{+1}(x) - m_{-1}(x)}{m(x)} \geq c > 0$$

and

$$\underline{m}_c(r) \to \infty \text{ as } r \to \infty$$

then $\tilde{\Delta}$ has empty essential spectrum.

Example 4.3.3. In the case of a tree, we note that if $m(x) \geq m > 2$ for all vertices x then the first assumption is satisfied and so if, in addition, $\underline{m}_c(r) \to \infty$ then $\tilde{\Delta}$ will have empty essential spectrum.

The proof of Theorem 4.3.2 will follow easily once we establish the following lemma which is analogous to [8, Proposition 2.1] and apply the second result of Theorem 4.2.2. Let $\tilde{\Delta}_{G \setminus B_r}$ denote the self-adjoint extension of the Laplacian acting on $C_0(V, B_r)$, the space of functions with finite support which vanish on B_r, to $\ell^2(V, B_r)$, the square summable functions which vanish on B_r. We then have that

Lemma 4.3.1. $\sigma_{ess}(\tilde{\Delta}) = \sigma_{ess}(\tilde{\Delta}_{G \setminus B_r})$.

Assuming the lemma for now we prove the theorem:

Proof of Theorem 4.3.2: By applying (4.6) from Theorem 4.2.2 it follows that

$$\lambda_0(\tilde{\Delta}_{G \setminus B_r}) \to \infty \text{ as } r \to \infty$$

since $\underline{m}_c(r) \to \infty$. Now, applying Lemma 4.3.1, since the essential spectrum of $\tilde{\Delta}$ is the same as that of $\tilde{\Delta}_{G\setminus B_r}$ and the bottom of the spectrum of $\tilde{\Delta}_{G\setminus B_r}$ is increasing to infinity it must follow that the essential spectrum of $\tilde{\Delta}$ is empty. $\square$

Proof of Lemma 4.3.1: Let $\lambda \in \sigma_{ess}(\tilde{\Delta}_{G\setminus B_r})$ and let $\{f_i\}_{i=0}^{\infty}$ be a sequence of orthonormal functions vanishing on B_r and satisfying

$$\lim_{i\to\infty} \|\tilde{\Delta}_{G\setminus B_r} f_i - \lambda f_i\|_{\ell^2} = 0.$$

Then, since,

$$\Delta_{G\setminus B_r} f_i(x) \neq \Delta f_i(x) \text{ only for } x \in \partial B_r$$

and, by orthonormality, for every vertex x, $f_i(x) \to 0$ as $i \to \infty$ it follows that

$$\lim_{i\to\infty} \|\tilde{\Delta} f_i - \lambda f_i\|_{\ell^2} = 0$$

so that $\lambda \in \sigma_{ess}(\tilde{\Delta})$.

Now, assume that $\lambda \in \sigma_{ess}(\tilde{\Delta})$ and $\{f_i\}_{i=0}^{\infty}$ is a sequence of orthonormal function in the domain of $\tilde{\Delta}$ such that

$$\|\tilde{\Delta} f_i - \lambda f_i\|_{\ell^2} \to 0 \text{ as } i \to \infty.$$

Let

$$\varphi_r(x) = \begin{cases} 0 & \text{if } x \in B_r(x_0) \\ 1 & \text{otherwise} \end{cases}$$

We claim that $\{\varphi_r f_i\}_{i=0}^{\infty}$ will be a sequence of bounded functions with no convergent subsequence satisfying

$$\|\tilde{\Delta}_{G\setminus B_r}(\varphi_r f_i) - \lambda(\varphi_r f_i)\|_{\ell_2} \to 0 \text{ as } i \to \infty.$$

To show that $\{\varphi_r f_i\}_{i=0}^{\infty}$ has no convergent subsequences we first note that, since $\{f_i\}_{i=0}^{\infty}$ are orthonormal, $\{f_i\}_{i=0}^{\infty}$ has no convergent subsequences. This follows since pointwise, using orthonormality as above,

$$f_i(x) \to 0 \text{ as } i \to \infty \quad \text{for all } x \in V$$

while $\|f_i\|_{\ell^2} = 1$ for all i. Now, assume that $\{\varphi_r f_i\}_{i=0}^{\infty}$ has a convergent subsequence, say

$$\varphi_r f_{i_k} \to f \text{ in } \ell_2 \text{ as } k \to \infty.$$

Since $f_i \in \ell_2(V)$, $\{f_{i_k}(x)\}_{k=0}^{\infty}$ has a convergent subsequence for each x. Because B_r has only finitely many vertices we can find a subsequence of $\{f_{i_k}\}_{k=0}^{\infty}$ which converges for each $x \in B_r$. We continue to denote this subsequence as $\{f_{i_k}\}_{k=0}^{\infty}$ and let $\hat{f}(x)$ be defined by

$$\hat{f}(x) = \begin{cases} f(x) & \text{for } x \notin B_r \\ \lim_{k\to\infty} f_{i_k}(x) & \text{for } x \in B_r \end{cases}$$

Then, it follows that

$$\begin{aligned} \|f_{i_k} - \hat{f}\|_{\ell_2}^2 &= \sum_{x \in V} \left(f_{i_k}(x) - \hat{f}(x)\right)^2 \\ &= \sum_{x \notin B_r} \left(f_{i_k}(x) - f(x)\right)^2 + \sum_{x \in B_r} \left(f_{i_k}(x) - \hat{f}(x)\right)^2 \\ &\to 0 \text{ as } k \to \infty. \end{aligned}$$

so that $\{f_i\}_{i=0}^{\infty}$ would have a convergent subsequence. The contradiction shows that $\{\varphi_r f_i\}_{i=0}^{\infty}$ cannot have a convergent subsequence.

It remains to show that $\|(\tilde{\Delta}_{G\setminus B_r} - \lambda I)(\varphi_r f_i)\|_{\ell^2} \to 0$ as $i \to \infty$. First, we calculate $\tilde{\Delta}_{G\setminus B_r}(\varphi_r f_i)$:

$$\begin{aligned}
\Delta_{G\setminus B_r}(\varphi_r f_i)(x) &= \sum_{y\sim x}\big(\varphi_r(x)f_i(x)-\varphi_r(y)f_i(y)\big)\\
&= \sum_{y\sim x}\big(\varphi_r(x)f_i(x)-\varphi_r(x)f_i(y)+\varphi_r(x)f_i(y)-\varphi_r(y)f_i(y)\big)\\
&= \varphi_r(x)\Delta_{G\setminus B_r}f_i(x)+\sum_{y\sim x}f_i(y)\big(\varphi_r(x)-\varphi_r(y)\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{x\in V}\big(\Delta_{G\setminus B_r}(\varphi_r f_i)(x)-\lambda(\varphi_r f_i)(x)\big)^2 \leq &\sum_{x\in V}\Big(\varphi_r(x)\big((\Delta_{G\setminus B_r}f_i)(x)-\lambda f_i(x)\big)\Big)^2\\
&+\sum_{x\in V}\sum_{y\sim x}\Big(f_i(y)\big(\varphi_r(x)-\varphi_r(y)\big)\Big)^2\\
=&\sum_{x\notin B_r}\big((\Delta f_i)(x)-\lambda f_i(x)\big)^2\\
&+\sum_{x\in\partial B_r}\sum_{\substack{y\sim x\\ y\notin B_r}}\big(f_i^2(x)+f_i^2(y)\big).
\end{aligned}$$

The first sum above converges to 0 as $i\to\infty$ by the assumption on f_i while, since f_i are orthonormal, $f_i(x)\to 0$ as $i\to\infty$ for each x, so it follows that the second sum also converges to 0. Therefore,

$$\|\tilde{\Delta}_{G\setminus B_r}(\varphi_r f_i)-\lambda(\varphi_r f_i)\|_{\ell_2}\to 0 \text{ as } i\to\infty$$

implying that $\lambda\in\sigma_{ess}(\tilde{\Delta}_{G\setminus B_r})$. □

Bibliography

[1] Cheeger J., Yau S.-T., *A Lower Bound for the Heat Kernel*, Communications in Pure and Applied Mathematics, **34**, No. 4 (1981) p. 465 – 480.

[2] Dodziuk J., *Maximum Principle for Parabolic Inequalities and the Heat Flow on Open Manifolds*, Indiana University Math Journal, **32**, No. 5 (1983) p. 703 – 716.

[3] Dodziuk J., *Difference Equations, Isoperimetric Inequality and Transience of Certain Random Walks*, Transactions of the American Mathematical Society, **284**, No. 2 (1984) p. 787 – 794.

[4] Dodziuk J., *Elliptic operators on infinite graphs*, in: Analysis, Geometry and Topology of Elliptic Operators, World Scientific Publishing, Hackensack, NJ (2006) p. 353 – 368.

[5] Dodziuk J., Karp L., *Spectral and Function Theory for Combinatorial Laplacians*, in: Geometry of Random Motion, Ithaca, NY, Contemporary Mathematics, **73**, Providence, RI (1988) p. 25 – 40.

[6] Dodziuk J., Kendall W.S., *Combinatorial Laplacians and Isoperimetric Inequality*, in: From Local Times to Global Geometry, Control and Physics, K.D. Ellworthy Ed., Pitman Research Notes in Mathematics, **150** (1986) p. 68 – 74.

[7] Dodziuk J., Mathai V. *Kato's Inequality and Asymptotic Spectral Properties for Discrete Magnetic Laplacians*, in: The Ubiquitous Heat Kernel, Contemporary Mathematics, American Mathematical Society, **398** Providence, RI (2006) p. 69 – 81.

[8] Donnelly H., Li P., *Pure Point Spectrum and Negative Curvature for Noncompact Manifolds*, Duke Mathematical Journal, **46**, No. 3 (1979) p. 497 – 503.

[9] Fischer-Colbrie D., Schoen R., *The Structure of Complete Stable Minimal Surfaces in 3-Manifolds of Nonnegative Scalar Curvature*, Communications in Pure and Applied Mathematics, **33**, No. 2 (1980) p. 199 – 211.

[10] Grigor'yan A., *Analytic and Geometric Background of Recurrence and Non-Explosion of the Brownian Motion on Riemannian Manifolds*, Bulletin of the American Math Society, **36**, No. 2 (1999) p. 135 – 249.

[11] Jost J., *Partial Differential Equations*, Springer-Verlag, 2002.

[12] Milnor J., *On Deciding Whether a Surface is Parabolic or Hyperbolic*, The American Mathematical Monthly, **84**, No. 1 (1977) p. 43 – 46.

[13] Reed M., Simon, B., *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press, New York, NY 1972.

[14] Reed M., Simon, B., *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness*, Academic Press, New York, NY 1975.

[15] Sullivan D., *Related Aspects of Positivity: λ-Potential Theory on Manifolds, Lowest Eigenstates, Hausdorff Geometry, Renormalized Markoff Processes...*, in: Aspects of Mathematics and its Applications, North-Holland Math Library, **34** North Holland, Amsterdam (1986) p. 747 – 779.